Parapositronium Decay and Dispersion Relations

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Abstract

Positronium decay rates are computed at the one-loop level, using convolution-type factorized amplitudes. The dynamics of this factorization is probed with dispersion relations, showing that unallowed approximations are usually made, and some $\mathcal{O}\left(\alpha^2\right)$ corrections missed. Further, we discuss the relevance of the Schrödinger wavefunction as the basis for perturbative calculations. Finally, we apply our formalism to the parapositronium two-photon decay.

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1 Introduction

Positronium is a bound state of electron and positron. In this paper, we will be interested in the singlet state, parapositronium, whose decay rate into 2γ has been precisely measured [1]:

$$\Gamma^{exp}\left(p\text{-}Ps \to 2\gamma\right) = (7.9909 \pm 0.0017) \times 10^9 \,\mathrm{sec}^{-1} \tag{1}$$

The corresponding theoretical predictions which include perturbative QED corrections to a non-relativistic treatment of the bound state wavefunction have been computed also with high accuracy (see for example [2], [3], [4]):

$$\begin{aligned}\Gamma\left(p\text{-}Ps \to 2\gamma\right) &= \frac{\alpha^5 m}{2}\left[1 - \left(5 - \frac{\pi^2}{4}\right)\frac{\alpha}{\pi} + 2\alpha^2 \log\frac{1}{\alpha} + 1.75\,(30)\left(\frac{\alpha}{\pi}\right)^2 - \frac{3\alpha^3}{2\pi}\log^2\frac{1}{\alpha}\right] \\ &= 7.98950\,(2) \times 10^9 \,\mathrm{sec}^{-1}\end{aligned}$$

with α the fine structure constant and m the electron mass. As can be observed, agreement between theory and experiment is good. However, the non-logarithmic $\mathcal{O}\left(\alpha^2\right)$ corrections, which have been obtained only recently, are not yet accessible experimentally.

Positronium is a test ground for bound state treatment in Quantum Field Theory. The first try dates back to the $40's$, with decay rates expressed through a factorized formula [5]

$$\Gamma\left(p\text{-}Ps \to 2\gamma\right) = \left|\phi_o\right|^2 \cdot \left(4 v_{rel} \sigma\left(e^+e^- \to 2\gamma\right)\right)_{v_{rel}\to 0}$$

with ϕ_o the Schrödinger positronium wavefunction at the origin, $\sigma\left(e^+e^- \to 2\gamma\right)$ the total cross section for $e^+e^- \to 2\gamma$ and v_{rel} the relative velocity of e^+ and e^- in their center of mass frame. Since then, more sophisticated decay amplitudes have been constructed, and systematic procedures for calculating corrections have been developed. However, the basic factorization of the bound state dynamics from the annihilation process has remained as a basic postulate. For low order corrections, this approximation is unquestionable, but for $\mathcal{O}\left(\alpha^2\right)$ corrections, factorization has to be tested. Indeed, non-perturbative phenomena responsible for the off-shellness of the electron and positron inside the positronium are of $\mathcal{O}\left(\alpha^2\right)$. In other words, to get a sensible theoretical prediction at $\mathcal{O}\left(\alpha^2\right)$, one must carefully analyze how binding energy effects enter the general factorization approach.

In the present paper, we propose a systematic procedure for factorizing the bound state dynamics from the annihilation process. From a fully relativistic model at the one-loop level, where off-shell constituents appear, we will recover the standard factorized amplitude used in the literature. Most importantly, we will show that those standard formulas involve some unnecessary approximations, and we will remove them. Since our derivation relies on well-established techniques of quantum field theory, we conclude that some $\mathcal{O}\left(\alpha^2\right)$ corrections have been forgotten. Our derivation of the factorized formula is particularized to the parapositronium two-photon decay for the sake of definiteness, but it is completely general, so equally valid for orthopositronium decays.

After those general considerations, an alternative and equivalent factorized amplitude is found for parapositronium, which allows some further analyses of the factorized formula. Using this last form, we compute lowest order plus binding energy corrections for parapositronium decay. Those calculations are carried with two different input forms for the bound state wavefunction; first the well-known Schrödinger momentum wavefunction, and then an improved form for this wavefunction. Finally, we comment on those two forms, and point towards the fact that it is possibly not the usual simple Schrödinger wavefunction, but rather the improved one that should be used, due to the dynamics of the factorization.

2 The Loop Model for Positronium Decay

In this section, we introduce the loop model we will use to describe positronium decay.

In that model, the positronium decays into a virtual electron-positron pair which subsequently annihilates into real or virtual photons (an odd number for ortho-states, an even number for para-states). The coupling of the positronium to its constituents is described by a form factor, denoted by F_B. It is not assumed to be a constant, since a constant form factor would amount to consider positronium as a point-like bound state. Specific forms for F_B will be discussed later; for now we just mention that it should somehow be related to the bound state wavefunction.

