Weak Invariance and Entropy *

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Abstract: *For continuous time control systems, this paper introduces invariance entropy as a measure for the amount of information necessary to achieve invariance of weakly invariant (or controlled invariant) compact subsets of the state space. Upper and lower bounds are derived, in particular, finiteness is proven. For linear control systems with compact control range, the invariance entropy is given by the sum of the real parts of the unstable eigenvalues of the uncontrolled system. A characterization via covers and corresponding feedbacks is provided.*

Keywords: Invariance entropy, feedbacks, topological entropy.

1 Introduction

This paper is concerned with the amount of information necessary to keep a continuous time control system in a given subset Q of the state space. We introduce 'invariance entropy' that measures, how fast open loop control functions must be readjusted in order to avoid exit from the subset Q. Due to the analysis of the open loop problem this information measure does not depend on any specific class of feedback strategies and hence is intrinsic.

The increasing relevance of control systems with restricted digital communication channels has spurred interest in the information necessary for accomplishing control tasks. Early contributions are due to Delchamps [5] who considered quantized feedbacks for stabilization; Wong and Brockett [10] study the influence of restricted communication channels. For the present paper, the work by Nair, Evans, Mareels, and Moran [7] is fundamental. They develop a method to describe data-rates necessary to render subsets Q of the state space invariant. Their approach is based on a notion describing for discrete time systems, how many feedbacks defined on open covers of Q are necessary in order to make Q invariant (or asymptotically

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stable) up to time N; then they let N tend to infinity and take the infimum over all covers and obtain what they call feedback entropy bearing some resemblance to the classical notion of topological entropy. In particular, they show that this number is equal to the minimum data rate for a symbolic controller rendering Q invariant.

The present paper introduces various versions of open loop entropies and discusses their relations. Since topological entropy is a property of dynamical systems (see e.g. Robinson [9] or Katok and Hasselblatt [6]), it would appear that a view of control systems as dynamical systems might be helpful. In fact, including the time shift along control functions to the dynamical system, one obtains a dynamical system, the control flow (cf. Colonius/Kliemann [3]). This point of view (though not necessarily the technical apparatus) is helpful in order to adapt several constructions traditionally used for topological entropy to control systems.

A preliminary definition of our information measure (see Section 3 for precise definitions of invariance entropy) is the following: For systems with compact control range let Q be a compact subset of the state space. Then, for $T > 0$, we let $r_{\text{inv}}(T; Q)$ be the minimal number of controls $u \in \mathcal{U}$ such that for every initial value $x \in Q$ there is u with corresponding trajectory $\varphi(t, x, u) \in Q$ for all $[0, T]$. Then we consider the exponential growth rate of these numbers as T tends to infinity,

$$\limsup_{T \to \infty} \frac{1}{T} \ln r_{\text{inv}}(T; Q).$$

A characteristic feature of this information measure is that no information on the present state of the system is involved. Our main results provide upper and lower bounds for the invariance entropy; in particular, it is shown that the invariance entropy is finite. For linear control systems (with compact control range) the invariance entropy is given by the sum of the real parts of the unstable eigenvalues. We remark that Nair, Evans, Mareels, and Moran [7] have a similar result for feedback entropy of systems linearized at an equilibrium, but with vanishing control range. We also give a characterization of invariance entropy in terms of covers and a feedback construction akin to the contribution in [7].

Section 2 recalls some basic properties of control systems (mainly for notational purposes). Section 3 introduces several variants of invariance entropy and their properties. Section 4 provides lower and upper bounds for the invariance entropy which can be computed directly from the right hand side of the system. One of these bounds, together with a classical result by Bowen [2] on topological entropy of linear maps, is used in Section 5 to compute the invariance entropy of linear control systems. Final Section 6 gives a characterization in terms of feedbacks defined on covers.

2 Preliminaries

In this preliminary section we recall some basic facts on nonlinear control systems, mainly to introduce some notation.

