Analytical Behaviour of Positronium Decay Amplitudes

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Abstract

Positronium annihilation amplitudes that are computed by assuming a factorization approximation with on-shell intermediate leptons do not exhibit good analytical behaviour. Using dispersion techniques, we find new contributions that interfere with the known results to restore analytical properties. Those new amplitudes which cannot be obtained using standard factorized amplitude formalism, contribute at $\mathcal{O}\left(\alpha^2\right)$. Therefore they have to be evaluated before any theoretical conclusion can be drawn upon the orthopositronium lifetime puzzle.

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1 Introduction

Positronium is a bound state of electron and positron. In this paper, we will be interested in the triplet state, orthopositronium, whose decay rate into 3γ has been precisely measured :

$$\Gamma^{exp}\left(o\text{-}Ps \rightarrow 3\gamma\right) = \begin{cases} 7.0398\,(29) \quad \mu sec^{-1} & \text{Tokyo[1]} \\ 7.0514\,(14) \quad \mu sec^{-1} & \text{Ann Arbor (Gas)[2]} \\ 7.0482\,(16) \quad \mu sec^{-1} & \text{Ann Arbor (Vacuum)[3]} \end{cases}$$

The corresponding theoretical predictions which include perturbative QED corrections to a non-relativistic treatment of the bound state wavefunction have been computed also with high accuracy (see for example [4], [5], [6], [7]):

$$\begin{aligned} \Gamma\left(o\text{-}Ps \rightarrow 3\gamma\right) &= \alpha^6 m \frac{2\left(\pi^2-9\right)}{9\pi}\left[1 - A\frac{\alpha}{\pi} - \frac{\alpha^2}{3}\log\frac{1}{\alpha} + B_o\left(\frac{\alpha}{\pi}\right)^2 - \frac{3\alpha^3}{2\pi}\log^2\frac{1}{\alpha}\right] \\ &= \left(7.0382 + B_o 0.39 \times 10^{-4}\right) \ \mu\,\mathrm{sec}^{-1} \end{aligned}$$

with $A = 10.286606\,(10)$, α the fine structure constant and m the electron mass. Recent theoretical efforts have focused on a more complete evaluation of the non-logarithmic $\mathcal{O}\left(\alpha^2\right)$ perturbative corrections, with the result $B_o = 44.52\,(26)$ [8] or, equivalently

$$\Gamma\left(o\text{-}Ps \rightarrow 3\gamma\right) = 7.039934\,(10) \quad \mu\,\mathrm{sec}^{-1}$$

This result renders the theoretical prediction still closer to the experimental measurement of Ref. [1].

Positronium is a test ground for bound state treatment in Quantum Field Theory. The first try dates back to the $40's$, with decay rates expressed through a factorized formula [9]

$$\Gamma\left(o\text{-}Ps \rightarrow 3\gamma\right) = \frac{1}{3}\left|\phi_o\right|^2 \cdot \left(4 v_{rel}\sigma\left(e^+e^- \rightarrow 3\gamma\right)\right)_{v_{rel}\rightarrow 0}$$

with ϕ_o the Schrödinger positronium wavefunction at the origin, $\sigma\left(e^+e^- \rightarrow 3\gamma\right)$ the total cross section for $e^+e^- \rightarrow 3\gamma$ and v_{rel} the relative velocity of e^+ and e^- in their center of mass frame. Since then, more sophisticated decay amplitudes have been constructed, and systematic procedures for calculating corrections have been developed [10]. However, the basic factorization of the bound state dynamics from the annihilation process has remained as a basic postulate. For low order corrections, this approximation is unquestionable, but for $\mathcal{O}\left(\alpha^2\right)$ corrections, factorization has to be tested. Indeed, non-perturbative phenomena responsible for the off-shellness of the electron and positron inside the positronium are of $\mathcal{O}\left(\alpha^2\right)$. In other words, to get a sensible theoretical prediction at $\mathcal{O}\left(\alpha^2\right)$, one must carefully analyze how binding energy effects enter the general factorization approach.

In a recent paper [11], we showed that the factorization of the bound state dynamics from the annihilation process can be given a firm grounding through dispersion relations. From a fully relativistic model, we recovered the standard factorized amplitudes used in the literature, and found some forgotten $\mathcal{O}\left(\alpha^2\right)$ contributions. Those amplitudes are gauge invariant, since they are expressed in terms of scattering amplitudes for *on-shell* e^+e^-. When applied to orthopositronium, however, the appearance of those on-shell intermediate leptons brings infrared singularities [12], giving an incorrect analytical behaviour to the amplitude $o\text{-}Ps \to 3\gamma$. Indeed, basic principles require that this amplitude vanishes in the soft photon limit, since it involves only neutral bosons [13]. One must conclude that the standard approaches are not complete and that they can only approximately describe the positronium decay.

The purpose of the present paper is to restore the analytical behaviour of positronium decay amplitudes. The relativistic model we used in [11] to describe $p\text{-}Ps \to \gamma\gamma$ has the correct analytical behaviour when applied to $o\text{-}Ps \to 3\gamma$ since only off-shell constituents appear. Indeed, we will find along with the known factorized-type amplitudes already found for para-positronium, a whole class of non-factorizable processes which enforce the positronium decay amplitude to vanish in the soft photon limit. Those new amplitudes were absent for $p\text{-}Ps \to \gamma\gamma$, but will contribute to $o\text{-}Ps \to 3\gamma$ at $\mathcal{O}\left(\alpha^2\right)$.

This paper is organized as follows. We first give a brief description of the derivation of factorized amplitudes from our relativistic model. Then we analyze the simple decay $p\text{-}Dm \to \gamma e^+e^-$, and show in detail how the analytical behaviour of the amplitude is restored by new contributions. Finally, we discuss the orthopositronium decay to three photons and describe all the new contributions. In the appendix, we present a short discussion of the decay $K_S^0 \to \gamma e^+e^-$, the hadronic counterpart of the decay $p\text{-}Dm \to \gamma e^+e^-$, in order to show the generality of the arguments developed for electromagnetic bound states.

2 Factorized Amplitude from a Loop Model

Here we briefly review the recently proposed method [11] to derive factorized amplitudes for bound state decays.

The decay rates for positronium are calculated in a loop model. The positronium first decays into a virtual electron-positron pair, which subsequently annihilates into real or virtual photons (an odd number for ortho-states, an even number for para-states). The coupling of the positronium to its constituents is described by a form factor, denoted by F_B. It is not assumed to be a constant, since a constant form factor would amount to consider positronium as a point-like state.

In our model, the decay amplitudes are (see for example figure 4)

$$\mathcal{M}^{\mu\nu\cdots}\left({}^{1}S\rightarrow A\right) = \int\frac{d^{4}q}{\left(2\pi\right)^{4}}F_{B}Tr\left\{\gamma_{5}\frac{i}{\not q-\frac{1}{2}\not P-m}\Gamma^{\mu\nu\cdots}\frac{i}{\not q+\frac{1}{2}\not P-m}\right\} \quad (1)$$

$$\mathcal{M}^{\mu\nu\cdots}\left({}^{3}S\rightarrow A\right) = e_{\alpha}\int\frac{d^{4}q}{\left(2\pi\right)^{4}}F_{B}Tr\left\{\gamma^{\alpha}\frac{i}{\not q-\frac{1}{2}\not P-m}\Gamma^{\mu\nu\cdots}\frac{i}{\not q+\frac{1}{2}\not P-m}\right\} \quad (2)$$

for parapositronium (pseudoscalar) and orthopositronium (vector, with polarization e_{α}) respectively . The $\Gamma^{\mu\nu\cdots}$ is the scattering amplitude for off-shell $e^{+}e^{-}$ with incoming momenta $\frac{1}{2}P+q$ and $\frac{1}{2}P-q$ into the final state A, m is the electron mass, P the positronium four-momentum and $F_{B}=F_{B}\left(q^{2},q\cdot P\right)$. From these amplitudes, the decay widths are calculated as

$$\Gamma\left({}^{2J+1}S\rightarrow A\right)=\frac{1}{2J+1}\frac{1}{2M}\int d\Phi_{A}\sum_{pol}\left|\mathcal{M}^{\mu\nu\cdots}\left({}^{2J+1}S\rightarrow A\right)\varepsilon_{\mu}^{*}\varepsilon_{\nu}^{*}\ldots\right|^{2}$$

with M the positronium mass.

What we have shown in our previous article [11] is that when one expresses the loop integration in (1) or (2) via a dispersion relation ([14], [15]), one ends up with a factorized convolution-type amplitude, i.e. for parapositronium:

$$\mathcal{M}^{\mu\nu\cdots}\left({}^{1}S\rightarrow A\right)=\frac{C}{2}\int\frac{d^{3}\mathbf{k}}{\left(2\pi\right)^{3}2E_{\mathbf{k}}}\psi\left(\mathbf{k}^{2}\right)Tr\left\{\gamma_{5}\left(-\not k'+m\right)\Gamma^{\mu\nu\cdots}\left(k,k',l_{1},\ldots\right)\left(\not k+m\right)\right\} \quad (3)$$

with $\psi\left(\mathbf{k}^{2}\right)$ the bound state wavefunction, $\Gamma^{\mu\nu\cdots}\left(k,k',l_{1},\ldots\right)$ the reduced scattering amplitude for *on-shell* $e^{+}e^{-}$ with momenta k and k', $\mathbf{k}=-\mathbf{k}'$ and $C=\sqrt{M}/m$ (M is the positronium mass). The analogous form for orthopositronium decay is obtained using the substitution $\gamma^{5}\rightarrow\not\varepsilon$. The form factor F_{B} is related with $\psi\left(\mathbf{k}^{2}\right)$ through

$$F_{B}=C\phi_{o}\mathcal{F}\left(\mathbf{k}^{2}\right)\left(\mathbf{k}^{2}+\gamma^{2}\right)$$

with $\phi_{o}\mathcal{F}\left(\mathbf{k}^{2}\right)=\psi\left(\mathbf{k}^{2}\right)$, ϕ_{o} being the Schrödinger wavefunction at the origin, and $\gamma^{2}=m^{2}-M^{2}/4$ (related to the binding energy and fine-structure constant through $E_{B}=M-2m=-m\alpha^{2}/4$).

