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On Electric Fields in Low Temperature Superconductors

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Abstract

The manifestly Lorentz covariant Landau-Ginzburg equations coupled to Maxwell's equations are considered as a possible framework for the effective description of the interactions between low temperature superconductors and magnetic as well as electric fields. A specific experimental set-up, involving a nanoscopic superconductor and only static applied fields whose geometry is crucial however, is described, which should allow to confirm or invalidate the covariant model through the determination of the temperature dependency of the critical magnetic-electric field phase diagram and the identification of some distinctive features it should display.

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1. Introduction. It is a widely held belief that (low T_c) superconductors cannot sustain electric fields in static configurations. The argument[1] is directly based on the first of the London equations,

$$\frac{\partial}{\partial t}\left(\Lambda\vec{J}\right) = \vec{E} \quad , \quad \vec{\partial}\times\left(\Lambda\vec{J}\right) = -\vec{B} \quad , \tag{1}$$

where Λ is a phenomenological parameter proper to the superconducting material, $\vec{J}$ is the supercurrent density, and $\vec{E}$, $\vec{B}$ are the electric and magnetic fields, respectively. Indeed, given that relation as well as the property of infinite conductivity, any nonvanishing electric field within the sample must set into motion a dissipationless supercurrent, hence a displacement of charges which very rapidly leads to an exact screening of any such electric field, certainly for time independent configurations (the case of stationary configurations with externally sustained supercurrents in the presence of magnetic vortices in type II superconductors is a different matter, see for example Refs.[2, 3, 4]).

However, such a situation raises a series of puzzles of varying degrees of concern. First, it is physically inconceivable that an applied electric field would discontinuously drop to zero from its external value when crossing the surface of a superconducting sample. There ought to exist some skin effect with a characteristic nonvanishing penetration depth however small. Nevertheless, none of the parameters appearing in the London equations provides for such an electric penetration length. Moreover, in the above picture for the screening of electric fields, mention is made of an electromagnetic charge density which also is not accounted for in the London equations, nor more generally in the Landau-Ginzburg (LG) equations (see below).

Another concern at a more formal level is the fact that the coupling of the London and LG equations to the electromagnetic fields is not spacetime covariant. Indeed, under Lorentz boosts, the supercurrent density $\vec{J}$ ought to transform as the space components of a 4-vector whose time component would then play the role of the aforementioned missing charge density, while the electric $\vec{E}$ and magnetic $\vec{B}$ fields transform as components of the two index antisymmetric field strength tensor $F_{\mu\nu}$. To illustrate this point more vividly perhaps, consider a flat infinite superconducting slab submitted to an external homogeneous magnetic field lying parallel to its surface. In such a case, the magnetic field will only partially penetrate the sample, with a characteristic penetration length related to the parameter Λ above[1]. Imagine now performing a Lorentz boost in a direction both parallel to the surface of the slab and perpendicular to the applied magnetic field. According to the Lorentz covariance of Maxwell's equations, in the boosted frame there appears now an electric field perpendicular to the surface of the sample, *also within the volume of the superconductor where the magnetic field in the initial frame is nonvanishing*. Thus Lorentz covariance requires the possibility of electric fields on the same footing as magnetic ones within superconductors, with an electric penetration length equal to the familiar magnetic one. Nevertheless, the existence of such electric fields within superconductors is incompatible with the London equations.

One may take issue with the above covariance argument, since the superconducting sample itself defines a preferred frame, thereby insisting that the physics should be described only with respect to that specific rest-frame. Even though that frame is obviously distinguished, the coupling of the London and LG equations to the electromagnetic fields should be consistent with the covariance properties of Maxwell's equations, even within that frame. In addition, one would also like to have available a manifestly covariant framework in which to study the interactions of electromagnetic fields and moving superconducting samples, an issue which the usual London and LG equations are unable to address, and which is certainly accounted for to anyone's satisfaction in the description of electromagnetic interactions with ordinary conductors. It remains true nevertheless that some physical characterizations of superconductors can be defined with respect only to the rest-frame, such as for example the frame dependent notion of the free energy whose value determines the occurrence of the superconducting-normal phase transition only when evaluated in the rest-frame.

Note that any spacetime covariant extension of the usual London and LG equations entails a time dependent LG (TDLG) equation in which space and time variations are on the same footing. Namely, a covariant TDLG equation is necessarily of second-order in time derivatives as it is in space derivatives, in sharp contrast with the usual non covariant first-order TDLG equations encountered in the literature[1]. In particular, in a covariant setting, the time scale associated to time dependent fluctuations is then naturally set by the time it takes light to travel the distance of some mean value of the penetration and coherence lengths. In contradistinction for first-order TDLG equations, this time scale is specified in terms of an additional parameter, the relaxation constant[1]. As a matter of fact, the latter quantity involves Boltzmann's constant, showing that the usual TDLG equations apply rather to the time dependency encured through thermodynamic fluctuations in the superconducting order parameter. As such, these TDLG equations do not provide a framework in which to study the intrinsically genuine time dependent dynamics of superconductors coupled to time varying electromagnetic fields, in the absence of thermodynamic fluctuations. A covariant extension of the London and LG equations would also provide such a dynamic framework, of relevance for instance to the dynamics of ensembles of magnetic vortices interacting among one another and with their electromagnetic environment.

A manifestly covariant extension of the LG equation which immediately comes to one's mind is of course the so-called U(1) Higgs model of particle physics, whose construction itself was motivated by the BCS and LG theories in the late 50's. Indeed, as an effective theory for superconductivity, this model coincides with the original LG formulation for stationnary configurations, and readily provides[5] a description of all the remarkable quantum phenomena of superconductivity. It may thus appear somewhat surprising that the covariant formulation has not been used to further explore superconducting phenomena possibly lying beyond the boundaries of the usual London and LG equations. The purpose of this Letter is to suggest examples of investigations along such lines.

