11-2020

Correlation-robust auction design

Wei HE

Jiangtao Li

Correlation-Robust Auction Design*

Wei He† Jiangtao Li‡

November 9, 2020

Abstract

We study the design of auctions when the auctioneer has limited statistical information about the joint distribution of the bidders' valuations. More specifically, we consider an auctioneer who has an estimate of the marginal distribution of a generic bidder's valuation but does not have reliable information about the correlation structure. We analyze the performance of mechanisms in terms of revenue guarantee, that is, the greatest lower bound of revenue across all joint distributions that are consistent with the marginals. A simple auction format, the second-price auction with no reserve price, is shown to be asymptotically optimal. Furthermore, for any finite number of bidders, we solve for the robustly optimal reserve price that generates the highest revenue guarantee among second-price auctions with reserve prices.

KEYWORDS: Robust mechanism design, correlation, second-price auction, low reserve price, duality approach, optimal transport.

---

*An earlier version of this paper was circulated under the title "Robustly Optimal Reserve Price." First version April 2019.

†Department of Economics, The Chinese University of Hong Kong, hewei@cuhk.edu.hk

‡School of Economics, Singapore Management University, jtli@smu.edu.sg

1 Introduction

Traditional models in mechanism design make strong assumptions about the detailed knowledge of the mechanism designer in the economic environment. Subsequently, the theoretical conclusions can sometimes be fragile; mechanisms that are optimized to perform well when the assumptions are exactly true may still fail miserably in the much more frequent cases when the assumptions are untrue. The so-called Wilson doctrine holds that practical mechanisms should be designed without assuming that the designer has precise knowledge about the economic environment.

This paper studies a robust version of the single-unit auction problem where we relax the assumption about the auctioneer's knowledge in the payoff environment—the auctioneer has limited statistical information about the joint distribution of the bidders' valuations. In particular, we consider an auctioneer who has an estimate of the marginal distribution of a generic bidder's valuation but has non-Bayesian uncertainty about the correlation structure. Lacking the knowledge of the correlation structure, our auctioneer ranks mechanisms according to their revenue guarantee, that is, the greatest lower bound of revenue across all joint distributions that are consistent with the marginals.

Several motivations can be offered for considering the robustness to the correlation structure:[1]

- First, while it is relatively easy to estimate the distribution of a generic bidder's valuation, it is significantly more difficult to estimate the joint distribution, which is a much higher-dimensional object; the computational and sampling complexity of learning the joint distribution is exponential in the number of bidders. In other words, obtaining an accurate statistical estimate of the joint distribution of bidders' values often requires the observation of unrealistically many examples of the joint value profiles.

- Second, besides the statistical aspect that the joint distribution is a much higher-dimensional object, there are many practical reasons why the joint distribution might be quite hard to observe and learn. For example, there are many instances in which the auctioneer cannot pin down the identities of the bidders (such as auctions that take place over the Internet or when bidders

[1] Of course, our results would not apply in settings in which the auctioneer has access to sufficiently rich data to obtain an accurate estimate of the joint distribution.

bid through proxies). In this case, the auctioneer has no way of estimating the correlation structure. It is reasonable to assume that each bidder has identical prior distribution, which can be deduced from an empirical study of a small random subgroup of the buyers' population.

- Third, given the known importance of the correlation structure (see for example Myerson (1981) and Crémer and McLean (1988)), understanding the robustness to the correlation structure is an especially useful exercise, at the very least in the sense of providing a robustness check for known mechanisms.[2]

- Fourth, while we could model the auctioneer's (lack of) knowledge of the payoff environment in many different ways, the correlation-robust framework seems to be a natural starting point. The source of the uncertainty, the correlation structure, is the same as that in Carroll (2017), who considers a multi-dimensional screening problem in which the seller knows the marginal distribution of the buyer's valuation for each good but does not know the joint distribution.[3]

- Finally, reserve prices observed in real-world auctions are substantially lower than the theoretically optimal ones (see for example Hasker and Sickles (2010)). Under the correlation-robust framework, for large markets, the robustly optimal mechanism is the simple and familiar second-price auction with no reserve price; for a finite number of bidders, we show that typically the auctioneer finds it optimal to use a low reserve price. Thus, both our analysis for large markets and a finite number of bidders could be perceived as supporting the use of a low reserve price from a novel robustness perspective.

To fix ideas and also to illustrate some of the motivations of our analysis, let us revisit the seminal paper of Myerson (1981) that studies optimal auction design in the independent private-value setting.

[2]For example, if the optimal mechanism derived under the independent private-value model also performs well under other correlation structures, then we would be comfortable using such a mechanism even without an accurate statistical estimate of the correlation structure. Example 1 shows that this is not the case.

[3]While the auctioneer is assumed to know the marginal distribution in the correlation-robust framework, our results for large markets does not depend on the knowledge of the marginal distribution; see Remark 2(b). For a finite number of bidders, we consider two variations of our model that relax the assumption about the auctioneer's knowledge of the marginal distribution and show that our results still hold true under alternative assumptions.

Example 1. In the independent private-value setting, Myerson (1981) shows that, under a regularity condition, the optimal mechanism can be implemented via a second-price auction with a reserve price (denoted r_M) that does not depend on the number of bidders. Suppose that each bidder's valuation is uniformly distributed on the $[0, 1]$ interval. Then $r_M = \frac{1}{2}$. For a thought experiment, suppose that there is a large number of bidders and the auctioneer needs to decide between two mechanisms: the second-price auction with reserve price $\frac{1}{2}$ and the second-price auction with no reserve price. We argue that the auctioneer should use the second-price auction with no reserve price. Consider two cases.

Case (1): If the bidders' valuations are indeed independent, then the second-price auction with reserve price $\frac{1}{2}$ is optimal and generates a strictly higher expected revenue than the second-price auction with no reserve price. However, as ex post revenue of these two mechanisms differs only in the region in which at most one bidder has a valuation above $\frac{1}{2}$, the difference in the expected revenue of these two mechanisms is vanishingly small as the number of bidders gets larger.

Case (2): Now consider an alternative scenario in which the bidders' valuations are maximally positively correlated (the assumption of independent types is untrue). Regardless of the number of bidders (provided that there are at least two bidders), the second-price auction with no reserve price is the optimal mechanism and generates an expected revenue of $\frac{1}{2}$, whereas the second-price auction with reserve price $\frac{1}{2}$ only generates an expected revenue of $\frac{3}{8}$. $\square$

While we only considered two particular correlation structures, the analysis already suggests that in large markets, the second-price auction with the optimally chosen reserve price under the independent private-value model is more vulnerable than the second-price auction with no reserve price. Intuitively, the optimality of the second-price auction with reserve price r_M in the independent private-value model depends on the intricate tradeoff of the following two events: (1) the largest valuation is less than r_M (so that the reserve price is not favorable); and (2) the largest valuation is weakly larger than r_M conditional on that the second largest valuation is less than r_M (so that the reserve price is favorable). Thus, the second-price auction with reserve price r_M may not perform well if the correlation structure is misspecified. In contrast, the second-price auction with no reserve price generates an expected revenue that is equal to the expectation of the second largest valuation regardless of the correlation structure. As such, one might expect that the second-price auction with no reserve price is a reasonable mechanism given non-Bayesian uncertainty about the correlation structure.

Indeed, our first result, Theorem 1, establishes the robust optimality of the second-price auction with no reserve price in large markets among all dominant-strategy mechanisms.[4,5] We show that the revenue guarantee of the second-price auction with no reserve price converges to the expectation of a generic bidder's valuation. Importantly, the expectation of a generic bidder's valuation is an *upper bound* of the highest revenue guarantee in our framework; our auctioneer could never rule out the maximally positive correlation as a candidate for the joint distribution, and the expectation of a generic bidder's valuation is the full surplus under this particular correlation structure.[6]

Although the robustness of a mechanism is a key concern, it is only one of several desiderata in practical mechanism design. Indeed, when selecting an auction format, the auctioneer might have to balance many different criteria. This perspective (of balancing multiple criteria) makes our result all the more appealing: besides having nice theoretical properties and being widely adopted in practice, the second-price auction with no reserve price is asymptotically optimal in the correlation-robust framework.[7] Our analysis supports the use of the second-price auction with no reserve price in large markets from a novel robustness perspective, complementary to existing reasons.

To show that the revenue guarantee of the second-price auction with no reserve price converges to the expectation of a generic bidder's valuation, we need

[4]Theorem 1 continues to hold even if the auctioneer uses a Bayesian mechanism, if we model the bidders' beliefs to be derived from the joint distribution; see Remark 1(a) for further discussion.

[5]One might think that the design of auctions in large markets is less interesting, on the ground that many auction formats including the second-price auction with any reserve price are optimal. While it is indeed the case that the choice of the reserve price does not matter under the assumption of independent values (since a large number of independent draws form some distribution would ensure that the second largest valuation approaches the upper bound of possible valuations), this is not the case in the correlation-robust framework, as illustrated in Example 1. We further wish to emphasize that the underlying logic of the optimality of the second-price auction in our correlation-robust framework and under the assumption of independent values is drastically different.

[6]Abusing the terminology slightly, we refer to the expectation of a generic bidder's valuation as the full surplus in the correlation-robust framework hereafter.

[7]English auctions have a number of properties that make them attractive for practical purposes. They are weakly group strategy-proof, preserve the privacy of trading agents, endow the bidders with obviously dominant strategies, and limit the information that agents and the designer must acquire prior to the auction. English auctions are also widely adopted in practice. Besides these properties that are common to all English auctions, the English auction with no reserve price is efficient and does not demand the commitment power that the auctioneer commits to permanently withholding an unsold object off the market; see Liu, Mierendorff, Shi, and Zhong (2019). English auctions and second-price auctions are strategically equivalent in our setting. To economize on notation, we work with second-price auctions.

to solve the minimization problem in which Nature minimizes the auctioneer's expected revenue by choosing a joint distribution that is consistent with the marginals. While this is a non-trivial task, due to the functional form of the ex post revenue function of the second-price auction with no reserve price, there is strong intuition about the properties of the worst-case correlation structure: if bidder i has the highest valuation and bidder j has the second highest valuation, then all the other bidders have the same valuation as that of bidder j. This is because (1) the choice of Nature for the other bidders' valuation does not matter for the ex post revenue for this particular realization; (2) since Nature is bounded by the marginal consistency constraint, choosing the same valuation as that of bidder j provides the maximum flexibility for Nature to reduce the auctioneer's ex post revenue for other realizations.

This intuition leads us to consider a candidate correlation structure that has a natural economic interpretation as the maximally positive correlation conditionally on the existence of a strong bidder. Let F denote the distribution of a generic bidder. One bidder, whom the seller believes is equally likely to be any one of the bidders, is a strong bidder whose value is drawn from F conditional on that her valuation is weakly higher than some threshold. Every other bidder is a weak bidder whose value is drawn from F conditional on that their values are weakly less than this threshold. Furthermore, all the bidders' valuations are maximally positively correlated.

To show that the candidate correlation structure is indeed a worst-case correlation structure, we adopt a duality approach. This step of our analysis is closely related to the optimal transport theory (see for example Villani (2003)). To wit, Nature's minimization problem can be interpreted as an optimal transportation problem in which Nature seeks to implement the transportation at minimal cost. A transportation plan is a joint distribution that is consistent with the marginals, and Nature's cost function is the ex post revenue function of the auctioneer. While the literature of optimal transport focuses on the case of two random variables, we work with multiple random variables. To be rigorous and self-contained, we prove an n-dimensional generalization of the weak duality property in the Kantorovich duality theorem (see Villani (2003, Theorem 1.1.3)). This generalization is straightforward and follows from a modification of the original proof.

There may be other sequences of mechanisms that are also asymptotically optimal. To further support the use of the second-price auction with no reserve price, we prove a complementary result to Theorem 1 on the rate of convergence.

Consider a class of *standard* mechanisms in which bidders who do not have the highest bid do not get the object. In Appendix C, we show that among all sequences of standard mechanisms, the revenue guarantee of the second-price auction with no reserve price converges to the full surplus with the fastest rate of convergence.

While Theorem 1 is a result on large markets, the second-price auction with no reserve price also performs well in small and moderate sized markets. For any marginal distribution, if there are n bidders, the difference of the full surplus and the revenue guarantee of the second-price auction with no reserve price is bounded above by $\frac{1}{n}$. For a numerical example, suppose that each bidder's valuation is uniformly distributed on the $[0, 1]$ interval. The revenue guarantee of the second-price auction with no reserve price is 75% of the full surplus with 4 bidders, and is 90% of the full surplus with 10 bidders.

The auctioneer could potentially do better using other mechanisms in markets with a finite number of bidders. For practical purposes and also for tractability, we focus on a familiar class of auction forms, second-price auctions with reserve prices, that are both theoretically appealing and widely adopted in practice.[8] Formally, we work with a maxmin optimization problem in which the auctioneer chooses a (random) reserve price to maximize the worst-case expected revenue, where the worst case is taken over all joint distributions that are consistent with the marginals. Theorem 2 and Theorem 3 solve for the robustly optimal (random) reserve price that generates the highest revenue guarantee among all reserve prices for any finite number of bidders.

Our basic model assumes that the auctioneer knows the marginal distribution of a generic bidder's valuation. In Section 7, we consider two variations of the basic model in which we relax this assumption, and show that our results persist when the model is made more realistic. Section 7.1 considers a model in which the auctioneer has local uncertainty about the marginal distribution. Section 7.2 considers a model in which the auctioneer has a conservative estimate of the marginal distribution.

Section 2 presents our model. Section 3 and Section 4 show that the second-price auction with no reserve price is asymptotically optimal. Section 5 and Section 6 solve for the robustly optimal (random) reserve price for any finite number of

[8] We emphasize that it is important to understand the revenue guarantee of standard auction formats such as second-price auctions with reserve prices. While second-price auctions with reserve prices might not provide the highest revenue guarantee among all mechanisms, they are nevertheless one of the most common forms of auctioning an object and have many other desirable features aside from revenue guarantee.

bidders. Section 7 discusses two variations of our basic model, Section 8 discussed related literature, and Section 9 concludes the paper.

2 Preliminaries

2.1 Notation

We first introduce some notation that will be used in the sequel. For any real-valued vector $x \in \mathbb{R}^n$, we write $x(k)$ for the k-th largest element of the vector. For any set S, we denote by $|S|$ its cardinality. If Y is a measurable set, then ΔY is the set of all probability measures on Y. If Y is a metric space, then we treat it as a measurable space with its Borel σ-algebra.

2.2 The auction environment

An auctioneer seeks to sell a single indivisible object. There are $n \geq 2$ risk-neutral bidders competing for the object. We denote by $I = \{1, 2, \ldots, n\}$ the set of bidders and i a typical bidder. Each bidder i holds private information about her valuation of the object, which is modeled as a random variable v_i with cumulative distribution function F_i. We denote by V_i the set of possible valuations of bidder i. The set of possible valuation profiles is $V = \times_{i \in I} V_i$ with a typical element v. We write v_{-i} for a valuation profile of bidder i's opponents; that is, $v_{-i} \in V_{-i} = \times_{j \neq i} V_j$. Apart from their private information, all bidders are identical. Hereafter, we shall write F for the common cumulative distribution function.[9] Without loss of generality, we normalize the support of F to be $[0, 1]$. We assume that F has a positive density f everywhere on the support.

The auctioneer has an estimate of the marginal distribution of a generic bidder's valuation, but has non-Bayesian uncertainty about the correlation structure. Any joint distribution is a plausible candidate as long as it is consistent with the marginals. We denote by

$$\Pi_n(F) = \left\{ \pi \in \Delta V : \forall i \in I, \forall A_i \subseteq V_i, \pi(A_i \times V_{-i}) = F(A_i) \right\}$$

the collection of such joint distributions in the setting with n bidders and marginal distribution F. When there is no confusion, we shall drop the dependence of

[9] Abusing notation slightly, we also use F to denote the probability measure that is consistent with the distribution F.

$\Pi_n(F)$ on the number of bidders n and/or the marginal distribution F. We make no assumption about the bidders' beliefs. In particular, we do not assume that a common prior exists, nor that the bidders' and the auctioneer's beliefs are consistent.

2.3 Dominant-strategy mechanisms

We focus on dominant-strategy mechanisms. The revelation principle holds, and we can restrict attention to direct mechanisms. A direct mechanism (q, t) consists of an allocation rule $q : V \to [0, 1]^n$ and a payment function $t : V \to \mathbb{R}^n$. Each bidder will report a valuation v_i, and based on the resulting profile of reports v, bidder i receives the object with probability $q_i(v)$ and pays $t_i(v)$ to the auctioneer.

A direct mechanism (q, t) is a dominant-strategy mechanism if for all $i \in I$, all $v \in V$, and all $v_i' \in V_i$,

$$v_i\, q_i(v) - t_i(v) \geq v_i\, q_i(v_i', v_{-i}) - t_i(v_i', v_{-i}),$$
$$v_i\, q_i(v) - t_i(v) \geq 0.$$

We denote by $\mathcal{M}_n$ the set of dominant-strategy mechanisms in the setting with n bidders, and we write M_n for a typical element of $\mathcal{M}_n$. For ease of notation, we shall drop the dependency of $\mathcal{M}_n$ and M_n on the number of bidders n when there is no confusion.

We are interested in the auctioneer's expected revenue in the dominant-strategy equilibrium in which each bidder truthfully reports her valuation of the object. Let $REV(M, \pi) = \int_V \sum_{i \in I} t_i(v)\, d\pi(v)$. That is, we use $REV(M, \pi)$ to denote the auctioneer's expected revenue by using the mechanism M under the joint distribution π.

2.4 Second-price auctions with reserve prices

Second-price auctions with reserve prices play an important role in our analysis. In the second-price auction with reserve price r, each bidder i submits a bid $m_i \in \mathbb{R}_+$. Conditional on the submitted bids $m = (m_1, m_2, \ldots, m_n)$, bidder i's probability of winning the object $q_i(m)$ and the payment from bidder i to the auctioneer $t_i(m)$

are given as follows:

$$q_i(m) = \begin{cases} \frac{1}{|W(m)|} & \text{if } i \in W(m) \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad t_i(m) = \begin{cases} \frac{\max(m(2),r)}{|W(m)|} & \text{if } i \in W(m) \\ 0 & \text{otherwise} \end{cases}$$

where $W(m) = \{i \in I : m_i = m(1), m_i \geq r\}$.

We are interested in the auctioneer's expected revenue in the dominant-strategy equilibrium in which each bidder submits a bid that is equal to her valuation of the object. For the second-price auction with reserve price r, let

$$REV(r, v) = \begin{cases} 0 & \text{if } v(1) < r; \\ r & \text{if } v(2) < r \leq v(1); \\ v(2) & \text{if } v(2) \geq r, \end{cases}$$

and let

$$REV(r, \pi) = \int_V REV(r, v)\, d\pi(v).$$

That is, we use $REV(r, v)$ to denote the auctioneer's ex post revenue by using the second-price auction with reserve price r when the realized valuation profile is v, and we use $REV(r, \pi)$ to denote the auctioneer's expected revenue by using the second-price auction with reserve price r under the joint distribution π.

2.5 Revenue guarantee as a criterion

We say that R is a revenue guarantee of mechanism M if for all $\pi \in \Pi$,

$$REV(M, \pi) \geq R.$$

We say that R is the revenue guarantee of mechanism M if it is a revenue guarantee and there is no higher revenue guarantee.