For parapositronium decay into two photons, our model is represented by figure 1. The corresponding amplitude is written

$$\mathcal{M}^{\mu\nu}\left(p\text{-}Ps \to 2\gamma\right) = \int \frac{d^4q}{\left(2\pi\right)^4} F_B Tr\left\{\gamma_5 \frac{i}{\not{q} - \frac{1}{2}\not{P} - m}\Gamma^{\mu\nu}\frac{i}{\not{q} + \frac{1}{2}\not{P} - m}\right\} \tag{2}$$

with m the electron mass and $F_B \equiv F_B\left(q^2, P\cdot q\right)$. The tensor $\Gamma^{\mu\nu}$ is the scattering amplitude for off-shell e^+e^-, with incoming momenta $\frac{1}{2}P - q$ and $\frac{1}{2}P + q$, into two photons :

$$\Gamma^{\mu\nu}\left(e^+e^- \to \gamma\gamma\right) = ie\gamma^\mu \frac{i}{\not{q} - \frac{1}{2}\not{P} + \not{k}_1 - m} ie\gamma^\nu + ie\gamma^\nu \frac{i}{\not{q} + \frac{1}{2}\not{P} - \not{k}_1 - m} ie\gamma^\mu$$

The decay width is expressed in terms of this amplitude

$$\Gamma\left(p\text{-}Ps \to 2\gamma\right) = \frac{1}{2!}\frac{1}{2M}\int d\Phi_{\gamma\gamma}\sum_{pol}\left|\mathcal{M}^{\mu\nu}\left(p\text{-}Ps \to 2\gamma\right)\varepsilon_{1\mu}^{*}\varepsilon_{2\nu}^{*}\right|^2$$

where $M < 2m$ is the positronium mass.

This model is readily extended to any final states by changing the scattering amplitude Γ to the proper one, and to orthopositronium states by the replacement $\gamma_5 \to \not{e}$ with e the orthopositronium polarization vector. Therefore, the conclusions reached in the next section will be of complete generality.

3 The Standard Approach as a Dispersion Relation

What we intend to show in this section is that the formula (2) leads to the expression for the decay amplitude found in the literature (see for example [2], [3], [6])

$$\mathcal{M}\left(p\text{-}Ps\to 2\gamma\right)\sim\int\frac{d^3\mathbf{k}}{\left(2\pi\right)^3 2E_{\mathbf{k}}}\psi\left(\mathbf{k}^2\right)Tr\left\{\left(1+\gamma^0\right)\gamma_5\Gamma^{\mu\nu}\left(k,k',l_1\right)\right\}\varepsilon_{1\mu}^*\varepsilon_{2\nu}^* \tag{3}$$

if we define the form factor as $F_B\equiv C\psi\left(\mathbf{k}^2\right)\left(\mathbf{k}^2+\gamma^2\right)$; $\psi\left(\mathbf{k}\right)$ is the bound state wavefunction and $\gamma^2=m^2-M^2/4$ (related to the binding energy through $E_B=M-2m\approx -m\alpha^2/4$). The scattering amplitude $\Gamma^{\mu\nu}$ describes the process $e^-\left(k\right)e^+\left(k'\right)\to 2\gamma$ with on-shell electron and positron of momenta $\mathbf{k}$ and $\mathbf{k}'=-\mathbf{k}$ and energies $E_{\mathbf{k}}=E_{\mathbf{k}'}=\sqrt{\mathbf{k}^2+m^2}$.

An interesting feature of this formula is that energy is apparently not conserved, since the electron and positron have energies $E_{\mathbf{k}}>M/2$. The justification of this formula as a simple convolution between the positronium wavefunction and the scattering amplitude for $e^+e^-\to 2\gamma$ is therefore inadequate. What we are going to demonstrate now is that one should understand (3) as a dispersion integral along the loop model branch cut. Further, the appearance of the spin wavefunction of the bound state $\left(1+\gamma^0\right)\gamma_5$ is questionable. Indeed, it is clear that it is the *moving* electron-positron pair which has to be projected onto the required spin state, so the projector cannot be simply $\left(1+\gamma^0\right)\gamma_5$. This is confirmed by the fact that (3) will be found by neglecting some momentum $|\mathbf{k}|$ dependence in the exact formula (2). Taking into account the exact projector will introduce some forgotten corrections.

Let us now demonstrate that by using a dispersion relation (see [8], [9]) to express the loop integration of (2), we will reach (3). Let us emphasize once again that the whole discussion of this section is readily extended to any para- or orthopositronium decay amplitude.

We first compute the imaginary part of (2), $\operatorname{Im}\mathcal{T}_{fi}\left(P^2\right)\equiv\operatorname{Im}\mathcal{M}^{\mu\nu}\left(p\text{-}Ps\to 2\gamma\right)\varepsilon_{1\mu}^*\varepsilon_{2\nu}^*$, for an arbitrary initial mass P^2. Considering the two possible cuts (figure 2), we obtain $\operatorname{Im}\mathcal{T}_{fi}$ by replacing the two propagators on each side of $\Gamma^{\mu\nu}$ by delta functions

$$\begin{aligned}\operatorname{Im}\mathcal{T}_{fi}\left(P^2\right) &= \int\frac{d^4q}{2\left(2\pi\right)^2}F_B\delta\left(\left(q-\frac{P}{2}\right)^2-m^2\right)\delta\left(\left(q+\frac{P}{2}\right)^2-m^2\right)\\ &\quad\times Tr\left\{\gamma_5\left(\not{q}-\frac{\not{P}}{2}+m\right)\Gamma^{\mu\nu}\left(\not{q}+\frac{\not{P}}{2}+m\right)\right\}\varepsilon_{1\mu}^*\varepsilon_{2\nu}^*\end{aligned}$$

After a straightforward integration over q^0 and $|\mathbf{q}|$, with $P=\left(\sqrt{P^2},\mathbf{0}\right)$, we reach

$$\begin{aligned}\operatorname{Im}\mathcal{T}_{fi}\left(P^2\right) &= \frac{1}{16\pi}\sqrt{1-\frac{4m^2}{P^2}}\theta\left(P^2-4m^2\right)\int\frac{d\Omega_{\mathbf{q}}}{4\pi}F_B\\ &\quad\times Tr\left\{\gamma_5\left(\not{q}-\frac{\not{P}}{2}+m\right)\Gamma^{\mu\nu}\left(\not{q}+\frac{\not{P}}{2}+m\right)\right\}\varepsilon_{1\mu}^*\varepsilon_{2\nu}^*\end{aligned}$$