After some considerations on the covariant London and LG equations presented in the next section, section 3 identifies a specific set-up which should enable to establish experimentally whether the covariant, rather than the usual noncovariant approach is relevant to superconducting phenomena in the presence of electric fields. What appears to us to be quite a remarkable circumstance is that this experimental confirmation of the covariant model should prove to be possible already using only a static, time independent configuration of external fields, through the observation of the superconducting-normal phase transition and the determination of the phase diagram in the (B, E) plane for a specific geometry of the applied fields.

2. The covariant LG equations. As mentioned already, the considered model, of application only to low temperature superconductors, is that of a U(1) gauge invariant coupling to the electromagnetic interactions of a complex scalar field ψ of charge $q = -2e < 0$ (the

Cooper pair charge) and with self-interactions determined through the usual LG potential $(|\psi|^2-1)^2$ properly normalized. The order parameter ψ is normalized to the square-root of the Cooper pair density in the bulk in the absence of any electromagnetic field (see Ref.[6] for some further details). Rather than listing all the relevant equations in terms of the physical quantities, let us already use the following choice of units. Space and time coordinates, namely $\vec{x}$ and the combination $x^0=ct$ with c being the speed of light in vacuum, are measured in units of the penetration length $\lambda(T)$. Similarly, magnetic $\vec{B}$ and electric $\vec{E}/c$ fields are measured in units of $\Phi_0/(2\pi\lambda^2(T))$, where $\Phi_0=2\pi\hbar/|q|$ is the usual quantum of flux. Note that these units are temperature dependent, since the penetration length $\lambda(T)$ is, with a dependency we shall model[1] through $\lambda(T)=\lambda(0)\left(1-(T/T_c)^4\right)^{-1/2}$, T_c being of course the critical temperature. Finally, the order parameter ψ is parametrized according to $\psi=fe^{i\theta}$, with f real and f^2 thus measuring the relative Cooper pair density. In terms of these units, space and time coordinates are denoted $\vec{u}$ and τ, and the magnetic and electric fields $\vec{b}$ and $\vec{e}$, respectively. Finally, let us also introduce the quantities,

$$j^0=\frac{q}{\hbar}\frac{\lambda^3(T)}{f^2}\,\mu_0\,c\rho_{\rm em}\quad,\quad \vec{j}=\frac{q}{\hbar}\frac{\lambda^3(T)}{f^2}\,\mu_0\,\vec{J}_{\rm em}\ , \tag{2}$$

where μ_0 is the usual vacuum magnetic permitivity, and $(c\rho_{\rm em},\vec{J}_{\rm em})$ are the superconducting electromagnetic charge and current densities (constructed in terms of ψ), indeed defining a 4-vector under Lorentz transformations. Note that these relations show that $(f^2j^0,f^2\vec{j})$ is in fact proportional to this electromagnetic 4-supercurrent, which must be locally conserved.

The latter remark is also confirmed by the inhomogeneous Maxwell equations (all space derivatives are of course with respect to $\vec{u}$),

$$\vec{\partial}\cdot\vec{e}=-f^2\,j^0\quad,\quad \vec{\partial}\times\vec{b}-\partial_\tau\vec{e}=-f^2\vec{j}\ , \tag{3}$$

which indeed, as is usual, imply the local conservation property $\partial_\tau\left(f^2j^0\right)+\vec{\partial}\cdot\left(f^2\vec{j}\right)=0$. The remaining electromagnetic equations of motion are given by,

$$\partial_\tau\vec{j}\,+\,\vec{\partial}j^0=-\vec{e}\quad,\quad \vec{\partial}\times\vec{j}=\vec{b}\ , \tag{4}$$

which are recognized as the appropriate covariant extension of the London equations in (1). Note that only the first London equation is modified by the inclusion of a contribution of the supercharge density, as was indeed required, while the second London equation, essential to the Meissner effect, retains its original form. The homogeneous Maxwell equations,

$$\vec{\partial}\times\vec{e}+\partial_\tau\vec{b}=\vec{0}\quad,\quad \vec{\partial}\cdot\vec{b}=0\ , \tag{5}$$

follow from the covariant London equations (4) (as they do also from the noncovariant ones in (1)).

The equations of motion for the order parameter ψ are given, on the one hand, by the covariant LG equation

$$\left[\vec{\partial}^2-\partial_\tau^2\right]f=\left[\vec{j}^2-j^{0\,2}\right]f-\kappa^2(1-f^2)f\ , \tag{6}$$

where the LG parameter $\kappa=\lambda(T)/\xi(T)$—$\xi(T)$ being the coherence length—is essentially temperature independent[1], and on the other hand, by the following conditions for the quantum phase θ,

$$\partial_\tau\theta=-j^0+\varphi\quad,\quad \vec{\partial}\theta=\vec{j}-\vec{a}\ . \tag{7}$$

In these latter relations, φ and $\vec{a}$ are the scalar and vector gauge potentials defined such that

$$\vec{e} = -\vec{\partial}\varphi - \partial_\tau \vec{a} \quad , \quad \vec{b} = \vec{\partial} \times \vec{a} \ . \tag{8}$$

Finally, these equations are subject to boundary conditions requiring vanishing values for those components of the vectors $\vec{\partial} f$ and $\vec{j}$ which are perpendicular to the surfaces of the superconducting sample in contact with an insulating material.

The advantage of using this representation of the system is that the number of gauge dependent variables is kept to a minimum[6]. Only the quantities θ, φ and $\vec{a}$ are defined up to the following gauge transformations,

$$\theta' = \theta + \chi \ \ , \ \ \varphi' = \varphi + \partial_\tau \chi \ \ , \ \ \vec{a}' = \vec{a} - \vec{\partial}\chi \ , \tag{9}$$

$\chi(\vec{u}, \tau)$ being an arbitrary function, while the absolute sign of f may also be subject to gauge transformations[7].