Our auctioneer ranks mechanism according to the revenue guarantee. That is, given the choice among a set of dominant-strategy mechanisms $\bar{\mathcal{M}}$, the auctioneer solves the following maxmin optimization problem:

$$\sup_{M \in \bar{\mathcal{M}}} \inf_{\pi \in \Pi} REV(M, \pi).$$

3 Large markets

We first consider the design of correlation-robust auctions in large markets. The auctioneer chooses a dominant-strategy mechanism to maximize the worst-case expected revenue, where the worst case is taken over all joint distributions that are consistent with the marginals. Formally, the auctioneer solves the following maxmin optimization problem:

$$\sup_{M \in \mathcal{M}} \inf_{\pi \in \Pi} REV(M, \pi). \tag{Maxmin}$$

We work with sequences of mechanisms $\{M_n\}_{n \geq 2}$ as the auctioneer may condition the choice of the mechanism on the number of bidders. That is, the auctioneer could use one mechanism M_2 when there are 2 bidders and use another mechanism M_{10} when there are 10 bidders. Abuse notation slightly, we use 0 (resp. r) to denote the second-price auction with no reserve price (resp. second-price auction with reserve price r) regardless of the number of bidders. This should not cause any confusion. When studying the asymptotic properties of a sequence of mechanisms, we write the second-price auction with no reserve price (resp. reserve price r) rather than the sequence of mechanisms where the designer uses the second-price auction with no reserve price (resp. reserve price r) for any number of bidders.

We say that a sequence of mechanisms $\{M_n\}_{n \geq 2}$ is asymptotically optimal if for any $\{M'_n\}_{n \geq 2}$, for any $\alpha < 1$, there exists N such that for all $n \geq N$, we have

$$\inf_{\pi \in \Pi} REV(M_n, \pi) > \alpha \inf_{\pi \in \Pi} REV(M'_n, \pi).$$

Theorem 1 below shows that a simple auction format, the second-price auction with no reserve price, is asymptotically optimal.

Theorem 1. *The second-price auction with no reserve price is asymptotically optimal.*

One may attempt to work with the maxmin optimization problem (Maxmin) directly. That is, we first ask, is there a systematic way of solving for the worst-case correlation structure for any dominant-strategy mechanism? In principle, if we have a way of identifying the worst-case correlation structure for any dominant-strategy mechanism, we could first calculate the worst-case expected revenue for any dominant-strategy mechanism, and then maximize the worst-case expected revenue (as a function of dominant-strategy mechanisms only) by choosing the mechanism. This approach is problematic, as it is far from clear (at least to us) what would be the worst-case correlation structure for any dominant-strategy mechanism.

We proceed indirectly. We first work with a simpler problem where we solve for the revenue guarantee of the second-price auction with no reserve price. We show that in the setting with n bidders, the revenue guarantee of the second-price auction with no reserve price is

$$\min_{\pi \in \Pi} REV(0, \pi) = \frac{n}{n-1} \int_0^{F^{-1}(\frac{n-1}{n})} x \, dF(x).$$

Therefore, as $n \to \infty$,

$$\min_{\pi \in \Pi} REV(0, \pi) \to \int_0^1 x \, dF(x).$$

Importantly, $\int_0^1 x \, dF(x)$ can be interpreted as the *full surplus* in the correlation-robust framework. This is because our auctioneer could never rule out the maximally positive correlation (defined by randomly drawing $q \sim U[0,1]$ and taking $v_1 = v_2 = \ldots = v_n = F^{-1}(q)$) as a candidate for the joint distribution. Therefore, for whatever dominant-strategy mechanism that the auctioneer might use, be it a second-price auction with some reserve price or a more complex mechanism, the expectation of a generic bidder's valuation is an upper bound of the revenue guarantee of the mechanism.

The most involved part of our analysis is to solve for the revenue guarantee of the second-price auction with no reserve price. Formally, we need to solve the following minimization problem:

$$\inf_{\pi \in \Pi} REV(0, \pi).$$

This is a non-trivial task, as the space of joint distributions that are consistent with the marginals is large. For this step, we adopt the duality approach. We construct the dual maximization problem of the primal minimization problem and show that the optimal value of the maximization problem is weakly less than the optimal value of the minimization problem. That is, we establish a weak duality property. We then proceed to construct the primal variables and dual variables such that the value of the objective function of the minimization problem under the constructed primal variables and the value of the objective function of the maximization problem under the constructed dual variables are the same.

This implies that the constructed primal variables is a solution to the primal minimization problem.[10]

Remark 1. (a) Theorem 1 shows that the second-price auction with no reserve price is asymptotically optimal among all sequences of dominant-strategy mechanisms. Among other things, working with dominant-strategy mechanisms spares us the need to model the bidders' hierarchies of beliefs about each other. A Bayesian approach that makes detailed assumptions about the bidders' hierarchies of beliefs about each other goes against the spirit of our exercise. Having said that, if we model the bidders' beliefs to be derived from the joint distribution, Theorem 1 continues to hold even if the auctioneer uses a Bayesian mechanism. This is because $\int_0^1 x\, dF(x)$ remains to be the full surplus in the Bayesian framework.

(b) Notably, Theorem 1 does not rely on the knowledge of the marginal distribution. Even if the auctioneer does not possess the knowledge of the marginal distribution, the auctioneer finds it robustly optimal to use the second-price auction with no reserve price in large markets. More formally, let $\mathcal{F}$ be an arbitrary collection of marginal distributions, and let $\Pi_n(\mathcal{F}) = \bigcup_{F \in \mathcal{F}} \Pi_n(F)$ denote the collection of joint distributions that the auctioneer considers plausible. Obviously, Theorem 1 still holds in this alternative setting.

(c) It is instructive to compare our model to that of Myerson (1981). As discussed in the introduction, the second-price auction with reserve price r_M (the optimal mechanism under the independence assumption) may not perform well if the correlation structure is misspecified. In contrast, we establish the robust optimality of the second-price auction with no reserve price in large markets. Intuitively, the second-price auction with no reserve price generates an expected revenue that is equal to the expectation of the second largest valuation regardless of the correlation structure. We formally show that in large markets, the worst case for the expectation of the second largest valuation is the expectation of a generic bidder's valuation. Another difference is that, while Myerson (1981) requires the regularity condition to establish the optimality of the second-price auction with

[10] For Theorem 1, it suffices to identify a lower bound of the worst-case revenue of the second-price auction with no reserve price that converges to the full surplus. While we are aware of alternative approaches of proving Theorem 1, we present the duality approach here as (1) the lower bound that we identify can be shown to be tight, which has the the added benefit of understanding the worst-case expected revenue and the worst-case correlation structure for the second-price auction with no reserve price for any finite number of bidders, and (2) this methodology will be used repeatedly throughout the paper, including the analysis of second-price auctions with (random) reserve prices for any finite number of bidders.

some reserve price, our model does not impose any condition on the marginal distribution.

(d) Theorem 1 is a result on large markets. The second-price auction with no reserve price also performs well in small and moderate sized markets. For any F, if there are n bidders, the difference of the full surplus and the revenue guarantee of the second-price auction with no reserve price is bounded above by $\frac{1}{n}$. Indeed,

$$\begin{aligned}
&\int_0^1 x\,dF(x) - \inf_{\pi\in\Pi} REV(0,\pi)\\
&= \int_0^1 x\,dF(x) - \frac{n}{n-1}\int_0^{F^{-1}(\frac{n-1}{n})} x\,dF(x)\\
&= \int_{F^{-1}(\frac{n-1}{n})}^1 x\,dF(x) - \frac{1}{n-1}\int_0^{F^{-1}(\frac{n-1}{n})} x\,dF(x)\\
&\leq \int_{F^{-1}(\frac{n-1}{n})}^1 1\,dF(x) = \frac{1}{n} \to 0.
\end{aligned}$$

For a numerical example, suppose that the marginal distribution F is the uniform distribution on the $[0,1]$ interval. In the setting with n bidders, the revenue guarantee of the second-price auction with no reserve price is

$$\min_{\pi\in\Pi} REV(0,\pi) = \frac{n}{n-1}\int_0^{F^{-1}(\frac{n-1}{n})} x\,dF(x) = \frac{n}{n-1}\int_0^{\frac{n-1}{n}} x\,dx = \frac{n-1}{2n},$$

whereas the full surplus is $\frac{1}{2}$. Thus, if there are n bidders, the revenue guarantee of the second-price auction with no reserve price is $\frac{n-1}{n}$ of the full surplus.

There may be other sequences of mechanisms that are also asymptotically optimal.[11] Consider a class of *standard* mechanisms in which bidders who do not have the highest bid do not get the object.[12] Formally, let

$$\hat{\mathcal{M}}_n = \left\{\hat{M}_n \in \mathcal{M}_n \colon \hat{q}_i(v_1, v_2, \ldots, v_n) = 0 \text{ if } v_i < \max_{1\leq j\leq n} v_j\right\}.$$

In Appendix C, we present a complementary result to Theorem 1 that among all sequences of standard mechanisms, the revenue guarantee of the second-price auction with no reserve price converges to the full surplus with the fastest rate of

[11] The second-price auction with any positive reserve price r is not asymptotically optimal. Regardless of the number of bidders, the revenue guarantee of the second-price auction with a positive reserve price r cannot exceed its expected revenue under the maximally positive correlation, which is $\int_r^1 x\,dF(x)$ and is strictly bounded away from $\int_0^1 x\,dF(x)$.

[12] Thus, the outcome of any standard mechanism is envy-free.

convergence. Following Remark 1(d), the revenue guarantee of the second-price auction with no reserve price converges to the full surplus at least in the rate of $O(\frac{1}{n})$. Appendic C shows that the revenue guarantee of any sequence of standard mechanisms converges to the full surplus at most in the rate of $O(\frac{1}{n})$.[13]

4 Proof of Theorem 1

In this section, we show that in the setting with n bidders, the revenue guarantee of the second-price auction with no reserve price is

$$\min_{\pi \in \Pi} REV(0, \pi) = \frac{n}{n-1} \int_0^{F^{-1}(\frac{n-1}{n})} x \, dF(x).$$

This, combined with the analysis in Section 3, establishes Theorem 1.

For the sake of clarity, we first consider the case in which there are only two bidders.

Observation 1. *Suppose that $n = 2$. A worst-case correlation structure for the second-price auction with no reserve price is the maximally negative correlation, defined by randomly drawing $q \sim U[0, 1]$ and taking*

$$v_1 = F^{-1}(q) \text{ and } v_2 = F^{-1}(1 - q).$$

To see why this is a worst-case correlation structure, note that for the second-price auction with no reserve price, the auctioneer's ex post revenue function

$$REV(0, v) = v(2) = \min(v_1, v_2)$$

is a supermodular function. Since Nature chooses a joint distribution to minimize the expected value of a supermodular function, a worst-case correlation structure for the auctioneer is indeed the maximally negative correlation.[14] It follows that

[13]It is common in computer science to evaluate an algorithm by bounding its error with a function of some measure m on the operation of the algorithm. A bound of this kind expresses the rate at which the error diminishes as m is relaxed, with the error converging to zero as m goes to infinity. An algorithm with a faster rate of convergence is deemed superior to an algorithm with a slower rate of convergence because it approximates the exact solution of the problem more accurately than the slower algorithm when m is sufficiently large. In economics, Satterthwaite and Williams (2002) rank market mechanisms according to how quickly inefficiency diminishes as the size of the market increases.

[14]A function $g : V \to \mathbb{R}$ is supermodular if

$$g(v \vee v') + g(v \wedge v') \geq g(v) + g(v')$$

the revenue guarantee of the second-price auction with no reserve price is

$$\min_{\pi \in \Pi} REV(0, \pi) = 2 \int_0^{F^{-1}(\frac{1}{2})} x \, dF(x).$$

Remark 2. (a) There are other worst-case correlation structures. Consider the correlation structure defined by randomly drawing $s \sim U\{1, 2\}$, $q \sim U[0, \frac{1}{2}]$, and taking $v_s = F^{-1}(q + \frac{1}{2})$ and $v_{-s} = F^{-1}(q)$. In words, a bidder, whom the auctioneer believes is equally likely to be any one of the bidders, is a strong bidder, the other bidder is a weak bidder, and the bidders' valuations are maximally positively correlated. Clearly, under this correlation structure, the auctioneer also obtains an expected revenue of $2 \int_0^{F^{-1}(\frac{1}{2})} x \, dF(x)$. Hence, this is also a worst-case correlation structure for the second-price auction with no reserve price.

(b) An equivalent but indirect way of defining the maximally negative correlation is as follows. The maximally negative correlation is the unique joint distribution such that

1. the probability concentrates on the following curve

$$L_0 : F(1) - F(v_2) = F(v_1) - F(0), \; v_1 \in [0, 1];$$

2. the joint distribution is consistent with the marginals.

While indirect, this alternative definition is somewhat more intuitive. Throughout the rest of the paper, we shall construct joint distributions indirectly.

Our analysis in the case of two bidders is particularly simple, as we could exploit the fact that the auctioneer's ex post revenue function $REV(0, v) = v(2) = \min(v_1, v_2)$ is a supermodular function. However, when there are more than two bidders, for the second-price auction with no reserve price, the ex post revenue function $REV(0, v) = v(2)$ is no longer a supermodular function. Thus, the worst-case correlation structure in the case of two bidders does not generalize in a straightforward manner. Nevertheless, for the second-price auction with no reserve price, since Nature's objective is to minimize the expectation of $v(2)$ by choosing a joint distribution that is consistent with the marginals, we have the following observation, which helps pin down a worst-case correlation structure when there are more than two bidders.

for all $v, v' \in V$, where $\vee$ denotes the component-wise maximum and $\wedge$ denotes the component-wise minimum. For detailed discussions on the ordering of joint distributions based on the integrals of supermodular functions, see for example Meyer and Strulovici (2012).

Observation 2. *Suppose that $n \geq 3$. Consider the thought experiment in which the values of $v(1)$ and $v(2)$ are fixed and Nature has the flexibility to choose $v(3), v(4), \ldots, v(n)$. For whatever values of $v(3), v(4), \ldots, v(n)$ that Nature chooses, the ex post revenue is $v(2)$, which is fixed by assumption.*

Observation 2 is about a specific scenario in which the values of $v(1)$ and $v(2)$ have been fixed. Thus, Nature's choice of $v(3), v(4), \ldots, v(n)$ does not matter. However, Nature's objective is not to minimize the ex post revenue for this particular realization of values, but to minimize the expectation of $v(2)$. Since Nature is constrained to choose a joint distribution that is consistent with the marginals, although the specific values of $v(3), \ldots, v(n)$ do not affect the ex post revenue for this particular realization, Nature would choose the other bidders' valuations to be as high as possible, that is, $v(3) = \ldots = v(n) = v(2)$. This choice gives Nature the maximum flexibility to minimize the expected revenue.

Motivated by these observations, we consider the following joint distribution, which is our candidate for a worst-case correlation structure. Define π^0 to be the unique joint distribution such that

1. the probability concentrates on n symmetric curves $L_0^1, L_0^2, \ldots, L_0^n$ where

$$L_0^i = \Big\{ v \in V : F(v_j) - F(0) = (n-1)(F(1) - F(v_i)), \, \forall j \neq i, \\ v_i \in [F^{-1}(\frac{n-1}{n}), 1] \Big\};$$

2. the joint distribution is consistent with the marginals.

The interpretation of the curve L_0^i is that in the region in which bidder i has the highest valuation, Nature puts probability in a way such that bidders other than i have the same valuation (motivated by Observation 2), and bidder i's valuation is maximally negatively correlated with the other bidders' valuation (motivated by Observation 1).

In what follows, we formally show that the correlation structure π^0 we construct above is a worst-case correlation structure for the second-price auction with no reserve price.

Proposition 1. *Suppose that $n \geq 2$.*[15] *Then*

$$\pi^0 \in \arg\min_{\pi \in \Pi} REV(0, \pi),$$

[15] This covers the case of two bidders as a special case.

and the revenue guarantee of the second-price auction with no reserve price is

$$\min_{\pi\in\Pi} REV(0,\pi) = \frac{n}{n-1}\int_0^{F^{-1}(\frac{n-1}{n})} x\,dF(x).$$

Remark 3. As in the case of two bidders, there are other worst-case correlation structures. Consider the correlation structure defined by randomly drawing $s \sim U\{1,2,\ldots,n\}$, $q \sim U[0,\frac{n-1}{n}]$, and taking $v_s = F^{-1}(\frac{q}{n-1}+\frac{n-1}{n})$ and $v_w = F^{-1}(q)$ for each $w \neq s$. In words, a bidder, whom the auctioneer believes is equally likely to be any one of the bidders, is a strong bidder, all the other bidders are weak bidders, and all the bidders' valuations are maximally positively correlated. It is easy to verify that the auctioneer obtains the same expected revenue under this correlation structure as that under π^0. Hence, this is also a worst-case correlation structure for the second-price auction with no reserve price.

To solve the minimization problem

$$\min_{\pi\in\Pi} REV(0,\pi), \tag{Primal-0}$$

we adopt a duality approach. We construct the dual maximization problem of the primal minimization problem and show that the optimal value of the maximization problem is weakly less than the optimal value of the minimization problem. We then proceed to construct the primal variables and dual variables such that the value of the objective function of the minimization problem under the constructed primal variables and the value of the objective function of the maximization problem under the constructed dual variables are the same. This implies that the constructed primal variables is a solution to the primal minimization problem.

Define $\mathbb{J}$ by

$$\mathbb{J} : L^1(F)\times L^1(F)\times\ldots\times L^1(F)\to\mathbb{R} \text{ and } \mathbb{J}(\mu_1,\mu_2,\ldots,\mu_n) = \sum_{i\in I}\int_{V_i}\mu_i(v_i)\,dF(v_i).$$

Consider the following dual maximization problem of the primal minimization problem:

$$\begin{aligned}\max_{\mu_1,\mu_2,\ldots,\mu_n\in L^1(F)} \quad & \mathbb{J}(\mu_1,\mu_2,\ldots,\mu_n) = \sum_{i\in I}\int_{V_i}\mu_i(v_i)\,dF(v_i) \\ \text{subject to} \quad & \text{for all } v\in V, \sum_{i\in I}\mu_i(v_i) \le REV(0,v).\end{aligned} \tag{Dual-0}$$

The duality approach has a natural economic interpretation (see for example Villani

(2003)). The optimal value of the primal minimization problem is the least cost for a planner that chooses a cost minimizing transport plan. The dual maximization problem may be interpreted as a decentralized solution. We can interpret $\mu_i(v_i)$ as the price paid per unit of mass to transport companies at the location v_i.[16] The optimal value of the dual maximization problem represents the maximal profit of transport companies with the profit being constrained by that the total cost paid to the transport sector for transporting one unit of goods $\sum_{i\in I}\mu_i(v_i)$ should not exceed the "if I do it myself" cost $REV(0,v)$.

Lemma 1. *The optimal value of the dual maximization problem (Dual-0) is weakly less than the optimal value of the primal minimization problem (Primal-0).*

Lemma 1 is essentially the n-dimensional generalization of the weak duality property in the Kantorovich duality theorem. The Kantorovich duality theorem establishes the strong duality in the case of two random variables. For our results, it suffices to prove the weak duality property. The extension to the case of n random variables is straightforward. To be self-contained, we present the short proof here.

Proof of Lemma 1. It suffices to show that for any feasible dual variables $\mu = (\mu_1, \mu_2, \ldots, \mu_N)$ of the dual maximization problem and any feasible primal variables π of the primal minimization problem, the value of the objective function of the maximization problem under μ is weakly less than the value of the objective function of the minimization problem under π. As we shall see below, this follows immediately from the feasibility constraint.

Let π be feasible variables of the primal minimization problem. That is, for all $i \in I$ and for all measurable sets $A_i \in V_i$,

$$\pi(A_i \times V_{-i}) = F(A_i). \tag{1}$$

Let $\mu = (\mu_1, \mu_2, \ldots, \mu_N)$ be feasible variables of the dual maximization problem. That is, for all $v \in V$,

$$\sum_{i\in I}\mu_i(v_i) \le REV(0,v). \tag{2}$$

[16] More formally, $\mu_i(v_i)$ can be interpreted as the shadow cost of the primal minimization problem as one perturbs the marginal distribution F at v_i.

