ABOUT LOCAL CONTINUITY WITH RESPECT TO L_2 INITIAL DATA FOR ENERGY SOLUTIONS OF THE NAVIER-STOKES EQUATIONS

TOBIAS BARKER

IN MEMORY OF MY STEPFATHER BRIAN RUDDLE (1950-2019)

Abstract In this paper we consider classes of initial data that ensure local-in-time Hadamard well-posedness of the associated weak Leray-Hopf solutions of the three-dimensional Navier-Stokes equations. In particular, for any solenodial L_2 initial data u_0 belonging to certain subsets of $VMO^{-1}(\mathbb{R}^3)$, we show that weak Leray-Hopf solutions depend continuously with respect to small divergence-free L_2 perturbations of the initial data u_0 (on some finite-time interval). Our main result is inspired and improves upon previous work of the author [4] and work of Jean-Yves Chemin [9]. Our method builds upon [4] and [9]. In particular our method hinges on decomposition results for the initial data inspired by Calderón [7] together with use of persistence of regularity results. The persistence of regularity statement presented may be of independent interest, since it does not rely upon the solution or the initial data being in the perturbative regime.

Keywords Navier-Stokes equations, Hadamard well-posedness, Fourier analysis, Littlewood-Paley theory, real interpolation, Besov spaces, persistence of regularity

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1. Introduction

At the beginning of the 20th century, Jacques Hadamard introduced a notion of well-posedness of partial differential equations. In particular, a evolutionary partial differential equation is said to be *Hadamard well-posed* if

(1) **(Existence)** A solution exists for all time.
(2) **(Uniqueness)** The solution is unique for all time.
(3) **(Continuous Dependence)** The solution depends continuously on the initial data.

The issue of Hadamard well-posedness depends not only on the equation under consideration, but also on the notion of 'solution' and the classes considered for the initial data.

For the Navier-Stokes equations, a popular notion of solution (with certain physical relevance) is that of *weak Leray-Hopf solutions*. In particular, for any $L_2(\mathbb{R}^3)$ divergence-free initial data we say that $u : \mathbb{R}^3 \times (0, \infty) \to \mathbb{R}^3$ is a weak Leray-Hopf solution associated to u_0 if

- $u \in C_w([0, \infty); J(\mathbb{R}^3)) \cap L^2(0, \infty; \dot{H}^1(\mathbb{R}^3))$[1].
- u solves the Navier-stokes equations in the distributional sense:

$$\partial_t u - \Delta u + u \cdot \nabla u + \nabla p = 0 \ \text{ in } \ \mathbb{R}^3 \times (0, \infty), \quad \operatorname{div} u = 0, \quad u(\cdot, 0) = u_0.$$

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[1] Throughout this paper $J(\mathbb{R}^3) := \{u_0 \in L_2(\mathbb{R}^3)\colon \operatorname{div} u_0 = 0\}$. $C_w([0, \infty); J(\mathbb{R}^3))$ denotes continuity in time with respect to the weak L_2 topology.

- u satisfies the energy inequality for all $t \geq 0$:
$$\|u(\cdot,t)\|^2_{L_2(\mathbb{R}^3)} + 2\int_0^t \int_{\mathbb{R}^3} |\nabla u(y,s)|^2 dyds \leq \|u_0\|^2_{L_2(\mathbb{R}^3)}.$$

For any $u_0 \in J(\mathbb{R}^3)$, global-in-time existence of an associated weak Leray-Hopf solution of the Navier-Stokes equations was established by Leray in [25] in 1934. Up to the present date, whether or not weak Leray-Hopf solutions are unique remains an outstanding open problem in mathematical fluid mechanics. Recently, sufficient conditions for nonuniqueness were provided in [20] and numerical evidence that these sufficient conditions hold was provided in [18].

Let us now give a definition that expresses the continuous dependence requirement for Hadamard well-posedness in the context of weak Leray-Hopf solutions.

Definition 1. Let $u_0 \in L_2(\mathbb{R}^3)$ be weakly divergence-free. We say that weak Leray-Hopf solutions are ‘***locally continuously dependent with respect to*** u_0’ if the following holds true.
There exists a finite positive T, $\varepsilon > 0$ and a continuous function Ψ with $\Psi(0) = 0$ such that if

- $v_0 \in B_{L_2}(u_0, \varepsilon) := \{w_0 \in J(\mathbb{R}^3) : \|w_0 - u_0\|_{L_2} < \varepsilon\}$
- $v(\cdot, v_0)$ and $u(\cdot, u_0)$ are global-in-time weak Leray-Hopf solutions associated to u_0 and v_0

then for all $t \in (0, T]$ one has the estimate
$$\|v(\cdot,t) - u(\cdot,t)\|^2_{L_2(\mathbb{R}^3)} + 2\int_0^t \int_{\mathbb{R}^3} |\nabla(v-u)|^2 dyds \leq \Psi(\|v_0 - u_0\|_{L_2(\mathbb{R}^3)}). \tag{1}$$

The above definition can be readily modified to give a notion of *global-in-time* Hadamard well-posedness for energy solutions of the Navier-Stokes equations. In this paper, we do not address the global-in-time case. We refer the reader to [26] for an interesting discussion relating to potential barriers for extending local-in-time Hadamard well-posedness to corresponding global-in-time versions. In [26], this potential barrier is referred to as *‘the real butterfly effect’*.

Whilst Definition 1 expresses the notion of Hadamard’s continuous dependence condition in the context of weak Leray-Hopf solutions, it also has ramifications for the regularity of solutions with initial data close to those which generate smooth solutions. In particular, suppose that

a) Weak Leray-Hopf solutions are locally continuously dependent with respect to u_0.
b) The weak Leray-Hopf solution $u(\cdot, u_0)$ (unique on $(0,T)$) belongs to $C^\infty(\mathbb{R}^3 \times (0,T])$.

Then a)-b) imply that for any compact set K contained in $\mathbb{R}^3 \times (0,T)$, there exists a positive $\varepsilon(K, \Psi)$ such that
$$\|v_0 - u_0\|_{L_2(\mathbb{R}^3)} < \varepsilon(K,\Psi) \Rightarrow \quad \text{any suitable[2] weak Leray-Hopf solution } v(\cdot, v_0) \in L^\infty_{x,t}(K).$$

[2] We say that a weak Leray-Hopf solution (v, q) is suitable on $\mathbb{R}^3 \times (0,T)$ if for every non-negative $\varphi \in C_0^\infty(\mathbb{R}^3 \times (0,T))$ v satisfies the local energy inequality
$$2\int_0^T \int_{\mathbb{R}^3} |\nabla v|^2 \varphi dxdt \leq \int_0^T \int_{\mathbb{R}^3} [|v|^2(\partial_t \varphi + \Delta\varphi) + (|v|^2 + 2q)v \cdot \nabla\varphi] dxdt.$$

Such an statements follow immediately from a contradiction argument and the '*persistence of singularities'* in [27].
In this paper, we are concerned with the following natural question:

> **(Q) Which $\mathcal{Z}\subset \mathcal{S}^{'}(\mathbb{R}^3)$ are such that $u_0\in J(\mathbb{R}^3)\cap\mathcal{Z}$ implies that weak Leray-Hopf solutions are locally continuously dependent with respect to u_0?**

From Definition 1, we see that positive answers to **(Q)** provide classes of initial data for which weak Leray-Hopf solutions are Hadamard well-posed locally in time.

In [4], the author provided the current widest[3]class of initial data for which the associated weak Leray-Hopf solutions are unique on some time interval. In particular the following Theorem was proven in [4].

Theorem 1.3 [4] . Suppose that there exists $q>3$ and $s\in(-1+\frac{2}{q},0)$ such that

$$u_0\in J(\mathbb{R}^3)\cap VMO^{-1}(\mathbb{R}^3)\cap \dot{B}^{s}_{q,q}(\mathbb{R}^3). \tag{2}$$

Then, there exists a $\hat{T}(u_0)>0$ such that all weak Leray-Hopf solutions on Q_∞, with initial data u_0, coincide on $Q_{\hat{T}(u_0)}:=\mathbb{R}^3\times(0,\hat{T}(u_0))$.

The main result of this paper, which we state below, shows that for such classes of initial data, weak Leray-Hopf solutions are Hadamard well-posed locally in time.

Theorem 1. *Suppose that there exists $q>3$ and $s\in(-1+\frac{2}{q},0)$ such that*

$$u_0\in J(\mathbb{R}^3)\cap VMO^{-1}(\mathbb{R}^3)\cap \dot{B}^{s}_{q,q}. \tag{3}$$

Let $\hat{T}(u_0)$ be as in the above Theorem and let u be the unique Leray-Hopf solution associated with u_0. Then for any positive $\eta\in(0,1)$ there exists $T(\eta,u_0,s,q)\in(0,\hat{T}]$ and $C(\eta,u_0,s,q)>0$ such that the following holds. For any weak Leray solution v associated with v_0 with

$$\|v_0-u_0\|_{L^2(\mathbb{R}^3)}<1, \tag{4}$$

we have that for all $t\in[0,T]$

$$\|v(t)-u(t)\|^2_{L_2}+\int\limits_0^t\|\nabla(v-u)(t')\|^2_{L_2}dt'\leq C\|v_0-u_0\|^{2-2\eta}_{L^2}. \tag{5}$$

1.1. **Comparison with previous literature.** The classical approach to determining $\mathcal{Z}$ such that (**Q**) holds true dates back to Leray in [25] (we refer to this as 'Leray's approach'). Let us now describe this in more detail.

Let $v(\cdot,v_0)$ and $u(\cdot,u_0)$ be two weak Leray-Hopf solutions with $u_0\in\mathcal{Z}\cap J(\mathbb{R}^3)$, $v_0\in J(\mathbb{R}^3)$[4] and $w\equiv u-v$. In Leray's approach, one requires the existence of $u(\cdot,u_0)$ in path spaces $\mathcal{X}_T$ possessing certain properties. In particular, b, $c\in C_w([0,T];J(\mathbb{R}^3))\cap L^2([0,T];\dot{H}^1(\mathbb{R}^3))$ and $a\in\mathcal{X}_T\Rightarrow$

$$F(a,b,c,t):=\int\limits_0^t\int\limits_{\mathbb{R}^3}(a\otimes b):\nabla c dx dt'<\infty$$

[3]For the subclass of weak Leray-Hopf solutions called '*local Leray solutions*', Lemarie Rieusset built upon ideas in [4] to show in [24] that short-time uniqueness holds for a wider class of initial data than those considered in [4].

[4]Throughout this paper $J(\mathbb{R}^3):=\{u_0\in L_2(\mathbb{R}^3)$: div $u_0=0\}$.

for $t\in[0,T]$ and F satisfies certain continuity estimates (see, for example, [12]). Once $\mathcal{X}_T$ satisfies this requirement, the approach in [25] gives a positive answer to **(Q)** by applying Gronwall's lemma to the energy inequality

$$\|w(\cdot,t)\|_{L^2_x}^2+2\int\limits_0^t\int\limits_{\mathbb{R}^3}|\nabla w|^2dxdt'\leq\|u_0-v_0\|_{L^2_x}^2+2\int\limits_0^t\int\limits_{\mathbb{R}^3}u\otimes w:\nabla wdxdt'. \tag{6}$$

Leray's approach was first used in [25] to show that for $\mathcal{Z}=H^1(\mathbb{R}^3)$ and $\mathcal{Z}=L_p(\mathbb{R}^3)$ $(3<p\leq\infty)$ we have local-in-time Hadamard well-posedness of 'turbulent solutions' (which are a subclass of Leray-Hopf solutions). Leray's approach has been applied to many other cases and we only attempt to list the cases most relevant to this paper. At the start of the 21^{st} century, [13] utilized Littlewood-Paley theory and Leray's approach to provide a positive answer for question **(Q)** for the homogeneous Besov spaces

$$\mathcal{Z}=\dot{B}^{-1+\frac{3}{p}}_{p,q}(\mathbb{R}^3)$$

with $2\leq p<\infty, 2<q<\infty$ and

$$\frac{3}{p}+\frac{2}{q}\geq 1.$$

Certain further extensions were provided in [12].

For the wider yet class[5] $\mathcal{Z}=\dot{\mathbb{B}}^{-1+\frac{3}{p}}_{p,\infty}$ $(p\in(3,\infty))$, arguments in [8][6] imply that there exists a $T(u_0)$ and a weak Leray-Hopf solution $u(\cdot,u_0)$ that is infinitely smooth on $\mathbb{R}^3\times(0,T(u_0))$. However, for this case the main difficulty is that it is unknown if Leray's approach is applicable. Specifically, when u_0 belongs to the above class and w belongs to the energy space (without assuming w solves an equation) it is not known that this trilinear term

$$\int\limits_0^T\int\limits_{\mathbb{R}^3}u(\cdot,u_0)\otimes w:\nabla wdxdt'$$

in (6) is even convergent.

These difficulties were tackled by Jean-Yves Chemin in [9], which provided a positive answer to **(Q)** for $\mathcal{Z}=\dot{H}^\alpha(\mathbb{R}^3)\cap\dot{\mathbb{B}}^{-1+\frac{3}{p}}_{p,\infty}$ $(\alpha>0$ and $p\in(3,\infty))$ by means of the following theorem.

Theorem 2. *Suppose that there exists $\alpha>0$ and $p\in(3,\infty)$ such that*

$$u_0\in J(\mathbb{R}^3)\cap\dot{H}^\alpha(\mathbb{R}^3)\cap\dot{\mathbb{B}}^{-1+\frac{3}{p}}_{p,\infty}. \tag{7}$$

Furthermore, let T be such that the strong solution u associated with u_0 is defined on $\mathbb{R}^3\times(0,T)$. Then for any positive η, a constant C exists such that, for any weak Leray solution v associated with v_0, we have that if $\|v_0-u_0\|_{L_2}$ is small enough that

$$\frac{1}{2}\|v(t)-u(t)\|_{L_2}^2+\int\limits_0^t\|\nabla(v-u)(t')\|_{L_2}^2dt'\leq C\|v_0-u_0\|_{L^2}^{2-2\eta}. \tag{8}$$

[5]We denote $\dot{B}^{-1+\frac{3}{p}}_{p,\infty}$ to be the homogeneous Besov space and $\dot{\mathbb{B}}^{-1+\frac{3}{p}}_{p,\infty}$ to be the subspace of closure of Schwartz functions.

[6]For an exposition of these arguments, we also refer to [4].

The heuristic idea of Jean-Yves Chemin is to split the strong solution u into a low frequency part[7] $S_j u$ and a high frequency part. Then $w_j := v - S_j u$ satisfies the equation

$$\partial_t w_j - \Delta w_j + S_j u \cdot \nabla w_j + w_j \cdot \nabla S_j u + w_j \cdot \nabla w_j = -\nabla p_j - \nabla \cdot F_j \tag{9}$$

$$\mathrm{div}\, w_j = 0, \quad w_j(\cdot, 0) = v_0 - S_j u_0 \tag{10}$$

$$R_j := \nabla \cdot F_j, \;\; F_j = S_j u \otimes S_j u - S_j(u \otimes u). \tag{11}$$

A major part of Chemin's work involves paraproduct type analysis in frequency space to estimate the Reynolds stress R_j of the strong solution u with initial data $\dot{\mathbb{B}}^{-1+\frac{3}{p}}_{p,\infty} \cap L_2$. He gets

$$\|F_j\|_{L_2(0,T;L_2(\mathbb{R}^3))} \leq C_{u,T} 2^{-\frac{j}{p-2}}. \tag{12}$$

When one applies Gronwall's lemma (which can be done since $S_j u$ belongs to subcritical spaces[8]) one gets

(13)
$$\|w_j(t)\|^2_{L_2} + \int\limits_0^t \|\nabla w_j(t')\|^2_{L_2} dt' \leq C\|v_0\|^2_{L_2} + C\|u_0 - S_j(u_0)\|^2_{L_2} + \int\limits_0^t \int\limits_{\mathbb{R}^3} S_j(u) \otimes w_j : \nabla w_j dx dt'.$$

He then uses that u 'just misses' by a logarithm belonging to a 'good' critical[9] space (that allows the trilinear term in (6) to be estimated in a way that allows Gronwall to be performed). In particular, $S_j(u)$ belongs to such 'good' spaces but has a corresponding norm which grows like εj (for arbitrary $\varepsilon > 0$) as the frequency parameter j grows. Once the Gronwall argument is performed this produces

$$\|w_j(t)\|^2_{L_2} \leq C(\|v_0\|^2_{L_2} + \|u_0 - S_j(u_0)\|^2_{L_2} + \|F_j\|^2_{L_2(0,T;L_2)}) \exp(\varepsilon j).$$

The fact that $u_0 \in \dot{H}^\alpha$ gives an exponential decay as j grows of $\|u_0 - S_j(u_0)\|^2_{L_2}$. This, in conjunction with the exponential decay for the Reynold's stress (12), crucially offsets the small exponential growth coming from Gronwall's lemma. With a bit more work, this allows Chemin to conclude the proof of Theorem 2.

