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A New Laplace Second-Order Autoregressive Time-Series Model—NLAR(2)

LEE S. DEWALD AND PETER A. W. LEWIS

Abstract—**A time-series model for Laplace (double-exponential) variables having second-order autoregressive structure (NLAR(2)) is presented. The model is Markovian and extends the second-order process in exponential variables, NEAR(2), to the case where the marginal distribution is Laplace. The properties of the Laplace distribution make it useful for modeling in some cases where the normal distribution is not appropriate. The time-series model has four parameters and is easily simulated. The autocorrelation function for the process is derived as well as third-order moments to further explore dependency in the process. The model can exhibit a broad range of positive and negative correlations and is partially time reversible. Joint distributions and the distribution of differences are presented for the first-order case NLAR(1).**

I. Introduction

IN STANDARD time-series analysis, one assumes the marginal distributions of $\{X_n\}$ are normal. However, a Gaussian distribution will not always be appropriate. In earlier works stationary non-Gaussian time-series models were developed for variables with positive and highly skewed marginal distributions [1]–[6].

Other situations still remain for which Gaussian marginals are inappropriate, i.e., where the marginal time-series variable being modeled, although not skewed or inherently positive valued, has a large kurtosis or long-tailed distribution. The position errors in a large navigation system have such a distribution. In particular, Hsu [7] modeled pooled position errors using the double exponential distribution. McGill [8] showed that the Laplace distribution provides a characterization of the error in a timing device under periodic excitation. Again, speech-waves are modeled using Laplace variables [9]. In the "speech-like" process given by the linear autoregressive (AR(1)) model

$$X_n = cX_{n-1} + (1 - c^2)^{1/2} E_n, \tag{1.1}$$

where $0.8 \leq c \leq 0.9$, the innovation sequence $\{E_n\}$ is independent and identically distributed (i.i.d.) Laplace [10]. In image coding systems using a two-dimensional discrete cosine transform, Reininger and Gibson [11] showed that the Laplace distribution gives the best approximation to the distribution of the non-DC coefficients. Recently Sethia and Anderson [12] required a stationary autoregressive process with Laplace marginals in their research in communications technology.

Some of the special properties of the double exponential distribution are discussed briefly in the next section.

The approach taken in this paper to develop a family of Laplacian time-series models (NLAR(2)) follows that of the earlier work for the new exponential autoregressive (NEAR) model in [4] and [6]. In Section III we establish the validity of the four-parameter, Markovian, random-coefficient linear model by analyzing the innovation structure. It is shown that a convex combination of three-scaled Laplace variables can be combined with an independent pair of Laplace variables to obtain another Laplace variable. Necessary and sufficient conditions for the existence of the NLAR(2) model are given using results of Nicholls and Quinn [13].

The random-coefficient approach is not the only way to generate Laplace variables with a specified correlation structure. The literature contains numerous articles on generation of random sequences. One approach put forth in several papers [14]–[18] involves passing white Gaussian noise through a linear filter followed by a zero-memory nonlinear transform. This is a general procedure that produces exactly the required marginal distribution and a good approximation to the autocorrelation structure. However, the scheme lacks the simplicity of the method being proposed, which is just a random-coefficient linear combination of Laplacian random variables. Moreover, the filtering approach produces, for example, in the first-order autoregressive case, only *one* process. It is important to note that in nonnormal time series there are infinitely many processes with a given marginal and autocorrelation structure. The NLAR(2) does this; the difference in the various NLAR(2) processes can, for instance, be explored through third and fourth joint moments.

The NLAR(2) time-series model provides great flexibility to systems modeling because of the broad range of correlations and dependence structure that can be obtained with the use of the four parameters. We demonstrate in Section V that the correlations $\{\rho(l)\}$ satisfy Yule–Walker-type equations as in the AR(2) process. We investigate the parameter space within which the NLAR(2) model is valid.

Finally, in Section VI we demonstrate the high degree of symmetry underlying the NLAR(2) model by showing that $E(X_i X_j X_k) = 0$ for all i, j, k. This property is useful in

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model fitting and in determination of reversibility/directionality in the model. We show that further analysis of the residuals as given by Lawrence and Lewis [19] is necessary to further address directionality in the NLAR(2) model.

Several cases of the NLAR(2) family are analogous to those developed in [6] for the NEAR(2) model, which will not be listed here. However, the results for the first-order NLAR(1) model presented in Section IV are new.

II. Special Properties of the Laplace Distribution

The Laplace distribution is also known as the double exponential. In general, the density of a Laplace distributed variable L has two parameters—a location parameter $-\infty < \mu < +\infty$ and a scale parameter $\lambda > 0$. The parameter μ is fixed here at zero. For $-\infty < x < \infty$ we have

$$f_L(x;\lambda) = \frac{1}{2\lambda}\exp\left(-|x|/\lambda\right). \tag{2.1}$$

In what follows we will define $\{L_n\}$ as a sequence of i.i.d. random variables of the Laplace distribution with $\lambda = 1$ (standard Laplace). The characteristic function of the standard Laplace variable is

$$\phi_L(\omega) = \frac{1}{1+\omega^2}, \qquad -\infty < \omega < \infty, \tag{2.2}$$

and we have

$$E(L^n) = \begin{cases} 0, & \text{if } n \text{ is odd} \\ n!, & \text{if } n \text{ is even,} \end{cases} \tag{2.3}$$

so that $E(L) = 0$, $\operatorname{var}(L) = 2$, skewness is zero, and kurtosis is 3. The value of the kurtosis indicates that the symmetric Laplace distribution has heavier tails than the normal distribution, for which the kurtosis is 0.