Let $d, m \in \mathbb{N}$, M an open subset of $\mathbb{R}^d$ and $U \subset \mathbb{R}^m$ compact. Let $f : M \times \mathbb{R}^m \to \mathbb{R}^d$ be a continuous mapping such that the partial derivative with respect to the first argument exists and depends continuously on both arguments. Define the set of *admissible control functions* by

$$\mathcal{U} := \{u : \mathbb{R} \to \mathbb{R}^m \mid u \text{ measurable and } u(t) \in U \text{ a.e.}\}.$$

The *shift flow* on $\mathcal{U}$ is given by

$$\Theta : \mathbb{R} \times \mathcal{U} \to \mathcal{U}, \quad \Theta(t,u) := \Theta_t u \quad \text{with } (\Theta_t u)(s) := u(t+s) \quad \text{for all } t,s \in \mathbb{R}.$$

We consider the control system

$$\dot{x}(t) = f(x(t), u(t)), \quad u \in \mathcal{U}. \tag{1}$$

For given initial value $x \in M$ and control function $u \in \mathcal{U}$ the solution of the initial value problem with $x(0) = x$ will be denoted by $\varphi(t,x,u)$. Throughout we assume that solutions are defined globally. This assumption is justified by the fact that we only consider trajectories which do not leave a compact subset of the state space M. Thus we obtain a cocycle $\varphi : \mathbb{R} \times M \times \mathcal{U} \to M$, i.e.

$$\varphi(t+s,x,u) = \varphi(s, \varphi(t,x,u), \Theta_t u) \quad \text{for all } t,s \in \mathbb{R}, \ x \in M, \ u \in \mathcal{U}. \tag{2}$$

The *reachable set or positive orbit* from $x \in M$ at time $t \geq 0$ is

$$\mathcal{O}_t^+(x) = \{\varphi(t,x,u) \mid u \in \mathcal{U}\}, \text{ and } \mathcal{O}^+(x) = \bigcup_{t \geq 0} \mathcal{O}_t^+(x).$$

A subset Q of the state space M is called *weakly invariant* (or *controlled invariant* or *viable*) if for all $x \in Q$ there is some $u \in \mathcal{U}$ with $\varphi(t,x,u) \in Q$ for all $t \geq 0$, and Q is called *strongly invariant* if $\mathcal{O}^+(x) \subset Q$ for all $x \in Q$.

3 Definition and Elementary Properties

This section presents the definition of several versions of invariance entropy. Basic properties of these notions are derived.

Consider the control system (1). Let $K, Q \subset M$ be nonvoid compact sets with $K \subset Q$ and Q weakly invariant. For given $T, \varepsilon > 0$ we call $\mathcal{S} \subset \mathcal{U}$ a (T,ε)*-spanning* set for (K,Q) if for every $x \in K$ there exists $v \in \mathcal{S}$ with

$$\varphi(t,x,v) \in N_\varepsilon(Q) := \{p \in M \mid \exists\, q \in Q : \ d(p,q) < \varepsilon\} \quad \text{for all } t \in [0,T];$$

here d denotes the Euclidean distance (note that this notion is different from the one used for topological entropy). By $r_{\text{inv}}(T,\varepsilon,K,Q)$ we denote the minimal cardinality of a (T,ε)-spanning set. A set $\mathcal{S}^* \subset \mathcal{U}$ is called T*-spanning* for (K,Q) if for every $x \in K$ there exists $v \in \mathcal{S}^*$ with

$$\varphi(t,x,v) \in Q \quad \text{for all } t \in [0,T].$$

The minimal cardinality of a T-spanning set is denoted by $r_{\text{inv}}^*(T,K,Q)$. If there is no finite T-spanning set we define $r_{\text{inv}}^*(T,K,Q) := \infty$. Let $0 < T_1 < T_2$. Since every

(T_2,ε)-spanning (T_2-spanning) set is obviously also (T_1,ε)-spanning (T_1-spanning), it follows that

$$r_{\text{inv}}(T_1,\varepsilon,K,Q) \le r_{\text{inv}}(T_2,\varepsilon,K,Q) \quad \text{and} \quad r^*_{\text{inv}}(T_1,K,Q) \le r^*_{\text{inv}}(T_2,K,Q).$$