For the case $u_0 = v_0$ the author has shown weak-strong uniqueness for wider yet classes of initial data than those considered by Chemin in [9]. Specifically, $u_0 \in J(\mathbb{R}^3) \cap VMO^{-1} \cap \dot{B}^s_{q,q}$ with $s \in (-1 + \frac{2}{q}, 0)$ and $q > 3$ (see also Lemarie-Rieusset [24] for recent extensions for the class of 'local Leray solutions'). For such classes of initial data, the author showed that

$$\|w(\cdot, t)\|^2_{L_2} \leq C t^\beta. \tag{14}$$

Such a decay depletes the singularity near the initial time for the trilinear term in (6) due to u having rough initial data. This allows us to infer that

$$\|w(\cdot, t)\|^2_{L^2_x} \leq C \int\limits_0^t \frac{(\sup_{0<s<T} s^{\frac{1}{2}} \|u(\cdot, s)\|_{L^\infty_x})^2}{s} \|w(\cdot, s)\|^2_{L^2_x} ds.$$

[7] For $j \geq 0$, the Fourier transform of $S_j u$ is compactly supported in $B(0, \frac{4}{3}2^j)$.

[8] We say $\mathcal{X} \subset \mathcal{S}'(\mathbb{R}^3 \times \mathbb{R})$ is a subcritical space for the Navier-Stokes equations if there exists a $\alpha > 0$ such that $\|u_\lambda\|_{\mathcal{X}} = \lambda^\alpha \|u\|_{\mathcal{X}}$ for any $\lambda > 0$. Here $u_\lambda(x,t) := \lambda u(\lambda x, \lambda^2 t)$.

[9] We say $\mathcal{Y} \subset \mathcal{S}'(\mathbb{R}^3 \times \mathbb{R})$ is a critical space for the Navier-Stokes equations if $\|u\|_{\mathcal{Y}} = \|u_\lambda\|_{\mathcal{Y}}$ for any $\lambda > 0$.

Then the conclusion of weak-strong uniqueness in reached in [4] by a comparison of the quantity[10]

$$\frac{\|w(\cdot,t)\|_{L_2}^2}{t^\beta}. \tag{15}$$

Unfortunately, such a strategy cannot prove Theorem 1, since w in that case isn't zero as $t \downarrow 0$. Hence w has no decay in time to deplete the singularity in time of u that occurs when estimating the trilinear term (6). Hence despite weak-strong uniqueness being known for such initial data, the stronger result of L_2 stability remained open. In this paper we settle this case by means of Theorem 1.

1.2. **Novelty of our results.** The proof of Theorem 1 requires involves two observations that differ from from Chemin's proof of Theorem 2

The first observation is somewhat similar to the author's work on weak-strong uniqueness [4], which was in turn inspired by the work of Calderón [7] (see also splitting arguments contained in [19] and [5]). The difference compared to the author's work on weak-strong uniqueness is that the space $\dot{H}^{\hat{\alpha}}$ was not used for the splittings, since L_2 instead played a prominent role. However, here this extra information must be kept to get the L_2 stability. Specifically if $u_0 \in J(\mathbb{R}^3) \cap VMO^{-1}(\mathbb{R}^3) \cap \dot{B}^s_{q,q}$ with $s \in (-1+\frac{2}{q}, 0)$ and $q \in (3,\infty)$, then we show that $u_0 = u_0^1 + u_0^2$ with

$$u_0^2 \in \dot{B}^{-1+\frac{3}{p}+\delta}_{p,p}(\mathbb{R}^3) \cap J(\mathbb{R}^3) \quad \text{and} \quad u_0^1 \in \dot{H}^{\hat{\alpha}} \cap VMO^{-1}(\mathbb{R}^3) \cap J(\mathbb{R}^3).$$

We then reduce to considering L_2 stability of energy solutions of the *perturbed* Navier-Stokes equations.

$$\partial_t U - \Delta U + U\cdot\nabla U + e^{t\Delta}u_0^2\cdot\nabla U + U\cdot\nabla e^{t\Delta}u_0^2 + \nabla P = -e^{t\Delta}u_0^2\cdot\nabla e^{t\Delta}u_0^2 \tag{16}$$

$$\operatorname{div} U = 0, \quad U(\cdot,0) = u_0^1 \tag{17}$$

In particular, the main goal reduces to showing an analogy of Theorem 2 but for this perturbed Navier-Stokes system and with initial data u_0^1.

Recall that in Chemin's proof the exponential decay of the Reynold's stress (11) is crucial to offset the exponential growth in frequency parameter coming from estimates of the low frequency part $S_j u$ used for the application of Gronwall's lemma. However, notice that the estimate of the Reynold's stress (12) does not possess any decay in j as p tends to infinity. Consequently, the main difficulty in proving Theorem 1 is that u_0^1 belongs to an L_∞ based critical space VMO^{-1}. In particular, the arguments in [9] seem to not give the required exponential decay for the Reynold's stress (11) for the spaces that u_0^1 belongs to.

The second observation and main new idea of this paper is to overcome this difficulty by using additional information about the strong solution $U(\cdot,u_0^1)$ which was not exploited in Chemin's paper. In particular, we use $u_0^1 \in \dot{H}^{\hat{\alpha}} \cap J(\mathbb{R}^3) \cap VMO^{-1}(\mathbb{R}^3)$ to show[11] (see Proposition 1) that there exists $\alpha(\hat{\alpha},\delta) \in (0,\hat{\alpha}]$ such that

$$U \in L_\infty(0,T;H^\alpha). \tag{18}$$

[10] Comparison of this quantity was previously exploited by Dong and Zhang to prove weak-strong uniqueness results in [11].

[11] We mention that persistency arguments proven in [17] applied to strong solutions to the perturbed Navier-Stokes equations would also suffice to show (18) with $\alpha = \hat{\alpha}$. By comparison the Proposition we show does not require initial data that generates the existence of a local-in-time strong solution (such as $u_0^1 \in VMO^{-1}$). Furthermore, the Proposition we give does not require that the solution U is in critical spaces.

This gives a decay of the $L^\infty_t L^2_x$ space-time norm involving the high frequencies of U, which gives that the associated Reynold's stress for the perturbed Navier-Stokes equations (16) has an exponential decay in j depending on α.

We finally mention that it is crucial that $e^{t\Delta}u_0^2$ belongs to subcritical spaces. In particular this means the extra terms in the perturbed Navier-Stokes equations do not destroy the arguments involving Gronwall's lemma.

1.3. **Points of Independent Interest and Further Remarks.**

1.3.1. *Partial Propagation of Regularity for the Perturbed Navier-Stokes Equations.* In Section 3.2, we prove the following propagation of regularity result.

Proposition 1. *Let $T>0$ be finite and $\delta\in(0,1)$. Suppose that V is divergence-free and*

$$V\in L^\infty_T L^2,\quad \sup_{0<t<T} t^{\frac12(1-\delta)}\|V(\cdot,t)\|_{L^\infty_x}<\infty. \tag{19}$$

Furthermore, suppose that there exists $\hat\alpha\in(0,1)$ such that

$$u_0^1\in J(\mathbb{R}^3)\cap \dot H^{\hat\alpha}(\mathbb{R}^3)\cap \dot B^{-1}_{\infty,\infty}(\mathbb{R}^3). \tag{20}$$

Assume that $U\in C_w([0,T];J(\mathbb{R}^3))\cap L^2((0,T);\dot H^1(\mathbb{R}^3))$ is a weak solution to the equation

$$\partial_t U-\Delta U+V\cdot\nabla U+U\cdot\nabla V+U\cdot\nabla U+V\cdot\nabla V+\nabla\Pi=0\ \text{ in }\ \mathbb{R}^3\times(0,T) \tag{21}$$

$$\operatorname{div} U=0,\quad U(\cdot,0)=u_0^1. \tag{22}$$

Furthermore, assume that U satisfies the energy inequality for $t\in[0,T]$:

$$\|U(\cdot,t)\|^2_{L^2(\mathbb{R}^3)}+2\int\limits_0^t\int\limits_{\mathbb{R}^3}|\nabla U(y,s)|^2dyds\le\|u_0^1\|^2_{L^2(\mathbb{R}^3)}+2\int\limits_0^t\int\limits_{\mathbb{R}^3}(V\otimes U+V\otimes V):\nabla U dyds. \tag{23}$$

In addition, assume that U satisfies

$$\sup_{0<t<T} t^{\frac12}\|U(\cdot,t)\|_{L^\infty(\mathbb{R}^3)}<\infty. \tag{24}$$

Then the above assumptions allow us to conclude that there exists $\alpha(\hat\alpha,\delta)\in(0,\hat\alpha]$ such that

$$U\in L^\infty(0,T;H^\alpha(\mathbb{R}^3)). \tag{25}$$

In [17] it is shown that when the initial data is in VMO^{-1}, the strong solution constructed by an iteration scheme propagates any additional regularity of the initial data on the homogeneous Besov scale. Furthermore, in [28] it is shown that when b is divergence-free and belongs to certain critical spaces that one can propagate the Hölder continuity of the initial data for the drift-diffusion equation with pressure

$$\partial_t u-\Delta u+b\cdot\nabla u+\nabla q=0,\ \ \operatorname{div} u=0,\ \ u(\cdot,0)=u_0^1. \tag{26}$$

Although the proof of Proposition 1 is concise and elementary, perhaps at first sight the statement seems somewhat unexpected. Indeed, it is not known if strong solutions can be constructed for u_0^1 satisfying (20). Furthermore, the result of Proposition 1 even holds true when the assumption (24) is replaced by certain supercritical[12] assumptions (see Remark 2). Such propagation results may be of independent interest and of use in other contexts.

[12] We say $\mathcal{X}\subset\mathcal{S}'(\mathbb{R}^3\times\mathbb{R})$ is a supercritical space for the Navier-Stokes equations if there exists a $\alpha>0$ such that $\|u_\lambda\|_{\mathcal{X}}=\lambda^{-\alpha}\|u\|_{\mathcal{X}}$ for any $\lambda>0$. Here $u_\lambda(x,t):=\lambda u(\lambda x,\lambda^2 t)$.

1.3.2. *Conjectures and remarks.* Arguments from [21] and the subsequent paper [23] show that when $u_0 \in J(\mathbb{R}^3) \cap VMO^{-1}(\mathbb{R}^3)$ there exists a $T(u_0)$ and a weak Leray-Hopf solution $u(\cdot, u_0)$ that is infinitely smooth on $\mathbb{R}^3 \times (0, T(u_0))$. Specifically, $u(\cdot, u_0)$ can be taken to belong to $\mathcal{E}_T$ defined below (29). We refer to such a u as a '*strong solution*', which is typically constructed by a Picard iteration scheme. If u_0 and v_0 both belong to $J(\mathbb{R}^3) \cap VMO^{-1}(\mathbb{R}^3)$, with $u(\cdot, u_0)$ and $v(\cdot, v_0)$ being associated strong solutions, then known arguments give the following. There exists a positive $T(u_0, v_0)$ such that for all $t \in (0, T(u_0, v_0))$ one has the estimate

$$\|v(\cdot,t)-u(\cdot,t)\|^2_{L_2(\mathbb{R}^3)}+2\int\limits_0^t\int\limits_{\mathbb{R}^3}|\nabla(v-u)|^2dyds \leq C\|u_0-v_0\|^2_{L^2(\mathbb{R}^3)}+C\|u_0-v_0\|_{L^2(\mathbb{R}^3)}\|u_0-v_0\|_{BMO^{-1}(\mathbb{R}^3)}.$$

Therefore, the class of strong solutions with initial data in $J(\mathbb{R}^3) \cap VMO^{-1}(\mathbb{R}^3)$ are Hadamard well-posed locally-in-time. Let us emphasize that in this paper we are concerned with Hadamard well-posedness issues for the wider class of weak Leray-Hopf solutions. In particular, the continuous dependence we examine is for $J(\mathbb{R}^3)$ perturbations of the initial data, as opposed to $J(\mathbb{R}^3) \cap VMO^{-1}(\mathbb{R}^3)$ perturbations.

The classes of initial data for which weak-strong uniqueness is proven in [4] (and for which local Hadamard well-posedness is proven by means of Theorem 1) *just miss* the case $u_0 \in J(\mathbb{R}^3) \cap VMO^{-1}(\mathbb{R}^3)$. In particular,

$$u_0 \in J(\mathbb{R}^3) \cap VMO^{-1}(\mathbb{R}^3) = J(\mathbb{R}^3) \cap VMO^{-1}(\mathbb{R}^3) \cap \dot{B}^s_{q,q}$$

with $s = -1 + \frac{2}{q}$ and $q \in (2, \infty)$, whereas Theorem 1.3 in [4] assumes $s \in (-1 + \frac{2}{q}, 0)$. Despite this, the following conjecture was made in [4]

> **(C)** If $u_0 \in J(\mathbb{R}^3) \cap VMO^{-1}(\mathbb{R}^3)$, the associated weak Leray-Hopf solutions coincide on some time interval.

Let us recap reasoning from [24] as to why such a conjecture seems plausible. For $u_0 \in J(\mathbb{R}^3)$, Leray proved existence of at least one global-in-time weak Leray-Hopf solution by first considering the mollified system

$$\partial_t u_\epsilon - \Delta u_\epsilon + (\varphi_\epsilon \star u_\epsilon) \cdot \nabla u_\epsilon + \nabla p_\epsilon = 0 \tag{27}$$

$$\operatorname{div} u_\epsilon = 0, \quad u_\epsilon(\cdot, 0) = u_0. \tag{28}$$

Here, $\varphi \in C_0^\infty(\mathbb{R}^3)$, $\int\limits_{\mathbb{R}^3} \varphi(x)dx = 1$ and $\varphi_\epsilon(x) := \frac{1}{\epsilon^3}\varphi(\frac{x}{\epsilon})$. Then Leray uses energy estimates and compactness arguments to obtain global-in-time weak Leray-Hopf solutions in the limit as $\varepsilon \downarrow 0$. In [24], solutions obtained in such a way are called '*restricted Leray solutions*'. Using arguments from [23] (see also [4] for an exposition of those arguments), one gets that for $u_0 \in VMO^{-1} \cap J(\mathbb{R}^3)$ there exists $\hat{T}(u_0)$ such that for all $T \in (0, \hat{T})$ the following holds true. Namely, if $u(\cdot, u_0)$ is a restricted Leray solution then

$$\|u\|_{\mathcal{E}_T} \leq 2\|e^{t\Delta}u_0\|_{\mathcal{E}_T}. \tag{29}$$

Here,

$$\|u\|_{\mathcal{E}_T} := \sup_{0<t<T} \sqrt{t}\|u(\cdot,t)\|_{L_\infty(\mathbb{R}^3)} +$$

$$+ \sup_{(x,t)\in\mathbb{R}^3\times(0,T)} \Big(\frac{1}{|B(0,\sqrt{t})|}\int\limits_0^t \int\limits_{|y-x|<\sqrt{t}} |u|^2dyds\Big)^{\frac{1}{2}} \tag{30}$$

and $e^{t\Delta}u_0$ represents the heat-flow acting on u_0. Then (29) implies that **(C)** holds true for restricted Leray-solutions[13] by means of the uniqueness of mild solutions constructed in [21].

In some sense the above reasoning justifies why conjecture **(C)** is plausible for weak Leray-Hopf solutions. However, it appears to be an open problem to even show local Hadamard well-posedness when $\mathcal{Z} = VMO^{-1}$, for restricted Leray solutions. In turn this makes corresponding conjectures surrounding **(Q)** when $\mathcal{Z} = VMO^{-1}$ seemingly more speculative than **(C)**.

2. Preliminaries

2.1. **General Notation.** Throughout this paper we adopt the Einstein summation convention. For arbitrary vectors $a = (a_i)$, $b = (b_i)$ in $\mathbb{R}^n$ and for arbitrary matrices $F = (F_{ij})$, $G = (G_{ij})$ in $\mathbb{M}^n$ we put

$$a\cdot b = a_i b_i,\ |a| = \sqrt{a\cdot a},$$
$$a\otimes b = (a_i b_j) \in \mathbb{M}^n,$$
$$FG = (F_{ik}G_{kj}) \in \mathbb{M}^n,\ F^T = (F_{ji}) \in \mathbb{M}^n,$$
$$F : G = F_{ij}G_{ij} \text{ and } |F| = \sqrt{F:F}.$$

Let $e^{t\Delta}u_0$ denote the heat kernel convoluted with u_0.

