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Capacity per Unit-Energy of Gaussian Random Many-Access Channels

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***Abstract*—We consider a Gaussian multiple-access channel with random user activity where the total number of users ℓ_n and the average number of active users k_n may be unbounded. For this channel, we characterize the maximum number of bits that can be transmitted reliably per unit-energy in terms of ℓ_n and k_n. We show that if $k_n \log \ell_n$ is sublinear in n, then each user can achieve the single-user capacity per unit-energy. Conversely, if $k_n \log \ell_n$ is superlinear in n, then the capacity per unit-energy is zero. We further demonstrate that orthogonal-access schemes, which are optimal when all users are active with probability one, can be strictly suboptimal.**

I. Introduction

Chen *et al.* [1] introduced the many-access channel (MnAC) as a multiple-access channel (MAC) where the number of users grows with the blocklength and each user is active with a given probability. This model is motivated by systems consisting of a single receiver and many transmitters, the number of which is comparable or even larger than the blocklength, a situation that may occur, *e.g.*, in a machine-to-machine communication system with many thousands of devices in a given cell that are active only sporadically. In [1], Chen *et al.* considered a Gaussian MnAC with ℓ_n users, each of which is active with probability α_n, and determined the number of messages M_n each user can transmit reliably with a codebook of average power not exceeding P. Since then, MnACs have been studied in various papers under different settings. For example, Polyanskiy [2] considered a Gaussian MnAC where the number of active users grows linearly in the blocklength and each user's payload is fixed. Zadik *et al.* [3] presented improved bounds on the tradeoff between user density and energy-per-bit of this channel. Low-complexity schemes for the MnAC were studied in [4], [5]. Generalizations to quasi-static fading MnACs can be found in [6]–[9]. Shahi *et al.* [10] studied the capacity region of strongly asynchronous MnACs.

Recently, we studied the capacity per unit-energy of the Gaussian MnAC as a function of the order of growth of users when all users are active with probability one [11]. We showed that if the order of growth is above $n/\log n$, then the capacity per unit-energy is zero, and if the order of growth is below $n/\log n$, then each user can achieve the singe-user capacity per unit-energy. Thus, there is a sharp transition between orders of growth where interference-free communication is feasible and orders of growth where reliable communication at a positive rate is infeasible. We further showed that the capacity per unit-energy can be achieved by an *orthogonal-access scheme* where the codewords of different users are orthogonal to each other.

In this paper, we extend the analysis of [11] to a random-access setting. In particular, we consider a setting where the total number of users ℓ_n may grow as an arbitrary function of the blocklength and the probability α_n that a user is active may be a function of the blocklength, too. Let $k_n = \alpha_n \ell_n$ denote the average number of active users. We demonstrate that if $k_n \log \ell_n$ is sublinear in n, then each user can achieve the single-user capacity per unit-energy. Conversely, if $k_n \log \ell_n$ is superlinear in n, then the capacity per unit-energy is zero. Hence, there is again a sharp transition between orders of growth where interference-free communication is feasible and orders of growth where reliable communication at a positive rate is infeasible, but the transition threshold depends on the behaviors of both ℓ_n and k_n. We further show that orthogonal-access schemes, which are optimal when $\alpha_n = 1$, are strictly suboptimal when $\alpha_n \to 0$.

The rest of the paper is organized as follows. Section II introduces the system model. Section III presents our main results. Section IV briefly discusses the capacity per unit-energy when the error probability is replaced by the so-called *per-user probability of error* considered, e.g., in [2]–[9].

