Towards Cereceda's conjecture for planar graphs

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Abstract

The reconfiguration graph $R_k(G)$ of the k-colourings of a graph G has as vertex set the set of all possible k-colourings of G and two colourings are adjacent if they differ on the colour of exactly one vertex.

Cereceda conjectured ten years ago that, for every k-degenerate graph G on n vertices, $R_{k+2}(G)$ has diameter $\mathcal{O}(n^2)$. The conjecture is wide open, with a best known bound of $\mathcal{O}(k^n)$, even for planar graphs. We improve this bound for planar graphs to $2^{\mathcal{O}(\sqrt{n})}$. Our proof can be transformed into an algorithm that runs in $2^{\mathcal{O}(\sqrt{n})}$ time.

1 Introduction

Let G be a graph, and let k be a non-negative integer. A k-colouring of G is a function $f : V(G) \to \{1, \dots, k\}$ such that $f(u) \neq f(v)$ whenever $(u, v) \in E(G)$. The reconfiguration graph $R_k(G)$ of the k-colourings of G has as vertex set the set of all k-colourings of G and two vertices of $R_k(G)$ are adjacent if they differ on the colour of exactly one vertex. Let d be a positive integer. Then G is said to be d-degenerate if every subgraph of G contains a vertex of degree at most d. Expressed differently, G is d-degenerate if there is an ordering $v_1, \dots, v_n$ of its vertices such that v_i has at most d neighbours

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v_j with $j < i$. Note that each d-degenerate graph G has a $(d+1)$-colouring, and so $R_k(G)$ is well-defined for each $k > d$.

In the past decade, the study of reconfiguration graphs for graph colourings has been the subject of much attention. One typically asks whether the reconfiguration graph is connected. If so, what is its diameter and, in case it is not, what is the diameter of its connected components? See [4, 8, 9] for some examples. Computational work has focused on deciding whether there is a path in the reconfiguration graph between a given pair of colourings [5, 10, 14]. Other structural considerations of the reconfiguration graph have also been investigated in [1, 2]. Reconfiguration graphs have also been studied for many other decision problems; see [16] for a recent survey.

We remark that reconfiguration problems for graph colourings do not have known results for which the reconfiguration graph is connected but has a diameter that is not polynomial in the order of the graph. (In nearly all cases, the diameter turns out to be quadratic in the number of vertices.) On the other hand, the problem of deciding whether a pair of colourings are in the same component of the reconfiguration graph tends to be PSPACE-complete whenever the reconfiguration graph is disconnected. There are exceptions to this pattern such as, for example, deciding whether a pair of 3-colourings of a graph belong to the same component [10].

Given a d-degenerate graph G, it is not difficult to show that $R_{d+2}(G)$ is connected [13]. The foregoing pattern motivated Cereceda [7] to conjecture that $R_{d+2}(G)$ has diameter that is quadratic in the order of G.

Conjecture 1. *Let d be a positive integer, and let G be a d-degenerate graph on n vertices. Then $R_{d+2}(G)$ has diameter $\mathcal{O}(n^2)$.*

Conjecture 1 has resisted some efforts and has only been verified (other than for trees) for graphs with degeneracy at least $\Delta - 1$ where Δ denotes the maximum degree of the graph [12]. It is also known to hold if degeneracy is replaced by tree-width [3].

In the expectation of the difficulty of Conjecture 1, Bousquet and Perarnau [6] have shown that for every $d \geq 1$ and $\epsilon > 0$ and every graph G with maximum average degree $d - \epsilon$, the diameter of $R_{d+1}(G)$ is $\mathcal{O}(n^c)$ for some constant $c = c(d, \epsilon)$ (see [11] for a short proof). Their result in particular implied that the reconfiguration graph of 8-colourings for planar graphs has a diameter that is polynomial in the order of the graph. Since planar graphs are 5-degenerate, the one outstanding case of Conjecture 1 restricted to planar graphs is thus $k = 7$ (aside, of course, from improving the constant term in the exponent of the diameter). On the other hand, the best known upper bound on the diameter in Conjecture 1 is $\mathcal{O}(d^n)$ – even for planar graphs – and this follows from [13]. In this note, we significantly improve this bound for planar graphs.