For $\lambda \in \mathbb{R}$, $\lfloor \lambda \rfloor$ denotes the greatest integer less than λ. Furthermore, $\lceil \lambda \rceil$ denotes the smallest integer greater than λ.

If X is a Banach space with norm $\|\cdot\|_X$, then $L_s(a,b;X)$, with $a<b$ and $s\in[1,\infty)$, will denote the usual Banach space of strongly measurable X-valued functions $f(t)$ on (a,b) such that

$$\|f\|_{L^s(a,b;X)} := \left(\int\limits_a^b \|f(t)\|_X^s dt\right)^{\frac{1}{s}} < +\infty.$$

The usual modification is made if $s=\infty$. Sometimes we will denote $L^p(0,T;L^q)$ by $L^p_T L^q$ or $L^p(0,T;L^p_x)$.

Let $C([a,b];X)$ denote the space of continuous X valued functions on $[a,b]$ with usual norm. In addition, let $C_w([a,b];X)$ denote the space of X valued functions, which are continuous from $[a,b]$ to the weak topology of X.

2.2. **Function spaces.** For a tempered distribution f, let

$$\mathcal{F}(f)(\xi) := \int\limits_{\mathbb{R}^3} \exp(-ix\cdot\xi) f(x)dx$$

denote its Fourier transform. Let $d,m\in\mathbb{N}\setminus\{0\}$. We begin by recalling the definition of the *homogeneous Besov spaces* $\dot{B}^s_{p,q}(\mathbb{R}^d;\mathbb{R}^m)$. There exists a non-negative radial function $\varphi\in C^\infty(\mathbb{R}^d)$ supported on the annulus $\{\xi\in\mathbb{R}^d : 3/4\le|\xi|\le 8/3\}$ and $\chi\in C_0^\infty(B(4/3))$ such that

$$\chi(\xi) + \sum_{j\ge 0}\varphi(2^{-j}\xi) = 1, \quad \xi\in\mathbb{R}^3, \tag{31}$$

$$\sum_{j\in\mathbb{Z}}\varphi(2^{-j}\xi) = 1, \quad \xi\in\mathbb{R}^3\setminus\{0\}. \tag{32}$$

[13]For analogous statements in bounded domains for initial data in Besov spaces, we refer to [14]-[16].

The high frequency cut off $\dot{S}_j$ and the homogeneous Littlewood-Paley projectors $\dot{\Delta}_j$ are defined by

$$\dot{\Delta}_j f = \varphi(2^{-j}D)f, \quad j \in \mathbb{Z}, \tag{33}$$

$$\dot{S}_j f = \chi(2^{-j}D)f, \quad j \in \mathbb{Z}, \tag{34}$$

for all tempered distributions f on $\mathbb{R}^d$ with values in $\mathbb{R}^m$. The notation $\varphi(2^{-j}D)f$ denotes convolution with the inverse Fourier transform of $\varphi(2^{-j}\cdot)$ with f. Notice that $\dot{S}_j = I - \sum_{k=j}^{\infty}\dot{\Delta}_k$. Furthermore, for tempered distributions such that $\sum_{k\in\mathbb{Z}}\dot{\Delta}_k f$ converges to f (in the sense of tempered distributions) we have that $\dot{S}_j f = \sum_{k=-\infty}^{k=j-1}\dot{\Delta}_k f$.

Let $p, q \in [1,\infty]$ and $s \in (-\infty, d/p)$.[14] The homogeneous Besov space $\dot{B}^s_{p,q}(\mathbb{R}^d;\mathbb{R}^m)$ consists of all tempered distributions f on $\mathbb{R}^d$ with values in $\mathbb{R}^m$ satisfying

$$\|f\|_{\dot{B}^s_{p,q}(\mathbb{R}^d;\mathbb{R}^m)} := \Big(\sum_{j\in\mathbb{Z}}\big(2^{js}\|\dot{\Delta}_j f\|_{L^p}\big)^q\Big)^{\frac{1}{q}}. \tag{35}$$

and such that $\sum_{j\in\mathbb{Z}}\dot{\Delta}_j f$ converges to f in the sense of tempered distributions on $\mathbb{R}^d$ with values in $\mathbb{R}^m$. In this range of indices, $\dot{B}^s_{p,q}(\mathbb{R}^d;\mathbb{R}^m)$ is a Banach space. When $s \geq 3/p$ and $q > 1$, the spaces must be considered *modulo polynomials*. Note that other reasonable choices of the function φ defining $\dot{\Delta}_j$ lead to equivalent norms.

It is known that if $1 \leq q_1 \leq q_2 \leq \infty$, $1 \leq p_1 \leq p_2 \leq \infty$ and $s \in \mathbb{R}$ then

$$\dot{B}^s_{p_1,q_1}(\mathbb{R}^3) \hookrightarrow \dot{B}^{s-3(\frac{1}{p_1}-\frac{1}{p_2})}_{p_2,q_2}(\mathbb{R}^3). \tag{36}$$

See Proposition 2.2 p.64 of [3], for example.

We now recall a particularly useful property of Besov spaces, i.e., their characterization in terms of the heat kernel. For all $s \in (-\infty, 0)$, there exists a constant $c := c(s) > 0$ such that for all tempered distributions f on $\mathbb{R}^3$,

$$c^{-1}\sup_{t>0} t^{-\frac{s}{2}}\|e^{t\Delta}f\|_{L^p(\mathbb{R}^3)} \leq \|f\|_{\dot{B}^s_{p,\infty}(\mathbb{R}^3)} \leq c\sup_{t>0} t^{-\frac{s}{2}}\|e^{t\Delta}f\|_{L^p(\mathbb{R}^3)}. \tag{37}$$

We will need the following Proposition, whose statement and proof can be found in the book [3] (Proposition 2.22 there). In the Proposition below we use the notation

$$\mathcal{S}'_h := \{\text{ tempered distributions } u \text{ such that } \lim_{j\to-\infty}\|S_j u\|_{L_\infty(\mathbb{R}^3)} = 0\}. \tag{38}$$

Proposition 2. *A constant C exists with the following properties. If s_1 and s_2 are real numbers such that $s_1 < s_2$ and $\theta \in (0,1)$, then we have, for any $p \in [1,\infty]$ and any $u \in \mathcal{S}'_h$,*

$$\|u\|_{\dot{B}^{\theta s_1+(1-\theta)s_2}_{p,1}(\mathbb{R}^3)} \leq \frac{C}{s_2-s_1}\Big(\frac{1}{\theta}+\frac{1}{1-\theta}\Big)\|u\|^{\theta}_{\dot{B}^{s_1}_{p,\infty}(\mathbb{R}^3)}\|u\|^{1-\theta}_{\dot{B}^{s_2}_{p,\infty}(\mathbb{R}^3)}. \tag{39}$$

Furthermore, we define the Chemin-Lerner norm[15]

$$\|u\|_{\tilde{L^r_T}(\dot{B}^s_{p,q})} := \|(2^{js}\|\dot{\Delta}_j u\|_{L^r(0,T;L^p)})\|_{l^q(\mathbb{Z})}. \tag{40}$$

The following useful Lemma was proven in [3] (Lemma 2.4 there). We state in below.

Lemma 1. *Let $\mathcal{C}$ be an annulus. Positive constants c and C exist such that for all $p \in [1,\infty]$ and any couple (t,λ) of positive real numbers, we have*

$$\textit{supp}\ \hat{u} \subset \lambda\mathcal{C} \Rightarrow \|e^{t\Delta}u\|_{L_p} \leq Ce^{-c\lambda^2 t}\|u\|_{L_p}.$$

[14] The choice $s = d/p$, $q = 1$ is also valid.

[15] This was introduced in [10] for the special case $s = \frac{d}{2}+1$, $r = 1$, $p = 2$ and $q = 2$.

Lemma 1 yields two useful estimates immediately. Namely, for $s\in\mathbb{R}$ and $p,q\in[1,\infty]^2$ we have

$$\|e^{t\Delta}u\|_{\tilde{L}^r_T(\dot{B}^{s+\frac{2}{r}}_{p,q})}\leq C(r,s,p,q)\|u\|_{\dot{B}^s_{p,q}}. \tag{41}$$

Second, by interpolation for homogeneous Sobolev spaces we have that for any $\alpha>0$

$$\|e^{t\Delta}u\|_{\dot{H}^\alpha}\leq\frac{C'(\alpha)\|u\|_{L_2}}{t^{\frac{\alpha}{2}}}. \tag{42}$$

We will also make use of the following Lemma contained in the book [3] (Corollary 2.54 there).

Lemma 2. *Suppose that $(s,p,r)\in(0,\infty)\times[1,\infty]^2$ with $s<\frac{3}{p}$. Then there exists a constant $C(s)$ such that*

$$\|uv\|_{\dot{B}^s_{p,q}(\mathbb{R}^3)}\leq\frac{C^{s+1}}{s}(\|v\|_{L_\infty(\mathbb{R}^3)}\|u\|_{\dot{B}^s_{p,q}(\mathbb{R}^3)}+\|u\|_{L_\infty(\mathbb{R}^3)}\|v\|_{\dot{B}^s_{p,q}(\mathbb{R}^3)}) \tag{43}$$

Finally, $BMO^{-1}(\mathbb{R}^3)$ is the space of all tempered distributions such that the following norm is finite:

$$\|u\|_{BMO^{-1}(\mathbb{R}^3)}:=\sup_{x\in\mathbb{R}^3,R>0}\frac{1}{|B(0,R)|}\int\limits_0^{R^2}\int\limits_{B(x,R)}|e^{t\Delta}u|^2dydt. \tag{44}$$

Note that $VMO^{-1}(\mathbb{R}^3)$ is the subspace that coincides with the closure of test functions $C^\infty_0(\mathbb{R}^3)$, with respect to the norm (44).

2.3. **Decompostions of Besov spaces.** Now, we can state a Lemma regarding decomposition of homogeneous Besov spaces taken from the authors paper [4] (Proposition 2.8 there). For more general decomposition results, we refer to [1].

Proposition 3. *For $i=1,2,3$ let $p_i\in(1,\infty)$, $s_i\in\mathbb{R}$ and $\theta\in(0,1)$ be such that $s_1<s_0<s_2$ and $p_2<p_0<p_1$. In addition, assume the following relations hold:*

$$s_1(1-\theta)+\theta s_2=s_0, \tag{45}$$

$$\frac{1-\theta}{p_1}+\frac{\theta}{p_2}=\frac{1}{p_0} \tag{46}$$

and

$$s_i<\frac{3}{p_i}. \tag{47}$$

Suppose that $u_0\in\dot{B}^{s_0}_{p_0,p_0}(\mathbb{R}^3)$. Then for all $\epsilon>0$, there exists $u^{1,\epsilon}\in\dot{B}^{s_1}_{p_1,p_1}(\mathbb{R}^3)$, $u^{2,\epsilon}\in\dot{B}^{s_2}_{p_2,p_2}(\mathbb{R}^3)$ such that

$$u=u^{1,\epsilon}+u^{2,\epsilon}, \tag{48}$$

$$\|u^{1,\epsilon}\|^{p_1}_{\dot{B}^{s_1}_{p_1,p_1}}\leq\epsilon^{p_1-p_0}\|u_0\|^{p_0}_{\dot{B}^{s_0}_{p_0,p_0}} \tag{49}$$

and

$$\|u^{2,\epsilon}\|^{p_2}_{\dot{B}^{s_2}_{p_2,p_2}}\leq C(s_1,s_2,p_0,p_1,p_2,\|\mathcal{F}^{-1}\varphi\|_{L_1})\epsilon^{p_2-p_0}\|u_0\|^{p_0}_{\dot{B}^{s_0}_{p_0,p_0}}. \tag{50}$$

The following corollary is essentially contained in the author's paper [4]. However, in that case we did not show that one piece of the initial data had better regularity than L_2. For completeness, we therefore provide further details.

Corollary 1. *Suppose that $q>3$,*

$$u_0\in \dot{B}^{s}_{q,q}(\mathbb{R}^3)\cap L_2(\mathbb{R}^3)\ \ \textit{with}\ \ s\in(-1+\frac{2}{q},0) \tag{51}$$

and div $u_0=0$ in the sense of distributions.
Then the above assumption imply that there exists $\max{(q,4)}<p<\infty$, $\delta\in(0,1-\frac{3}{p})$ and $\hat{\alpha}\in(0,\frac{3}{2})$ such that for any $\epsilon>0$ there exists weakly divergence-free functions $\bar{u}^{1,\epsilon}\in \dot{B}^{-1+\frac{3}{p}+\delta}_{p,p}(\mathbb{R}^3)\cap L_2(\mathbb{R}^3)$ and $\bar{u}^{2,\epsilon}\in \dot{H}^{\hat{\alpha}}(\mathbb{R}^2)\cap L_2(\mathbb{R}^3)$ such that

$$u_0=\bar{u}^{1,\epsilon}+\bar{u}^{2,\epsilon}, \tag{52}$$

$$\|\bar{u}^{1,\epsilon}\|^p_{\dot{B}^{-1+\frac{3}{p}+\delta}_{p,p}}\leq \epsilon^{p-q}\|u_0\|^q_{\dot{B}^{s}_{q,q}}, \tag{53}$$

$$\|\bar{u}^{2,\epsilon}\|^2_{\dot{H}^{\hat{\alpha}}(\mathbb{R}^3)}\leq C(s,\alpha,p,q,\|\mathcal{F}^{-1}\varphi\|_{L_1})\epsilon^{2-q}\|u_0\|^q_{\dot{B}^{s}_{q,q}} \tag{54}$$

and

$$\|\bar{u}^{2,\epsilon}\|_{L_2},\|\bar{u}^{1,\epsilon}\|_{L_2}\leq C(\|\mathcal{F}^{-1}\varphi\|_{L_1})\|u_0\|_{L_2}. \tag{55}$$

Proof. Take $p>\max(q,4)$. The assumption that $s\in(-1+\frac{2}{q},0)$ implies that there exists $\tilde{\varepsilon}\in(0,1-\frac{2}{q})$ such that

$$s:=-1+\frac{2}{q}+\tilde{\varepsilon}. \tag{56}$$

Then,

$$\frac{\frac{2}{p}-1}{\frac{2}{q}-1}s-(-1+\frac{3}{p})=-\frac{1}{p}+\frac{-\frac{2}{p}+1}{-\frac{2}{q}+1}\tilde{\varepsilon}.$$

So

$$\lim_{p\uparrow\infty}\Big(\frac{\frac{2}{p}-1}{\frac{2}{q}-1}s-(-1+\frac{3}{p})\Big)=\frac{1}{-\frac{2}{q}+1}\tilde{\varepsilon}>0.$$

Thus, there exists a p sufficiently large and a $\hat{\delta}>0$ such that

$$\frac{1-\frac{2}{p}}{1-\frac{2}{q}}s=-1+\frac{3}{p}+\hat{\delta}. \tag{57}$$

Referring to the previous proposition, let $p_0=q$, $p_1=p$ and $p_2=2$ and let θ be such that

$$\frac{1-\theta}{p}+\frac{\theta}{2}=\frac{1}{q}.$$

Thus

$$1-\theta=\frac{1-\frac{2}{q}}{1-\frac{2}{p}}.$$

Thus, (57) implies

$$s=(1-\theta)\Big(-1+\frac{3}{p}+\hat{\delta}\Big)=(1-\theta)\Big(-1+\frac{3}{p}+\hat{\delta}-\frac{\theta\hat{\alpha}}{1-\theta}\Big)+\theta\hat{\alpha}. \tag{58}$$

Choose

$$\hat{\alpha}\in\Big(0,\min\Big(\frac{3}{2},\frac{\hat{\delta}(1-\theta)}{\theta}\Big)\Big) \tag{59}$$

Now, define $\delta := \hat{\delta} - \frac{\theta\hat{\alpha}}{1-\theta} > 0$, $s_1 := -1 + \frac{3}{p} + \delta$, $s_0 = s$ and $s_2 := \hat{\alpha}$. The above relations allow us to apply Proposition 3 to obtain the following decomposition: (we note that for $\hat{\alpha} < \frac{3}{2}$, $\dot{B}^{\hat{\alpha}}_{2,2}(\mathbb{R}^3)$ coincides with $\dot{H}^{\hat{\alpha}}(\mathbb{R}^3)$ with equivalent norms)

$$u_0 = u^{1,\epsilon} + u^{2,\epsilon}, \tag{60}$$

$$\|u^{1,\epsilon}\|^p_{\dot{B}^{-1+\frac{3}{p}+\delta}_{p,p}} \leq \epsilon^{p-q}\|u_0\|^q_{\dot{B}^s_{q,q}} \tag{61}$$

and

$$\|u^{2,\epsilon}\|^2_{\dot{H}^{\hat{\alpha}}} \leq C(\alpha, p, q, \|\mathcal{F}^{-1}\varphi\|_{L_1})\epsilon^{2-q}\|u_0\|^q_{\dot{B}^s_{q,q}}. \tag{62}$$

For $j \in \mathbb{Z}$ and $m \in \mathbb{Z}$, it can be seen that

$$\|\dot{\Delta}_m\left((\dot{\Delta}_j u_0)\chi_{|\dot{\Delta}_j u_0| \geq N(j,\epsilon)}\right)\|_{L_2},\ \|\dot{\Delta}_m\left((\dot{\Delta}_j u_0)\chi_{|\dot{\Delta}_j u_0| \leq N(j,\epsilon)}\right)\|_{L_2} \leq C(\|\mathcal{F}^{-1}\varphi\|_{L_1}))\|\dot{\Delta}_j u_0\|_{L_2}. \tag{63}$$

It is known that $u_0 \in L_2$ implies

$$\|u_0\|^2_{L_2} = \sum_{j\in\mathbb{Z}} \|\dot{\Delta}_j u_0\|^2_{L_2}.$$

Using this, (63) and the expression of $u^{1,\epsilon}$ given by Proposition 2.8 of the author's paper [4] , we can infer that

$$\|u^{2,\epsilon}\|_{L_2},\ \|u^{1,\epsilon}\|_{L_2} \leq C(\|\mathcal{F}^{-1}\varphi\|_{L_1})\|u_0\|_{L_2}.$$

The Leray projector $\mathbb{P}$, which projects onto divergence free vector fields, is defined as

$$\mathbb{P}f := f + \nabla(-\Delta)^{-1}(\text{div}\, f).$$

To establish the decomposition of the Corollary, we apply the Leray projector to each of $u^{1,\epsilon}$ and $u^{2,\epsilon}$, which is a continuous linear operator on the homogeneous Besov spaces under consideration.