3 Paradimuonium Decay into $\gamma e^{+}e^{-}$

The present section concerns paradimuonium, the singlet $\mu^{+}\mu^{-}$ bound state [16]. This state has not been observed yet. The reason to consider the decay p-$Dm\rightarrow\gamma e^{+}e^{-}$ is that it is the simplest process where a photon is kinematically allowed to have a vanishing energy. Therefore, we will be able to test whether the proposed factorization procedure (3) gives a correct analytical behaviour to the amplitude in the soft photon limit. The simplicity of the process comes from the pseudoscalar nature of the paradimuonium, which allows a manifestly gauge invariant treatment throughout. Note that the positronium decay $p\text{-}Ps \rightarrow \gamma\nu\overline{\nu}$ via a Z^0 has the same dynamics. After having analyzed this simple decay, we will be ready to tackle the more interesting process $o\text{-}Ps \rightarrow \gamma\gamma\gamma$.

The decay $p\text{-}Dm \rightarrow e^+e^-\gamma$ is shown in figure 1. When the form factor F_B appearing at the vertex $p\text{-}Dm \rightarrow \mu^+\mu^-$ allows a change of variable $q \rightarrow -q$, the two amplitudes can be combined into

$$\mathcal{M}\left(p\text{-}Dm \rightarrow e^+e^-\gamma\right) = 8me^3\varepsilon^{\mu\nu\rho\sigma}k_\rho t_\sigma \varepsilon_\mu^*\left(k\right)\frac{\left\{\overline{u}\left(p\right)\gamma_\nu v\left(p'\right)\right\}}{t^2+i\varepsilon}\mathcal{I}\left(M^2,x\right) \tag{4}$$

with $x = 2\omega/M$ the reduced photon energy, M the dimuonium mass and m the muon mass. The loop integral, which can be viewed as an effective form factor, is given by

$$\mathcal{I}\left(P^2,x\right) = \eta\int\frac{d^4q}{\left(2\pi\right)^4}F_B\frac{1}{\left(q-\frac{1}{2}P\right)^2-m^2}\frac{1}{\left(q+\frac{1}{2}P\right)^2-m^2}\frac{1}{\left(q-\frac{1}{2}P+k\right)^2-m^2}$$

where $\eta = P^2/M^2$ and $t^2 = P^2(1-x)$. Gauge invariance is ensured by the factorized tensor structure, i.e. by the antisymmetric Levi-Civita tensor. $\mathcal{I}\left(P^2,x\right)$ is given by the same expression as the two-photon decay integral we encountered previously [11], the only difference being that $t^2 \neq 0$.

The differential width is given by

$$\frac{d\Gamma\left(p\text{-}Dm \rightarrow e^+e^-\gamma\right)}{dx} = \frac{16\alpha^3}{3}m^2\,M^3\left|\mathcal{I}\left(M^2,x\right)\right|^2\rho\left(x,a\right)$$

with the phase space factor

$$\rho\left(x,a\right) = \sqrt{1-\frac{a}{1-x}}\left[a+2\left(1-x\right)\right]\frac{x^3}{\left(1-x\right)^2}$$

where $a = 4m_e^2/M^2$, and the bounds on x are $[0, 1-a]$.

We are going to calculate the integral $\mathcal{I}\left(M^2,x\right)$ using dispersion relations. The main difference with the two-photon decay case is the appearance of two different cuts (figure 2). We have shown [11] that considering the vertical cuts is strictly equivalent to the decay amplitude calculation done using formula (3) where Γ^μ is the scattering amplitude $\mu^+\left(k'\right)\mu^-\left(k\right) \rightarrow \gamma\gamma^* \rightarrow e^+e^-\gamma$ with on-shell muons. Standard approaches found in the literature are then obtained by making an $\mathcal{O}\left(\alpha^2\right)$ approximation in (3). Obviously, those approaches completely miss the oblique cuts. What we are going to show is that those forgotten cuts contribute at the order of $\gamma^2 = m^2 - M^2/4 \approx \alpha^2/4$, and more importantly, that *the amplitude for* $p\text{-}Dm \rightarrow e^+e^-\gamma$ *has a correct* *soft-photon limit behaviour only if we take all the cuts into account.* By this we mean that each cut gives a contribution to the amplitude that behaves as a constant when the photon energy goes to zero. The combination of the vertical and oblique cuts, on the contrary, forces the amplitude to vanish in that limit. We thus recover the analytical behaviour expected from Low's theorem [13] for the decay of $p\text{-}Dm \to \gamma\gamma^*$ involving only neutral bosons (for $p\text{-}Dm \to \gamma e^+ e^-$, infrared divergences cannot arise from electron bremsstrahlung processes due to selection rules).

3.1 Soft photon limit property of the absorptive part

To keep the discussion as general as possible, we extract the imaginary part of $\mathcal{I}\left(P^2, t^2\right)$ without specifying the form factor F_B. The absorptive part is found by cutting the relevant propagators as

$$\begin{aligned}
\operatorname{Im}\mathcal{I}_1\left(P^2,x\right) &= \eta\int\frac{d^4q}{2\left(2\pi\right)^4}F_B\frac{2\pi i\delta\left(\left(q-\frac{1}{2}P\right)^2-m^2\right)2\pi i\delta\left(\left(q+\frac{1}{2}P\right)^2-m^2\right)}{\left(q-\frac{1}{2}P+k\right)^2-m^2}\\
\operatorname{Im}\mathcal{I}_2\left(P^2,x\right) &= \eta\int\frac{d^4q}{2\left(2\pi\right)^4}F_B\frac{2\pi i\delta\left(\left(q+\frac{1}{2}P\right)^2-m^2\right)2\pi i\delta\left(\left(q-\frac{1}{2}P+k\right)^2-m^2\right)}{\left(q-\frac{1}{2}P\right)^2-m^2}
\end{aligned}$$

for the vertical and oblique cuts, respectively. By a straightforward integration, the first expression gives ($s = P^2$):

$$\operatorname{Im}\mathcal{I}_1\left(s,x\right)=\frac{\eta}{s}\frac{F_B\left(q_0=0,|\mathbf{q}|=\sqrt{s/4-m^2}\right)}{16\pi x}\ln\left[\frac{1+\sqrt{1-4m^2/s}}{1-\sqrt{1-4m^2/s}}\right]\theta\left(s-4m^2\right) \tag{5}$$

while the second one cannot be completely integrated without specifying F_B:

$$\operatorname{Im}\mathcal{I}_2\left(s,x\right)=\frac{\eta}{s}\frac{1}{16\pi x}\int_{q_{\min}}^{q_{\max}}dq_0\frac{F_B\left(q_0,|\mathbf{q}|=\sqrt{q_0^2+q_0\sqrt{s}+s/4-m^2}\right)}{q_0}\theta\left(s-\frac{4m^2}{1-x}\right) \tag{6}$$

with the bounds given by

$$q_{\min}=-\frac{x\sqrt{s}}{4}\left(1+\sqrt{1-\frac{4m^2}{s}\frac{1}{1-x}}\right),q_{\max}=-\frac{x\sqrt{s}}{4}\left(1-\sqrt{1-\frac{4m^2}{s}\frac{1}{1-x}}\right) \tag{7}$$

Interestingly, the second cuts contribute for $q_0 \neq 0$. This is in sharp contrast with the two-photon decay, since there only the first cuts exist. Further, approximated bound state wavefunction where a $\delta\left(q_0\right)$ appears cannot be used ([5], [6], [7]), and one should revert to the full four-dimensional Bethe-Salpeter wavefunction (for example the Barbieri-Remiddi one [17]).

Let us now demonstrate an important point, i.e. that in the soft photon limit, the combination $\mathrm{Im}\,\mathcal{I}_1\left(s,x\right)+\mathrm{Im}\,\mathcal{I}_2\left(s,x\right)$ behaves as a constant when $x\rightarrow 0$ despite the fact that each cut diverges in that limit. This will guarantee that the imaginary part of the whole amplitude vanishes in the soft photon limit thanks to the presence of k_ρ in the tensor structure of (4). This behaviour is what one expects from Low's theorem [13]. For all the form factors we considered, the behaviour of the combination $\mathrm{Im}\,\mathcal{I}_1\left(s,x\right)+\mathrm{Im}\,\mathcal{I}_2\left(s,x\right)$ when $x\rightarrow 0$ is summarized by :

$$\mathrm{Im}\,\mathcal{I}_1\left(s,x\right)+\mathrm{Im}\,\mathcal{I}_2\left(s,x\right)\overset{x\rightarrow 0}{\rightarrow}\frac{\eta}{s}\frac{1}{16\pi}\left[\frac{F_B\left(0\right)}{\sqrt{1-\frac{4m^2}{s}}}+\frac{\sqrt{s-4m^2}}{2}\frac{\partial F_B}{\partial q}\left(0\right)\right]$$

Specifically, the form factors of interest are constructed from the Schrödinger wavefunction (see [11]). Let us recall their definition and give the limit obtained when using each of them :

$$\begin{aligned} F_B^I &\equiv C\phi_o\frac{8\pi\gamma}{\mathbf{q}^2+\gamma^2} \qquad (8)\\ &\Rightarrow \mathrm{Im}\,\mathcal{I}_1\left(s,x\right)+\mathrm{Im}\,\mathcal{I}_2\left(s,x\right)\overset{x\rightarrow 0}{\rightarrow}\frac{\eta}{s}\frac{C\phi_o}{2}\gamma\frac{\left(s/4-\gamma^2-m^2\right)}{\sqrt{1-\frac{4m^2}{s}}\left(\gamma^2-m^2+\frac{s}{4}\right)^2}\end{aligned}$$

$$\begin{aligned} F_B^{II} &\equiv C\phi_o\frac{32\pi\gamma^3}{\left(\mathbf{q}^2+\gamma^2\right)^2} \qquad (9)\\ &\Rightarrow \mathrm{Im}\,\mathcal{I}_1\left(s,x\right)+\mathrm{Im}\,\mathcal{I}_2\left(s,x\right)\overset{x\rightarrow 0}{\rightarrow}\frac{\eta}{s}\frac{C\phi_o}{2}\gamma^3\frac{\left(4\gamma^2-3\left(s-4m^2\right)\right)}{\sqrt{1-\frac{4m^2}{s}}\left(\gamma^2-m^2+\frac{s}{4}\right)^3}\end{aligned}$$

i.e. constant limits. Remark that for a constant form factor $F_B=F_B^{\mathrm{Const}}$, the limit is simply

$$\mathrm{Im}\,\mathcal{I}_1\left(s,x\right)+\mathrm{Im}\,\mathcal{I}_2\left(s,x\right)\overset{x\rightarrow 0}{\rightarrow}\frac{\eta}{s}\frac{F_B^{\mathrm{Const}}}{16\pi}\frac{1}{\sqrt{1-4m^2/s}}$$