In fact, this decoupling of gauge variant and gauge invariant quantities may be rendered complete when substituting the covariant London equations (4) into the inhomogeneous Maxwell equations (3). In addition to the covariant LG equation (6), one then finds for the $(j^0, \vec{j})$ 4-supercurrent,

$$\left[\vec{\partial}^2 - \partial_\tau^2\right] j^0 = f^2 j^0 - \partial_\tau \left[\partial_\tau j^0 + \vec{\partial} \cdot \vec{j}\right] \quad , \quad \left[\vec{\partial}^2 - \partial_\tau^2\right] \vec{j} = f^2 \vec{j} + \vec{\partial} \left[\partial_\tau j^0 + \vec{\partial} \cdot \vec{j}\right] \ . \tag{10}$$

Any solution to this set of three coupled equations for j^0, $\vec{j}$ and f then leads to specific values for $\vec{b}$ and $\vec{e}$ through the covariant London equations (4), and in turn, once the gauge potentials φ and $\vec{a}$ related to these fields determined up to the gauge transformations parametrized by χ, the corresponding solution for the quantum phase θ is also finally obtained from (7)[7].

Although nonlinear, the equations (6) and (10) also establish that these covariant LG equation admit progressive wave solutions with covariant dispersion relations in a linear regime. As a matter of fact, the phase and group velocities for variations in the 4-supercurrent and the order parameter f are different, unless the LG parameter κ takes the same critical value $\kappa_c = 1/\sqrt{2}$ as the one which is so crucial to the understanding of the stability and interaction properties of magnetic vortices. Indeed, for a fluctuation of wave number k (in units of $1/|\vec{u}|$) around the vacuum solution $(j^0 = 0, \vec{j} = \vec{0}, f = 1)$, the group velocities are, respectively,

$$v_j = \frac{k}{\sqrt{k^2 + 1}} \quad , \quad v_f = \frac{k}{\sqrt{k^2 + 2\kappa^2}} \ , \tag{11}$$

thus showing that when $\kappa > \kappa_c$ (resp. $\kappa < \kappa_c$), waves in the 4-supercurrent will overtake (resp. be overtaken by) those in the order parameter f. In other words, within the superconductor, fluctuations in the electromagnetic fields will propagate more rapidly (resp. slowly) than those in the order parameter, in accordance with the relative magnetic and superconducting rigidities that the parameters $\lambda(T)$ and $\xi(T)$ characterize.

Clearly, such properties are totally different in the case of the usual noncovariant first-order TDLG equation, in which time scales are then normalized with respect to the relaxation parameter, which itself is temperature dependent. The ensuing dispersion relations are then linear in frequency, implying that the group velocities of fluctuations in the supercurrent $\vec{j}$ and in the order parameter f are then also identical, independently of the value for κ. Such differences between the covariant and noncovariant frameworks must lead to distinct physical properties in the case of time dependent configurations in ultra-high frequency regimes, $\nu \sim c/\lambda(T), c/\xi(T)$, an issue which, however, is beyond the scope of this work.

To conclude this general discussion, let us also give the expression for the free energy E of the system in the covariant form,

$$\left(\frac{\lambda^3(T)}{2\mu_0}\left(\frac{\Phi_0}{2\pi\lambda^2(T)}\right)^2\right)^{-1} E = \int_{(\infty)} d^3\vec{u}\left\{[\vec{e}-\vec{e}_{\rm ext}]^2 + \left[\vec{b}-\vec{b}_{\rm ext}\right]^2\right\} +$$

$$+ \int_\Omega d^3\vec{u}\left\{(\partial_\tau f)^2 + \left(\vec{\partial} f\right)^2 + f^2\left(j^{0\,2}+\vec{j}^2\right) + \frac{1}{2}\kappa^2\left(1-f^2\right)^2 - \frac{1}{2}\kappa^2\right\}\ , \qquad (12)$$

where the normalization factor related to our choice of units is displayed together with E in the l.h.s., $\vec{e}_{\rm ext}$ and $\vec{b}_{\rm ext}$ are externally applied electric and magnetic fields, respectively, and Ω stands for the volume of the superconducting sample.

The same expression is also of application to the noncovariant model, in which case one has $j^0 = 0$ and $\vec{e} = \vec{0}$ within the superconductor, and the quadratic term in $\partial_\tau f$ is to be replaced by a linear term while the time coordinate is then also measured in units of the relaxation parameter for the TDLG equation. The term in $[\vec{e}-\vec{e}_{\rm ext}]^2$ measures the energy required to expulse the electric field from the superconductor. In the noncovariant case, and in accordance with the first London equation, we shall thus assume that the associated penetration depth is essentially vanishing for all practical purposes. For physics reasons, such an approximation cannot be very reliable when it comes to nanoscopic superconductors, but we shall use it as a working hypothesis anyway. Note that the free energy E is defined here in such a way that it vanishes at the superconducting-normal phase transition. And as a last remark, clearly, in the case of stationary configurations and in the absence of any electric fields, the equations of both approaches coincide with the usual noncovariant time independent LG equations.

3. Characterizing the phase transition. In order to identify a specific geometry of applied fields which could help discriminate experimentally between the two approaches already in a static configuration, consider again the situation of the flat infinite slab of the Introduction, this time subjected not only to the homogeneous magnetic field parallel to its surface, but also to an homogeneous electric field applied perpendicularly to its surface (an electric field parallel to the slab does not induce a supercharge distribution j^0, hence neither a feature distinctive from the noncovariant model). The slab is taken to be of thickness $2a$, while the external electric field $\vec{e}_{\rm ext}$ is aligned along the x axis, and the external magnetic field $\vec{b}_{\rm ext}$ along the y axis, with components $e_{\rm ext}$ and $b_{\rm ext}$, respectively (the origin of this coordinate system is of course positioned in the center of the slab). For this specific geometry, the expulsion of the magnetic field is achieved through an induced supercurrent $\vec{j}$ circulating along the z axis, also parallel to the slab, while that of the electric field is achieved through the appearance of a nonvanishing supercharge density j^0, an occurrence which simply cannot arise in the noncovariant approach. Both these effects imply a deviation from its canonical value of unity for the order parameter f. In view of the symmetries of the problem, both $j^0(u)$ and $j^z(u)$ are odd functions of the normalized $u = x/\lambda(T)$ coordinate along the x axis, while $f(u)$ is even.

