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A relativistic mass tensor with geometric interpretation

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We derive a relativistic mass tensor (dyadic or matrix) whose origin and properties have a direct geometric interpretation in terms of projection operators related to the particle's world line and local inertial frame in Minkowski space, yet whose eigenvalues are simply the longitudinal (m_l) and the transverse (m_t) mass. Writing the noncovariant equations of motion (EOM) for a point particle in terms of this mass tensor bridges the gap between the compact but sterile form of the Lorentz covariant EOM and the usual ("unwieldy") noncovariant EOM in which m_l and m_t appear. General expressions for 3- and 4-space mass (inverse mass) tensors are presented in terms of the system Lagrangian (Hamiltonian).

I. INTRODUCTION

A well-known result of the special theory of relativity is that in the relativistic motion of a point particle the force and acceleration are generally noncollinear.[1-5] The relativistic (but not explicitly covariant) equations of motion (EOM) of a point particle with rest mass m, in an inertial frame traveling with velocity $-\mathbf{v}$ relative to the rest frame of the particle can be written as,

$$\mathbf{F} = \frac{d\mathbf{p}}{dt}, \tag{1}$$

where $\mathbf{p} = m\gamma\mathbf{v}$,

$$\gamma = 1/\sqrt{(1 - v^2/c^2)},$$

and m is the particle rest mass.

Having written the EOM in a specific inertial frame of reference, the noncollinearity of $\mathbf{F}$ and $\mathbf{a}$ is often expressed as a manifestation of unequal transverse and longitudinal mass parameters m_t and m_l, respectively. Expansion of the derivative on the right-hand side of Eq. (1) yields, for the ith component, "the unwieldy expression,"[1]

$$F_i = \gamma^3 m \left(\frac{\delta_{ij}}{\gamma^2} + \frac{dx_i}{dt}\frac{dx_j}{dt}\Big/c^2\right)\frac{d^2x_j}{dt^2}, \tag{2}$$

where we follow the Einstein convention of summing over repeated indices; $i,j = 1,2,3$ (Greek letters take the values 1–4, Latin letters range over 1–3).

When the force is parallel to the velocity Eq. (2) reduces to

$$F_i = \gamma^3 m \frac{d^2x_i}{dt^2},$$

$$= m_l \frac{d^2x_i}{dt^2}, \tag{3}$$

and when the force is perpendicular to the velocity it reduces to

$$F_i = \gamma m \frac{d^2x_i}{dt^2},$$

$$= m_t \frac{d^2x_i}{dt^2}. \tag{4}$$

Equation (2) implies that, in general, the particle acceleration is always parallel to the force only in the rest frame (local inertial frame) of the particle.

The concepts of longitudinal (m_l) and transverse (m_t) relativistic mass have experienced varying levels of popularity[2] depending on whether (1) the unity and compactness of the covariant EOM, or (2) insight into the details of the motion in a specific inertial frame, were felt to be more important. Recently, detailed analyses of the 3-space relation between force and acceleration, specifically the angle between $\mathbf{F}$ and $\mathbf{a}$ as a function of the angle between $\mathbf{F}$ and $\mathbf{v}$,[6] and the interpretation of a "paradoxical" negative acceleration (velocity components orthogonal to the force may diminish in magnitude)[7,8] have been reported. As shown in the references, it is not necessary to refer to a relativistic mass to discuss the latter effects. Our derivation of a relativistic mass tensor will provide another perspective on these phenomena.

We present an alternative way to express the noncovariant (by this we mean "not explicitly covariant") EOM which shows the physical results ascribable to m_l and m_t in a more general formalism, yet retains much of the compactness characteristic of the covariant EOM. Moreover, our expression for the mass tensor (dyadic or matrix) admits of an intuitive geometrical interpretation in terms of projection operators in three- and four-dimensional spacetime. The 3-space mass tensor derived below can be interpreted as γ times the spatial components of a 4-space mass tensor. The latter is shown to be m times the projection operator which projects onto the 3-space perpendicular to the particle world line. Further decomposition of the mass tensor into projection operators parallel and perpendicular to the particle 3-velocity yields the usual definitions of longitudinal and transverse mass. The 3- and 4-space projection operators are shown to arise as a natural consequence of the constant length of the particle 4 velocity. The relations among kinematic constraints, their associated projection operators, and generalized mass tensors is a central theme of the paper. The kinematic origin, formulas, and properties of relativistic mass and motion are elucidated by the analysis. This new perspective complements the usual presentation.

The plan of the paper is the following. In Sec. II we review the constraints on relativistic particle motion resulting from the constancy of the particle 4 velocity. Section III presents the properties of the associated projection operator in Minkowski space. In Sec. IV we derive a suggestive form of the particle equations of motion, identify the 3-space mass tensors, and make the connection between the mass tensors and 3- and 4-space projection operators. In Sec. V we formulate a general method for deriving mass tensors and projection operators in the Lagrangian and Hamiltonian formulations of quite general dynamical systems. We then apply this general approach in Sec. VI to derive the covariant equations of motion in terms of mass and inverse mass tensors and projection operators, demonstrating explicitly that the 3-space mass tensor is directly related to the spatial components of a 4-space projection operator. In Appendices A and B we apply our method to classical rigid body rotation and to relativistic particle motion, respectively. The standard results are seen to follow naturally from considerations of projectors and the mass tensors. In Sec. VII we discuss and summarize our results.