It is easy to verify that this result is indeed what can be calculated from the imaginary parts :
$\mathrm{Im}\,\mathcal{I}_1\left(s,x\right)+\mathrm{Im}\,\mathcal{I}_2\left(s,x\right)=$

$$\frac{\eta}{s}\frac{F_B^{\mathrm{Const}}}{16\pi x}\left[\ln\left[\frac{1+\sqrt{1-\frac{4m^2}{s}}}{1-\sqrt{1-\frac{4m^2}{s}}}\right]\theta\left(s-4m^2\right)-\ln\left[\frac{1+\sqrt{1-\frac{4m^2}{s(1-x)}}}{1-\sqrt{1-\frac{4m^2}{s(1-x)}}}\right]\theta\left(s-\frac{4m^2}{1-x}\right)\right]$$

In conclusion, the absorptive part of the amplitude is seen to vanish. Thus, we expect that the dispersive part will also vanish in that limit. The oblique cuts, omitted from standard analyzes, are seen to be essential to maintain good analytical properties of the amplitude.

In the appendix, the restoration of analyticity by a cancellation among the vertical and oblique cuts is presented in the context of the kaon decay $K_S^0 \to e^+e^-\gamma$ via a charged pion loop. In this analysis, the form factor describing the $K \to \pi\pi$ vertex is taken as a constant. This decay is interesting since there is a close similarity between this hadronic decay process and the present QED bound state decay. The main conclusion is that even if charged particles are present in intermediate states, the amplitude has to vanish in the soft photon limit, as expected from the fact that the decays $K_S^0 \to \gamma\gamma^*$ or $p\text{-}Dm \to \gamma\gamma^*$ involve only neutral bosons. Further, this K_S^0 decay provides a physically sensible process where one can analyze the constant form factor assumption, since for such a loosely bound system as the $p\text{-}Dm$ a constant form factor cannot be realistic.

We will now evaluate the contribution of each cut for the case of the Schrödinger momentum wavefunction form factor F_B^I in (8). In doing so, we will gain a better understanding of the analyticity restoration at the level of the differential decay rate for $p\text{-}Dm \to e^+e^-\gamma$. The purpose of the next sections is only illustrative, and the discussion can be straightforwardly transcribed for the advocated improved form factor F_B^{II} (9) (see[11]).

3.2 The vertical cuts reproduce standard approach results

To precisely show what happens when the oblique cuts are forgotten, we compute here the rate with the vertical cuts only. The present derivation is therefore equivalent to standard analyses.

For our computation, we take the first cut imaginary part given by (5). The real part is found through an unsubstracted dispersion relation with $\eta = s/M^2$ (see [11], [14], [15]) :

$$\begin{aligned}
\mathcal{I}_1\left(M^2,x\right) &= \operatorname{Re}\mathcal{I}_1\left(M^2,x\right) = \frac{C\phi_o}{\pi M^2}\frac{2\gamma}{x}\int_{4m^2}^{\infty}\frac{ds}{\left(s-M^2\right)^2}\ln\left[\frac{1+\sqrt{1-4m^2/s}}{1-\sqrt{1-4m^2/s}}\right] \\
&= \frac{C\phi_o}{M^3}\frac{1}{x}\left[\frac{2}{\pi}\arctan\frac{M}{2\gamma}\right] \qquad (10)
\end{aligned}$$

where the first equality holds since $M^2 < 4m^2$. Remark that for $x \to 1$, the second cuts vanish (see 6) and (10) reproduce the result obtained in [11] for the parapositronium two-photon decay.

Inserting the result for $\mathcal{I}_1$ into the expression of the differential rate with $C^2 = M/m^2$ and $|\phi_o|^2 = \alpha^3 m^3/8\pi$, we get:

$$\frac{d\Gamma\left(p\text{-}Dm \to \gamma e^+e^-\right)}{dx} = \frac{\alpha^6 m}{6\pi}\left(\frac{4m^2}{M^2}\right)\left|\frac{2}{\pi}\arctan\frac{M}{2\gamma}\right|^2\frac{\rho\left(x,a\right)}{x^2}$$

Note that, independently of the value for γ, the spectrum is always linear in x due to the $1/x$ in (10). This is an incorrect soft photon behaviour since Low's theorem requires a cubic spectrum instead [12] (when the amplitude behaves as x, the rate $\sim x^2$ and an additional x comes from phase-space) . By expanding the differential rate around $\gamma = 0$, we get

$$\frac{d\Gamma\left(p\text{-}Dm \to \gamma e^+ e^-\right)}{dx} = \frac{\alpha^6 m}{6\pi}\left(1 - \frac{\alpha}{\pi} + \frac{1}{4}\alpha^2 - \frac{25}{96\pi}\alpha^3 + \frac{3}{64}\alpha^4 + \mathcal{O}\left(\alpha^5\right)\right)\frac{\rho\left(x,a\right)}{x^2}$$

As lowest order, we recover the standard result:

$$\frac{d\Gamma\left(p\text{-}Dm \to \gamma e^+ e^-\right)}{dx} = \frac{\alpha^6 m_\mu}{6\pi}\frac{\rho\left(x,a\right)}{x^2} \tag{11}$$

Note that the order α corrections disappear when using the form factor F_B^{II} (see [11]). For completeness, the total rate is simply :

$$\Gamma\left(p\text{-}Dm \to \gamma e^+ e^-\right) = \frac{\alpha^6 m}{6\pi}F\left(a\right)\left(\frac{4m^2}{M^2}\right)\left|\frac{2}{\pi}\arctan\frac{M}{2\gamma}\right|^2$$

with

$$F\left(a\right) = \left[\frac{4}{3}\sqrt{1-a}\left(a-4\right) + 2\log\left(\frac{1+\sqrt{1-a}}{1-\sqrt{1-a}}\right)\right]$$

3.3 The oblique cut contribution to the rate

We have just seen that the vertical cuts suffice to reproduce the lowest order decay rate. It is interesting to investigate how the oblique cuts can restore the analytical behaviour of the spectrum without affecting this lowest order evaluation.

The imaginary part corresponding to the oblique cuts is, from (6),

$$\begin{aligned}
\operatorname{Im}\mathcal{I}_2\left(s,x\right) &= \frac{\eta}{s}\frac{C\phi_o}{16\pi x}\int_{q_{\min}}^{q_{\max}}\frac{dq_0}{q_0}\frac{8\pi\gamma}{q_0^2 + q_0\sqrt{s} + s/4 - m^2 + \gamma^2}\theta\left(s - \frac{4m^2}{1-x}\right) \\
&= \frac{\eta}{s}\frac{C\phi_o}{16\pi x}8\pi\gamma\left[\mathcal{H}\left(q_{\max}\right) - \mathcal{H}\left(q_{\min}\right)\right]\theta\left(s - \frac{4m^2}{1-x}\right)
\end{aligned}$$

with $q_{\min}$, $q_{\max}$ given in (7) and

$$\mathcal{H}\left(q\right) = -\frac{2\sqrt{s}}{M\left(M^2-s\right)}\ln\left[\frac{M+\sqrt{s}+2q}{M-\sqrt{s}-2q}\right] + \frac{2}{M^2-s}\ln\left[\frac{\left(\sqrt{s}+2q\right)^2 - M^2}{q^2}\right]$$

We should now use an unsubstracted dispersion relation to get $\mathcal{I}_2\left(M^2,x\right)$. This integral is quite complicated and not very interesting for the present purpose. Instead, we revert to numerical evaluation of the dispersion integral for $\mathcal{I}_2\left(M^2,x\right)=\text{Re}\mathcal{I}_2\left(M^2,x\right)$ as a function of x, and compare it to (10). In the figure 3, we plot the vertical cuts contribution (dashed line)

$$x\cdot\frac{M^3}{C\phi_o}\mathcal{I}_1\left(M^2,x\right)=\frac{2}{\pi}\arctan\frac{M}{2\gamma}$$

and the complete result (solid line)

$$\begin{aligned}x\cdot\frac{M^3}{C\phi_o}\mathcal{I}\left(M^2,x\right) &= x\cdot\frac{M^3}{C\phi_o}\left[\mathcal{I}_1\left(M^2,x\right)+\mathcal{I}_2\left(M^2,x\right)\right]\\ &= \frac{2}{\pi}\arctan\frac{M}{2\gamma}+\frac{M\gamma}{2\pi}\int_{\frac{4m^2}{1-x}}^{+\infty}\frac{ds}{s-M^2}\left[\mathcal{H}\left(q_{\max}\left(x\right)\right)-\mathcal{H}\left(q_{\min}\left(x\right)\right)\right]\end{aligned}$$

for $m=1,M=1.999$ (i.e. $\gamma\approx0.03$) and for $m=1,M=1.99994$ ($\gamma\approx0.0077$). For $x\rightarrow0$, one can verify that $x\cdot\mathcal{I}\left(M^2,x\right)\rightarrow0$. The figures show that near zero the two cuts interfere destructively in order to maintain a correct analytical behaviour for the whole amplitude. Away from $x=0$, the oblique cut contributions are strongly suppressed relatively to the vertical one, and this suppression increases as γ decreases ($\mathcal{I}_2\rightarrow0$ when $\gamma\rightarrow0$). As can be seen on the graph, it is typically for $x\lesssim\gamma$ that the oblique cut contributes. Therefore, we can summarize by giving a simple representation of the different contributions to the amplitude. From the figure 3, one can see that the behaviours of the vertical cuts, the oblique cuts and their combination are quite precisely modelled as

$$\mathcal{I}_1\sim\frac{1}{x},\mathcal{I}_2\sim-\frac{\gamma/M}{x\left(x+\gamma/M\right)}\Rightarrow\mathcal{I}=\mathcal{I}_1+\mathcal{I}_2\sim\left(\frac{1}{x+\gamma/M}\right)$$

As a consequence, the spectrum behaves as

$$\frac{d\Gamma\left(p\text{-}Dm\rightarrow\gamma e^+e^-\right)}{dx}\sim\left|\mathcal{I}\right|^2\rho\left(x,a\right)\sim x^3\left(\frac{1}{x+\gamma/M}\right)^2\sim x\left(\frac{x}{x+\gamma/M}\right)^2 \tag{12}$$

i.e. a linear spectrum when $\gamma\rightarrow0$ (11), and a x^3 spectrum when x is small. The effect of $\gamma\neq0$ is therefore to soften the photon spectrum and slightly reduce the total width. This behaviour is exactly what we postulated in a previous work [12].