The sum $Y = \sum_{i=1}^{n} L_i$ of $n \geq 2$ i.i.d. Laplace variables can be written as the difference of two i.i.d. random variables Y_1, Y_2 with Gamma distribution, shape parameter n, and scale parameter 1. This follows immediately from the characteristic function

$$\begin{aligned}\phi_Y(\omega) &= (1+\omega^2)^{-n} = (1+i\omega)^{-n}(1-i\omega)^{-n} \\ &= \phi_{Y_1}(\omega)\phi_{Y_2}(-\omega).\end{aligned} \tag{2.4}$$

When $n = 1$ a Laplace variable is the difference of two i.i.d. exponential variables. This makes it simple to generate Laplace variates in computer simulations. Replacing n by $t > 0$ in (2.4), we see that $[\phi_L(\omega)]^t$ is the characteristic function for the variable $Y_1 - Y_2$, where $Y_i \sim \text{Gamma}(t,1)$, $i = 1,2$ and Y_1 and Y_2 are independent. This demonstrates that the Laplace distribution is infinitely divisible.

Random variables with a standard Laplace distribution are self-decomposable. Let

$$\phi_\epsilon(\omega) = \phi_L(\omega)/\phi_L(\rho\omega), \qquad 0 \leq \rho < 1. \tag{2.5}$$

According to Feller [20, p. 588], if $\phi_\epsilon(\omega)$ is the transform of a random variable for each $0 \leq \rho < 1$, then L is said to be self-decomposable. But

$$\begin{aligned}\phi_\epsilon(\omega) &= \left\{1 + (\rho\omega)^2\right\}(1+\omega^2)^{-1} \\ &= \left\{\rho + (1-\rho)(1-i\omega)^{-1}\right\} \\ &\quad \cdot \left\{\rho + (1-\rho)(1+i\omega)^{-1}\right\} \qquad (2.6) \\ &= \rho^2 + (1-\rho^2)(1+\omega^2)^{-1}. \qquad (2.7)\end{aligned}$$

We recognize (2.6) as the product of the characteristic functions of two i.i.d. innovation variables ϵ_1 and $-\epsilon_2$ as described in the EAR(1) process in [1]. Also, from (2.7)

$$\epsilon = \begin{cases} 0, & \text{w.p. } \rho^2, \\ L, & \text{w.p. } 1-\rho^2. \end{cases} \tag{2.8}$$

Thus ϵ is the solution of a first-order liner autoregressive equation $X_n = \rho X_{n-1} + \epsilon_n$, where $\{X_n\}$ is a stationary time series with double exponential marginal distribution for all n. We call this the LAR(1) model. It has the same properties as the EAR(1) model in [1] with two important differences. First, if $-1 < \rho < 0$ negative serial correlations for odd lags are obtained. Secondly it is partially time reversible in the sense that for all l and n, both of the following are true:

$$E\left(X_n^2 X_{n+l}\right) = E\left(X_n X_{n+l}^2\right) = 0, \tag{2.9}$$

$$P\left(X_n \geq X_{n-1}\right) = P\left(X_n \leq X_{n-1}\right) = 1/2. \tag{2.10}$$

These results are derived in Sections IV and VI. Note, however, that since LAR(1) is a linear AR(1) model with non-Gaussian innovation $\{\epsilon_n\}$, it is not fully time reversible [21]. Finally, note that this LAR(1) model has the zero-defect property; when $\epsilon_n = 0$ then $X_n/X_{n-1} = \rho$ and ρ can be determined exactly in long enough runs of the series $\{X_n\}$. This property is generally undesirable, but the broader NLAR(2) model developed in the next section is free of this defect, except for the special parameter values for which it reduces to the LAR(1) model.

III. A Second-Order Autoregressive Laplace Time-Series Model (NLAR(2))

Following the terminology in [4], [5], and [6], we propose the following time-series model, called the new Laplace second-order autoregressive model (NLAR(2)). The NLAR(2) model has four parameters, double-exponential marginal distribution for $\{X_n\}$, second-order autoregressive Markov dependence, and autocorrelations satisfying Yule–Walker-type equations.