II. CONSTRAINTS ON PARTICLE MOTION

Consider a nonrelativistic point mass connected to a weightless rigid rod of length r whose other end is attached to the origin. The motion of the mass is then constrained to the surface of the unit sphere about the origin. If $\mathbf{r}$ represents the present position of the particle, then an incremental change must satisfy the requirement that $\delta\mathbf{r}$ be perpendicular to $\mathbf{r}$, i.e., $\mathbf{r}\cdot\delta\mathbf{r}=0$. It is shown in Appendix A that the effective mass for this case can be written in terms of the projection operator onto the plane perpendicular to $\mathbf{r}$.

This analogy helps motivate the following analysis. The four velocity of a point particle satisfies the following (constraint) equation,[1-5]

$$u^\mu u^\mu = -c^2, \tag{5}$$

where we use the $(+\ +\ +\ -)$ convention for the metric tensor (realized using an imaginary fourth component of tensors, where convenient). The particle four momentum is

$$p^\mu = mu^\mu. \tag{6}$$

Combining Eqs. (5) and (6) yields

$$p^\mu p^\mu = -m^2c^2. \tag{7}$$

Note that this is a constant in time as well as under Lorentz transformations. Differentiating Eq. (5),

$$u^\mu\, du^\mu = 0, \tag{8}$$

shows that any change in the four velocity must be orthogonal to u^μ. Since the four velocity is tangent to the particle world line, any change in particle motion is also orthogonal to the world line. In other words, the only acceptable transformation of the constant length four-velocity vector is a rotation in 4 space.

The (covariant) Minkowski force is defined through the equation,

$$K^\mu = dp^\mu/d\tau, \tag{9}$$

where τ is the proper time (i.e., local time t measured in the rest frame of the particle). Differentiating Eq. (7) with respect to τ for constant rest mass and using Eqs. (6) and (9), we also obtain the well known results,

$$K^\mu p^\mu = 0, \tag{10}$$

and

$$K^\mu u^\mu = 0. \tag{11}$$

Hence, the Minkowski force also lies entirely in the 3-space orthogonal to the world line.

III. PROJECTION OPERATORS

Equation (8) implies that δu^μ lies in the 3-plane perpendicular to the world line. Hence, we require the particle motion be such that δu^μ is an eigenvector of the projection operator onto the 3-plane perpendicular to u^μ. Note that in the particle rest frame (local inertial frame) u^μ is parallel to the time axis. Therefore, this projection operator also projects onto the local three-dimensional space of the particle. Although the projection is onto the local 3-plane it generally will be expressed in terms of coordinates of an arbitrary inertial frame of reference. Hence, it may have a nonzero fourth (timelike) component. In the space of 4 vectors the projection operator onto the 3-plane perpendicular to an arbitrary 4 vector u^μ can be written,

$$P^{\mu\sigma} = \delta^{\mu\sigma} - u^\mu w^\sigma/(u^\alpha w^\alpha). \tag{12}$$

Repeated application on any vector by a projector P has no further effect, i.e.,

$$P^2 = P. \tag{13}$$

It is clear that projection onto a lower dimensional subspace cannot possess an inverse. Taking the determinant of Eq. (13) shows that $\det(P)$ must equal either one or zero. The nonexistence of an inverse implies that zero is the correct answer.

Because the determinant of an operator (or matrix) equals the product of its eigenvalues, it follows that at least one of the eigenvalues of P equals zero. For any vector A

lying completely in the projected subspace we have,

$$PA = A. \qquad (14)$$

Hence A is an eigenvector of P with eigenvalue 1, and P is a unit operator when operating on vectors in the projected subspace. It follows that the projection operator can be written (in matrix form) as,

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad (15)$$

when written in an inertial frame in which $W^{\mu}_{\cdot}$ (timelike) points along the fourth coordinate axis. The Lorentz Invariant eigenvalues of P are 1,1,1,0, and its trace is equal to three, the dimensionality of the projected subspace. Now, consider an inertial frame in which a point particle travels with velocity $\mathbf{v}$ parallel to the x axis. The projector onto the 3-plane perpendicular to the world line (the local frame), can be obtained by using u^{μ} in Eq. (12). After simplification, we have,

$$P = \begin{pmatrix} \gamma^2 & 0 & 0 & i\gamma^2 v/c \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ i\gamma^2 v/c & 0 & 0 & -\gamma^2 v/c \end{pmatrix}. \qquad (16)$$

In this expression we have used Eq. (12), substituting u for W, and imaginary fourth components, for convenience. Clearly Eq. (16) reduces to Eq. (15) when v goes to zero.

Although the spatial diagonal elements have the same ratio as m_l and m_t, our discussion implies that the nonzero (invariant) eigenvalues are all equal to 1, as might be expected for the ratios of mass parameters in the local inertial frame.

IV. THE NONCOVARIANT EQUATIONS OF MOTION

Using the expression for $\mathbf{p}$ following Eq. (1) and the well-known relation for the particle energy,

$$E = mc^2\gamma, \qquad (17)$$

we find [cf., Eq. (6)],

$$\mathbf{v} = \mathbf{p}c^2/E. \qquad (18)$$

Taking the derivative of Eq. (18) with respect to time we obtain,

$$\frac{d\mathbf{v}}{dt} = \frac{d\mathbf{p}}{dt}\frac{c^2}{E} - \frac{\mathbf{p}c^2}{E^2}\frac{dE}{dt}. \qquad (19)$$

Now use the relation

$$\frac{dE}{dt} = \mathbf{F}\cdot\mathbf{v}, \qquad (20)$$

Eq. (18) and Eq. (1) to transform Eq. (19) into,[5,9]

$$\frac{d\mathbf{v}}{dt} = \mathbf{F}\frac{c^2}{E} - \frac{\mathbf{v}}{E}(\mathbf{F}\cdot\mathbf{v}). \qquad (21)$$