□

3. Properties of strong solutions of the Navier-Stokes equations

In this section, we first describe certain properties of the strong solution $u(\cdot, u_0)$ with initial data $u_0 \in VMO^{-1}(\mathbb{R}^3) \cap J(\mathbb{R}^3) \cap \dot{B}^s_{q,q}(\mathbb{R}^3)$ ($s \in (-1+\frac{2}{q}, 0)$ and $q > 3$) that will be needed to prove Theorem 1. We must mention that the first subsection is mostly a collection of results already contained in the literature, gathered for the reader's convenience. Where the context is slightly different to the previous literature, or when fixes are needed, we provide the reader with details.

In the second subsection we prove Proposition 1. This will imply that some of the Sobolev regularity of u_0^1 persists for $U(x,t) := u - e^{t\Delta}u_0^2$. This will be a crucial ingredient in proving Theorem 1.

3.1. **Regularity properties of strong solutions.**

First, we discuss the construction and regularity of the strong solution u described in Theorem 1. For initial data in $VMO^{-1}(\mathbb{R}^3)$, local-in-time strong solutions to the Navier-Stokes equations were shown to exist by Koch and Tataru in [21]. Such solutions belong to the pathspace $\mathcal{P}_T$ where

$$\mathcal{P}_T := \{u \in \mathcal{S}'(\mathbb{R}^3 \times (0,T)) : \|u\|_{\mathcal{E}_T} < \infty\}. \tag{64}$$

Here,

$$\|u\|_{\mathcal{E}_T} := \sup_{0<t<T} \sqrt{t}\|u(\cdot,t)\|_{L_\infty(\mathbb{R}^3)} +$$

$$+\sup_{(x,t)\in\mathbb{R}^3\times(0,T)}\Big(\frac{1}{|B(0,\sqrt{t})|}\int\limits_0^t\int\limits_{|y-x|<\sqrt{t}}|u|^2dyds\Big)^{\frac{1}{2}}. \tag{65}$$

Furthermore, the Koch-Tataru solutions satisfy the integral formulation of the Navier-Stokes equations

$$u(x,t):=e^{t\Delta}u_0+\int\limits_0^t e^{(t-s)\Delta}\mathbb{P}\nabla\cdot(u\otimes u)ds. \tag{66}$$

Here, $e^{t\Delta}$ denotes the heat semigroup in $\mathbb{R}^3$ and $\mathbb{P}$ denotes the projection of vector fields onto divergence-free vector fields. Throughout this paper, we denote the bilinear term by

$$B(f,g):=\int\limits_0^t e^{(t-s)\Delta}\mathbb{P}\nabla\cdot(f\otimes g)ds. \tag{67}$$

From (44), we see that for $0<T\leq\infty$

$$u_0\in BMO^{-1}(\mathbb{R}^3)\Rightarrow\|S(t)u_0\|_{\mathcal{E}_T}\leq C\|u_0\|_{BMO^{-1}}. \tag{68}$$

Since $C_0^\infty(\mathbb{R}^3)$ is dense in $VMO^{-1}(\mathbb{R}^3)$, we can see from the above that for $u_0\in VMO^{-1}(\mathbb{R}^3)$

$$\lim_{T\to 0^+}\|S(t)u_0\|_{\mathcal{E}_T}=0. \tag{69}$$

It was shown in [21] that there exists a universal constant C such that for all $f,\ g\in\mathcal{E}_T$

$$\|B(f,g)\|_{\mathcal{E}_T}\leq C\|f\|_{\mathcal{E}_T}\|g\|_{\mathcal{E}_T}. \tag{70}$$

Here is the needed proposition related to the construction of the 'strong solution'. The statement and references can also be found in [4].

Proposition 4. *Suppose that $u_0\in VMO^{-1}(\mathbb{R}^3)\cap J(\mathbb{R}^3)$. There exists a universal constant $\varepsilon_0>0$ such that if*

$$\|S(t)u_0\|_{\mathcal{E}_T}<\varepsilon_0, \tag{71}$$

then there exists a $u\in\mathcal{E}_T$, which solves the Navier-Stokes equations in the sense of distributions and satisfies the following properties. The first property is that u solves the following integral equation:

$$u(x,t):=S(t)u_0+B(u,u)(x,t) \tag{72}$$

in $\mathbb{R}^3\times(0,T)$, along with the estimate

$$\|u\|_{\mathcal{E}_T}<2\|S(t)u_0\|_{\mathcal{E}(T)}. \tag{73}$$

The second property is that u is a weak Leray-Hopf solution on $\mathbb{R}^3\times(0,T)$.

If $\pi_{u\otimes u}$ is the associated pressure we have (here, $\lambda\in(0,T)$ and $p\in(2,\infty)$):

$$\pi_{u\otimes u}\in L^{\frac{5}{3}}(\mathbb{R}^3\times(0,T))\cap L^\infty(\lambda,T;L^{\frac{p}{2}}(\mathbb{R}^3)). \tag{74}$$

Furthermore for $\lambda\in(0,T)$ and $k=0,1\ldots$, $l=0,1\ldots$:

$$\sup_{(x,t)\in\mathbb{R}^3\times(\lambda,T)}|\partial_t^l\nabla^k u|+|\partial_t^l\nabla^k\pi_{u\otimes u}|\leq c(p_0,\lambda,\|u_0\|_{BMO^{-1}},\|u_0\|_{L_2},k,l). \tag{75}$$

Finally, when ε_0 is sufficiently small there exists constants $C(k)$ such that for $k=0,1,\ldots$

$$\sup_{0<t<T}(\sqrt{t})^{k+1}\|\nabla^k u(\cdot,t)\|_{L_\infty(\mathbb{R}^3)}\leq C(k). \tag{76}$$

Proof. Conclusions (72) and (73) are due to [21]. The fact that u is a weak-Leray Hopf solution follows from ideas in [23] (see also the appendix of [4]). The proof of (74)-(75) are also described in [4].

Let us focus on proving (76), which will be required to prove Theorem 1. First, we recall the known fact that $\pi_{u\otimes u}$ is a composition of Riesz transforms acting on $u\otimes u$. Thus, using (73) and the Calderoón-Zygmund theory we get that for $r\in(0,\sqrt{t})$

$$\|\pi_{u\otimes u}\|_{L^{\infty}(t-r^2,t;BMO(\mathbb{R}^3))}\leq C\|u\|^2_{L^{\infty}(\mathbb{R}^3\times(t-r^2,t))}\leq\frac{C'\varepsilon_0^2}{t-r^2}. \tag{77}$$

Using this, we obtain that for all $x\in\mathbb{R}^3$

$$\frac{1}{r^2}\int\limits_{t-r^2}^{t}\int\limits_{B(x,r)}|u|^3+|\pi_{u\otimes u}-(\pi_{u\otimes u})_{B(x,r)}|^{\frac32}dxdt'\leq\frac{C''r^3\varepsilon_0^3}{(t-r^2)^{\frac32}}. \tag{78}$$

Here, C'' is a universal constant. Taking $r:=\frac{\sqrt{t}}{2}$ we get

$$\frac{4}{t}\int\limits_{t-\frac{t}{4}}^{t}\int\limits_{B(x,\frac{\sqrt{t}}{2})}|u|^3+|\pi_{u\otimes u}-(\pi_{u\otimes u})_{B(x,r)}|^{\frac32}dxdt'\leq C'''\varepsilon_0^3. \tag{79}$$

If $C'''\varepsilon_0^3\leq\varepsilon_{CKN}$, we can apply the Caffarelli-Kohn-Nirenberg theory [6] to immediately infer (76). □

Next will discuss some further regularity properties of Koch and Tataru's strong solution that will be needed to prove Theorem 1. First we begin with a lemma taken from [29] (Lemma 2.1 there).

Lemma 3. *Let $q\in\mathbb{Z}$, $(x,t)\in\mathbb{R}^3\times(0,\infty)$, $i,j,n\in\{1,2,3\}$ and φ be as in the definition of the Littlewood Paley projectors*[16]*. Define*

$$g_q^{i,j,n}(x,t):=\frac{1}{8\pi^3}\int\limits_{\mathbb{R}^3}e^{ix\cdot\xi}\varphi(2^{-q}\xi)e^{-t|\xi|^2}\Big(\delta_{ij}-\frac{\xi_i\xi_j}{|\xi|^2}\Big)\xi_n d\xi \tag{80}$$

$$g_{1,q}^{i,j}(x,t):=\frac{1}{8\pi^3}\int\limits_{\mathbb{R}^3}e^{ix\cdot\xi}\varphi(2^{-q}\xi)e^{-t|\xi|^2}\Big(\delta_{ij}-\frac{\xi_i\xi_j}{|\xi|^2}\Big)d\xi \tag{81}$$

$$g_{2,q}(x,t):=\frac{1}{8\pi^3}\int\limits_{\mathbb{R}^3}e^{ix\cdot\xi}\varphi(2^{-q}\xi)e^{-t|\xi|^2}d\xi. \tag{82}$$

Then, we have the estimates

$$|g_q^{i,j,n}(x,t)|\leq\frac{C_{univ}2^{4q}e^{-ct2^{2q}}}{1+(2^q|x|)^6} \tag{83}$$

and

$$|g_{1,q}^{i,j}(x,t)|+|g_{2,q}(x,t)|\leq\frac{C_{univ}2^{3q}e^{-ct2^{2q}}}{1+(2^q|x|)^6} \tag{84}$$

Now, we state the main further regularity properties of the solutions in Proposition 4.

[16] See section 2.2 'Function Spaces'

Proposition 5. *Let u be the strong solution as in Proposition 4, with initial data $u_0 \in VMO^{-1}(\mathbb{R}^3)\cap J(\mathbb{R}^3)$. Then one also has that*

$$\|u\|_{L^{\infty}(0,T;\dot{B}^{-1}_{\infty,\infty})} \leq C_{univ}(\|u_0\|_{BMO^{-1}}+\|u\|^2_{\mathcal{E}_T}), \tag{85}$$

$$\|u\|_{\tilde{L}^1(0,T;\dot{B}^1_{\infty,\infty})} \leq C_{univ}(\|u\|_{\mathcal{E}_T}+\|e^{t\Delta}u_0\|_{\mathcal{E}_T}+\|u\|_{\mathcal{E}_T}\sup_{0<s<T}s\|\nabla u(\cdot,s)\|_{L_\infty})+\|e^{t\Delta}u_0\|_{\tilde{L}^1(0,T;\dot{B}^1_{\infty,\infty})}$$
$$\leq C'_{univ}(\|e^{t\Delta}u_0\|_{\mathcal{E}_T}+\|e^{t\Delta}u_0\|_{\tilde{L}^1(0,T;\dot{B}^1_{\infty,\infty})}). \tag{86}$$

Proof. The proof of (85) is proven in [29] (Proposition 2.2 there) for the case when $T=\infty$. The adjustment for T finite is not difficult and we omit it. In [29], a proof of (86) is also presented for $T=\infty$ (Proposition 2.3 there) but there seems to be a minor error in the argument. We find it instructive to present their arguments here, but with the minor fix.

From the definition of the Chemin-Lerner spaces (40), we need to show

$$\sup_{q\in\mathbb{Z}}2^q\int\limits_0^T\|\dot{\Delta}_q u(\cdot,t)\|_{L_\infty}dt \leq C_{univ}(\|u\|_{\mathcal{E}_T}+\|e^{t\Delta}u_0\|_{\mathcal{E}_T}+\|u\|_{\mathcal{E}_T}\sup_{0<s<T}s\|\nabla u(\cdot,s)\|_{L_\infty})$$
$$+\|e^{t\Delta}u_0\|_{\tilde{L}^1(0,T;\dot{B}^1_{\infty,\infty})}. \tag{87}$$

For the case of q such that $2^{-2q}\geq T$, we have

$$2^q\int\limits_0^T\|\dot{\Delta}_q u(\cdot,t)\|_{L_\infty}dt\leq 2^q\|u\|_{\mathcal{E}_T}\int\limits_0^{2^{-2q}}\frac{1}{t^{\frac12}}dt\leq C\|u\|_{\mathcal{E}_T}. \tag{88}$$

We are left to consider the case $T>2^{-2q}$ and we see that for this case, it suffices to show

$$2^q\int\limits_{2^{-2q}}^T\|\dot{\Delta}_q B(u,u)(\cdot,t)\|_{L_\infty}dt\leq C_{univ}(\|u\|_{\mathcal{E}_T}+\|e^{t\Delta}u_0\|_{\mathcal{E}_T}+\|u\|_{\mathcal{E}_T}\sup_{0<s<T}s\|\nabla u(\cdot,s)\|_{L_\infty}). \tag{89}$$

Following the proof given in [29], we have that

$$\dot{\Delta}_q B(u,u)(x,t)=\int\limits_0^{\frac t2}\dot{\Delta}_q e^{(t-s)\Delta}\mathbb{P}\nabla\cdot(u\otimes u)ds+\int\limits_{\frac t2}^t\int\limits_{\mathbb{R}^3}g_{1,q}(y,t-s)\nabla\cdot(u(x-y,s)\otimes u(x-y,s))dsdy$$
$$=F_{1,q}(x,t)+F_{2,q}(x,t). \tag{90}$$

We treat $F_{1,q}$ in the same way as in [29]. First observe that

$$F_{1,q}(x,t)=\int\limits_{\mathbb{R}^3}g_{2,q}(y,\frac t2)(u(x-y,\frac t2)-e^{\frac t2\Delta}u_0(x-y))dy.$$

Using Lemma 3, we get that

$$2^q\int\limits_{2^{-2q}}^T\|F_{1,q}(\cdot,t)\|_{L^\infty}dt\leq C_{univ}2^q(\|u\|_{\mathcal{E}_T}+\|e^{t\Delta}u_0\|_{\mathcal{E}_T})\int\limits_{2^{-2q}}^T\int\limits_{\mathbb{R}^3}\frac{C_{univ}2^{3q}e^{-ct2^{2q}}}{t^{\frac12}(1+(2^q|y|)^6)}dydt$$
$$\leq C_{univ}(\|u\|_{\mathcal{E}_T}+\|e^{t\Delta}u_0\|_{\mathcal{E}_T}). \tag{91}$$

The very minor fix required for the proof in [29] regards $F_{2,q}$. In [29], the integral is split into the regions $B(0,2\sqrt{t})$ and $\mathbb{R}^3\setminus B(0,2\sqrt{t})$. In the outer region the authors in [29] integrate by parts but seem to not account for the boundary traces on the sphere of radius $2\sqrt{t}$. The very minor fix we propose is to not integrate by parts and then to proceed with similar arguments to the estimates used in [29] for the integral over the ball $B(0,2\sqrt{t})$.