II. Problem Formulation and Preliminaries

A. Model and Definitions

Consider a network with ℓ users that, if they are active, wish to transmit their messages $W_i, i = 1, \ldots, \ell$ to one common receiver. The messages are assumed to be independent and uniformly distributed on $\{1, \ldots, M_n^{(i)}\}$. To transmit their messages, the users send a codeword of n symbols over the channel, where n is referred to as the *blocklength*. We consider a many-access scenario where the number of users ℓ grows with n, hence, we denote it as ℓ_n. We further assume that a user is active with probability α_n, where $\alpha_n \to \alpha \in [0, 1]$ as

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n tends to infinity. Since an inactive user is equivalent to a user transmitting the all-zero codeword, we can express the distribution of the i-th user's message as

$$\Pr\{W_i = w\} = \begin{cases} 1-\alpha_n, & w = 0 \\ \frac{\alpha_n}{M_n^{(i)}}, & w \in \{1, \ldots, M_n^{(i)}\} \end{cases} \tag{1}$$

and assume that the codebook is such that message 0 is mapped to the all-zero codeword. We denote the average number of active users at blocklength n by k_n, i.e., $k_n = \alpha_n \ell_n$.

We consider a Gaussian channel model where the received vector $\mathbf{Y}$ is given by

$$\mathbf{Y} = \sum_{i=1}^{\ell_n} \mathbf{X}_i(W_i) + \mathbf{Z}.$$

Here $\mathbf{X}_i(W_i)$ is the n-length transmitted codeword from user i for message W_i and $\mathbf{Z}$ is a vector of n i.i.d. Gaussian components $Z_j \sim \mathcal{N}(0, N_0/2)$ independent of $\mathbf{X}_i$.

Definition 1: For $0 \leq \epsilon < 1$, an $(n, \{M_n^{(\cdot)}\}, \{E_n^{(\cdot)}\}, \epsilon)$ code for the Gaussian many-access channel consists of:

1) Encoding functions $f_i : \{0, 1, \ldots, M_n^{(i)}\} \to \mathbb{R}^n$, $i = 1, \ldots, \ell_n$ which map user i's message to the codeword $\mathbf{X}_i(W_i)$, satisfying the energy constraint
$$\sum_{j=1}^{n} x_{ij}^2(w_i) \leq E_n^{(i)} \tag{2}$$
where x_{ij} is the j-th symbol of the transmitted codeword. If $W_i = 0$, then $x_{ij} = 0$ for $j = 1, \ldots, n$.
2) Decoding function $g : \mathbb{R}^n \to \{0, 1, \ldots, M_n^{(1)}\} \times \ldots \times \{0, 1, \ldots, M_n^{(\ell_n)}\}$ which maps the received vector $\mathbf{Y}$ to the messages of all users and whose probability of error $P_e^{(n)}$ satisfies
$$P_e^{(n)} \triangleq \Pr\{g(\mathbf{Y}) \neq (W_1, \ldots, W_{\ell_n})\} \leq \epsilon. \tag{3}$$

An $(n, \{M_n^{(\cdot)}\}, \{E_n^{(\cdot)}\}, \epsilon)$ code is said to be *symmetric* if $M_n^{(i)} = M_n$ and $E_n^{(i)} = E_n$ for all $i = 1, \ldots, \ell_n$. For compactness, we denote such a code by (n, M_n, E_n, ϵ). In this paper, we restrict ourselves to symmetric codes.

Definition 2: For a symmetric code, the rate per unit-energy $\dot{R}$ is said to be ϵ-achievable if for every $\delta > 0$ there exists an n_0 such that if $n \geq n_0$, then an (n, M_n, E_n, ϵ) code can be found whose rate per unit-energy satisfies $\frac{\log M_n}{E_n} > \dot{R} - \delta$. Furthermore, $\dot{R}$ is said to be achievable if it is ϵ-achievable for all $0 < \epsilon < 1$. The capacity per unit-energy $\dot{C}$ is the supremum of all achievable rates per unit-energy.

B. Order Notations

Let $\{a_n\}$ and $\{b_n\}$ be two sequences of nonnegative real numbers. We write $a_n = O(b_n)$ if there exists an n_0 and a positive real number S such that for all $n \geq n_0$, $a_n \leq S b_n$. We write $a_n = o(b_n)$ if $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$, and $a_n = \Omega(b_n)$ if $\liminf_{n\to\infty} \frac{a_n}{b_n} > 0$. Similarly, $a_n = \Theta(b_n)$ indicates that there exist $0 < l_1 < l_2$ and n_0 such that $l_1 b_n \leq a_n \leq l_2 b_n$ for all $n \geq n_0$. We write $a_n = \omega(b_n)$ if $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$.