The stationary NLAR(2) model has the same form as the stationary NEAR(2) model in [6]. Writing the time series $\{X_n\}$ in the form of an additive, linear, random-coefficient autoregressive process, we have for all n that

$$X_n = \beta_1 K_n' X_{n-1} + \beta_2 K_n'' X_{n-2} + \epsilon_n, \tag{3.1}$$

where $\{K_n', K_n''\}$ is a sequence of i.i.d. discrete bivariate random variables with distribution

$$\{K_n', K_n''\} = \begin{cases} (1,0), & \text{w.p. } \alpha_1, \\ (0,1), & \text{w.p. } \alpha_2, \\ (0,0), & \text{w.p. } 1-\alpha_1-\alpha_2, \end{cases} \tag{3.2}$$

$n = 0, \pm 1, \pm 2, \cdots$; $\{\epsilon_n\}$ is an i.i.d. innovation sequence whose distribution is given in (3.7), and $\{\epsilon_n\}$ and $\{K_n', K_n''\}$ are mutually independent and independent of $\{X_n\} = X_m$, $m = n-1, n-2, \cdots$. The parameter space is defined by $0 \le |\beta_i| \le 1$ and $0 \le \alpha_i \le 1$, $i = 1, 2$; $\alpha_1 + \alpha_2 \le 1$. Graphs of the admissible regions in the parameter space and the correlation space are presented in Section V.

Equations (3.1) and (3.2) have a direct physical interpretation. The observed process at time n, X_n, is only one of three possibilities: 1) X_n is some multiple of what it was at time $n-1$, $\beta_1 X_{n-1}$, plus some independent random noise ϵ_n; 2) X_n is some multiple (possibly different than β_1) of its value at time $n-2$, $\beta_2 X_{n-2}$, plus some independent random noise; and 3) X_n is just random noise ϵ_n independent of everything up to time n.

The work of Nicholls and Quinn [13] on random-coefficient autoregressive models is relevant to the NLAR(2) process. They have given the necessary and sufficient conditions for the existence of the *unique* covariance stationary solution to the following class of univariate random-coefficient autoregressive (RCA) models of order k, RCA(k),

$$Z_n = \sum_{i=1}^{k} \{\gamma_i + B_n(i)\} Z_{n-i} + \epsilon_n, \tag{3.3}$$

$n = 0, \pm 1, \pm 2, \cdots$, where the following conditions hold.

1) The γ_i are real constants.

2) $\{\boldsymbol{B}_n\}$ is a k-vector, second-order stationary, independent process with $E(\boldsymbol{B}_n) = \boldsymbol{0}$ and constant covariance matrix.

3) $\{\epsilon_n\}$ is a scalar second-order stationary, independent process, independent of $\{\boldsymbol{B}_n\}$, with $E(\epsilon_n^2) = \sigma^2$ for all n.

They also have shown that if $\{\boldsymbol{B}_n\}$ and $\{\epsilon_n\}$ are i.i.d. processes, then the solution $\{Z_n\}$ is strictly stationary and ergodic.

Let $\gamma_i = \alpha_i \beta_i$ for $i = 1, 2$ and $B_n(1) = \beta_1(K_n' - \alpha_1)$ and $B_n(2) = \beta_2(K_n'' - \alpha_2)$. Then (3.1) and (3.3) have the same form. That is, (3.1) forms an RCA(2) model if the innovation of NLAR(2) satisfies condition 3). Thus applying the results in [13, pp. 31, 37], there exists a *unique* strictly stationary and ergodic solution to (3.3) for γ_i and $B_n(i)$ as defined above, if and only if all of the roots of the characteristic equation

$$\left(t^2 - \alpha_1\beta_1^2 t - \alpha_2\beta_2^2\right)\left(t^2 - \alpha_2\beta_2^2\right) = 0 \tag{3.4}$$

are within the unit circle, i.e., if and only if $\alpha_1\beta_1^2 + \alpha_2\beta_2^2 < 1$. This is satisfied for the conditions on the parameters defining NLAR(2), thus establishing the existence of the model (3.1).

No marginal distribution is ascribed to solutions of the general RCA(k) models in [13]. It is, in fact, determined by the independent choices of the innovation and the random coefficients. However, by specifying the marginal distribution and the random coefficients, we restrict the innovation more than the RCA(k) model does. If the X_n in (3.1) or Z_n in (3.3) have a standard Laplace marginal distribution, then all their moments are given by (2.3). From (3.1) or (3.3) it follows that for all $k = 1, 2, \cdots$

$$E(\epsilon_n^{2k}) = \{(2k)!\}\Bigg[1 - \left(\alpha_1\beta_1^{2k} + \alpha_2\beta_2^{2k}\right) - \sum_{i=1}^{k-1}\left\{\left(\alpha_1\beta_1^{2(k-i)} + \alpha_2\beta_2^{2(k-i)}\right) \cdot E\left(\epsilon_n^{2(k-i)}\right)/(2i)!\right\}\Bigg] > 0, \tag{3.5}$$

and for this to be true it is necessary that

$$\alpha_1\beta_1^{2k} + \alpha_2\beta_2^{2k} < 1. \tag{3.6}$$

Since α_1 and α_2 are probabilities it is necessary that $|\beta_i| \le 1$ for $i = 1, 2$ for (3.6) to hold. If not there exists for every α_1 and α_2 an integer m such that $\alpha_1\beta_1^{2m}$ or $\alpha_2\beta_2^{2m}$ is greater than 1.

We have now established the necessary conditions on the innovation $\{\epsilon_n\}$, and on β_1 and β_2 for the existence of a unique strictly stationary solution to (3.3) with a marginal Laplace distribution and with the random coefficients given by (3.2). In Theorem 1 we show that $|\beta_i| \le 1$ for $i = 1, 2$ is also a sufficient condition and that such an innovation random variable ϵ_n exists. We also give its explicit form—a convex combination of Laplace random variables. For simplicity we regard the parameter space as being described by strict inequalities for α_i and β_i.