Theorem 1. *For every planar graph G on n vertices, $R_7(G)$ has diameter at most $2^{\mathcal{O}(\sqrt{n})}$.*

1.1 An overview of the proof

We sketch the key steps of the proof. The main idea is based on the approach by Feghali, Johnson, and Paulusma [12]. Namely, to find a recolouring sequence that goes through a colouring that uses less colours. More precisely, given two 7-colourings α and β of a planar graph G, to prove the theorem it suffices to describe a sequence of (sub-exponentially many) recolourings between α and β. Our general strategy will be to describe, in a first stage, a recolouring sequence σ_1 from α to some 5-colouring γ_1 of G and a recolouring sequence σ_2 from β to some 5-colouring γ_2. In a second and final stage, we will use the two missing colours in γ_1 and γ_2 to find a recolouring sequence σ_3 from γ_1 to γ_2. The final solution is then obviously the concatenation of σ_1, σ_3, and (the reverse of) σ_2. We now give the details.

The most difficult part of the proof is Lemma 2, which states that we can find a recolouring sequence from a 7-colouring of G to some 6-colouring by subexponentially many recolourings. To achieve this, we first show, in Lemma 1, that there always exists a recolouring sequence from a 7-colouring of G to some 6-colouring by at most πn^2 recolourings, where π is the product of degrees of vertices having 'high' degree. Since the maximum average degree of a planar graph is strictly less than 6, we can then show, in Lemma 2, that this implies $\pi \leq 2^{\mathcal{O}(\sqrt{n})}$. We note that this part of the proof does not exploit at all the planarity of G – all that is needed is for our graph to have maximum average degree bounded away by one from the number of colours. Now, to further recolour the 6-colouring obtained in the previous step to some 5-colouring, we essentially follow the approach from [11], that is, to find a 'large' independent set I of G such that each vertex of I has 'low' degree and then apply induction on $G - I$; some difficulties that are not present in [11] must, however, be deal with. Namely, to extend the recolouring sequence in $G - I$ to a recolouring sequence in G, it is important to ensure that the neighbourhood of every vertex in I contains at most 5 colours. It is here that we crucially rely on the planarity of G.

In the final step, we will use the two missing colours in γ_1 and γ_2 to recolour a subset $S \subseteq V(G)$ such that $G - S$ has a nice structure that can be easily used to quickly finish the proof. Specifically, we combine known results by Mihók, Thomassen, and Wood [15, 17, 18] to find two independent sets I_1 and I_2 such that $G - I_1 - I_2$ is 2-degenerate. Then we recolour I_1 with colour 6, I_2 with colour 7 and then use a result of Cereceda [7] to recolour γ_1 to γ_2 in $G - I_1 - I_2$ by $\mathcal{O}(n^2)$ recolourings.

2 The proof of Theorem 1

We begin with the following three lemmas. In the first lemma, we obtain a crude bound on the number of recolourings required for degenerate graphs to reduce the number of colours by one.

Lemma 1. *Let $k \geq 1$, and let G be a k-degenerate graph on n vertices. Let $\{u_1, \ldots, u_s\}$ be the set of vertices of G of degree at least $k+2$. If α is a $(k+2)$-colouring of G, then we can recolour α to some $(k+1)$-colouring of G by at most $\mathcal{O}(n^2 \prod_{i=1}^{s} deg(u_i))$ recolourings.*

Proof. The proof is a generalisation of the one found in [12].

Fix a k-degenerate ordering $\sigma = v_1, \ldots, v_n$ of G and assume without loss of generality that u_i appears before u_j in σ whenever $i < j$. To avoid any confusion, when considering a subset of vertices $v_{i_1}, \ldots, v_{i_t}$ with $i_1 < i_2 < \cdots < i_t$ we refer to v_{i_1} as the *leftmost* vertex in the subset and to v_{i_t} as the *rightmost* vertex.