We can rewrite this in tensor, matrix, or dyadic form, depending on which is more convenient or instructive. For the moment, we write it in dyadic form as,[9]

$$\frac{d\mathbf{v}}{dt} = c^2/E\,\mathbf{F}\cdot(\mathbf{I} - \mathbf{vv}/c^2), \qquad (22)$$

where $\mathbf{I}$ is the unit dyadic in 3 space (i.e., δ^{ij}; $i,j = 1,2,3$). Equation (22) shows that there are two components of acceleration, the part proportional to $\mathbf{F}\cdot\mathbf{I} = \mathbf{F}$ is in the direction of the force. The part proportional to $-(\mathbf{F}\cdot\mathbf{v})\mathbf{v}$ is antiparallel (parallel) to the velocity when the force makes an acute (obtuse) angle with the velocity. In other words, if the force is tending to speed the particle up then the orthogonal components of velocity must shrink to accommodate the rotation of u^{μ} (and vice versa). This is the physical (kinematic) origin of the "paradoxical" negative acceleration and is discussed further in Appendix B.

The form of Eq. (22) suggests defining[9] an inverse mass dyadic

$$\mathsf{M}^{-1} = (c^2/E)(\mathbf{I} - \mathbf{vv}/c^2), \qquad (23)$$

or, using Eq. (17) for E,

$$\mathsf{M}^{-1} = 1/(m\gamma)(\mathbf{I} - \mathbf{vv}/c^2).$$

After a little algebra, this can be put in the form

$$\mathsf{M}^{-1} = [\mathsf{P}_l/m_l + \mathsf{P}_t/m_t], \qquad (23a)$$

where

$$\mathsf{P}_t = [\mathbf{I} - \mathbf{vv}/v^2]$$

is the projector onto the 2-space orthogonal to $\mathbf{v}$ and

$$\mathsf{P}_l = \mathbf{vv}/v^2$$

is the (3-space complement) projector onto the 1-space parallel to $\mathbf{v}$. Equation (23a) shows that m_l and m_t remain meaningful mass parameters even for quite general angles between the force and particle velocity.

Equation (22) can now be written in the suggestive form

$$\frac{d\mathbf{v}}{dt} = \mathbf{F}\cdot\mathsf{M}^{-1}. \qquad (24)$$

Note that the inverse mass dyadic [Eq. (23)] is reminiscent of the form of a projection operator. In fact, in the limit as v approaches c it becomes the projection operator onto the 2-plane perpendicular to the particle velocity. This allows only rotations of $\mathbf{v}$, and thereby insures that v does not exceed c. Because v is less than c the diagonal elements in Eq. (23) are strictly positive (although approaching 0 as v approaches c).

To find the mass dyadic itself we can either take the inverse of Eq. (23), directly (it's not really a projection operator for $v < c$ so the inverse exists) or rewrite Eq. (2) in dyadic form. Either method leads in a straightforward manner to

$$\mathsf{M} = m\gamma(\mathbf{I} + \gamma^2\mathbf{vv}/c^2), \qquad (25)$$

or, with a little algebra,

$$\mathsf{M} = (m_l\mathsf{P}_l + m_t\mathsf{P}_t). \qquad (25a)$$

[Using the orthogonality of P_l and P_t and the fact that $\mathsf{P}_l + \mathsf{P}_t = \mathbf{I}$ it is clear that Eq. (25a) is the inverse of Eq. (23a).]

Hence, Eq. (2) can be written in vector and dyadic notation as

$$\mathbf{F} = \mathsf{M}\cdot\mathbf{a}. \qquad (26)$$

One easily verifies that Eqs. (23) and (25) satisfy

$$\mathsf{M}\cdot\mathsf{M}^{-1} = \mathbf{I}. \qquad (27)$$

Because $\mathbf{u} = \mathbf{v}\gamma$, where $\mathbf{u}$ is the spatial part of the 4 velocity, we can rewrite the mass dyadic [Eq. (25)] as

$$\mathsf{M} = E/c^2(\mathbf{I} + \mathbf{uu}/c^2), \qquad (28)$$

or, using Eq. (5), more suggestive of a projector as

$$\mathsf{M} = m\gamma[\mathbf{I} - \mathbf{uu}/(u^\mu u^\mu)]. \qquad (29)$$

Hence, in component notation the mass tensor is,

$$M^{ij} = m\gamma[\delta^{ij} - u^i u^j/(u^\mu u^\mu)], \qquad (30)$$

or in terms of the projector onto the local 3 space, it becomes

$$M^{ij} = m\gamma P^{ij}. \qquad (31)$$

This (matrix) is diagonalized by rotating the spatial coordinate axes so that the velocity is along the x axis, yielding [cf., Eqs. (16) and (31)],

$$M^{ij} = \begin{pmatrix} m\gamma^3 & 0 & 0 \\ 0 & m\gamma & 0 \\ 0 & 0 & m\gamma \end{pmatrix}. \qquad (32)$$

This is just Eq. (25a) in matrix notation and in diagonal form. We therefore have the quite reasonable result that the eigenvalues of the (inverse) mass dyadic are the ordinary (inverse) longitudinal relativistic mass and (inverse) transverse mass (the latter with multiplicity two). The eigenvalues and eigenvectors follow immediately from the form of Eqs. (23a), (25a), and (32).

V. GENERAL LAGRANGIAN AND HAMILTONIAN ANALYSIS

We now formulate a general Lagrangian method for obtaining mass tensors and a Hamiltonian method for inverse mass tensors, for quite general dynamical systems (the Lagrangian method is applied in Appendix A to rigid body rotation). We first derive the noncovariant EOM with this method, reproducing the results obtained in the previous section. The equations are then applied in Sec. VI to derive the covariant (4-space) mass tensors.