In conclusion, the oblique cuts have a small contribution to the decay rate comparatively to the vertical cuts. However, their presence is essential to guarantee the analytical properties of the amplitude expected from Low's theorem. Further, when tackling $\mathcal{O}\left(\alpha^2\right)$ corrections, one must include contributions from the oblique cuts.

4 Orthopositronium Decay to three Photons

We now turn to the interesting decay $o\text{-}Ps\rightarrow\gamma\gamma\gamma$. We start as usual with the loop model amplitude. Dispersion techniques express that amplitude into separate contributions, arising from different cuts in figure 4. Since only off-shell intermediate leptons appear in the loop, the complete amplitude has a correct soft photon behaviour. This implies that one must consider all the cuts to preserve the analytical properties of the amplitude. This is one conclusion of the previous section.

Let us give the loop model amplitude

$$\mathcal{M}^{\mu\nu\rho}\left(o\text{-}Ps\rightarrow\gamma\gamma\gamma\right) = e^{3}e_{\alpha}\left(P\right)\int\frac{d^{4}q}{\left(2\pi\right)^{4}}F_{B}Tr\left\{\gamma^{\alpha}\frac{1}{\not q-\frac{1}{2}\not P-m}\Gamma^{\mu\nu\rho}\frac{1}{\not q+\frac{1}{2}\not P-m}\right\} \tag{13}$$

where the vertex is given by six amplitudes grouped as

$$\Gamma^{\mu\nu\rho}=\Gamma_{1}^{\mu\nu\rho}+\Gamma_{2}^{\mu\nu\rho}+\Gamma_{3}^{\mu\nu\rho}$$

with

$$\begin{aligned}
\Gamma_{1}^{\mu\nu\rho} &= \gamma^{\nu}\frac{1}{\not q-\frac{1}{2}\not P+\not l_{2}-m}\gamma^{\rho}\frac{1}{\not q+\frac{1}{2}\not P-\not l_{1}-m}\gamma^{\mu} \\
&\quad+\gamma^{\mu}\frac{1}{\not q-\frac{1}{2}\not P+\not l_{1}-m}\gamma^{\rho}\frac{1}{\not q+\frac{1}{2}\not P-\not l_{2}-m}\gamma^{\nu} \\
\Gamma_{2}^{\mu\nu\rho} &= \gamma^{\nu}\frac{1}{\not q-\frac{1}{2}\not P+\not l_{2}-m}\gamma^{\mu}\frac{1}{\not q+\frac{1}{2}\not P-\not l_{3}-m}\gamma^{\rho} \\
&\quad+\gamma^{\rho}\frac{1}{\not q-\frac{1}{2}\not P+\not l_{3}-m}\gamma^{\mu}\frac{1}{\not q+\frac{1}{2}\not P-\not l_{2}-m}\gamma^{\nu} \\
\Gamma_{3}^{\mu\nu\rho} &= \gamma^{\mu}\frac{1}{\not q-\frac{1}{2}\not P+\not l_{1}-m}\gamma^{\nu}\frac{1}{\not q+\frac{1}{2}\not P-\not l_{3}-m}\gamma^{\rho} \\
&\quad+\gamma^{\rho}\frac{1}{\not q-\frac{1}{2}\not P+\not l_{3}-m}\gamma^{\nu}\frac{1}{\not q+\frac{1}{2}\not P-\not l_{1}-m}\gamma^{\mu}
\end{aligned}$$

and $P=l_{1}+l_{2}+l_{3}$. The amplitudes in $\Gamma_{1}^{\mu\nu\rho}$ are shown on figure 4. Using charge-conjugation, we can show that the two drawn amplitudes are equal. The proof of this is in close analogy with the demonstration of Furry's theorem, and requires a change of integration variable $q\rightarrow -q$. This change is allowed for the form factor (see (8), (9)). Therefore, one can forget one term in each $\Gamma_{i}^{\mu\nu\rho}$ and multiply the other by 2. In the following, to keep the combinational as clear as possible, we will continue to consider all the six diagrams.

Let us now review the contributions aarising from the possible cuts in figure 4.

4.1 The vertical cut reproduces standard approach results

First we have the six vertical cuts shown on figure 5. As demonstrated in the paper [11], those vertical cuts reproduce the known result: their combination is what is calculated with (3), i.e. at lowest order [4]

$$\Gamma\left(o\text{-}Ps \to \gamma\gamma\gamma\right) = \frac{2\left(\pi^2-9\right)}{9\pi}\alpha^6 m$$

This result is obtained in the static limit, i.e. with $F_B \propto \delta^{(3)}(\mathbf{q})$.

If this limit is not taken, one has to go through the integration of the convolution-type amplitude (3). As explained in [11], the order of the corrections that will arise from that integration depends on the explicit form for F_B. Let us recall that for the parapositronium decay to two photons, we found that the form factors (8) led to $\mathcal{O}(\alpha)$ corrections and higher, while (9) gave corrections starting at $\mathcal{O}\left(\alpha^2\right)$.

Finally, it is important to note that the six vertical cuts are separately gauge invariant since the scattering amplitude for on-shell $e^+e^- \to \gamma\gamma\gamma$ is.

4.2 The oblique cuts and the structure dependent contributions

The oblique cuts are depicted in figure 6 for the photon 1 on the positronium side. Similar cuts for the photon 2 or 3 singled out are easily drawn. The oblique cuts are not gauge invariant, contrary to the vertical ones. To see this, it suffices to note that each of these oblique cuts is, from the optical theorem, a product of a scattering amplitude $e^+e^- \to \gamma\gamma$ times a bremsstrahlung amplitude $o\text{-}Ps \to e^+e^-\gamma$. Gauge invariance is broken by the bremsstrahlung amplitude, because the form factor is evaluated at different momenta for different cuts. To visualize the situation, let us change the momentum parametrization and draw amplitudes for $o\text{-}Ps \to e^+e^-\gamma(l_1)$ as shown in figure 7. Obviously, when contracted by l_1^μ, the two amplitudes fail to cancel each other, due to the different momentum dependences of F_B :

$$\begin{aligned}
\mathcal{M}^\mu_{Brem1} &= eF_B\left(p_1+l_1,p_2\right)\left\{\overline{u}\left(p_1\right)\gamma^\mu\frac{i}{\not{p}_1+\not{l}_1-m}\gamma^\alpha v\left(p_2\right)\right\}e_\alpha\left(P\right)\\
&\quad +eF_B\left(p_1,p_2+l_1\right)\left\{\overline{u}\left(p_1\right)\gamma^\alpha\frac{i}{-\not{p}_2-\not{l}_1-m}\gamma^\mu v\left(p_2\right)\right\}e_\alpha\left(P\right)\\
&= ieF_B\left(p_1+l_1,p_2\right)\left\{\overline{u}\left(p_1\right)\frac{2p_1^\mu+\gamma^\mu\not{l}_1}{2p_1\cdot l_1}\gamma^\alpha v\left(p_2\right)\right\}e_\alpha\left(P\right)\\
&\quad +ie^2F_B\left(p_1,p_2+l_1\right)\left\{\overline{u}\left(p_1\right)\gamma^\alpha\frac{-2p_2^\mu-\not{l}_1\gamma^\mu}{2p_2\cdot l_1}v\left(p_2\right)\right\}e_\alpha\left(P\right)
\end{aligned}$$

and when contracted by $l_{1,\mu}$:

$$l_{1,\mu}\mathcal{M}^\mu_{Brem1} = ie\left[F_B\left(p_1+l_1,p_2\right)-F_B\left(p_1,p_2+l_1\right)\right]\left\{\overline{u}\left(p_1\right)\gamma^\alpha v\left(p_2\right)\right\}e_\alpha\left(P\right)$$

Therefore, one should add to these diagrams the structure dependent amplitude $\mathcal{M}^\mu_{SD1}$(figure 7). The new direct coupling between $o\text{-}Ps, e^+, e^-$ and

γ must be such that the combination of the bremsstrahlung graphs and this structure graph is gauge invariant. For the paradimuonium, we have not considered such structure terms because the amplitude was gauge invariant throughout thanks to the factorized tensor structure (see (4)). This is a peculiar feature of the two-photon decay of pseudoscalar positronium or dimuonium state.