Theorem 1: Let $\{X_n\}$ be a stationary process with standard Laplace marginal distribution. For all n let (3.1) and (3.2) hold with $0 < |\beta_i| < 1$, $0 < \alpha_i < 1$ for $i = 1, 2$ and $\alpha_1 + \alpha_2 < 1$. Then

$$\epsilon_n = K_n L_n = \begin{cases} L_n, & \text{w.p. } 1 - p_2 - p_3, \\ |b_2| L_n & \text{w.p. } p_2, \\ |b_3| L_n & \text{w.p. } p_3, \end{cases} \tag{3.7}$$

where $\{L_n\}$ are i.i.d. standard Laplace variates; the K_n have values in $\{1, |b_2|, |b_3|\}$ and are independent of $\{X_n\}$, $\{L_n\}$, $\{K_n', K_n''\}$ for all n. Furthermore

$$p_2 = \frac{\left\{\left(\alpha_1\beta_1^2 + \alpha_2\beta_2^2\right)b_2^2 - (\alpha_1+\alpha_2)\beta_1^2\beta_2^2\right\}}{\left(b_2^2 - b_3^2\right)\left(1 - b_2^2\right)}, \tag{3.8}$$

$$p_3 = \frac{\left\{(\alpha_1+\alpha_2)\beta_1^2\beta_2^2 - \left(\alpha_1\beta_1^2 + \alpha_2\beta_2^2\right)b_3^2\right\}}{\left(b_2^2 - b_3^2\right)\left(1 - b_3^2\right)}, \tag{3.9}$$

$$1 > b_2^2 = \tfrac{1}{2}\left\{s + \left(s^2 - 4r\right)^{1/2}\right\} > b_3^2 = \tfrac{1}{2}\left\{s - \left(s^2 - 4r\right)^{1/2}\right\} > 0, \tag{3.10}$$

$$s = (1-\alpha_1)\beta_1^2 + (1-\alpha_2)\beta_2^2, \tag{3.11}$$

$$r = (1 - \alpha_1 - \alpha_2)\beta_1^2\beta_2^2. \tag{3.12}$$

The proof of Theorem 1 closely follows the one in [8] for the NEAR(2) model and is not given here.

Many special cases of the NLAR(2) model could be mentioned. The following have one or more of the parameters at their boundary value and have valid but less complicated results for the distribution of $\{\epsilon_n\}$ in (3.7). If $\alpha_1 = \alpha_2 = 0$ then $\{\epsilon_n\}$ is the i.i.d. sequence $\{L_n\}$ and $X_n = \epsilon_n$. If $\alpha_1 = 1$ then $\{\epsilon_n\}$ is the innovation of the LAR(1) model derived from (2.7) and (2.8). If $|\beta_1| = |\beta_2| = 1$ and $\alpha_1 + \alpha_2 < 1$, then each ϵ_n is distributed as a scaled Laplace random variable, $\sqrt{1-\alpha_1-\alpha_2}\,L_n$. This model is called the TLAR(2) model, which is easily extendable to higher-order autoregressions. If $\alpha_1 < 1$ and $\alpha_2 = 0$ or $\beta_2 = 0$, then $\{\epsilon_n\}$ is the innovation of the new first-order autoregressive model NLAR(1). This model is the subject of the next section.

IV. The NLAR(1) Model

The NLAR(1) model is the first-order autoregressive version of NLAR(2). The two-parameter model ($\alpha_2 = 0$ and, or $\beta_2 = 0$ in (3.1)) is

$$X_n = K_n'\beta_1 X_{n-1} + \epsilon_n, \tag{4.1}$$

where

$$K_n' = \begin{cases} 1, & \text{w.p. } \alpha_1 \\ 0, & \text{w.p. } 1-\alpha_1 \end{cases}$$

$$\epsilon_n = \begin{cases} L_n, & \text{w.p. } 1-p_2 \\ \sqrt{1-\alpha_1}\,|\beta_1| L_n, & \text{w.p. } p_2 \end{cases} \tag{4.2}$$

$$p_2 = \alpha_1\beta_1^2/\{1-(1-\alpha_1)\beta_1^2\}. \tag{4.3}$$

From (4.2) and (4.3) we see that the inversion of the characteristic function for ϵ_n, letting $\lambda = (1-\alpha_1)^{-1/2}|\beta_1|^{-1}$, gives

$$f_{\epsilon_n}(x) = \tfrac{1}{2}(1-p_2)e^{-|x|} + \tfrac{1}{2}\lambda p_2 e^{-\lambda|x|}, \tag{4.4}$$

which is a convex mixture of Laplace densities.