Thus, we have

$$\begin{aligned}
\mathbb{J}(\mu) &= \sum_{i\in I}\int_{V_i} \mu_i(v_i)\, dF(v_i) \\
&= \sum_{i\in I}\int_{V} \mu_i(v_i)\, d\pi(v) \\
&= \int_{V}\sum_{i\in I} \mu_i(v_i)\, d\pi(v) \\
&\leq \int_{V} REV(0,v)\, d\pi(v) \\
&= REV(0,\pi),
\end{aligned}$$

where the second line follows from (1) and the fourth line follows from (2). □

We are now ready to show that for the second-price auction with no reserve price, π^0 is a worst-case correlation structure. The proof proceeds as follows. Step (1) calculates the value of the objective function of the primal minimization problem under π^0. Step (2) constructs dual variables. Step (3) verifies that the value of the objective function of the dual maximization problem under the constructed dual variables is the same as the value of the objective function of the primal minimization problem under π^0.

Step (1). The value of the objective function of the primal minimization problem under π_0^* is

$$\frac{n}{n-1}\int_0^{c_n(0)} x\, dF(x)$$

where $c_n(0) = F^{-1}(\frac{n-1}{n})$ denotes the threshold for the reserve price 0.[17]

Step (2). For each $i \in I$, let

$$\mu_i(v_i) = \begin{cases} \frac{v_i}{n-1} - \frac{c_n(0)}{n(n-1)}, & \text{if } v_i < c_n(0); \\ \frac{c_n(0)}{n}, & \text{if } v_i \geq c_n(0). \end{cases}$$

It is easy to verify that these dual variables satisfy the constraints of the dual maximization problem. Indeed, since $\mu_i(v_i)$ is a weakly increasing function of v_i,

1. if $v(2) \geq c_n(0)$, then

$$\sum_{i\in I}\mu_i(v_i) \leq n\frac{c_n(0)}{n} = c_n(0) \leq v(2) = REV(0,v);$$

[17] In Section 5, we shall use the notation $c_n(r)$ to denote the threshold for the second-price auction with reserve price r.

2. if $v(2) < c_n(0)$, then

$$\sum_{i \in I} \mu_i(v_i) \leq (n-1)(\frac{v(2)}{n-1} - \frac{c_n(0)}{n(n-1)}) + \frac{c_n(0)}{n} = v(2) = REV(0, v).$$

Step (3). We now calculate the value of the objective function of the dual maximization problem under the constructed dual variables as follows:

$$\begin{aligned}
\mathbb{J}(\mu_1, \mu_2, \ldots, \mu_n) &= \sum_{i \in I} \int_{V_i} \mu_i(v_i)\, dF(v_i) \\
&= n \int_{V_1} \mu_1(v_1)\, dF(v_1) \\
&= n \int_0^{c_n(0)} \frac{v_1}{n-1} - \frac{c_n(0)}{n(n-1)}\, dF(v_1) + n \int_{c_n(0)}^1 \frac{c_n(0)}{n}\, dF(v_1) \\
&= \frac{n}{n-1} \int_0^{c_n(0)} v_1\, dF(v_1) - \frac{c_n(0)}{n-1} \int_0^{c_n(0)} 1\, dF(v_1) + c_n(0) \int_{c_n(0)}^1 1\, dF(v_1) \\
&= \frac{n}{n-1} \int_0^{c_n(0)} v_1\, dF(v_1),
\end{aligned}$$

where the last line follows from the definition of $c_n(0)$. This completes the proof of Proposition 1.

5 Robustly optimal reserve price

Theorem 1 establishes the robust optimality of the second-price auction with no reserve price in large markets. In markets with a finite number of bidders, the auctioneer could potentially do better using other mechanisms. For practical purposes and also for tractability, in this section and Section 6, we study an important class of auction forms, second-price auctions with reserve prices, that are both theoretically appealing and widely adopted in practice.[18] We consider deterministic reserve prices here, and analyze the case in which the auctioneer is allowed to randomize over reserve prices in Section 6.

Our auctioneer chooses a reserve price to maximize the worst-case expected revenue, where the worst case is taken over all joint distributions that are

[18] As discussed in the introduction, revenue guarantee is not the only criterion when selecting an auction format; the auctioneer might have to balance many different criteria. Thus, it is important to understand the revenue guarantee of standard auction formats such as second-price auctions with reserve prices. For other papers that study the robustness of standard auction formats, see for example Bergemann, Brooks, and Morris (2017) and Bergemann, Brooks, and Morris (2019).

consistent with the marginals. Formally, the auctioneer solves the following maxmin optimization problem:

$$\sup_{r\in[0,1]} \inf_{\pi\in\Pi} REV(r,\pi). \tag{Maxmin-r}$$

We refer to the solution to this maxmin optimization problem as the robustly optimal reserve price.

The maxmin optimization problem (Maxmin-r) can be interpreted as a two-player zero-sum game. The two players are the auctioneer and Nature. The auctioneer first chooses a reserve price $r \in [0,1]$. After observing the choice of the reserve price, Nature chooses a correlation structure $\pi \in \Pi$. The auctioneer's payoff is $REV(r,\pi)$, and Nature's payoff is $-REV(r,\pi)$.

It is not clear (at least to us) for an arbitrary reserve price what would be the worst-case correlation structure. To bypass this difficulty, we take an indirect approach. In the maxmin optimization problem (Maxmin-r), for each reserve price r, Nature can choose any joint distribution that is consistent with the marginals. This is not easy to work with, as the space of such joint distributions is very large. The novelty in our analysis is that we work with an auxiliary problem that has the interpretation that we impose a restriction on what Nature can do. For each reserve price r, we construct a particular correlation structure π^r that is consistent with the marginals. We can easily solve the following auxiliary problem:

$$\max_{r\in[0,1]} REV(r,\pi^r).$$ [19]

The auxiliary problem then corresponds to an extreme restriction on Nature's strategies in the sense that if the auctioneer chooses a reserve price r, Nature has no choice but to choose π^r. We show that the solution to this auxiliary problem is also the solution to the maxmin optimization problem (Maxmin-r).

The key step of our analysis is thus the construction of $\{\pi^r\}_{r\in[0,1]}$. The construction of $\{\pi^r\}_{r\in[0,1]}$ depends on the number of bidders and the marginal distribution, and will be made clear in the formal analysis. Before we move on to the formal analysis, we wish to provide a sketch of our analysis. The sketch highlights the requirements on $\{\pi^r\}_{r\in[0,1]}$ and should also make our approach more transparent.

In the first step, for each reserve price r, we explicitly construct a joint

[19]Our construction of $\{\pi^r\}_{r\in[0,1]}$ ensures that a solution to this auxiliary problem exists.

distribution π^r that is consistent with the marginals. At this stage, we do not know whether the joint distribution π^r that we construct is a worst-case correlation structure for the reserve price r. Nevertheless, since π^r is consistent with the marginals, the worst-case expected revenue of the reserve price r is weakly lower than its expected revenue under the correlation structure π^r. That is, for any r,

$$\inf_{\pi\in\Pi} REV(r,\pi) \leq REV(r,\pi^r).$$

In the second step, we solve the following auxiliary maximization problem:

$$\max_{r\in[0,1]} REV(r,\pi^r).$$

Let r^* denote a solution to the auxiliary maximization problem. Thus,

$$REV(r^*,\pi^{r^*}) \geq REV(r,\pi^r)$$

for all r.

In the third step, we show that for the reserve price r^*, the correlation structure π^{r^*} is a worst-case correlation structure. Formally, we show that

$$REV(r^*,\pi^{r^*}) = \min_{\pi\in\Pi} REV(r^*,\pi).$$

Our logic can be succinctly summarized via a series of inequalities and equalities. For any r,

$$\inf_{\pi\in\Pi} REV(r,\pi) \leq REV(r,\pi^r) \leq REV(r^*,\pi^{r^*}) = \min_{\pi\in\Pi} REV(r^*,\pi).$$

It follows that r^* is a solution to the maxmin optimization problem (Maxmin-r).

We are now ready to present the formal analysis. For the sake of clarity, we first consider the setting with only two bidders in Section 5.1. Section 5.2 studies the general setting with n bidders.[20]

For ease of exposition, we introduce one more notation. For any $r \in [0,1]$ and any subset of bidders $S \subseteq I$, let

$$V^{r,S} = \{v \in V : v_i \geq r \text{ if and only if } i \in S\}.$$

[20] To be clear, our analysis in the case of n bidders can be easily adopted in the setting with only two bidders. We organize in this way so as to present our analysis in the clearest way possible.

In words, for any valuation profile $v \in V^{r,S}$, bidders in S have valuations weakly higher that r and bidders not in S have valuations lower than r. When S consists of a single bidder i, we write $V^{r,i}$ rather than $V^{r,\{i\}}$.

5.1 Two bidders

Suppose that there are only two bidders. Recall that for the second-price auction with no reserve price, a worst-case correlation structure is the maximally negative correlation (Observation 1). It is less clear what would be the worst-case correlation structure for an arbitrary reserve price r. Nevertheless, if Nature can only put positive probability in the regions $V^{r,\emptyset}$ and $V^{r,\{1,2\}}$, we have a similar observation as in the case of the second-price auction with no reserve price.

Observation 3. *Fix an arbitrary reserve price $r \in [0,1]$. In the constrained minimization problem in which Nature can only put positive probability in the regions $V^{r,\emptyset}$ and $V^{r,\{1,2\}}$, a worst-case correlation structure is the unique joint distribution such that*

1. *in the region $V^{r,\{1,2\}}$, the probability concentrates on the following curve*

$$L_r : F(1) - F(v_2) = F(v_1) - F(r),\ v_1 \in [r,1];$$

2. *in the region $V^{r,\emptyset}$, the probability concentrates on the following curve*

$$v_2 = v_1,\ v_1 \in [0,r);$$

3. *the joint distribution is consistent with the marginals.*

We denote this joint distribution by π^r (see Figure 1 for a graphical illustration of π^r in the case in which F is the uniform distribution on the $[0,1]$ interval).

To see why this is a worst-case correlation structure when Nature is constrained to only put positive probability in the regions $V^{r,\emptyset}$ and $V^{r,\{1,2\}}$, note that we can think of Nature's constrained minimization problem as two sub-problems, namely, the choice of the joint distribution in the region $V^{r,\emptyset}$ and the choice of the joint distribution in the region $V^{r,\{1,2\}}$. We can safely treat these two sub-problems separately, since these two choices do not interact with each other in terms of the consistency requirement. In the region $V^{r,\{1,2\}}$, the auctioneer's ex post revenue function is $REV(r,v) = v(2) = \min(v_1, v_2)$, which is a supermodular function. Therefore, our logic in Observation 1 applies here. In the region $V^{r,\emptyset}$, the joint distribution does not matter as long as it is consistent with the marginals, as the ex post revenue for any valuation profile in this region is zero. For concreteness, when constructing π^r, we pick the joint distribution such that the probability in the region $V^{r,\emptyset}$ concentrates on the curve $v_2 = v_1, v_1 \in [0, r)$. This particular choice plays no role in our analysis.

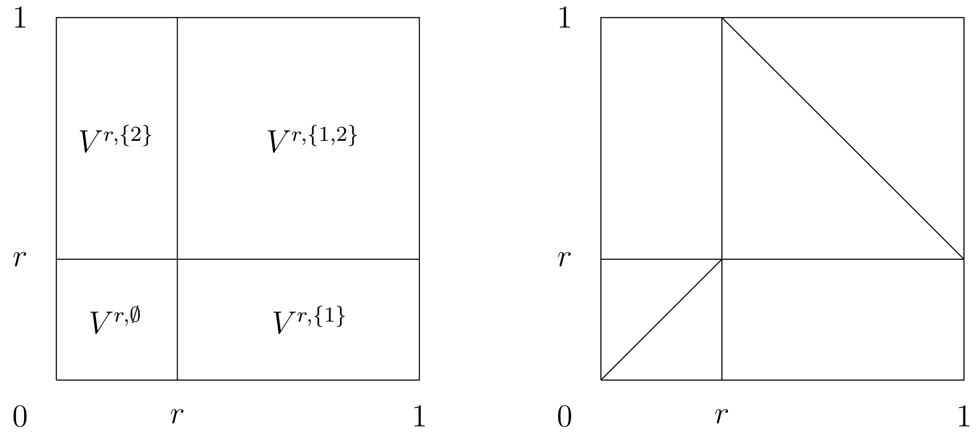

Figure 1: The figure on the left depicts the four regions given by $V^{r,\emptyset}$, $V^{r,1}$, $V^{r,2}$, and $V^{r,\{1,2\}}$. The figure on the right depicts the correlation structure π^r that we construct in the case in which F is the uniform distribution on the $[0, 1]$ interval.

While we have solved the constrained minimization problem in which Nature can only put positive probability in the regions $V^{r,\emptyset}$ and $V^{r,\{1,2\}}$, our logic so far is incomplete for the purpose of identifying a worst-case correlation structure when Nature can choose any joint distribution that is consistent with the marginals, as Nature may want to allocate some probability to the regions $V^{r,1}$ and $V^{r,2}$.

Nevertheless, our analysis above leads us to consider an auxiliary maximization problem that we formulate below. Now that we have constructed the correlation structure π^r for each $r \in [0, 1]$, we can easily calculate the expected revenue of the reserve price r under π^r as follows:

$$REV(r, \pi^r) = \int_{[r,1]^2} v(2)\, d\pi^r(v) = 2 \int_r^{c(r)} x\, dF(x),$$

where $c(r) = F^{-1}(\frac{1+F(r)}{2})$. Consider the following auxiliary maximization problem:

$$\max_{r \in [0,1]} REV(r, \pi^r) = 2 \int_r^{c(r)} x\, dF(x). \tag{Max-2}$$

Proposition 2 below shows that the solution to the maximization problem (Max-2) is the robustly optimal reserve price.

Proposition 2. *Suppose that $n = 2$. Let r^* denote a solution to the maximization problem (Max-2). Then,*

$$\pi^{r^*} \in \arg\min_{\pi \in \Pi} REV(r^*, \pi).$$

This further implies that r^* is the robustly optimal reserve price and generates the highest revenue guarantee of $REV(r^*, \pi^{r^*})$.

Proposition 2 is a special case of Theorem 2, which solves for the robustly optimal reserve price in the general setting with n bidders. For this reason, we omit its proof.

The auxiliary maximization problem (Max-2) is easy to solve. In particular, by the first-order condition, we have $F(2r^*) = \frac{1+F(r^*)}{2}$. For any F, it is straightforward to solve for r^*. Example 2 below applies Proposition 2 to the case in which F is the uniform distribution on the $[0, 1]$ interval.

Example 2 (Two bidders and uniform distribution)**.** Suppose that $n = 2$ and F is the uniform distribution on the $[0, 1]$ interval. From our analysis above, r^* necessarily satisfies that $2r^* = \frac{1+r^*}{2}$. The robustly optimal reserve price is $\frac{1}{3}$ and generates the highest revenue guarantee of $\frac{1}{3}$. □

Remark 4. The key step in our analysis is the construction of the correlation structures $\{\pi^r\}_{r\in[0,1]}$. Proposition 2 shows that for r^*, the correlation structure π^{r^*} is a worst-case correlation structure. This suffices for our purpose of solving for the robustly optimal reserve price, sparing us the need to solve for the worst-case correlation structure for any $r \in [0, 1]$.[21]

5.2 n bidders

Next, we solve for the robustly optimal reserve price in the general setting with n bidders. Motivated by the worst-case correlation structure for the second-price auction with no reserve price (Proposition 1) and our analysis in Section 5.1, we

[21] While the correlation structure π^{r^*} is the worst case for r^*, π^r is not the worst case for any $r \in [0, 1]$. Readers might wonder what makes r^* special. In Appendix D, we provide a direct proof of Proposition 2. While notationally intensive, the direct proof highlights the role of the first-order condition that the robustly optimal reserve price necessarily satisfies. While we believe that a direct proof for any number of bidders could be given, we do not proceed along this direction. This alternative approach would inevitably be much more tedious when there are more bidders, compared to our duality approach.

construct the correlation structures $\{\pi^r\}_{r\in[0,1]}$ as follows. For each reserve price $r \in [0,1]$, we define π^r to be the unique joint distribution such that

1. it only puts positive probability in the regions $V^{r,\emptyset}$ and $V^{r,I}$;
2. in the region $V^{r,I}$, the probability concentrates on n symmetric curves $L_r^1, L_r^2, \ldots, L_r^n$ where

$$L_r^i = \Big\{v \in V^{r,I} : F(v_j) - F(r) = (n-1)(F(1) - F(v_i)),\ \forall j \neq i,$$
$$v_i \in [F^{-1}(\frac{(n-1)+F(r)}{n}), 1]\Big\};$$

3. in the region $V^{r,\emptyset}$, the probability concentrates on the following curve

$$v_i = v_1,\ \forall i \neq 1,\ v_1 \in [0,r];$$

4. the joint distribution is consistent with the marginals.

The interpretation of the curve L_r^i is that in the region in which every bidder's valuation is weakly higher than r and bidder i has the highest valuation, Nature puts probability in a way such that bidders other than i have the same valuation, and bidder i's valuation is maximally negatively correlated with the other bidders' valuation. In the region $V^{r,\emptyset}$, the joint distribution does not matter as long as it is consistent with the marginals, as the ex post revenue for any valuation profile in this region is zero. For concreteness, when constructing π^r, we pick the joint distribution such that the probability in the region $V^{r,\emptyset}$ concentrates on the curve $v_i = v_1$, $\forall i \neq 1$, $v_1 \in [0,r]$. This particular choice plays no role in our analysis.

Consider the following auxiliary maximization problem:

$$\max_{r\in[0,1]} REV(r,\pi^r) = \int_{[r,1]^n} v(2)\, d\pi^r(v) = \frac{n}{n-1}\int_r^{c_n(r)} x\, dF(x) \tag{Max-n}$$

where $c_n(r) = F^{-1}(\frac{(n-1)+F(r)}{n})$. Theorem 2 below shows that the solution to the maximization problem (Max-n) is the robustly optimal reserve price.

Theorem 2. *Suppose that there are n bidders. Let r_n^* denote a solution to the maximization problem (Max-n). Then,*

$$\pi^{r_n^*} \in \arg\min_{\pi\in\Pi} REV(r_n^*, \pi).$$

This further implies that r_n^* is the robustly optimal reserve price and generates the highest revenue guarantee of $REV(r_n^*, \pi^{r_n^*})$.

It suffices to show that for the reserve price r_n^*, $\pi^{r_n^*}$ is a worst-case correlation structure. Since r_n^* is a solution to the maximization problem (Max-n), for any reserve price r,

$$\inf_{\pi \in \Pi} REV(r, \pi) \le REV(r, \pi^r) \le REV(r^*, \pi^{r^*}) = \min_{\pi \in \Pi} REV(r^*, \pi).$$

Thus, r_n^* is the robustly optimal reserve price and generates the highest revenue guarantee of $REV(r_n^*, \pi^{r_n^*})$. To show that $\pi^{r_n^*}$ is a worst-case correlation structure for r_n^*, as in the case of the second-price auction with no reserve price, we adopt the duality approach. The proof can be found in Appendix A.

The auxiliary maximization problem (Max-n) is easy to solve. In particular, by the first-order condition, we have $F(nr_n^*) = F(\frac{(n-1)+F(r_n^*)}{n})$. Example 3 below illustrates how to apply Theorem 2 to the case in which F is the uniform distribution on the $[0, 1]$ interval.

Example 3 (n bidders and uniform distribution)**.** Suppose that there are n bidders and F is the uniform distribution on the $[0, 1]$ interval. From our analysis above, r_n^* necessarily satisfies that $nr_n^* = \frac{(n-1)+r_n^*}{n}$. The robustly optimal reserve price is $r_n^* = \frac{1}{n+1}$ and generates the highest revenue guarantee of $\frac{n}{2(n+1)}$. $\square$

Theorem 2 solves for the robustly optimal reserve price for any finite number of bidders. We now briefly discuss the case in which the number of bidders is large. We first revisit the example where the marginal distribution is the uniform distribution on $[0, 1]$.