In the course of the derivation, the delta functions forced $q^0 = 0$ and $|\mathbf{q}| = \sqrt{P^2/4 - m^2}$. In other words, the electron momenta are

$$\frac{1}{2}P \pm q = \left(\sqrt{\frac{P^2}{4}}, \pm\mathbf{q}\right) \text{ with } \left(\frac{1}{2}P \pm q\right)^2 = \frac{P^2}{4} - |\mathbf{q}|^2 = m^2 \qquad (4)$$

This kinematics is to be understood in the trace evaluation. The angular dependence arises from the relative orientations of $\mathbf{q}$ and the photon momenta $\mathbf{l}_1$. Note also that the relation (4) cannot be satisfied for the physical value $P^2 = M^2 < 4m^2$. This is obvious since the loop cannot have an imaginary part for the physical bound states, its constituents being always off-shell. From the kinematics (4) one can prove that the factors on both sides of $\Gamma^{\mu\nu}$ are true projectors, which serves to enforce gauge invariance in the expression

$$\left(\not{q} - \frac{\not{P}}{2} + m\right)\Gamma^{\mu\nu}\left(\not{q} + \frac{1}{2}\not{P} + m\right)$$

Indeed, those two projectors play exactly the same role as external spinors when demonstrating Ward identities. Remark also that the dependence of the form factor on the loop energy can be taken arbitrarily since $q^0 = 0$.

The real part will now be calculated using an unsubstracted dispersion relation

$$\operatorname{Re}\mathcal{T}_{fi}\left(M^2\right) = \frac{P}{\pi}\int_{4m^2}^{+\infty}\frac{ds}{s - M^2}\operatorname{Im}\mathcal{T}_{fi}\left(s = P^2\right) \qquad (5)$$

where it is understood that P^2 should be replaced by s everywhere, i.e. scalar products that will appear when evaluating the trace should be expressed with the kinematics defined for an initial energy s. Since $M^2 < 4m^2$, the principal part can be omitted and $\mathcal{T}_{fi}\left(M^2\right) = \operatorname{Re}\mathcal{T}_{fi}\left(M^2\right)$. Now let us write the form factor in the general form

$$F_B \equiv C\phi_o\mathcal{F}\left(\mathbf{q}^2\right)\left(\mathbf{q}^2 + \gamma^2\right) = C\phi_o\mathcal{F}\left(s/4 - m^2\right)\cdot\left(s - M^2\right)/4 \qquad (6)$$

with $\gamma^2 \equiv m^2 - M^2/4$ and ϕ_o the bound state wavefunction at zero separation. Then Eq. (5) can be written as

$$\begin{aligned}\mathcal{T}_{fi}\left(M^2\right) &= C\phi_o\int_{4m^2}^{+\infty}ds\int\frac{d\Omega_{\mathbf{q}}}{4\pi}\mathcal{F}\left(s/4 - m^2\right)\frac{\sqrt{1 - 4m^2/s}}{64\pi^2} \\ &\quad\times Tr\left\{\gamma_5\left(\not{q} - \frac{\not{P}}{2} + m\right)\Gamma^{\mu\nu}\left(\not{q} + \frac{\not{P}}{2} + m\right)\right\}\varepsilon^*_{1\mu}\varepsilon^*_{2\nu}\end{aligned}$$

Let us transform the s integral back into a $|\mathbf{q}|$ integral, keeping in mind the constraints obtained when extracting the imaginary part. Using $\mathbf{q}^2 = s/4 - m^2$, $ds = 8|\mathbf{q}|\,d|\mathbf{q}|$, the decay amplitude dispersion integral is

$$\mathcal{T}_{fi}\left(M^2\right) = \frac{C}{2}\phi_o\int\frac{d^3\mathbf{q}}{(2\pi)^3}\frac{\mathcal{F}\left(\mathbf{q}^2\right)}{\sqrt{P^2(\mathbf{q})}}Tr\left\{\gamma_5\left(\not{q} - \frac{\not{P}(\mathbf{q})}{2} + m\right)\Gamma^{\mu\nu}\left(\not{q} + \frac{\not{P}(\mathbf{q})}{2} + m\right)\right\}\varepsilon^*_{1\mu}\varepsilon^*_{2\nu}$$

where, as the notation suggests, it is understood that any P^2 appearing in the amplitude must be replaced by $4|\mathbf{q}|^2 + 4m^2$. In particular, $\sqrt{P^2(\mathbf{q})}$ can be replaced by $2E_{\mathbf{q}}$ with $E_{\mathbf{q}} = \sqrt{|\mathbf{q}|^2 + m^2}$. This amounts to consider the scattering amplitude with incoming on-shell electron-positron having momenta $\left(\frac{1}{2}P(\mathbf{q}) \pm q\right)^2 = m^2$ (since $q^0 = 0$). Note the fact that $E_{\mathbf{q}} > M/2$, apparently the energy is not conserved. This is not surprising since the present formula is a dispersion integral, done along the cut where $P^2(\mathbf{q}) > 4m^2$. Finally, in view of the kinematics, we introduce $k = \frac{1}{2}P(\mathbf{q}) + q$ and $k' = \frac{1}{2}P(\mathbf{q}) - q$ (hence $E_k = E_{k'} = E_{\mathbf{q}}$ and $\mathbf{k} = -\mathbf{k}' = \mathbf{q}$) to write the amplitude simply as