A. Lagrangian formulation

The Euler–Lagrange differential equations for a general system are[2]

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = \frac{\partial L}{\partial q_i}, \qquad (33)$$

where q_i is the ith generalized coordinate. Using the chain rule the time derivative can be written as (sum over repeated indices)

$$\frac{d}{dt} = \frac{\partial \dot{q}_i}{\partial t}\frac{\partial}{\partial \dot{q}_j} + \frac{\partial q_i}{\partial t}\frac{\partial}{q_j} + \frac{\partial}{\partial t}. \qquad (34)$$

Initially, assume that the canonical momentum, ($p_i \equiv \partial L/\partial \dot{q}_i$), is not an explicit function of q_j or of the time t. Hence, Eq. (33) can be written

$$\ddot{q}_j\left(\frac{\partial^2 L}{\partial \dot{q}_j\, \partial \dot{q}_i}\right) = \frac{\partial L}{\partial q_i}. \qquad (34)$$

The right-hand side of this equation is just the generalized force. Therefore, by analogy with Newton's second law, $F = ma$, we identify the (symmetric) effective mass tensor as

$$M^{ij} = \frac{\partial^2 L}{\partial \dot{q}_j\, \partial \dot{q}_i}. \qquad (35)$$

For a nonrelativistic point particle this is usually just m times the unit dyadic (however, cf., our comments below regarding solid state physics). For a free, relativistic, point particle with Lagrangian (t is the independent variable),[1-5]

$$L = -mc^2\sqrt{(1 - v^2)} - V, \qquad (36)$$

we obtain, using Eqs. (35) and (36),

$$M^{ij} = m\gamma(\delta^{ij} + \gamma^2 v^i v^j/c^2), \qquad (37)$$

the same as Eq. (25), except written here in component form.

To treat the electromagnetic interaction we allow the canonical momenta to be functions of q and t. Then, in place of Eq. (34) we obtain, from Eqs. (33) and (34),

$$\ddot{q}_j\left(\frac{\partial^2 L}{\partial \dot{q}_j\, \partial \dot{q}_i}\right) = \frac{\partial L}{\partial q_i} - \dot{q}_j\left(\frac{\partial^2 L}{\partial q_j\, \partial \dot{q}_i}\right) - \frac{\partial^2 L}{\partial t \partial \dot{q}_i}. \qquad (38)$$

Applying this more general form to the Lagrangian of a charged point particle interacting with a given electromagnetic field (charge q, scalar and vector potentials ϕ,$\mathbf{A}$),

$$L = -mc^2\sqrt{(1 - v^2)} + q/c\mathbf{v}\cdot\mathbf{A} - q\phi, \qquad (39)$$

we obtain, using Cartesian coordinates written in dyadic form,

$$\begin{aligned}\mathbf{a}\cdot[m\gamma(\mathbf{I} + \gamma^2\mathbf{vv}/c^2)] &= [-q\nabla\phi + q/c\nabla(\mathbf{v}\cdot\mathbf{A})] \\ &\quad - \left[q/c\mathbf{v}\cdot\nabla\mathbf{A} + q/c\frac{\partial\mathbf{A}}{\partial t}\right]. \end{aligned} \qquad (40)$$

Rearranging terms, and using the vector identity

$$\mathbf{v}\times(\nabla\times\mathbf{A}) = \nabla(\mathbf{v}\cdot\mathbf{A}) - (\mathbf{v}\cdot\nabla)\mathbf{A}$$

and the well-known expressions for $\mathbf{E}$ and $\mathbf{B}$ in terms of ϕ and $\mathbf{A}$, we obtain

$$\mathbf{a}\cdot[m\gamma(\mathbf{I} + \gamma^2\mathbf{vv}/c^2)] = q\mathbf{E} + q/c\mathbf{v}\times\mathbf{B}. \qquad (41)$$

This is equivalent to Eqs. (25) and (26) written for a particular ($E - M$) force. The mass tensor is not changed by this type of interaction. Now postmultiply this equation by M^{-1}, obtaining,

$$\mathbf{a} = q/m_t(\mathsf{P}_t\cdot\mathbf{E} + \mathbf{v}\times\mathbf{B}/c) + q/m_l\mathsf{P}_l\cdot\mathbf{E}. \qquad (41a)$$

Hence, not surprisingly, the transverse mass governs the particle response to the full magnetic force and to the components of electric field transverse (orthogonal) to the velocity.

One could interpret the "extra" components of the mass dyadic as giving rise to a relativistic (kinematic) constraint "force"

$$-[m\gamma^3(\mathbf{a}\cdot\mathbf{v})\mathbf{v}/c^2],$$

which is antiparallel (parallel) to $\mathbf{v}$ when the angle between $\mathbf{a}$ and $\mathbf{v}$ is acute (obtuse). Hence, when the particle is speeding up ($\mathbf{a}$ makes an acute angle with $\mathbf{v}$) the component of $\mathbf{v}$ perpendicular to $\mathbf{a}$ will undergo a "negative acceleration" (cf., discussion in Refs. 6–8). This is discussed further in Appendix B.

B. Hamiltonian formulation

The Hamiltonian formulation of the general equations of motion are[2]

$$\dot{p}_i = -\frac{\partial H}{\partial q_i} \qquad (42)$$

and

$$\dot{q}_i = \frac{\partial H}{\partial p_i}. \tag{43}$$

The canonical momentum p_i is defined, as usual, as $\partial L/\partial \dot{q}_i$.