Given an index $h \in [n]$, we will describe an algorithm RECOLOUR(h) that outputs a sequence of recolourings with the following properties:

(i) for $i < h$, v_i is not recoloured,

(ii) for $i > h$, v_i is recoloured at most $\prod_{j=\ell}^{s} \deg(u_j)$ times, where u_ℓ is the leftmost vertex of degree at least $k+2$ with index at least h in σ, and

(iii) v_h is recoloured once to a different colour.

Note that the algorithm takes $\mathcal{O}(n \prod_{i=1}^{s} \deg(u_i))$ recolourings to recolour v_h. Hence, by repeatedly using such a sequence on the lowest index of a vertex with colour $k+2$, we can obtain a colouring in which colour $k+2$ does not appear by $\mathcal{O}(n^2 \prod_{i=1}^{s} \deg(u_i))$ recolourings, and the lemma follows. The description of the algorithm is found below.

ALGORITHM 1: RECOLOUR(h)

Data: A graph G, a k-degenerate ordering σ of G and index $h \in [n]$
Result: A sequence of recolourings satisfying the properties (i), (ii), and (iii)

1 **if** *there is a colour c that is not used on v_h or any of its neighbours* **then**
2   recolour v_h to c and terminate;
3 **end**
4 **else if** *v_h has degree exactly $k+1$* **then**
5   let v_i be the rightmost neighbour of v_h;
6   let c be the colour of v_i;
7   call RECOLOUR(i);
8   recolour v_h to c;
9 **end**
10 **else if** *v_h has degree at least $k+2$* **then**
11   let c be a colour not appearing on v_h or any of its neighbours v_j with $j < h$;
12   let $v_{i_1}, \dots, v_{i_t}$ be the neighbours of v_h with $h < i_1 < i_2 < \cdots < i_t$;
13   **for** $j \in [t]$ *in ascending order* **do**
14   **if** *the colour of v_{i_j} is c* **then**
15   call RECOLOUR(i_j);
16   **end**
17   **end**
18   recolour v_h to c;
19 **end**

We prove the correctness and properties (i)–(iii) of the algorithm by induction on $n - h$. Obviously, whenever the condition on Line 1 of RECOLOUR(h) applies, we only recolour v_h once to a colour not appearing on it or any of its neighbours. Since σ is a k-degenerate ordering, this is the case when $h = n$ and the algorithm is correct and satisfies (i)–(iii) for $n - h = 0$.

For the induction step, assume that $h < n$, every colour is used either on v_h or its neighbour, and for all $i > h$ the algorithm RECOLOUR(i) is correct and satisfies properties (i)–(iii).

Since every colour appear either on v_h or its neighbour, the degree of v_h is at least $k + 1$.

Case 1: v_h has degree $k+1$. Then the condition on Line 4 applies and RECOLOUR(h) performs the steps on Lines 5–8. Since there are $k+2$ colours and each colour appears either on v_h or on one of its neighbours, each colour appears precisely once in the closed neighbourhood of v_h. As v_i is the right-most neighbour of v_h and σ is k-degenerate ordering, we have that $i > h$. This implies that RECOLOUR(i) recolours v_i, but none of the other neighbours of v_h; therefore the colour c does not appear on any neighbour of v_h and we can safely recolour it to c on Line 8. This also implies that (i) and (ii) and (iii) follow from (i) and (ii) for RECOLOUR(i).

Case 2: v_h has degree at least $k+2$. Then RECOLOUR runs the steps on Lines 10–18. Since the algorithm applies the recursive calls only on the vertices with greater index than h in σ, (i) holds.

After the execution of the for-loop between Line 13 and Line 17, colour c no longer appears on v or any of its neighbours, after which v_h is recoloured to c on Line 18. So the algorithm is correct and (iii) holds. To prove (ii), note that the algorithm calls RECOLOUR(i) at most $\deg(v_h)$ times and only when $i > h$. Hence, each vertex will get recoloured at most

$$\deg(v_h) \prod_{j=\ell+1}^{s} \deg(u_j) = \prod_{j=\ell}^{s} \deg(u_j),$$

since $u_\ell = v_h$, and (ii) follows. This completes the proof of the lemma. □

The maximum average degree of a graph G is defined as

$$\mathrm{mad}(G) = \max\left\{\frac{\sum_{v\in V(H)}\deg(v)}{|V(H)|} : H \subseteq G\right\},$$

By Euler's formula, the maximum average degree of a planar graph is strictly less than six. By definition, if a graph has maximum average degree strictly less than k for some positive integer k, then this graph is also $(k-1)$-degenerate.