Indeed, using Lemma 3 we get that

(92)
$$2^q\int\limits_{2^{-2q}}^{T}\|F_{2,q}(\cdot,t)\|_{L^\infty}dt\leq C_{univ}2^q(\|u\|_{\mathcal{E}_T}\sup_{0<s<T}s\|\nabla u(\cdot,s)\|_{L_\infty})\int\limits_{2^{-2q}}^{T}\int\limits_{\frac{t}{2}}^{t}\int\limits_{\mathbb{R}^3}\frac{C_{univ}2^{3q}e^{-c(t-s)2^{2q}}}{s^{\frac{3}{2}}(1+(2^q|y|)^6)}dydsdt$$
$$\leq C_{univ}\|u\|_{\mathcal{E}_T}\sup_{0<s<T}s\|\nabla u(\cdot,s)\|_{L_\infty}.$$

This completes the proof. □

Remark 1. Using that BMO^{-1} is continuously embedded into $\dot{B}^{-1}_{\infty,\infty}$, we get from (41) that

$$\|e^{t\Delta}u_0\|_{\tilde{L}^1(\dot{B}^1_{\infty,\infty})},\ \|e^{t\Delta}u_0\|_{L^\infty(\dot{B}^{-1}_{\infty,\infty})}\leq\|u_0\|_{BMO^{-1}}. \tag{93}$$

Furthermore, if u_0 is smooth and compactly supported we have

$$\|e^{t\Delta}u_0\|_{\tilde{L}^1_T(\dot{B}^1_{\infty,\infty})}=\sup_{q\in\mathbb{Z}}\Big(2^q\int\limits_0^T\|\dot{\Delta}_q e^{t\Delta}u_0\|_{L^\infty(\mathbb{R}^3)}dt\Big)\leq cT\|u_0\|_{\dot{B}^1_{\infty,\infty}}.$$

Using this, (93) and a density argument allows us to infer that

$$u_0\in VMO^{-1}\Rightarrow\lim_{T\downarrow 0}\|e^{t\Delta}u_0\|_{\tilde{L}^1_T(\dot{B}^1_{\infty,\infty})}=0. \tag{94}$$

3.2. **Propagation of Sobolev Regularity for the Navier-Stokes Equations.**

Lemma 4. *Define*

$$L(f)(\cdot,t):=\int\limits_0^t e^{(t-s)\Delta}\mathbb{P}\nabla\cdot f(\cdot,s)ds. \tag{95}$$

Assume that for some finite $T>0$ and $p\in(2,\infty)$

$$f\in L^p_TL^2_x. \tag{96}$$

Then for every $\beta\in(0,1-\frac{2}{p})$ we have

$$\|L(f)\|_{L^\infty_T H^\beta}\leq C(\beta,T)\max\Big(\|f\|_{L^p_TL^2_x},\|f\|_{L^2_TL^2_x}\Big) \tag{97}$$

Proof. It is classical that

$$\|L(f)\|_{L^\infty_TL^2_x}\leq\|f\|_{L^2_TL^2_x}. \tag{98}$$

Applying (42) and using the continuity of the Leray Projector $\mathbb{P}$ on homogeneous Sobolev spaces, we get that

$$\|e^{(t-s)\Delta}\mathbb{P}\nabla\cdot f(\cdot,s)\|_{\dot{H}^\beta}\leq C\|e^{(t-s)\Delta}f(\cdot,s)\|_{\dot{H}^{\beta+1}}\leq\frac{C'\|f(\cdot,s)\|_{L^2_x}}{(t-s)^{\frac{1+\beta}{2}}}.$$

We then apply Hölder's inequality in time to get

$$\|L(f)(\cdot,t)\|_{\dot{H}^{\beta}} \leq \|f\|_{L^p_T L^2_x}\Big(\int_0^t \frac{ds}{(t-s)^{\frac{(1+\beta)p}{2(p-1)}}}\Big)^{1-\frac{1}{p}}.$$

This converges for $p\in(2,\infty)$ and $\beta\in(0,1-\frac{2}{p})$. □

Lemma 5. *Let $T>0$ be finite and $\delta\in(0,1)$. Suppose that V is divergence-free and*

$$V\in L^{\infty}_T L^2,\quad \sup_{0<t<T} t^{\frac{1}{2}(1-\delta)}\|V(\cdot,t)\|_{L^{\infty}_x}<\infty. \tag{99}$$

Furthermore, suppose that there exists $q\in(3,\infty)$ and $s\in(-1+\frac{2}{q},0)$ with

$$u_0\in J(\mathbb{R}^3)\cap \dot{B}^{s}_{q,q}. \tag{100}$$

Assume that $U\in C_w([0,T];J(\mathbb{R}^3))\cap L^2(0,T;\dot{H}^1(\mathbb{R}^3))$ is a weak solution to the equation

$$\partial_t U-\Delta U+V\cdot\nabla U+U\cdot\nabla V+U\cdot\nabla U+V\cdot\nabla V+\nabla\Pi=0 \ \ in \ \ \mathbb{R}^3\times(0,T) \tag{101}$$

$$div\, U=0,\ \ U(\cdot,0)=u_0. \tag{102}$$

Furthermore, assume that U satisfies the energy inequality for $t\in[0,T]$:

(103)

$$\|U(\cdot,t)\|^2_{L^2(\mathbb{R}^3)}+2\int\limits_0^t\int\limits_{\mathbb{R}^3}|\nabla U(y,s)|^2dyds\leq\|u_0\|^2_{L^2(\mathbb{R}^3)}+2\int\limits_0^t\int\limits_{\mathbb{R}^3}(V\otimes U+V\otimes V):\nabla U dyds.$$

Then the above assumptions imply that there exists a $\gamma(s,q,\delta)>0$ such that

$$\sup_{0<t<T}\frac{\|U(\cdot,t)-e^{t\Delta}u_0\|_{L^2_x}}{t^{\frac{\gamma}{2}}}<\infty. \tag{104}$$

Proof. Since $V\in L^{\infty}_t L^2_x$ and V belongs to subcritical spaces, the proof of the above can be completed using similar reasoning in the author's paper on weak-strong uniqueness (in particular Lemma 1.5 in [4], which treated the case $V\equiv 0$). For the case $V\equiv 0$, we also refer to section C.3.2 in [2]. Since the adjustments required due to V are insignificant, we omit the details. □

Proof of Proposition 1

First, it is known that U satisfies the mild formulation of the Navier-Stokes equations

$$U(\cdot,t)=e^{t\Delta}u_0^1+L(V\otimes U+U\otimes V+V\otimes V)(\cdot,t)+L(U\otimes U)(\cdot,t). \tag{105}$$

First, using (20), it is immediate that

$$e^{t\Delta}u_0^1\in L^{\infty}(0,\infty;H^{\hat{\alpha}}). \tag{106}$$

Using that $U\in C_w([0,T];J(\mathbb{R}^3))$ and (19), it easily follows that there exists a $p(\delta)>2$ such that

$$V\otimes U+U\otimes V+V\otimes V\in L^p_T L^2_x.$$

We can therefore apply Lemma 4 to get that there exists a positive $\hat{\alpha_1}(p)$ such that

$$L(V\otimes U+U\otimes V+V\otimes V)(\cdot,t)\in L^{\infty}(0,T;H^{\hat{\alpha_1}}). \tag{107}$$

The term $L(U\otimes U)(\cdot,t)$ requires more work. First, notice that by Lebesgue interpolation

$$(2^{j(\frac{\hat{\alpha}}{2}-\frac{1}{2})}\|\dot{\Delta}_j u_0^1\|_{L_4})^4\leq(2^{-j}\|\dot{\Delta}_j u_0^1\|_{L^{\infty}_x})^2(2^{j\hat{\alpha}}\|\dot{\Delta}_j u_0^1\|_{L^2_x})^2.$$

Thus, $u_0^1 \in \dot{B}_{4,4}^{\frac{\hat{\alpha}}{2}-\frac{1}{2}}$. Since $\hat{\alpha} \in (0,1)$, we may apply Lemma 5 to infer that there exists $\gamma(\hat{\alpha}) > 0$ such that

$$\sup_{0<t<T} \frac{\|U(\cdot,t) - e^{t\Delta}u_0^1\|_{L^2_x}}{t^{\frac{\gamma}{2}}} < \infty. \tag{108}$$

By the heat-flow characterization of homogeneous Besov spaces, we have

$$\|e^{t\Delta}u_0^1\|_{L^4(\mathbb{R}^3)} \leq \frac{C\|u_0^1\|_{\dot{B}_{4,4}^{\frac{\hat{\alpha}}{2}-\frac{1}{2}}}}{t^{\frac{1}{4}(1-\hat{\alpha})}}. \tag{109}$$

We also get that $\sup_{0<t} t^{\frac{1}{2}}\|e^{t\Delta}u_0^1\|_{L^\infty(\mathbb{R}^3)} \leq C_{univ}\|u_0^1\|_{\dot{B}^{-1}_{\infty,\infty}(\mathbb{R}^3)}$. Using this in conjunction with (24), we infer

$$\sup_{0<t<T} t^{\frac{1}{2}}\|U(\cdot,t) - e^{t\Delta}u_0^1\|_{L^\infty(\mathbb{R}^3)} < \infty. \tag{110}$$

Now, we write

(111)

$$L(U\otimes U)(\cdot,t) = L((U - e^{t\Delta}u_0^1)\otimes(U - e^{t\Delta}u_0^1) + e^{t\Delta}u_0^1\otimes(U - e^{t\Delta}u_0^1) + (U - e^{t\Delta}u_0^1)\otimes e^{t\Delta}u_0^1))(\cdot,t)$$
$$+ L(e^{t\Delta}u_0^1\otimes e^{t\Delta}u_0^1)(\cdot,t).$$

Using (108)-(110), we infer that there exists $q(\hat{\alpha},\gamma) > 2$ such that

$$(U - e^{t\Delta}u_0^1)\otimes(U - e^{t\Delta}u_0^1) + e^{t\Delta}u_0^1\otimes(U - e^{t\Delta}u_0^1) + (U - e^{t\Delta}u_0^1)\otimes e^{t\Delta}u_0^1 + e^{t\Delta}u_0^1\otimes e^{t\Delta}u_0^1 \in L^q_T L^2_x.$$

We then can apply Lemma 4 to deduce that there exists $\hat{\alpha}_2(q) > 0$ such that

$$L(U\otimes U)(\cdot,t) \in L^\infty(0,T;H^{\hat{\alpha}_2(q)}). \tag{112}$$

Using this, along with (105)- (107), we get the desired conclusion with $\alpha = \min(\hat{\alpha},\hat{\alpha_1},\hat{\alpha_2}) > 0$.

Remark 2. Let $\gamma \in (0,\frac{1}{2})$ be as in (108). Note that the Proposition still holds if the critical assumption (24) is replaced by the supercritical assumption

$$\sup_{0<t<T} t^{\frac{\tilde{\gamma}}{2}}\|U(\cdot,t)\|_{L^\infty(\mathbb{R}^3)} < \infty \quad \text{with} \quad \tilde{\gamma} \in (1,\gamma+1).$$

4. REYNOLD'S STRESS

4.1. **Reynold's stress expression.** Suppose that a and b are Schwartz functions. In the proof of Theorem 2 in [9], Chemin evaluates the Reynold's stress (11) of the form

$$B_j(a,b) := \dot{S}_j(a)\dot{S}_j(b) - \dot{S}_j(ab).$$

To do this, Chemin uses a decomposition of the Reynold's stress that somewhat resembles the classical paraproduct decomposition. We recall Chemin's expression now. First if N_0 is a large enough integer, it is argued in [9] that

$$\dot{S}_{j-N_0}(a)\dot{S}_{j-N_0}(b) = \dot{S}_j(\dot{S}_{j-N_0}(a)\dot{S}_{j-N_0}(b)). \tag{113}$$

Indeed, $\dot{S}_{j-N_0}a$ and $\dot{S}_{j-N_0}b$ have Fourier transforms supported in the ball $B(0,\frac{4}{3}2^{j-N_0})$. Thus, in Fourier space $\dot{S}_{j-N_0}a\dot{S}_{j-N_0}b$ is supported in $B(0,\frac{8}{3}2^{j-N_0}) \subset B(0,2^{j-N_0+2})$. Next the Fourier multiplier of $\dot{\Delta}_{j'}$ is compactly supported on the annulus $\{\xi\in\mathbb{R}^d : 2^{j'}\frac{3}{4} \leq |\xi| \leq 2^{j'}\frac{8}{3}\}$. Thus

$$2^{j'}\frac{3}{4} \geq 2^{j-N_0+2} \Rightarrow \dot{\Delta}_{j'}(\dot{S}_{j-N_0}(a)\dot{S}_{j-N_0}(b)) = 0.$$

If N_0 is a fixed large integer (which can be chosen to be independent of j, for example $N_0 = 3$), the above is satisfied for all $j' \geq j$. In particular this gives (113).

Using (113), in [9] Chemin writes

$$B_j(a,b) = \dot{S}_j(a)\dot{S}_j(b) - \dot{S}_{j-N_0}(a)\dot{S}_{j-N_0}(b) + \dot{S}_j(\dot{S}_{j-N_0}(a)\dot{S}_{j-N_0}(b) - ab).$$

So $B_j(a,b) = \sum_{k=1}^{4} B_j^k(a,b)$ Here,

$$B_j^1(a,b) := (\dot{S}_j - \dot{S}_{j-N_0})(a)\dot{S}_j(b) \tag{114}$$

$$B_j^2(a,b) := \dot{S}_{j-N_0}(a)(\dot{S}_j - \dot{S}_{j-N_0})(b) \tag{115}$$

$$B_j^3(a,b) := \dot{S}_j((\dot{S}_{j-N_0} - I)(a)\dot{S}_{j-N_0}(b)) \tag{116}$$

$$B_j^4(a,b) := \dot{S}_j(a(\dot{S}_{j-N_0} - I)(b)) \tag{117}$$

Furthermore, $B_j^4(a,b) = B_j^3(b,a) - B_j^{41}(a,b)$. Here,

$$B_j^{41}(a,b) := \dot{S}_j((I - \dot{S}_{j-N_0})(a)(I - \dot{S}_{j-N_0})(b)) \tag{118}$$

In order to reduce the frequency interactions in $B_j^{41}(a,b)$, Chemin uses that for a large enough fixed integer N_1

$$j' \geq j + N_1, \quad j'' \leq j' - 2 \Rightarrow \dot{S}_j(\dot{\Delta}_{j'} a \dot{\Delta}_{j''} b) = 0. \tag{119}$$

To see this, notice that

$$\operatorname{supp} \mathcal{F}(\dot{\Delta}_{j'} a \dot{\Delta}_{j''} b) \subset 2^{j'} C + 2^{j''} C$$

with $C = \{\xi : \frac{3}{4} \leq |\xi| \leq \frac{8}{3}\}$. Now for $j'' \leq j' - 2$ we have

$$2^{j'} C + 2^{j''} C \subset 2^{j'} C + B(0, 2^{j'-2} 8/3) \subset 2^{j'} C'.$$

Here, $C' = \{\xi : \frac{1}{12} \leq |\xi| \leq \frac{10}{3}\}$. Next the Fourier multiplier of $\dot{S}_j$ is compactly supported on the ball $B(0, \frac{4}{3} 2^j)$. Thus,

$$2^j \frac{4}{3} < \frac{2^{j'}}{12} \Rightarrow \dot{S}_j(\dot{\Delta}_{j'} a \dot{\Delta}_{j''} b) = 0.$$

If $j' \geq j + N_1$, the above is true provided $N_1 \geq 5$. This then gives (119). Using (113), Chemin writes

$$B_j^{41}(a,b) := \dot{S}_j\Big(\sum_{j',j''>j+N_1, |j''-j'|<2} \dot{\Delta}_{j'} a \dot{\Delta}_{j''} b\Big) + \dot{S}_j\Big(\sum_{j',j''=j-N_0}^{j+N_1} \dot{\Delta}_{j'} a \dot{\Delta}_{j''} b\Big). \tag{120}$$

4.2. **Reynold's stress estimate.** Let $u_0 \in J(\mathbb{R}^3) \cap VMO^{-1}(\mathbb{R}^3) \cap \dot{B}^s_{q,q}$ with $s \in (-1 + \frac{2}{q}, 0)$ and $q > 3$, then we apply Corollary 1 show that $u_0 = u_0^1 + u_0^2$ with

$$u_0^2 \in \dot{B}_{p,p}^{-1+\frac{3}{p}+\delta}(\mathbb{R}^3) \cap J(\mathbb{R}^3) \quad \text{and} \quad u_0^1 \in \dot{H}^{\hat{\alpha}} \cap J(\mathbb{R}^3).$$