Since every (T,ε_1)-spanning set is also (T,ε_2)-spanning if $\varepsilon_1 < \varepsilon_2$, we obtain

$$r_{\text{inv}}(T,\varepsilon_1,K,Q) \ge r_{\text{inv}}(T,\varepsilon_2,K,Q) \quad \text{for} \quad \varepsilon_1 < \varepsilon_2. \tag{3}$$

We define the *invariance entropy* $h_{\text{inv}}(K,Q)$ and the *strict invariance entropy* $h^*_{\text{inv}}(K,Q)$ by

$$h_{\text{inv}}(\varepsilon,K,Q) := \limsup_{T\to\infty} \frac{1}{T} \ln r_{\text{inv}}(T,\varepsilon,K,Q), \;\; h_{\text{inv}}(K,Q) := \lim_{\varepsilon\searrow 0} h_{\text{inv}}(\varepsilon,K,Q),$$
$$h^*_{\text{inv}}(K,Q) := \limsup_{T\to\infty} \frac{1}{T} \ln r^*_{\text{inv}}(T,K,Q).$$

From (3) it follows that the limit $\lim_{\varepsilon\searrow 0} h_{\text{inv}}(\varepsilon,K,Q)$ is well defined. If $K = Q$ we often suppress the argument K. Thus we write e.g. $r_{\text{inv}}(T,\varepsilon,Q)$ instead of $r_{\text{inv}}(T,\varepsilon,Q,Q)$.

Remark 1. *In general, it is not true that for the strict invariance entropy the numbers $r^*_{\text{inv}}(T,K,Q)$ are finite (compare the example at the end of Section 5). Hence we introduce the weaker version $h_{\text{inv}}(K,Q)$. In Section 4 we will show that $h_{\text{inv}}(K,Q)$ as defined above is finite.*

The following proposition summarizes the basic properties of these quantities.

Proposition 1. *Let $K,Q \subset M$ be nonvoid compact sets with $K \subset Q$ and Q weakly invariant for system (1). Then (i) $r_{\text{inv}}(T,\varepsilon,K,Q) < \infty$ for all $T,\varepsilon > 0$; (ii) $r^*_{\text{inv}}(T,Q)$ is either finite for all $T > 0$ or for none; (iii) the function $T \mapsto \ln r^*_{\text{inv}}(T,Q)$ is subadditive and consequently*

$$h^*_{\text{inv}}(Q) = \lim_{T\to\infty} \frac{1}{T} \ln r^*_{\text{inv}}(T,Q) = \inf_{T>0} \frac{1}{T} \ln r^*_{\text{inv}}(T,Q);$$

(iv) $h_{\text{inv}}(K,Q) \le h^*_{\text{inv}}(K,Q)$.

Remark 2. *From Proposition 1 (ii) and (iii) it follows that $h^*_{\text{inv}}(Q) < \infty$ if and only if $r^*_{\text{inv}}(T,Q) < \infty$ for one $T > 0$ if and only if $r^*_{\text{inv}}(T,Q) < \infty$ for all $T > 0$.*

In order to compute upper bounds for $h_{\text{inv}}(K,Q)$ it will be useful to define another quantity which will be called the *strong invariance entropy* for (K,Q): We define the *lift* of Q by

$$\mathcal{Q} := \{(x,u) \in Q \times \mathcal{U} \mid \varphi(t,x,u) \in Q \text{ for all } t \ge 0\}. \tag{4}$$

For given $T,\varepsilon > 0$ a set $\mathcal{S}^+ \subset \mathcal{Q}$ is called *strongly* (T,ε)*-spanning* for (K,Q) if for every $x \in K$ there exists $(y,v) \in \mathcal{S}^+$ with

$$d_{T,v}(x,y) := \max_{t\in[0,T]} d(\varphi(t,x,v),\varphi(t,y,v)) < \varepsilon.$$