In our next lemma, we show that we can reduce the number of colours by one using subexponentially many recolourings if we further assume our graph to have bounded maximum average degree.

Lemma 2. *Suppose $k \geq 2$ is an integer and G is a a graph with n vertices and mad(G) $< k+1$. If α is a $(k+2)$-colouring of G, then we can recolour α to some $(k+1)$-colouring of G by $k^{\mathcal{O}(k^2\sqrt{n})}$ recolourings per vertex.*

Proof. We proceed by induction on $n = |V(G)|$. (Our proof can be transformed into a recursive algorithm running in time $k^{\mathcal{O}(k^2\sqrt{n})}$.)

Suppose H is a graph with $\mathrm{mad}(H) < k+1$ that contains at most $2(k+1)\sqrt{|V(H)|}$ vertices of degree at most k. Let α^H be a $(k+2)$-colouring of H, and let $h = |V(H)|$.

Claim 1. *We can recolour α^H to some $(k+1)$-colouring of H by $k^{\mathcal{O}(k^2\sqrt{h})}$ recolourings.*

Proof of Claim. Let $U = \{u_1, \ldots, u_s\}$ be the set of vertices of degree at least $k+2$ in H, let $W = \{w_1, \ldots, w_t\}$ be the set of vertices of degree less than or equal to k, and let $Z = V(H) \setminus (U \cup W) = \{z_1, \ldots, z_{h-s-t}\}$ be the set of vertices of degree precisely $k+1$. Due to Lemma 1, to prove the claim it suffices to show that $\prod_{i=1}^{s} \deg(u_i) \leq k^{\mathcal{O}(k^2\sqrt{h})}$.

We can assume that H is connected, because otherwise we can prove the claim for each connected component of H. Thus,

$$\sum_{i=1}^{s} \deg(u_i) + \sum_{i=1}^{t} \deg(w_i) \;\geq\; \sum_{i=1}^{s} \deg(u_i) + t. \tag{1}$$

On the other hand, since $\mathrm{mad}(G) < k+1$,

$$\begin{aligned}
& \sum_{i=1}^{s} \deg(u_i) + \sum_{i=1}^{t} \deg(w_i) + \sum_{i=1}^{h-s-t} \deg(z_i) < (k+1)h \\
\Longleftrightarrow \quad & \sum_{i=1}^{s} \deg(u_i) + \sum_{i=1}^{t} \deg(w_i) + (k+1)(h-s-t) < (k+1)(h-s-t+s+t) \\
\Longleftrightarrow \quad & \sum_{i=1}^{s} \deg(u_i) + \sum_{i=1}^{t} \deg(w_i) < (k+1)(s+t). \qquad (2)
\end{aligned}$$

Combine Inequalities (1) and (2):

$$t + \sum_{i=1}^{s} \deg(u_i) < (k+1)(s+t) \Longrightarrow (k+2)s + t < (k+1)(s+t) \Longleftrightarrow s < kt.$$

Since, by assumption, $t < 2(k+1)\sqrt{h}$, it follows that $s < 2k(k+1)\sqrt{h}$. Substituting these bounds into $(k+1)(s+t) - t$ gives us

$$\sum_{i=1}^{s} \deg(u_i) < k(k+1)^2 2\sqrt{h} + k(k+1)2\sqrt{h} \leq 4k(k+1)^2\sqrt{h} = a. \tag{3}$$

By the AM-GM inequality of arithmetic and geometric means, it holds that

$$\frac{\sum_{i=1}^{s} \deg(u_i)}{s} \geq \left(\prod_{i=1}^{s} \deg(u_i) \right)^{s^{-1}}. \tag{4}$$