To derive our general expression for the mass tensor we begin by differentiating Eq. (43) with respect to time. We use an expression for d/dt analogous to the one employed above but remembering that H is a function of q,p (and perhaps t) as a result of the Legendre transformation $H = p_i \dot{q}_i - L$. Hence we have,

$$\ddot{q}_i = \left(\frac{\partial^2 H}{\partial p_i \partial p_j}\right)\dot{p}_j + \left(\frac{\partial^2 H}{\partial p_i \partial q_j}\right)\dot{q}_j + \left(\frac{\partial^2 H}{\partial p_i \partial t}\right). \tag{44}$$

Now use Hamilton's equations once more to replace $\dot{p}$ and $\dot{q}$,

$$\ddot{q}_i = \left(\frac{\partial^2 H}{\partial p_i \partial p_j}\right)\left(\frac{-\partial H}{\partial q_j}\right) + \left(\frac{\partial^2 H}{\partial p_i \partial q_j}\right)\left(\frac{\partial H}{\partial p_j}\right) + \left(\frac{\partial^2 H}{\partial p_i \partial t}\right). \tag{45}$$

For particles in a conservative potential we can drop the last two terms. H is the sum of kinetic and potential energy ($H = T + V$) so that $-\partial H/\partial q_j$ is the jth component of the conservative force acting on the particle. The acceleration with respect to the ith generalized coordinate can now be written as,

$$\ddot{q}_i = \left(\frac{\partial^2 H}{\partial p_i \partial p_j}\right) F_j, \tag{46}$$

and the definition of a generalized inverse effective mass follows immediately, viz.,

$$M_{ij}^{-1} = \frac{\partial^2 H}{\partial p_i \partial p_j}. \tag{47}$$

A number of comments are in order here.

(1) Replacing p_i with $\hbar k_i$ in Eq. (47) yields a standard expression for the effective mass in solid-state physics.[10]

(2) Starting with the Lagrangian in Eq. (36) yields, after solving for $\mathbf{v}$ in terms of $\mathbf{p}$ and performing the Legendre transformation, the Hamiltonian, $H = \sqrt{(\mathbf{p}^2 c^2 + m^2 c^4)} + V$. It is left as an exercise for the reader to verify that substituting this Hamiltonian into Eq. (47) reproduces our expression for the inverse relativistic mass, Eq. (23), in component form (after replacing $\mathbf{p}$ with $m\mathbf{v}\gamma$). Hence Eq. (46), when written in dyadic form, becomes identical to Eq. (22).

(3) Determining H using the Lagrangian [Eq. (39)] for a particle interacting with an electromagnetic field, Eq. (47) will also reproduce our expression for the inverse relativistic mass dyadic, if one remembers to replace p with its definition in terms of v at the end.

(4) For those cases in which M is nonsingular ($\det M^{ij} \neq 0$) our results imply the analogue of Eq. (27), namely,

$$\left(\frac{\partial^2 L}{\partial \dot{q}_k \partial \dot{q}_i}\right) \cdot \left(\frac{\partial^2 H}{\partial p_i \partial p_j}\right) = \delta_{kj}. \tag{48}$$

VI. COVARIANT RELATIVISTIC MASS TENSOR

In this section we derive the covariant EOM where the proper time τ is the independent variable. The corresponding (invariant) Lagrangian and Hamiltonian will be denoted by L_τ and H_τ, respectively. Hamilton's principle is now

$$\delta \int L_\tau d\tau = 0, \tag{49}$$

yielding the covariant version of the Euler–Lagrange equations

$$\frac{d}{d\tau}\left(\frac{\partial L_\tau}{\partial q^{\mu\prime}}\right) = \frac{\partial L_\tau}{\partial q^\mu}. \tag{50}$$

We denote derivatives with respect to τ with a prime, e.g., $q' = dq/d\tau$. The quantity in the parentheses is the covariant canonical momentum, conjugate to q^μ.

With the same assumption of a conservative (nonelectromagnetic) force, repeat the analysis leading to Eq. (34), obtaining

$$q^{\nu\prime\prime}\left(\frac{\partial^2 L_\tau}{\partial q^{\nu\prime}\partial q^{\mu\prime}}\right) = \frac{\partial L_\tau}{\partial q^\mu}. \tag{51}$$

Identify the covariant mass tensor N as

$$N^{\mu\nu} = \frac{\partial^2 L_\tau}{\partial q^{\mu\prime}\partial q^{\nu\prime}}. \tag{52}$$

Now, to apply this formula, we need to select a suitable invariant Lagrangian. For a relativistic point particle in Cartesian coordinates, the most natural Lagrangian,[1,2] up to an arbitrary additive constant, is

$$L_\tau = -mc\sqrt{(-u^\gamma u^\gamma)} - V_\tau. \tag{53}$$

The first derivative with respect to u^μ gives the canonical momentum

$$p^\mu = mcu^\mu/\sqrt{(-u^\gamma u^\gamma)}, \tag{54}$$

which yields the usual result, Eq. (6), when the identity Eq. (5) is used. The second derivative (wrt u^ν) yields the covariant mass tensor

$$N^{\mu\nu} = m(\delta^{\mu\nu} - u^\mu u^\nu/u^\gamma u^\gamma). \tag{55}$$

This is recognized as simply the product of the particle rest mass m and the projection operator onto the particles local 3 space. Hence, the eigenvalues of N are just 0 and m (the latter triply degenerate). If we take the velocity along the x axis and use an imaginary fourth component of u^μ, Eq. (55) reduces to m times Eq. (16).