Let us now turn to the soft photon behaviour. The structure terms do not alter the cancellation among cuts in the soft photon limit (see the corresponding discussion for K_S^0 decay in the appendix). Indeed, Low's theorem applied to the bremsstrahlung plus structure dependent amplitudes gives the following expansion around $l_1 = 0$ (see [13], [18]):

$$\mathcal{M}^{\mu}_{Brem} + \mathcal{M}^{\mu}_{Struct} =$$

$$iF_B\left(p_1,p_2\right)\left\{\overline{u}\left(p_1\right)\left[\frac{2p_1^{\mu}+\gamma^{\mu}\not{l}_1}{2p_1\cdot l_1}\gamma^{\alpha}-\gamma^{\alpha}\frac{2p_2^{\mu}+\not{l}_1\gamma^{\mu}}{2p_2\cdot l_1}\right]v\left(p_2\right)\right\}e_{\alpha}\left(P\right)+\mathcal{O}\left(l_1\right)$$

i.e. terms of order $\left(l_1\right)^0$ cancel because $\mathcal{M}^{\mu}_{Brem} + \mathcal{M}^{\mu}_{Struct}$ is gauge invariant. Importantly, note that the form factor is now evaluated at $\left(p_1^2 = m^2, p_2^2 = m^2\right)$, i.e. at the same point as for the vertical cuts. Since for a constant form factor the combination of all the cuts necessarily behaves correctly in the soft photon limit, and since a momentum dependence in the form factor only modifies $\mathcal{O}\left(l_1\right)$ terms, the conclusion follows.

Remark that the appearance of these new structure contributions could have been guessed from the start, since the loop model amplitude (13) fails to be gauge invariant due to momentum dependences in the form factor. More precisely, the Ward identity $l_{1,\mu}\mathcal{M}^{\mu\nu\rho}\left(o\text{-}Ps \rightarrow \gamma\gamma\gamma\right) = 0$ is verified only if the form factor allows for linear changes of the integration variable like $q \rightarrow q + l_1$. A general form factor will not allow such shifts and one must supplement the loop model with new structure dependent amplitudes, as noted in [12]. The dispersion technique we followed here is interesting since we get a more precise information on their origin, and we are able to constraint them through Low's theorem.

The same discussion can be made for the photons 2 and 3. One ends up with twelve oblique cuts, to which six structure dependent amplitudes must be added. All these new contributions should become important at the order $\gamma^2 \approx \alpha^2/4$, so that one can really question the completeness of the results given in the literature [8].

5 Conclusions

We have shown that the standard approaches used to calculate positronium decay rates cannot be used to fully evaluate $\mathcal{O}\left(\alpha^2\right)$ corrections. In the case of orthopositronium, many new contributions arise at that order from processes where the electron emerging from the bound state decay is off-shell. Those contributions are essential to preserve the basic property of analyticity of the positronium decay amplitude, i.e. its vanishing in the soft photon limit. Further, gauge invariance requires some interplay between the annihilation process and the bound state dynamics through structure dependent amplitudes.

All these new contributions vanish in the static limit, i.e. when the form factor is replaced by a delta function $\delta^{(3)}(\mathbf{q})$. Therefore, one can view these as non-perturbative effects arising from the binding of the constituents in the bound state. Indeed, the size of the corrections is fixed by the γ^2 appearing in the bound state wavefunction. Concerning this wavefunction, forms where the dependence on the energy q_0 is replaced by a $\delta(q_0)$ should not be used because it is specifically for non-zero value of the energy that the additional diagrams contribute. This raises the question of the form for F_B to be used. Let us recall that in [11], we advoquate the use of the improved form factor F_B^{II} instead of the usual Schrödinger momentum wavefunction.

In conclusion, it should now be clear that $\mathcal{O}(\alpha^2)$ corrections as given in the literature are rather incomplete. A lot of work is still needed before a definite theoretical prediction up to that order can be compared to experiment. Even if the theoretical basis seems settled, the orthopositronium life-time puzzle remains an open question.

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6 Appendix : Kaon Radiative Decay $K_S^0 \to \gamma e^+ e^-$

The amplitude for $K_S^0 \to \gamma e^+ e^-$ is computed at lowest order in a pion loop model. The pion is treated as a point-like charged particle, which allows simple scalar QED treatment. Special attention is paid to the low energy behaviour of the amplitude.

6.1 Decay Amplitude and Loop Integration

The amplitude for kaon decay into $\gamma e^+ e^-$ is given by

$$\begin{aligned} \mathcal{M}\left(K_S^0 \to \gamma e^+ e^-\right) &= -2ie^3 \mathcal{M}\left(K_S^0 \to \pi^+ \pi^-\right) \varepsilon_\mu^* \left(k\right) \frac{\left\{\overline{u}\left(p\right) \gamma_\nu v\left(p'\right)\right\}}{t^2} \\ & \times \int \frac{d^d q}{\left(2\pi\right)^d} \mathcal{M}^{\mu\nu}\left(\pi^+ \pi^- \to \gamma\gamma\right) \end{aligned}$$

with $t = p + p' = P - k$. The $\pi^+ \pi^- \to \gamma\gamma$ amplitude arises from the one-photon (figure 8a) and two-photon (the so-called seagull graph, figure 8b) coupling amplitudes as

$$\mathcal{M}^{\mu\nu}\left(\pi^+ \pi^- \to \gamma\gamma\right) = \frac{\left(2q - k\right)^\mu \left(2q + t\right)^\nu - g^{\mu\nu}\left(q^2 - m^2\right)}{\left(\left(q + t\right)^2 - m^2\right)\left(\left(q - k\right)^2 - m^2\right)\left(q^2 - m^2\right)}$$

with m the pion mass. The amplitude $\mathcal{M}\left(K_S^0 \to \pi^+ \pi^-\right)$, taken as a constant, has been factored out. The integration is done using dimensional regularization to preserve gauge invariance and get a finite result. We obtain

$$\begin{aligned} & \int \frac{d^d q}{\left(2\pi\right)^d} \mathcal{M}^{\mu\nu}\left(\pi^+ \pi^- \to \gamma\gamma\right) \\ & \quad = \frac{-i}{\left(4\pi\right)^2} \int_0^1 dx \int_0^{1-x} dy \left[\tfrac{4xy}{\Delta}\left[g^{\mu\nu}\left(k \cdot t\right) - t^\mu k^\nu\right] + \tfrac{y(1-2y)}{\Delta} g^{\mu\nu} t^2\right] \end{aligned}$$

The denominator function is $\Delta = m^2\left(1 - 4\left(a - b\right)xy + 4by\left(y - 1\right)\right)$ with the definitions $a = M^2/4m^2, b = t^2/4m^2$ and M the kaon mass. In the kaon rest-frame, $b = M^2\left(1 - \omega\right)/4m^2$ with ω the reduced photon energy $2k^0/M$. When integrating over Feynman parameters, the last term vanishes. Therefore, the gauge invariance of the amplitude becomes manifest:

$$\begin{aligned} \mathcal{M}\left(K_S^0 \to \gamma e^+ e^-\right) = \frac{-2e^3}{\left(4\pi\right)^2} \mathcal{M}\left(K_S^0 \pi^+ \pi^-\right) \frac{1}{m^2} F\left(a, b\right) \times \\ \varepsilon_\mu^*\left(k\right)\left\{\overline{u}\left(p\right) \gamma_\nu v\left(p'\right)\right\} \frac{g^{\mu\nu}\left(k \cdot t\right) - t^\mu k^\nu}{t^2} \end{aligned} \tag{14}$$

$F\left(a, b\right)$ is the Feynman parameter integral

$$F\left(a, b\right) = \int_0^1 dy \int_0^{1-y} dx \frac{4xy}{1 - 4\left(a - b\right)xy + 4by\left(y - 1\right) + i\varepsilon}$$

The prescription $i\varepsilon$ gives the sign of the imaginary part.

6.2 Feynman Parameter Integration via Dispersion Relations

To calculate $F(a,b)$, we shall use dispersion relations (see for example [14], [15]). A direct integration is possible, but we shall gain insight into the dynamics of the process by using dispersion techniques.

Absorptive Part Extraction

A first integration over the Feynman parameters gives

$$F(a,b)=\int_0^1 dy\left[\frac{y-1}{a-b}+\frac{\ln K(b)-\ln K(a)}{4y(a-b)^2}+\frac{b(y-1)(\ln K(b)-\ln K(a))}{(a-b)^2}\right] \tag{15}$$

where $\ln K(x)=\ln(1+4x(y-1)y-i\varepsilon)$. The imaginary part of $F(a,b)$ comes from the values of y for which the argument of a logarithm is negative. There the logarithm imaginary part is $-i\pi$. Integrating on the relevant y values, we find

$$\begin{aligned}\operatorname{Im}F(a,b) &= \frac{\pi\left(\ln\left[\sqrt{a}+\sqrt{a-1}\right]-b\sqrt{\frac{a-1}{a}}\right)}{2(a-b)^2}\theta(a-1)\\ &-\frac{\pi\left(\ln\left[\sqrt{b}+\sqrt{b-1}\right]-b\sqrt{\frac{b-1}{b}}\right)}{2(a-b)^2}\theta(b-1)\end{aligned} \tag{16}$$

or, recalling the definition of a and b (see [19])

$$\begin{aligned}\operatorname{Im}F(a,b) &= \left\{\frac{4\pi m^4}{M^4\omega^2}\ln\left[\frac{1+\sqrt{1-\frac{4m^2}{M^2}}}{1-\sqrt{1-\frac{4m^2}{M^2}}}\right]-\frac{2\pi m^2}{M^2}\frac{1-\omega}{\omega^2}\sqrt{1-\frac{4m^2}{M^2}}\right\}\theta\left(M^2-4m^2\right)\\ &-\left\{\frac{4\pi m^4}{M^4\omega^2}\ln\left[\frac{1+\sqrt{1-\frac{4m^2}{M^2(1-\omega)}}}{1-\sqrt{1-\frac{4m^2}{M^2(1-\omega)}}}\right]-\frac{2\pi m^2}{M^2}\frac{1-\omega}{\omega^2}\sqrt{1-\frac{4m^2}{M^2(1-\omega)}}\right\}\theta\left(M^2-\frac{4m^2}{1-\omega}\right)\end{aligned} \tag{17}$$

Let us consider the figure 9. Obviously, the first line in (17) corresponds to the vertical cuts (which contribute only if the decaying particle's mass M^2 is greater than $(2m)^2$) while the second line corresponds to the oblique cuts (which contribute only if the virtual photon energy t^2 is greater than $(2m)^2$).