To find the conditional density of X_n, given X_{n-1}, we use (4.1) through (4.4) to evaluate $P(X_n < x_n | X_{n-1})$. We have

$$\begin{aligned} P(X_n < x_n|X_{n-1}) &= P(K_n'\beta_1 X_{n-1} + \epsilon_n < x_n | X_{n-1}) \\ &= \alpha_1 P(\epsilon_n < x_n - \beta_1 x_{n-1}) \\ &\quad + (1-\alpha_1)P(\epsilon_n < x_n). \end{aligned} \tag{4.5}$$

Differentiating (4.5) with respect to x_n yields

$$f_{X_n|X_{n-1}}(x_n|x_{n-1}) = \alpha_1 f_{\epsilon_n}(x_n - \beta_1 x_{n-1}) + (1-\alpha_1) f_{\epsilon_n}(x_n). \tag{4.6}$$

Now we can write the joint density $f_{X_n X_{n-1}}$ as the product $f_{X_n|X_{n-1}} f_{X_{n-1}}$. In fact, the n-dimensional distribution of $X_1, \cdots, X_n$ is obtained by using this product recursively to obtain the density $f_{X_n \cdots X_1} = f_{X_n|X_{n-1}} f_{X_{n-1}|X_{n-2}} \cdots f_{X_2|X_1} f_{X_1}$.

We now consider the distribution of the difference $Z_n = X_n - X_{n-1}$. Using (4.1)–(4.4) and the fact that ϵ_n is a convex mixture of Laplacian random variables, we used partial-fraction decomposition to invert the characteristic function of Z_n to obtain, for the density,

$$\begin{aligned} f_{Z_n}(y) = {} & \exp\{-|y|/(1-\beta_1)\}\left(\frac{\alpha_1(1-\beta_1)}{2}\right) \\ & \cdot\left(\frac{p_2}{\{(1-\beta_1)^2 - \sigma^2\}} - \frac{(1-p_2)}{\beta_1(2-\beta_1)}\right) \\ & + \exp(-|y|/\sigma)(\sigma p_2/2) \\ & \cdot\left(\frac{\alpha_1}{\sigma^2 - (1-\beta_1)^2} - \frac{(1-\alpha_1)}{1-\sigma^2}\right) \\ & + \frac{1}{2}\exp(-|y|) \\ & \cdot\left(\frac{(1-\alpha_1)p_2}{1-\sigma^2} + \frac{(1-\alpha_1)(1-p_2)}{2} + \frac{\alpha_1(1-p_2)}{\beta_1(2-\beta_1)}\right) \\ & + (1-p_2)(1-\alpha_1)|y|\exp(-|y|)/4. \end{aligned} \tag{4.7}$$

One immediate result using (4.7) is that $f_{Z_n}(y)$ is symmetric about zero and therefore $P(Z_n < 0) = P(Z_n > 0) = 1/2$. This demonstrates one feature of the partial time reversibility of the NLAR models; i.e., probabilities of a run down ($X_n > X_{n-1}$) and a run up ($X_n < X_{n-1}$) are the same. To evaluate probabilities of higher-order runs would require the joint distribution of the sequence $\{Z_n\}$. More on reversibility in the sense of directional moments is presented in Section VI.

V. Autocorrelation Structure of the NLAR(2) Model

In this section we show that the autocorrelations $\rho(l) = \mathrm{Corr}(X_n, X_{n-l})$, $l = 0, \pm 1, \pm 2, \cdots$ of the NLAR(2) model satisfy the Yule–Walker-type difference equations; thus, the second-moment dependency aspects are indistinguishable in form from those for the AR(2) process. We also compare the admissible regions of an AR(2) with an NLAR(2) with four parameters and an NLAR(2) with only two parameters.

From the independence of $\{K_n\}$ and $\{K_n', K_n''\}$ and (3.1), (3.2), and (3.7), we see that $E(K_n') = \alpha_1$, $E(K_n'') = \alpha_2$, and $E(\epsilon_n) = E(K_n)E(L_n) = 0$. Multiplying (3.1) on both sides by X_{n-l}, we have for $l \geq 1$, $E(X_n X_{n-l}) = \alpha_1\beta_1 E(X_{n-1}X_{n-l}) + \alpha_2\beta_2 E(X_{n-2}X_{n-l})$. Dividing by $\mathrm{var}(X_n)$ we have $\rho(-l) = \alpha_1\beta_1\rho(l-1) + \alpha_2\beta_2\rho(l-2)$, since $\rho(-l) = \rho(l)$. Substituting $\alpha_i\beta_i = a_i$ for $i = 1, 2$ and $\rho(0) = 1$, we have