Combining Inequalities (3) and (4) we get

$$\frac{a}{s} > \left(\prod_{i=1}^{s} \deg(u_i) \right)^{s^{-1}},$$

or, since both sides of the inequality are positive, equivalently

$$f(s) = \left(\frac{a}{s} \right)^{s} > \prod_{i=1}^{s} \deg(u_i).$$

It remains to find an upper bound for the expression $f(s)$ when s is between 1 and $k(k+1)2\sqrt{h}$. The derivative $f'(s) = \partial f(s)/\partial s$ of $f(s)$ at s is given by

$$f'(s) = \left(\frac{a}{s} \right)^{s} \cdot \left(\log \left(\frac{a}{s} \right) - 1 \right),$$

and since $f'(s)$ is positive for each $s \in [1, 2k(k+1)\sqrt{h}]$, it follows that $f(s)$ is maximized when $s = k(k+1)2\sqrt{h}$. Therefore, we obtain

$$(2(k+1))^{k(k+1)2\sqrt{h}} > \prod_{i=1}^{s} \deg(u_i),$$

finishing the proof of the claim. $\square$

For the inductive step, suppose that G contains more that $2(k+1)\sqrt{n}$ vertices of degree at most k and that we can recolour any subgraph H of G with $h < n$ vertices to some $(k+1)$-colouring γ^H such that each vertex gets recoloured at most $k^{\mathcal{O}(k^2\sqrt{h})}$ times. Let S be an independent set in G containing only vertices of degree at most k of size at least $2\sqrt{n}$. Since G can be greedily coloured with $(k+1)$ colours using its k-degenerate ordering and G contains more than $2(k+1)\sqrt{n}$ vertices of degree at most k, such a set S exists and can be found in polynomial time. By the inductive hypothesis we can recolour the graph $H = G - S$ to some $(k+1)$-colouring such that each vertex get recoloured at most $k^{ck^2\sqrt{h}}$ times for some constant $c > 1$. We can extend this sequence of recolourings of H to a sequence in G by recolouring a vertex $u \in S$ whenever some neighbour of u gets recoloured to its colour (this is possible because the number of colours is $k+2$ and u has at most k neighbours in G). At the end of the sequence, we can recolour each vertex of S to a colour other than $k+2$. It follows that the maximum number $f(n)$ of times a vertex of G is recoloured satisfies

$$f(n) \leq k \cdot \left(k^{ck^2\sqrt{h}}\right) + 1 \leq k \cdot \left(k^{ck^2\sqrt{n-2\sqrt{n}}}\right) + 1 = k^{ck^2\sqrt{n-2\sqrt{n}}+1} + 1. \quad (5)$$

Since $k \geq 2$, to show that $f(n) \leq k^{ck^2\sqrt{n}}$, it suffice to show that $ck^2\sqrt{n} > ck^2\sqrt{n-2\sqrt{n}} + 1$ for each $n \geq 4$. Adding -1 to both sides of (5) and then squaring yields the result. □

In our next lemma, we adapt the proof method introduced in [11] to show that we can further reduce the number of colours by one for planar graphs.

Lemma 3. *Let G be a planar graph, and let γ be a 6-colouring of G. Then we can recolour γ to some 5-colouring of G using seven colours by at most $\mathcal{O}(n^c)$ recolourings for some constant $c > 1$.*

Proof. Let H be any subgraph of G, and let $h = |V(H)|$. An independent set I of H is said to be *special* if it contains at least $h/49$ vertices and every vertex of I has at most 6 neighbours in $G - I$. Let S be the set of vertices of H of degree at most 6. Then S has at least $h/7$ vertices since otherwise

$$\sum_{v \in V(H)} \deg(v) \geq \sum_{v \in V(H)-S} \deg(v) > 7\left(h - \frac{h}{7}\right) = 6h,$$

which contradicts that $\text{mad}(G) < 6$. Let $I \subseteq S$ be a maximal independent subset of S. Each vertex of I has at most 6 neighbours in S and every vertex of $S - I$ has at least one neighbour in I. Therefore, $|I| + 6|I| \geq |S|$ and so I is a special independent set as needed.

Let us prove by induction on the order of G that there is a sequence of recolourings from a 6-colouring γ of G to some 5-colouring of G. We will then argue that at most $\mathcal{O}(n^c)$ recolourings have been performed for some constant $c > 1$, thereby finishing the proof.