III. Capacity per Unit-Energy

In this section, we discuss our results on the behavior of capacity per unit-energy for Gaussian random MnACs. Our main result is Theorem 1, which characterizes the capacity per unit-energy in terms of ℓ_n and k_n. In Theorem 2, we characterize the behavior of the largest rate per unit-energy that can be achieved by an orthogonal-access scheme. These results are presented in Subsection III-A. The proofs of Theorems 1 and 2 are given in Subsections III-B and III-C, respectively.

Before presenting our results, we first note that the case where k_n vanishes as $n \to \infty$ is uninteresting. Indeed, this case only happens if $\alpha_n \to 0$. Then, the probability that all the users are inactive, given by $\left((1-\alpha_n)^{\frac{1}{\alpha_n}}\right)^{k_n}$, tends to one since $(1-\alpha_n)^{\frac{1}{\alpha_n}} \to 1/e$ and $k_n \to 0$. Consequently, a code with $M_n = 2$ and $E_n = 0$ for all n and a decoding function that always declares that all users are inactive achieve an error probability $P_e^{(n)}$ that vanishes as $n \to \infty$. This implies that $\dot{C} = \infty$. In the following, we avoid this trivial case and assume that ℓ_n and α_n are such that k_n is bounded away from zero.

A. Our Main Results

Theorem 1: Assume that $k_n = \Omega(1)$. Then the capacity per unit-energy of the Gaussian random MnAC has the following behavior:

1) If $k_n \log \ell_n = o(n)$, then $\dot{C} = (\log e)/N_0$.
2) If $k_n \log \ell_n = \omega(n)$, then $\dot{C} = 0$.

Proof: See Subsection III-B. ■

Theorem 1 demonstrates that there is a sharp transition between orders of growth where interference-free communication is feasible and orders of growth where no positive rate per unit-energy is feasible. The same behavior was observed for the non-random-access case, where the transition threshold seperating these two regimes is at $n/\log n$ [11]. When α_n converges to a positive value, the order of growth of $k_n \log \ell_n$ coincides with that of both $k_n \log k_n$ and $\ell_n \log \ell_n$. In this case, the transition threshold in the random-access case is also at $n/\log n$. However, when $\alpha_n \to 0$, the orders of growth of k_n and ℓ_n are different and the transition threshold for ℓ_n is in general larger than $n/\log n$, so random user-activity enables interference-free communication at an order of growth above the limit $n/\log n$ of the non-random-access case. Similarly, when $\alpha_n \to 0$, the transition threshold for k_n is in general smaller than $n/\log n$, so treating a random MnAC with ℓ_n users as a non-random MnAC with k_n users may be overly-optimistic.

In [11], it was shown that, when $k_n = o(n/\log n)$ and $\alpha_n = 1$, an orthogonal-access scheme is sufficient to achieve the capacity per unit-energy. It turns out that this is not the case anymore when $\alpha_n \to 0$.

Theorem 2: Assume that $k_n = \Omega(1)$. The largest rate per unit-energy $\dot{C}_\perp$ achievable with an orthogonal-access scheme satisfies the following:

1) If $\ell_n = o(n/\log n)$, then $\dot{C}_\perp = (\log e)/N_0$.
2) If $\ell_n = \omega(n/\log n)$, then $\dot{C}_\perp = 0$.

Proof: See Subsection III-C. ■

Observe that there is again a sharp transition between the orders of growth of ℓ_n where interference-free communication is feasible and orders of growth where no positive rate per unit-energy is feasible. In contrast to the optimal transmission scheme, the transition threshold for orthogonal-access schemes happens at $n/\log n$, irrespective of the behavior of α_n. Thus, by using an orthogonal-access scheme, we treat the random MnAC as if it were a non-random MnAC. Theorem 2 also implies that there are orders of growth of ℓ_n and k_n where non-orthogonal-access schemes are necessary to achieve the capacity per unit-energy.