$$\mathcal{T}_{fi}\left(M^2\right) = \frac{C}{2}\int \frac{d^3\mathbf{k}}{(2\pi)^3\, 2E_{\mathbf{k}}}\left[\phi_o \mathcal{F}\left(\mathbf{k}^2\right)\right] Tr\left\{\gamma_5\left(-\not{k}' + m\right)\Gamma^{\mu\nu}\left(k, k', l_1\right)\left(\not{k} + m\right)\right\}\varepsilon^*_{1\mu}\varepsilon^*_{2\nu} \tag{7}$$

where $\Gamma^{\mu\nu}(k, k', l_1)$ is the amplitude for on-shell $e^-(k)\, e^+(k')$ scattering into 2γ. Gauge invariance is present due to the two projectors, well defined since $k^2 = k'^2 = m^2$. Hereafter, the equation (7) will be referred as the standard approach expression. Indeed, we can recognize (7) as the standard decay amplitude for bound states [6] :

$$\mathcal{M}\left(p\text{-}Ps \to 2\gamma\right) = \sqrt{2M}\int \frac{d^3\mathbf{k}}{(2\pi)^3}\psi(\mathbf{k})\frac{1}{\sqrt{2E_{\mathbf{k}}}}\frac{1}{\sqrt{2E_{\mathbf{k}}}}\mathcal{M}\left(e^-(\mathbf{k}), e^+(-\mathbf{k}) \to 2\gamma\right) \tag{8}$$

with the amplitude constrained to the required spin state given by

$$\mathcal{M}\left(e^-(\mathbf{k}), e^+(-\mathbf{k}) \to 2\gamma\right) = \frac{1}{2\sqrt{2m}} Tr\left\{\gamma_5\left(-\not{k}' + m\right)\Gamma^{\mu\nu}\left(k, k', l_1\right)\left(\not{k} + m\right)\right\}\varepsilon^*_{1\mu}\varepsilon^*_{2\nu}$$

Matching the present expression for the amplitude (7) with the expression (8), we obtain the constant C

$$C = \sqrt{M}/m \tag{9}$$

and the form factor $\mathcal{F}$ is identified with the wavefunction as $\phi_o\mathcal{F}\left(\mathbf{k}^2\right) = \psi(\mathbf{k})$. This means that the function $\mathcal{F}\left(\mathbf{k}^2\right)$ is normalized to unity and behaves as a delta of the momentum in the limit of vanishing binding energy :

$$\int \frac{d^3\mathbf{k}}{(2\pi)^3}\mathcal{F}\left(\mathbf{k}^2\right) = 1, \qquad \lim_{\gamma \to 0}\mathcal{F}\left(\mathbf{k}^2\right) = (2\pi)^3\,\delta^{(3)}(\mathbf{k}) \tag{10}$$

From the expression (7), it is clear that (3) is an approximation. Indeed, neglecting $\mathbf{k}$ dependences in the two projectors $(-\not{k}' + m)$ and $(\not{k} + m)$, we reach (3) from (7). The projectors appearing in (7) for particles in motion will introduce some new corrections to the positronium decay rate of the order of the binding energy γ^2, i.e. α^2. Further, when generalizing to orthopositronium decay amplitudes, the expression (7) is gauge invariant while (3) is not. The fact that for the parapositronium decay into two photons (3) is gauge invariant is due to the peculiar feature of γ^5 appearance in the trace (see (11) below).

In conclusion, the equivalence between the loop model and the standard expression (7) for positronium decay amplitudes opens new possibilities for explicit computations. This is precisely what we are going to exploit in the following sections.

4 Form Factor Dispersion Relation

We have established the correspondence between (2) and (7). Let us construct an alternative, but equivalent, dispersion procedure specific to the two-photon case that will be used in explicit calculations. By computing the trace in (2), the tensor structure factorizes

$$\mathcal{M}\left(p\text{-}Ps \rightarrow \gamma\gamma\right) = 8me^2\varepsilon^{\mu\nu\rho\sigma}l_{1,\rho}l_{2,\sigma}\varepsilon_{1\mu}^*\varepsilon_{2\nu}^*\mathcal{I}\left(M^2\right) \tag{11}$$

In this equation, $\mathcal{I}\left(M^2\right)$ can be viewed as an effective form factor, modeled as the electron-positron loop with the coupling F_B. There is only one term in $\mathcal{I}\left(M^2\right)$ since the direct and crossed amplitudes are equal under $q \rightarrow -q$, i.e. an allowed variable change as $F_B\left(q^2, P\cdot q\right) = F_B\left(q^2, -P\cdot q\right)$, and we write

$$\mathcal{I}\left(P^2\right) = \eta\int\frac{d^4q}{\left(2\pi\right)^4}F_B\frac{1}{\left(q-\frac{1}{2}P\right)^2 - m^2}\frac{1}{\left(q+\frac{1}{2}P\right)^2 - m^2}\frac{1}{\left(q-\frac{1}{2}P+l_1\right)^2 - m^2} \tag{12}$$

It is to evaluate the effective form factor $\mathcal{I}\left(P^2\right)$ that we will now use dispersion techniques. Remark that the factorization of the tensor part is interesting, since gauge invariance is manifest, and that $\mathcal{I}\left(P^2\right)$ is convergent while the amplitude (2) is superficially divergent.