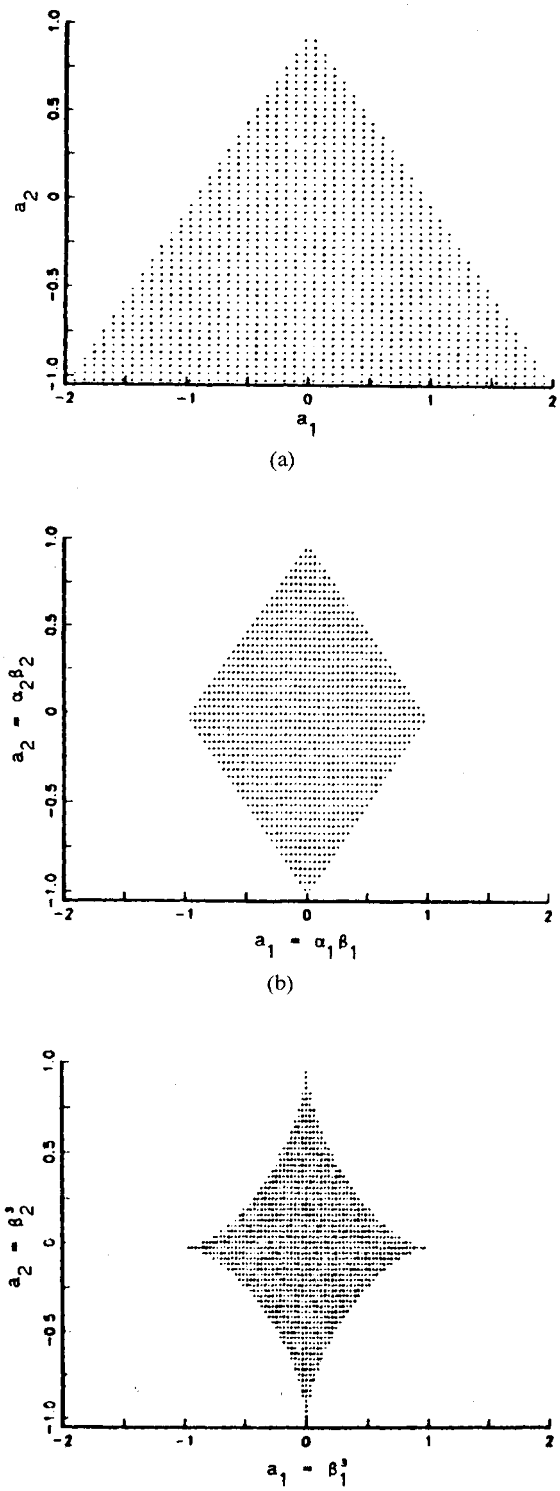
$$\begin{aligned} \rho(1) &= a_1 + a_2\rho(1) \\ \rho(2) &= a_1\rho(1) + a_2, \end{aligned} \tag{5.1}$$

which are the same equations as those that occur for the AR(2) process.

Since $|\beta_i| \leq 1$ for $i = 1, 2$ and $\alpha_1 + \alpha_2 \leq 1$ in NLAR(2), the usual triangular admissible region for AR(2) given in

[22, p. 61] shrinks to the interior of a diamond-shaped area in ($a_1 = \alpha_1\beta_1$, $a_2 = \alpha_2\beta_2$) coordinates: $|a_1| + |a_2| \le 1$ (See Figs. 1(a) and (b)). In ($\rho(1), \rho(2)$) coordinates the equation $\rho(1)^2 = (1 + \rho(2))/2$ defining allowable combinations of $\rho(1)$ and $\rho(2)$ in AR(2) also changes. For NLAR(2) the space in ($\rho(1), \rho(2)$) coordinates becomes a triangular region bounded below by $|\rho(1)| = (1/2)\{1 + \rho(2)\}$. (See Figs. 2(a) and (b)).

The reduction in allowable parameter or correlation combinations for NLAR(2) is not large. This encouraged us to consider a 2-parameter NLAR(2) model by specifying $\alpha_i = \beta_i^2$, for $i = 1, 2$, so that $a_i = \beta_i^3$. The parameter space