B. Proof of Theorem 1

To prove Part 1), we use an achievability scheme with a decoding process consisting of two steps. First, the receiver determines which users are active. If the number of estimated active users is less than or equal to ξk_n for some positive integer ξ, then the receiver decodes the messages of all active users. If the number of estimated active users is greater than ξk_n, then it declares an error. The total error probability of this scheme is upper-bounded by

$$P(\mathcal{D}) + \sum_{k'_n=1}^{\xi k_n} \Pr\{K'_n = k'_n\} P\big(\mathcal{E}_m(k'_n)\big) + \Pr\{K'_n > \xi k_n\}$$

where K'_n is the number of active users, $P(\mathcal{D})$ is the probability of a detection error, and $P\big(\mathcal{E}_m(k'_n)\big)$ is the probability of a decoding error when the receiver has correctly detected that there are k'_n users active. In the following, we show that these probabilities vanish as $n \to \infty$ for any fixed, positive integer ξ. Furthermore, by Markov's inequality, we have that $\Pr\{K'_n > \xi k_n\} \le 1/\xi$. It thus follows that the total probability of error vanishes as we let first $n \to \infty$ and then $\xi \to \infty$.

To enable user detection at the receiver, out of n channel uses, each user uses the first n'' channel uses to send its signature and $n' = n - n''$ channel uses for sending the message. Furthermore, the signature uses energy E''_n out of E_n, while the energy used for sending message is given by $E'_n = E_n - E''_n$.

Let $\mathbf{s}_i$ denote the signature of user i and $\tilde{\mathbf{x}}_i(w_i)$ denote the codeword of length n' for sending the message w_i, where $w_i = 1, \dots, M_n$. Then the codeword $\mathbf{x}_i(w_i)$ is given by

$$\mathbf{x}_i(w_i) = (\mathbf{s}_i, \tilde{\mathbf{x}}_i(w_i)).$$

Explicitly, for a given arbitrary $0 < b < 1$, we let

$$n'' = bn, \tag{4}$$

and

$$E''_n = bE_n, \quad E_n = c_n \ln \ell_n \tag{5}$$

with $c_n = \ln(\frac{n}{k_n \ln \ell_n})$.

Based on the first n'' received symbols, the receiver detects which users are active. We need the following lemma to show that the detection error probability vanishes as $n \to \infty$.

Lemma 1: If $k_n \log \ell_n = o(n)$, then there exist signatures $\mathbf{s}_i, i = 1, \dots, \ell_n$ with n'' channel uses and energy E''_n such that $P(\mathcal{D})$ vanishes as $n \to \infty$.

Proof: The proof follows along similar lines as that of [1, Theorem 2]. For details, see the extended version of this paper [12]. ■

We next use the following lemma to show that $P\big(\mathcal{E}_m(k'_n)\big)$ vanishes as $n \to \infty$.

Lemma 2: Let $A_n \triangleq \frac{1}{k'_n} \sum_{i=1}^{k'_n} \mathbf{1}(\hat{W}_i \neq W_i)$ and $\mathcal{A}_n \triangleq \{1/k'_n, \dots, 1\}$, where $\mathbf{1}(\cdot)$ denotes the indicator function. Then for any arbitrary $0 < \rho \le 1$, we have

$$\Pr\{A_n = a\} \le \binom{k'_n}{ak'_n} M_n^{ak'_n\rho} e^{-nE_0(a,\rho)}, \quad a \in \mathcal{A}_n$$

where

$$E_0(a,\rho) \triangleq \frac{\rho}{2} \ln\left(1 + \frac{a2k'_nE'_n}{n'(\rho+1)N_0}\right).$$

Proof: See [13, Theorem 2]. ■

The probability of error $P\big(\mathcal{E}_m(k'_n)\big)$ can be written as

$$P\big(\mathcal{E}_m(k'_n)\big) = \sum_{a \in \mathcal{A}_n} \Pr\{A_n = a\}. \tag{6}$$

