Algebraic quantum theory

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Abstract

The main objective consists in endowing the elementary particles with an algebraic space-time structure in the perspective of unifying quantum field theory and general relativity: this is realized in the frame of the Langlands global program based on the infinite dimensional representations of algebraic groups over adele rings. In this context, algebraic quanta, strings and fields of particles are introduced.

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Introduction

This work is a first attempt for endowing the elementary particles with an algebraic space-time structure. The quantum essence of the quantum (field) theory is then of algebraic nature and the most adequate mathematical frame envisaged to carry out this project is the Langlands program which sets up bijections between the set of equivalence classes of representations of the Weil-Deligne group and the equivalence classes of cuspidal representations of the general linear group.

The algebraic part of the Langlands program is realized by the Galois cohomology and more particularly by the Eisenstein cohomology which, being in one-to-one correspondence with the representation of the general linear group, constitutes a representation of the Weil-Deligne group while the analytic part of the Langlands program is given by the cuspidal representations of the general linear group. The cuspidal representation of a general linear group is constituted by the sum of its irreducible representations inflated from the corresponding unitary irreducible representations of the additive group of $\mathbb{R}$ or $\mathbb{C}$.

But, instead of working with a linear mathematical frame as commonly envisaged in quantum theories, a bilinear mathematical frame will be considered for describing the structure of the elementary particles. This is justified mathematically in the sense that the enveloping algebra of a given algebra allows to define a representation of this algebra. This leads us to consider that if an algebra is supposed to be tractable physically, for example by a system of observations and measures, its mathematical management will only be reached by considering its enveloping algebra, i.e. by tensoring the given algebra by its opposite algebra. Generally, if a given algebra can become "measurable", its opposite algebra will not be and its enveloping algebra will constitute the manageable algebra corresponding to an objectivable reality. But then, the quantum theories should be of bilinear nature as the invariants of the group theory and, more particularly, as the invariants of special relativity. If bilinearity is taken into account, then the quantum theories can be merged fashionably with the special relativity: this allows to find a simple solution to the problem of the interactions between elementary particles in the sense that the inextricable N-body problem [Der] becomes easily solvable if it is replaced by a N-bibody problem. Owing to that, a new light is brought to the gravitational force which then results from diagonal interactions between bibodies.

An elementary particle must then be considered as a biobject, called a bisemiparticle: it is composed of the union of a left and a right semiparticle localized respectively in the upper and in the lower half space: this constitutes the basis of the model worked out by the author since the end of the seventies. The fundamental algebraic (space-)time structure of a bisemiparticle is then given by a "physical field" consisting of a sheaf of rings on a bisemimodule defined by the tensor product of a right and a left semimodule respectively over a right and a left adele (semi)ring.

Bisemiparticles find a physical conceptual basis in the fact that every action implies and needs a reaction. This concept of action-reaction was worked out by several authors in the frame of quantum mechanics: let us mention the famous paper of R.P. Feynman and J.A. Wheeler [F-W] on the absorber theory of radiation and other papers [Whe] such as, for example, the paper of C.W. Rietdyk [Rie] conjecturing a retroactive influence based on the EPR paradox.

The vacuum implicit in the Dirac theory by negative energy levels [Dir1], [Dir2] corresponds in this bisemiparticle model to the existence of a right semiparticle associated to a left semiparticle. It is likely that P.A.M. Dirac had several times the presentiment of this hidden reality, especially in his celebrated paper "the quantum theory of the electrons" [Dir1] and in a more recent paper [Dir6] on the epistemology of relativity and quantum mechanics. Notice that the set of dual right semiparticles associated to the left semiparticles in the bisemiparticles could be a candidate for the dark matter.

On the other hand, it is conceptually acceptable to think that elementary particles have an internal structure which looks pointlike to the observer but which must be very complex in order to explain their transformations and decays. Furthermore, a spinning particle cannot be pointlike.

The idea then consists in endowing elementary (semi)particles with an internal algebraic space-time structure from which their "mass" shell could be generated. Indeed, a way of bridging the gap between quantum field theory and general relativity is to consider that the expanding space-time, to which the cosmological constant of the general relativity equations can correspond, could constitute the fundamental structure of the vacuum of quantum field theory [Pie1]. As, this vacuum of QFT is generator of matter, it is natural to admit that its space-time structure will generate matter, i.e. massive elementary particles due to the fluctuations of these elementary vacua associated to local strong curvatures of the space-time at the origin of degenerated singularities. So, the vacuum of the QFT becomes peopled of massless bisemiparticles potentially able to generate their mass shells due to the fluctuations of these bisemiparticles internal vacua which could contribute to the dark energy. If the geometry of general relativity is envisaged at the elementary particle level and if elementary particle internal vacua are taken into account, then the vacuum quantum fields correspond to them in the perspective of the Langlands program. Indeed, the generation of the "discontinued" algebraic space-time obtained by the (representation of) algebraic groups resulting from the Eisenstein cohomology of Shimura varieties is in bijection with its analytic "continued" counterpart given by global elliptic modules (which are truncated Fourier series) included in the corresponding automorphic forms. So, the "discontinued" behavior of the space-time of quantum field theory is in bijection with the "continued" geometry of space-time of classical general relativity by means of the Langlands correspondences.

More concretely, the internal vacuum space-time structure of an elementary bisemiparticle will originate from its internal time structure which will be localized in the orthogonal complement space with respect to its space structure. So, the internal time structure of a bisemiparticle, which is its vacuum time field, will be assumed to be of algebraic nature and will correspond to a bisemisheaf of rings over a general bilinear algebraic semigroup.

The "time" bilinear algebraic semigroup will decompose into conjugacy "bi"classes which are in one-to-one correspondence with the "bi"places of the considered algebraic extension (bisemi)field. The functional representation space of the "time" bilinear algebraic semigroup is given by the bilinear Eisenstein cohomology which decomposes following the bicosets of the Shimura bisemivariety such that the conjugacy "bi"classes of the bilinear algebraic semigroup correspond to the Shimura bisemivariety bicosets which are $G_t(\mathbb{A}_R \times \mathbb{A}_L)$-subbisemimodules decomposing into one-dimensional irreducible (bi)components. Note that the conjugacy classes of $G_t(\mathbb{A}_R \times \mathbb{A}_L)$ are defined with respect to its smallest ramified normal bilinear subsemigroup which implies that the equivalent representatives of the μ-th conjugacy class of $G_t(\mathbb{A}_R \times \mathbb{A}_L)$ can be cut into $(p)+\mu$ equivalent conjugacy subclass representatives having a rank equal to N^2 and interpreted as time biquanta, i.e. the products of right quanta by left quanta. So, the representations of the one-dimensional components of the right and left conjugacy classes of $G_t(\mathbb{A}_R \times \mathbb{A}_L)$ are one-dimensional subsemimodules whose rank is a multiple of the rank N of a time quantum constituting also the representation of the global inertia subgroup of the considered conjugacy class. These one-dimensional subsemimodules are isomorphic to one-dimensional (semi)tori constituting the irreducible analytic representations of $G_t(\mathbb{A}_R \times \mathbb{A}_L)$ associated to the considered right or left places of the algebraic real number semifield and are interpreted physically as elementary "time" waves and strings.

So the vacuum algebraic time structure of a left and of a right semiparticle will be given by a set of correlated left and right waves represented by a left and a right time semisheaf of rings generated by Eisenstein cohomology from a $1D$-time symmetric splitting semifield $L^{\mp}$. The union of these left and right semisheaves of rings is the vacuum time "physical field" of the particle.

But, the Eisenstein cohomology needs a cuspidal automorphic representation allowing to give an analytic representation to the bilinear algebraic semigroup $G_t(\mathbb{A}_R \times \mathbb{A}_L)$. In this purpose, it is assumed that the space of global elliptic semimodules is included into the space of cusp forms so that the ring of endomorphisms acting on global elliptic bisemimodules is generated by the tensor product of Hecke operators whose coset representatives are given in function of the decomposition group associated to the split Cartan subgroup.

These global elliptic bisemimodules are expanded in formal power series whose coefficients can be obtained from the eigenvalues of the coset representatives of the tensor product of Hecke operators. Each term of a global elliptic semimodule is a one-dimensional irreducible torus whose radius can be obtained from the coefficient mentioned above.

In fact, only global elliptic bisemimodules have a real meaning and decompose into a sum of pairs of one-dimensional tori constituting irreducible analytic representations on pairs of places of the considered algebraic extension (bisemi)field.

The algebraic space structure of a semiparticle can also be constructed as the functional representation of a bilinear algebraic semigroup in the context of the Langlands program or can be generated from its $1D$-time wave structure by a $(\gamma_{t\to r}\circ E)$ morphim where:

a) E is an endomorphism based upon a Galois antiautomorphism which transforms the Eisenstein cohomology in Eisenstein homology and in a complementary Eisenstein cohomology associated to the generation of a disconnected complementary $1D$-time wave structure.

b) $\gamma_{t\to r}$ is a morphism transforming partially the complementary $1D$-time wave structure which is a $1D$-time semisheaf into a complementary $3D$-space semisheaf representing the structure of a spatial wave.

If the smooth endomorphism E_t is such that the complementary Eisenstein cohomology is associated to the generation of a complementary connected semisheaf which is proved to be three dimensional, then the resulting complementary $3D$-semisheaf will constitute the basic time structure of the three semiquarks of a semibaryon.

The fundamental $1D$-time structure of a semibaryon is thus composed of a $1D$-core time semisheaf generated by Eisenstein homology on the basis of the smooth endomorphism E_t and of three $1D$-time complementary semisheaves generated by the complementary Eisenstein cohomology and constituting the time structure of the three semiquarks. The space structure of the semiquarks is then obtained by $(\gamma_{t_i\to r_i}\circ E_i)$ morphisms, $1\le i\le 3$, as explained above.

Semiphotons result from the nearly complete transformation of $1D$-time wave semisheaves into $1D$-complementary space wave semisheaves by the $(\gamma_{t\to r}\circ E)$ morphism.

The vacuum space-time structure of semiparticles is thus assumed to be given by:

a) the number of sections of the space-time semisheaves representing their structure;

b) the set of ranks of these sections and especially the set of parameters $c_{t\to r}(\rho)_{R,L}$ measuring the generation of the complementary $3D$-space semisheaf with respect to the reduced $1D$-time semisheaf.