Example 4 (large n and uniform distribution)**.** *Suppose that there are n bidders and F is the uniform distribution on the $[0, 1]$ interval. As $n \to \infty$, the robustly optimal reserve price $r_n^* = \frac{1}{n+1} \to 0$, and the revenue guarantee $\frac{n}{2(n+1)} \to \frac{1}{2}$ which is the expectation of a generic bidder's valuation.*

These features generalize to any marginal distribution. Indeed, Theorem 2 has immediate implications as follows.

Corollary 1. *For any marginal distribution F,*

1. *$r_n^* < \frac{1}{n}$ for any n;*
2. *$\lim_{n\to\infty} r_n^* \to 0$;*
3. *$\lim_{n\to\infty} REV(r_n^*, \pi^{r_n^*}) \to \int_0^1 x\, dF(x)$; and*
4. *The second-price auction with no reserve price is asymptotically optimal.*

The first statement follows from the first-order condition of the auxiliary maximization problem (Max-n) that

$$F(nr_n^*) = F(\frac{(n-1)+F(r_n^*)}{n}) < 1.$$

The second statement then trivially follows. By Theorem 2,

$$REV(r_n^*, \pi^{r_n^*}) = \frac{n}{n-1}\int_{r_n^*}^{c_n(r_n^*)} x\,dF(x) \to \int_0^1 x\,dF(x)$$

as $n \to \infty$. The fourth statement follows from the observation that $\int_0^1 x\,dF(x)$ is the full surplus in our framework.

6 Random reserve price

In this section, we extend our analysis to the case in which the auctioneer is allowed to randomize over reserve prices and solve for the robustly optimal random reserve price that generates the highest revenue guarantee among all random reserve prices. While we view the use of a random reserve price to be less practical, this is interesting from a theoretical perspective, as the auctioneer may want to use the randomization in reserve prices to hedge against the uncertainty in the correlation structure. Indeed, we show that the auctioneer could achieve a strictly higher revenue guarantee using a random reserve price.

We consider the case in which the auctioneer is allowed to randomize over reserve prices. By allowing the auctioneer to randomize over reserve prices, we are enlarging the auctioneer's strategy space. Let $\mathcal{G}$ denote the set of all cumulative distribution functions on the $[0, 1]$ interval. The auctioneer chooses a distribution $G \in \mathcal{G}$ rather than a deterministic reserve price $r \in [0, 1]$. For any random reserve price G, let

$$REV(G, v) = \int_0^1 REV(r, v)\,dG(r)$$

and let

$$REV(G, \pi) = \int_V REV(G, v)\,d\pi(v).$$

That is, we use $REV(G, v)$ to denote the auctioneer's ex post revenue by using the random reserve price G when the realized valuation profile is v, and we use $REV(G, \pi)$ to denote the auctioneer's expected revenue by using the random reserve price G under the joint distribution π.

The auctioneer solves the following maxmin optimization problem:

$$\sup_{G \in \mathcal{G}} \inf_{\pi \in \Pi} REV(G, \pi).$$

We refer to the solution of this maxmin optimization problem as the robustly optimal random reserve price.

Our approach is to identify a saddle point (G^*, π^*) such that

$$REV(G^*, \pi) \geq REV(G^*, \pi^*) \geq REV(G, \pi^*)$$

for all $G \in \mathcal{G}$ and $\pi \in \Pi$. Since

$$\begin{aligned}
\max_{G \in \mathcal{G}} \min_{\pi \in \Pi} REV(G, \pi) &\geq \min_{\pi \in \Pi} REV(G^*, \pi) \\
&= REV(G^*, \pi^*) \\
&= \max_{G \in \mathcal{G}} REV(G, \pi^*) \\
&\geq \min_{\pi \in \Pi} \max_{G \in \mathcal{G}} REV(G, \pi) \\
&\geq \max_{G \in \mathcal{G}} \min_{\pi \in \Pi} REV(G, \pi),
\end{aligned}$$

we can conclude that G^* is the robustly optimal random reserve price, and $REV(G^*, \pi^*)$ is the highest revenue guarantee.

For the sake of clarity, we first study the case in which there are n bidders and each bidder's valuation is uniformly distributed on the $[0, 1]$ interval in Section 6.1. We then extend our analysis to general distributions in Section 6.2.

6.1 Uniform distribution

We construct a particular random reserve price G^* and a correlation structure π^* such that (G^*, π^*) is a saddle point.

We first construct the correlation structure π^*. The correlation structure π^* has the feature that given that the auctioneer knows the joint distribution is π^*, the auctioneer is indifferent among a range of reserve prices.

Construction of π^*. Let $\bar{b} = \frac{2n-1}{2n}$. The correlation structure π^* is such that

1. it only puts positive probability in the regions $V^{\bar{b},\emptyset}$ and $V^{\bar{b},i}$ for each $i \in I$;

2. in the region $V^{\bar{b},i}$, π^* is uniformly distributed on the following region

$$D_i := \Big\{(v_1, v_2, \ldots, v_n) : \bar{b} \leq v_i \leq 1, \\ 0 \leq v_1 = v_2 = \ldots = v_{i-1} = v_{i+1} = \ldots = v_n < \bar{b}\Big\}$$

with total measure $1 - \bar{b}$;

3. in the region $V^{\bar{b},\emptyset}$, π^* is uniformly distributed on the following line

$$D_0 := \{(v_1, v_2, \ldots, v_n) : 0 \leq v_1 = \ldots = v_n < \bar{b}\}$$

with total measure $\frac{1}{2}$.

In words, in the region $V^{\bar{b},i}$, bidders other than bidder i have the same valuation which is independent of v_i. In the region $V^{\bar{b},\emptyset}$, all the bidders have the same valuation. It is straightforward to verify that π^* is consistent with the marginals. Figure 2 illustrates π^* in the case in which there are two bidders.

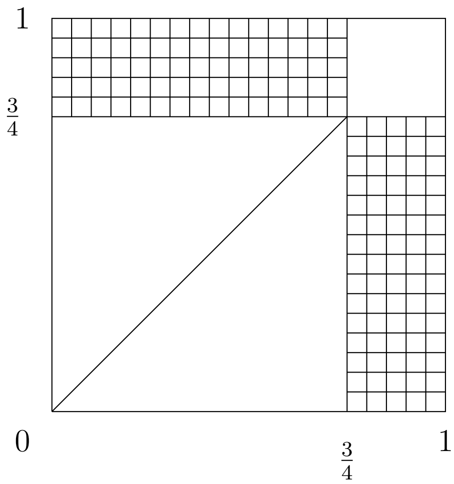

Figure 2: The figure depicts the correlation structure π^* that we construct in the case in which there are two bidders and F is the uniform distribution on $[0, 1]$. In this case, $\bar{b} = \frac{3}{4}$.

Before we proceed, let us discuss some intuition behind the construction of the correlation structure π^*. One intuition is that π^* creates a lot of indifferences for the choice of the reserve price. The other intuition is described as follows. We focus on the region in which only bidder i's valuation is weakly larger than $\bar{b}$. The ex post revenue for each v in this region and each realization of the random reserve price is then the maximum of $v_{-i}(1)$ and the realization of the random reserve price, which is a submodular function. To minimize the expectation of a submodular function, in this region, bidders other than bidder i have the same valuation.

We now calculate the expected revenue for each reserve price $r \in [0,1]$ against π^*. It is straightforward to calculate that

$$REV(r, \pi^*) = \begin{cases} \frac{2n-1}{4n}, & \text{if } r \in [0, \bar{b}]; \\ nr(1-r), & \text{if } r \in (\bar{b}, 1]. \end{cases}$$

Since $\bar{b} = \frac{2n-1}{2n} \geq \frac{1}{2}$, $nr(1-r) < n\bar{b}(1-\bar{b}) = \frac{2n-1}{4n}$ whenever $r > \bar{b}$. Thus,

$$\arg\max_{r \in [0,1]} REV(r, \pi^*) = [0, \bar{b}].$$

Construction of G^*. Let

$$G^*(r) = \bar{b}^{-\frac{1}{n-1}} r^{\frac{1}{n-1}}$$

with support $[0, \bar{b}]$. Since every reserve price in the support of G^* maximizes the auctioneer's expected revenue against π^*,

$$G^* \in \arg\max_{G \in \mathcal{G}} REV(G, \pi^*).$$

Thus, $REV(G^*, \pi^*)$ is an upper bound of the revenue guarantee.

Proposition 3. *Suppose that there are n bidders and each bidder's valuation is uniformly distributed on the $[0,1]$ interval. Then, G^* is the robustly optimal random reserve price, and generates the highest revenue guarantee of $\frac{2n-1}{4n}$.*

It remains to show that

$$\pi^* \in \arg\min_{\pi \in \Pi} REV(G^*, \pi).$$

Here, as in the case of a deterministic reserve price, we adopt the duality approach. In words, the revenue guarantee of G^* is $REV(G^*, \pi^*)$. Since we have established that $REV(G^*, \pi^*)$ is an upper bound of the revenue guarantee, G^* is the robustly optimal random reserve price, and achieves the highest revenue guarantee $REV(G^*, \pi^*)$.

Proof of Proposition 3. In what follows, we show that

$$\pi^* \in \arg\min_{\pi \in \Pi} REV(G^*, \pi).$$

We first calculate the ex post revenue of the auctioneer as follows:

$$REV(G^*, v) = \begin{cases} \bar{b}^{-\frac{1}{n-1}} \left[\frac{1}{n} v(1)^{\frac{n}{n-1}} + \frac{n-1}{n} v(2)^{\frac{n}{n-1}} \right], & \text{if } v(1) \leq \bar{b}; \\ \frac{\bar{b}}{n} + \frac{n-1}{n} \bar{b}^{-\frac{1}{n-1}} v(2)^{\frac{n}{n-1}}, & \text{if } v(2) \leq \bar{b} < v(1); \\ v(2), & \text{if } v(2) > \bar{b}. \end{cases}$$

Let

$$u(x) = \begin{cases} \frac{1}{n} \bar{b}^{-\frac{1}{n-1}} x^{\frac{n}{n-1}}, & \text{if } x \leq \bar{b}; \\ \frac{\bar{b}}{n}, & \text{if } x > \bar{b}. \end{cases}$$

One can easily verify that

$$REV(G^*, v) \geq \sum_{i \in I} u(v_i)$$

for all $v \in V$. It follows that for any $\pi \in \Pi$,

$$\begin{aligned} REV(G^*, \pi) &= \int_V REV(G, v) \, d\pi(v) \\ &\geq \int_V \sum_{i \in I} u(v_i) \, d\pi(v) \\ &= \sum_{i \in I} \int_V u(v_i) \, d\pi(v) \\ &= n \int_0^1 u(x) \, dx \\ &= \frac{2n-1}{4n}. \end{aligned}$$

Since $REV(G^*, \pi^*) = \frac{2n-1}{4n}$, we obtain the desired result. $\square$

6.2 General distribution

We now extend our analysis to a large class of marginal distributions. We make the following assumption: $xf(x)$ is weakly increasing in x. In words, this assumption says that the density function does not decrease too fast. Our analysis here parallels that in the case of uniform distribution.

We first present the following technical lemma which is used in the construction of the saddle point. The lemma is a consequence of the assumption that $xf(x)$ is weakly increasing in x.

Lemma 2. *Fix a marginal distribution F such that $xf(x)$ is weakly increasing in*

x. Let $\psi(x) := x - \frac{1-F(x)}{f(x)}$, and let

$$\gamma(x) := 1 - F(x) - \frac{1}{n-1} x^{-\frac{n}{n-1}} \int_0^x y^{\frac{n}{n-1}} f(y)\, dy.$$

Then,

1. *$\lim_{x\to 0} xf(x) = 0$;*
2. *there exists a unique $b^* \in (0,1)$ such that $\psi(b^*) = 0$; and*
3. *there exists $x \in [b^*, 1]$ such that $\gamma(x) = 0$.*

Let $\bar{b}_F$ be such that $\bar{b}_F \in [b^*, 1]$ and $\gamma(\bar{b}_F) = 0$. We are now ready to construct a particular random reserve price G_F^* and a correlation structure π_F^* such that (G_F^*, π_F^*) is a saddle point.

Construction of π_F^*. As in the case of uniform distribution, π_F^* only puts positive probability in the regions $V^{\bar{b}_F,\emptyset}$ and $V^{\bar{b}_F,i}$ for each $i \in I$.

In $V^{\bar{b}_F,i}$, π_F^* concentrates on the following region

$$\begin{aligned} D_i = \Big\{ &(v_1, v_2, \ldots, v_n) : \bar{b}_F \le v_i \le 1, \\ &0 \le v_1 = v_2 = \ldots = v_{i-1} = v_{i+1} = \ldots = v_n < \bar{b}_F \Big\}. \end{aligned}$$

The marginal of π_F^* coincides with the restriction of F on $[\bar{b}_F, 1] \subseteq V_i$. Restricted in $V^{\bar{b}_F,i}$, all the $\{v_j\}_{j\neq i}$ are maximally positively correlated with the marginal of π_F^* being H on $[0, \bar{b}_F) \subseteq V_j$ for each $j \neq i$, where

$$H(x) = \frac{1}{n-1} x^{-\frac{n}{n-1}} \int_0^x y^{\frac{n}{n-1}} f(y)\, \mathrm{d}y.$$

Then the restriction of π_F^* on $V^{\bar{b}_F,i}$ is the product measure on V_i and $\prod_{j\neq i} V_j$; that is, v_i and $(v_1, v_2, \ldots, v_{i-1}, v_{i+1}, \ldots, v_n)$ are independently distributed. Note that H is a feasible measure on $[0, \bar{b}_F)$, because

1. $\lim_{x\to 0} H(x) = 0$ and $H(\bar{b}_F) = 1 - F(\bar{b}_F)$;
2. H is continuous;
3. H is weakly increasing since the derivative of H is

$$\begin{aligned} h(x) &= \frac{1}{n-1} \left[-\frac{n}{n-1} \cdot x^{-\frac{2n-1}{n-1}} \cdot \int_0^x y^{\frac{n}{n-1}} f(y)\, \mathrm{d}y + f(x) \right] \\ &\ge \frac{1}{n-1} \left[-\frac{n}{n-1} \cdot x^{-\frac{2n-1}{n-1}} \cdot x f(x) \cdot \int_0^x y^{\frac{1}{n-1}}\, \mathrm{d}y + f(x) \right] \\ &= 0. \end{aligned}$$

In the region $V^{\bar{b}_F,\emptyset}$, π_F^* concentrates on the following line

$$D_0 = \{(v_1, \ldots, v_n) \colon 0 \leq v_1 = \ldots = v_n < \bar{b}_F\},$$

with the density on any dimension V_j being $f(v_i) - (n-1)h(v_j)$. The density is well defined since by the construction of H, we have that $f(v_i) - (n-1)h(v_j) \geq 0$ for $v_j \in [0, \bar{b}_F]$.

In words, in the region $V^{\bar{b}_F,i}$, bidders other than bidder i have the same valuation which is independent of v_i. In the region $V^{\bar{b}_F,\emptyset}$, all the bidders have the same valuation. It is straightforward to verify that π_F^* is consistent with the marginal distribution F. The intuition behind the construction here is similar to the case of uniform distribution, and we shall not repeat the arguments.

Now that we have constructed the correlation structure π_F^*, we can calculate $REV(r, \pi_F^*)$ for all $r \in [0,1]$. If $r \in (\bar{b}_F, 1]$, then $REV(r, \pi_F^*) = nr(1 - F(r))$. If $r \in [0, \bar{b}_F]$, then

$$REV(r, \pi_F^*) = \int_r^{\bar{b}_F} x\Big[f(x) - (n-1)h(x)\Big]\, dx + n\Big[\int_r^{\bar{b}_F} xh(x)\, dx + rH(r)\Big],$$

which is the sum of the expected revenue in the region $V^{\bar{b}_F,\emptyset}$ and the expected revenue in the n symmetric regions $\{V^{\bar{b}_F,i}\}_{i\in I}$. The expected revenue in each of the n symmetric regions is the sum of $\int_r^{\bar{b}_F} xh(x)\, dx$ and $rH(r)$, where $\int_r^{\bar{b}_F} xh(x)\, dx$ (resp. $rH(r)$) is the expected revenue from valuations profiles such that the second highest valuation is weakly higher than (resp. lower than) the reserve price r. This simplifies to

$$\begin{aligned}
REV(r, \pi_F^*) &= \int_r^{\bar{b}_F} x\Big[f(x) + h(x)\Big]\, dx + nrH(r) \\
&= n\int_r^{\bar{b}_F} \Big[xh(x) + H(x)\Big]\, dx + nrH(r) \\
&= n\int_r^{\bar{b}_F} xh(x)\, dx + n\Big[\bar{b}_F H(\bar{b}_F) - rH(r) - \int_r^{\bar{b}_F} x\, dH(x)\Big] + nrH(r) \\
&= n\bar{b}_F H(\bar{b}_F) \\
&= n\bar{b}_F(1 - F(\bar{b}_F)).
\end{aligned}$$

The second equality holds since by the construction of H, $nH(x) + (n-1)xh(x) = xf(x)$ for any $x \in [0, \bar{b}_F]$. The third equality uses integration by parts, and the last equality follows from that $H(\bar{b}_F) = (1 - F(\bar{b}_F))$.

Note that the derivative of $x(1 - F(x))$ is $1 - F(x) - xf(x)$, which is negative for $x > b^*$. Since $\bar{b}_F \geq b^*$, for any $r > \bar{b}_F$, $nr(1 - F(r)) < n\bar{b}_F(1 - F(\bar{b}_F))$. Thus,

$$\arg\max_{r\in[0,1]} REV(r, \pi_F^*) = [0, \bar{b}_F].$$

Construction of G_F^*. Let

$$G_F^*(r) = \bar{b}_F^{-\frac{1}{n-1}} r^{\frac{1}{n-1}}$$

with support $[0, \bar{b}_F]$. Since every reserve price in the support of G_F^* maximizes the auctioneer's expected revenue against π_F^*,

$$G_F^* \in \arg\max_{G\in\mathcal{G}} REV(G, \pi_F^*).$$

Thus, $REV(G_F^*, \pi_F^*)$ is an upper bound of the revenue guarantee.

Theorem 3. *Suppose that there are n bidders and each bidder's valuation is distributed according to F. Then, G_F^* is the robustly optimal random reserve price, and generates the highest revenue guarantee of*

$$REV(G_F^*, \pi_F^*) = n\bar{b}_F(1 - F(\bar{b}_F)).$$

It remains to show that

$$\pi_F^* \in \arg\min_{\pi\in\Pi} REV(G_F^*, \pi).$$

In words, the revenue guarantee of G_F^* is $REV(G_F^*, \pi_F^*)$. Since we have established that $REV(G_F^*, \pi_F^*)$ is an upper bound of the revenue guarantee, G_F^* is the robustly optimal random reserve price, and achieves the highest revenue guarantee $REV(G_F^*, \pi_F^*)$.