The factor η is introduced because there is a subtlety in the above factorization. Indeed, there is an arbitrariness in the choice of variable for the dispersion integral. This situation is well-known for the photon vacuum polarization :

$$\Pi^{\mu\nu}\left(k^2\right) = \left(k^2g^{\mu\nu} - k^\mu k^\nu\right)\Pi_1\left(k^2\right) = \left(g^{\mu\nu} - \frac{k^\mu k^\nu}{k^2}\right)\Pi_2\left(k^2\right) \tag{13}$$

where one writes a dispersion relation for $\Pi_1\left(k^2\right)$, which is less divergent than $\Pi_2\left(k^2\right)$ due to the factorization of the tensor structure. The analogue of (13) here is

$$\mathcal{M}\left(p\text{-}Ps \rightarrow \gamma\gamma\right) = 8me^2\varepsilon^{\mu\nu\rho\sigma}l_{1,\rho}l_{2,\sigma}\varepsilon_{1\mu}^*\varepsilon_{2\nu}^*\mathcal{I}_1\left(M^2\right) = 8me^2\varepsilon^{\mu\nu\rho\sigma}\frac{l_{1,\rho}}{M}\frac{l_{2,\sigma}}{M}\varepsilon_{1\mu}^*\varepsilon_{2\nu}^*\mathcal{I}_2\left(M^2\right)$$

Here we will choose to write a dispersion relation for $\mathcal{I}_2\left(M^2\right)$, and this corresponds to the choice $\eta = P^2/M^2$. This choice seems arbitrary, but is in fact necessary to recover the results of the first section, i.e. the standard expression (8). To understand this, consider the factorized amplitude (11). In the dispersion procedure used to recover the standard approach in the previous section, the dispersion relation was built on the whole amplitude, so it is clear that the photon momenta appearing in the tensor structure were also incorporated, i.e. they were reduced as $l_{1,\rho} \rightarrow \overline{l_{1,\rho}} \times \sqrt{P^2}/2$. That is the reason why we must include a factor P^2 into the effective form factor $\mathcal{I}\left(P^2\right)$.

Let us give a general expression for $\mathcal{I}\left(P^2\right)$ as a dispersion integral. The loop integral $\mathcal{I}\left(P^2\right)$ has an imaginary part obtained by cutting the propagators

$$\mathrm{Im}\,\mathcal{I}\left(P^2\right) = \frac{1}{2}\frac{P^2}{M^2}\int\frac{d^4q}{\left(2\pi\right)^4}F_B\frac{2\pi i\delta\left(\left(q-\frac{1}{2}P\right)^2 - m^2\right)2\pi i\delta\left(\left(q+\frac{1}{2}P\right)^2 - m^2\right)}{\left(q-\frac{1}{2}P+l_1\right)^2 - m^2}$$

$$= \frac{F_B}{8\pi M^2} \int \frac{d\Omega_{\mathbf{q}}}{4\pi} \frac{\sqrt{1-4m^2/s}}{1+\sqrt{1-4m^2/s}\cos\theta} \theta\left(s-4m^2\right) \tag{14}$$

with $s = P^2$. The form factor is evaluated for $|\mathbf{q}|^2 = s/4 - m^2$ and $q^0 = 0$. The unsubstracted dispersion relation is

$$\mathrm{Re}\,\mathcal{I}\left(M^2\right) = \frac{P}{\pi} \int_{-\infty}^{+\infty} \frac{ds}{s - M^2} \,\mathrm{Im}\,\mathcal{I}\left(s\right) \tag{15}$$

In our domain $M^2 < 4m^2$ and $\mathcal{I}\left(M^2\right) = \mathrm{Re}\,\mathcal{I}\left(M^2\right)$. With the change of variable $|\mathbf{q}|^2 = s/4 - m^2$, $ds = 8\,|\mathbf{q}|\,d\,|\mathbf{q}|$, we write

$$\mathcal{I}\left(M^2\right) = \mathrm{Re}\,\mathcal{I}\left(M^2\right) = \frac{C\phi_o}{2M^2} \int \frac{d^3\mathbf{q}}{(2\pi)^3} \mathcal{F}\left(\mathbf{q}^2\right) \frac{1}{\sqrt{|\mathbf{q}|^2 + m^2} + |\mathbf{q}|\cos\theta} \tag{16}$$

where $F_B \equiv C\phi_o \mathcal{F}\left(\mathbf{q}^2\right)\left(\mathbf{q}^2 + \gamma^2\right)$ (see 6).

The final expression of the amplitude is therefore

$$\mathcal{M}\left(p\text{-}Ps \rightarrow \gamma\gamma\right) = 8me^2 \frac{C\phi_o}{2M^2} \left[\varepsilon^{\mu\nu\rho\sigma} l_{1,\rho} l_{2,\sigma} \varepsilon^*_{1\mu} \varepsilon^*_{2\nu}\right] \int \frac{d^3\mathbf{q}}{(2\pi)^3} \mathcal{F}\left(\mathbf{q}^2\right) \frac{1}{\sqrt{|\mathbf{q}|^2 + m^2} + |\mathbf{q}|\cos\theta} \tag{17}$$

and the decay rate is expressed as

$$\Gamma\left(p\text{-}Ps \rightarrow \gamma\gamma\right) = \frac{4\pi\alpha^2 m^2}{M} C^2 \left|\phi_o\right|^2 \left| \int \frac{d^3\mathbf{q}}{(2\pi)^3} \mathcal{F}\left(\mathbf{q}^2\right) \frac{1}{\sqrt{|\mathbf{q}|^2 + m^2} + |\mathbf{q}|\cos\theta} \right|^2 \tag{18}$$

This is our third representation for the same decay amplitude : the first is the loop integral (2), the second is the well-known amplitude (7) or (8) (with no approximation for projectors) viewed as a dispersion integral for the amplitude, and the third is the present dispersion integral (17) for the effective loop form factor $\mathcal{I}\left(P^2\right)$. All three procedures are strictly equivalent to each other.

5 Positronium Decay to two Photons

Before going through the rate calculation, we will analyze the delta limit for the form factor as in (10). Then we will go through two different calculations of $\Gamma\left(p\text{-}Ps \rightarrow \gamma\gamma\right)$, obtained for specific choices of F_B (or equivalently, $\mathcal{F}\left(\mathbf{q}^2\right)$).