By $r_{\rm inv}^{+}(T,\varepsilon,K,Q)$ we denote the minimal cardinality of a strongly (T,ε)-spanning set. As for $r_{\rm inv}(T,\varepsilon,K,Q)$ it follows by continuous dependence on initial conditions that $r_{\rm inv}^{+}(T,\varepsilon,K,Q)$ is finite. We define

$$h_{\rm inv}^{+}(\varepsilon,K,Q) := \limsup_{T\to\infty}\frac{1}{T}\ln r_{\rm inv}^{+}(T,\varepsilon,K,Q), \quad h_{\rm inv}^{+}(K,Q) := \lim_{\varepsilon\searrow 0} h_{\rm inv}^{+}(\varepsilon,K,Q).$$

Obviously $r_{\rm inv}^{+}(T,\varepsilon,K,Q)$, considered as a function of T and ε, has the same monotonicity properties as $r_{\rm inv}(T,\varepsilon,K,Q)$. Again, for $K=Q$ we drop the corresponding argument.

Proposition 2. *Let $K,Q\subset M$ be nonvoid compact sets with $K\subset Q$ and Q weakly invariant for system (1). Then $h_{\rm inv}(K,Q)\le h_{\rm inv}^{+}(K,Q)$.*

The following proposition summarizes some more properties of both invariance entropy and strict invariance entropy.

Proposition 3. *Let $K,Q\subset M$ be nonvoid compact sets with $K\subset Q$ and Q weakly invariant for system (1).*

(i) If there exist finitely many controls $u_1,\dots,u_n\in\mathcal{U}$ such that for every point $x\in K$ there exists $i\in\{1,\dots,n\}$ with $\varphi(\mathbb{R}_0^{+},x,u_i)\subset Q$, then

$$h_{\rm inv}(K,Q)=h_{\rm inv}^{*}(K,Q)=0.$$

In particular this holds if K is finite or if Q is strongly invariant.

(ii) For all $\varepsilon>0$ and $\tau>0$

$$h_{\rm inv}(\varepsilon,K,Q)=\limsup_{n\to\infty}\frac{1}{n\tau}\ln r_{\rm inv}(n\tau,\varepsilon,K,Q). \tag{5}$$

(iii) Let $K_i\subset K$, $i=1,\dots,N$, be closed subsets of K with $K=\bigcup_{i=1}^{N}K_i$. Then

$$h_{\rm inv}(K,Q)=\max_{i=1,\dots,N} h_{\rm inv}(K_i,Q).$$

Assertions (ii) and (iii) remain valid for the strict invariance entropy.

Remark 3. *Proposition 3 (ii) shows that for all time steps $\tau>0$ one obtains the same result. Hence from the invariance entropy one cannot deduce any information on maximum allowable time steps (cf. also Nesic/Teel [8]).*

The next result shows that the invariance entropy cannot increase under semiconjugation.

Proposition 4. *Consider two control systems $\dot{x}=f(x,u)$ and $\dot{y}=g(y,v)$ on M and N with corresponding solutions $\varphi(t,x,u)$ and $\psi(t,y,v)$ and control spaces $\mathcal{U}$ and $\mathcal{V}$. Let $\pi:M\to N$ be a continuous map and $h:\mathcal{U}\to\mathcal{V}$ any map the semiconjugation property*

$$\pi(\varphi(t,x,u))=\psi(t,\pi(x),h(u)) \text{ for all } x\in M,\ u\in\mathcal{U},\ t\ge 0. \tag{6}$$

Then

$$h_{\text{inv}}(\pi(K), \pi(Q)) \leq h_{\text{inv}}(K, Q),$$

if $K \subset Q \subset M$ are compact and Q is weakly invariant. The analogous statement holds for the strict invariance entropy.

4 General Bounds

For simplicity we assume throughout this section that $M = \mathbb{R}^d$. Again, $K, Q \subset \mathbb{R}^d$ are supposed to be nonvoid compact sets with $K \subset Q$ and Q being weakly invariant. We will provide rough bounds for $h_{\text{inv}}(K, Q)$ – one lower and one upper bound – which can be computed directly from the right hand side of the system. Since the upper bound is always finite we also prove finiteness of $h_{\text{inv}}(K, Q)$.