Here, $p \in (4,\infty)$, $\delta \in (0, 1 - \frac{3}{p})$ and $\hat{\alpha} \in (0, \frac{3}{2})$. From (36) and Proposition 2, we see that $u_0^2 \in \dot{B}_{p,p}^{-1+\frac{3}{p}} \subset VMO^{-1}$. Thus,

$$u_0^1 \in \dot{H}^{\hat{\alpha}} \cap VMO^{-1}(\mathbb{R}^3) \cap J(\mathbb{R}^3).$$

Furthermore, by (41) and the Sobolev embedding we have

$$e^{t\Delta} u_0^2 \in \tilde{L}_T^\infty(\dot{B}_{p,\infty}^{-1+\frac{3}{p}}) \cap \tilde{L}_T^2(\dot{B}_{p,\infty}^{\frac{3}{p}}) \cap \tilde{L}_T^1(\dot{B}^1_{\infty,\infty}) \cap L_T^\infty L^2.$$

Recall we consider $U = u - e^{t\Delta}u_0^2$ which solve the *perturbed* Navier-Stokes equations.

$$\partial_t U - \Delta U + U\cdot\nabla U + e^{t\Delta}u_0^2\cdot\nabla U + U\cdot\nabla e^{t\Delta}u_0^2 + \nabla P = -e^{t\Delta}u_0^2\cdot\nabla e^{t\Delta}u_0^2 \tag{121}$$

$$\operatorname{div} U = 0, \quad U(\cdot,0) = u_0^1 \tag{122}$$

Then $\dot{S}_jU$ satisfies the equation

$$\partial_t\dot{S}_jU - \Delta\dot{S}_jU + \dot{S}_jU\cdot\nabla\dot{S}_jU + e^{t\Delta}u_0^2\cdot\nabla\dot{S}_jU + \dot{S}_jU\cdot\nabla e^{t\Delta}u_0^2 + \nabla P_j = \nabla\cdot F_j - e^{t\Delta}u_0^2\cdot\nabla e^{t\Delta}u_0^2 \tag{123}$$

$$\operatorname{div}\dot{S}_jU = 0, \quad \dot{S}_jU(\cdot,0) = \dot{S}_ju_0^1. \tag{124}$$

Using that U and $e^{t\Delta}u_0^2$ are divergence free, we can write[17]

$$F_j := \sum_{k=1}^{6} F_j^{(k)} \tag{125}$$

$$F_j^{(1)} := \dot{S}_j(U)\otimes\dot{S}_j(U) - \dot{S}_j(U\otimes U) \tag{126}$$

$$F_j^{(2)} := (e^{t\Delta}u_0^2 - \dot{S}_j(e^{t\Delta}u_0^2))\otimes\dot{S}_j(U) \tag{127}$$

$$F_j^{(3)} := \dot{S}_j(e^{t\Delta}u_0^2)\otimes\dot{S}_j(U) - \dot{S}_j(e^{t\Delta}u_0^2\otimes U) \tag{128}$$

$$F_j^{(4)} := \dot{S}_j(U)\otimes(e^{t\Delta}u_0^2 - \dot{S}_j(e^{t\Delta}u_0^2)) \tag{129}$$

$$F_j^{(5)} := \dot{S}_jU\otimes\dot{S}_j(e^{t\Delta}u_0^2)) - \dot{S}_j(U\otimes(e^{t\Delta}u_0^2)) \tag{130}$$

$$F_j^{(6)} := e^{t\Delta}u_0^2\otimes e^{t\Delta}u_0^2 - \dot{S}_j(e^{t\Delta}u_0^2\otimes e^{t\Delta}u_0^2)). \tag{131}$$

Now, we state a proposition regarding estimates of F_j, which is a crucial ingredient in proving Theorem 1.

Proposition 6. *Suppose that there exists $\alpha\in(0,\frac{3}{2})$ and finite $T>0$ such that*

$$U\in \tilde{L}^\infty_T\dot{B}^{-1}_{\infty,\infty}\cap\tilde{L}^1_T\dot{B}^1_{\infty,\infty}\cap L^\infty_T H^\alpha. \tag{132}$$

Furthermore, suppose that there exists $p>4$ and a $\delta\in(0,1-\frac{3}{p})$ such that

$$u_0^2\in\dot{B}^{-1+\frac{3}{p}+\delta}_{p,p}\cap J(\mathbb{R}^3). \tag{133}$$

With this U and u_0^2 let F_j be defined by (125)-(131). Then we conclude that

$$\begin{aligned}\|F_j\|_{L^2_TL^2_x} &\leq c(\alpha)2^{-j\alpha}\|U\|_{L^\infty_T\dot{H}^\alpha}(j\|U\|_{\tilde{L}^2_T(\dot{B}^0_{\infty,\infty})} + T^{\frac{1}{2}}\|U\|_{L^\infty_T(\dot{B}^{-1}_{\infty,\infty})})\\ &+ c(\alpha)2^{-j\alpha}\|U\|_{L^\infty_T\dot{H}^\alpha}(j\|e^{t\Delta}u_0^2\|_{\tilde{L}^2_T(\dot{B}^0_{\infty,\infty})} + T^{\frac{1}{2}}\|e^{t\Delta}u_0^2\|_{L^\infty_T(\dot{B}^{-1}_{\infty,\infty})})\\ &+ C(p)2^{-\frac{j}{p}}\|e^{t\Delta}u_0^2\|_{\tilde{L}^2_T(\dot{B}^{\frac{3}{p}}_{p,\infty})}\|U\|^{\frac{2}{p}}_{\tilde{L}^\infty_T(\dot{B}^{-1}_{\infty,\infty})}\|U\|^{1-\frac{2}{p}}_{L^\infty_T(L^2)} + C(\delta)T^{\frac{\delta}{4}}2^{-j\frac{\delta}{2}}\|u_0^2\|_{L_2}\|u_0^2\|_{\dot{B}^{-1+\frac{3}{p}+\delta}_{p,p}}.\end{aligned} \tag{134}$$

[17] Here, $a\otimes b$ denotes a matrix with $(a\otimes b)_{ij} = a_ib_j$. Furthermore, $(\nabla\cdot(a\otimes b))_i = \partial_l(a_lb_i)$. Here we adopt the Einstein summation convention.

Remark 3. In proving Theorem 1, we are concerned with an estimate for F_j that exhibits exponential decay for large frequencies. The precise estimate above is not required, so we will often use the more compact estimate

$$\|F_j\|_{L^2_T L^2_x} \leq C_{U,u_0,p,\alpha,T,\delta} 2^{-j\gamma_{\alpha,p,\delta}}. \tag{135}$$

Here, $\gamma_{\alpha,p,\delta} := \min(\frac{\alpha}{2}, \frac{1}{p}, \frac{\delta}{2}) > 0$.

Proposition 6 will immediately follow as the result of three lemmas, the first of which handles the estimate of $F_j^{(1)}$.

Lemma 6. *Suppose that for some $\alpha > 0$ and some finite T*

$$b \in \tilde{L}^\infty_T \dot{B}^{-1}_{\infty,\infty} \cap \tilde{L}^1_T \dot{B}^1_{\infty,\infty} \cap L^\infty_T H^\alpha. \tag{136}$$

Then we conclude

$$\|B_j(b,b)\|_{L^2_T L^2_x} \leq c(\alpha) 2^{-j\alpha} \|b\|_{L^\infty_T \dot{H}^\alpha} (j \|b\|_{\tilde{L}^2_T(\dot{B}^0_{\infty,\infty})} + T^{\frac{1}{2}} \|b\|_{L^\infty_T(\dot{B}^{-1}_{\infty,\infty})}) \tag{137}$$

Remark 4. Notice that if $b \in \tilde{L}^\infty_T \dot{B}^{-1}_{\infty,\infty} \cap \tilde{L}^1_T \dot{B}^1_{\infty,\infty}$ then

$$\| \|\dot{\Delta}_j b\|_{L^\infty_x} \|_{L^2_T} \leq \|2^j \|\dot{\Delta}_j b\|_{L^\infty_x}\|^{\frac{1}{2}}_{L^1_T} \| \|2^{-j} \dot{\Delta}_j b\|_{L^\infty_x}\|^{\frac{1}{2}}_{L^\infty_T}.$$

Thus

$$\|b\|_{\tilde{L}^2_T(\dot{B}^0_{\infty,\infty})} \leq \|b\|^{\frac{1}{2}}_{\tilde{L}^\infty_T(\dot{B}^{-1}_{\infty,\infty})} \|b\|_{\tilde{L}^1_T(\dot{B}^{-1}_{\infty,\infty}})^{\frac{1}{2}}$$

Proof. First we estimate $\dot{S}_j b = \sum_{k \leq j-1} \dot{\Delta}_k b$. In particular we have,

$$\begin{aligned} \|\dot{S}_j b\|_{L^2_T L^\infty_x} &\leq \sum_{k \leq 0} \|\dot{\Delta}_k b\|_{L^2_T L^\infty_x} + \sum_{k=0}^{j-1} \|\dot{\Delta}_k b\|_{L^2_T L^\infty_x} \\ &\leq j \|b\|_{\tilde{L}^2_T(\dot{B}^0_{\infty,\infty})} + T^{\frac{1}{2}} \sum_{k \leq 0} 2^k \|2^{-k} \|\dot{\Delta}_k b\|_{L^\infty_x}\|_{L^\infty_T} \\ &\leq C(j \|b\|_{\tilde{L}^2_T(\dot{B}^0_{\infty,\infty})} + T^{\frac{1}{2}} \|b\|_{\tilde{L}^\infty_T(\dot{B}^{-1}_{\infty,\infty})}). \end{aligned} \tag{138}$$

Now, let N_0 be a fixed integer as in the expression of the Reynold's stress in the previous subsection. Next we estimate

$$\dot{S}_j b - \dot{S}_{j-N_0} b = \sum_{k=j-N_0}^{j-1} \dot{\Delta}_k b.$$

In particular,

$$\begin{aligned} \|\dot{S}_j b - \dot{S}_{j-N_0} b\|_{L^\infty_T L^2} &\leq \sum_{k=j-N_0}^{j-1} 2^{-k\alpha} (2^{k\alpha} \|\dot{\Delta}_k b\|_{L^\infty_T L^2}) \\ &\leq \|b\|_{L^\infty_T \dot{H}^\alpha} \sum_{k=j-N_0}^{j-1} 2^{-k\alpha} \leq C_{N_0,\alpha} \|b\|_{L^\infty_T \dot{H}^\alpha} 2^{-j\alpha}. \end{aligned} \tag{139}$$

Next, we estimate $b-\dot{S}_{j-N_0}b=\sum_{k=j-N_0}^{\infty}\dot{\Delta}_k b$. We get

$$\|b-\dot{S}_{j-N_0}b\|_{L^\infty_T L^2}\leq \sum_{k=j-N_0}^{\infty}2^{-k\alpha}(2^{k\alpha}\|\dot{\Delta}_k b\|_{L^\infty_T L^2})$$
(140)
$$\leq \|b\|_{L^\infty_T \dot{H}^\alpha}\sum_{k=j-N_0}^{\infty}2^{-k\alpha}\leq C_{N_0,\alpha}\|b\|_{L^\infty_T\dot{H}^\alpha}2^{-j\alpha}.$$

Using (138)-(140), together with the continuity of $\dot{S}_j$ on Lebesgue spaces (with bounds independent of j), it is not difficult to see that the following holds. Namely for $i=1,2,3$ ($B^i_j(b,b)$ as defined in the previous subsection), we have

$$\|B^i_j(b,b)\|_{L^2_T L^2}\leq c(\alpha,N_0)2^{-j\alpha}\|b\|_{L^\infty_T\dot{H}^\alpha}(j\|b\|_{\tilde{L}^2_T(\dot{B}^0_{\infty,\infty})}+T^{\frac12}\|b\|_{L^\infty_T(\dot{B}^{-1}_{\infty,\infty})}).$$

Now, we must estimate $B^{41}_j(b,b)$. Using the continuity of $\dot{S}_j$ on Lebesgue spaces (with bounds independent of j) and Hölder's inequality, we get

(141)
$$\|B^{41}_j(b,b)\|_{L^2_TL^2_x}\leq \sum_{j',j''>j+N_1,|j''-j'|<2}\|\dot{\Delta}_{j'}b\|_{L^\infty_TL^2_x}\|\dot{\Delta}_{j''}b\|_{L^2_TL^\infty_x}+\sum_{j',j''=j-N_0}^{j+N_1}\|\dot{\Delta}_{j'}b\|_{L^\infty_TL^2_x}\|\dot{\Delta}_{j''}b\|_{L^2_TL^\infty_x}$$
$$\leq C_{N_1,N_0}\|b\|_{\tilde{L}^2_T(\dot{B}^0_{\infty,\infty})}\Big(\sum_{j'\geq j-N_0}\|\dot{\Delta}_{j'}b\|_{L^\infty_TL^2_x}\Big)\leq C_{N_1,N_0}\|b\|_{\tilde{L}^2_T(\dot{B}^0_{\infty,\infty})}\Big(\sum_{j'\geq j-N_0}2^{-j'\alpha}2^{j'\alpha}\|\dot{\Delta}_{j'}b\|_{L^\infty_TL^2_x}\Big)$$
$$\leq C_{N_1,N_0,\alpha}2^{-j\alpha}\|b\|_{\tilde{L}^2_T(\dot{B}^0_{\infty,\infty})}\|b\|_{L^\infty_T\dot{H}^\alpha}.$$

This estimate is of the required form. Thus the proof is completed. □

The following Lemma treats the estimates of $F^{(2)}_j-F^{(5)}_j$ in Proposition 6. Specifically, (144) handles $F^{(3)}_j$ and $F^{(5)}_j$, whilst (145) deals with $F^{(2)}_j$ and $F^{(4)}_j$.

Lemma 7. *Suppose that there exists $p\in(4,\infty)$ such that*

(142)
$$a\in\tilde{L}^\infty_T(\dot{B}^{-1+\frac3p}_{p,\infty})\cap\tilde{L}^2_T(\dot{B}^{\frac3p}_{p,\infty})\cap\tilde{L}^1_T(\dot{B}^1_{\infty,\infty})\cap L^\infty_TL^2.$$

Suppose that there exists $\alpha\in(0,\frac32)$ such that

(143)
$$b\in\tilde{L}^\infty_T\dot{B}^{-1}_{\infty,\infty}\cap\tilde{L}^1_T\dot{B}^1_{\infty,\infty}\cap L^\infty_TH^\alpha.$$

Then we conclude

$$\|B_j(a,b)\|_{L^2_TL^2_x}\leq C(p)2^{-\frac jp}\|a\|_{\tilde{L}^2_T(\dot{B}^{\frac3p}_{p,\infty})}\|b\|^{\frac2p}_{\tilde{L}^\infty_T(\dot{B}^{-1}_{\infty,\infty})}\|b\|^{1-\frac2p}_{L^\infty_T(L^2)}$$
(144)
$$+c(\alpha)2^{-j\alpha}\|b\|_{L^\infty_T\dot{H}^\alpha}(j\|a\|_{\tilde{L}^2_T(\dot{B}^0_{\infty,\infty})}+T^{\frac12}\|a\|_{L^\infty_T(\dot{B}^{-1}_{\infty,\infty})}),$$

(145)
$$\|(\dot{S}_j(b))(a-\dot{S}_ja)\|_{L^2_TL^2_x}\leq C(p)2^{-\frac jp}\|a\|_{\tilde{L}^2_T(\dot{B}^{\frac3p}_{p,\infty})}\|b\|^{\frac2p}_{\tilde{L}^\infty_T(\dot{B}^{-1}_{\infty,\infty})}\|b\|^{1-\frac2p}_{L^\infty_T(L^2)}$$

Remark 5. Notice that by Bernstein's inequality

$$\|a\|_{\tilde{L}^\infty_T(\dot{B}^{-1}_{\infty,\infty})}\leq C\|a\|_{\tilde{L}^\infty_T(\dot{B}^{-1+\frac3p}_{p,\infty})}.$$

Thus, by the previous remark $a\in\tilde{L}^2_T(\dot{B}^0_{\infty,\infty})$.