5.1 Decay Rate in the Static Limit

To compute the decay rate in the limit $\gamma^2 \rightarrow 0$ for the form factor, we do not need to specify F_B. We just need to know that

$$F_B = C\phi_o \mathcal{F}\left(\mathbf{q}^2\right)\left(\mathbf{q}^2 + \gamma^2\right) \overset{\gamma^2 \rightarrow 0+}{\rightarrow} C\phi_o \left(2\pi\right)^3 \delta^{(3)}\left(\mathbf{q}\right)\left(\mathbf{q}^2 + \gamma^2\right)$$

By setting $\mathcal{F}\left(\mathbf{q}^2\right) = (2\pi)^3\,\delta^{(3)}\left(\mathbf{q}\right)$ in (12), we must recover exactly the lowest order decay rate $\Gamma\left(p\text{-}Ps \to \gamma\gamma\right) = \frac{1}{2}\alpha^5 m$. Using (18), we get

$$\int \frac{d^3\mathbf{q}}{(2\pi)^3}\,(2\pi)^3\,\delta^{(3)}\left(\mathbf{q}\right)\frac{1}{\sqrt{\left|\mathbf{q}\right|^2 + m^2} + \left|\mathbf{q}\right|\cos\theta} = \frac{1}{m}$$

Importantly, this result is independent of the binding energy : the loop do not introduce any corrections in the static limit. The decay rate in that limit is therefore :

$$\Gamma\left(p\text{-}Ps \to \gamma\gamma\right) = \frac{1}{2}\alpha^5 m\left(\frac{m^2}{M}C^2\right) \tag{19}$$

with $\left|\phi_0\right|^2 = \alpha^3 m^3/8\pi$. It remains to match C such that purely kinematic corrections vanish (M factors in the above formula arise from products like $\left(l_1 \cdot l_2\right) = M^2/2$ and from the $1/2M$ decay width factor, while m comes from electron propagators in the loop and from the wavefunction ϕ_0). With the definition (9) $C = \sqrt{M}/m$, the decay rate is exactly $\Gamma\left(p\text{-}Dm \to \gamma\gamma\right) = \frac{1}{2}\alpha^5 m$ as it should. In other words, the value for C obtained by matching (7) and (8) is such that no correction arise from factor M/m in the static limit.

To conclude, let us repeat that we have not specified the form factor. This means that any form factor which has a three-dimensional delta function limit for $\gamma^2 \to 0$ gives the correct lowest order decay rate $\frac{1}{2}\alpha^5 m$. In the following, we shall present two forms, both built on the Schrödinger momentum wavefunction.

5.2 Schrödinger Form Factor

We can now apply the formulas of the preceding section to write down the dispersion integral for the form factor

$$\mathcal{F}_I\left(\mathbf{q}^2\right) = \frac{8\pi\gamma}{\left(\mathbf{q}^2 + \gamma^2\right)^2} \tag{20}$$

with γ^2 related to the binding energy and the fine structure constant through $E_B = M - 2m = -m\alpha^2/4$. This form factor is just the Shrödinger momentum wavefunction for the bound state (note that it satisfies the properties (10) as used in [3]).

By using the formula (14) and (6), the imaginary part is after the angular integration

$$\operatorname{Im}\mathcal{I}\left(s\right) = \frac{C\phi_o}{16\pi M^2}\times\left[\frac{8\pi\gamma}{\left(\mathbf{q}^2+\gamma^2\right)^2}\left(\mathbf{q}^2+\gamma^2\right)\right]_{\mathbf{q}^2 = s/4 - m^2}\times\ln\left[\frac{1+\sqrt{1-\frac{4m^2}{s}}}{1-\sqrt{1-\frac{4m^2}{s}}}\right]\times\theta\left(s-4m^2\right)$$

We now integrate the imaginary part using the unsubstracted dispersion relation (15). As always, $M^2 < 4m^2$ so that the principal part can be forgotten and $\mathcal{I}\left(M^2\right) = \operatorname{Re}\mathcal{I}\left(M^2\right)$. For the given form factor $\mathcal{F}_I\left(\mathbf{q}^2\right)$, the calculation of integral $\mathcal{I}\left(M^2\right)$ is now straightforward, and we get

$$\mathcal{I}\left(M^2\right) = \frac{C\phi_o}{M^3}\frac{2}{\pi}\arctan\frac{M}{2\gamma} \tag{21}$$

The integral needed for this calculation is

$$\int_0^1 \frac{dx}{x_o - x} \ln\left[\frac{1+\sqrt{1-x}}{1-\sqrt{1-x}}\right] \overset{x_o \geq 1}{=} 2\arctan^2 \frac{1}{\sqrt{x_o - 1}} \tag{22}$$

and its derivatives $(\partial/\partial x_o)^n$. With the result for $\mathcal{I}$, the decay rate is

$$\Gamma\left(p\text{-}Ps \to \gamma\gamma\right) = \frac{1}{2}\alpha^5 m \left(\frac{4m^2}{M^2}\right)\left(\frac{2}{\pi}\arctan\frac{M}{2\gamma}\right)^2$$

where we used $|\phi_0|^2 = \alpha^3 m^3/8\pi$ and $C = \sqrt{M}/m$. By expanding this result around $\gamma = 0$, and expressing corrections as a series in the fine structure constant α, we recover the standard result as zeroth order :

$$\begin{aligned} \Gamma\left(p\text{-}Ps \to \gamma\gamma\right) &= \frac{1}{2}\alpha^5 m \left(1 - \frac{\alpha}{\pi} + \frac{1}{8}\alpha^2 - \frac{13}{96\pi}\alpha^3 + \frac{1}{64}\alpha^4 + \mathcal{O}\left(\alpha^5\right)\right)^2 \\ &\approx \frac{1}{2}\alpha^5 m \left(1 - 0.637\alpha + 0.351\alpha^2 - 0.166\alpha^3 + 0.074\alpha^4 + \mathcal{O}\left(\alpha^5\right)\right) \end{aligned}$$