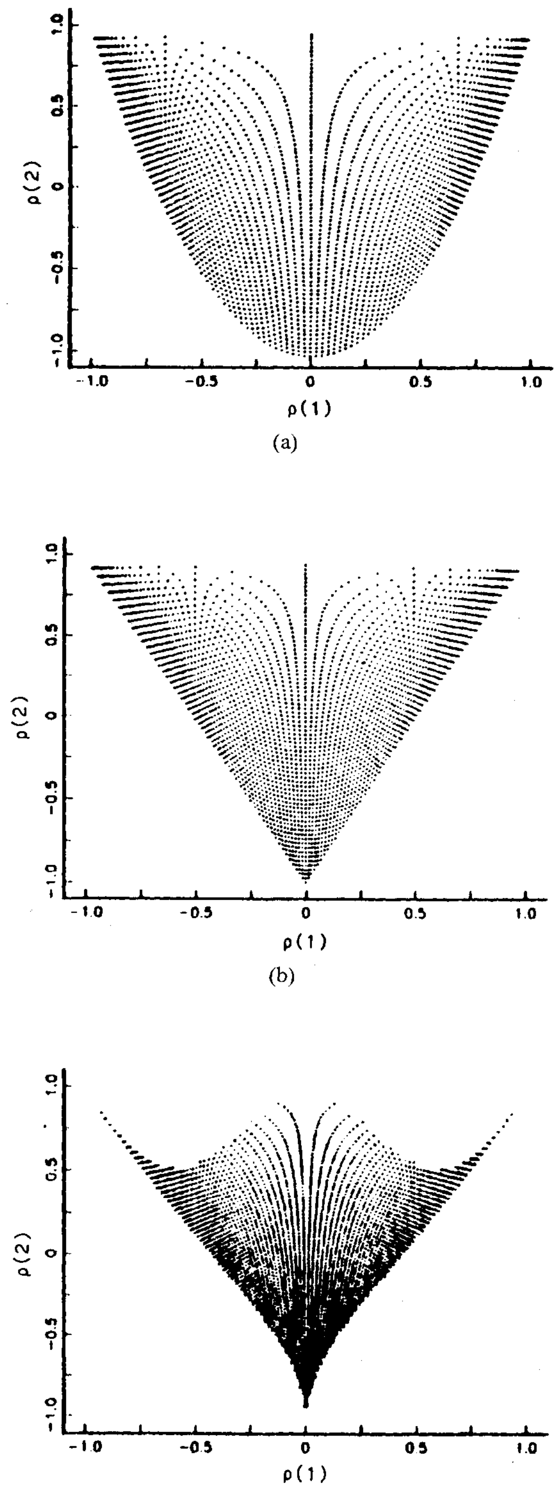


Fig. 1. (a) Admissible region in parameter coordinates for linear AR(1) model. (b) Admissible region in parameter coordinates for NLAR(2) model with four parameters. (c) Admissible region in parameter coordinates for NLAR(2) model with only two parameters.

Fig. 2. (a) Admissible region for $\rho(1)$ and $\rho(2)$ for linear AR(2) model. (b) Admissible region for $\rho(1)$ and $\rho(2)$ for NLAR(2) model with four parameters. (c) Admissible region for $\rho(1)$ and $\rho(2)$ for NLAR(2) model with two parameters.

in (a_1, a_2) coordinates becomes the symmetric region bounded by the curves $\beta_2^3 = \pm(1-\beta_1^2)^{3/2}$ (see Fig. 1(c)). In (β_1, β_2) coordinates the admissible region of the two-parameter model is bounded by the unit circle $\beta_1^2 + \beta_2^2 = 1$. Using only two parameters leads to the admissible region in Fig. 2(c) for $(\rho(1), \rho(2))$ space. The $(\rho(1), \rho(2))$ space was obtained by transforming the lines $\beta_2^3 = a_2 = c$, $-1 \le c \le 1$, in Fig. 1(c), to $\rho(2) = (1-a_2)\rho(1)^2 + a_2$, where $|\rho(1)| \le a_1/(1-a_2) = \beta_1^3/(1-\beta_2^3)$ and $\beta_2^3 = (1-\beta_1^2)^{3/2}$ if $a_2 \ge 0$ and $\beta_2^3 = -(1-\beta_1^2)^{3/2}$ if $a_2 < 0$.

All the plots in Fig. 1 were generated from a grid of equally spaced values of a_1 and a_2. In Fig. 1(a) the points satisfy the Yule–Walker equations (5.1). In Figs. 1(b) and (c) the points also satisfy the conditions of Theorem 1. In Fig. 2 the feasible combinations of $\rho(1)$ and $\rho(2)$ are plotted for those values of a_1 and a_2 from Fig. 1 using the Yule-Walker equations (5.1).

VI. Time Reversibility Assessed by Third-Order Moments in NLAR(2)

In Section V we demonstrated that the second-moment dependency aspects of the NLAR(2) model were indistinguishable in form from those of the ordinary AR(2) model. Also, it is well known that if the linear autoregressive model is not Gaussian, then the process is not completely determined by the first and second moments. Thus in model identification it becomes necessary to examine third-order moments to further identify the process. Special third-order moments $E(X_n^2 X_{n+l})$, for all l, are known as directional moments. If the directional moments for all l are equal, which is necessary for a process to be fully time reversible, we say the process is partially time reversible in the sense of directional moments.

A process is fully time reversible [23] if the joint distribution of $X_n, X_{n+1}, \cdots, X_{n+r}$, is the same as that for $X_{n+r}, X_{n+r-1}, \cdots, X_n$ for all r and for all n. Since LAR(1), a special case of NLAR(2), is not fully time reversible, NLAR(2) is in general not time reversible.

In this section we show by induction arguments that all the third-order moments of NLAR(2) are the same as those for Gaussian AR(2), i.e., $E(X_i X_j X_k) = 0$ for i, j, k. This implies particularly that the directional moments of NLAR(2) are equal and therefore that NLAR(2) is always partially time reversible.

In Section II we found that $E(X_i^3) = 0$ for all i since X_i is marginally standard Laplace. It is easy to establish the following two equations,

$$E(X_n X_{n-1}^2) = \beta_2 \alpha_2 E(X_n^2 X_{n-1}) \tag{6.1}$$

$$E(X_n^2 X_{n-1}) = \{(\beta_2^2 \alpha_2)/(1 - 2\beta_1\beta_2\alpha_1\alpha_2)\} E(X_n X_{n-1}^2). \tag{6.2}$$

Solving (6.1) and (6.2) simultaneously yields $E(X_n X_{n-1}^2) = E(X_n^2 X_{n-1}) = 0$.

Now, using separate induction arguments and the stationarity assumption, we establish that $E(X_n X_{n-l}^2) = 0$ for all $l \ge 1$, and $E(X_n^2 X_{n-k}) = 0$ for all $k \ge 1$.

The proof of $E(X_n X_{n-l}^2) = 0$ is straightforward.