Proof. The first estimate we need to prove Lemma 7 is

$$\|\dot{S}_j b\|_{L^\infty_T(L^{\frac{2p}{p-2}})} \leq C(p)2^{\frac{2j}{p}}\|b\|^{\frac{2}{p}}_{\tilde{L}^\infty_T(\dot{B}^{-1}_{\infty,\infty})}\|b\|^{1-\frac{2}{p}}_{L^\infty_T L^2_x}. \tag{146}$$

By interpolation

(147)

$$\|\dot{\Delta}_k b\|_{L^\infty_T(L^{\frac{2p}{p-2}})} \leq C\|\dot{\Delta}_k b\|^{\frac{2}{p}}_{L^\infty_T(L^\infty)}\|\dot{\Delta}_k b\|^{1-\frac{2}{p}}_{L^\infty_T(L^2)} \leq 2^{\frac{2k}{p}}(2^{-k}\|\dot{\Delta}_k b\|_{L^\infty_T(L^\infty)})^{\frac{2}{p}}\|\dot{\Delta}_k b\|^{1-\frac{2}{p}}_{L^\infty_T(L^2)}$$
$$\leq 2^{\frac{2k}{p}}\|b\|^{\frac{2}{p}}_{L^\infty_T(\dot{B}^{-1}_{\infty,\infty})}\|b\|^{1-\frac{2}{p}}_{L^\infty_T(L^2)}.$$

Here, we used

$$\|\dot{\Delta}_k b\|_{L^2_x} \leq C_{univ}\|b\|_{L^2_x}.$$

Summation over $k \leq j-1$ then yields (146).

Now we proceed with the main part of the proof of Lemma 7. We start with $B^1_j(a,b)$ as defined by (114). Using Hölder's inequality and (146) gives

$$\|B^1_j(a,b)\|_{L^2_T L^2} \leq C(p)2^{\frac{2j}{p}}\|b\|^{\frac{2}{p}}_{\tilde{L}^\infty_T(\dot{B}^{-1}_{\infty,\infty})}\|b\|^{1-\frac{2}{p}}_{L^\infty_T L^2_x}\|(\dot{S}_j-\dot{S}_{j-N_0})a\|_{L^2_T L^p}. \tag{148}$$

Now,

$$\|(\dot{S}_j-\dot{S}_{j-N_0})a\|_{L^2_T L^p} \leq \sum_{k=j-N_0}^{j-1}\|\dot{\Delta}_k a\|_{L^2_T L^p} \leq \sum_{k=j-N_0}^{j-1}(2^{\frac{3k}{p}}\|\dot{\Delta}_k a\|_{L^2_T L^p})2^{-\frac{3k}{p}} \leq C(p,N_0)2^{-\frac{3j}{p}}\|a\|_{\tilde{L}^2_T(\dot{B}^{\frac{3}{p}}_{p,\infty})}.$$

Combining this with (148) gives

$$\|B^1_j(a,b)\|_{L^2_T L^2} \leq C(p,N_0)2^{-\frac{j}{p}}\|a\|_{\tilde{L}^2_T(\dot{B}^{\frac{3}{p}}_{p,\infty})}\|b\|^{\frac{2}{p}}_{\tilde{L}^\infty_T(\dot{B}^{-1}_{\infty,\infty})}\|b\|^{1-\frac{2}{p}}_{L^\infty_T L^2_x}. \tag{149}$$

Next, verbatim reasoning to Lemma 6 gives

$$\|B^2_j(a,b)\|_{L^2_T L^2} \leq c(\alpha,N_0)2^{-j\alpha}\|b\|_{L^\infty_T \dot{H}^\alpha}(j\|a\|_{\tilde{L}^2_T(\dot{B}^0_{\infty,\infty})}+T^{\frac{1}{2}}\|a\|_{L^\infty_T(\dot{B}^{-1}_{\infty,\infty})}). \tag{150}$$

Now, we estimate $B^3_j(a,b)$ defined by (116). Using that $\dot{S}_j$ is a bounded operator on Lebesgue spaces with bound independent on j, Hölder's inequality and (146), we get that

$$\begin{aligned}\|B^3_j(a,b)\|_{L^2_T L^2_x} &\leq C_{univ}\|\dot{S}_{j-N_0}a-a\|_{L^2_T L^p_x}\|\dot{S}_{j-N_0}b\|_{L^\infty_T(L_x^{\frac{2p}{p-2}})} \leq \\ &\leq C(N_0,p)2^{\frac{2j}{p}}\sum_{k\geq j-N_0}\|\dot{\Delta}_k a\|_{L^2_T L^p_x}\|b\|^{\frac{2}{p}}_{\tilde{L}^\infty_T(\dot{B}^{-1}_{\infty,\infty})}\|b\|^{1-\frac{2}{p}}_{L^\infty_T L^2_x}.\end{aligned} \tag{151}$$

Now,

$$\sum_{k\geq j-N_0}\|\dot{\Delta}_k a\|_{L^2_T L^p_x} \leq \sum_{k\geq j-N_0}(2^{\frac{3k}{p}}\|\dot{\Delta}_k a\|_{L^2_T L^p_x})2^{-\frac{3k}{p}} \leq C(p,N_0)2^{-\frac{3j}{p}}\|a\|_{\tilde{L}^2_T(\dot{B}^{\frac{3}{p}}_{p,\infty})}.$$

Combining this with (151) gives

$$\|B^3_j(a,b)\|_{L^2_T L^2_x} \leq C(N_0,p)2^{\frac{-j}{p}}\|a\|_{\tilde{L}^2_T(\dot{B}^{\frac{3}{p}}_{p,\infty})}\|b\|^{\frac{2}{p}}_{\tilde{L}^\infty_T(\dot{B}^{-1}_{\infty,\infty})}\|b\|^{1-\frac{2}{p}}_{L^\infty_T L^2_x}. \tag{152}$$

Next, we must estimate $B^3_j(b,a)$. We get by Hölder's inequality and the continuity of $\dot{S}_j$ that

$$\|B^3_j(b,a)\|_{L^2_T L^2_x} \leq C_{univ}\|(\dot{S}_{j-N_0}b-b)\|_{L^\infty_T L^2_x}\|\dot{S}_{j-N_0}a\|_{L^2_T L^\infty_x}.$$

We then use (138) and (140) to conclude

$$\|B^3_j(b,a)\|_{L^2_T L^2_x} \leq c(\alpha,N_0)2^{-j\alpha}\|b\|_{L^\infty_T \dot{H}^\alpha}(j\|a\|_{\tilde{L}^2_T(\dot{B}^0_{\infty,\infty})}+T^{\frac{1}{2}}\|a\|_{L^\infty_T(\dot{B}^{-1}_{\infty,\infty})}). \tag{153}$$

Next, by identical reasoning to the previous lemma we obtain

$$\|B_j^{41}(a,b)\|_{L^2_T L^2_x} \leq C_{N_1,N_0,\alpha} 2^{-j\alpha} \|a\|_{\tilde{L}^2_T(\dot{B}^0_{\infty,\infty})} \|b\|_{L^\infty_T \dot{H}^\alpha}. \tag{154}$$

Combining the above estimates gives (144).

Finally, we mention that the proof of (145) follows from verbatim arguments as those used for (151)-(152). □

The final lemma below estimates $F_j^{(6)}$ in Proposition 6 and thus completes the proof.

Lemma 8. *Suppose that for some* $\delta \in (0,1)$

$$u_0^2 \in \dot{B}^{-1+\delta}_{\infty,\infty}(\mathbb{R}^3) \cap L_2. \tag{155}$$

Then for finite $T > 0$

$$\|e^{t\Delta}u_0^2 e^{t\Delta}u_0^2 - \dot{S}_j(e^{t\Delta}u_0^2 e^{t\Delta}u_0^2)\|_{L^2_T L^2} \leq C(\delta) T^{\frac{\delta}{4}} 2^{-j\frac{\delta}{2}} \|u_0^2\|_{L_2} \|u_0^2\|_{\dot{B}^{-1+\delta}_{\infty,\infty}}. \tag{156}$$

Proof. By the heat flow characterization of Besov spaces (37), we have

$$\|e^{t\Delta}u_0^2\|_{L_\infty} \leq \frac{C(\delta)\|u_0^2\|_{\dot{B}^{-1+\delta}_{\infty,\infty}}}{t^{\frac{1}{2}(1-\delta)}}. \tag{157}$$

Applying (42) with $\alpha = \frac{\delta}{2}$ gives

$$\|e^{t\Delta}u_0^2\|_{\dot{H}^{\frac{\delta}{2}}} \leq \frac{C(\delta)\|u_0^2\|_{L_2}}{t^{\frac{\delta}{4}}}. \tag{158}$$

Using Lemma 2 and (157)-(158), we see that

$$\|e^{t\Delta}u_0^2 e^{t\Delta}u_0^2\|_{\dot{H}^{\frac{\delta}{2}}} \leq C(\delta)\|e^{t\Delta}u_0^2\|_{\dot{H}^{\frac{\delta}{2}}}\|e^{t\Delta}u_0^2\|_{L_\infty} \leq \frac{C(\delta)\|u_0^2\|_{\dot{B}^{-1+\delta}_{\infty,\infty}}\|u_0^2\|_{L_2}}{t^{\frac{1}{2}-\frac{\delta}{4}}}. \tag{159}$$

Using this, we see that

$$\begin{aligned} \|e^{t\Delta}u_0^2 e^{t\Delta}u_0^2 - \dot{S}_j(e^{t\Delta}u_0^2 e^{t\Delta}u_0^2)\|_{L^2_x} &\leq \sum_{k\geq j} 2^{-k\frac{\delta}{2}}(2^{k\frac{\delta}{2}}\|\dot{\Delta}_k(e^{t\Delta}u_0^2 e^{t\Delta}u_0^2)\|_{L^2_x}) \\ &\leq C(\delta)2^{-j\frac{\delta}{2}}\|e^{t\Delta}u_0^2 e^{t\Delta}u_0^2\|_{\dot{H}^{\frac{\delta}{2}}} \leq \frac{C(\delta)2^{-j\frac{\delta}{2}}\|u_0^2\|_{\dot{B}^{-1+\delta}_{\infty,\infty}}\|u_0^2\|_{L_2}}{t^{\frac{1}{2}-\frac{\delta}{4}}}. \end{aligned} \tag{160}$$

Integrating over $(0,T)$ then gives (156). □

5. Proof of Theorem 1

Step 1: collecting properties of the strong solution $u(\cdot, u_0)$
Recall that there exists $q > 3$ and $s \in (-1+\frac{2}{q}, 0)$ such that

$$u_0 \in J(\mathbb{R}^3) \cap VMO^{-1}(\mathbb{R}^3) \cap \dot{B}^s_{q,\infty}. \tag{161}$$

Applying Theorem 1.3 in [4], Proposition 4, Proposition 5 and Remark 1 we conclude that for all $\varepsilon > 0$ there exists $\hat{T}(\epsilon, u_0) > 0$ and weak Leray-Hopf solution $u(\cdot, u_0)$ (unique on $\mathbb{R}^3 \times (0, \hat{T})$) with the following properties. Namely,

$$\sup_{0<s<\hat{T}} (s^{\frac{1}{2}}\|u(\cdot,s)\|_{L^\infty(\mathbb{R}^3)} + s\|\nabla u(\cdot,s)\|_{L^\infty(\mathbb{R}^3)}) < \infty, \tag{162}$$

$$u \in L^\infty(0,\hat{T}; \dot{B}^{-1}_{\infty,\infty}) \tag{163}$$

and

$$\|u\|_{\tilde{L}^1(0,\hat{T};\dot{B}^1_{\infty,\infty})} < \frac{\varepsilon}{2}. \tag{164}$$

Step 2: splitting the initial data and properties of the solution to the perturbed equation

Using (161), we apply Corollary 1 to show that there exists $p(s,q) > \max(4,q)$, $\hat{\alpha}(s,q) \in (0,\frac{3}{2})$ and $\delta(s,q) \in (0,1-\frac{3}{p})$ such that

$$u_0 = u_0^1 + u_0^2,\ \ u_0^2 \in \dot{B}^{-1+\frac{3}{p}+\delta}_{p,p}(\mathbb{R}^3) \cap J(\mathbb{R}^3) \quad \text{and} \quad u_0^1 \in \dot{H}^{\hat{\alpha}} \cap J(\mathbb{R}^3). \tag{165}$$

From the continuous embedding $L^2_x \hookrightarrow \dot{B}^{-\frac{3}{2}+\frac{3}{p}}_{p,p}$ and Proposition 2 we see that $u_0^2 \in \dot{B}^{-1+\frac{3}{p}}_{p,p} \hookrightarrow VMO^{-1} \hookrightarrow \dot{B}^{-1}_{\infty,\infty}$. Hence,

$$u_0^1 \in \dot{H}^{\hat{\alpha}} \cap VMO^{-1}(\mathbb{R}^3) \cap J(\mathbb{R}^3).$$

Define $U := u - e^{t\Delta}u_0^2$. Using that $u_0^2 \in VMO^{-1}$, the heat-flow characterisation of homogeneous Besov spaces, Remark 1 and the properties of u in step 1, we see that there exists $T(\hat{T},\varepsilon,u_0^2) \in (0,\hat{T}]$ such that U satisfies the following properties. Namely,

$$\sup_{0<s<T} s^{\frac{1}{2}}\|U(\cdot,s)\|_{L^\infty(\mathbb{R}^3)} < \infty, \tag{166}$$

$$U \in L^\infty(0,T;\dot{B}^{-1}_{\infty,\infty}) \tag{167}$$

and

$$\|U\|_{\tilde{L}^1(0,T;\dot{B}^1_{\infty,\infty})} < \varepsilon. \tag{168}$$

From (165), the continuous embedding and the heat flow characterization of homogeneous Besov spaces, we have

$$\sup_{t>0} t^{\frac{1}{2}(1-\delta)}\|e^{t\Delta}u_0^2\|_{L^\infty(\mathbb{R}^3)} \leq C\|u_0^2\|_{\dot{B}^{-1+\frac{3}{p}+\delta}_{p,p}} \text{ and } \|e^{t\Delta}u_0^2\|_{L^2(\mathbb{R}^3)} + 2\int_0^t\int_{\mathbb{R}^3} |\nabla e^{s\Delta}u_0^2|^2 dyds = \|u_0^2\|_{L^2(\mathbb{R}^3)}. \tag{169}$$

The fact that $u(\cdot,u_0)$ is a weak Leray-Hopf solution, together with (165) and (169), allows us to infer that $U := u - e^{t\Delta}u_0^2 \in C_w([0,T];J(\mathbb{R}^3)) \cap L^2(0,T;\dot{H}^1(\mathbb{R}^3))$ is a solution of the following system:

$$\partial_t U - \Delta U + U\cdot\nabla U + e^{t\Delta}u_0^2\cdot\nabla U + U\cdot\nabla e^{t\Delta}u_0^2 + \nabla P = -e^{t\Delta}u_0^2\cdot\nabla e^{t\Delta}u_0^2 \tag{170}$$

$$\operatorname{div} U = 0, \quad U(\cdot,0) = u_0^1 \in \dot{H}^{\hat{\alpha}} \cap VMO^{-1}(\mathbb{R}^3) \cap J(\mathbb{R}^3). \tag{171}$$

Using that $u(\cdot,u_0)$ is a weak Leray-Hopf solution, (165) and (169), we can use known arguments[18] to infer that

$$\|U(\cdot,t)\|^2_{L^2(\mathbb{R}^3)} + 2\int_0^t\int_{\mathbb{R}^3} |\nabla U(y,s)|^2 dyds \leq \|u_0^1\|^2_{L^2(\mathbb{R}^3)} + 2\int_0^t\int_{\mathbb{R}^3} (e^{s\Delta}u_0^2 \otimes U + e^{s\Delta}u_0^2 \otimes e^{s\Delta}u_0^2) : \nabla U dyds. \tag{172}$$

The above properties of U, together with (166) and (169), allows us to apply Proposition 1 (taking $V := e^{t\Delta}u_0^2$). Consequently, there exists $\alpha(\hat{\alpha},\delta) \in (0,\hat{\alpha}]$ such that

$$U \in L^\infty(0,T;H^\alpha(\mathbb{R}^3)). \tag{173}$$

[18] See [1], [5] or [22] for example.