Each one-dimensional section representative μ of the semiparticle time or space semisheaf of rings has thus a rank $n_\mu=(p+\mu)N$ and is composed of $(p+\mu)$ quanta having a rank N . And, a one-dimensional spatial section representative μ , which is a string, is interpreted as the internal vacuum structure of a semiphoton at $(p+\mu)$ quanta. By this way, the vacuum time and space structure of semiparticles is quantified.

As the vacuum fundamental space-time structure of semiparticles is strongly perturbed because it is assumed to have likely a spatial extension of the order of the Planck length, singularities are generated on the sections of the space-time semisheaves $\theta^{1-3}_{R,L}(t,r)_{ST}$. Consequently, these sections are submitted to versal deformations and spreading-out isomorphisms. Recall that a spreading-out isomorphism constitutes an algebraic extension of the quotient algebra of the corresponding versal deformation such that the base sheaves of the quotient algebra can be pulled out partially or completely by a blowing-up morphism, called spreading-out map which depends on a smooth endomorphism based on a Galois antiautomorphism.

As the base sheaves of the versal deformation do not necessarily cover compactly the fundamental space-time semisheaf, a gluing-up of these base sheaves is envisaged so that they cover it by patches.

Taking into account the codimension of the singularities on the fundamental space-time semisheaf $\theta^{1-3}_{R,L}(t,r)_{ST}$, it is proved that a maximum of two successive spreading-out isomorphisms consecutive to versal deformations can occur leading to the generation of two embedded semisheaves covering the fundamental space-time semisheaf of a semiparticle.

This allows to give a new light on the nature of the quantum field theory vacuum which is a state of zero-energy from which elementary particles are created by pairs. Indeed, one of the objectives of the present algebraic quantum theory is to consider that the vacuum of QFT must be viewed as being part of the internal structure of bisemiparticles in the sense that the fundamental $4D$-space-time semisheaf $\theta^{1-3}_{R,L}(t,r)_{ST}$ of a semiparticle and the first covering semisheaf $\theta^{1-3}_{R,L}(t,r)_{MG}$, obtained from $\theta^{1-3}_{R,L}(t,r)_{ST}$ by versal deformation and spreading-out isomorphism and called the "middle-ground" structure, constitute the "vacuum physical semifield" of the semiparticle from which the second covering semisheaf $\theta^{1-3}_{R,L}(t,r)_{M}$, obtained from $\theta^{1-3}_{R,L}(t,r)_{MG}$ by versal deformation and spreading-out isomorphism, constitute the mass physical semifield of the semiparticle. Note that this way of generating the mass of a (semi)particle replaces advantageously the Higgs mechanism.

These two space-time and middle-ground structures have likely a spatial extension of the order of the Planck length to which it is well known that there is a breakdown of the standard quantum field theory [Pen].

Matter is then created from space-time. This reflects as aspiration of A. Einstein (June 9, 1952): "I wish to show that space-time is not necessarily something to which one can ascribe a separate existence independently of the actual objects of physical reality. Physical objects are not in space, but these objects are spatially extended. In this way, the concept of "empty space" loses its meaning".

The quantification of the $4D$-semisheaves " ST ", " MG " and " M ", given by their algebraic structure, involves that the frequencies of vibration of the semisheaves " ST ", " MG " and " M " are quantified. This allows to demonstrate that the $4D$-semisheaf " M " of a semiparticle is observable while the $4D$-semisheaves " ST " and " MG " are unobservable because the vibration frequency of the semisheaf " M " is inferior to the vibration frequencies of the semisheaves " ST " and " MG ".

The semisheaves $\theta^{1-3}_{R,L}(t,r)_{ST}$, $\theta^{1-3}_{R,L}(t,r)_{MG}$ and $\theta^{1-3}_{R,L}(t,r)_{M}$ constitute commutative algebras while one of these algebras extended by versal deformation(s) and spreading-out isomorphism(s) becomes non-commutative.

As it was briefly taken up above, the algebraic structure of the " ST ", " MG " and " M " levels of an elementary particle is crudely given by bisemisheaves over $10D$-bisemimodules of the corresponding bisemiparticle composed of the union of a left and of a right semiparticle. It is then demonstrated that a " ST ", " MG " or " M " bisemimodule, noted $(M_{R;ST,MG,M}\otimes M_{L;ST,MG,M})$ can break down under a blowing-up morphism into:

a) a diagonal bisemimodule $(M_{R;ST,MG,M}\otimes_D M_{L;ST,MG,M})$ of dimension 4, characterized by a flat geometry and a diagonal orthogonal basis: it gives the diagonal central bistructure of the " ST ", " MG " or " M " level of the bisemiparticle;

b) a magnetic bisemimodule $(M^S_{R;ST,MG,M}\otimes_m M^S_{L;ST,MG,M})$ of dimension 3, characterized by a non-orthogonal basis and a metric called magnetic. It is composed of " ST ", " MG " or " M " magnetic biquanta and constitutes the magnetic moment of the corresponding level of the bisemiparticle; this magnetic bisemimodule results from the off-diagonal spatial interactions between a left and a right semiparticle;

c) an electric bisemimodule $(M^{T-(S)}_{R;ST,MG,M}\otimes_e M^{S-(T)}_{L;ST,MG,M})$ of dimension 3, characterized by a metric called electric. It is composed of time-space or space-time biquanta which constitute the electric charge of the corresponding level of the bisemiparticle. This electric bisemimodule results from off-diagonal interactions between the time (resp. space) components of the right semiparticle and the space (resp. time) components of the associated left semiparticle.

The diagonal, magnetic and electric bisemimodules are defined respectively by the "diagonal", "magnetic" and "electric" tensor products between the right and left semimodules M_R and M_L (resp. M^S_R and M^S_L or M^S_R and M^T_L , ...).

If we consider the projection of the right (resp. left) semimodule on the left (resp. right) semimodule, then the right (resp. left) semimodule becomes the dual semimodule of the left (resp. right) semimodule.

Furthermore, a bijective linear isometric map from the projected right (resp. left) semimodule to the left (resp. right) semimodule transforms each covariant element into a contravariant element and gives rise to a left (resp. right) diagonal, magnetic or electric bisemimodule.

Associated with the appropriate internal bilinear form, the left (resp. right) diagonal, magnetic or electric bisemimodule allows to define a left (resp. right) bilinear internal Hilbert, magnetic or electric space.

The algebras of operators acting on bilinear Hilbert, magnetic and electric spaces are bialgebras of bioperators owing to the bilinearity of these spaces. We have thus to consider bialgebras of von Neumann of bounded bioperators acting on these bilinear spaces. As the representation space of a given algebra is isomorphic to its enveloping algebra, the extended bilinear Hilbert spaces characterized by a nonorthogonal Riemanian metric will be taken as natural representation spaces for the bialgebras of bounded operators.

In this context, an (elliptic differential) bioperator $(T_R \otimes T_L)$ maps the bisemisheaf $\widetilde{M}_R \otimes \widetilde{M}_L$ on the $GL_n(\mathbb{A}_R \times \mathbb{A}_L)$-bisemimodule $M_R \otimes M_L$ into the bisemisheaf $\widetilde{M}^a_R \otimes \widetilde{M}^a_L$ on the shifted $GL_{n[m]}((\mathbb{A}_R \otimes \mathbb{C}) \times (\mathbb{A}_L \otimes \mathbb{C}))$-bisemimodule $(M^a_R \otimes M^a_L)$. $\widetilde{M}^a_R \otimes \widetilde{M}^a_L$ is a perverse bisemisheaf whose sections are defined over the conjugacy classes μ with multiplicities m_μ of $GL_{n[m]}((\mathbb{A}_R \otimes \mathbb{C}) \times (\mathbb{A}_L \otimes \mathbb{C}))$. Now, the Langlands program, setting up bijections between the algebraic $GL_n(\mathbb{A}_R \times \mathbb{A}_L)$-bisemimodule $(M_R \otimes M_L)$ and the corresponding analytic global elliptic bisemimodule $((\phi_R(s_R) \otimes \phi_L(s_L))$, can be extended, under the action of a bioperator, into a shifted Langlands program establishing bijections between the shifted bisemimodule $(M^a_R \times M^a_L)$ and the corresponding analytic shifted global elliptic bisemimodule $(\phi^a_R(s_R) \otimes \phi^a_L(s_L))$. An eigenbivalue equation directly follows from the shifted global elliptic bisemimodule such that its eigenbivalues form an embedded sequence in one-to-one correspondence with the embedded eigenbifunctions of $T_R \otimes T_L$ which form an increasing sequence of truncated global elliptic bisemimodules, i.e. products, right by left, of truncated Fourier series. In this perspective, a bisemiparticle spatial wave bifunction will be given by the analytical representation of the $GL_2(\mathbb{A}_R \times \mathbb{A}_L)$-bisemimodule and will consist in an increasing set of products, right by left, of truncated Fourier series whose number of terms corresponds to the number of considered conjugacy class representatives of $GL_2(\mathbb{A}_R \times \mathbb{A}_L)$: so, a bisemiparticle wave bifunction is described in terms of its spectral representation which corresponds to the classical assertion saying that the spectral theorem is equivalent to consider that any unitary representation of a compact Lie group is a direct sum of irreducible representations.

As the consequence of the algebraic spectral representation of a wave bifunction, the group of automorphisms of an analytic von Neumann algebra is isomorphic to the group of "shifted" automorphisms of Galois which has for consequence that the entire dimensions of the von Neumann algebras are in fact the integers labelling the classes of degrees of Galois extensions.

It is then proved that the discrete spectrum of a bioperator is obtained by means of an isomorphism from the bialgebra of von Neumann on an extended bilinear Hilbert space to the corresponding bialgebra of von Neumann on a bilinear diagonal Hilbert space.

The bilinear structure of this quantum theory involves that:

1) the traditional calculus with the amplitudes of probability of quantum field theories is replaced by a calculus with intensities of probability;

2) the rotation of the sections of the semisheaves of a right (resp. left) semiparticle with respect to

the sections of the semisheaves of its associated left (resp. right) semiparticle allows to define the internal angular momentum of the right (resp. left) semiparticle from which it results that a right and a left semiparticles rotate in opposite senses and have only two possible spin states.