To prove $E(X_n^2 X_{n-k}) = 0$ we first show that the expectation of special third-order moments of the form $X_n X_{n-1} X_{n-k}$ for $k \ge 2$ is zero. Define $\mu_k = E(X_n X_{n-1} X_{n-k})$ and assume $E(X_n^2 X_{n-j}) = 0$, $j \le k-1$. From (3.1)

$$\begin{aligned}\mu_k = E(X_n X_{n-1} X_{n-k}) &= \alpha_1\beta_1 E\left(X_n^2 X_{n-(k-1)}\right) \\ &\quad + \alpha_2\beta_2\left(X_n X_{n-1} X_{n-(k-1)}\right) \\ &= \alpha_2\beta_2\mu_{k-1} = \cdots = (\alpha_2\beta_2)^{k-1}\mu_1.\end{aligned} \tag{6.3}$$

Now from (6.1) and (6.2) we have $\mu_1 = E(X_n X_{n-1}^2) = \alpha_2\beta_2 E(X_n^2 X_{n-1}) = 0$. Therefore $\mu_k = 0$.

We now proceed to show that $E(X_i X_j X_k) = 0$ for all i, j, k. Without loss of generality let $i < j < k$ so that $k = i + n$, $j = i + m$ and $n > m$. Therefore by stationarity $E(X_i X_j X_k) = E(X_i X_{i+m} X_{i+n}) = E(X_i X_{i-(n-m)} X_{i-n})$. Fixing m so that $0 < m < n$, we use induction on n. Let $n = 2$, implying $m = 1$. The first step in the induction follows from $E(X_i X_{i-1} X_{i-2}) = \mu_2 = 0$. Next assume that for $m < n \le K$, $E(X_i X_{i-(n-m)} X_{i-n}) = 0$. Now we show that $E(X_i X_{i-(K+1-m)} X_{i-(K+1)}) = 0$. Using (3.1) we write

$$\begin{aligned}E&\left(X_i X_{i-(K+1-m)} X_{i-(K+1)}\right) \\ &= \alpha_1\beta_1 E\left(X_{i-1} X_{i-(K+1-m)} X_{i-(K+1)}\right) \\ &\quad + \alpha_2\beta_2 E\left(X_{i-2} X_{i-(K+1-m)} X_{i-(K+1)}\right) \\ &\quad + E\left(\epsilon_i X_{i-(K+1-m)} X_{i-(K+1)}\right).\end{aligned}$$

Now

$$\begin{aligned}E&\left(\epsilon_i X_{i-(K+1-m)} X_{i-(K+1)}\right) \\ &= E(\epsilon_i) E\left(X_{i-(K+1-m)} X_{i-(K+1)}\right) = 0\end{aligned}$$

and

$$\begin{aligned}E&\left(X_{i-1} X_{i-(K+1-m)} X_{i-(K+1)}\right) \\ &= E\left(X_i X_{i-(K-m)} X_{i-K}\right) = 0\end{aligned}$$

by stationarity and the assumption. Likewise,

$$\begin{aligned}E&\left(X_{i-2} X_{i-(K+1-m)} X_{i-(K+1)}\right) \\ &= E\left(X_i X_{i-\{(K-1)-m\}} X_{i-(K-1)}\right) = 0.\end{aligned}$$

This completes the induction.

An immediate result from the argument about third moments is that $Z_n = X_n - X_{n-1}$ for $\{X_n\}$ of the NLAR(2) is not skewed.

The residual analysis in [6] and [19] using cross correlations between linear autoregressive residuals $R_n = X_n - a_1 X_{n-1} - a_2 X_{n-2}$, and their squares R_n^2, does not shed any new light on the directionality/reversibility in the NLAR(2) model or help identify the appropriateness of the Laplacian model. This is because all third moments have zero expectation. Thus, we see that $E(R_n^2 R_{n+l}) = E(R_n R_{n+l}^2) = 0$ for all l.

Note that the basis for the residual analysis using the $\{R_n\}$ process is that this process is uncorrelated, but not necessarily independent. The moment results show that the

R_n have zero skewness. In fact, it is easy to show that the distribution of R_n is the same as the distribution of $-R_n$. Thus, the R_n are symmetric, although they will, of course, not have Laplacian distributions.

VII. Conclusions and Further Analysis

We have demonstrated that like the other canonical distributions, the Laplace distribution has several special properties. We presented and justified the use of a very broad Markovian model that has four parameters and the correlation structure and third-order behavior of an AR(2) model. It is easy to simulate on a computer.

There are many other uses of the NLAR(2) construction within the context of time series analysis. A moving average model (NLMA(1)) and a mixed model (NLARMA(1,1)) have been derived. A detailed discussion of these models along with other possibilities will be reported elsewhere.

If the residual analysis for nonlinear autoregressive processes suggested in [6] and [19] is to be useful in modeling with NLAR(2), it must be extended to consider at least some special fourth-order moments, such as $E(X_{n-l}^2 R_n^2)$, $E(R_{n-l}^2 R_n^2)$, or $E(X_{n-l}^3 R_n)$, in order to distinguish the process from other candidates.

Finally, the joint probability density function for the NLAR(1) model will be used elsewhere to investigate the important problem of parameter estimation in the model. A likelihood analysis for the NLAR(2) model appears to be much more difficult, but is also possible.

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