Using Lemma 2, we upper-bound $\Pr\{A_n = a\}$ as

$$\begin{aligned}\Pr\{A_n = a\} &\le \binom{k'_n}{ak'_n} M_n^{ak'_n\rho} \exp[-n'E_0(a,\rho)] \\ &\le \exp\left[k'_n H_2(a) + a\rho k'_n \log M_n - n'E_0(a,\rho)\right] \\ &= \exp\left[-E'_n f_n(a,\rho)\right]\end{aligned}$$

where

$$f_n(a,\rho) \triangleq \frac{n'E_0(a,\rho)}{E'_n} - \frac{a\rho k'_n \log M_n}{E'_n} - \frac{k'_n H_2(a)}{E'_n}.$$

We next show that, for sufficiently large n, we have

$$\Pr\{A_n = a\} \le \exp\left[-E'_n f_n(1/k'_n, \rho)\right], \quad a \in \mathcal{A}_n. \tag{7}$$

To this end, we first note that for any fixed value of ρ and our choices of E'_n and $\dot{R}$,

$$\liminf_{n\to\infty} \frac{df_n(a,\rho)}{da} > 0, \quad a \in \mathcal{A}_n.$$

This follows from [14, Eq. (11)] and the fact that $\frac{k'_n E'_n}{n'} \to 0$ as $n \to \infty$, which in turn follows from our choice of E'_n and since $k'_n = o(n/\log n)$. Hence, there exists an $n_0 > 0$ such that

$$\min_{a \in \mathcal{A}_n} f_n(a,\rho) \ge f_n(1/k'_n, \rho), \quad n \ge n_0.$$

Next we show that, for our choice of E'_n and $\dot{R} = \frac{(1-b)\log e}{(1+\rho)N_0} - \delta$ (for some arbitrary $0 < \delta < \frac{(1-b)\log e}{(1+\rho)N_0}$), we have

$$\liminf_{n\to\infty} f_n(1/k'_n, \rho) > 0. \tag{8}$$

Let

$$i_n(1/k'_n,\rho) \triangleq \frac{n' E_0(1/k'_n,\rho)}{E'_n}$$
$$j(\rho) \triangleq \frac{\rho \dot{R}}{(1-b)\log e}$$
$$h_n(1/k'_n) \triangleq \frac{k'_n H_2(1/k'_n)}{E'_n}.$$

Note that $\frac{h_n(1/k'_n)}{j(\rho)}$ vanishes as $n \to \infty$ for our choice of E'_n. Consequently,

$$\liminf_{n\to\infty} f_n(1/k'_n,\rho) = j(\rho)\left\{\liminf_{n\to\infty} \frac{i_n(1/k'_n,\rho)}{j(\rho)} - 1\right\}.$$

The term $j(\rho) = \rho\dot{R}/(1-b)\log e$ is bounded away from zero for our choice of $\dot{R}$ and $\delta < \frac{(1-b)\log e}{(1+\rho)N_0}$. Furthermore, since $E'_n/n' \to 0$, we get

$$\lim_{n\to\infty} \frac{i_n(1/k'_n,\rho)}{j(\rho)} = \frac{(1-b)\log e}{(1+\rho)N_0\dot{R}}$$

which is strictly larger than 1 for our choice of $\dot{R}$. So, (8) follows. We conclude that there exist two positive constants n_0 and γ such that, for $n \geq n_0$,

$$\Pr\{A_n = a\} \leq e^{-E'_n\gamma}, \quad a \in \mathcal{A}_n. \tag{9}$$

Since $|\mathcal{A}_n| = k'_n$, it follows from (6) and (9) that

$$P\big(\mathcal{E}_m(k'_n)\big) \leq k'_n e^{-E'_n\gamma}. \tag{10}$$

Noting that $E'_n = (1-b)c_n \ln \ell_n$ and $k'_n = O(\ell_n)$, it follows that $P\big(\mathcal{E}_m(k'_n)\big)$ tends to 0 as $n \to \infty$ for our choice of $\dot{R} = \frac{(1-b)\log e}{(1+\rho)N_0} - \delta$. Since ρ, δ, and b are arbitrary, any rate $\dot{R} < \frac{\log e}{N_0}$ is thus achievable. This proves Part 1) of Theorem 1.