Writing out the EOM in Eq. (51), using our result for the mass tensor N we have

$$\frac{d^2 x^\nu}{d\tau^2} m(\delta^{\mu\nu} - u^\mu u^\nu/u^\gamma u^\gamma) = K^\mu, \tag{56}$$

where K^μ is the Minkowski force, equal to $\partial L_\tau/\partial x^\mu$. It may at first appear that this provides no new insight, inasmuch as the left-hand side reduces to $m\, d^2x^\mu/d\tau^2$ because of the orthogonality of the 4 velocity and 4 acceleration. Hence, we simply recover the usual covariant form of $f = ma$,

$$\frac{md^2x^\mu}{d\tau^2} = K^\mu. \tag{57}$$

However, if we use the well-known relation

$$d\tau = \frac{dt}{\gamma}, \tag{58}$$

for $d\tau$ in Eq. (56), we obtain

$$\left[\frac{\gamma^2 d^2 x^\nu}{dt^2} + \left(\frac{\gamma d\gamma}{dt}\right)u^\nu\right] N^{\mu\nu} = K^\mu. \tag{59}$$

The second term within the brackets drops out because $u^{\nu}N^{\mu\nu}=0$, yielding,

$$\frac{\gamma^2 d^2x^{\nu}}{dt^2}N^{\mu\nu}=K^{\mu}. \quad (59a)$$

Divide both sides of this equation by γ, and recall the relation between K^i and the ordinary force vector F^i, viz.,

$$F^i = K^i/\gamma. \quad (60)$$

Also, note that when the index $\mu\neq 0$ only spatial components contribute to the left-hand side of Eq. (59a) because $d^2t/dt^2=0$. Hence, we obtain for the spatial components of the EOM,

$$\frac{d^2x^j}{dt^2}\gamma N^{ij}=F^i \quad (61)$$

or

$$\frac{d^2x^j}{dt^2}\gamma m\left(\delta^{ij}-\frac{u^iu^j}{u^{\mu}u^{\mu}}\right)=F^i. \quad (62)$$

Comparison with Eq. (30) shows that we have recovered our noncovariant EOM and exactly the same expression for the mass dyadic. Moreover, we now see that the 3-space mass dyadic or tensor is simply obtained from the spatial components of the 4-space mass tensor by,

$$M^{ij}=\gamma N^{ij}. \quad (63)$$

In addition, through Eq. (55), a natural connection with the projection operator onto the local 3 space of the particle has finally been established.

The Hamiltonian analysis can be carried through in the same manner as before, but now using τ as the independent variable. As an example, we use the following Lagrangian, corresponding to a particle of mass m, and charge q, interacting with a prescribed electromagnetic field,

$$L_{\tau} = -mc\sqrt{(-u^{\mu}u^{\mu})}+q/cu^{\mu}A^{\mu}. \quad (64)$$

The canonical momentum is easily found to be,

$$p^{\mu}=mu^{\mu}+q/cA^{\mu}, \quad (65)$$

and the usual Legendre transformation yields the invariant Hamiltonian,

$$H_{\tau}=(p^{\mu}-q/cA^{\mu})^2/m+\sqrt{[-(p^{\mu}-q/cA^{\mu})^2]}, \quad (66)$$

written in an obvious short-hand notation. Note that, like the Lagrangian, the invariant Hamiltonian is really a constant, in fact equal to zero. However, as usual, it is only its functional form that is of interest. The completely covariant form of Hamilton's equations can be manipulated in exactly the same manner as before to provide an expression for an "inverse" mass 4 tensor N^*

$$N^{*\mu\nu}=\frac{\partial^2H_{\tau}}{\partial p^{\mu}\partial p^{\nu}}. \quad (67)$$

Using our expression for H_{τ} we readily obtain (after transforming expressions in p^{μ} into expressions in u^{μ}),

$$N^{*\mu\nu}=1/m(\delta^{\mu\nu}-u^{\mu}u^{\nu}/u^{\gamma}u^{\gamma}). \quad (68)$$

This is seen to be simply $1/m$ times the (by now familiar) projection operator onto the 3-space orthogonal to the four velocity. Moreover, we see that,

$$NN^*=P. \quad (69)$$

Hence, the two mass tensors are, in a sense, "inverses" of each other within the local 3 space (although N cannot have a true inverse). We infer that, in general

$$\left(\frac{\partial^2L_{\tau}}{\partial q^{\mu}\partial q^{\nu}}\right)\left(\frac{\partial^2H_{\tau}}{\partial p^{\nu}\partial p^{\alpha}}\right)=P^{\mu\alpha}. \quad (70)$$

Some expressions for the kinematic portion of the invariant Lagrangian L_{τ} other than the square root form appearing in Eqs. (53) and (64) (such as, e.g., $1/2mu^{\mu}u^{\mu}$) don't always yield the "correct" form of covariant mass and inverse mass tensors. We argue[11] that the form of Lagrangian we use is, in fact, the preferred form.

VII. DISCUSSION

The noncovariant EOM we derived in Eq. (22) is suggestive of the form of a projection operator times the force as v approaches c, forcing **a** to become orthogonal to **v** in that limit. This is a reflection of the requirement that v must remain less than c for a massive particle. This observation has motivated our efforts to try to understand the relation among (1) the kinematic constraint $u^{\mu}u^{\mu}=-c^2$, (2) the longitudinal and transverse mass parameters, and (3) projection operators in 3 and 4 space.

We have obtained quite general expressions for M, M^{-1}, and their 4-space analogues in terms of the system Lagrangian and Hamiltonian. The longitudinal and transverse mass parameters m_l and m_t appear not just for forces parallel and perpendicular to the velocity, but as eigenvalues of M relevant for any forces. Components of acceleration resulting from any impressed force are therefore quite easily calculated with the aid of M^{-1}. Evaluation of these tensors leads directly to intuitively meaningful projection operators, and also displays explicitly the noncollinear nature of the response to an arbitrary (and unspecified) force. The appearance of the projectors has been shown, both by direct analysis and by analogy with nonrelativistic rigid body rotation, to be a consequence of the fact that any variation in u^{μ} must lie completely within the local 3 space of the particle.