Proof of Theorem 3. In what follows, we show that

$$\pi_F^* \in \arg\min_{\pi\in\Pi} REV(G_F^*, \pi).$$

We calculate the ex post revenue of the auctioneer as follows:

$$REV(G_F^*, v) = \begin{cases} \bar{b}_F^{-\frac{1}{n-1}} \left[\frac{1}{n} v(1)^{\frac{n}{n-1}} + \frac{n-1}{n} v(2)^{\frac{n}{n-1}}\right], & \text{if } v(1) \leq \bar{b}_F; \\ \frac{\bar{b}_F}{n} + \frac{n-1}{n} \bar{b}_F^{-\frac{1}{n-1}} v(2)^{\frac{n}{n-1}}, & \text{if } v(2) \leq \bar{b}_F < v(1); \\ v(2), & \text{if } v(2) > \bar{b}_F. \end{cases}$$

Let

$$u(x) = \begin{cases} \frac{1}{n}\bar{b}_F^{-\frac{1}{n-1}} x^{\frac{n}{n-1}}, & \text{if } x \le \bar{b}_F; \\ \frac{\bar{b}_F}{n}, & \text{if } x > \bar{b}_F. \end{cases}$$

One can easily verify that

$$REV(G_F^*, v) \ge \sum_{i \in I} u(v_i)$$

for all $v \in V$. It follows that for any $\pi \in \Pi$,

$$\begin{aligned} REV(G_F^*, \pi) &\ge \sum_{i \in I} \int_V u(v_i)\, d\pi(v) \\ &= n \int_0^1 u(x)\, dF(x) \\ &= n \left[\int_0^{\bar{b}_F} \frac{1}{n} \bar{b}_F^{-\frac{1}{n-1}} x^{\frac{n}{n-1}}\, dF(x) + \frac{\bar{b}_F}{n}(1 - F(\bar{b}_F)) \right] \\ &= n\bar{b}_F(1 - F(\bar{b}_F)), \end{aligned}$$

where the last equality follows from the construction of $\bar{b}_F$. Since $REV(G_F^*, \pi_F^*) = n\bar{b}_F(1 - F(\bar{b}_F))$, we obtain the desired result. □

Remark 5. As the number of bidders gets large, $\bar{b}_F$ converges to 1, the robustly optimal random reserve price is $G_F^*(r) = \bar{b}_F^{-\frac{1}{n-1}} r^{\frac{1}{n-1}}$ which converges to the Dirac measure on zero, and the highest revenue guarantee is

$$REV(G_F^*, \pi_F^*) = n\bar{b}_F(1 - F(\bar{b}_F)) = n\bar{b}_F \frac{1}{n-1} \bar{b}_F^{-\frac{n}{n-1}} \int_0^{\bar{b}_F} y^{\frac{n}{n-1}} f(y)\, dy,$$

which converges to $\int_0^1 x\, dF(x)$.

7 Extensions

In this section, we consider two variations of our basic model. The common theme of these two models is that we relax the assumption of the auctioneer's knowledge of the marginal distribution, and the purpose is to study how the results persist when the model is made more realistic. As discussed in Remark 1(b), Theorem 1 does not rely on the knowledge of the marginal distribution. Thus, our analysis in this section focuses on the robustly optimal reserve price for any finite number of bidders. Section 7.1 considers a model in which the auctioneer has local uncertainty about the marginal distribution. That is, the auctioneer does not know the marginal distribution but she believes F is a good enough estimate. Section 7.2 interprets F as a conservative estimate of the marginal distribution. That is, the auctioneer does not know the marginal distribution but she believes that the marginal distribution first-order stochastically dominates F.

7.1 Local uncertainty about the marginal distribution

We first consider a model in which the auctioneer has local uncertainty about the marginal distribution. The auctioneer has uncertainty about the marginal distribution, but she is confident that the marginal distribution is sufficiently close to F. The rationale for studying this alternative model is clear. Rather than assuming the exact knowledge of the marginal distribution, we only require that the auctioneer has a good enough estimate of the marginal distribution. Thus, we further relax the assumption of the auctioneer's knowledge. We denote by $\hat{F}$ the true marginal distribution. Let $d(F, \hat{F})$ denote the total variation distance between two distributions F and $\hat{F}$.

We now argue that Theorem 2 is robust to the local uncertainty about the marginal distribution in the following sense. Let r_F^* denote the robustly optimal reserve price calculated under F, and let $r_{\hat{F}}^*$ denote the robustly optimal reserve price calculated under $\hat{F}$. We claim that for any $\epsilon > 0$, there exists $\delta > 0$ such that for $d(F, \hat{F}) < \delta$,

$$\inf_{\pi \in \Pi(\hat{F})} REV(r_F^*, \pi) > \inf_{\pi \in \Pi(\hat{F})} REV(r_{\hat{F}}^*, \pi) - \epsilon.$$

In words, even if the auctioneer does not know the marginal distribution $\hat{F}$, the revenue guarantee of r_F^* is close to the revenue guarantee of $r_{\hat{F}}^*$, provided that F is a good enough estimate of $\hat{F}$. Thus, our auctioneer is protected from slight misspecification of the marginal distribution.

We provide the intuition below without presenting the formal proof. The key observation is that, for $d(F, \hat{F})$ sufficiently small, for any joint distribution $\pi \in \Pi(F)$ (resp. $\hat{\pi} \in \Pi(\hat{F})$), there exists a joint distribution $\hat{\pi} \in \Pi(\hat{F})$ (resp. $\pi \in \Pi(F)$) such that for any reserve price r, $REV(r, \pi)$ and $REV(r, \hat{\pi})$ are close to each other. Thus, for any reserve price, r_F^* and $r_{\hat{F}}^*$ in particular, the worst-case expected revenue when Nature chooses $\pi \in \Pi(F)$ and when Nature chooses $\pi \in \Pi(\hat{F})$ cannot be far apart. Formally, for any $\epsilon > 0$, there exists $\delta > 0$ such that for $d(F,\hat{F}) < \delta$,

$$\left| \inf_{\pi \in \Pi(F)} REV(r_F^*, \pi) - \inf_{\pi \in \Pi(\hat{F})} REV(r_F^*, \pi) \right| < \frac{\epsilon}{2}, \tag{3}$$

$$\left| \inf_{\pi \in \Pi(F)} REV(r_{\hat{F}}^*, \pi) - \inf_{\pi \in \Pi(\hat{F})} REV(r_{\hat{F}}^*, \pi) \right| < \frac{\epsilon}{2}. \tag{4}$$

By the definition of r_F^* (resp. $r_{\hat{F}}^*$), the worst-case expected revenue of r_F^* (resp. $r_{\hat{F}}^*$) is weakly higher than the worst-case expected revenue of $r_{\hat{F}}^*$ (resp. r_F^*) when Nature chooses $\pi \in \Pi(F)$ (resp. $\pi \in \Pi(\hat{F})$). Formally,

$$\inf_{\pi \in \Pi(F)} REV(r_F^*, \pi) \geq \inf_{\pi \in \Pi(F)} REV(r_{\hat{F}}^*, \pi), \tag{5}$$

$$\inf_{\pi \in \Pi(\hat{F})} REV(r_{\hat{F}}^*, \pi) \geq \inf_{\pi \in \Pi(\hat{F})} REV(r_F^*, \pi). \tag{6}$$

It is an elementary exercise to show that the claim follows from (3), (4), (5), and (6).

7.2 F as a conservative estimate

Next, we study another model that relaxes the assumption of the auctioneer's knowledge. Consider a setting in which the auctioneer does not know the marginal distribution but she believes that the marginal distribution first-order stochastically dominates F. In other words, F is the auctioneer's conservative estimate of the marginal distribution.

Write $\tilde{F} \succeq_{FOSD} F$ if $\tilde{F}$ first-order stochastically dominates F. Let

$$\tilde{\Pi}(F) = \cup_{\tilde{F} \succeq_{FOSD} F} \Pi(\tilde{F}).$$

Consider the following problem in which the auctioneer chooses a reserve price to maximize the worst-case expected revenue, where the worst case is taken over all joint distributions in $\tilde{\Pi}(F)$:

$$\sup_{r \in [0,1]} \inf_{\pi \in \tilde{\Pi}(F)} REV(r, \pi). \tag{FOSD}$$

Let r_F^* denote the robustly optimal reserve price calculated under F. We claim that r_F^* is also a solution to the above maxmin optimization problem (FOSD).

We provide the intuition below without presenting the formal proof. Note that for any reserve price r, the ex post revenue function $REV(r, v)$ is weakly increasing in v_i for any i. This leads to our key observation: for any $\tilde{F}$ that first-order stochastically dominates F and $\tilde{\pi}$ that is consistent with $\tilde{F}$, there exists a joint distribution π that is consistent with F such that for any reserve price r, $REV(r, \pi) \leq REV(r, \tilde{\pi})$. It follows that for any r,

$$\inf_{\pi \in \tilde{\Pi}(F)} REV(r, \pi) = \inf_{\pi \in \Pi(F)} REV(r, \pi). \tag{7}$$

The claim then follows from the definition of r_F^* and (7).

8 Related literature

This paper joins the burgeoning literature of robust mechanism design.[22] A large body of papers focus on the case in which the designer does not have reliable information about the agents' hierarchies of beliefs about each other while assuming the knowledge of the payoff environment; see, for example, Bergemann and Morris (2005), Chung and Ely (2007), Chen and Li (2018), Yamashita and Zhu (2020), Bergemann, Brooks, and Morris (2016, 2017, 2019), Du (2018), Brooks and Du (2020), and Libgober and Mu (2020).[23]

The focus of this paper is on the uncertainty about the payoff environment, that is, the distribution of the bidders' valuations. More explicitly, our auctioneer has an estimate of the distribution of a generic bidder's valuation, but has non-Bayesian uncertainty about the correlation structure. Thus, the closest to our paper in terms of the source of uncertainty is Carroll (2017), who considers a multi-dimensional screening setting in which the seller knows the marginal distribution of the buyer's valuation for each good but does not know the joint distribution. Each mechanism is evaluated by its worst-case expected profit, over all joint distributions that are consistent with the known marginals. In this setting, Carroll (2017) shows that the optimal mechanism for the seller is simply to screen along each component separately.

Several papers are similar in spirit to ours in that the auctioneer is assumed to have some limited information about payoff environment and evaluates mechanisms using the worst-case criterion. These papers assume that the auctioneer only knows some moment conditions (for example, the mean) that the marginal distribution

[22]See Carroll (2019) for a recent survey on robust mechanism design and references therein.

[23]Börgers and Li (2019) propose a notion of strategic simplicity that can be interpreted as a form of robustness—the outcome implemented in strategically simple mechanism does not depend on higher-order beliefs.

needs to satisfy. Neeman (2003) considers an auctioneer who knows (a lower bound of) the mean of each bidder's valuation. He works with the notion of "effectiveness," which is the ratio of the revenue generated by the English auction and the benchmark of full-surplus extraction (this benchmark may not be feasible even for the optimal mechanism). Koçyiğit, Iyengar, Kuhn, and Wiesemann (2020) study, among other settings, second-price auctions with reserve price when there are n ex ante symmetric bidders with a known lower bound for the mean. They characterize the optimal reserve price, and also show that randomized mechanisms yield strictly more revenue in this setting. Suzdaltsev (2020a) considers an auctioneer who knows that the bidders' valuations are independent draws from some unknown distribution F, and solves for the reserve price in a second-price auction to maximize worst-case expected revenue among all deterministic reserve prices under two specifications: (1) the seller knows the mean of F and an upper bound on values; (2) the seller knows the mean of F and an upper bound on its variance. He shows that it is optimal to set the reserve price to seller's own valuation. Suzdaltsev (2020b) considers an auctioneer who knows only the means and an upper bound for valuations. He shows that among all deterministic and dominant-strategy mechanisms, a linear version of Myersonian optimal auction generates the highest revenue guarantee. Che (2020) considers an auctioneer who only knows the mean of the marginal distribution of each bidder's valuation and the range, and shows that a second-price auction with an optimal, random reserve price obtains the optimal revenue guarantee within a broad class of mechanisms. As in our paper, the optimal reserve price in Che (2020) also converges to zero as the number of bidders goes to infinity.[24]

There is also a large literature in computer science that shows that simple mechanisms can perform reasonable well in a variety of settings. The most closely related to our work is Bei, Gravin, Lu, and Tang (2019) that consider the design of auctions in the correlation-robust framework. They use a different performance measure from the revenue guarantee. They focus on the sequential posted-price mechanism (SPM) and (among other results) show that SPM achieves a constant ($2\ln 4+2 \approx 4.78$) approximation to the optimal correlation-robust mechanism, that is, the revenue guarantee of the optimal sequential posted-price mechanism is weakly larger than $\frac{1}{2\ln 4+2}$ times the revenue guarantee of the optimal dominant-strategy mechanism.

[24] Also see Carrasco, Luz, Kos, Messner, Monteiro, and Moreira (2018) that study the revenue maximization problem of a seller who is partially informed about the distribution of buyer's valuation, only knowing its first n moments.

9 Conclusion

We consider a robust version of the single-unit auction problem in which the auctioneer has an estimate of the marginal distribution of a generic bidder's valuation but has non-Bayesian uncertainty about the correlation structure. A simple auction format, the second-price auction with no reserve price, is shown to be asymptotically optimal. Furthermore, the revenue guarantee of the second-price auction with no reserve price converges to the full surplus with the fastest rate of convergence among all sequences of standard mechanisms. In settings with a finite number of bidders, we focus on second-price auctions with reserve prices and solve for the robustly optimal reserve price that generates the highest revenue guarantee among all reserve prices. We show that typically the auctioneer finds it optimal to use a low reserve price. Both our analysis for large markets and a finite number of bidders could be perceived as supporting the use of a low reserve price from a novel robustness perspective.

From a theoretical perspective, it would be interesting to understand which mechanism generates the highest revenue guarantee among all dominant-strategy mechanisms. Further research might also consider additional restrictions on the joint distributions that the auctioneer perceives plausible. While classical papers such as Myerson (1981) consider one extreme formulation of the single-unit auction problem in the sense that the auctioneer knows the exact correlation structure, we consider the other extreme formulation in the sense that the auctioneer has no additional information besides the marginal distribution. It might be fruitful to investigate settings in which the auctioneer has some additional information besides the marginals, such as the knowledge that the bidders' valuations are positively correlated.

A Proof of Theorem 2

It suffices to show that for the reserve price r_n^*, $\pi^{r_n^*}$ is a worst-case correlation structure. The proof proceeds as follows. We first show that r_n^* necessarily satisfies

$$F(nr_n^*) = \frac{(n-1) + F(r_n^*)}{n}.$$

We then show that for any r such that $F(nr) = \frac{(n-1)+F(r)}{n}$, π^r is a worst-case correlation structure.

The first step is straightforward. This requirement on r_n^* is an immediate implication of the first-order condition. Consider the maximization problem:

$$\max_{r\in[0,1]} REV(r,\pi^r) = \frac{n}{n-1}\int_r^{c_n(r)} x\,dF(x).$$

By the first-order condition, we have $\frac{dREV(r,\pi^r)}{dr} = \frac{n}{n-1} f(r)(\frac{c_n(r)}{n} - r)$. Let

$$\begin{aligned}R_n :=&\{r\in[0,1] : \frac{n}{n-1} f(r)(\frac{c_n(r)}{n} - r) = 0\}\\ =&\{r\in[0,1] : F(nr) = \frac{(n-1)+F(r)}{n}\}\end{aligned}$$

denote the set of stationary points. Since the first-order derivative has a positive value at $r=0$ and has a negative value at $r=1$, the maximization problem has an interior solution. Thus, it must be that $r_n^* \in R_n$.

In what follows, we show that for any reserve price $r\in R_n$, π^r is a worst-case correlation structure. That is, π^r is a solution to the following minimization problem:

$$\min_{\pi\in\Pi} REV(r,\pi). \tag{Primal-r}$$

We adopt a duality approach. We construct the dual maximization problem (Dual-r) of the primal minimization problem (Primal-r) as follows:

$$\begin{aligned}\max_{\mu_1,\mu_2,\ldots,\mu_n\in L^1(F)} \quad & \mathbb{J}(\mu_1,\mu_2,\ldots,\mu_n) = \sum_{i\in I}\int_{V_i}\mu_i(v_i)\,dF(v_i) && \text{(Dual-r)}\\ \text{subject to} \quad & \text{for all } v\in V, \sum_{i\in I}\mu_i(v_i) \le REV(r,v).\end{aligned}$$

As in the proof of Theorem 1, one can easily show that the optimal value of the maximization problem (Dual-r) is weakly less than the optimal value of the minimization problem (Primal-r).

We are now ready to show that for any reserve price $r\in R_n$, π^r is a worst-case correlation structure. Step (1) calculates the value of the objective function of the primal minimization problem (Primal-r) under π^r. Step (2) constructs dual variables and calculates the value of the objective function of the dual maximization problem (Dual-r) under the constructed dual variables. Step (3) verifies that these two values are the same for any $r\in R_n$.

Step (1). The value of the objective function of the primal minimization problem (Primal-r) under π^r is

$$\frac{n}{n-1}\int_r^{c_n(r)} x\,dF(x)$$

where $c_n(r) = F^{-1}(\frac{(n-1)+F(r)}{n})$.

Step (2). For each $i \in I$, let

$$\mu_i(v_i) = \begin{cases} 0, & \text{if } v_i < r; \\ \frac{1}{n-1}(v_i - r), & \text{if } r \le v_i < nr; \\ r, & \text{if } v_i \ge nr. \end{cases}$$

It is easy to verify that these dual variables satisfy the constraints of the dual maximization problem (Dual-r). Indeed, since $\mu_i(v_i)$ is a weakly increasing function of v_i,

1. if $v(2) \ge nr$, then $\sum_{i\in I} \mu_i(v_i) \le nr \le v(2) = REV(r, v)$;
2. if $v(1) \ge nr > v(2) \ge r$, then

$$\sum_{i\in I} \mu_i(v_i) \le r + (n-1)\,\frac{1}{n-1}\,(v(2) - r) = v(2) = REV(r, v);$$

3. if $v(1) \ge nr$ and $r > v(2)$, then $\sum_{i\in I} \mu_i(v_i) = r = REV(r, v)$;
4. if $nr > v(1) \ge v(2) \ge r$, then

$$\sum_{i\in I} \mu_i(v_i) \le \frac{1}{n-1}\,(nr - r) + (n-1)\,\frac{1}{n-1}\,(v(2) - r) = v(2) = REV(r, v);$$

5. if $nr > v(1) \ge r > v(2)$, $\sum_{i\in I} \mu_i(v_i) = \frac{1}{n-1}(v(1) - r) < r = REV(r, v)$;
6. if $r > v(1)$, then $\sum_{i\in I} \mu_i(v_i) = 0 = REV(r, v)$.

We now calculate the value of the objective function of the dual maximization problem (Dual-r) under the constructed dual variables as follows:

$$\begin{aligned} \mathbb{J}(\mu_1, \mu_2, \ldots, \mu_n) &= \sum_{i\in I} \int_{V_i} \mu_i(v_i)\,dF(v_i) \\ &= n\int_{V_1} \mu_1(v_1)\,dF(v_1) \\ &= n\int_r^{nr} \frac{1}{n-1}\,(v_1 - r)\,dF(v_1) + n\int_{nr}^1 r\,dF(v_1) \end{aligned}$$

$$= \frac{n}{n-1}\int_{r}^{nr} v_1\, dF(v_1) - \frac{n}{n-1}\int_{r}^{nr} r\, dF(v_1) + n\int_{nr}^{1} r\, dF(v_1). \tag{8}$$

Step (3). Recall that $R_n = \{r \in [0,1] : F(nr) = \frac{(n-1)+F(r)}{n}\}$. Thus, for any $r \in R_n$,

$$c_n(r) = F^{-1}(\frac{(n-1)+F(r)}{n}) = nr.$$

The value of the objective function of the primal minimization problem (Primal-r) under π^r is

$$\frac{n}{n-1}\int_{r}^{c_n(r)} x\, dF(x) = \frac{n}{n-1}\int_{r}^{nr} x\, dF(x).$$

The value of the objective function of the dual maximization problem (Dual-r) under the constructed dual variables is also

$$\frac{n}{n-1}\int_{r}^{nr} x\, dF(x),$$

since the last two terms in (8) cancel off. This completes the proof that for any $r \in R_n$, π^r is a solution to the primal minimization problem (Primal-r).

B Proof of Lemma 2

1. Suppose that $\lim_{x\to 0} xf(x) = c > 0$. Since $xf(x)$ is weakly increasing in x, for any $x > 0$, we have that $xf(x) \geq c$ and $f(x) \geq \frac{c}{x}$. But then $F(x) \geq \int_0^x \frac{c}{y}\, dy = \infty$ for any $x > 0$. We have a contradiction.