To each $1D$- and $3D$-" ST ", " MG " and " M " semisheaf corresponds a phase space which has the structure of a F-Steenrod bundle whose basis is given by the considered semisheaf.

The bisections of the bisemisheaves are tensor products of differentiable functions for which wave equations are studied: they are degenerated second order elliptic differential bilinear equations.

The right (resp. left) wave function, solution of a wave equation, is proved to have a spectral decomposition in terms of eigenfunctions having an algebraic representation as mentioned above. The statistical interpretation of the wave function is the same as in quantum theories. This allows to reconcile the Bohr and Einstein points of view about quantum theories.

Every elementary bisemiparticle has a "mass" central algebraic structure composed of pairs of right and left one-dimensional sections (which are in fact one-dimensional waves or open strings) behaving like (damped) harmonic oscillators. Thus, a "field" (in the physical sense, but having an algebraic structure) can be associated to the mass structure of each elementary bisemiparticle. And, as each one-dimensional subsection is an open string, this algebraic quantum model has also a "string" aspect.

The mass equation for a bisection " μ " of the bisemielectron is especially considered: it is the equation of a damped harmonic oscillator whose general solution consists in the superposition of two damped waves in phase opposition with frequencies given by $E_\mu = \frac{\hbar}{c}\nu_\mu s$ and whose general motion corresponds to a damped sinusoidal motion whose dephasage is proportional to the linear momentum of the considered left (resp. right) section of the left (resp. right) semielectron.

It is proved that the energy of a section s_μ at $\mu_p = p+\mu$ quanta can be given in function of the energy E^I_μ of a quantum on this section following $E_\mu = \mu_p E^I_\mu$. And, the energy E^I_μ of a quantum $\widetilde{M}^I_\mu \in s_\mu$ can be calculated from the analytic development of the corresponding nontrivial zero of the Riemann Zeta function $\zeta(s)$.

The internal machinery of a bisemiparticle allows to justify the absorption and the emission of right and left quanta. In fact, each spatial one-dimensional bisection of the " ST ", " MG " and " M " bisemisheaf behaves globally like two adjacent gyroscopes having opposite torques which allows to understand that diagonal biquanta are emitted under the action of a diagonal centrifugal biforce represented mathematically by a diagonal smooth biendomorphism corresponding to an inverse deformation and that magnetic biquanta are emitted under the action of a Coriolis biforce represented mathematically by a magnetic smooth biendomorphism.

The emission and reabsorption of left and right magnetic quanta by left and right semisheaves having different magnitudes of rotational velocities generate by reaction a global movement of translation of the bisemiparticle.

The structure of bisemiparticles is given by bisemisheaves so that an action-reaction process is generated by the interactions between the right semisheaves of the right semiparticle and the left semisheaves of the left semiparticle. Generalizing this concept to a set of bisemiparticles, it can be easily demonstrated that the interactions between a set of bisemiparticles result from the interactions between the right and left semisheaves belonging to different bisemiparticles, leading to a set of mixed action-reaction processes of bilinear nature associated to interferences. This allows to get rid of the inextricable problems of Physics originated from linearity as A. Einstein outlined in [Ein5]: "Linear laws have solutions which satisfy the superposition principle but they do not describe the interactions between elementary particles".

The general mathematical frame allowing to describe the interactions between a set of N bisemiparticles is the Langlands reducible program as developed in [Pie9]. In this context, the spatial mass structure (i.e. the "mass" field) of N interacting bisemiparticles is given by the nonorthogonal completely reducible functional representation space of $GL_{2N}(\mathbb{A}_R \times \mathbb{A}_L)$ as introduced in chapter 5.

The bilinear interactions generate gravitational, magnetic and electric biquanta giving rise to a gravito-electro-magnetic field such that the gravitation results from diagonal interactions between bisemiparticles while the electromagnetism originates from off-diagonal interactions.

It is then proved that:

a) a set of bisemifermions interact by means of a gravito-electro-magnetic field;

b) a set of bisemiphotons interact by means of a gravito-magnetic field;

c) a set of bisemifermions and of bisemiphotons interact by means of a gravito-electro-magnetic field.

The biwave equation of N interacting bisemiparticles separates automatically into N_q biwave equations of the N_q bisections of the N bisemiparticles and into $((N_q)^2 - N_q)$ biwave equations referring to the interactions between the right and left sections of these N bisemiparticles.

In this context, the antisymmetric electromagnetic field tensor is replaced by a gravito-electro-magnetic tensor whose diagonal components are the components of a gravitational field. This leads to a new conceptual approach of the electromagnetism and of the quantum gravity.

In this algebraic quantum model, the strong interactions and the cause of the confinement of the semiquarks result from the new structure proposed for the semibaryons. Indeed, the confinement of the semiquarks originates from the generation of the three semiquarks from the core time semisheaf of the semibaryon by a smooth endomorphism E_t . The core time structure of a semibaryon is physically justified by the fact that the quarks contribute only to about 15% of the spin of the nucleon [Ash].

We then have that a right and a left semibaryon of a given bisemibaryon interact by means of:

a) the electric charges and the magnetic moments of the three bisemiquarks;

b) a gravito-electro-magnetic field resulting from the bilinear interactions between the right and the left semiquarks of different bisemiquarks;

c) a strong gravitational and electric fields resulting from the bilinear interactions between the central core structures of the left and right semibaryons and the right and left semiquarks.

The leptonic decay of a bisemibaryon results essentially from the diagonal emission of a bisemilepton throughout a diagonal biendomorphism. The emitted bisemineutrino allows to take into account the bilinear interactions between the emitting bisemiquark and the emitted bisemilepton.

The nonleptonic decay of a bisemibaryon consists essentially in the emission of a meson by a bisemiquark throughout a nonorthogonal biendomorphism.

Finally, it is shown that the EPR paradox receives a new lighting because the linear frame of quantum theories is replaced by a bilinear frame so that two elementary bisemiparticles interact through the space by means of a gravito-magneto-(electric) field and nonlocally through the time by means of a $1D$-time gravitational field.

Let us also make the following last remarks:

1) This algebraic quantum theory is an algebraic quantum field theory describing the structure of elementary particles in terms of bisemiparticles from their internal algebraic space-time structures interpreted as elementary internal vacua whose union corresponds to the essential part of the vacuum of QFT.

   This algebraic quantum theory lies on the Langlands mathematical program and does not proceed from Lagrangian methods of classical mechanisms as the quantum field theories. This theory is thus not a priori directed towards the description of the trajectories of particles. However, the study of the structure of bisemiparticles corresponds to bringing up to light the existence of an internal dynamics with respect to internal variables of which the most popular are the proper time and the proper mass. But, this elementary "internal dynamics" also evolves with respect to an external time variable throughout the equivalent of the Stone theorem and leads to an "external dynamics" corresponding to the classical or quantum dynamics.

2) In this AQT, all observables are quantified due to the algebraic nature of the theory: thus, the internal time, the internal space, the mass, the energy, the linear momentum, the charge, the electromagnetism and the gravitation are quantified.

3) The internal structure of a massive bisemiparticle is composed of biwave packets localized into the $1D$-time and $3D$-spatial orthogonal spaces: the $3D$-spatial structure of a bisemiparticle thus has a wave aspect which becomes evident when it interacts with other bisemiparticles by interference process. The corpuscular aspect of the $3D$-spatial structure of a bisemiparticle can become apparent when its $3D$-spatial biwave packet is flattened into two dimensions as resulting from a collision.

4) The internal time structures of semiparticles are perhaps not localized in a traditional one-dimensional time space but in a three-dimensional space. Then, a magnetic moment and/or field related to $3D$-time structures ought to be envisaged as resulting from off-diagonal interactions between time right-semisheaves and time-left semisheaves.

5) Some of the difficulties of the standard model seem to have been solved in this algebraic quantum model, as for example the origin of the mass, the nature of the dark matter and energy and the existence of three families of elementary particles.

1 Algebraic representation of the fundamental $4D$-space-time structure of semiparticles

1.1 Generation of $1D$-semisheaves of rings by Eisenstein cohomology

The aim of this section consists in the generation of two symmetric right and left $1D$-time semisheaves of rings $\theta^1_R(-t)$ and $\theta^1_L(+t)$ whose q right and left sections are continuous functions over the completions of finite Galois extensions of global number field K of characteristic zero. The finite Galois extensions of K are the splitting field over K of the polynomial ring $K[t]$ in the time indeterminate " t ".

The dimension corresponding to the time variable will be called the *generative dimension*.

Notation : " R,L " means " R (respectively L)".

Definition 1.1.1 (Symmetric polynomial ring) The polynomial ring $K[t]$ is assumed to have for elements the polynomials $P_{\mu_\nu}(t)$ and $P_{\mu_\nu}(-t)$ which are such that :

a) All the polynomials $P_{\mu_\nu}(t)$, $1\le\mu\le q$, $1\le\nu\le\infty$, have a same number of positive simple (real) roots " $N^+_{\mu L}$ " and a same number of negative simple (real) roots " $N^+_{\mu R}$ ", i.e. $N^+_{1L}=\cdots=N^+_{\mu L}=\cdots=N^+_{qL}$ and $N^+_{1R}=\cdots=N^+_{\mu R}=\cdots=N^+_{qR}$ with $N^+_{\mu L}$ in general not equal to $N^+_{\mu R}$ and $N^\pm_{\mu R,L}\in I\!\!N$.

b) The polynomials $P_{\mu_\nu}(-t)$, $1\le\mu\le q$, $1\le\nu\le\infty$, have a number of positive simple (real) roots $N^-_{\mu L}$ equal to the number of negative simple (real) roots $N^+_{\mu R}$ of the polynomials $P_\mu(t)$ and vice versa.

   So we have that

   1. $N^+_{\mu L}=N^-_{\mu R}$, $N^+_{\mu R}=N^-_{\mu L}$, $\forall\ \mu$, $1\le\mu\le q$;
   2. $N^+_{\mu L}+N^-_{\mu L}=N^-_{\mu R}+N^+_{\mu R}$.