Numerically, the corrections at the one-loop level are

$$\Gamma^{Form\,Fact.}\left(p\text{-}Ps \to \gamma\gamma\right) = \Gamma_o \times 0.9954 \approx 7.9956 \times 10^9 \,\text{sec}^{-1}$$

If we combine the present correction with radiative corrections up to order $\alpha^2 \ln\alpha$ as found in the literature [2], the theoretical value is modified as

$$\Gamma^{Rad.Corr.+Form\,Fact.}\left(p\text{-}Ps \to \gamma\gamma\right) = 7.9527 \times 10^9 \,\text{sec}^{-1}$$

while the experimental value is (1). We can therefore note that the present form factor leads to a too small value for $\Gamma\left(p\text{-}Ps \to \gamma\gamma\right)$, since it introduces new order α corrections. The problem is in the form factor, which do not converge fast enough towards the static limit delta. In other words, for a given γ^2, $\mathcal{F}_I\left(\mathbf{q}^2\right)$ is not enough peaked around $\mathbf{q} = 0$.

5.3 Improved Schrödinger form factor

The second possibility we analyze is

$$\mathcal{F}_{II}\left(\mathbf{q}^2\right) = \frac{32\pi\gamma^3}{\left(\mathbf{q}^2 + \gamma^2\right)^3} \tag{23}$$

This form factor has the same delta limit property (10), but converges faster than (20). We can therefore expect much smaller corrections than with $\mathcal{F}_I\left(\mathbf{q}^2\right)$. Repeating the derivation of the preceding section, we get from (14) :

$$\operatorname{Im}\mathcal{I}\left(s\right) = \frac{C\phi_o}{16\pi M^2} \times \left[\frac{32\pi\gamma^3}{\left(\mathbf{q}^2+\gamma^2\right)^3}\left(\mathbf{q}^2+\gamma^2\right)\right]_{\mathbf{q}^2 = s/4 - m^2} \times \ln\left[\frac{1+\sqrt{1-\frac{4m^2}{s}}}{1-\sqrt{1-\frac{4m^2}{s}}}\right] \times \theta\left(s - 4m^2\right)$$

The unsubstracted dispersion integral is (15), and we get using the result (22) :

$$\mathcal{I}\left(M^2\right) = \mathrm{Re}\,\mathcal{I}\left(M^2\right) = \frac{C\phi_o}{M^3}\left(\frac{32\gamma^3}{\pi M^3}\right)\left(\frac{M^2}{8\gamma^2} - \frac{M}{4\gamma}\left(1 - \frac{M^2}{4\gamma^2}\right)\arctan\frac{M}{2\gamma}\right)$$

The rate is then

$$\Gamma\left(p\text{-}Ps \to \gamma\gamma\right) = \frac{1}{2}\alpha^5 m\left(\frac{4m^2}{M^2}\right)\left[\left(\frac{32\gamma^3}{\pi M^3}\right)\left(\frac{M^2}{8\gamma^2} - \frac{M}{4\gamma}\left(1 - \frac{M^2}{4\gamma^2}\right)\arctan\frac{M}{2\gamma}\right)\right]^2$$

Thus, transcribing into a series in α :

$$\Gamma\left(p\text{-}Ps \to \gamma\gamma\right) = \frac{1}{2}\alpha^5 m\left(1 - \frac{1}{4}\alpha^2 + \frac{2}{3\pi}\alpha^3 - \frac{7}{64}\alpha^4 + \mathcal{O}\left(\alpha^5\right)\right)$$

Numerically, the corrections starting at order α^2 are

$$\Gamma\left(p\text{-}Ps \to \gamma\gamma\right) = \left(1 - 1.32 \times 10^{-5}\right) \times \Gamma_o$$

i.e. very small. If we combine the present correction with radiative corrections up to order $\alpha^2 \ln\alpha$ as found in the literature, the theoretical value is modified as

$$\Gamma^{rad.Corr.+Form\,Fact.}\left(p\text{-}Ps \to \gamma\gamma\right) = 7.9894 \times 10^9\ \mathrm{sec}^{-1}$$

As announced, this form factor leads to acceptable corrections, contrary to the $\mathcal{F}_I\left(\mathbf{q}^2\right)$ form factor.

Before closing this section, let us discuss how the form factor $\mathcal{F}_{II}\left(\mathbf{q}^2\right)$ could arise as the right form factor. First, one can see that it implies the F_B form

$$F_B = C\phi_o \frac{32\pi\gamma^3}{\left(\mathbf{q}^2 + \gamma^2\right)^3}\left(\mathbf{q}^2 + \gamma^2\right) = C\phi_o\left[\frac{8\pi\gamma}{\left(\mathbf{q}^2 + \gamma^2\right)^2}\right]4\gamma^2$$