Given the equations and the different conditions imposed on these quantities at the boundaries $u = \pm u_a \equiv \pm a/\lambda(T)$, it proves possible as well as useful to express both $j^0(u)$ and $j^z(u)$ in terms of a single function $j(u)$

$$j^0(u) = -e_{\rm ext} j(u) \quad , \quad j^z(u) = -b_{\rm ext} j(u) \ , \tag{13}$$

so that one has for the electric and magnetic fields within the sample,

$$e(u) = e_{\rm ext} \frac{d}{du} j(u) \quad , \quad b(u) = b_{\rm ext} \frac{d}{du} j(u) \ , \tag{14}$$

thus showing once again that Lorentz covariance implies that the penetration lengths for both types of fields are identical. The set of equations to be considered then reduces to,

$$\frac{d^2}{du^2} j(u) = f^2(u) j(u) \quad , \quad \frac{d^2}{du^2} f(u) = \left(b^2_{\rm ext} - e^2_{\rm ext}\right) j^2(u) f(u) - \kappa^2 \left(1 - f^2(u)\right) f(u) \ , \tag{15}$$

subject to the boundary conditions

$$\frac{d}{du} j(u)_{|u=\pm u_a} = 1 \quad , \quad \frac{d}{du} f(u)_{|u=\pm u_a} = 0 \ . \tag{16}$$

Note already the subtle interplay between the magnetic and electric field contributions to the LG equation for $f(u)$, which leads to values larger than unity for $f(u)$ in the electric regime $e^2_{\rm ext} > b^2_{\rm ext}$, while, as is usual, $f(u)$ remains less than unity in the magnetic regime $b^2_{\rm ext} > e^2_{\rm ext}$. The existence of these two regimes is a direct and distinctive consequence of manifest Lorentz covariance; only the magnetic one arises in the noncovariant approach (see below). Note also that in view of these equations, the solutions for $j(u)$ and $f(u)$ are necessarily functions of the specific combination $(b^2_{\rm ext} - e^2_{\rm ext})$ only, indeed justifying this notion of electric or magnetic regimes.

Up to the normalisation factor displayed in (12) as well as the infinite surface of the slab, the free energy $\mathcal{E}$ of configurations obeying these equations is simply given by

$$\mathcal{E} = 2u_a \left\{ \left[1 - \frac{1}{u_a} j(u_a)\right] \left(b^2_{\rm ext} + e^2_{\rm ext}\right) \ - \ \frac{1}{u_a} \int_0^{u_a} du \left[\left(b^2_{\rm ext} - e^2_{\rm ext}\right) j^2 f^2 + \frac{1}{2} \kappa^2 f^4 \right] \right\} \ . \tag{17}$$

Consequently, the curve in the (b, e) phase diagram which characterizes the superconducting-normal phase transition obeys the following equation in the covariant approach,

$$b^2 + e^2 = \frac{1}{\left[1 - \frac{1}{u_a} j(u_a)\right]} \frac{1}{u_a} \int_0^{u_a} du \left[\left(b^2 - e^2\right) j^2 f^2 + \frac{1}{2} \kappa^2 f^4 \right] \ . \tag{18}$$

The noteworthy property of this relation is that the l.h.s. involves only the combination $(b^2 + e^2)$, while the r.h.s. is only a function of the combination $(b^2 - e^2)$ of the external fields (since the solutions $j(u)$ and $f(u)$ also share that property).

Before addressing the specific consequences of this equation for the (b, e) phase diagram, let us consider the corresponding expressions in the noncovariant approach. In that case, one has of course $j^0(u) = 0$ (which also implies $e(u) = 0$ within the superconductor, as follows from the first London equation) as well as $j^z(u) = -b_{\rm ext} j(u)$. The equations then remain as given in (15), including the boundary conditions, with the only but important difference that the factor $(b_{\rm ext}^2-e_{\rm ext}^2)$ appearing in the LG equation is of course replaced by $b_{\rm ext}^2$ only. Hence, the noncovariant LG equations only admit the magnetic regime of solutions. Note that in this case, the solutions for $j(u)$ and $f(u)$ are then also functions of the $b_{\rm ext}^2$ external field only. Consideration of the expression for the free energy then leads to the following condition of criticality in the (b,e) phase diagram in the noncovariant case,

$$b^2+\frac{1}{\left[1-\frac{1}{u_a}j(u_a)\right]}e^2=\frac{1}{\left[1-\frac{1}{u_a}j(u_a)\right]}\,\frac{1}{u_a}\int_0^{u_a}du\left[b^2j^2f^2+\frac{1}{2}\kappa^2f^4\right]\ . \tag{19}$$

In spite of the apparent similarity with (18), recall however that the r.h.s. of this expression is a function of b^2 only, while the l.h.s. is no longer the simple combination b^2+e^2 characteristic of a circle since the coefficient multiplying the term in e^2 is also a function of b^2.

Note that the conditions (18) and (19) coincide in the limit that no electric field is applied, $e=0$, as they should of course. Moreover in the absence of any magnetic field, $b=0$, (19) implies the existence of a nonvanishing critical electric field, $e_0=\kappa/\sqrt{2}$, or in physical units $E_0(T)/c=(\lambda(0)/\lambda(T))^2\,B_c^\infty(0)$, $B_c^\infty(0)=\Phi_0/(2\sqrt{2}\pi\lambda(0)\xi(0))$ being the usual thermodynamic critical magnetic field in the bulk at zero temperature. Clearly, the existence of such a critical electric field even in the noncovariant approach is consequence of our definition for the free energy in (12) which accounts for the expulsed electric energy density through the term in $[\vec{e}-\vec{e}_{\rm ext}]^2$. In particular, this critical electric field $E_0(T)$ vanishes at the critical temperature T_c, as does the critical magnetic field $B_0(T)$ in the absence of any electric field, $e=0$.