Remark that, when the polynomials $P^c_{\mu_\nu}(t)$ and $P^c_{\mu_\nu}(-t)$ of the polynomial ring $K[t]$ have simple complex roots, $K[t]$ is manifestly a symmetric polynomial ring if all the polynomials $P^c_{\mu_\nu}(t)$ and $P^c_{\mu_\nu}(-t)$, $1\le\mu\le q$, $1\le\nu\le\infty$, have a same number of simple complex roots.

Definition 1.1.2 (Symmetric splitting semifield) This polynomial ring $K[t]$ is then composed of a set of pairs of polynomials $\{P_{\mu_\nu}(t),P_{\mu_\nu}(-t)\}_{\mu,\nu}$. Each set of pairs of polynomials for the index μ generates the symmetric splitting subfield L_μ which is composed of the set of positive simple roots, noted L^+_μ , and of the symmetric set of negative simple roots, noted L^-_μ . L_μ is thus characterized by the properties:

a) $L_\mu=L^-_\mu\cup L^+_\mu$.

b) $L^-_\mu\cap L^+_\mu=\emptyset$.

c) To each positive simple root $\alpha_{\mu+}\in L^+_\mu$ corresponds the symmetric negative simple root $\alpha_{\mu-}\in L^-_\mu$.

L^+_μ and L^-_μ are respectively a left and a right algebraic extension semisubfields. They are semisubfields because they are commutative division semisubrings. They are "*semisubrings*" because $(L^+_\mu,+)$ and

$(L_\mu^-,+)$ are abelian semisubgroups $[H-N]$ lacking for inverses with respect to the addition and endowed with associative multiplication and distributive laws.

Similarly, each set of pairs of polynomials $\{P^c_{\mu_\nu}(t),P^c_{\mu_\nu}(-t)\}_{\mu,\nu}\in K[t]$ generates a complex symmetric splitting subfield L^c_μ composed of the set of complex simple roots, noted L^{c+}_μ , and of the symmetric set of complex conjugate simple roots, noted L^{c-}_μ . L^{c+}_μ and L^{c-}_μ are also respectively a left and a right algebraic extension semisubfields.

Definition 1.1.3 (Right and left specializations) We consider the right and left specializations [Wei1] of the right and left semisubrings $A_{\mu R}$ and $A_{\mu L}$ (included respectively in the semirings A_R and A_L) from the polynomial subring $\{P_{\mu_\nu}(t),P_{\mu_\nu}(-t)\}\in K[t]$.

The right (resp. left) specialization of $A_{\mu R}$ (resp. $A_{\mu L}$) is completely determined by $p_{\mu R}$ (resp. $p_{\mu L}$) which is a nonzero prime right (resp. left) specialization ideal of $A_{\mu R}$ (resp. $A_{\mu L}$), i.e. the set of all negative (resp. positive) nonunits of $A_{\mu R}$ (resp. $A_{\mu L}$), such that $p_{\mu R}\cap p_{\mu L}=\emptyset$.

We denote by $B_{\mu R}$ (resp. $B_{\mu L}$) the integral closure of $A_{\mu R}$ (resp. $A_{\mu L}$) in L^-_μ (resp. L^+_μ) (i.e. the set of elements of L^-_μ (resp. L^+_μ) which are integral over $A_{\mu R}$ (resp. $A_{\mu L}$)). Then, the right (resp. left) semisubring $B_{\mu R}$ (resp. $B_{\mu L}$) is a finitely generated $A_{\mu R}$-right semimodule (resp. $A_{\mu L}$-left semimodule) [Ser3].

Let $b_{\mu R_{1_\mu}}\subset\cdots\subset b_{\mu R_{n_\mu}}$ (resp. $b_{\mu L_{1_\mu}}\subset\cdots\subset b_{\mu L_{n_\mu}}$) be a chain of distinct prime right (resp. left) ideals of $B_{\mu R}$ (resp. $B_{\mu L}$) obtained under the right (resp. left) action of the right (resp. left) Galois group $\Gamma_{\mu R}=\mathrm{Aut}_K L^-_\mu$ (resp. $\Gamma_{\mu L}=\mathrm{Aut}_K L^+_\mu$).

If $p_{\mu R}=b_{\mu R_{i_\mu}}\cap A_{\mu R}$ (resp. $p_{\mu L}=b_{\mu L_{i_\mu}}\cap A_{\mu L}$), $1\le i_\mu\le n_\mu$, then $b_{\mu R_{i_\mu}}$ (resp. $b_{\mu L_{i_\mu}}$) divides (or is above) $p_{\mu R}$ (resp. $p_{\mu L}$).

Then, $B_{\mu R}/b_{\mu R_{n_\mu}}$ (resp. $B_{\mu L}/b_{\mu L_{n_\mu}}$) is an extension of $A_{\mu R}/p_{\mu R}$ (resp. $A_{\mu L}/p_{\mu L}$) of finite degree, called the right (resp. left) global residue degree of $b_{\mu R}\equiv b_{\mu R_{n_\mu}}$ (resp. $b_{\mu L}\equiv b_{\mu L_{n_\mu}}$) and noted $f_{b_{\mu R}}$ (resp. $f_{b_{\mu L}}$).

1.1.4 Inertia subgroups and adele semirings

If the right (resp. left) ideal b_{μ_R} (resp. b_{μ_L}) is assumed to be unramified, we have more precisely that:

$$[L^{-(nr)}_\mu:K]=f_{b_{\mu_R}}\qquad(\text{resp.}\quad[L^{+(nr)}_\mu:K]=f_{b_{\mu_L}}\ )$$

where $L^{-(nr)}_\mu$ (resp. $L^{+(nr)}_\mu$) is a right (resp. left) unramified algebraic extension.

Let $\mathrm{Gal}(L^{-(nr)}_\mu/K)$ (resp. $\mathrm{Gal}(L^{+(nr)}_\mu/K)$) denote the Galois subgroup of the unramified right (resp. left) extension $L^{-(nr)}_\mu$ (resp. $L^{+(nr)}_\mu$) of K and let $\mathrm{Gal}(L^-_\mu/K)$ (resp. $\mathrm{Gal}(L^+_\mu/K)$) be the Galois subgroup of the corresponding ramified right (resp. left) extension L^-_μ (resp. L^+_μ).

If $I_{L^-_\mu}$ (resp. $I_{L^+_\mu}$) denotes the global inertia subgroup of $\mathrm{Gal}(L^-_\mu/K)$ (resp. $\mathrm{Gal}(L^+_\mu/K)$), then the equalities follow:

$$\begin{aligned}\mathrm{Gal}(L^-_\mu/K)\big/I_{L^-_\mu}&=\mathrm{Gal}(L^{-(nr)}_\mu/K)\ ,\\ (\text{resp.}\quad\mathrm{Gal}(L^+_\mu/K)\big/I_{L^+_\mu}&=\mathrm{Gal}(L^{+(nr)}_\mu/K)\ ),\end{aligned}$$

leading to the exact sequences:

$$\begin{array}{rl}
& 1 \longrightarrow I_{L^-_\mu} \longrightarrow \mathrm{Gal}(L^-_\mu/K) \longrightarrow \mathrm{Gal}(L^{-(nr)}_\mu/K) \longrightarrow 1\\
(\text{resp.} & 1 \longrightarrow I_{L^+_\mu} \longrightarrow \mathrm{Gal}(L^+_\mu/K) \longrightarrow \mathrm{Gal}(L^{+(nr)}_\mu/K) \longrightarrow 1\ ).
\end{array}$$

On the other hand, it was seen in [Pie9] that

$$[L^{-(nr)}_\mu : K] = f_{b_{\mu_R}} = kp+\mu' = p+\mu\ , \quad 1 \le \mu \le q \le \infty\ ,$$

where $k \le p-1$ is an integer referring to congruence classes modulo p such that $kp+\mu'=p+\mu$

$$(\text{resp.}\quad [L^{+(nr)}_\mu : K] = f_{b_{\mu_L}} = kp+\mu' = p+\mu\ ).$$

If the global residue degree $f_{b_{\mu_R}}$ (resp. $f_{b_{\mu_L}}$) is an integer and not an integer modulo p , then $p=0$ and $f_{b_{\mu_R}} = f_{b_{\mu_L}} = \mu$.

If N denotes the order of the global inertia subgroups $I_{L^-_\mu}$ (resp. $I_{L^+_\mu}$), $1 \le \mu \le q \le \infty$, then the degrees of the right (resp. left) ramified extensions L^-_μ (resp. L^+_μ) are given by integers modulo N :

$$\begin{array}{rl}
& n_{\mu_R} = [L^-_\mu : K] = * + f_{b_{\mu_R}} \cdot N \approx f_{b_{\mu_R}} \cdot N = (p+\mu)N\\
(\text{resp.} & n_{\mu_L} = [L^+_\mu : K] = * + f_{b_{\mu_L}} \cdot N \approx f_{b_{\mu_L}} \cdot N = (p+\mu)N\ )
\end{array}$$

where $*$ denotes an integer inferior to N .

Let $L_{\overline{v}_\mu}$ (resp. L_{v_μ}) be the μ-th completion corresponding to the right (resp. left) ramified algebraic extension L^-_μ (resp. L^+_μ) and associated to the place $\overline{v}_\mu$ (resp. v_μ).

The completion $L_{\overline{v}_\mu}$ (resp. L_{v_μ}), which is a one-dimensional K-semimodule, is assumed to be generated from an irreducible (central) K-semimodule $L_{\overline{v}^1_\mu}$ (resp. $L_{v^1_\mu}$) of rank (or degree) N such that $L_{\overline{v}^1_\mu} \simeq p_{\mu_R}$ (resp. $L_{v^1_\mu} \simeq p_{\mu_L}$).

As a result, $L_{\overline{v}_\mu}$ (resp. L_{v_μ}) is cut into a set of $(p+\mu)$ equivalent real subcompletions $L_{\overline{v}^{\mu'}_\mu}$ (resp. $L_{v^{\mu'}_\mu}$), $1 \le \mu' \le \mu$, of rank N : since the rank of $L_{\overline{v}_\mu}$ (resp. L_{v_μ}) is also given by:

$$\begin{array}{rl}
& n_{\mu_R} = [L_{\overline{v}_\mu} : K] \simeq f_{b_{\mu_R}} \cdot N = (p+\mu)\ N\\
(\text{resp.} & n_{\mu_L} = [L_{v_\mu} : K] \simeq f_{b_{\mu_L}} \cdot N = (p+\mu)\ N\ ).
\end{array}$$

So, the ranks or degrees of the real completions $L_{\overline{v}_\mu}$ (resp. L_{v_μ}), $1 \le \mu \le q$, are integers of $\mathbb{Z} \big/ p\ N\ \mathbb{Z}$, noted in condensed form $\overline{\overline{\mathbb{Z}}}_{p_q}$.