Let I be a special independent set of G, and let G^* be the graph obtained from G by

- removing all vertices of degree at most 5 in I from G and
- for each vertex v in I of degree 6, deleting v and identifying a pair of neighbours of v that are coloured alike in γ (such a pair always exists since at most 6 colours appear on v and its neighbours).

Notice that G^* is planar. Indeed, one can think of some embedding of G in the plane and then note that the neighbours of any vertex v form part of the boundary of a face F in $G - v$; thus, indentifying a pair of neighbours of v inside the interior of F in the graph $G - v$ does not break the planarity.

Let γ' denote the colouring of G^* that agrees with γ on $V(G^*) \cap V(G)$ and such that, for each $z \in V(G^*) \setminus V(G)$, if z is the vertex obtained by the identification of some vertices x and y of G, then $\gamma'(z) = \gamma(x)(= \gamma(y))$. Graph G^* has less vertices than G, so can we apply our induction hypothesis to find a sequence of recolourings from γ' to some 5-colouring γ'' of G^*.

We let $\gamma^\star$ be the 5-colouring of $G-I$ that agrees with γ'' on $V(G)\cap V(G^*)$ and such that, for each pair of vertices $x, y \in V(G)$ identified into a new vertex z, $\gamma^\star(z) = \gamma''(x) = \gamma''(y)$. We can transform γ' to $\gamma^\star$ by

- recolouring x and y using the same recolouring as z for every pair $x, y \in V(G) - I$ identified into a vertex $z \in V(G^*)$;
- recolouring each $v \in V(G^*) \cap V(G)$ using the same recolouring.

We can extend this sequence to G by recolouring each vertex of I to a colour from $\{1, \ldots, 7\}$ not appearing on it or its neighbours (this is possible since each vertex of I either has degree at most 5 or has degree 6 but with at least two neighbours that are in some sense always coloured alike). At the end of this sequence, we recolour each vertex of I of colour 7 to another colour (this is again possible by the same reasoning). So our aim of transforming into a 5-colouring is achieved unless some vertex of I has colour 6.

Suppose that there is a vertex v of I with colour 6. We emulate the proof of the 5-Colour Theorem to show that we can recolour v to a colour from $\{1, \ldots, 5\}$ without introducing *new* vertices of colour 6 or 7. By repeating the same procedure at most $|I|$ times, we can transform γ into a 5-colouring of G, as needed. For this, we require some definitions.

Let i and j be two colours. Then a component C of a subgraph of G induced by colours i and j is called an (i, j)-component. Suppose that C is an (i, j)-component, $7 \notin \{i, j\}$. Then colours i and j are said to be *swapped* on C if the vertices coloured j are recoloured 7, then the vertices coloured i are recoloured j, and finally the vertices initially coloured j are recoloured i. Since no vertex coloured i or j in C is adjacent to a vertex of colour 7, it is clear that each colouring is proper and that no new vertices of colour 7 are introduced.

If for a vertex v at least one colour in $\{1, \dots, 5\}$ does not appear in its neighbourhood, we can immediately recolour v. So we can assume that v has either degree 5 or 6 with precisely five neighbours $v_1, \dots, v_5$ coloured distinctly; if a colour appears more than once in the neighbourhood of v, then we assume without loss of generality it is the colour of v_5. Suppose also that these neighbours appear in this order in a plane embedding of G. Let i denote the colour of v_i ($i = 1, \dots, 5$). If the $(1, 3)$-component $C_{1,3}$ that contains v_1 does not contain v_3, we swap colours 1, 3 on $C_{1,3}$ (this is possible since colour 7 is not used on G), which in turn allows us to recolour v to 1. So we can assume that $C_{1,3}$ contains both v_1 and v_3. In the same vein, the vertices v_2 and v_4 must be contained in the same $(2, 4)$-component $C_{2,4}$. By the Jordan Curve Theorem, this is impossible. Hence, either $C_{1,3}$ does not contain both v_1 and v_3 or $C_{2,4}$ does not contain both v_2 and v_4 and we are able to recolour v, as required.