Further, one can see that in the soft photon limit $\omega\to 0$ (or $b\to a$), $\operatorname{Im}F(a,b)$ behaves as a constant. Indeed, applying L'Hospital's rule twice, we get

$$\operatorname{Im}F\left(\frac{M^2}{4m^2},\frac{M^2(1-\omega)}{4m^2}\right)\overset{\omega\to 0}{=}\pi\frac{2m^4}{M^4}\frac{1}{\sqrt{1-4m^2/M^2}} \tag{18}$$

while individually the contribution of each cut is divergent as $1/(a-b)^2 \sim 1/\omega^2$. We will come back to this point later.

Dispersive Part Integral

In order to write down the unsubstracted dispersion integral, let us express $\mathrm{Im}\, F(a,b)$ as a function of the available energy through $a(s) = s/4m^2$ and $b(s) = s(1-\omega)/4m^2$. Then

$$\mathrm{Re}\, F(a(s_o), b(s_o)) = \frac{P}{\pi} \int \frac{ds}{s - s_o} \mathrm{Im}\, F(a(s), b(s)) \tag{19}$$

with $s_o < 4m^2$, such that we can omit the principal part. For such a kinematics $F(a(s_o), b(s_o)) = \mathrm{Re}\, F(a(s_o), b(s_o))$. In the next section we shall analytically continue F to the physical value $s_o = M^2$. The result of (19) is easily obtained in terms of the integrals

$$\int_0^1 \frac{dy}{y_o - y} \ln\left[\tfrac{1+\sqrt{1-y}}{1-\sqrt{1-y}}\right] = 2\arcsin^2\sqrt{\frac{1}{y_o}}, \quad \int_0^1 \frac{dy}{y_o - y}\sqrt{1-y} = 2 - 2\sqrt{y_o - 1}\arcsin\sqrt{\frac{1}{y_o}}$$

valid for $y_o > 1$ (see [14]). We can write $F(a,b)$ with $0 < a < 1$ and $0 < b < 1$ as

$$F(a,b) = -\frac{1}{2(a-b)} + \frac{1}{(a-b)^2}\left(\frac{1}{2}\left(f(a) - f(b)\right) + b\left(g(a) - g(b)\right)\right) \tag{20}$$

in terms of $f(x) \equiv \arcsin^2\sqrt{x}$ and $g(x) \equiv \sqrt{\frac{1-x}{x}}\arcsin\sqrt{x}$.

Analytic Continuation

The results for $a > 1$ and $b > 1$, are obtained by analytic continuation of (20). The analytic continuation of $f(a)$ and $g(a)$ are

$$\begin{aligned} f(x) &= \arcsin^2\left(\sqrt{x}\right) = -\left(\ln\left(\sqrt{x} + \sqrt{x-1}\right) - \frac{1}{2}i\pi\right)^2 \\ g(x) &= \sqrt{\frac{1-x}{x}}\arcsin\sqrt{x} = \sqrt{\frac{x-1}{x}}\left(\ln\left(\sqrt{x} + \sqrt{x-1}\right) - \frac{1}{2}i\pi\right) \end{aligned}$$

The equalities hold for $x \in C, \mathrm{Im}\, x > 0$. The right hand sides define the analytical continuation for $x > 1$. Hence the result is conveniently expressed as (20) with

$$\begin{aligned} f(x) &= \begin{cases} \arcsin^2\left(\sqrt{x}\right) & 0 < x < 1 \\ -\left(\ln\left(\sqrt{x} + \sqrt{x-1}\right) - \frac{1}{2}i\pi\right)^2 & x > 1 \end{cases} \\ g(x) &= \begin{cases} \sqrt{\frac{1-x}{x}}\arcsin\left(\sqrt{x}\right) & 0 < x < 1 \\ \sqrt{\frac{x-1}{x}}\left(\ln\left(\sqrt{x} + \sqrt{x-1}\right) - \frac{1}{2}i\pi\right) & x > 1 \end{cases} \end{aligned} \tag{21}$$

Had we chosen to analytically continue f and g in the lower half complex plane, their imaginary parts would have had the opposite sign. One can verify from (21) that the upper half complex plane analytic continuation reproduces $\mathrm{Im}\, F(a,b)$ as given in (16). This result corresponds to [20] where a sign mistake has to be corrected, and to [21], [22].

6.3 Decay Width, Differential Width and Low's Theorem

The differential decay width is given by $d\Gamma\left(K_S^0 \to e^+e^-\gamma\right) = \frac{1}{2M}\sum_{spin} |\mathcal{M}|^2 \, d\Phi_3$. After a straightforward integration over the electron-positron phase-space, we get the differential rate in terms of the reduced photon energy ω:

$$\frac{d\Gamma\left(K_S^0 \to e^+e^-\gamma\right)/d\omega}{\Gamma\left(K_S^0 \to \pi^+\pi^-\right)} = \frac{\alpha^3}{3\pi^3}\frac{\left|F\left(\frac{1}{a_\pi},\frac{1-\omega}{a_\pi}\right)\right|^2}{a_\pi^2\sqrt{1-a_\pi}}\sqrt{1-\frac{a_e}{1-\omega}}\left[a_e + 2\left(1-\omega\right)\right]\frac{\omega^3}{\left(1-\omega\right)^2}$$

with $a_e = 4m_e^2/M^2$ and $a_\pi = 4m^2/M^2 = 1/a$. The bounds on the ω integration are $\omega_{\min} = 0$ and $\omega_{\max} = 1 - a_e$. A numerical integration of the differential rate gives the prediction for the rate $K_S^0 \to e^+e^-\gamma$ relatively to $K_S^0 \to \pi^+\pi^-$

$$R_{\pi^+\pi^-} = \frac{\Gamma\left(K_S^0 \to e^+e^-\gamma\right)}{\Gamma\left(K_S^0 \to \pi^+\pi^-\right)} \simeq 4.70 \times 10^{-8} \tag{22}$$

where the real and imaginary part of $F\left(\frac{1}{a_\pi},\frac{1-\omega}{a_\pi}\right)$ contribute for 1.26×10^{-8} and 3.43×10^{-8}, respectively.

Let us turn to the soft photon behaviour of the differential width. We have to analyze the loop integral function $F\left(\frac{1}{a_\pi},\frac{1-\omega}{a_\pi}\right)$. This function tends to a constant for very low ω:

$$F\left(\frac{1}{a_\pi},\frac{1-\omega}{a_\pi}\right) \overset{\omega \to 0}{\to} -\frac{a_\pi}{4} + \frac{a_\pi^2}{4\left(a_\pi - 1\right)} g\left(\frac{1}{a_\pi}\right) \tag{23}$$

(note that for $a_\pi < 1$, the imaginary part of (23) is the same as in (18)). Let us emphasize the strong cancellation between the two cuts in the soft photon limit. Individually, each cut diverges as ω^2, but their combination is convergent. Since the rest of the amplitude tends to zero like ω (due to the $\left[g^{\mu\nu}\left(k \cdot t\right) - t^\mu k^\nu\right]$ factor), *Low's theorem is verified: the amplitude behaves as* ω *near* $\omega = 0$. The resulting spectrum is thus in ω^3 (ω^2 from the squared amplitude and a ω from phase space). In other words, if we had forgotten one cut, the resulting spectrum behaviour would have been divergent as $1/\omega$ near zero instead of vanishing like ω^3. In fact, this ω^3 spectrum is exactly the same as in the $\pi^0 \to e^+e^-\gamma$ differential decay rate.

Comparison with the Two-Photon Decay Mode

From the amplitude (14), we readily obtain the amplitude for $K_S^0 \to \gamma\gamma$ by removing the electron current, the photon propagator and by taking the limit $\omega \to 1$:

$$\mathcal{M}\left(K_S^0 \to \gamma\gamma\right) = \frac{-2\alpha}{4\pi} \frac{\mathcal{M}\left(K_S^0 \to \pi^+\pi^-\right)}{m^2} \varepsilon_\mu^* (k)\, \varepsilon_\nu^* \left(k'\right) \left[g^{\mu\nu} \left(k \cdot k'\right) - k'^\mu k^\nu\right] F\left(\frac{M^2}{4m^2}, 0\right)$$

with $F\left(\frac{1}{a_\pi}, 0\right) = \frac{1}{2} a_\pi^2 \left\{ f\left(\frac{1}{a_\pi}\right) - \frac{1}{a_\pi} \right\}$ i.e.

$$F\left(\frac{M^2}{4m^2}, 0\right) = \frac{4m^4}{M^2} \left(\frac{i\pi}{M^2} \ln\left[\frac{1+\sqrt{1-\frac{4m^2}{M^2}}}{1-\sqrt{1-\frac{4m^2}{M^2}}}\right] - \frac{1}{2} \left(\frac{1}{m^2} - \frac{\pi^2}{M^2} + \frac{1}{M^2} \ln^2 \left[\frac{1+\sqrt{1-\frac{4m^2}{M^2}}}{1-\sqrt{1-\frac{4m^2}{M^2}}}\right] \right) \right) \tag{24}$$

which corresponds to the result given in [14], [21], [23]. The relative decay rate is :

$$\frac{\Gamma\left(K_S^0 \to \gamma\gamma\right)}{\Gamma\left(K_S^0 \to \pi^+\pi^-\right)} = \frac{\alpha^2}{\pi^2} \frac{1}{a_\pi^2 \sqrt{1-a_\pi}} \left| F\left(\frac{1}{a_\pi}, 0\right) \right|^2 \simeq 2.94 \times 10^{-6}$$

while the experimental value for this ratio is $(3.5 \pm 1.3) \times 10^{-6}$ [24].