Next we prove Part 2). Let $\hat{W}_i$ denote the receiver's estimate of W_i, and denote by $\mathbf{W}$ and $\hat{\mathbf{W}}$ the vectors $(W_1,\ldots,W_{\ell_n})$ and $(\hat{W}_1,\ldots,\hat{W}_{\ell_n})$, respectively. The messages $W_1,\ldots,W_{\ell_n}$ are independent, so it follows from (1) that

$$H(\mathbf{W}) = \ell_n H(\mathbf{W}_1) = \ell_n\left(H_2(\alpha_n) + \alpha_n \log M_n\right)$$

where $H_2(\cdot)$ denotes the binary entropy function. Since $H(\mathbf{W}) = H(\mathbf{W}|\mathbf{Y}) + I(\mathbf{W};\mathbf{Y})$, we obtain

$$\ell_n\left(H_2(\alpha_n) + \alpha_n \log M_n\right) = H(\mathbf{W}|\mathbf{Y}) + I(\mathbf{W};\mathbf{Y}). \tag{11}$$

To bound $H(\mathbf{W})$, we use the upper bounds [1, Lemma 2]

$$H(\mathbf{W}|\mathbf{Y}) \leq \log 4 + 4P_e^{(n)}\big(k_n \log M_n + k_n + \ell_n H_2(\alpha_n) + \log M_n\big) \tag{12}$$

and [1, Lemma 1]

$$I(\mathbf{W};\mathbf{Y}) \leq \frac{n}{2}\log\left(1 + \frac{2k_nE_n}{nN_0}\right). \tag{13}$$

Using (12) and (13) in (11), rearranging terms, and dividing by k_nE_n, yields

$$\Big(1 - 4P_e^{(n)}(1+1/k_n)\Big)\dot{R} \leq \frac{\log 4}{k_nE_n} + \frac{H_2(\alpha_n)}{\alpha_nE_n}\Big(4P_e^{(n)} - 1\Big) + 4P_e^{(n)}(1/E_n + 1/k_n) + \frac{n}{2k_nE_n}\log\left(1 + \frac{2k_nE_n}{nN_0}\right). \tag{14}$$

We next show that if $k_n \log \ell_n = \omega(n)$, then the right-hand side (RHS) of (14) tends to a non-positive value. To this end, we need the following lemma.

Lemma 3: If $\dot{R} > 0$, then $P_e^{(n)}$ vanishes as $n \to \infty$ only if $E_n = \Omega(\log \ell_n)$.

Proof: The proof of this lemma follows along similar lines as that of [11, Lemma 2]. For details, see [12]. ■

Part 2) of Theorem 1 follows now by contradiction. Indeed, let us assume that $k_n \log \ell_n = \omega(n)$, $P_e^{(n)} \to 0$, and $\dot{R} > 0$. Then, Lemma 3 together with the assumption that $k_n = \Omega(1)$ implies that $k_nE_n = \omega(n)$. It follows that the last term on the RHS of (14) tends to zero as $n \to \infty$. The assumption $k_n \log \ell_n = \omega(n)$ in turn implies that $\ell_n \to \infty$ as $n \to \infty$. So, by Lemma 3, $E_n \to \infty$. Together with the assumption that $k_n = \Omega(1)$, this implies that the first and third term on the RHS of (14) vanish as $n \to \infty$. Finally, $\frac{H_2(\alpha_n)}{\alpha_nE_n}$ is a sequence of non-negative numbers and $(4P_e^{(n)} - 1) \to -1$ as $n \to \infty$, so the second term converges to a non-positive value. Thus, we obtain that $\dot{R}$ tends to a non-positive value as $n \to \infty$. This contradicts the assumption $\dot{R} > 0$, so Part 2) of Theorem 1 follows.