In the following we denote by $\text{div}_{\text{x}} f(x, u)$ the divergence of the function f with respect to the first variable, i.e.

$$\text{div}_{\text{x}} f(x, u) = \sum_{i=1}^{d} \frac{\partial f_i}{\partial x_i}(x, u) = \text{tr} \frac{\partial f}{\partial x}(x, u),$$

where $f_1, \ldots, f_d : \mathbb{R}^d \times \mathbb{R}^m \to \mathbb{R}$ are the coordinate functions of f.

Theorem 5. *If the Lebesgue measure $\lambda^d(K)$ of K is positive, then the following estimate holds.*

$$h_{\text{inv}}(K, Q) \geq \max \left\{ 0, \min_{(x,u) \in Q \times U} \text{div}_{\text{x}} f(x, u) \right\}. \quad (7)$$

The proof of this result is based on the Liouville trace formula. The next theorem, whose proof is a modification of [6, Theorem 3.3.9, p. 124], provides an upper bound for the strong invariance entropy and hence for the invariance entropy.

Theorem 6. *With $L := \max_{(x,u) \in Q \times U} \| \frac{\partial f}{\partial x}(x, u) \|$ the following estimate in terms of the fractal dimension $\dim_F(K)$ holds.*

$$h^+_{\text{inv}}(K, Q) \leq L \dim_F(K) \leq Ld. \quad (8)$$

Example 1. *For one-dimensional control systems Theorems 5 and 6 yield*

$$h_{\text{inv}}(K, Q) \in \left[\min_{(x,u) \in Q \times U} \frac{\partial f}{\partial x}(x, u), \max_{(x,u) \in Q \times U} \frac{\partial f}{\partial x}(x, u) \right],$$

if K has positive Lebesgue measure.

5 Linear Control Systems

In this section we compute the invariance entropy for control systems in $\mathbb{R}^d$ of the form

$$\dot{x}(t) = Ax(t) + Bu(t), \quad u \in \mathcal{U} \tag{9}$$

with matrices $A \in \mathbb{R}^{d\times d}$ and $B \in \mathbb{R}^{d\times m}$ and compact control range U.

Theorem 7. *Let $K, Q \subset \mathbb{R}^d$ be nonvoid compact sets with $K \subset Q$ and Q being weakly invariant. Denote the eigenvalues of A by $\lambda_1, \dots, \lambda_d$. Then the following estimate involving the real parts $\Re(\lambda_i)$ of the eigenvalues holds:*

$$h^+_{\text{inv}}(K, Q) \leq \sum_{i:\ \Re(\lambda_i)>0} \Re(\lambda_i).$$

If, in addition, K has positive Lebesgue measure, we have

$$h_{\text{inv}}(K, Q) = h^+_{\text{inv}}(K, Q) = \sum_{i:\ \Re(\lambda_i)>0} \Re(\lambda_i).$$

Remark 4. *The existence of a nonvoid compact weakly invariant subset for the linear control system (9) can be guaranteed, if the matrix A is hyperbolic and the control range U is compact and convex with nonvoid interior. Then there exists a unique control set D and its closure $Q = \text{cl}(D)$ is compact (see Colonius/Spadini [4, Theorem 4.1]). It is easily seen to be weakly invariant.*

At the end of this section we want to show by an example that $h^*_{\text{inv}}(Q) = \infty$ is possible even if $h_{\text{inv}}(Q) = 0$.

Example 2. *Consider the linear control system $\dot{x} = -x + u(t)$ on $\mathbb{R}$ with control range $U = [-1, 1]$ ($d = m = 1$). Let $Q \subset [-1, 1]$ be an infinite compact set which is totally disconnected (e.g. a Cantor set). Then for every $x \in Q$ there exists a unique constant control function $u_x \in \mathcal{U}$ with $\varphi(t, x, u_x) = x$ for all $t \geq 0$, namely $u_x(t) \equiv x$. Thus, Q is weakly invariant. Since Q is totally disconnected, each point $x \in Q$ can be kept in Q for some positive time $T > 0$ only by making it a stationary point, i.e. by using the constant control function u_x. Consequently, since Q is infinite, one needs infinitely many control functions to obtain a T-spanning set for Q. By Theorem 7 one has $h_{\text{inv}}(Q) = 0$ in this case.*

6 Characterization via Finite Covers and Relation to Feedback Entropy

In this last section we will give an alternative characterization of the strict invariance entropy $h^*_{\text{inv}}(Q)$ via finite covers of the set Q. Again, for simplicity we assume that $M = \mathbb{R}^d$. This definition will reveal a connection to the topological feedback entropy defined in [7], and will also provide a clearer view on what is measured by the quantity $h^*_{\text{inv}}(Q)$. Again consider the general control system (1).