Our alternative way of writing the not explicitly covariant EOM in terms of mass and inverse mass dyadics complements other ways of analyzing the noncollinearity of force and acceleration in relativistic particle dynamics. For example, the "paradoxical" negative acceleration components recently discussed in the literature are now understood more intuitively in geometric terms, following an analysis with mass dyadics and tensors, kinematic constraints, and the associated projection operators.

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APPENDIX A: PROJECTION OPERATORS AND RIGID BODY ROTATION

Consider the Lagrangian for a nonrelativistic point particle of mass m constrained to rotate about the origin at a fixed distance r

$$L=1/2m(\boldsymbol{\omega}\times\mathbf{r})^2-V, \quad (A1)$$

where **r** and $\boldsymbol{\omega}$ (i.e., $d\boldsymbol{\theta}/dt$) are the position and angular velocity vectors, respectively. With the aid of the identity,[13]

$$(\mathbf{A}\times\mathbf{B})(\mathbf{C}\times\mathbf{D})=\begin{vmatrix}\mathbf{A}\cdot\mathbf{C} & \mathbf{B}\cdot\mathbf{C}\\ \mathbf{A}\cdot\mathbf{D} & \mathbf{B}\cdot\mathbf{D}\end{vmatrix}, \quad (A2)$$

for the cross product in Eq. (A1), we obtain

$$\frac{\partial L}{\partial \omega} = mr^2(\mathbf{I} - \mathbf{rr}/r^2)\cdot\omega, \tag{A3}$$

for the canonical momentum.

Using the identity

$$\frac{d}{dt} = \frac{d\omega}{dt}\cdot\frac{\partial}{\partial \omega},$$

the Euler–Lagrange equations can be written,

$$\frac{d\omega}{dt}\cdot\left(\frac{\partial^2 L}{\partial\omega\partial\omega}\right) = \text{Torque}. \tag{A4}$$

Taking the derivative of Eq. (A3) we obtain, for the (generalized) "effective mass,"

$$\frac{\partial^2 L}{\partial\omega\partial\omega} = mr^2(\mathbf{I} - \mathbf{rr}/r^2). \tag{A5}$$

We see that the effective mass is proportional to the projection operator corresponding to the constraint equation $\mathbf{r}^2 = r^2$.

For a multiparticle rigid body θ remains the appropriate generalized coordinate. In that case Eq. (A4) becomes

$$\frac{d^2\theta}{dt^2}\cdot\left[\sum_i m_i\, r_i^2(\mathbf{I} - \mathbf{r}_i\mathbf{r}_i/r_i^2)\right] = \text{Torque}, \tag{A6}$$

where the sum is over all the particles in the rigid body. The weighted sum of projection operators within the brackets is simply the usual moment of inertia matrix[2,14] written in dyadic notation.

APPENDIX B: COORDINATE FREE DISCUSSION OF RELATIVISTIC ACCELERATION

In this Appendix we look at the various components of particle acceleration relative to components parallel and perpendicular to **v** and **F**. We define the projection operators

$$\mathsf{Q}_l = \mathbf{FF}/F^2 \tag{B1}$$

and

$$\mathsf{Q}_t = \mathbf{I} - \mathsf{Q}_l, \tag{B2}$$

which project onto the one and two dimensional subspaces parallel and perpendicular to **F**, respectively.

We first study the components of acceleration **a** relative to the velocity vector. Consider a general force in Eq. (24), using Eq. (23a) for the inverse mass dyadic, viz.,

$$\mathbf{a} = \mathbf{F}\cdot(\mathsf{P}_l/m_l + \mathsf{P}_t/m_t). \tag{B3}$$

From the meaning of the projection operators, the component of acceleration parallel to **v** is simply,

$$\mathbf{a}_{v,l} = \mathbf{F}\cdot\mathsf{P}_l/m_l = \mathbf{F}_{v,l}/m_l, \tag{B4}$$

showing the obvious result that the acceleration will tend to speed the particle up if the angle between the force and the velocity is acute. Similarly, the component of acceleration perpendicular to **v** is

$$\mathbf{a}_{v,t} = \mathbf{F}\,\mathsf{P}_t/m_t, \quad = \mathbf{F}_{v,t}/m_t. \tag{B5}$$

Using Eqs. (B4) and (B5) one calculates the components of acceleration more simply than with Eq. (2) inasmuch as m_l and m_t remain the only parameters needed for arbitrary directions of the force.

To obtain the component of acceleration parallel to the force vector **F** postmultiply Eq. (22) by Q_l,

$$\mathbf{a}_{F,l} = c^2/E\,\mathbf{F}\cdot(\mathbf{I} - \mathbf{vv}/c^2)\cdot\mathbf{FF}/F^2 = 1/(F^2E)\mathbf{F}[F^2c^2 - (\mathbf{F}\cdot\mathbf{v})^2]. \tag{B6}$$

We see that for all $v<c$ the component of acceleration in the direction of the force is positive. However, Eq. (B6) shows that if there existed positive energy particles traveling faster than the speed of light (tachyons) then for sufficiently acute values of the angle between force and velocity the expression within the brackets could become negative, causing the acceleration to be in a direction opposite to that of an applied force.[12,9]

To obtain the component of acceleration orthogonal to the force vector, postmultiply Eq. (22) by Q_t obtaining

$$\mathbf{a}_{F,t} = -c^2/E(\mathbf{F}\cdot\mathbf{v})[\mathbf{v}\cdot(\mathbf{I} - \mathbf{FF}/F^2)]. \tag{B7}$$

The expression within the brackets is simply the projection of the velocity vector onto the plane orthogonal to the force vector. One readily sees that if the angle between the force and velocity is obtuse (acute), leading to a negative (positive) value of $(\mathbf{F}\cdot\mathbf{v})$ then the acceleration components orthogonal to the force will be in the same (opposite) direction as $\mathbf{v}_{F,t}$. In other words, if the force is slowing the particle down, any nonzero components of velocity orthogonal to the force will increase in magnitude, and conversely. This is the so-called "negative acceleration."