We now estimate the number of recolourings of a vertex $v \in I$ in terms of the number of recolourings of vertices of $G - I$. When recolouring γ to a 6-colouring β that uses only colours 1 to 5 on $G - I$, v is recoloured at most five more times than any of its neighbours (this bound is achieved if v is recoloured every time one of its neighbours is recoloured and these neighbours are recoloured the same number of times). Moreover, recolouring β to a 5-colouring of G contributes an additional $\mathcal{O}(n)$ recolourings per vertex. Therefore, the maximum number $f(n)$ of recolourings per vertex satisfies the recurrence relation

$$f(n) \leq 5 \cdot f\left(n - \frac{n}{49}\right) + \mathcal{O}(n),$$

and the lemma follows by the master theorem. □

We also require some auxiliary results whose algorithmic versions (running in polynomial time) is implicit in the respective papers.

Lemma 4 ([7]). *Let d and k be positive integers, $k \geq 2d + 1$, and let G be a d-degenerate graph on n vertices. Then $R_k(G)$ has diameter $\mathcal{O}(n^2)$.*

Lemma 5 ([17]). *Let $G = (V, E)$ be a planar graph. There is a partition $V = I \cup D$ such that $G[I]$ is an independent set and $G[D]$ is a 3-degenerate graph.*

Lemma 6 ([15, 18]). *Let k be a positive integer, and let $G = (V, E)$ be a k-degenerate graph. There is a partition $V = I \cup F$ such that $G[I]$ is an independent set and $G[F]$ is a $(k-1)$-degenerate graph.*

We combine Lemmas 5 and 6 to obtain the following corollary.

Corollary 2.1. *Let $G = (V, E)$ be a planar graph. Then there is a partition $V = I_1 \cup I_2 \cup A$ such that $G[I_1]$ and $G[I_2]$ are independent sets and $G[A]$ is a 2-degenerate graph.*

We are now ready to prove Theorem 1.

Proof of Theorem 1. Let α and β be two 7-colourings of G. To prove the theorem, it suffices to show that we can recolour α to β by $2^{\mathcal{O}(\sqrt{n})}$ recolourings.

Combining Lemmas 2 and 3, we can recolour α to some 5-colouring γ_1 of G and β to some 5-colourings γ_2 by $2^{\mathcal{O}(\sqrt{n})}$ recolourings. We apply Corollary 2.1 to find a partition $V = I_1 \cup I_2 \cup A$ such that $G[I_1]$ and $G[I_2]$ are independent sets and $H = G[A]$ is a 2-degenerate graph. From γ_1 and γ_2 we recolour the vertices in I_1 to colour 7 and those in I_2 to colour 6 (the colours that are not used in either γ_1 or γ_2). Let γ_1^H and γ_2^H denote, respectively, the restrictions of γ_1 and γ_2 to H. We focus on H and as long as we do not use colours 6 and 7 we can recolour γ_1^H to γ_2^H without worrying about adjacencies between A and $I_1 \cup I_2$. Since H is 2-degenerate, we can apply Lemma 4 with $k = 5$ and $d = 2$ to recolour γ_1^H to γ_2^H by $\mathcal{O}(n^2)$ recolourings. This completes the proof. □

3 Final remarks

From the proof of Theorem 1, notice that to obtain a polynomial bound on the diameter of the reconfiguration graph for 7-colourings of planar graphs, it would suffice to show that we can recolour any 7-colouring of a planar graph to some 6-colouring by polynomially many recolourings.

Problem 1. *Given a planar graph G and a 7-colouring α of G, can we recolour α to some 6-colouring of G by $O(n^c)$ recolourings for some constant $c > 0$?*

In order to obtain a sub-exponential bound on the diameter of reconfiguration graphs of colourings for graphs with any bounded maximum average degree, it would suffice to find a positive answer to the following problem. (The proof of this fact follows by combining Lemma 2 with an affirmative answer to Problem 2 in the same way that Lemmas 8, 9 and 10 in [12] are combined to obtain Theorem 6 in [12].)

Problem 2. *Let $k \geq 2$, and let $G = (V, E)$ be a graph with $mad(G) < k$. Then there exists a partition $\{V_1, V_2\}$ of G such that $G[V_1]$ is an independent set and $mad(G[V_2]) < k - 1$.*

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