For a finite cover $\mathcal{A}$ of Q let $c(\mathcal{A}|Q)$ denote the minimal cardinality of a subcover. We say that a triple $(\mathcal{A}, v, \tau)$ is *invariantly covering* Q if τ is a positive real number, $\mathcal{A}$ is a finite cover of Q and $v : \mathcal{A} \to \mathcal{U}$ is a map assigning a control function $v_A \in \mathcal{U}$ to each $A \in \mathcal{A}$ with $\varphi(t, A, v_A) \subset Q$ for all $t \in [0, \tau]$. If Q is invariantly covered by a triple $(\mathcal{A}, v, \tau)$, where $\mathcal{A} = \{A_1, A_2, \ldots, A_q\}$ is ordered, then we set $v_a := v_{A_a}$ for $a = 1, \ldots, q$. For every $N \in \mathbb{N}$ and every N-tuple $(a_0, a_1, \ldots, a_{N-1}) \in \{1, \ldots, q\}^N$ we define the control function

$$v_{a_0,a_1,\ldots,a_{N-1}}(t) := v_{a_j}(t - j\tau), \quad \text{for all } t \in [j\tau, (j+1)\tau), \quad j = 0, 1, \ldots, N-1,$$

and the set

$$Q_{a_0,a_1,\ldots,a_{N-1}} := \left\{x \in Q \mid \varphi(j\tau, x, v_{a_0,a_1,\ldots,a_{N-1}}) \in A_{a_j},\ j = 0, 1, \ldots, N-1\right\}. \tag{10}$$

For every $a \in \{1, \ldots, q\}$ we define the diffeomorphism $f_a : \mathbb{R}^d \to \mathbb{R}^d$, $f_a(x) := \varphi(\tau, x, v_a)$. Then the cocycle property (2) yields

$$Q_{a_0,a_1,\ldots,a_{N-1}} = A_{a_0} \cap \bigcap_{j=1}^{N-1} (f_{a_{j-1}} \circ \cdots \circ f_{a_1} \circ f_{a_0})^{-1}(A_{a_j}). \tag{11}$$

Let $\mathcal{A}_N := \left\{Q_{a_0,a_1,\ldots,a_{N-1}} \,|(a_0, a_1, \ldots, a_{N-1}) \in \{1, \ldots, q\}^N\right\}$ be the family of these sets. Then $\mathcal{A}_N$ is also a finite cover of Q (moreover, it is an open cover, if $\mathcal{A}$ is an open cover, since in this case openness follows immediately from equation (11)): For every $x \in Q$ we find at least one N-tuple $(a_0, a_1, \ldots, a_{N-1})$ (which may be not unique) with $\varphi(j\tau, x, v_{a_0,a_1,\ldots,a_{N-1}}) \in A_{a_j}$ for $j = 0, 1, \ldots, N-1$, which follows by the invariant covering property of $(\mathcal{A}, v, \tau)$. Now we define

$$h^*_{\text{inv}}(\mathcal{A}, v, \tau) := \frac{1}{\tau} \lim_{N\to\infty} \frac{\ln c(\mathcal{A}_N|Q)}{N}. \tag{12}$$

It can easily be shown that $h^*_{\text{inv}}(\mathcal{A}, v, \tau)$ does not depend on the ordering of the set $\mathcal{A}$. The existence of the limit above follows from a subadditivity argument.

Theorem 8. *For the control system (1) the strict invariance entropy and the entropy (12) defined via covers satisfy*

$$h^*_{\text{inv}}(Q) = \inf_{(\mathcal{A},v,\tau)} h^*_{\text{inv}}(\mathcal{A}, v, \tau), \tag{13}$$

where the infimum is taken over all triples which are invariantly covering Q.

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