On the other hand, as a place is an equivalence class of completions, we have to consider at a place $\overline{v}_\mu$ (resp. v_μ) a set of real completions $\{L_{\overline{v}_\mu,m_\mu}\}$ (resp. $\{L_{v_\mu,m_\mu}\}$), $m_\mu \in \mathbb{N}$, equivalent to the basic completion $L_{\overline{v}_\mu}$ (resp. L_{v_μ}) and having the same rank n_{μ_R} (resp. n_{μ_L}) as $L_{\overline{v}_\mu}$ (resp. L_{v_μ}); the integer $m^{(\mu)} = \sup(m_\mu)$ is interpreted as the multiplicity of $L_{\overline{v}_\mu}$ and L_{v_μ} .

Then, a right (resp. left) "ramified" adele semiring $\mathbb{A}_{L_{\overline{v}}}$ (resp. $\mathbb{A}_{L_v}$) can be introduced by:

$$\begin{array}{rl}
& \mathbb{A}_{L_{\overline{v}}} = \prod\limits_\mu L_{\overline{v}_\mu} \prod\limits_{m_\mu} L_{\overline{v}_\mu,m_\mu}\\
(\text{resp.} & \mathbb{A}_{L_v} = \prod\limits_\mu L_{v_\mu} \prod\limits_{m_\mu} L_{v_\mu,m_\mu}\ ).
\end{array}$$

1.1.5 Representations of the algebraic bilinear general semigroup

Let $T_2^t(\mathbb{A}_{L_{\overline{v}}})$ (resp. $T_2(\mathbb{A}_{L_v})$) denote the matrix algebra of lower (resp. upper) triangular matrices of order 2 over the adele semiring $\mathbb{A}_{L_{\overline{v}}}$ (resp. $\mathbb{A}_{L_v}$) . Then, according to [Pie10], an algebraic bilinear general semigroup over the product of $\mathbb{A}_{L_{\overline{v}}}$ by $\mathbb{A}_{L_v}$ can be introduced by:

$$GL_2(\mathbb{A}_{L_{\overline{v}}} \times \mathbb{A}_{L_v}) = T_2^t(\mathbb{A}_{L_{\overline{v}}}) \times T_2(\mathbb{A}_{L_v})$$

such that:

1) $GL_2(\mathbb{A}_{L_{\overline{v}}} \times \mathbb{A}_{L_v})$ has a bilinear Gauss decomposition:

$$GL_2(\mathbb{A}_{L_{\overline{v}}} \times \mathbb{A}_{L_v}) = [(D_2(\mathbb{A}_{L_{\overline{v}}}) \times D_2(\mathbb{A}_{L_v})][UT_2(\mathbb{A}_{L_v}) \times UT_2^t(\mathbb{A}_{L_{\overline{v}}})]$$

   where

   - $D_2(\cdot)$ is a subgroup of diagonal matrices,
   - $UT_2(\cdot)$ is a subgroup of unitriangular matrices;

2) $GL_2(\mathbb{A}_{L_{\overline{v}}} \times \mathbb{A}_{L_v})$ has for modular representation space $\text{Repsp}(GL_2(\mathbb{A}_{L_{\overline{v}}} \times \mathbb{A}_{L_v}))$ given by the tensor product $M_R \otimes M_L$ of a right $T_2^t(\mathbb{A}_{L_{\overline{v}}})$-semimodule M_R by a left $T_2(\mathbb{A}_{L_v})$-semimodule M_L .

M_R (resp. M_L) decomposes into $T_2^t(L_{\overline{v}_\mu})$-subsemimodules $M_{\overline{v}_{\mu,m_\mu}}$ (resp. $T_2(L_{v_\mu})$-subsemimodules $M_{v_{\mu,m_\mu}}$) following:

$$M_R = \bigoplus_{\mu=1}^{q} \bigoplus_{m_\mu} M_{\overline{v}_{\mu,m_\mu}} \qquad (\text{resp.} \quad M_L = \bigoplus_{\mu=1}^{q} \bigoplus_{m_\mu} M_{v_{\mu,m_\mu}} \ ).$$

Each $T_2^t(L_{\overline{v}_\mu})$-subsemimodule $M_{\overline{v}_{\mu,m_\mu}}$ (resp. $T_2(L_{v_\mu})$-subsemimodule $M_{v_{\mu,m_\mu}}$) constitutes an equivalent representative of the μ-th conjugacy class of $T_2^t(\mathbb{A}_{L_{\overline{v}}})$ (resp. $T_2(\mathbb{A}_{L_v})$) with respect to the fixed global inertia subgroup I_{L_μ} and has a rank given by $n_{\mu_R} = (p+\mu)\cdot N$ (resp. $n_{\mu_L} = (p+\mu)\cdot N$). So, the $T_2^t(\mathbb{A}_{L_{\overline{v}}})$-semimodule M_R (resp. $T_2(\mathbb{A}_{L_v})$-semimodule M_L) has a rank:

$$\begin{aligned} n_R &= \bigoplus_{\mu=1}^{q} \bigoplus_{m_\mu} n_{\mu_R} = \bigoplus_{\mu} \bigoplus_{m_\mu} (p+\mu)\cdot N \\ (\text{resp.} \quad n_L &= \bigoplus_{\mu=1}^{q} \bigoplus_{m_\mu} n_{\mu_L} = \bigoplus_{\mu} \bigoplus_{m_\mu} (p+\mu)\cdot N \ ). \end{aligned}$$

On the other hand, the right (resp. left) global inertia subgroup $I_{L_{\overline{v}_\mu}}$ (resp. $I_{L_{v_\mu}}$) has a representation space given by $\text{Repsp}(T_2^t(L_{\overline{v}^1_\mu}))$ (resp. $\text{Repsp}(T_2^t(L_{v^1_\mu}))$) where $L_{\overline{v}^1_\mu}$ (resp. $L_{v^1_\mu}$) is an irreducible completion of rank N as introduced in section 1.1.4.

1.1.6 Quanta, strings and field are introduced

Consequently, each representative $M_{\overline{v}_{\mu,m_\mu}}$ (resp. $M_{v_{\mu,m_\mu}}$) of the μ-th conjugacy class of $T_2^t(\mathbb{A}_{L_{\overline{v}}})$ (resp. $T_2(\mathbb{A}_{L_v})$) is cut into $(p+\mu)$ equivalent conjugacy subclass representatives $M_{\overline{v}^{\mu'}_{\mu,m_\mu}}$ (resp. $M_{v^{\mu'}_{\mu,m_\mu}}$), $1 \le \mu' \le \mu$, having a rank equal to N and being in one-to-one correspondence with the $(p+\mu)$ equivalent subcompletions $L_{\overline{v}^{\mu'}_{\mu,m_\mu}}$ (resp. $L_{v^{\mu'}_{\mu,m_\mu}}$) of $L_{\overline{v}_\mu}$ (resp. L_{v_μ}).

These conjugacy subclass representatives $M_{\overline{v}^{\mu'}_{\mu,m_\mu}}$ (resp. $M_{v^{\mu'}_{\mu,m_\mu}}$) are interpreted as right (resp. left) time quanta which are thus closed irreducible $1D$ algebraic sets of degree N .

Each representative $M_{\overline{v}_{\mu,m_\mu}}$ (resp. $M_{v_{\mu,m_\mu}}$), being a one-dimensional $T_2^t(L_{\overline{v}_\mu})$-subsemimodule (resp. $T_2(L_{v_\mu})$-subsemimodule), is a string localized in the lower (resp. upper) half space. So, each string $M_{\overline{v}_{\mu,m_\mu}}$ (resp. $M_{v_{\mu,m_\mu}}$) is composed of $(p+\mu)$ quanta, where $(p+\mu)$ is the global residue degree $f_{b_{\mu_R}}$ (resp. $f_{b_{\mu_L}}$) referring to the dimension of a quantum class representative.

On the other hand, we want to introduce the set of smooth continuous (bi)functions on the representation space $M_R \otimes M_L = \mathrm{Repsp}(\mathrm{GL}_2(\mathbb{A}_{L_{\overline{v}}} \times \mathbb{A}_{L_v}))$ of the algebraic bilinear semigroup $\mathrm{GL}_2(\mathbb{A}_{L_{\overline{v}}} \times \mathbb{A}_{L_v})$. Due to the bilinear Gauss decomposition of $\mathrm{GL}_2(\mathbb{A}_{L_{\overline{v}}} \times \mathbb{A}_{L_v})$, we have to envisage the set of smooth continuous functions $\phi_{G_R}(x_{g_R})$, $x_{g_R} \in T_2^t(\mathbb{A}_{L_{\overline{v}}})$, on $M_R = \mathrm{Repsp}(T_2^t(\mathbb{A}_{L_{\overline{v}}}))$ and localized in the lower half space as well as the corresponding symmetric set of smooth continuous functions $\phi_{G_L}(x_{g_L})$, $x_{g_L} \in T_2(\mathbb{A}_{L_v})$, on $M_L = \mathrm{Repsp}(T_2(\mathbb{A}_{L_v}))$ and localized in the upper half space.

On $M_R \otimes M_L$, the tensor products $\phi_{G_R}(x_{g_R}) \otimes \phi_{G_L}(x_{g_L})$ of smooth continuous functions have to be considered: the are called bifunctions.

But, as $\mathrm{GL}_2(\mathbb{A}_{L_{\overline{v}}} \times \mathbb{A}_{L_v})$ is partitioned into conjugacy classes, we have to take into account the bifunctions $\phi_{G_{\mu,m_\mu,R}}(x_{\mu_R}) \otimes \phi_{G_{\mu,m_\mu,L}}(x_{\mu_L})$ on the conjugacy class representatives $M_{\overline{v}_{\mu,m_\mu}} \otimes M_{v_{\mu,m_\mu}}$.