C. *Proof of Theorem 2*

To prove Part 1), we present a scheme that is similar to the one given in [11] for the non-random-access case. Specifically, each user is assigned n/ℓ_n channel uses out of which the first one is used for sending a pilot signal and the rest are used for sending the message. Out of the available energy E_n, tE_n for some arbitrary $0 < t < 1$ is used for the pilot signal and $(1-t)E_n$ is used for sending the message. Let $\tilde{\mathbf{x}}(w)$ denote the codeword of length $\frac{n}{\ell_n} - 1$ for sending message w. Then user i sends in his assigned slot the codeword

$$\mathbf{x}(w_i) = \left(\sqrt{tE_n}, \tilde{\mathbf{x}}(w_i)\right).$$

The receiver first detects from the pilot signal whether user i is active or not. If the user is estimated as active, then it decodes the user's message. Let $P_i = \Pr\{\hat{W}_i \neq W_i\}$ denote the probability that user i's message is decoded erroneously. Since all users follow the same coding scheme, the probability of correct decoding is given by

$$P_c^{(n)} = \left(1 - P_1\right)^{\ell_n}. \tag{15}$$

By employing the transmission scheme that was used to prove [11, Theorem 2], we get an upper bound on the probability of error P_1 as follows.

Lemma 4: For $n \geq n_0$ and sufficiently large n_0, the probability of error in decoding user 1's message can be upper-bounded as:

$$P_1 \leq \frac{2}{n^2}.$$

Proof: See [12]. ■

From Lemma 4 and (15),

$$P_c^{(n)} \geq \left(1 - \frac{2}{n^2}\right)^{\ell_n}$$
$$\geq \left(1 - \frac{2}{n^2}\right)^{\frac{n}{\log n}}$$

which tends to one as $n \to \infty$. Thus, Part 1) of Theorem 2 follows.

To prove Part 2), we first note that we consider symmetric codes, i.e., the pair (M_n, E_n) is the same for all users. However, each user may be assigned different numbers of channel uses. Let n_i denote the number of channel uses assigned to user i. For an orthogonal-access scheme, if $\ell_n = \omega(n/\log n)$, then there exists at least one user, say $i = 1$, such that $n_i = o(\log n)$. Using that $H(W_1|W_1 \neq 0) = \log M_n$, it follows from Fano's inequality that

$$\log M_n \leq 1 + P_1 \log M_n + \frac{n_1}{2} \log\left(1 + \frac{2E_n}{n_1 N_0}\right).$$

This implies that the rate per unit-energy $\dot{R} = (\log M_n)/E_n$ for user 1 is upper-bounded by

$$\dot{R} \leq \frac{\frac{1}{E_n} + \frac{n_1}{2E_n}\log\left(1 + \frac{2E_n}{n_1 N_0}\right)}{1 - P_1}. \tag{16}$$

Since $\ell_n = \omega(n/\log n)$, it follows from Lemma 3 that $P_e^{(n)}$ goes to zero only if

$$E_n = \Omega(\log n). \tag{17}$$

In contrast, (16) implies that $\dot{R} > 0$ only if $E_n = O(n_1)$. Since $n_1 = o(\log n)$, this further implies that

$$E_n = o(\log n). \tag{18}$$

No sequence $\{E_n\}$ can satisfy both (18) and (17) simultaneously. We thus obtain that if $\ell_n = \omega(n/\log n)$, then the capacity per unit-energy is zero. This is Part 2) of Theorem 2.

IV. Per-User Probability of Error

Many works in the literature on many-access channels, including [2]–[9], consider a *per-user probability of error*

$$P_{e,A}^{(n)} \triangleq \frac{1}{\ell_n}\sum_{i=1}^{\ell_n} \Pr\{\hat{W}_i \neq W_i\} \tag{19}$$

rather than the joint error probability (3). In the following, we briefly discuss the behavior of the capacity per unit-energy when the error probability is $P_{e,A}^{(n)}$, which in this paper we shall refer to as *average probability of error (APE)*. To this end, we define an $(n, \{M_n^{(\cdot)}\}, \{E_n^{(\cdot)}\}, \epsilon)$ code under APE with the same encoding and decoding functions defined in Section II, but with the probability of error (3) replaced with (19). We denote the capacity per unit-energy under APE by $\dot{C}^A$.