2. Let $\eta(x) := xf(x) - (1 - F(x))$. Since $xf(x)$ is weakly increasing in x, $\eta(x)$ is increasing in x. Since $\lim_{x\to 0} \eta(x) < 0$ and $\eta(1) > 0$, there exists a unique b^* such that

$$\eta(x)\begin{cases} < 0, x < b^*; \\ = 0, x = b^*; \\ > 0, x > b^*. \end{cases}$$

Since $\psi(x) = \frac{\eta(x)}{f(x)}$, we have that

$$\psi(x)\begin{cases} < 0, x < b^*; \\ = 0, x = b^*; \\ > 0, x > b^*. \end{cases}$$

3. We show that (1) $\lim_{x\to 0}\gamma(x) > 0$; and (2) for any $x \le b^*$ such that $\gamma(x) \le 0$, we have that $\gamma'(x) \ge 0$. It then follows that $\gamma(b^*) \ge 0$. Since $\gamma(1) < 0$, there exists $x \in [b^*, 1]$ such that $\gamma(x) = 0$.

For (1),

$$\begin{aligned}
\lim_{x\to 0}\gamma(x) &= 1 - \frac{1}{n-1}\lim_{x\to 0}\frac{\int_0^x y^{\frac{n}{n-1}} f(y)\,\mathrm{d}y}{x^{\frac{n}{n-1}}} \\
&= 1 - \frac{1}{n-1}\lim_{x\to 0}\frac{x^{\frac{n}{n-1}} f(x)}{\frac{n}{n-1}x^{\frac{1}{n-1}}} \\
&= 1 - \frac{1}{n}\lim_{x\to 0} x f(x) \\
&= 1.
\end{aligned}$$

For (2), for any $x \le b^*$ and $\gamma(x) \le 0$,

$$\begin{aligned}
\gamma'(x) &= -\frac{n}{n-1}f(x) + \frac{n}{(n-1)^2}x^{-\frac{2n-1}{n-1}}\int_0^x y^{\frac{n}{n-1}} f(y)\,\mathrm{d}y \\
&\ge -\frac{n}{n-1}f(x) + \frac{n}{(n-1)}\frac{1-F(x)}{x} \\
&\ge 0,
\end{aligned}$$

where the first inequality follows from the definition of the function γ and the assumption that $\gamma(x) \le 0$, and the second inequality is due to the fact that $\psi(x) \le 0$ for $x \le b^*$.

References

Bei, X., N. Gravin, P. Lu, and Z. G. Tang (2019): "Correlation-Robust Analysis of Single Item Auction," in *Proceedings of the Thirtieth Annual ACM-SIAM Symposium on Discrete Algorithms*, pp. 193–208.

Bergemann, D., B. Brooks, and S. Morris (2016): "Informationally Robust Optimal Auction Design," mimeo.

——— (2017): "First-Price Auctions With General Information Structures: Implications for Bidding and Revenue," *Econometrica*, 85(1), 107–143.

——— (2019): "Revenue Guarantee Equivalence," *American Economic Review*, 109(5), 1911–1929.

Bergemann, D., and S. Morris (2005): "Robust Mechanism Design," *Econometrica*, 73, 1771–1813.

Börgers, T., and J. Li (2019): "Strategically Simple Mechanisms," *Econometrica*, 87(6), 2003–2035.

Brooks, B., and S. Du (2020): "Optimal Auction Design with Common Values: An Informationally-Robust Approach," forthcoming at Econometrica.

Carrasco, V., V. F. Luz, N. Kos, M. Messner, P. Monteiro, and H. Moreira (2018): "Optimal Selling Mechanisms under Moment Conditions," *Journal of Economic Theory*, 177, 245–279.

Carroll, G. (2017): "Robustness and Separation in Multidimensional Screening," *Econometrica*, 85(2), 453–488.

——— (2019): "Robustness in Mechanism Design and Contracting," *Annual Review of Economics*, 11, 139–166.

Che, E. W. (2020): "Distributionally Robust Optimal Auction Design under Mean Constraints," mimeo.

Chen, Y.-C., and J. Li (2018): "Revisiting the Foundations of Dominant-Strategy Mechanisms," *Journal of Economic Theory*, 178, 294–317.

Chung, K.-S., and J. C. Ely (2007): "Foundations of Dominant-Strategy Mechanisms," *Review of Economic Studies*, 74(2), 447–476.

Crémer, J., and R. P. McLean (1988): "Full Extraction of the Surplus in Bayesian and Dominant Strategy Auctions," *Econometrica*, 56(6), 1247–1257.

Du, S. (2018): "Robust Mechanisms under Common Valuation," *Econometrica*, 86(5), 1569–1588.

Hasker, K., and R. Sickles (2010): "eBay in the Economic Literature: Analysis of an Auction Marketplace," *Review of Industrial Organization*, 37(1), 3–42.

Koçyiğit, Ç., G. Iyengar, D. Kuhn, and W. Wiesemann (2020): "Distributionally Robust Mechanism Design," *Management Science*, 66(1), 159–189.

Libgober, J., and X. Mu (2020): "Informational Robustness in Intertemporal Pricing," forthcoming at the Review of Economic Studies.

LIU, Q., K. MIERENDORFF, X. SHI, AND W. ZHONG (2019): "Auctions with Limited Commitment," *American Economic Review*, 109(3), 876–910.

MEYER, M., AND B. STRULOVICI (2012): "Increasing Interdependence of Multivariate Distributions," *Journal of Economic Theory*, 147(4), 1460–1489.

MYERSON, R. (1981): "Optimal Auction Design," *Mathematics of Operations Research*, 6(1), 58–71.

NEEMAN, Z. (2003): "The Effectiveness of English Auctions," *Games and Economic Behavior*, 43(2), 214–238.

SATTERTHWAITE, M. A., AND S. R. WILLIAMS (2002): "The Optimality of a Simple Market Mechanism," *Econometrica*, 70(5), 1841–1863.

SUZDALTSEV, A. (2020a): "Distributionally Robust Pricing In Independent Private Value Auctions," mimeo.

——— (2020b): "An Optimal Distributionally Robust Auction," mimeo.

VILLANI, C. (2003): *Topics in Optimal Transportation*, no. 58. American Mathematical Society.

YAMASHITA, T., AND S. ZHU (2020): "On the Foundations of Ex Post Incentive Compatible Mechanisms," mimeo.

Supplemental Material

Appendix C shows that among a class of sequences of dominant-strategy mechanisms, the revenue guarantee of the second-price auction with no reserve price converges to the full surplus with the fastest rate of convergence. Appendix D presents a direct proof of Proposition 2, which highlights the role of the first-order condition that the robustly optimal reserve price necessarily satisfies.

C Fastest rate of convergence

Theorem 1 shows that the second-price auction with no reserve price is asymptotically optimal among all sequences of dominant-strategy mechanisms. Here, we present a complementary result to Theorem 1 that among a class of sequences of dominant-strategy mechanisms, the revenue guarantee of the second-price auction with no reserve price converges to the full surplus with the fastest rate of convergence.

We focus on *standard* mechanisms in which bidders who do not have the highest bid do not get the object. Formally, let

$$\hat{\mathcal{M}}_n = \left\{ \hat{M}_n \in \mathcal{M}_n \colon \hat{q}_i(v_1, v_2, \ldots, v_n) = 0 \text{ if } v_i < \max_{1 \le j \le n} v_j \right\}.$$

Obviously, the second-price auction with no reserve price is a standard mechanism.

The notion of optimality is captured by the following definition.

Definition 1. *We say that the revenue guarantee of $\{\hat{M}_n\}_{n\ge 2}$ converges to $\int_0^1 x\,dF(x)$ with the fastest rate of convergence if*

1. *the revenue guarantee of $\{\hat{M}_n\}_{n\ge 2}$ converges to $\int_0^1 x\,dF(x)$; and*
2. *for any $\{\hat{M}_n'\}_{n\ge 2}$, there exists some $\alpha > 0$ such that for all $n \ge 2$,*

$$\int_0^1 x\,dF(x) - \inf_{\pi_n \in \Pi_n} REV(\hat{M}_n, \pi_n) \le \alpha \left(\int_0^1 x\,dF(x) - \inf_{\pi_n \in \Pi_n} REV(\hat{M}_n', \pi_n) \right).$$

Proposition 4. *The revenue guarantee of the second-price auction with no reserve price converges to $\int_0^1 x\,dF(x)$ with the fastest rate of convergence.*

Recall that the revenue guarantee of the second-price auction with no reserve price converges to $\int_0^1 x\,dF(x)$ at least in the rate of $O(\frac{1}{n})$ (see Remark 1(e)). Thus, to prove Proposition 4, it suffices to show that for any sequence of standard mechanisms, the revenue guarantee converges to $\int_0^1 x\,dF(x)$ at most in the rate of $O(\frac{1}{n})$.

While the proof is a bit long and somewhat technical, the logic is clear. We first argue that it suffices to work with symmetric joint distributions and symmetric mechanisms. For any symmetric $\hat{M}_n$, we explicitly construct a symmetric joint distribution and establish a lower bound of the difference between $\int_0^1 x\,dF(x)$ and the expected revenue of the mechanism $\hat{M}_n$ under the constructed joint distribution. Since the joint distribution that we construct is symmetric, this bound is also the bound between $\int_0^1 x\,dF(x)$ and the worst-case expected revenue of the mechanism $\hat{M}_n$ when Nature chooses among all symmetric joint distributions. We break down our analysis into easily digestible steps.

Step 1. Symmetric joint distributions and symmetric mechanisms.

We denote by $S_n(F)$ the collection of symmetric joint distributions that are consistent with the marginal distribution F when there are n bidders. For ease of notation, we shall drop the dependency of $S_n(F)$ on F when there is no confusion.

Since $S_n \subseteq \Pi_n$, for any $\hat{M}_n$,

$$\inf_{\pi_n \in S_n} REV(\hat{M}_n, \pi_n) \geq \inf_{\pi_n \in \Pi_n} REV(\hat{M}_n, \pi_n).$$

Thus, to show that $\inf_{\pi_n \in \Pi_n} REV(\hat{M}_n, \pi_n)$ converges to $\int_0^1 x\,dF(x)$ at most in the rate of $O(\frac{1}{n})$, it suffices to show that $\inf_{\pi_n \in S_n} REV(\hat{M}_n, \pi_n)$ converges to $\int_0^1 x\,dF(x)$ at most in the rate of $O(\frac{1}{n})$.

For now on, we shall work with S_n. As such, it is without loss of generality to focus on symmetric mechanisms. In what follows, we show that for any symmetric $\{\hat{M}_n\}_{n \geq 2}$, there exists some $\beta > 0$ such that for any $n \geq 2$,

$$\int_0^1 x\,dF(x) - \inf_{\pi_n \in S_n} REV(\hat{M}_n, \pi_n) \geq \frac{\beta}{n}.$$

Step 2. Technical preparation.

For any $c \in (0,1)$, let $z_c \in (0,1)$ be such that $F([z_c, 1]) = \frac{c}{2}$, and let $b_c \in (0,1)$ be such that $F([0, b_c]) = c$. Also let

$$a_c = \frac{1}{F([z_c, 1])} \int_{[z_c, 1]} x\,dF(x).$$

Clearly, for c sufficiently close to 1, we have $a_c < b_c$. Hereafter, we fix some c_1^* such that $0 < a_{c_1^*} < b_{c_1^*} < 1$.

We first prove two technical lemmas that are used to establish a bound of the auctioneer's expected revenue in Step (3). Lemma 1 is used in Step (3a), and Lemma 2 is used in Step (3b). We suggest that readers proceed to Step (3) directly and refer to these technical lemmas when necessary.

For any measurable set $E \subseteq [0,1]$, for any $n \geq 2$, let

$$x_n^*(E) = \sup \left\{ x \in E : F(E \cap [0,x]) = \frac{n-1}{n} F(E) \right\}.$$

Lemma 1. *There exists some $\kappa > 0$ such that for any measurable set $E \subseteq [0,1]$*

with $F(E) \geq c_1^*$*, for any* $n \geq 2$*,*

$$\int_{E \cap (x_n^*(E), 1]} x \, dF(x) - \frac{1}{n-1} \int_{E \cap [0, x_n^*(E)]} x \, dF(x) \geq \frac{\kappa}{n}. \tag{9}$$

Lemma 2. *For any* $\epsilon > 0$ *and any measurable set* E *with* $F(E) > 0$*, there exist two measurable sets* $C_1, C_2 \subseteq E$ *and a mapping* $g : E \to E$ *such that*

1. $F(C_1) = F(C_2) > \frac{1}{2} F(E)$*;*
2. $x \in C_1$ *if and only if* $g(x) \in C_2$*;*
3. *for any* $x \in C_1$*,*
$$F([0, x] \cap C_1) = F([0, g(x)] \cap C_2);$$
4. *for any* $x \in E \setminus C_1$*,*
$$F([0, x] \cap (E \setminus C_1)) = F([0, g(x)] \cap (E \setminus C_2));$$
5. *for any* $x \in C_1$*,* $0 < g(x) - x < \epsilon$*;*
6. *for any* $x \in E \setminus C_1$*,* $g(x) \leq x$*.*

Proof of Lemma 1. Fix an arbitrary measurable set E with $F(E) \geq c_1^*$. For any $n \geq 2$, let $z_n \in (0, 1)$ and $y_n \in (0, 1)$ be such that

$$F([z_n, 1]) = \frac{n-1}{n} c_1^* \text{ and } F([y_n, 1]) = \frac{n-1}{n} F(E).$$

By construction, $z_2 \geq z_n$ for all $n \geq 2$. Since $F(E) \geq c_1^*$, $z_n \geq y_n$ for all $n \geq 2$.

Since $F(E \cap [0, x_n^*(E)]) = \frac{n-1}{n} F(E) = F([y_n, 1])$, we have

$$\int_{E \cap [0, x_n^*(E)]} x \, dF(x) \leq \int_{[y_n, 1]} x \, dF(x).$$

Thus,

$$\begin{aligned} \frac{1}{F(E \cap [0, x_n^*(E)])} \int_{E \cap [0, x_n^*(E)]} x \, dF(x) &\leq \frac{1}{F([y_n, 1])} \int_{[y_n, 1]} x \, dF(x) \\ &\leq \frac{1}{F([z_n, 1])} \int_{[z_n, 1]} x \, dF(x) \\ &\leq \frac{1}{F([z_2, 1])} \int_{[z_2, 1]} x \, dF(x). \end{aligned} \tag{10}$$

The second inequality holds since $z_n \geq y_n$ for all $n \geq 2$. The third inequality holds since $z_2 \geq z_n$ for all $n \geq 2$.

Let $\gamma = \frac{1}{F([z_2,1])} \int_{[z_2,1]} x\, dF(x)$. It follows from (10) and the definition of $x_n^*(E)$ that

$$\frac{1}{n-1} \int_{E\cap[0,x_n^*(E)]} x\, dF(x) \leq \frac{\gamma}{n-1} F(E \cap [0, x_n^*(E)]) = \frac{\gamma}{n} F(E). \tag{11}$$

Since $F([z_{c_1^*}, 1]) = \frac{c_1^*}{2} = F([z_2, 1])$, $z_{c_1^*} = z_2$. It follows that

$$b_{c_1^*} > a_{c_1^*} = \frac{1}{F([z_{c_1^*}, 1])} \int_{[z_{c_1^*},1]} x\, dF(x) = \frac{1}{F([z_2, 1])} \int_{[z_2,1]} x\, dF(x) = \gamma. \tag{12}$$

For any $n \geq 2$, let $s_n \in (0, 1)$ be such that

$$F([0, s_n]) = \frac{n-1}{n} c_1^*.$$

By construction, $\{s_n\}_{n\geq 2}$ is a strictly increasing sequence. As n goes to infinity, $F([0, s_n])$ converges to $c_1^* = F([0, b_{c_1^*}])$. Thus, s_n converges to $b_{c_1^*}$. It follows from (12) that there exists some sufficiently large integer $N \geq 4$ such that $s_N > \gamma$. Since

$$F(E \cap [0, x_n^*(E)]) = \frac{n-1}{n} F(E) \geq \frac{n-1}{n} c_1^* = F([0, s_n]),$$

$x_n^*(E) \geq s_n$ for all $n \geq 2$. For all $n \geq N$,

$$\int_{E\cap(x_n^*(E),1]} x\, dF(x) \geq s_n\, F(E \cap (x_n^*(E), 1]) \geq s_N\, F(E \cap (x_n^*(E), 1]) = \frac{s_N}{n} F(E). \tag{13}$$

The first inequality holds since $x \geq s_n$ for any $x \in E \cap (x_n^*(E), 1]$. The second inequality holds since $\{s_n\}_{n\geq 2}$ is a strictly increasing sequence. The equality follows from the definition of $x_n^*(E)$.

It follows from (11) and (13) that for $n \geq N$,

$$\int_{E\cap(x_n^*(E),1]} x\, dF(x) - \frac{1}{n-1} \int_{E\cap[0,x_n^*(E)]} x\, dF(x) \geq \frac{s_N - \gamma}{n} F(E) \geq \frac{s_N - \gamma}{n} c_1^*.$$

This proves (9) for $n \geq N$.

Next, we prove (9) for $2 \leq n < N$. Let $\Upsilon_k = \{x\colon f(x) \leq k\}$.[25] Fix some ϵ^*

[25]Recall that f is the density of F.

such that $0 < \epsilon^* < \frac{c_1^*}{N}$ (recall that $N \geq 4$). There exists a sufficiently large positive integer k^* such that $F(\Upsilon_{k^*}) \geq 1 - \epsilon^*$. For $2 \leq n < N$,

$$\begin{aligned}
\int_{E\cap(x_n^*(E),1]} x\,dF(x) &= \int_{E\cap(x_n^*(E),1]\cap\Upsilon_{k^*}} x\,dF(x) + \int_{(E\cap(x_n^*(E),1])\setminus\Upsilon_{k^*}} x\,dF(x) \\
&\geq \int_{E\cap(x_n^*(E),1]\cap\Upsilon_{k^*}} x\,dF(x) + F((E\cap(x_n^*(E),1])\setminus\Upsilon_{k^*})\,x_n^*(E) \\
&\geq k^* \int_{[x_n^*(E),x_n^*(E)+\frac{F(E)}{k^*n}-\frac{\epsilon^*}{k^*}]} x\,dx + \epsilon^*\,x_n^*(E) \\
&= \frac{1}{2}\,(2x_n^*(E) + \frac{F(E)}{k^*n} - \frac{\epsilon^*}{k^*})\,(\frac{F(E)}{n} - \epsilon^*) + \epsilon^*\,x_n^*(E). \qquad (14)
\end{aligned}$$

The first inequality holds since $x > x_n^*(E)$ for any $x \in (E\cap(x_n^*(E),1])\setminus\Upsilon_{k^*}$. For the second inequality, the key observation is that $x > x_n^*(E)$ for any $x \in E\cap(x_n^*(E),1]\cap\Upsilon_{k^*}$. Since

$$F(E\cap(x_n^*(E),1]\cap\Upsilon_{k^*}) + F(E\cap(x_n^*(E),1])\setminus\Upsilon_{k^*}) = F(E\cap(x_n^*(E),1]) = \frac{1}{n}\,F(E),$$

to obtain a lower bound, we set $F((E\cap(x_n^*(E),1])\setminus\Upsilon_{k^*})$ to be as large as possible. Since $F(\Upsilon_{k^*}) \geq 1-\epsilon^*$,

$$F((E\cap(x_n^*(E),1])\setminus\Upsilon_{k^*}) \leq \epsilon^*.$$

Thus, we set $F((E\cap(x_n^*(E),1])\setminus\Upsilon_{k^*}) = \epsilon^*$ and $F(E\cap(x_n^*(E),1]\cap\Upsilon_{k^*}) = \frac{F(E)}{n} - \epsilon^*$. Moreover, since $\Upsilon_{k^*} = \{x\colon f(x) \leq k^*\}$,

$$\int_{E\cap(x_n^*(E),1]\cap\Upsilon_{k^*}} x\,dF(x) \geq k^* \int_{[x_n^*(E),x_n^*(E)+\frac{F(E)}{k^*n}-\frac{\epsilon^*}{k^*}]} x\,dx,$$

where the right hand side is the integral of a random variable over this region $[x_n^*, x_n^* + \frac{F(E)}{k^*n} - \frac{\epsilon^*}{k^*}]$ with the constant density k^* and total measure $\frac{F(E)}{n} - \epsilon^*$. Intuitively, the lower bound is obtained by concentrating the total measure $\frac{F(E)}{n} - \epsilon^*$ on the smallest value possible.