4. The (B,E) phase diagram. A complete unravelling of the consequences of the criticality conditions (18) and (19) requires of course a numerical approach. Nevertheless, an analysis in some limiting situations already suffices to gain insight into the differences implied by the two models. An obvious such situation is obtained in the macroscopic limit, namely when the slab half-thickness a is much larger than both the penetration and coherence lengths. For all practical purposes, the function $j(u)$ then essentially vanishes whereas the order parameter retains its canonical value of unity within most of the volume of the sample, except for a small region close to the surface. Hence in the above expressions of criticality, in the limit that $a\rightarrow\infty$, only the contribution in $\kappa^2f^4/2$ tends to dominate, leading in both cases to the condition,

$$b^2+e^2\simeq\frac{1}{2}\kappa^2\quad,\quad a\gg\lambda(T),\xi(T)\ . \tag{20}$$

Since this will prove to be useful, let us normalize the measurement of these fields to the value b_0 of the critical magnetic field in the absence of any electric field (in the macroscopic limit, we thus have $b_0=\kappa/\sqrt{2}$). In terms of the physical quantities, one then obtains the following approximation to the criticality condition in the (B,E) phase diagram

$$\left(\frac{B}{B_0}\right)^2+\left(\frac{E/c}{B_0}\right)^2\simeq 1\quad,\quad a\gg\lambda(T),\xi(T)\ . \tag{21}$$

Hence in the macroscopic limit, the two models are not distinguished in their (B,E) phase diagrams. In particular, both their critical magnetic, B_0, and electric, E_0, fields (in the absence each time of the other field) reach a vanishing value at the critical temperature T_c.

Consider now the nanoscopic limit, namely when $a \ll \lambda(T), \xi(T)$. In practice, this situation may be encountered indeed for nanoscopic samples close to the critical temperature T_c. In such a case, one may develop series expansion solutions in u for the functions $j(u)$ and $f(u)$ in order to evaluate the criticality conditions (18) and (19). However, since contributions of order b^2 and e^2 appear on both sides of these equations, in order to be of sufficient accuracy, the expansion in u must at the same time include at least the first order corrections in b^2 and e^2 in the r.h.s. of (18) and (19) as well. For this reason, it is more relevant to consider a weak field expansion for the solutions independently of whether u_a is small or not, to be used to compute to first order in b^2 and e^2 the r.h.s. of the criticality conditions above, and then eventually take the nanoscopic limit. Note that given the result (20), critical fields are at least of the order of $\kappa/\sqrt{2}$, so that such a weak field expansion should be warranted for small values of the LG parameter κ, namely for type I superconductors.

After some work, one then finds that the criticality conditions (18) and (19), evaluated to first order in b^2 and e^2, imply the following constraint on the physical fields in the (B, E) phase diagram,

$$\left(\frac{B}{B_0}\right)^2 + C\left(\frac{E/c}{B_0}\right)^2 \simeq 1 \ , \tag{22}$$

where as before B_0 stands for the critical magnetic field value in the absence of any electric field, $E = 0$, which is in general a function of temperature and of a of course, while C is a factor given by the following expressions,

$$\begin{aligned} \text{covariant model} \quad &: \quad C = \left(\frac{1+\beta}{1-\beta}\right) \ , \\ \text{noncovariant model} \quad &: \quad C = \left(\frac{u_a}{u_a - \tanh u_a}\right)\left(\frac{1}{1-\beta}\right) \ , \end{aligned} \tag{23}$$

where

$$\beta = \frac{u_a}{16(u_a - \tanh u_a)^2}\frac{1}{(\kappa^2-2)^2}\left\{8\kappa\sqrt{2}\frac{\tanh^2 u_a}{\tanh(\kappa\sqrt{2}u_a)} - (3\kappa^4 - 10\kappa^2 + 16)\tanh u_a + \right.$$
$$\left. + (5\kappa^4 - 22\kappa^2 + 16)\tanh^3 u_a + (\kappa^2-2)(3\kappa^2-4)\frac{u_a}{\cosh^4 u_a}\right\} \ . \tag{24}$$

These expressions are valid in the weak field approximation to first order whatever the value for a. Taking now the nanoscopic limit as well, one finds $(1 - \tanh(u_a)/u_a)^{-1} = 3/u_a^2\left[1 + \mathcal{O}(u_a^2)\right]$ and $\beta = 1/2\left[1 + \mathcal{O}(u_a^2)\right]$, leading finally to the following criticality conditions in the (B, E) phase diagram in the weak field limit,

$$\begin{aligned} \text{covariant model} \quad &: \quad \left(\frac{B}{B_0}\right)^2 + 3\left(\frac{E/c}{B_0}\right)^2 \simeq 1 \quad , \quad a \ll \lambda(T), \xi(T) \ , \\ \text{noncovariant model} \quad &: \quad \left(\frac{B}{B_0}\right)^2 + 6\left(\frac{\lambda(0)}{a}\right)^2 \frac{1}{1-\left(\frac{T}{T_c}\right)^4}\left(\frac{E/c}{B_0}\right)^2 \simeq 1 \quad , \quad a \ll \lambda(T), \xi(T) \ . \end{aligned} \tag{25}$$