We can now compare the two electromagnetic modes :

$$R_{\gamma\gamma} = \frac{\Gamma\left(K_S^0 \to e^+e^-\gamma\right)}{\Gamma\left(K_S^0 \to \gamma\gamma\right)} \simeq 0.016$$

in accordance with [21] and [19]. This ratio $R_{\gamma\gamma}$ is similar to [24]

$$\frac{\Gamma\left(\pi^0 \to e^+e^-\gamma\right)}{\Gamma\left(\pi^0 \to \gamma\gamma\right)} \simeq 0.012$$

The small difference is due to the phase-space factor. The similarity between a constant coupling model like for $\pi^0 \to e^+e^-\gamma$ and the present loop model can be understood from the behaviour of the photon energy spectrum. Indeed, the pure phase-space spectrum is very strongly peaked at high ω, and therefore the decay rate is quite insensible to the detail of the $F\left(\frac{1}{a_\pi}, \frac{1-\omega}{a_\pi}\right)$ function. Specifically, if we replace $F\left(\frac{1}{a_\pi}, \frac{1-\omega}{a_\pi}\right)$ by its value for $\omega = 1$ (24) we find the ratio $R_{\pi^+\pi^-} \simeq 4.671 \times 10^{-8}$, very close to (22).

6.4 Soft Photon Behaviour with a non-constant Form Factor

We now turn to the case of a non-constant amplitude $\mathcal{M}\left(K_S^0 \to \pi^+\pi^-\right) \equiv \mathcal{F}\left((q+k)^2, (q-t)^2\right)$. We analyze the process from the point of view of

Low's theorem. The constraint of gauge invariance will be seen to require some new structure dependent contributions. We will prove that those new contributions do not affect the interference between absorptive parts observed in previous sections, responsible for the vanishing of the amplitude in the soft photon limit.

Let us concentrate on the absorptive part of the amplitude for $K_S^0 \to \gamma e^+ e^-$. Using the optical theorem (see figure 2), we write

$$2\,\mathrm{Im}\,\mathcal{M}^{\mu}\left(K_S^0 \to \gamma e^+ e^-\right) = \mathcal{V}^{\mu} + \mathcal{D}^{\mu} \tag{25}$$

with $\mathcal{V}^{\mu}$ the vertical cut contribution and $\mathcal{D}^{\mu}$ the diagonal one :

$$\begin{aligned}
\mathcal{V}^{\mu} &= \int d\Phi_{\pi\pi}\ (2\pi)^4\,\delta^{(4)}\left(P - p_1 - p_2\right)\,\mathcal{M}\left(K_S^0 \to \pi^+(p_1)\,\pi^-(p_2)\right) \\
&\quad \times\mathcal{M}^{\mu}\left(\pi^+(p_1)\,\pi^-(p_2) \to \gamma\gamma^* \to \gamma e^+ e^-\right) \\
\mathcal{D}^{\mu} &= \int d\Phi_{\pi\pi}\ (2\pi)^4\,\delta^{(4)}\left(P - p_1 - p_2 - k\right)\,\mathcal{M}^{\mu}\left(K_S^0 \to \pi^+(p_1)\,\pi^-(p_2)\,\gamma(k)\right) \\
&\quad \times\mathcal{M}\left(\pi^+(p_1)\,\pi^-(p_2) \to \gamma^* \to e^+ e^-\right)
\end{aligned}$$

The $\pi\pi \to \gamma e^+ e^-$ amplitude is gauge invariant, as can be easily verified, and so is $\mathcal{V}^{\mu}$, independently of the form of $\mathcal{F}\left(p_1^2, p_2^2\right)$. Therefore $\mathcal{D}^{\mu}$ has to be gauge invariant too. The two bremsstrahlung contributions to $\mathcal{D}^{\mu}$ are $\mathcal{M}^{\mu}_{IB}\left(K_S^0 \to \pi^+(p_1)\,\pi^-(p_2)\,\gamma(k)\right) =$

$$\mathcal{F}\left((p_1+k)^2, p_2^2\right)\frac{p_1^{\mu}}{p_1\cdot k} - \mathcal{F}\left(p_1^2, (p_2+k)^2\right)\frac{p_2^{\mu}}{p_2\cdot k}$$

Trivially, gauge invariance is verified if the form factor is constant. Problems arise if $\mathcal{F}(p_1+k, p_2) \neq \mathcal{F}(p_1, p_2+k)$, since when contracted by k^{μ} the two terms fail to cancel. To maintain gauge invariance, one must supplement the amplitude $K_S^0 \to \pi^+\pi^-\gamma$ with a structure dependent term, for example $\mathcal{M}^{\mu}_{SD}\left(K_S^0 \to \pi^+(p_1)\,\pi^-(p_2)\,\gamma(k)\right) =$

$$\left[\mathcal{F}\left(p_1^2, (p_2+k)^2\right) - \mathcal{F}\left((p_1+k)^2, p_2^2\right)\right]\frac{(p_1-p_2)^{\mu}}{p_1\cdot k - p_2\cdot k} \tag{26}$$

such that $k^{\mu}\mathcal{M}^{\mu}_{IB} + k^{\mu}\mathcal{M}^{\mu}_{SD} = 0$.

Let us now analyze the soft photon behaviour of this process. The form factor is expanded around $k = 0$ as

$$\begin{aligned}
\mathcal{F}\left((p_1+k)^2, p_2^2\right) &= \mathcal{F}\left(p_1^2, p_2^2\right) + 2p_1\cdot k\times\mathcal{F}' + \mathcal{O}\left(k^2\right) \\
\mathcal{F}\left(p_1^2, (p_2+k)^2\right) &= \mathcal{F}\left(p_1^2, p_2^2\right) + 2p_2\cdot k\times\mathcal{F}' + \mathcal{O}\left(k^2\right)
\end{aligned}$$

since we expect $\mathcal{F}' = \frac{\partial\mathcal{F}}{\partial p_1^2}\left(p_1^2, p_2^2\right) = \frac{\partial\mathcal{F}}{\partial p_2^2}\left(p_1^2, p_2^2\right)$ when evaluated on-shell, i.e. at $p_1^2 = p_2^2 = m^2$. The amplitude is now

$$\mathcal{M}^{\mu}_{IB+SD} = \mathcal{F}\left(p_1^2, p_2^2\right)\left[\frac{p_1^{\mu}}{p_1\cdot k} - \frac{p_2^{\mu}}{p_2\cdot k}\right] + \mathcal{O}(k) \tag{27}$$

i.e. the terms of order $(k)^0$ disappeared. This result is valid in full generality, it does not depend on the present structure dependent term (26) chosen for illustration (see [13], [18]).

Now let us analyze the soft photon limit for the imaginary part (25). We have obtained for the two cuts the structures

$$\begin{aligned} \mathcal{V}^\mu &= \mathcal{F}\left(p_1^2, p_2^2\right)\left[\frac{A^\mu}{k} + B^\mu\right] \\ \mathcal{D}^\mu &= \mathcal{F}\left(p_1^2, p_2^2\right)\left[\frac{C^\mu}{k}\right] + \mathcal{O}\left(k\right) \end{aligned}$$

Except for the $\mathcal{O}\left(k\right)$ terms in $\mathcal{D}^\mu$, this is exactly the soft photon expansion one would get for a constant form factor. Hence we expect from the analyzes of previous sections that the combination will vanish when $k \to 0$. Note that it is essential that no constant term (due to the momentum dependence of $\mathcal{F}$) appears in $\mathcal{M}^\mu_{IB+SD}$. This was guaranteed by the gauge invariance of $\mathcal{M}^\mu_{IB+SD}$, which in turn necessitates the contribution of some structure dependent amplitude.

Paradimuonium Decay to γe^+e^- : Figures

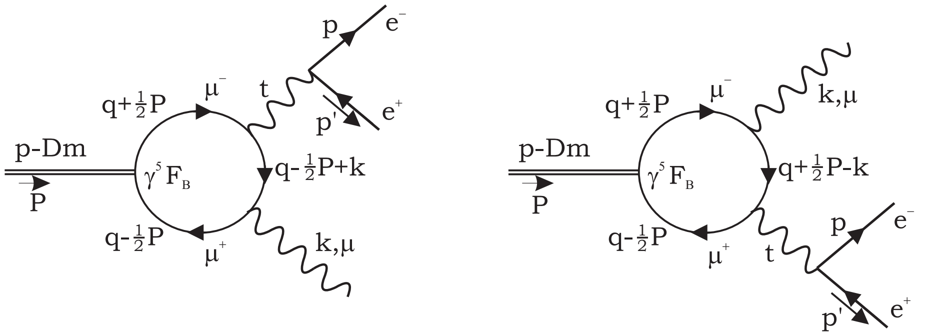


Figure 1 : The loop model direct and crossed diagrams for the decay p-Dm $\rightarrow \gamma e^+e^-$

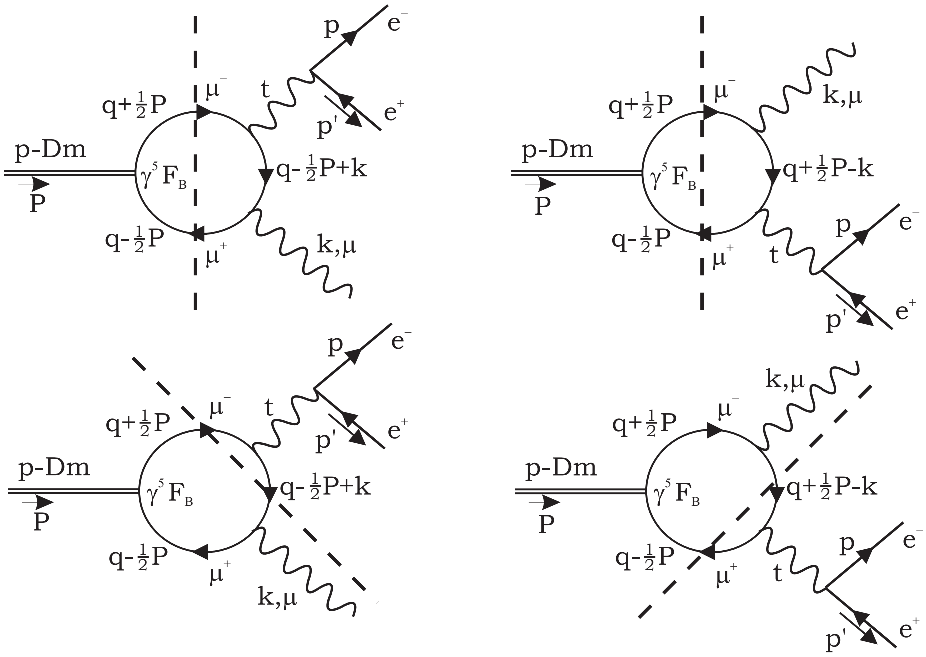


Figure 2 : The vertical and oblique cuts contributing to the amplitude imaginary part