Using similar arguments as above, one can show that for $2 \leq n < N$,

$$\begin{aligned}
&\frac{1}{n-1}\int_{E\cap[0,x_n^*(E)]} x\,dF(x) \\
&= \frac{1}{n-1}\int_{E\cap[0,x_n^*(E)]\cap\Upsilon_{k^*}} x\,dF(x) + \frac{1}{n-1}\int_{E\cap[0,x_n^*(E)]\setminus\Upsilon_{k^*}} x\,dF(x) \\
&\leq \frac{k^*}{n-1}\int_{[x_n^*(E)-\frac{(n-1)F(E)}{k^*n}+\frac{\epsilon^*}{k^*},x_n^*(E)]} x\,dx + \frac{\epsilon^*}{n-1}\,x_n^*(E)
\end{aligned}$$

$$= \frac{1}{2}\,(2x_n^*(E) - \frac{(n-1)F(E)}{k^* n} + \frac{\epsilon^*}{k^*})\,(\frac{F(E)}{n} - \frac{\epsilon^*}{n-1}) + \frac{\epsilon^*}{n-1}\,x_n^*(E). \qquad (15)$$

It follows from (14) and (15) that for $2 \le n < N$,

$$\int_{E\cap(x_n^*(E),1]} x\,dF(x) - \frac{1}{n-1}\int_{E\cap[0,x_n^*(E)]} x\,dF(x) \ge \frac{F(E)}{2k^* n}\,(F(E) - 4\epsilon^*) + \frac{(\epsilon^*)^2}{2k^*}\,\frac{n}{n-1}$$
$$\ge \frac{c_1^*}{2k^* n}\,(c_1^* - 4\epsilon^*).$$

The lemma follows by setting $\kappa = \min\{(s_N - \gamma)c_1^*,\ \frac{c_1^*}{2k^*}\,(c_1^* - 4\epsilon^*)\}$. $\square$

Proof of Lemma 2. For any $\delta \in (0, F(E))$, let x_δ be the smallest x such that

$$F([0,x]\cap E) = F(E) - \delta,$$

and $y_\delta \in E$ be the largest y such that

$$F([y,1]\cap E) = F(E) - \delta.$$

Consider the mapping $\phi_\delta : [0, x_\delta]\cap E \to [y_\delta, 1]\cap E$ defined as follows: for any $x \in [0, x_\delta]\cap E$,

$$F([0,x]\cap E) = F([y_\delta, \phi_\delta(x)]\cap E).^{26}$$

By construction, ϕ_δ is an increasing mapping. Let

$$A_\delta = \Big\{x \in [0, x_\delta]\cap E\colon \phi_\delta(x) - x \ge \epsilon\Big\}.$$

Then $F(A_\delta) \to 0$ as $\delta \to 0$. Fix some sufficiently small $\delta^* > 0$ such that

$$F\Big(([0, x_{\delta^*}]\cap E)\setminus A_{\delta^*}\Big) > \frac{1}{2}\,F(E).$$

Let $C_1 = ([0, x_{\delta^*}]\cap E)\setminus A_{\delta^*}$, $g = \phi_{\delta^*}$ on C_1, and $C_2 = \{g(x)\colon x \in C_1\}$. For any $x \in E\setminus C_1$, let $g(x) \in E\setminus C_2$ be such that

$$F\Big([0,x]\cap(E\setminus C_1)\Big) = F\Big([0,g(x)]\cap(E\setminus C_2)\Big).$$

By construction, Conditions (1) - (5) in the lemma hold.

Next, we show that Condition (6) in the lemma holds. We first show that for

[26] There may be multiple $z \in [y_\delta, 1]\cap E$ such that $F([0,x]\cap E) = F([y_\delta, z]\cap E)$. We just arbitrarily fix one such point.

any $x \in E \setminus C_1$,

$$F([0,x] \cap C_1) \geq F([0,x] \cap C_2).$$

- This is obviously true if $F([0,x] \cap C_2) = 0$.

- Consider the case that $F([0,x] \cap C_2) > 0$. Fix some $\epsilon_1 > 0$. Let $C_3 = C_2 \cup \{x\}$. Since $F([0,y] \cap C_2)$ is continuous and weakly increasing in $y \in C_3$, there exists some $y \in C_2$ such that $y < x$ and

$$F([0,y] \cap C_2) > F([0,x] \cap C_2) - \epsilon_1. \tag{16}$$

  Then there exists some $z \in C_1$ such that $g(z) = y$ and $z < y < x$. By Condition (3) in the lemma,

$$F([0,y] \cap C_2) = F([0,g(z)] \cap C_2) = F([0,z] \cap C_1) \leq F([0,x] \cap C_1). \tag{17}$$

  It follows from (16) and (17) that

$$F([0,x] \cap C_1) > F([0,x] \cap C_2) - \epsilon_1. \tag{18}$$

  Since (18) holds for any $\epsilon_1 > 0$, we have

$$F([0,x] \cap C_1) \geq F([0,x] \cap C_2).$$

  This completes the proof that for any $x \in E \setminus C_1$,

$$F([0,x] \cap C_1) \geq F([0,x] \cap C_2). \tag{19}$$

It follows from the construction of $g(x)$ and (19) that

$$F([0,g(x)] \cap (E \setminus C_2)) = F([0,x] \cap (E \setminus C_1)) \leq F([0,x] \cap (E \setminus C_2)).$$

As a result, $g(x) \leq x$. $\square$

Step (3). Establishing bounds.

It follows from Lemma 1 that we can fix some $\kappa^* > 0$ such that for any measurable set $E \subseteq [0,1]$ with $F(E) \geq c_1^*$, for any $n \geq 2$,

$$\int_{E \cap (x_n^*(E),1]} x \, dF(x) - \frac{1}{n-1} \int_{E \cap [0,x_n^*(E)]} x \, dF(x) \geq \frac{\kappa^*}{n}.$$

Fix some sufficiently large integer $d \geq 2$ such that

1. $2^d \kappa^* > 24$;

2. $c_1^* < 1 - F([1 - \frac{1}{2^{d-1}}, 1])$.

Fix any $n \geq 2$ and an arbitrary symmetric $\hat{M}_n$ with allocation rule $(\hat{q}_1, \hat{q}_2, \ldots, \hat{q}_n)$. For any $y \in [0,1]$, let

$$h_1^n(y) = \inf \left\{ z \in [y,1] \colon \hat{q}_1(z, y, \ldots y) \geq \frac{2n-1}{2n} \right\}$$

if there exists some $z \in [y,1]$ such that $\hat{q}_1(z, y, \ldots y) \geq \frac{2n-1}{2n}$. In words, $h_1^n(y)$ is the lowest type of bidder 1 that gets the object with at least probability $\frac{2n-1}{2n}$, when every other bidder's valuation is y. If there is no such type, let $h_1^n(y) = 1$.

For each $1 \leq k \leq n^d - 2$, let

$$C_k = \left\{ y \in [\frac{k-1}{n^d}, \frac{k}{n^d}) \colon \; h_1^n(y) \in [y, \frac{k+1}{n^d}) \right\},$$

Also let

$$\begin{aligned} W_1^n &= \cup_{1 \leq k \leq n^d - 2} C_k, \\ W_2^n &= [0, 1 - \frac{2}{n^d}) \setminus W_1^n, \\ W_3^n &= [1 - \frac{2}{n^d}, 1]. \end{aligned}$$

By construction, $W_1^n \subseteq [0, 1 - \frac{2}{n^d})$, and

$$\hat{q}_1(\frac{k+1}{n^d}, y, \ldots, y) \geq \frac{2n-1}{2n}$$

for $y \in W_1^n \cap [\frac{k-1}{n^d}, \frac{k}{n^d})$, $1 \leq k \leq n^d - 2$.

We classify our analysis into two cases based on the measure of the set W_1^n. In the first case, the measure of the set W_1^n is weakly higher than c_1^*. In the second case, the measure of W_1^n is lower than c_1^*. In each case, we construct a joint distribution in S_n and establish a lower bound of the difference between $\int_0^1 x \, dF(x)$ and the auctioneer's expected revenue under the constructed joint distribution. Since this joint distribution lies in S_n, this bound is also a lower bound of

$$\int_0^1 x \, dF(x) - \inf_{\pi_n \in S_n} REV(\hat{M}_n, \pi_n).$$

Our construction of the joint distribution for each of these two cases is technical (and may seem at hoc). Before we proceed, we wish to offer some intuition behind our construction. In the first case, the measure of the set W_1^n is sufficiently large. Recall that if all bidders other than bidder 1 have valuation y, any type of bidder 1 that is larger than $h_1^n(y)$ gets the object with at least probability $\frac{2n-1}{2n}$. For any $y \in F(W_1^n)$, by definition, $h_1^n(y)$ and y are not far apart. Loosely speaking, the allocation rule in this region is close to that of the second-price auction. We exploit this observation in the construction of our joint distribution in this case.

In the second case, the measure of W_2^n is sufficiently large. For any $y \in F(W_2^n)$, by definition, $h_1^n(y)$ and y are sufficiently far apart. Thus, if all bidders other than bidder 1 have valuation y, and bidder 1's valuation is larger than but sufficiently close to y, bidder 1 obtains the object with probability less than $\frac{2n-1}{2n}$, and all other bidders do not get the object. Due to the participation constraint, the ex post revenue of the auctioneer is bounded by $\frac{2n-1}{2n}$ multiplied by bidder 1's valuation. We exploit exactly this inefficiency in allocation in the construction of our joint distribution in this case.

Step (3a). The first case: $F(W_1^n) \geq c_1^*$.

We first consider the case in which the measure of the set W_1^n is weakly higher than c_1^*.

Let

$$L_1^n = W_1^n \times W_1^n \times \ldots \times W_1^n,$$
$$L_2^n = W_2^n \times W_2^n \times \ldots \times W_2^n,$$
$$L_3^n = W_3^n \times W_3^n \times \ldots \times W_3^n.$$

Define $\bar{\pi}_n$ to be the unique joint distribution such that

1. it only puts positive probability in the regions L_1^n and $L_2^n \cup L_3^n$;
2. in the region L_1^n: Let

$$\bar{x}^n = \sup \left\{ x \in W_1^n : F(W_1^n \cap [0, x]) = \frac{n-1}{n} F(W_1^n) \right\}.$$

The probability concentrates on n symmetric curves $K_1, K_2, \ldots, K_n$ where

$$K_i = \Big\{ v \in L_1^n \colon F(W_1^n \cap [0, v_j]) \\ = (n-1)\Big[F(W_1^n) - F(W_1^n \cap [0, v_i])\Big], \forall j \neq i, v_i \in [\bar{x}^n, 1] \Big\};$$

3. in the region $L_2^n \cup L_3^n$, v_1, v_2, …, v_n are maximally positively correlated;

4. the joint distribution is consistent with the marginals.

Obviously, $\bar{\pi}_n \in S_n$.

The interpretation of the curve K_i is that in the subset of L_1^n in which bidder i has the highest valuation, Nature puts probability in a way such that bidders other than i have the same valuation, and bidder i's valuation is maximally negatively correlated with the other bidders' valuations.

Define a mapping

$$\psi(y) = \frac{k+1}{n^d} \text{ for } y \in W_1^n \cap [\frac{k-1}{n^d}, \frac{k}{n^d}).$$

By construction, $y + \frac{1}{n^d} < \psi(y) \leq y + \frac{2}{n^d}$. Note that for any $y \in W_1^n \cap [0, \bar{x}^n]$, there exists some $\mu(y) \in W_1^n \cap [\bar{x}^n, 1]$ such that $(\mu(y), y, \ldots, y) \in K_1$. Formally,

$$F(W_1^n \cap [0, y]) = (n-1)\left[F(W_1^n) - F(W_1^n \cap [0, \mu(y)])\right] = (n-1)\, F(W_1^n \cap [\mu(y), 1]).$$

In what follows, we first show that for any $y \in W_1^n \cap [0, \bar{x}^n]$,

$$\hat{t}_1(\mu(y), y, \ldots, y) \leq \frac{1}{2n}\, \mu(y) + \frac{2n-1}{2n}\, (y + \frac{2}{n^d}).$$

We then use this bound on $t_1(\mu(y), y, \ldots, y)$ to show that the auctioneer's expected revenue by using the mechanism $\hat{M}_n$ when the joint distribution is $\bar{\pi}_n$ is bounded above by

$$\int_0^1 y\, dF(y) - \frac{\kappa^*}{4n}.$$

1. If $\mu(y) \geq \psi(y)$, then

$$\begin{aligned} &\mu(y)\, \hat{q}_1(\mu(y), y, \ldots, y) - \hat{t}_1(\mu(y), y, \ldots, y) \\ &\geq \mu(y)\, \hat{q}_1(\psi(y), y, \ldots, y) - \hat{t}_1(\psi(y), y, \ldots, y) \\ &= (\mu(y) - \psi(y))\, \hat{q}_1(\psi(y), y, \ldots, y) + \psi(y)\, \hat{q}_1(\psi(y), y, \ldots, y) - \hat{t}_1(\psi(y), y, \ldots, y) \end{aligned}$$

$$
\begin{aligned}
&\geq (\mu(y) - \psi(y))\, \hat{q}_1(\psi(y), y, \ldots, y) + \psi(y)\, \hat{q}_1(y, y, \ldots, y) - \hat{t}_1(y, y, \ldots, y) \\
&= (\mu(y) - \psi(y))\, \hat{q}_1(\psi(y), y, \ldots, y) + (\psi(y) - y)\, \hat{q}_1(y, y, \ldots, y) \\
&+ y\, \hat{q}_1\, (y, y, \ldots, y) - \hat{t}_1(y, y, \ldots, y) \\
&\geq (\mu(y) - \psi(y))\, \hat{q}_1(\psi(y), y, \ldots, y) + (\psi(y) - y)\, \hat{q}_1(y, y, \ldots, y) \\
&\geq \frac{2n-1}{2n}\, (\mu(y) - \psi(y)) + \frac{1}{n^d}\, \hat{q}_1(y, y, \ldots, y) \\
&\geq \frac{2n-1}{2n}\, (\mu(y) - \psi(y)). \qquad (20)
\end{aligned}
$$

The first inequality is the incentive constraint of type $\mu(y)$ of bidder 1. The second inequality follows from the incentive constraint of type $\psi(y)$ of bidder 1. The third inequality follows from the participation constraint of type y of bidder 1. The fourth inequality holds since $\hat{q}_1(\psi(y), y, \ldots, y) \geq \frac{2n-1}{2n}$ and $\psi(y) > y + \frac{1}{n^d}$.

Thus, we have

$$
\begin{aligned}
\hat{t}_1(\mu(y), y, \ldots, y) &\leq \mu(y) \left[\hat{q}_1(\mu(y), y, \ldots, y) - \frac{2n-1}{2n} \right] + \frac{2n-1}{2n}\, \psi(y) \\
&\leq \frac{1}{2n}\, \mu(y) + \frac{2n-1}{2n}\, \psi(y) \\
&\leq \frac{1}{2n}\, \mu(y) + \frac{2n-1}{2n}\, (y + \frac{2}{n^d}).
\end{aligned}
$$

The first inequality follows from (20). The second inequality follows from the feasibility constraint $\hat{q}_1(\mu(y), y, \ldots, y) \leq 1$. The third inequality holds since $\psi(y) \leq y + \frac{2}{n^d}$.

2. If $\mu(y) < \psi(y)$, then

$$
\hat{t}_1(\mu(y), y, \ldots, y) \leq \mu(y) < \frac{1}{2n}\, \mu(y) + \frac{2n-1}{2n} \psi(y) \leq \frac{1}{2n}\, \mu(y) + \frac{2n-1}{2n}\, (y + \frac{2}{n^d}).
$$

The first inequality follows from the participation constraint of type $\mu(y)$ of bidder 1 and the feasibility constraint $\hat{q}_1(\mu(y), y, \ldots, y) \leq 1$. The second inequality follows from the assumption that $\mu(y) < \psi(y)$. The third inequality holds since $\psi(y) \leq y + \frac{2}{n^d}$.

This completes the proof that

$$
\hat{t}_1(\mu(y), y, \ldots, y) \leq \frac{1}{2n}\, \mu(y) + \frac{2n-1}{2n}\, (y + \frac{2}{n^d}).
$$

The auctioneer's expected revenue by using the mechanism $\hat{M}_n$ when the joint distribution is $\bar{\pi}_n$ is

$$
\begin{aligned}
& REV(M_n, \bar{\pi}_n) \\
\leq & \frac{n}{n-1} \int_{W_1^n \cap [0,\bar{x}^n]} \left[\frac{\mu(y)}{2n} + \frac{2n-1}{2n} (y + \frac{2}{n^d}) \right] dF(y) + \int_{W_2^n \cup W_3^n} x \, dF(x) \\
\leq & \frac{1}{2(n-1)} \int_{W_1^n \cap [0,\bar{x}^n]} \mu(y) \, dF(y) + \frac{2n-1}{2(n-1)} \int_{W_1^n \cap [0,\bar{x}^n]} y \, dF(y) + \frac{2n-1}{n^d(n-1)} + \int_{W_2^n \cup W_3^n} x \, dF(x) \\
= & \frac{1}{2} \int_{W_1^n \cap [\bar{x}^n,1]} y \, dF(y) + \frac{2n-1}{2(n-1)} \int_{W_1^n \cap [0,\bar{x}^n]} y \, dF(y) + \frac{2n-1}{n^d(n-1)} + \int_{W_2^n \cup W_3^n} x \, dF(x) \\
= & \int_{[0,1]} y \, dF(y) - \frac{1}{2} \int_{W_1^n \cap [\bar{x}^n,1]} y \, dF(y) + \frac{1}{2(n-1)} \int_{W_1^n \cap [0,\bar{x}^n]} y \, dF(y) + \frac{2n-1}{n^d(n-1)} \\
< & \int_{[0,1]} y \, dF(y) - \frac{\kappa^*}{2n} + \frac{\kappa^*}{4n} \\
= & \int_{[0,1]} y \, dF(y) - \frac{\kappa^*}{4n}.
\end{aligned}
$$

The first equality holds since $F(W_1^n \cap [0, y]) = (n-1) \, F(W_1^n \cap [\mu(y), 1])$. The second equality holds since

$$
\int_{[0,1]} y \, dF(y) = \int_{W_1^n \cap [0,\bar{x}^n]} y \, dF(y) + \int_{W_1^n \cap [\bar{x}^n,1]} y \, dF(y) + \int_{W_2^n \cup W_3^n} x \, dF(x).
$$

The third inequality follows from Lemma 1 and that

$$
\frac{2n-1}{n^d(n-1)} \leq \frac{3}{n^d} \leq \frac{3}{2^{d-1}n} < \frac{\kappa^*}{4n}.
$$

Let $\beta_1 = \frac{\kappa^*}{4}$. We conclude that if $F(W_1^n) \geq c_1^*$, then

$$
\int_0^1 x \, dF(x) - \inf_{\pi_n \in S_n} REV(\hat{M}_n, \pi_n) \geq \int_0^1 x \, dF(x) - REV(\hat{M}_n, \bar{\pi}_n) \geq \frac{\beta_1}{n}.
$$

Step (3b). The second case: $F(W_1^n) < c_1^*$.