Since the critical magnetic field B_0 does vanish towards the critical temperature $T = T_c$, so do all the critical fields B and E which are defined by either of these relations, and thus in particular also the critical electric field E_0 in the absence of any magnetic field, $B = 0$, as was already remarked previously in the noncovariant case. However, by having chosen to normalize the measurements of these fields to B_0, a very distinctive feature appears for the covariant model when compared to the noncovariant one. Indeed, the ratio $E/(cB_0)$ always retains a finite and nonvanishing value, whatever the critical values for B and E within the intervals $[0, B_0]$ and $[0, E_0]$, *even in the limit of the critical temperature* T_c, whereas in the noncovariant model, that same ratio $E/(cB_0)$ must vanish like $\sqrt{1-(T/T_c)^4}$ (given our chosen model for $\lambda(T)$). In particular, in the weak field approximation and including the result (20) valid for macroscopic samples, one thus derives in the covariant case the following bounds for the critical electric field E_0,

$$\sqrt{\frac{1-\beta}{1+\beta}} < \frac{E_0/c}{B_0} < 1 \ , \tag{26}$$

with $E_0/(cB_0)$ moving towards lower values within that interval when the critical temperature T_c is approached (recall that β is also temperature dependent through u_a). In the nanoscopic limit $a \ll \lambda(T), \xi(T)$, these same bounds reduce to

$$\frac{1}{\sqrt{3}} < \frac{E_0/c}{B_0} < 1 \ . \tag{27}$$

In contradistinction in the noncovariant case, the lower bound on $E_0/(cB_0)$ always vanishes, since one then finds,

$$\sqrt{\left(1-\frac{1}{u_a}\tanh u_a\right)(1-\beta)} < \frac{E_0/c}{B_0} < 1 \ , \tag{28}$$

reducing in the nanoscopic limit $a \ll \lambda(T), \xi(T)$ to

$$\frac{1}{\sqrt{6}}\left(\frac{a}{\lambda(0)}\right)\sqrt{1-\left(\frac{T}{T_c}\right)^4} < \frac{E_0/c}{B_0} < 1 \ . \tag{29}$$

As a matter of fact, this type of consideration may be refined further still in the covariant case. Indeed, an obvious solution to the covariant LG equations is $j(u) = \sinh u/\cosh u_a$, $f(u) = 1$ in the case that $e_{\rm ext} = b_{\rm ext}$, a fact which, as was remarked previously, is a distinctive feature of the covariant approach, since this solution defines precisely the boundary between the magnetic and electric regimes of superconductivity, and as such its existence is a direct consequence of Lorentz covariance. Hence, the critical condition (18) for the corresponding fields b_1 and e_1 simplifies in this specific instance to the exact result, valid under all circumstances,

$$b_1^2 + e_1^2 = 2b_1^2 = 2e_1^2 = \frac{1}{2}\kappa^2 \frac{1}{1-\frac{1}{u_a}\tanh u_a} \ . \tag{30}$$

When the weak field approximation is also warranted for the evaluation of the critical magnetic field B_0, this result combines with those above to lead to the following bounds,

$$\frac{1}{\sqrt{2}}\sqrt{1-\beta} < \frac{B_1}{B_0} = \frac{E_1/c}{B_0} < \frac{1}{\sqrt{2}} \ , \tag{31}$$

and in the nanoscopic limit,

$$\frac{1}{2} < \frac{B_1}{B_0} = \frac{E_1/c}{B_0} < \frac{1}{\sqrt{2}} \quad , \quad a \ll \lambda(T), \xi(T) \ , \tag{32}$$

whereas in the noncovariant case, one finds similarly in the nanoscopic limit

$$\frac{1}{\sqrt{1+6\left(\frac{\lambda(0)}{a}\right)^2\frac{1}{1-\left(\frac{T}{T_c}\right)^4}}}<\frac{B_1}{B_0}=\frac{E_1/c}{B_0}<\frac{1}{\sqrt{2}}\quad,\qquad a\ll\lambda(T),\xi(T)\ . \tag{33}$$

Hence here again for those specific configurations such that $B_{\rm ext}=E_{\rm ext}/c$, the lower bound on B_1/B_0 reaches a vanishing value at the critical temperature in the noncovariant case, whereas that lower bound remains finite and is only mildly temperature dependent in the covariant case.

The existence of such finite bounds on the values for $E_1/(cB_1)$ as a function of temperature in the covariant case, translates into the following nice characterization in terms of the $(B/B_0, E/(cB_0))$ phase diagram. Indeed, the limits (32) (which are more refined in (31)) imply that the phase boundary curve in that diagram must always cross the diagonal line $B=E/c$ within the interval of B/B_0 or $E/(cB_0)$ values defined by these bounds in the covariant model, whatever the value for the temperature (see Fig.1). Such a property is simply not met in the noncovariant model (see Fig.2). Similarly, the lower bounds (26) or (27) on $E_0/(cB_0)$ imply that, when approaching the critical temperature, the same phase boundary curve at $B/B_0=0$ cannot move below a specific finite value in the covariant model, while it must necessarily do so in the noncovariant one.

As a conclusion thus, which should remain valid beyond the specific limits considered here, it appears that by choosing to normalize the measurement of critical electric and magnetic fields to the critical magnetic field in the absence of any electric field, for a given nanoscopic sample with this specific geometry of applied fields and by approaching the critical temperature, the (B, E) phase diagram provides the necessary distinctive features which should enable to discriminate experimentally between the covariant and noncovariant models, and in any case confirm or invalidate the description offered by the Lorentz covariant LG equations. Indeed, as was remarked previously, the ordinary noncovariant framework is not physically realistic when it comes to nanoscopic samples in the presence of electric fields, since it ignores the partial penetration, albeit small, of the electric field into the sample's surface. The present analysis has concentrated on the weak field approximation, essentially in the nanoscopic limit. Similar distinctive differences between the two models should also exist for larger values of κ, in ways still to be investigated requiring then a detailed numerical study which is not pursued in this Letter.