The set of smooth continuous bifunctions $\{\phi_{G_{\mu,m_\mu,R}}(x_{\mu_R}) \otimes \phi_{G_{\mu,m_\mu,L}}(x_{\mu_L})\}_{m_\mu}^{\mu}$ on the $\mathrm{GL}_2(\mathbb{A}_{L_{\overline{v}}} \times \mathbb{A}_{L_v})$-bisemimodule $M_R \otimes M_L$ is a bisemisheaf of rings, noted $\mathcal{C}_{M_R} \otimes \mathcal{C}_{M_L}$ or $\widetilde{M}_R \otimes \widetilde{M}_L$, in such a way that the set of continuous bifunctions are the (bi)sections of $\mathcal{C}_{M_R} \otimes \mathcal{C}_{M_L}$.

Note that $\mathcal{C}_{M_R}$ (resp. $\mathcal{C}_{M_L}$), having as sections the smooth continuous functions $\phi_{G_{\mu,m_\mu,R}}(x_{\mu_R})$ (resp. $\phi_{G_{\mu,m_\mu,L}}(x_{\mu_L})$), is a semisheaf of rings because it is a sheaf of abelian semigroups $\mathcal{C}_{M_R}(x_{\mu_R})$ (resp. $\mathcal{C}_{M_L}(x_{\mu_L})$) for every right (resp. left) point x_{μ_R} (resp. x_{μ_L}) of the topological semispace $M_R = \mathrm{Repsp}(T_2^t(\mathbb{A}_{L_{\overline{v}}}))$ (resp. $M_L = \mathrm{Repsp}(T_2(\mathbb{A}_{L_v}))$) where $\mathcal{C}_{M_R}(x_{\mu_R})$ (resp. $\mathcal{C}_{M_L(x_{\mu_L}}$) has the structure of a semiring [Ser1], [G-D].

Remark that the pair $\{\mathcal{C}_{M_R}, \mathcal{C}_{M_L}\}$ of semisheaves of ring or their product $\mathcal{C}_{M_R} \otimes \mathcal{C}_{M_L}$ is what the physicists call a field because each pair $\{\phi_{G_{\mu,m_\mu,R}}(x_{\mu_R}), \phi_{G_{\mu,m_\mu,L}}(x_{\mu_L})\}$ of smooth continuous symmetric functions behaves like a harmonic oscillator as it will be seen in the following.

As each representative $M_{\overline{v}_{\mu,m_\mu}}$ (resp. $M_{v_{\mu,m_\mu}}$) of the μ-th conjugacy class of $T_2^t(\mathbb{A}_{L_{\overline{v}}})$ (resp. $T_2(\mathbb{A}_{L_v})$) has a rank equal to $n_{\mu_R} = (p+\mu)N$ (resp. $n_{\mu_L} = (p+\mu)N$), we will say by abuse of language that the function $\phi_{G_{\mu,m_\mu,R}}(x_{\mu_R})$ (resp. $\phi_{G_{\mu,m_\mu,L}}(x_{\mu_L})$) on $M_{\overline{v}_{\mu,m_\mu}}$ (resp. $M_{v_{\mu,m_\mu}}$) is characterized by a rank n_{μ_R} (resp. n_{μ_L}).

If $\phi_{G^{\mu'}_{\mu,m_\mu,R}}(x_{G^{\mu'}_{\mu_R}})$ (resp. $\phi_{G^{\mu'}_{\mu,m_\mu,L}}(x_{G^{\mu'}_{\mu_L}})$) denotes of the smooth continuous function on the μ'-th equivalent conjugacy subclass representative $M_{\overline{v}^{\mu'}_{\mu,m_\mu}}$ (resp. $M_{v^{\mu'}_{\mu,m_\mu}}$), then $(M_{\overline{v}^{\mu'}_{\mu,m_\mu}}, \phi_{G^{\mu'}_{\mu,m_\mu,R}}(x_{G^{\mu'}_{\mu_R}}))$ (resp. $(M_{v^{\mu'}_{\mu,m_\mu}}, \phi_{G^{\mu'}_{\mu,m_\mu,L}}(x_{G^{\mu'}_{\mu_L}}))$) is a closed irreducible one-dimensional subscheme of rank N associated to the right (resp. left) quantum $M_{\overline{v}^{\mu'}_{\mu,m_\mu}}$ (resp. $M_{v^{\mu'}_{\mu,m_\mu}}$) and noted $\widetilde{M}_{\overline{v}^{\mu'}_{\mu,m_\mu}}$ (resp. $\widetilde{M}_{v^{\mu'}_{\mu,m_\mu}}$).

1.1.7 Emergent projection and Borel-Serre compactification

1) As the right (resp. left) subsemimodules $M_{\overline{v}_{\mu,m_\mu}}$ (resp. $M_{v_{\mu,m_\mu}}$) are not necessarily closed strings, the emergent toroidal projective isomorphisms:

$$\gamma_{\mu_R} : \quad M_{\overline{v}_{\mu,m_\mu}} \longrightarrow T^1_{\mu_R} \qquad (\text{resp.} \quad \gamma_{\mu_L} : \quad M_{\mu,m_\mu} \longrightarrow T^1_{\mu_L})$$

are introduced such that [Pie3]:

a) the geometric points of $M_{\overline{v}_{\mu,m_\mu}}$ (resp. $M_{v_{\mu,m_\mu}}$) are mapped onto the origin, called the emergence point which can be viewed as the point at infinity of the resulting projective variety;

b) these geometric points are then projected symmetrically from the origin into the affine connected compact algebraic varieties $\widetilde{M}^T_{\overline{v}_{\mu,m_\mu}}$ (resp. $\widetilde{M}^T_{v_{\mu,m_\mu}}$) [C-S] which "are" $1D$-(semi)tori $T^1_{\mu_R}$ (resp. $T^1_{\mu_L}$) [B-T] characterized by a radius of ejection $r_{\mu_{R,L}}$ and such that:

- $\widetilde{M}^T_{\overline{v}_{\mu,m_\mu}}$ (resp. $\widetilde{M}^T_{v_{\mu,m_\mu}}$) are localized in the lower (resp. upper) half space with respect to the time variable " t ";

- each time quantum $M^T_{\overline{v}^{\mu'}_{\mu,m_\mu}}$ is localized on a closed affine subset of $M^T_{\overline{v}_{\mu,m_\mu}}$, taking into account the $\gamma_\mu : M_{\overline{v}^{\mu'}_{\mu,m_\mu}} \to M^T_{v^{\mu'}_{\mu,m_\mu}}$ morphism.

  Remark that it will also be considered in the following that $\widetilde{M}^T_{\overline{v}^{\mu'}_{\mu,m_\mu}}$ (resp. $\widetilde{M}^T_{v^{\mu'}_{\mu,m_\mu}}$) are isomorphic to $1D$-(semi)tori, the distinction between the two cases being in general evident.

2) the space $X = GL_2(I\!R)/GL_2(\mathbb{Z})$ corresponds to the set of lattices of $I\!R$. In this perspective, we have introduced in [Pie9] a lattice bisemispace $X_{S_{R\times L}} = GL_2(\mathbb{A}_{L_{\overline{\omega}}} \times \mathbb{A}_{L_\omega})/GL_2(\overline{\mathbb{Z}}^2_{p_q})$, where $\mathbb{A}_{L_\omega}$ is a ramified adele semiring over a complex semifield L_ω , such that the boundary $\partial\overline{X}_{S_{R\times L}}$ of the compactified bisemispace $\overline{X}_{S_{R\times L}}$ corresponds to the boundary of the Borel-Serre compactification [B-S] and is given by:

$$\partial\overline{X}_{S_{R\times L}} = GL_2(\mathbb{A}_{L^T_{\overline{v}}} \times \mathbb{A}_{L^T_v})/GL_2(\overline{\mathbb{Z}}^2_{p_q})$$

where $\mathbb{A}_{L^T_{\overline{v}}}$ (resp. $\mathbb{A}_{L^T_v}$) is the right (resp. left) ramified adele semiring with respect to the "toroidal" completions of the $L^T_{\overline{v}_{\mu,m_\mu}}$ (resp. $L^T_{v_{\mu,m_\mu}}$) : $\mathbb{A}_{L^T_{\overline{v}}} = \prod\limits_\mu L^T_{\overline{v}_\mu} \prod\limits_{m_\mu} L^T_{\overline{v}_{\mu,m_\mu}}$.

Let us note that there exits an isomorphism $\gamma_{R\times L} : \quad \overline{X}_{S_{R\times L}} \longrightarrow \partial\overline{X}_{S_{R\times L}}$ between the compactified lattice bisemispace $\overline{X}_{S_{R\times L}}$ and its boundary $\partial\overline{X}_{S_{R\times L}}$ such that a one-to-one correspondence exists between the complex "bipoints" of $\overline{X}_{S_{R\times L}}$ and the real "bipoints" of $\partial\overline{X}_{S_{R\times L}}$ (a bipoint being defined as the product of a right point localized in the lower half space by a left point localized in the upper half space).

3) The double coset decomposition $\partial\overline{S}_{K^D_{R\times L}}$ of the boundary $\partial\overline{X}_{S_{R\times L}}$ of the compactified lattice bisemispace corresponds to a Shimura bisemivariety and is given by:

$$\partial\overline{S}_{K^D_{R\times L}} = P_2(\mathbb{A}_{L^T_{\overline{v}^1}} \times \mathbb{A}_{L^T_{v^1}}) \setminus GL_2(\mathbb{A}_{L^T_{\overline{v}}} \times \mathbb{A}_{L^T_v})/GL_2(\overline{\mathbb{Z}}^2_{p_q})$$

where

- $P_2(\mathbb{A}_{L^T_{v^1}})$ is the standard parabolic subgroup over the adele subsemiring $\mathbb{A}_{L^T_{v^1}} = \prod\limits_\mu L^T_{v^1_\mu} \prod\limits_{m_\mu} L^T_{v^1_{\mu,m_\mu}}$ where $L^T_{v^1_{\mu,m_\mu}}$ denotes the μ-th irreducible toroidal central subcompletion of $L^T_{v_{\mu,m_\mu}}$ having a rank equal to N .