The factor in brackets is the momentum Schrödinger wavefunction while the additional factor γ^2 gives to the coupling F_B the desirable property of vanishing when $\gamma \to 0$. From known Bethe-Salpeter analyses (see for example [3],[7],[4]), one constructs an approximated bound state wavefunction by considering only coulombic photon exchanges in the Bethe-Salpeter kernel. Let $\Psi(q)$ be this wavefunction (the Barbieri-Remiddi wavefunction [7]), but with its spin wavefunction part omitted. Then, if one defines the coupling as

$$F_B = C \times 4\gamma^2 \times \left(\mathbf{q}^2 + \gamma^2\right) \times \Psi(q) \qquad (24)$$

by going through the dispersion analyses of section 3, one ends up with $\psi\left(\mathbf{q}^2\right) = \mathcal{F}_{II}\left(\mathbf{q}^2\right)$. The reason for this is quite technical, but let us just mention that usually one "uses" some part of $\Psi(q)$ to set the energy q^0 to zero to go from a four-dimensional towards a three-dimensional convolution-type decay amplitude (see [3], [4]), while here it is the dispersion relation which enforces $q^0 = 0$, leaving $\Psi(q)$ unaltered. This brings a supplementary $1/\left(\mathbf{q}^2 + \gamma^2\right)$ factor. Further, the formula (24) is the analogue of standard expressions making the connection between Bethe-Salpeter vertex and bound state wavefunction via some propagators. Even if the present remarks do not constitute a rigorous proof, they point towards $\mathcal{F}_{II}\left(\mathbf{q}^2\right)$ as the appropriated form factor, rather than the widely used $\mathcal{F}_I\left(\mathbf{q}^2\right)$. In conclusion, we would like to stress that the well-known Barbieri-Remiddi wavefunction, used as a basis for decay rate calculations, is not simply the Schrödinger momentum wavefunction, contrary to what is often seen in the literature.

6 Conclusions

The result of our analyses is three-fold. First, we have shown how the standard convolution-type factorized amplitude can be given a coherent grounding from dispersion relations. This demonstrates that the formula usually quoted is an approximation, missing some of those $\mathcal{O}\left(\alpha^2\right)$ corrections it is meant to evaluate. Then we have calculated those corrections for p-$Ps \rightarrow \gamma\gamma$ using our exact factorized formula, or equivalently, its effective form factor realization, for two different couplings F_B. Finally, we comment on those couplings.

Concerning this last point, it should now be clear that the dispersion method used to factorize the bound state dynamics from the annihilation process imposes some constraints. By this we mean that there is a definite procedure for gluing together the bound state wavefunction and the decay amplitude.

More precisely, the energy dependences and the spinor part of the Bethe-Salpeter wavefunction are usually incorrectly introduced. As we have discussed, the energy dependences introduce a modification of the naive Schrödinger wavefunction, while the spinor part should be introduced in the decay process rather that in the bound state wavefunction, in order to properly project the constituents onto the required spin state.

Finally, we have shown that the static limit reproduces the first approximation to decay rates. This approximation will be modified by higher order corrections only, provided the form factor has a three-dimensional delta limit for $\gamma \rightarrow 0$. Whether this limit property is shared by quarkonia wavefunctions is still an open question.

In a following paper, the present formalism will be applied to the orthopositronium decay into $\gamma\gamma\gamma$. The analytical defects of this decay channel, noticed in [10], will be completely removed thanks to dispersion techniques. As a result, in addition to the new $\mathcal{O}\left(\alpha^2\right)$ corrections found here, a whole set of new amplitudes contributing to that order will be presented. Those amplitudes are non-factorizable contributions to the decay rate that could not appear in the standard factorized procedures, but unavoidable from the requirements of gauge invariance and analyticity.

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Parapositronium Decay to $\gamma\gamma$: Figures

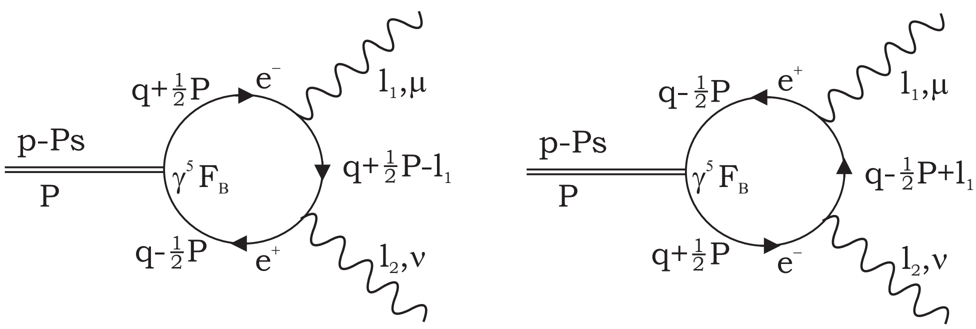

Figure 1 : The loop model direct and crossed diagrams for the decay p-Ps$\rightarrow\gamma\gamma$

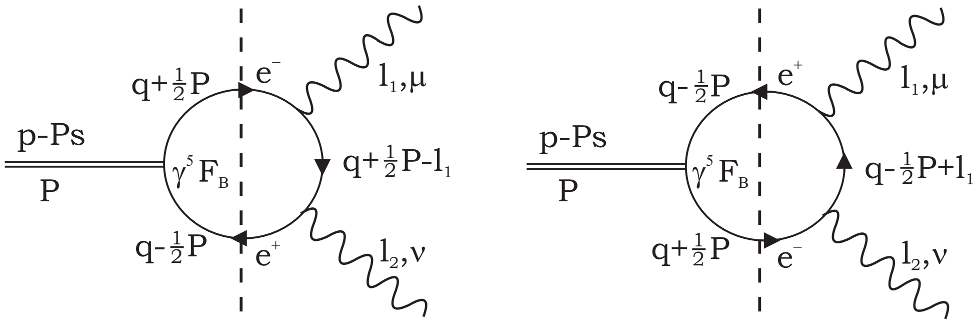

Figure 2 : The two cuts contributing to the imaginary part