Step 3: properties of the high frequency cut-off operator acting on U

Using (167)-(168) we see that for $j\in\mathbb{N}$

(174)
$$\|\nabla\dot{S}_jU\|_{L^1_TL^\infty_x}\leq C_{univ}\sum_{j'\leq j-1}2^{j'}\|\dot{\Delta}_{j'}U\|_{L^1_TL^\infty_x}$$
$$\leq C_{univ}\Big(j\|U\|_{\tilde{L}^1(0,T;\dot{B}^1_{\infty,\infty})}+\sum_{j'\leq 0}T2^{2j'}(2^{-j'}\|\dot{\Delta}_{j'}U\|_{L^\infty_TL^\infty_x}\Big)\leq C_{univ}j\varepsilon+C_{univ}T\|U\|_{L^\infty(0,T;\dot{B}^{-1}_{\infty,\infty})}.$$

By redefining ε, $\hat{T}$ and T appropriately, we have

$$\|\nabla\dot{S}_jU\|_{L^1_TL^\infty_x}\leq j\varepsilon\log 2+C_{U,T}. \tag{175}$$

Since U belongs to the global energy class, the high frequency cut-off $\dot{S}_j(U)$ also belongs to $C_w([0,T];J(\mathbb{R}^3))\cap L^2(0,T;\dot{H}^1(\mathbb{R}^3))$. Additionally, $\dot{S}_jU$ satisfies the equation in $\mathbb{R}^3\times(0,T)$

(176)
$$\partial_t\dot{S}_jU-\Delta\dot{S}_jU+\dot{S}_jU\cdot\nabla\dot{S}_jU+e^{t\Delta}u_0^2\cdot\nabla\dot{S}_jU+\dot{S}_jU\cdot\nabla e^{t\Delta}u_0^2+\nabla P_j=\nabla\cdot F_j-e^{t\Delta}u_0^2\cdot\nabla e^{t\Delta}u_0^2$$

$$\operatorname{div}\dot{S}_jU=0,\quad \dot{S}_jU(\cdot,0)=\dot{S}_ju_0^1. \tag{177}$$

Here, F_j is defined by (125)-(131). The properties (165), (167)- (168) and (173) allow us apply Proposition 6 and Remark 3. From this one infers that

$$\|F_j\|_{L^2_TL^2_x}\leq C_{U,\delta,u_0,p,\alpha,T}2^{-j\gamma_{\alpha,p,\delta}}. \tag{178}$$

Here, $\gamma_{\alpha,p,\delta}:=\min(\frac{\alpha}{2},\frac{1}{p},\frac{\delta}{2})>0$.

Finally, it is immediate that since U belongs to the energy class we have $\dot{S}_jU\in L^\infty(\mathbb{R}^3\times(0,T))\cap L^\infty(0,T;L^2_x)$. Together with (169) and (178), we get that

$$\dot{S}_jU\otimes\dot{S}_jU+e^{t\Delta}u_0^2\otimes\dot{S}_jU+\dot{S}_jU\otimes e^{t\Delta}u_0^2-F_j+e^{t\Delta}u_0^2\otimes e^{t\Delta}u_0^2\in L^2_TL^2_x.$$

This implies that

$$\dot{S}_jU\in C([0,T];J(\mathbb{R}^3)) \tag{179}$$

and that the following energy equality holds for $t\in[0,T]$:

$$\|\dot{S}_j(U)(\cdot,t)\|^2_{L^2(\mathbb{R}^3)}+2\int_0^t\int_{\mathbb{R}^3}|\nabla\dot{S}_j(U)(y,s)|^2dyds=\|\dot{S}_ju_0^1\|^2_{L^2(\mathbb{R}^3)}$$
$$+2\int_0^t\int_{\mathbb{R}^3}(e^{s\Delta}u_0^2\otimes\dot{S}_j(U)+e^{s\Delta}u_0^2\otimes e^{s\Delta}u_0^2-F_j):\nabla\dot{S}_j(U)dyds. \tag{180}$$

Step 4: Comparing $\dot{S}_jU$ with other weak Leray-Hopf solutions

Let $v(\cdot,v_0)$ be *any* weak Leray-Hopf solution to the Navier-Stokes equations with initial data $v_0\in J(\mathbb{R}^3)$. Define $v_0^1:=v_0-u_0^2\in J(\mathbb{R}^3)$ and $V:=v(\cdot,v_0)-e^{t\Delta}u_0^2$. Utilizing the same reasoning applied to U in Step 2, we see that $V\in C_w([0,T];J(\mathbb{R}^3))\cap L^2_T\dot{H}^1$ satisfies the following properties in $\mathbb{R}^3\times(0,T)$. Namely,

$$\partial_tV-\Delta V+V\cdot\nabla V+e^{t\Delta}u_0^2\cdot\nabla V+V\cdot\nabla e^{t\Delta}u_0^2+\nabla\Pi=-e^{t\Delta}u_0^2\cdot\nabla e^{t\Delta}u_0^2, \tag{181}$$

$$\operatorname{div}V=0,\quad V(\cdot,0)=v_0^1\in J(\mathbb{R}^3) \tag{182}$$

and for $t \in [0,T]$

(183)

$$\|V(\cdot,t)\|^2_{L^2(\mathbb{R}^3)}+2\int\limits_0^t\int\limits_{\mathbb{R}^3}|\nabla V(y,s)|^2dyds \leq \|v_0^1\|^2_{L^2(\mathbb{R}^3)}+2\int\limits_0^t\int\limits_{\mathbb{R}^3}(e^{s\Delta}u_0^2\otimes V+e^{s\Delta}u_0^2\otimes e^{s\Delta}u_0^2):\nabla V dyds.$$

Now notice that $v(\cdot,v_0)-u(\cdot,u_0)\equiv V(\cdot,v_0^1)-U(\cdot,u_0^1)$ and $v_0-u_0\equiv v_0^1-u_0^1$. Therefore, to prove Theorem 1 it is sufficient to show that for all $\eta\in(0,1)$ there exists positive $T(\eta,u_0,U,s,q)$ and $C(T,U,u_0,s,q)$ such that for all $t\in[0,T]$ we have

$$\frac{1}{2}\|V(t)-U(t)\|^2_{L_2}+\int\limits_0^t\|\nabla(V-U)(t')\|^2_{L_2}dt'\leq C(T,U,u_0,s,q)\|v_0^1-u_0^1\|^{2-2\eta}_{L^2}. \tag{184}$$

Following Chemin's idea in [9], we now compare V with the high frequency cut-off of U $\dot{S}_j(U)$. Specifically, define $W_j:=V-\dot{S}_jU$. Then $W_j\in C_w([0,T];J(\mathbb{R}^3))\cap L^2(0,T;\dot{H}^1(\mathbb{R}^3))$ is a weak solution to the following equation. Namely,

(185)

$$\partial_t W_j-\Delta W_j+W_j\cdot\nabla W_j+e^{t\Delta}u_0^2\cdot\nabla W_j+W_j\cdot\nabla e^{t\Delta}u_0^2+W_j\cdot\nabla\dot{S}_jU+\dot{S}_jU\cdot\nabla W_j+\nabla\Pi_j=-\nabla\cdot F_j$$

$$\operatorname{div}W_j=0,\quad W_j(\cdot,0)=v_0^1-\dot{S}_ju_0^1\in J(\mathbb{R}^3). \tag{186}$$

Since $U\in L^\infty_T L^2_x$ we have that for every $k\in\mathbb{N}$ that

$$\nabla^k\dot{S}_j(U)\in L^\infty(\mathbb{R}^3\times(0,T)). \tag{187}$$

Using $U:=u-e^{t\Delta}u_0^2$, $u_0^2\in J(\mathbb{R}^3)$ and Proposition 4, we see that for $\lambda\in(0,T)$ and $k=0,1\ldots$, $l=0,1\ldots$:

$$\partial_t^l\nabla^k\dot{S}_jU\in L^\infty(\mathbb{R}^3\times(\lambda,T)). \tag{188}$$

From (179), (187)-(188) and the fact that V and $\dot{S}_jU$ satisfy global energy inequalities, standard arguments (see [22], [5] or [1] for example) imply the following. Namely that for $t\in[0,T]$, W_j satisfies the global energy inequality

(189)

$$\frac{1}{2}\|W_j(\cdot,t)\|^2_{L^2(\mathbb{R}^3)}+\int\limits_0^t\int\limits_{\mathbb{R}^3}|\nabla W_j|^2dyds$$

$$\leq\frac{1}{2}\|v_0^1-\dot{S}_ju_0^1\|^2_{L^2(\mathbb{R}^3)}+\int\limits_0^t\int\limits_{\mathbb{R}^3}(F_j+(e^{t\Delta}u_0^2\otimes W_j)):\nabla W_jdyds-\int\limits_0^t\int\limits_{\mathbb{R}^3}(W_j\cdot\nabla\dot{S}_jU)\cdot W_jdyds.$$

Step 5: Conclusion

Applying the Hölder and Young inequality to (189) yields

(190)

$$\|W_j(\cdot,t)\|^2_{L^2(\mathbb{R}^3)}+\int\limits_0^t\int\limits_{\mathbb{R}^3}|\nabla W_j|^2dyds\leq C_{univ}\|v_0^1-\dot{S}_ju_0^1\|^2_{L^2(\mathbb{R}^3)}+C_{univ}\int\limits_0^t\int\limits_{\mathbb{R}^3}|F_j(y,s)|^2dyds$$

$$+C_{univ}\int\limits_0^t\|W_j(\cdot,s)\|^2_{L^2(\mathbb{R}^3)}\Big(\|e^{t\Delta}u_0^2\|^2_{L^\infty(\mathbb{R}^3)}+\|\nabla\dot{S}_jU(\cdot,s)\|_{L^\infty(\mathbb{R}^3)}\Big)ds.$$

Using (169) and (178), we have that for $t \in (0,T)$

(191)

$$\|W_j(\cdot,t)\|^2_{L^2(\mathbb{R}^3)} + \int\limits_0^t \int\limits_{\mathbb{R}^3} |\nabla W_j|^2 dyds \leq C_{univ}\|v_0^1 - \dot{S}_j u_0^1\|^2_{L^2(\mathbb{R}^3)} + C_{U,\delta,u_0,p,\alpha,T} 2^{-2j\gamma_{\alpha,p,\delta}}$$

$$+ C_{univ}\int\limits_0^t \|W_j(\cdot,s)\|^2_{L^2(\mathbb{R}^3)}\Big(\frac{\|u_0^2\|^2_{\dot{B}^{-1+\delta}_{\infty,\infty}}}{s^{1-\delta}} + \|\nabla \dot{S}_j U(\cdot,s)\|_{L^\infty(\mathbb{R}^3)}\Big) ds.$$

Applying Gronwall's lemma gives that for $t \in [0,T]$

$$\|W_j(\cdot,t)\|^2_{L^2(\mathbb{R}^3)} + \int\limits_0^t \int\limits_{\mathbb{R}^3} |\nabla W_j|^2 dyds \leq (C_{univ}\|v_0^1 - \dot{S}_j u_0^1\|^2_{L^2(\mathbb{R}^3)}$$

$$+ C_{U,\delta,u_0,p,\alpha,T} 2^{-2j\gamma_{\alpha,p,\delta}}) \exp\Big(C'_{univ}\int\limits_0^t \frac{\|u_0^2\|^2_{\dot{B}^{-1+\delta}_{\infty,\infty}}}{s^{1-\delta}} + \|\nabla \dot{S}_j U(\cdot,s)\|_{L^\infty(\mathbb{R}^3)} ds\Big) \tag{192}$$

$$\leq C_{U,u_0,p,\delta,\alpha,T}(\|v_0^1 - \dot{S}_j u_0^1\|^2_{L^2(\mathbb{R}^3)} + 2^{-2j\gamma_{\alpha,p,\delta}}) \exp\Big(C'_{univ}\|\nabla \dot{S}_j U\|_{L^1_T L^\infty}\Big).$$

Putting $\hat{\varepsilon} := C'_{univ}\varepsilon$ and using (175) gives

(193)

$$\|W_j(\cdot,t)\|^2_{L^2(\mathbb{R}^3)} + \int\limits_0^t \int\limits_{\mathbb{R}^3} |\nabla W_j|^2 dyds \leq C'_{U,u_0,p,\delta,\alpha,T}(\|v_0^1 - \dot{S}_j u_0^1\|^2_{L^2(\mathbb{R}^3)} + 2^{-2j\gamma_{\alpha,p,\delta}})2^{j\hat{\varepsilon}}$$

Notice that the same reasoning as we used to get (193) applies to the high frequencies $U - \dot{S}_j(U)$ of U. In that case one has

(194)

$$\|U-\dot{S}_j(U)(\cdot,t)\|^2_{L^2(\mathbb{R}^3)} + \int\limits_0^t \int\limits_{\mathbb{R}^3} |\nabla U-\dot{S}_j(U)|^2 dyds \leq C'_{U,u_0,p,\delta,\alpha,T}(\|u_0^1-\dot{S}_j u_0^1\|^2_{L^2(\mathbb{R}^3)} + 2^{-2j\gamma_{\alpha,p,\delta}})2^{j\hat{\varepsilon}}$$

Noting that $V - U \equiv W_j - (U - \dot{S}_j(U))$ and $v_0^1 - u_0^1 \equiv v_0^1 - \dot{S}_j u_0^1 + (\dot{S}_j u_0^1 - u_0^1)$, we can combine (193)-(194) to get

$$\|(V-U)(\cdot,t)\|^2_{L^2(\mathbb{R}^3)} + \int\limits_0^t \int\limits_{\mathbb{R}^3} |\nabla(V-U)|^2 dyds$$

$$\leq C''_{U,u_0,p,\delta,\alpha,T}(\|u_0^1 - \dot{S}_j u_0^1\|^2_{L^2(\mathbb{R}^3)} + \|u_0^1 - v_0^1\|^2_{L^2_x} + 2^{-2j\gamma_{\alpha,p,\delta}})2^{j\hat{\varepsilon}}. \tag{195}$$

Recall from (165) that $u_0^1 \in \dot{H}^{\hat{\alpha}}$ with $\hat{\alpha} \in (0, \frac{3}{2})$. Thus,

$$\|u_0^1-\dot{S}_j u_0^1\|_{L^2(\mathbb{R}^3)} \leq \sum_{j'\geq j}(2^{j'\hat{\alpha}}\|\dot{\Delta}_{j'} u_0^1\|_{L^2_x})2^{-j'\hat{\alpha}} \leq \|u_0^1\|_{\dot{H}^{\hat{\alpha}}}\sum_{j'\geq j} 2^{-j'\hat{\alpha}} \leq C(\hat{\alpha})\|u_0^1\|_{\dot{H}^{\hat{\alpha}}}2^{-j\hat{\alpha}}.$$

Combining this with (195) gives that for all $j \in \mathbb{N}$ we have

(196)

$$\|(V-U)(\cdot,t)\|^2_{L^2(\mathbb{R}^3)} + \int\limits_0^t \int\limits_{\mathbb{R}^3} |\nabla(V-U)|^2 dyds \leq C'''_{U,u_0,p,\delta,\alpha,T}(\|u_0^1-v_0^1\|^2_{L^2(\mathbb{R}^3)} + 2^{-2j\min(\gamma_{\alpha,p,\delta},\hat{\alpha})})2^{j\hat{\varepsilon}}.$$

For any fixed $\eta \in (0,1)$, we take $\hat{\varepsilon}$ to satisfy

$$\hat{\varepsilon} = 2\eta \min(\gamma_{\alpha,p,\delta}, \hat{\alpha})). \tag{197}$$

Now, we treat two cases

(1) $u_0^1 = v_0^1$
(2) $0 < \|u_0^1 - v_0^1\|_{L^2_x} < 1$.

In the first case, we have

$$\|(V-U)(\cdot,t)\|^2_{L^2(\mathbb{R}^3)} + \int_0^t \int_{\mathbb{R}^3} |\nabla(V-U)|^2 dyds \leq C'''_{U,u_0,p,\delta,\alpha,T} 2^{-2j\min(\gamma_{\alpha,p,\delta},\hat{\alpha})+j\hat{\varepsilon}}. \tag{198}$$

With the smallness assumption (197), we see that taking $j \uparrow \infty$ results in $V \equiv U$ in $\mathbb{R}^3 \times (0,T)$. This recovers the author's weak-strong uniqueness result in [4].

In the second case, take

$$j := \left\lceil \frac{-\log_2(\|u_0^1 - v_0^1\|_{L^2_x})}{\min(\gamma_{\alpha,p,\delta}, \hat{\alpha})} \right\rceil > 0. \tag{199}$$

We then get that

$$2^{-2j\min(\gamma_{\alpha,p,\delta},\hat{\alpha})} \leq \|u_0^1 - v_0^1\|^2_{L^2_x} \tag{200}$$

and

$$2^{j\hat{\varepsilon}} \leq \frac{2^{\hat{\varepsilon}}}{\|u_0^1 - v_0^1\|^{2\eta}_{L^2_x}}. \tag{201}$$

Substituting (200)-(201) into (196) gives

$$\|(V-U)(\cdot,t)\|^2_{L^2(\mathbb{R}^3)} + \int_0^t \int_{\mathbb{R}^3} |\nabla(V-U)|^2 dyds \leq C_{U,u_0,p,\delta,\alpha,T,\hat{\varepsilon}} \|u_0^1 - v_0^1\|^{2(1-\eta)}_{L^2_x} \tag{202}$$

Putting cases 1 and 2 together gives (184). As explained in Step 4, this implies the conclusion of Theorem 1.

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(T. Barker) DMA, ÉCOLE NORMALE SUPÉRIEURE, CNRS, PSL RESEARCH UNIVERSITY, 75 005 PARIS

Email address: tobiasbarker5@gmail.com