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$$L_\tau = 1/2mu^\mu u^\mu - V,$$

also reproduces the correct covariant EOM but not our expression for $N^{\mu\nu}$. We argue that the form of the Lagrangian in Eqs. (53) and (64) is, in fact, a preferred one for a number of reasons:

(a) Eq. (39) is the only relativistic Lagrangian, with parameter t, which produces gauge invariant equations (see Ref. 1, p. 117).

(b) The Lagrangian with parameter τ [Eq. (64)] is obtained most directly from Eq. (39) by changing the variable of integration from t to τ in the not explicitly covariant Lagrangian (see Ref. 2, p. 209).

(c) Using a form other than $-mc\sqrt{(-u^\mu u^\mu)}$ for the particle portion of the Lorentz covariant Lagrangian with our equations for the mass tensor does not always lead to the "correct" form of the covariant mass tensor. Other reasons are in O. D. Johns, Am. J. Phys. **53**(10), 982 (1985).

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The depolarization field inside a homogeneous dielectric: A new approac

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A simple argument is presented to make plausible the theorem that a homogeneous dielectric placed in a uniform external field is uniformly polarized only if its shape is ellipsoidal. Expressions for the depolarization factors are displayed in a form which makes immediately apparent the well-known sum rule. An additional symmetry relation between the depolarization factors is used to write integral expressions for them. It is shown that these factors reduce to the standard form. Neither Dirichlet's integral representation nor ellipsoidal coordinates are used in this derivation.

I. INTRODUCTION

The problem of determining the electric field inside a homogenous dielectric (and its magnetic analog) was discussed almost 100 years ago by Maxwell in his famous Treatise.[1] The well-known result is that, if the external applied field is uniform, *and the dielectric has an ellipsoidal shape*, then the internal field (and, therefore, the polarization) is uniform. This is an intriguing result, and students often ask if there is a simple way of proving it. I have not found an explicit treatment of this problem in the most popular modern textbooks; however, a thorough treatment is given in the classic texts by Becker and Sauter[2] and Stratton,[3] and depolarization factors have been evaluated numerically and analytically for certain limiting cases by Osborn[4] and Stoner.[5] They reproduce Maxwell's calculation, which, in turn rests on the integral expression first derived by Dirichlet for the potential inside and outside a uniformly charged ellipsoid. Sommerfeld[6] derives the result somewhat more directly by solving Laplace's equation in ellipsoidal coordinates. Ellipsoidal Harmonics[7] can also be used to solve the problem.

All of these derivations introduce mathematics which is somewhat apart from the "mainstream" techniques which we teach to our beginning graduate students, and they are all specific to the ellipsoid; while these works[1–6] strongly suggest that the ellipsoid is unique in having a uniform depolarization field, none of them offers a proof.[8] Portis[9] gives a nice discussion (following Newton) which shows that the uniform polarization of the ellipsoid arises naturally from the same property of the sphere. Recall that, at any point inside a uniformly charged spherical shell, the net contribution to the field from two charge elements located at opposite ends of any chord drawn through that point and which subtend the same solid angle, is zero. This property of "cancellation by pairs" is preserved under uniform dilations ($x \to \alpha x$, $y \to \beta y$, $z \to \gamma z$) of the sphere with respect to any three orthogonal axes, and such dilations distort the sphere continuously into an ellipsoid. Since a polarized ellipsoid can be thought of as a superposition of infinitesima ly displaced uniformly charged ellipsoids of opposite sig the argument works for polarization also. While not uniqueness theorem in itself, this argument certainly e hances the physical plausibility of such a theorem. I a indebted to one of the referees for calling Ref. 9 to n attention.

In this paper I shall first show that, of a broad class symmetric shapes, the ellipsoid is the only one which a mits a uniform depolarization field; I shall then rederi the well known sum rule for the depolarization factors, ar simple expressions for these factors which, I believe, a somewhat more physical than the standard ones. In part cular, in the form in which they are displayed, these facto exhibit an additional symmetry relation which I exploit calculate all three of them, given an analytical expressic for just one of them. Finally, I will show that my resul reduce to the standard expressions for the general case, ar give the usual analytic expressions for spheroidally syn metric shapes. My derivation relies only on some wel known properties of surface integrals of homogeneoı functions, and the spherical harmonics.

II. THE POTENTIAL INSIDE A SYMMETRIC DIELECTRIC

Consider a piece of dielectric with constant scalar su ceptibility χ, which is bounded by a surface determined t the equation $f(x,y,z) = 1$, with

$$f(x,y,z) = \left(\frac{x}{a}\right)^k + \left(\frac{y}{b}\right)^k + \left(\frac{z}{c}\right)^k, \qquad (1$$

where k is an even positive integer. For $k = 2$ and $a \neq b \neq$ this is the triaxial ellipsoid. In the presence of a unifor applied external field $\mathbf{E}_0$, the interior field is given by tl superposition of $\mathbf{E}_0$ and the so-called depolarization fie $\mathbf{E}_1$ produced by a surface charge $\mathbf{P} \cdot \mathbf{n}$, where $\mathbf{P}$ is the pola ization and $\mathbf{n}$ is a unit normal vector directed out of tl