  $P_2(\mathbb{A}_{L^T_{\overline{v}^1}} \times \mathbb{A}_{L^T_{v^1}})$ is a bilinear parabolic subgroup and is considered as the smallest normal ramified bilinear subsemigroup of the bilinear algebraic semigroup $GL_2(\mathbb{A}_{L^T_{\overline{v}}} \times \mathbb{A}_{L^T_v})$. The bilinear quotient semigroup $P_2(\mathbb{A}_{L^T_{\overline{v}^1}} \times \mathbb{A}_{L^T_{v^1}})/GL_2(\mathbb{A}_{L^T_{\overline{v}}} \times \mathbb{A}_{L^T_v})$ has its (bi)cosets which are in one-to-one correspondence with the modular conjugacy classes of $GL_2(\mathbb{A}_{L^T_{\overline{v}}} \times \mathbb{A}_{L^T_v})$ with respect to fixed bielements which correspond to the product $L^T_{\overline{v}^1_\mu} \times L^T_{v^1_\mu}$ of irreducible subcompletions.

- the general bilinear semigroup $GL_2(\mathbb{A}_{L^T_{\overline{v}}} \times \mathbb{A}_{L^T_v})$ is a bilinear algebraic semigroup [Che1], also noted $G_{t_{R\times L}}(\mathbb{A}_{R\times L})$ in abbreviated form, to which corresponds the bilinear semigroup of modular automorphisms of $G_{t_{R\times L}}(\mathbb{A}_{R\times L})$, such that the set of products, right by left, of orbits of $G_{t_{R\times L}}(\mathbb{A}_{R\times L})$ coincide with its modular conjugacy (bi)classes.

The fixed bielements of a modular conjugacy class of $GL_2(\mathbb{A}_{L^T_{\overline{v}}}\times\mathbb{A}_{L^T_v})$ are the elements the bilinear parabolic subgroup $P_2(\mathbb{A}_{L^T_{\overline{v}^1}}\times\mathbb{A}_{L^T_{v^1}})$, representing the product of global inertia subgroups [Pie9] $(I_{L_{\overline{v}_\mu}}\times I_{L_{v_\mu}})$.

- the modular conjugacy classes of $G_{t_{R\times L}}(\mathbb{A}_R\times\mathbb{A}_L)$ correspond to the (bi)cosets of $G_{t_{R\times L}}(\mathbb{A}_R\times\mathbb{A}_L)/GL_2(\overline{\mathbb{Z}}^2_{p_q})$ since the subgroup $GL_2(\overline{\mathbb{Z}}^2_{p_q})$ constitutes the representation of the (bi)cosets of the tensor product of Hecke operators as it will be seen in definition 1.1.18: it is also noted $K^D_{R\times L}(\overline{\mathbb{Z}}^2_{p_q})$. The bilinear quotient semigroup $GL_2(\mathbb{A}_{L^T_{\overline{v}}}\times\mathbb{A}_{L^T_v})/GL_2(\overline{\mathbb{Z}}^2_{p_q})$ consists in a double symmetric tower of conjugacy class representatives characterized by increasing ranks, i.e. by increasing numbers of quanta or strings.

4) The double coset decomposition $\partial\overline{S}_{K^D_{R\times L}}$ restricted to the lower (resp. upper) half space then becomes:

$$\begin{aligned}\partial\overline{S}_{K_{t_R}}&=P_2(\mathbb{A}_{L^T_{\overline{v}^1}})\setminus T^t_2(\mathbb{A}_{L^T_{\overline{v}}})/T^t_2(\overline{\mathbb{Z}}_{p_q})\\ \text{(resp.}\quad\partial\overline{S}_{K_{t_L}}&=P_2(\mathbb{A}_{L^T_{v^1}})\setminus T_2(\mathbb{A}_{L^T_v})/T_2(\overline{\mathbb{Z}}_{p_q})\ ).\end{aligned}$$

It will also be noted:

$$\begin{aligned}\partial\overline{S}_{K_{t_R}}&=P_{t_R}(\mathbb{A}_{L^T_{\overline{v}^1}})\setminus G_{t_R}(\mathbb{A}_R)/K_R(\overline{\mathbb{Z}}_{p_q})\\ \text{(resp.}\quad\partial\overline{S}_{K_{t_L}}&=P_{t_L}(\mathbb{A}_{L^T_{v^1}})\setminus G_{t_L}(\mathbb{A}_L)/K_L(\overline{\mathbb{Z}}_{p_q})\ ).\end{aligned}$$

1.1.8 Right and left semisheaves of rings

The set of products, right by left, of toroidal projective isomorphisms:

$$\{\gamma_{\mu_R}\times\gamma_{\mu_L}:M_{\overline{v}_{\mu,m_\mu}}\otimes M_{v_{\mu,m_\mu}}\longrightarrow M^T_{\overline{v}_{\mu,m_\mu}}\otimes M^T_{v_{\mu,m_\mu}}\equiv T^1_{\mu_R}\otimes T^1_{\mu_L}\}$$

transforms the $\mathrm{GL}_2(\mathbb{A}_{L_{\overline{v}}}\times\mathbb{A}_{L_v})$-bisemimodule $M_R\otimes M_L=\bigoplus\limits_\mu\bigoplus\limits_{m_\mu}(M_{\overline{v}_{\mu,m_\mu}}\otimes M_{v_{\mu,m_\mu}})$ into the $\mathrm{GL}_2(\mathbb{A}_{L^T_{\overline{v}}}\times\mathbb{A}_{L^T_v})$-bisemimodule $M^T_R\otimes M^T_L=\bigoplus_\mu\bigoplus_{m_\mu}(M^T_{\overline{v}_{\mu,m_\mu}}\otimes M^T_{v_{\mu,m_\mu}})$.

Each representative $M^T_{\overline{v}_{\mu,m_\mu}}$ (resp. $M^T_{v_{\mu,m_\mu}}$) of M^T_R (resp. M^T_L) is a semitorus localized in the lower (resp. upper) half space. In fact, we shall be essentially interested in right (resp. left) one-dimensional tori: so, we have to double the representatives $M^T_{\overline{v}_{\mu,m_\mu}}$ (resp. $M^T_{v_{\mu,m_\mu}}$), i.e. to consider representatives $M^T_{2\overline{v}_{\mu,m_\mu}}$ (resp. $M^T_{2v_{\mu,m_\mu}}$) characterized by double ranks and by double quanta in such a way that $M^T_{2\overline{v}_{\mu,m_\mu}}$ (resp. $M^T_{2v_{\mu,m_\mu}}$) be closed strings [Del→Wit]. But, in the following, we shall maintain the condensed notation $M^T_{\overline{v}_{\mu,m_\mu}}$ (resp. $M^T_{v_{\mu,m_\mu}}$) for the two cases, the distinction being evident by itself. On the representation space $M^T_R=\mathrm{Repsp}(T^t_2(\mathbb{A}_{L^T_{\overline{v}}}))$ (resp. $M^T_L=\mathrm{Repsp}(T_2(\mathbb{A}_{L^T_v}))$) of $T^t_2(\mathbb{A}_{L^T_{\overline{v}}})$ (resp. $T_2(\mathbb{A}_{L^T_v})$) $\subset\mathrm{GL}_2(\mathbb{A}_{L^T_{\overline{v}}}\times\mathbb{A}_{L^T_v})$, we can consider the set of differentiable smooth functions $\phi_{G^T_{\mu,m_{\mu_R}}}(x_{\mu_R})$ (resp. $\phi_{G^T_{\mu,m_{\mu_L}}}(x_{\mu_L})$) on the representatives $M^T_{\overline{v}_{\mu,m_\mu}}$ (resp. $M^T_{v_{\mu,m_\mu}}$) of M^T_R (resp. M^T_L): it is a semisheaf of rings noted θ^1_R (resp. θ^1_L) or $\widetilde{M}^T_R$ (resp. $\widetilde{M}^T_L$).

The differentiable functions $\phi_{G^T_{\mu,m_{\mu_R}}}(x_{\mu_R})$ (resp. $\phi_{G^T_{\mu,m_{\mu_L}}}(x_{\mu_L})$) are the sections of the semisheaf of rings θ^1_R (resp. θ^1_L): they are noted in condensed notation s_{μ_R} (resp. s_{μ_L}).

Remark that the following developments will essentially deal with the semisheaf of rings θ^1_R (resp. θ^1_L) because they naturally lead to automorphic representations (see sections 1.15 to 1.23) and to Langlands global correspondences (similar developments can be envisaged on the semisheaves of rings $\mathcal{C}_{M_R}$ (resp. $\mathcal{C}_{M_L}$) or $\widetilde{M}_R$ and $\widetilde{M}_L$ (see section 1.1.6)).

It is then possible to define a graded algebra on the set of right (resp. left) sections $s_{\mu R,L}$ of the right (resp. left) semisheaf of rings $\theta^1_{R,L}$.

Proposition 1.1.9 *Let $n_{\mu R,L}$ be the right (resp. left) rank of the right (resp. left) section $s_{\mu R,L}$ and $n_{(\mu+1)R,L}$ the right (resp. left) rank of the right (resp. left) section $s_{(\mu+1)R,L}$. Then, the inequality $n_{(\mu+1)R,L} > n_{\mu R,L}$ leads to the topological embedding $s_{\mu R,L} \subset s_{(\mu+1)R,L}$ between the μ-th and the $(\mu+1)$-th section.*

Proof. If the inequality $n_{(\mu+1)R,L} > n_{\mu R,L}$ holds, then $s_{(\mu+1)R,L} \supset s_{\mu R,L}$. Indeed, if $r(x_{\mu+1})_{R,L}$ and $r(x_\mu)_{R,L}$ denote respectively the radii of ejection of the points $x_{(\mu+1)R,L} \in s_{(\mu+1)R,L}$ and $x_{\mu R,L} \in s_{\mu R,L}$, it is evident that $r(x_{\mu+1})_{R,L} > r(x_\mu)_{R,L}$. ■

Corollary 1.1.10 *Let $s_{1R,L} \subset \cdots \subset s_{qR,L}$ be the increasing filtration of the q sets of sections of the semisheaf of rings $\theta^1_{R,L}$ to which is associated the sequence of ranks*

$$n_{\theta^1_{R,L}} = \{n_{1R,L}, \cdots, n_{\mu R,L}, \cdots, n_{qR,L}\} .$$

Then, the right (resp. left) semisheaf of rings $\theta^1_{R,L}$ is characterized by the global rank given by the set $n_{\theta^1_{R,L}}$.