5. Numerical solutions. Here, we present the results of the numerical resolution of the LG equations and of the criticality conditions (18) and (19) for only one situation, which is close enough both to the discussion of the previous section and to an experimentally realistic situation. Namely, we take the following parameter values

$$\frac{a}{\lambda(0)}=5\quad,\quad\kappa=0.02\ . \tag{34}$$

Indeed, this value for κ is typical for aluminium (Al), while tabulated values of $\lambda(0)$ for Al—$\lambda(0)=16-50$ nm with $T_c=1.18$ K—would imply that the slab is then a few hundred nanometers thick, within reach of present lithographic techniques for Al on a SiO_2 substrate. Moreover, the critical magnetic field $B_c^\infty(0)$ for Al is also on the order of 100 Gauss, so that the required electric field values for a measurement of the (B, E) phase diagram would reach into 3 MV/m, namely 3 V/μm, certainly also a reasonable range of values for such a nanoscopic device. Of course, compared to the infinite slab model, such a device will be subjected to finite size corrections. Presumably, such corrections would imply that the role played by $\lambda(T)$ and κ in our analysis would be replaced by some effective quantities whose values would not differ to a great extent from those of Al in the bulk. Such corrections may be assessed only once a specific device is designed.

In Fig.1 (resp. Fig.2), we present the $(B/B_0, E/(cB_0))$ phase diagram for the covariant (resp. noncovariant) model, given the values in (34), for a series of temperatures in the range from $T = 0$ to $T = T_c$. The general behaviour of the phase diagram as a function of temperature is indeed the one described in the previous section. In particular in the covariant model, and as a function of temperature, the critical electric field values $E_0/(cB_0)$ and $E_1/(cB_0)$ obey the different finite lower (and upper bounds) derived from the analytical discussion, including those given in (26) and (31) when considering the associated values for β. In contradistinction, in the noncovariant case, the ratio $E/(cB_0)$ reaches a vanishing value when approaching the critical temperature, while in this case as well it way be checked that the different lower bounds (28), (29) and (33) are indeed also obeyed.

Such results, as well as the other considerations of this Letter show that it should be possible to experimentally discriminate between the ordinary noncovariant LG equations and the covariant ones advocated here, by determining the critical (B, E) phase diagram of a nanoscopic superconducting sample for temperatures approaching its critical temperature. The geometry of the applied fields is crucial for this purpose, with the external magnetic field parallel to the sample's surface and the external electric field perpendicular to it. By normalizing the measurement of fields to that of the critical magnetic field in the absence of any electric field, distinctive differences between the two approaches are best brought to the fore, and should enable to confirm or invalidate the covariant approach. Moreover, if the experiment should also allow for an absolute calibration of the applied fields, the comparison between the two models may be refined still further by considering the temperature dependency of the critical value for applied magnetic B and electric E/c fields of equal magnitude, this temperature dependency being constrained to lie within a specific interval whose existence is a direct consequence of the manifest Lorentz covariance of the covariant model. We hope to be able to report on such measurements in the future, but lithographic problems have hindered any progress until now.

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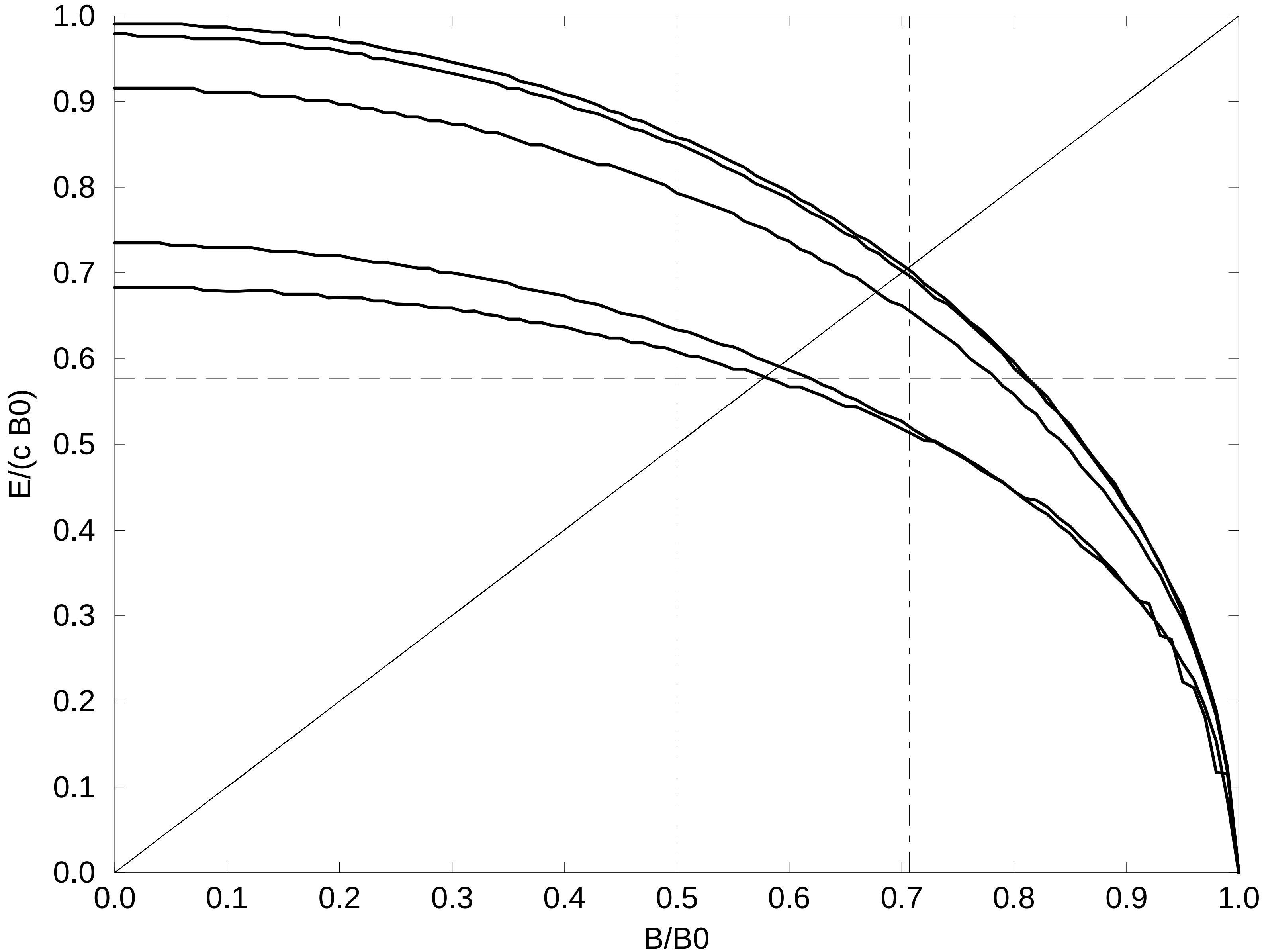
Figure Captions