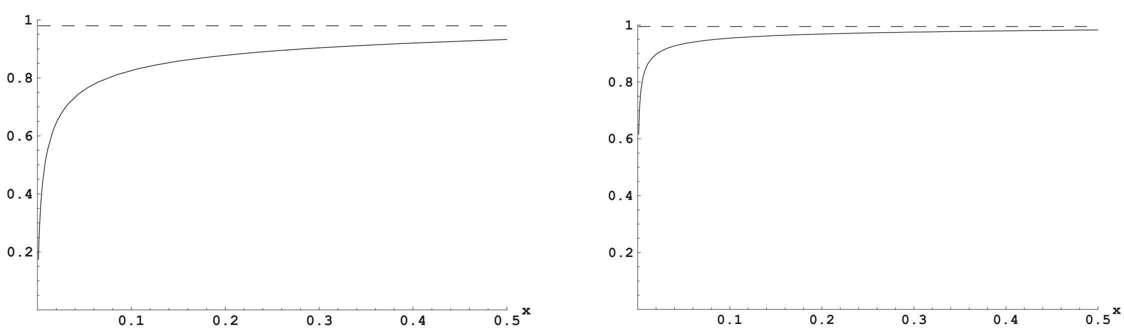


Figure 3 : The vertical cuts (dashed line) and vertical plus oblique cuts (continuous line) contributions, for γ=0.03 and γ=0.0077 as a function of the photon energy x

Orthopositronium Decay to three Photons : Figures

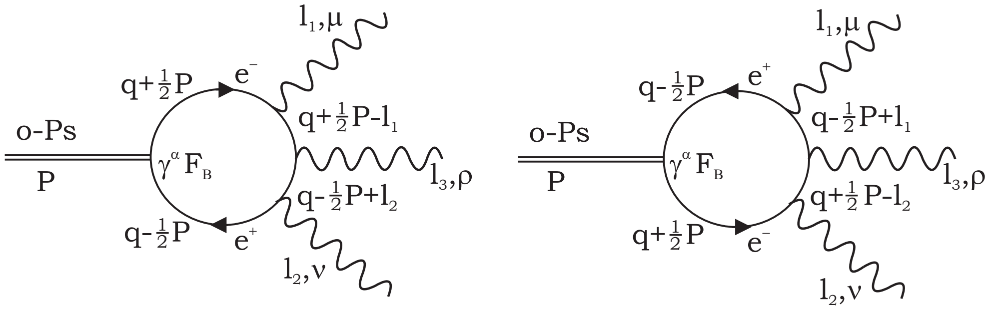
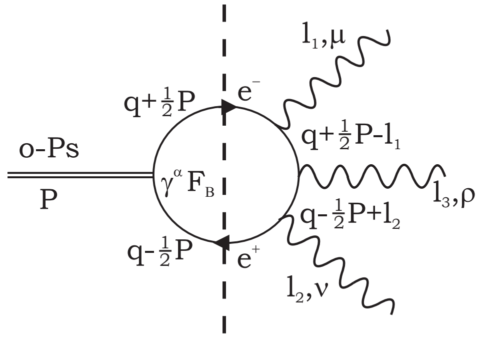


Figure 4 : The loop model diagrams $\Gamma_1^{\mu\nu\rho}$



+ 5 photon permutations

Figure 5 : The vertical cuts, known to reproduce standard results for the o-Ps decay rate

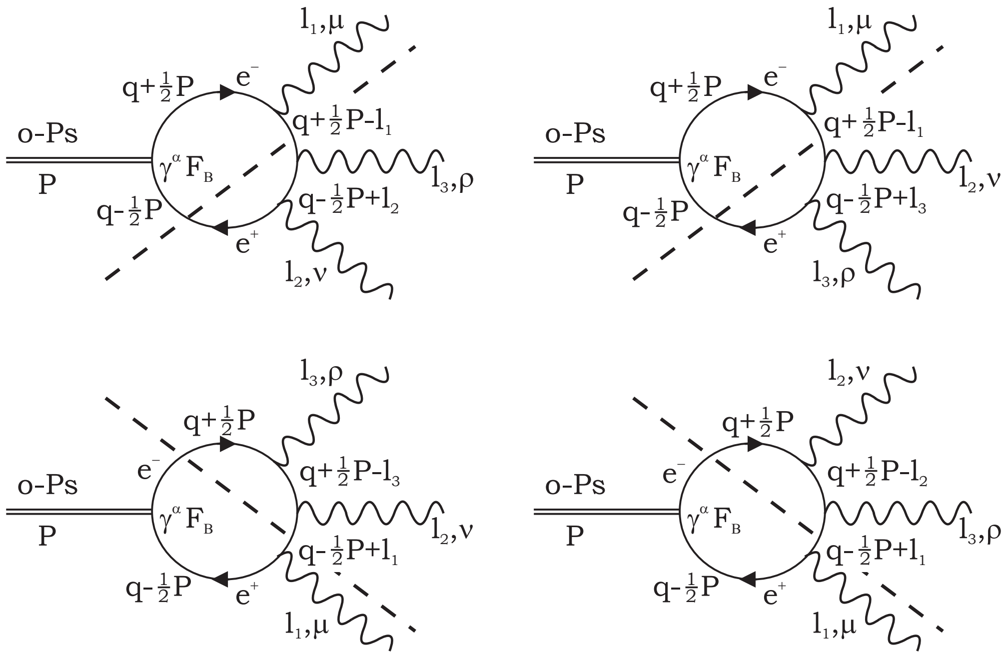


Figure 6 : The oblique cuts. Some new $\mathcal{O}(\alpha^2)$ contributions to the o-Ps decay rate

Orthopositronium Decay (contd.) : Figures

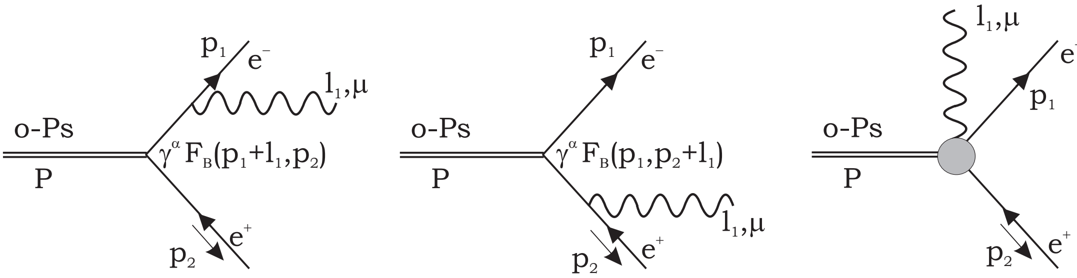


Figure 7 : The Bremsstrahlung and structure amplitudes, contributing to the oblique cuts

Kaon Decay to γe^+e^- : Figures

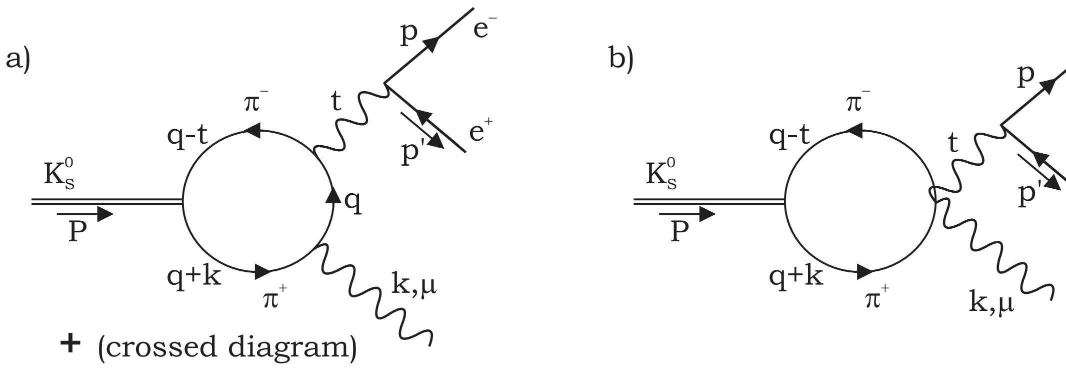


Figure 8 : The amplitude for K_S radiative decay to γe^+e^-, in a pion loop model

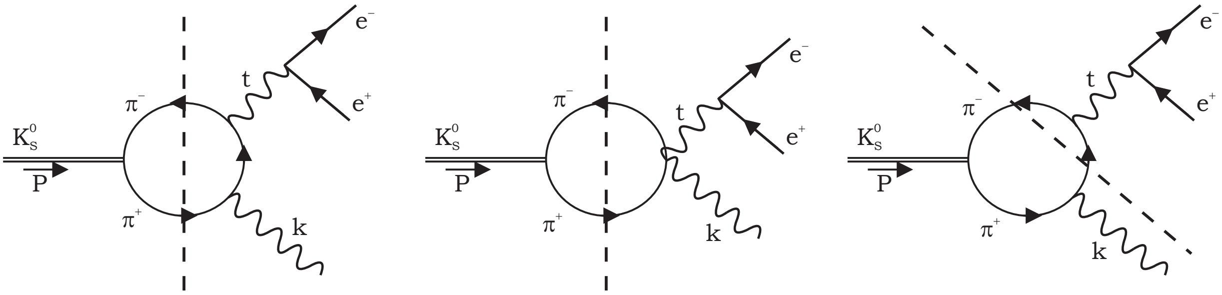


Figure 9 : The vertical and oblique cuts contributing to ImF(a,b)