Let $c_2^* = 1 - F([1 - \frac{1}{2^{d-1}}, 1]) - c_1^*$. If $F(W_1^n) < c_1^*$, then

$$
F(W_2^n) = F([0, 1 - \frac{2}{n^d}) \backslash W_1^n) = F([0, 1 - \frac{2}{n^d})) - F(W_1^n) > 1 - F([1 - \frac{1}{2^{d-1}}, 1]) - c_1^* = c_2^*.
$$

Let $\overline{x} \in (0,1)$ be such that $F([0, \overline{x}]) = \frac{c_2^*}{2}$. Fix some sufficiently large integer $\overline{d} \geq d$ such that

$$
\frac{1}{2^{\overline{d}-3}} \leq \int_{[0,\overline{x}]} x \, dF(x).
$$

By Lemma 2, there exists two measurable sets $C_1, C_2 \subseteq W_2^n$ and a mapping

$g: W_2^n \to W_2^n$ such that

1. $F(C_1) = F(C_2) > \frac{1}{2} F(W_2^n)$;

2. $x \in C_1$ if and only if $g(x) \in C_2$;

3. for any $x \in C_1$,
$$F([0,x] \cap C_1) = F([0,g(x)] \cap C_2);$$

4. for any $x \in W_2^n \setminus C_1$,
$$F([0,x] \cap (W_2^n \setminus C_1)) = F([0,g(x)] \cap (W_2^n \setminus C_2));$$

5. for any $x \in C_1$, $0 < g(x) - x < \frac{1}{n^{\bar{\bar{d}}}}$;

6. for any $x \in W_2^n \setminus C_1$, $g(x) \le x$.

Define $\bar{\bar{\pi}}_n$to be the unique symmetric joint distribution such that

1. it only puts positive probability in the regions L_2^n and $L_1^n \cup L_3^n$;

2. in the region L_2^n, the probability concentrates on n symmetric curves $J_1, J_2, \ldots, J_n$ where
$$J_i = \{v \in L_2^n \colon g(v_j) = v_i, \forall j \neq i\}.$$

3. in the region $L_1^n \cup L_3^n$, v_1, v_2, …, v_n are maximally positively correlated;

4. the joint distribution is consistent with the marginals.

Obviously, $\bar{\bar{\pi}}_n \in S_n$.

We first consider the seller's expected revenue from the curve J_1. By symmetry, the expected revenue from any other curve J_i is the same. On the curve J_1, bidder 1 has the highest valuation if $v_1 \in C_2$ and does not have the highest valuation if $v_1 \in W_2^n \setminus C_2$.

- For $v_1 \in C_2$, there exists some $v_2 \in C_1$ such that $v_1 = g(v_2)$. By the definition of W_2^n,
$$h_1^n(v_2) - v_2 > \frac{1}{n^d} \ge \frac{1}{n^{\bar{\bar{d}}}}.$$
Thus,
$$v_1 = g(v_2) < v_2 + \frac{1}{n^{\bar{\bar{d}}}} < h_1^n(v_2). \tag{21}$$

By the definition of h_1^n, we have

$$\hat{q}_1(v_1, v_2, \ldots, v_2) < \frac{2n-1}{2n}. \tag{22}$$

Since only bidder 1 has the highest valuation, only bidder 1 makes a payment to the auctioneer. The auctioneer's revenue when the valuation profile is $(v_1, v_2, \ldots, v_2)$ is

$$\hat{t}_1(v_1, v_2, \ldots, v_2) \leq v_1\, \hat{q}_1(v_1, v_2, \ldots, v_2) < \frac{2n-1}{2n}\, v_1 < \frac{2n-1}{2n}\,(v_2 + \frac{1}{n^{\bar{d}}}).$$

The first inequality follows from the participation constraint of type v_1 of bidder 1. The second inequality follows from (22). The last inequality follows from (21).

- For $v_1 \notin C_2$, bidders other than bidder i have the same value v_2, while bidder i has the valuation $v_1 = g(v_2) < v_2$. Thus, the auctioneer's revenue when the valuation profile is $(v_1, v_2, \ldots, v_2)$ is at most v_2.

The auctioneer's expected revenue from using the mechanism $\hat{M}_n$ when the joint distribution is $\bar{\bar{\pi}}_n$ is

$$\begin{aligned}
&REV(M_n, \bar{\bar{\pi}}_n)\\
&\leq \int_{W_2^n \setminus C_1} x\, dF(x) + \int_{C_1} \frac{2n-1}{2n}\,(x + \frac{1}{n^{\bar{d}}})\, dF(x) + \int_{W_1^n \cup W_3^n} x\, dF(x)\\
&= \int_{W_2^n} x\, dF(x) - \int_{C_1} x\, dF(x) + \int_{C_1} \frac{2n-1}{2n}\,(x + \frac{1}{n^{\bar{d}}})\, dF(x) + \int_{W_1^n \cup W_3^n} x\, dF(x)\\
&= \int_{[0,1]} x\, dF(x) - \frac{1}{2n} \int_{C_1} x\, dF(x) + \frac{2n-1}{2n^{\bar{d}+1}}\, F(C_1)\\
&\leq \int_{[0,1]} x\, dF(x) - \frac{1}{2n} \int_{C_1} x\, dF(x) + \frac{2n-1}{2n^{\bar{d}+1}}\\
&\leq \int_{[0,1]} x\, \mathrm{d}F(x) - \frac{1}{2n} \int_{[0,\bar{x}]} x\, \mathrm{d}F(x) + \frac{2n-1}{2n^{\bar{d}+1}}\\
&\leq \int_{[0,1]} x\, \mathrm{d}F(x) - \frac{1}{2n} \int_{[0,\bar{x}]} x\, \mathrm{d}F(x) + \frac{1}{4n} \int_{[0,\bar{x}]} x\, \mathrm{d}F(x)\\
&= \int_{[0,1]} x\, \mathrm{d}F(x) - \frac{1}{4n} \int_{[0,\bar{x}]} x\, \mathrm{d}F(x).
\end{aligned}$$

The third inequality holds since

$$F(C_1) > \frac{1}{2} F(W_2^n) \geq \frac{c_2^*}{2} = F([0, \bar{x}]).$$

The fourth inequality holds since

$$\frac{2n-1}{2n^{\bar{d}+1}} < \frac{1}{n^{\bar{d}}} \leq \frac{1}{2^{\bar{d}-1}} \frac{1}{n} \leq \frac{1}{4n} \int_{[0,\bar{x}]} x \, \mathrm{d}F(x).$$

Let $\beta_2 = \frac{1}{4} \int_{[0,\bar{x}]} x \, dF(x)$. We conclude that if $F(W_1^n) < c_1^*$, then

$$\int_0^1 x \, dF(x) - \inf_{\pi_n \in S_n} REV(\hat{M}_n, \pi_n) \geq \int_0^1 x \, dF(x) - REV(\hat{M}_n, \bar{\bar{\pi}}_n) \geq \frac{\beta_2}{n}.$$

Step (4). Wrapping up the proof.

Let $\beta = \min\{\beta_1, \beta_2\}$. It follows from our analysis in Step (3a) and Step (3b) that for any $n \geq 2$ and any symmetric $\hat{M}_n$,

$$\int_0^1 x \, dF(x) - \inf_{\pi_n \in S_n} REV(M_n, \pi_n) \geq \frac{\beta}{n}.$$

Thus, for any sequence of symmetric $\{\hat{M}_n\}_{n \geq 2}$, the revenue guarantee converges to $\int_0^1 x \, dF(x)$ at most in the rate of $O(\frac{1}{n})$. This completes the proof.

D Proof of Proposition 2

We first show that r^* necessarily satisfies that $F(2r^*) = \frac{1+F(r^*)}{2}$. Consider the auxiliary maximization problem:

$$\max_{r \in [0,1]} REV(r, \pi^r) = 2 \int_r^{c(r)} x \, dF(x).$$

By the first-order condition, we have that $\frac{dREV(r,\pi^r)}{dr} = 2f(r)(\frac{c(r)}{2} - r)$. Let

$$\begin{aligned} R := &\{r \in [0,1] : 2f(r)(\frac{c(r)}{2} - r) = 0\} \\ = &\{r \in [0,1] : c(r) = 2r\} \\ = &\{r \in [0,1] : F^{-1}(\frac{1+F(r)}{2}) = 2r\} \\ = &\{r \in [0,1] : F(2r) = \frac{1+F(r)}{2}\} \end{aligned}$$

denote the set of stationary points. Since the first-order derivative takes a positive value at $r = 0$, and takes a negative value at $r = 1$, the auxiliary problem has an interior solution. Thus, $r^* \in R$.

We proceed to show that for any $r \in R$, π^r is the worst-case correlation structure. Without loss of generality, we consider only symmetric joint distributions. We show that for any $\pi \in \Pi$ that is symmetric, there exists π' such that

1. $\pi' \in \Pi$;
2. π' puts zero probability in the regions $V^{r,1}$ and $V^{r,2}$; and
3. $REV(r, \pi') \leq REV(r, \pi)$.

Thus, to solve for the worst-case correlation structure, we only have to consider joint distributions that are consistent with the marginals and only put positive probability in the regions $V^{r,\emptyset}$ and $V^{r,\{1,2\}}$. This, combined with Observation 3, implies that π^r is indeed the worst-case correlation structure.

The idea behind the construction of π' for any symmetric π is intuitive. Unfortunately, the formal analysis requires quite a bit of notation. For ease of reference, we define nine segments as follows (see Figure 3):

$$\begin{aligned}
&A_1 = [0, r] \times [0, r]; && A_2 = [r, c(r)] \times [0, r]; && A_3 = [c(r), 1] \times [0, r]; \\
&A_4 = [0, r] \times [r, c(r)]; && A_5 = [0, r] \times [c(r), 1]; && A_6 = [r, c(r)] \times [r, c(r)]; \\
&A_7 = [c(r), 1] \times [r, c(r)]; && A_8 = [r, c(r)] \times [r, 1]; && A_9 = [r, 1] \times [r, 1].
\end{aligned}$$

For $1 \leq j \leq 9$, we also write $A_j = [\underline{x}^j, \bar{x}^j] \times [\underline{y}^j, \bar{y}^j]$. For example, $\underline{x}^2 = r$, $\bar{x}^2 = c(r)$, $\underline{y}^2 = 0$, and $\bar{y}^2 = r$.

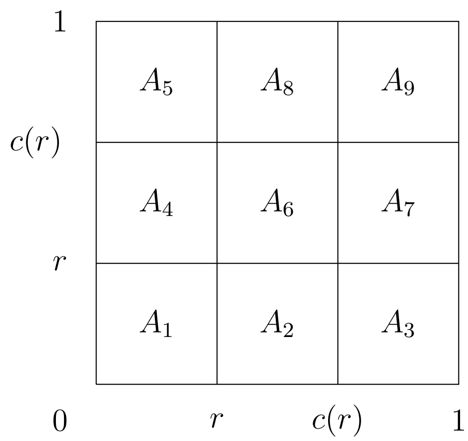

Figure 3: The nine segments that we define on the basis of r and $c(r)$.

Suppose that π has positive measures on $\cup_{2 \leq j \leq 5} A_j$.

Fix any $\pi \in \Pi$ that is symmetric. For $1 \leq j \leq 9$, let $a_j := \pi(A_j)$ denote the total measure of π on A_j. For any $[c_1, c_2] \subseteq [0, 1]$, $[d_1, d_2] \subseteq [0, 1]$ and any

$1 \leq j \leq 9$, let

$$\pi_x^j([c_1, c_2]) = \pi([c_1, c_2] \times [\underline{y}^j, \bar{y}^j]) \text{ and } \pi_y^j([d_1, d_2]) = \pi([\underline{x}^j, \bar{x}^j] \times [d_1, d_2]).$$

We consider two cases. In the first case, $a_2 \geq a_3$. In the second case, $a_2 < a_3$. Suppose that $a_2 \geq a_3$. Since π is symmetric, $a_4 \geq a_3$. If $a_3 \neq 0$, we construct a correlation structure π' from π by shifting all the measure from A_3 to A_7 and shifting the same measure from A_4 to A_1 in a way that respects the marginals. Otherwise, we skip this step. This weakly decreases the auctioneer's expected revenue, since *the ex post revenue is* r *for any* $v \in A_3 \cup A_4$ *and the ex post revenue is capped at* $c(r) = 2r$ *for any* $v \in A_7$. Formally, π' is such that

1. π' coincides with π on A_2, A_5, A_6, A_8, and A_9;
2. $\pi'(A_3) = 0$;
3. for any $[c_1, c_2] \times [d_1, d_2] \subseteq A_7$,
$$\pi'([c_1, c_2] \times [d_1, d_2]) = \pi([c_1, c_2] \times [d_1, d_2]) + \frac{\pi_x^3([c_1, c_2]) \cdot \pi_y^4([d_1, d_2])}{a_4};$$
4. for any $[c_1, c_2] \times [d_1, d_2] \subseteq A_4$,
$$\pi'([c_1, c_2] \times [d_1, d_2]) = \pi([c_1, c_2] \times [d_1, d_2]) - \frac{a_3}{a_4} \cdot \pi([c_1, c_2] \times [d_1, d_2]);$$
5. for any $[c_1, c_2] \times [d_1, d_2] \subseteq A_1$,
$$\pi'([c_1, c_2] \times [d_1, d_2]) = \pi([c_1, c_2] \times [d_1, d_2]) + \frac{\pi_x^4([c_1, c_2]) \cdot \pi_y^3([d_1, d_2])}{a_4}.$$

Analogously, one can construct a correlation structure π'' from π' by shifting all the measure from A_5 to A_8 and shifting the same measure from A_2 to A_1 in a way that respects the marginals and weakly decreases the auctioneer's expected revenue. Note that

$$\pi''(A_3) = \pi''(A_5) = 0 \text{ and } \pi''(A_2) = \pi''(A_4) = a_2 - a_3.$$

If $a_2 = a_3$, then we have proved the desired result. If $a_2 > a_3$, then the last step in this case is to construct a correlation structure $\hat{\pi}$ from π'' by shifting all the measure from A_2 to A_6 and shifting the same measure from A_4 to A_1 in a way that respects the marginals. This weakly decreases the expected revenue, since *the ex post revenue is r for any $v \in A_2 \cup A_4$ and the ex post revenue is capped at $c(r) = 2r$ for any $v \in A_6$.* Formally, $\hat{\pi}$ is such that

1. $\hat{\pi}$ coincides with π'' on A_3, A_5, A_7, A_8, and A_9;
2. $\hat{\pi}(A_2) = \hat{\pi}(A_4) = 0$;
3. for any $[c_1, c_2] \times [d_1, d_2] \subseteq A_1$,
$$\hat{\pi}([c_1, c_2] \times [d_1, d_2]) = \pi''([c_1, c_2] \times [d_1, d_2]) + \frac{\pi''^4_x([c_1, c_2]) \cdot \pi''^2_y([d_1, d_2])}{a_2 - a_3};$$
4. for any $[c_1, c_2] \times [d_1, d_2] \subseteq A_6$,
$$\hat{\pi}([c_1, c_2] \times [d_1, d_2]) = \pi''([c_1, c_2] \times [d_1, d_2]) + \frac{\pi''^2_x([c_1, c_2]) \cdot \pi''^4_y([d_1, d_2])}{a_2 - a_3}.$$

This completes the proof for the first case since

$$\hat{\pi}(A_2) = \hat{\pi}(A_3) = \hat{\pi}(A_4) = \hat{\pi}(A_5) = 0.$$

Next, we study the case in which $a_2 < a_3$. Since π is consistent with the marginals and $F(2r) = \frac{1+F(r)}{2}$, $a_2 + a_6 + a_8 = F(c(r)) - F(r) = 1 - F(c(r)) = a_3 + a_7 + a_9$. Since π is symmetric, $a_7 = a_8$. Thus, $a_2 + a_6 = a_3 + a_9 \geq a_3$, which implies $a_6 \geq a_3 - a_2$. We further divide A_6 by the 45-degree line into three regions: $A_6^u = \{v \in A_6 : v_1 < v_2\}$ (above the 45-degree line), $A_6^m = \{v \in A_6 : v_1 = v_2\}$ (the 45-degree line), and $A_6^d = \{v \in A_6 : v_1 > v_2\}$ (below the 45-degree line). Without loss of generality, we can work with π such that $\pi(A_6^m) = 0$.[27] Thus, $\pi(A_6^u) = \pi(A_6^d) = \frac{a_6}{2}$. For any $[c_1, c_2] \times [d_1, d_2] \subseteq A_6$, let

$$\pi_x^d([c_1, c_2]) = \pi\left(([c_1, c_2] \times [\underline{y}^6, \bar{y}^6]) \cap A_6^d\right)$$

and

$$\pi_y^d([d_1, d_2]) = \pi\left(([\underline{x}^6, \bar{x}^6] \times [d_1, d_2]) \cap A_6^d\right).$$

and

$$\pi_x^m([c_1, c_2]) = \pi\left(\{(z, z) \colon c_1 \leq z \leq c_2\}\right), \quad \pi_y^m([d_1, d_2]) = \pi\left(\{(z, z) \colon d_1 \leq z \leq d_2\}\right).$$

[27] Otherwise, let π_6^m be the restriction of π on A_6^m, and π_x^m and π_y^m be the marginal of π_6^m on V_1 and V_2, respectively. Then we can construct another finite measure $\bar{\pi}_6^m$ having the same marginals π_x^m and π_y^m as follows: $\bar{\pi}_6^m$ concentrates on the curve with the maximally negative correlation on A_6: $\pi_x^m[\underline{x}^6, v_1] = \pi_y^m[v_2, \bar{y}^6]$ for $(v_1, v_2) \in A_6$. Let $\bar{\pi}'$ be the finite measure by restricting π on $V \setminus A_6^m$, and $\bar{\pi} = \bar{\pi}' + \bar{\pi}_6^m$. Then $\bar{\pi}$ respects the marginals, $\bar{\pi}(A_6^m) = 0$, and $REV(r, \bar{\pi}) \leq REV(r, \pi)$.

We construct a correlation structure π' from π by shifting measure $\frac{a_3-a_2}{2}$ from A_3 to A_2 and shifting the same measure from A_6^d to A_7 in a way that respects the marginals and does not change the expected revenue. Formally, π' is such that

1. π' coincides with π on A_1, A_4, A_5, A_6^u, A_6^m, A_8, and A_9;
2. for any $[c_1, c_2] \times [d_1, d_2] \subseteq A_2$,
$$\pi'([c_1, c_2] \times [d_1, d_2]) = \pi([c_1, c_2] \times [d_1, d_2]) + (a_3 - a_2) \cdot \frac{\pi_x^d([c_1, c_2]) \cdot \pi_y^3([d_1, d_2])}{a_3 \cdot a_6};$$
3. for any $[c_1, c_2] \times [d_1, d_2] \subseteq A_3$,
$$\pi'([c_1, c_2] \times [d_1, d_2]) = \pi([c_1, c_2] \times [d_1, d_2]) - \frac{a_3 - a_2}{2} \cdot \frac{\pi([c_1, c_2] \times [d_1, d_2])}{a_3};$$
4. for any $[c_1, c_2] \times [d_1, d_2] \subseteq A_6^d$,
$$\pi'([c_1, c_2] \times [d_1, d_2]) = \pi([c_1, c_2] \times [d_1, d_2]) - (a_3 - a_2) \cdot \frac{\pi([c_1, c_2] \times [d_1, d_2])}{a_6};$$
5. for any $[c_1, c_2] \times [d_1, d_2] \subseteq A_7$,
$$\pi'([c_1, c_2] \times [d_1, d_2]) = \pi([c_1, c_2] \times [d_1, d_2]) + (a_3 - a_2) \cdot \frac{\pi_x^3([c_1, c_2]) \cdot \pi_y^d([d_1, d_2])}{a_3 \cdot a_6}.$$

Analogously, one can construct a correlation structure π'' from π' by shifting measure from A_5 to A_4 and shifting the same measure from A_6^u to A_8 in a way that respects the marginals and does not change the expected revenue. Note that $\pi''(A_2) = \pi''(A_3) = \pi''(A_4) = \pi''(A_5) = \frac{a_2+a_3}{2}$. We can then adopt our approach in the first case. This completes the proof.