Figure 1: The phase diagram $(B/B_0, E/(cB_0))$ for the covariant LG equations with the values (34). Shown from top to bottom are the curves associated to the following increasing temperature values, $T/T_c = 0, 0.8766, 0.9659, 0.9935, 0.9996$. The diagonal line determines those configurations such that $B_{\rm ext} = E_{\rm ext}/c$, the two vertical dot-dashed lines at $B/B_0 = 1/2$ and $B/B_0 = 1/\sqrt{2}$ correspond to the lower and upper bounds (32) obeyed by the critical electric E_1/c and magnetic B_1 fields of equal strength in the nanoscopic limit of the weak field approximation for the covariant LG equations, while the horizontal dashed line at $E/(cB_0) = 1/\sqrt{3}$ corresponds to the lower bound (27) on the critical electric field $E_0/(cB_0)$ in the same approximation. The existence of these finite bounds is the distinctive prediction of the covariant model and a direct consequence of its manifest Lorentz covariance.

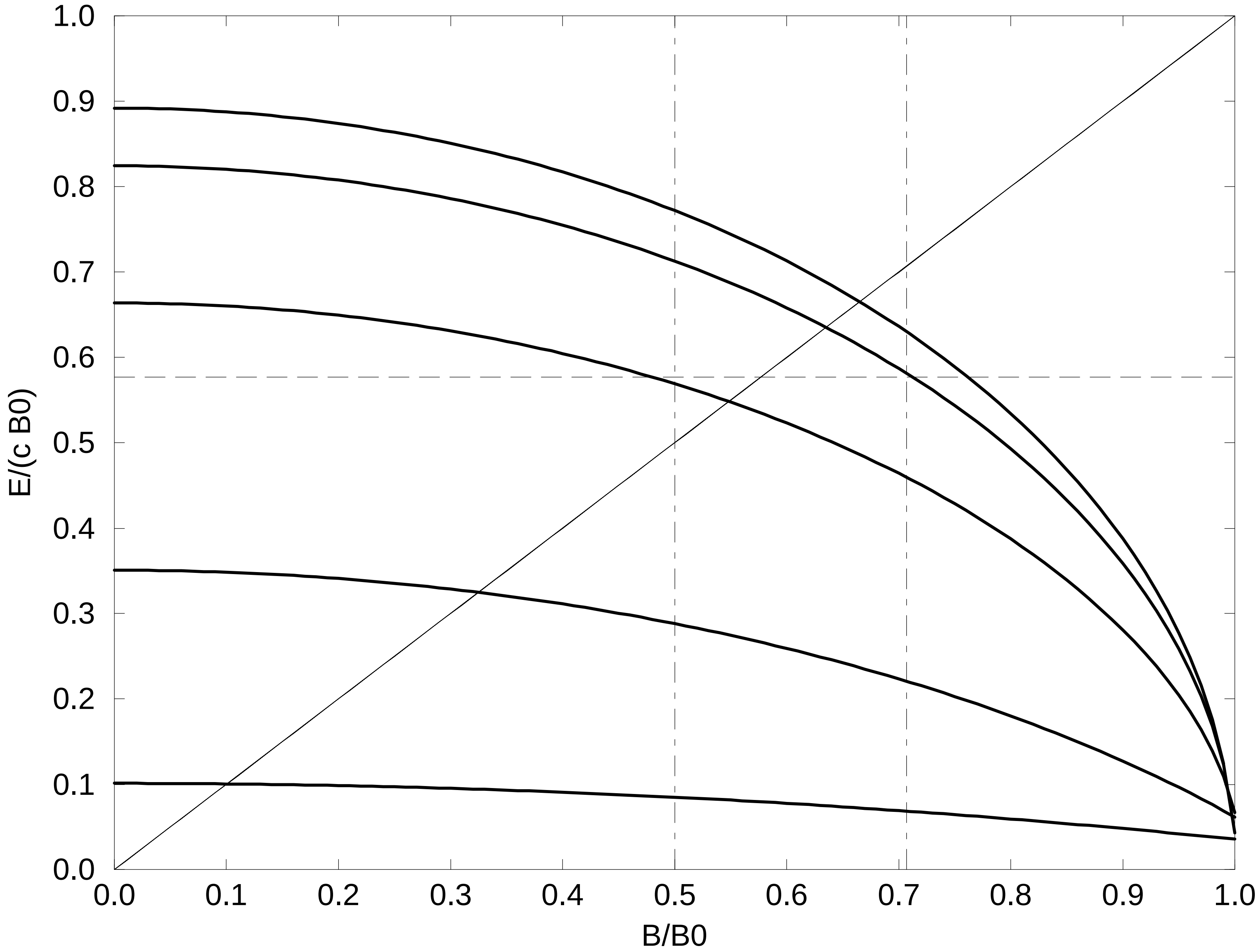
Figure 2: The same as in Fig.1 for the noncovariant model. In this case, the horizontal and two vertical lines are displayed only for the purpose of comparison with the covariant model.

Figure 1



1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0.0
E/(c B0)
0.0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0
B/B0

Figure 2


1.0
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E/(c B0)
0.0 0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1.0
B/B0