Under APE, if $\alpha_n \to 0$ as $n \to \infty$, then $\Pr\{W_i = 0\} \to 1$ for all $i = 1, \ldots, \ell_n$. Consequently, a code with $M_n = 2$ and $E_n = 0$ for all n and a decoding function that always declares that all users are inactive achieves an APE that vanishes as $n \to \infty$. This implies that $\dot{C}^A = \infty$ for vanishing α_n. In the following, we avoid this trivial case and assume that α_n is bounded away from zero.

For a Gaussian MnAC with APE and $\alpha_n = 1$ (non-random-access case), we showed in [14] that if the number of users grows sublinear in n, then each user can achieve the single-user capacity per unit-energy, and if the order of growth is linear or superlinear, then the capacity per unit-energy is zero. Perhaps not surprisingly, the same result holds in the random-access case since, when α_n is bounded away from zero, k_n is of the same order as ℓ_n.

Theorem 3: If $k_n = \Theta(\ell_n)$ and $\alpha_n \to \alpha \in (0, 1]$, then $\dot{C}^A$ has the following behavior:

1) If $\ell_n = o(n)$, then $\dot{C}^A = \frac{\log e}{N_0}$. Moreover, the capacity per unit-energy can be achieved by an orthogonal-access scheme where each user uses a codebook with orthogonal codewords.
2) If $\ell_n = \Omega(n)$, then $\dot{C}^A = 0$.

Proof: To prove Part 1), we first argue that $P_{e,A}^{(n)} \to 0$ only if $E_n \to \infty$. Indeed, we have

$$P_{e,A}^{(n)} \geq \min_i \Pr\{\hat{W}_i \neq W_i\}$$
$$\geq \alpha_n \Pr(\hat{W}_i \neq W_i | W_i \neq 0) \quad \text{for some } i.$$

Since $\alpha_n \to \alpha > 0$, this implies that $P_{e,A}^{(n)}$ vanishes only if $\Pr(\hat{W}_i \neq W_i|W_i \neq 0)$ vanishes. We next note that $\Pr(\hat{W}_i \neq W_i|W_i \neq 0)$ is lower-bounded by the error probability of the Gaussian single-user channel. By following the arguments in the proof of [14, Theorem 2, Part 1)], we obtain that $P_{e,A}^{(n)} \to 0$ only if $E_n \to \infty$, which also implies that $\dot{C}^A \leq \frac{\log e}{N_0}$.

We next show that any rate per unit-energy $\dot{R} < \frac{\log e}{N_0}$ is achievable by an orthogonal-access scheme where each user uses an orthogonal codebook of blocklength n/ℓ_n. Out of these n/ℓ_n channel uses, the first one is used for sending a pilot signal to convey whether the user is active or not, and the remaining channel uses are used to send the message. Specifically, to transmit message w_i, user i sends in his assigned slot the codeword $\mathbf{x}(w_i) = (x_1(w_1), \ldots, x_{n/\ell_n}(w_i))$, which is given by

$$x_k(w_i) = \begin{cases} \sqrt{tE_n}, & \text{if } k = 1 \\ \sqrt{(1-t)E_n}, & \text{if } k = w_i + 1 \\ 0, & \text{otherwise.} \end{cases}$$

From the pilot signal, the receiver first detects whether the user is active or not. As shown in the proof of Lemma 4, the detection error vanishes as $n \to \infty$. Using the upper bound on the decoding-error probability for an orthogonal code with M codewords and rate per unit-energy $\dot{R}$ given in [15, Lemma 3], we can then show that $P_i, i = 1, \ldots, \ell_n$ vanishes as n tends to infinity. This implies that also $P_{e,A}^{(n)}$ vanishes as $n \to \infty$. More details can be found in [12].

The proof of Part 2) follows from Fano's inequality and is similar to that of [14, Theorem 2, Part 2)]. Details can be found in [12]. ∎

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