We are now concerned with the cohomology of the boundary of the Borel-Serre compactification [B-S], [Sch2] of semispace $\partial\, \overline{S}_{K_{t_{R,L}}}$: it is the Eisenstein cohomology, as nicely developed by G. Harder [Har1], J. Schwermer [Sch1] and others, which becomes the so-called right (resp. left) Eisenstein cohomology when it leads to the generation of a right (resp. left) semisheaf of rings $\theta^1_{R,L}$ on the $G_{t_{R,L}}(\mathbb{A}_{R,L})$-semimodule.

Definition 1.1.11 (Nilpotent fibration on the right (resp. left) Shimura semivariety) If we take into account:

- the Gauss decomposition of the bilinear algebraic semigroup

$$\begin{aligned} G_{t_{R\times L}}(\mathbb{A}_R \times \mathbb{A}_L) \equiv GL_2(\mathbb{A}_R \times \mathbb{A}_L) &= T^t_2(\mathbb{A}_R) \times T_2(\mathbb{A}_L) \\ &= [D_2(\mathbb{A}_R) \times UT^t_2(\mathbb{A}_R)][D_2(\mathbb{A}_L) \times UT_2(\mathbb{A}_L)] , \end{aligned}$$

  as developed in 1.1.5. where $\mathbb{A}_R \equiv \mathbb{A}_{L^T_{\overline{v}}}$ and $\mathbb{A}_L \equiv \mathbb{A}_{L^T_v}$;

- the Levi decomposition of the right (resp. left) parabolic subgroup

$$\begin{aligned} P_{t_R}(\mathbb{A}_{L^T_{\overline{v}^1}}) &\equiv P_2(\mathbb{A}_{L^T_{\overline{v}^1}}) = D_2(\mathbb{A}_{L^T_{\overline{v}^1}}) \cdot UT^t_2(\mathbb{A}_{L^T_{\overline{v}^1}}) \\ (\text{resp.}\quad P_{t_L}(\mathbb{A}_{L^T_{v^1}}) &\equiv P_2(\mathbb{A}_{L^T_{v^1}}) = D_2(\mathbb{A}_{L^T_{v^1}}) \cdot UT^t_2(\mathbb{A}_{L^T_{v^1}})) \end{aligned}$$

- and the similar decomposition of

$$\begin{aligned} & K_R(\overline{\mathbb{Z}}_{p_q}) \equiv T_2^t(\overline{\mathbb{Z}}_{p_q}) = D_2(\overline{\mathbb{Z}}_{p_q}) \cdot UT_2^t(\overline{\mathbb{Z}}_{p_q}) \\ \text{(resp.}\quad & K_L(\overline{\mathbb{Z}}_{p_q}) \equiv T_2(\overline{\mathbb{Z}}_{p_q}) = D_2(\overline{\mathbb{Z}}_{p_q}) \cdot UT_2(\overline{\mathbb{Z}}_{p_q})\ ), \end{aligned}$$

  introduced in 1.1.7.,

into product of unitriangular matrices of nilpotent subsemigroups by diagonal matrices of centralizers $\mathrm{Z}(\cdot)$, noted here $M(\cdot)$ to respect the notations of [Sch1] and [Har2], we are led to define, following G. Harder and J. Schwermer, the fibration:

$$\begin{aligned} \partial\overline{S}_{K_{t_R}} &= P_2(\mathbb{A}_{L_{\overline{v}^1}^T}) \setminus T_2^t(\mathbb{A}_R) \Big/ T_2^t(\overline{\mathbb{Z}}_{p_q}) \\ &\longrightarrow S^{M_R}_{K^{M_R}} = M(\mathbb{A}_{L_{\overline{v}^1}^T}) \setminus M(\mathbb{A}_R) \Big/ K_R^{M_R}(\overline{\mathbb{Z}}_{p_q}) \\ &\qquad\qquad \equiv D_2(\mathbb{A}_{L_{\overline{v}^1}^T}) \setminus D_2(\mathbb{A}_R) \Big/ D_2(\overline{\mathbb{Z}}_{p_q}) \end{aligned}$$

$$\begin{aligned} \text{(resp.}\quad \partial\overline{S}_{K_{t_L}} &= P_2(\mathbb{A}_{L_{v^1}^T}) \setminus T_2(\mathbb{A}_L) \Big/ T_2(\overline{\mathbb{Z}}_{p_q}) \\ &\longrightarrow S^{M_L}_{K^{M_L}} = M(\mathbb{A}_{L_{v^1}^T}) \setminus M(\mathbb{A}_L) \Big/ K_L^{M_L}(\overline{\mathbb{Z}}_{p_q}) \\ &\qquad\qquad \equiv D_2(\mathbb{A}_{L_{v^1}^T}) \setminus D_2(\mathbb{A}_L) \Big/ D_2(\overline{\mathbb{Z}}_{p_q}\ ) \end{aligned}$$

having as right (resp. left) fiber, the right (resp. left) nilpotent fiber

$$\begin{aligned} & N(\mathbb{A}_{L_{\overline{v}^1}^T}) \setminus N(\mathbb{A}_R) \Big/ K_R^N \equiv UT_2^t(\mathbb{A}_{L_{\overline{v}^1}^T}) \setminus UT_2^t(\mathbb{A}_R) \Big/ UT_2^t(\overline{\mathbb{Z}}_{p_q}) \\ \text{(resp.}\quad & N(\mathbb{A}_{L_{v^1}^T}) \setminus N(\mathbb{A}_L) \Big/ K_L^N \equiv UT_2(\mathbb{A}_{L_{v^1}^T}) \setminus UT_2(\mathbb{A}_L) \Big/ UT_2(\overline{\mathbb{Z}}_{p_q})\ ). \end{aligned}$$

Proposition 1.1.12 *The right (resp. left) Eisenstein cohomology associated with the generation of a right (resp. left) semisheaf of rings $\theta^1_{R,L}$ decomposes into:*

$$\begin{aligned} & H^*_{R,L}(\partial\ \overline{S}_{K_{tR,L}}, \theta^1_{R,L}) \\ &= \quad H^*(P_{tR,L}(\mathbb{A}_{L_{v^1}^T}) \setminus G_{tR,L}(\mathbb{A}_{R,L})/K_{R,L}(\overline{\mathbb{Z}}_{p_q}), \theta^1_{R,L}) \\ &\simeq \bigoplus_{\xi_{R,L}\in\Xi_{K_{R,L}}} H^*(S^{M_{R,L}}_{K^{M_{L,R}(\xi_{R,L})}}, H^*(\tilde{u}_{R,L}, \theta^1_{R,L})) \end{aligned}$$

where $S^{M_{R,L}}_{K^{M_{R,L}}} = M(\mathbb{A}_{L_{v^1}^T}) \setminus M(\mathbb{A}_{R,L})/K^{M_{R,L}}_{R,L}(\overline{\mathbb{Z}}_{p_q})$.

Proof. The right (resp. left) Eisenstein cohomology $H^*_{R,L}(\partial\ \overline{S}_{K_{tR,L}}, \theta^1_{R,L})$ decomposes into the direct sum of right (resp. left) cohomology classes referring to right (resp. left) cosets $\xi_{R,L}$ of $(G_{tR,L}(\mathbb{A}_{R,L})/K_{R,L})$ such that the right (resp. left) coefficient system be given by the right (resp. left) semisimple Lie algebra cohomology $H^*_{R,L}(\tilde{u}_{R,L}, \theta^1_{R,L})$ which is a right (resp. left) semimodule for the right (resp. left) algebraic semigroup $M_{R,L}(\mathbb{A}_{R,L})$.

If $u_{P_{R,L}}$ is the right (resp. left) unipotent algebraic semigroup, then $\tilde{u}_{R,L} = \mathrm{Lie}\ (u_{P_{R,L}})$ is its right (resp. left) nilpotent Lie algebra.

Note that this decomposition of the right (resp. left) Eisenstein cohomology is an adaptation of the developments of G. Harder [Har2]. ∎

Definition 1.1.13 (Algebraic Hecke characters) If $\Gamma_{\mu R,L}$ denotes the right (resp. left) Galois subgroup $\mathrm{Gal}(L^{\mp}_{\mu}/K)$, let

$$\lambda_{\mu R,L}=\{\lambda_{1R,L},\cdots,\lambda_{\mu R,L},\cdots,\lambda_{qR,L}\}_{\Gamma_{\mu R,L}:L^T_{\overline{v}1}\to L^T_{\overline{v}_\mu}}$$

be given by

$$X(T_{R,L})=\mathrm{Hom}(T_{R,L}\times_K\ L^T_v,G_m)=\bigoplus_{\Gamma_{\mu R,L}:L^T_{v1}\to L^T_{v_\mu}}X(T^1_{\mu R,L})\ ,$$

where $G_m\equiv GL_1$.

Then, the set $\lambda_{R,L}=\{\lambda_{1R,L},\cdots,\lambda_{\mu R,L},\cdots,\lambda_{qR,L}\}$ is the sequential set of weights in $X(\theta^1_{R,L})$ referring to the q basic right (resp. left) sections of the right (resp. left) semisheaf of rings $\theta^1_{R,L}$.

Let $\omega_{R,L}=\{\omega_{1R,L},\cdots,\omega_{qR,L}\}$ be the set of q right (resp. left) actions of the Weyl groups on $\lambda_{R,L}\in X(\theta^1_{R,L})$.

Remark that $\omega_{R,L}$ is a set of Weyl subgroups because this set acts on the set of right (resp. left) characters $\lambda_{\mu R,L}$.

Consequently, the maximal convex right (resp. left) subsets of $X(\theta^1_{R,L})$ will be in negative (resp. positive) Weyl chambers.

Let finally $\phi_{R,L}=\omega_{R,L}\cdot\lambda_{R,L}$ be the set of right (resp. left) algebraic Hecke characters [Clo] on $\theta^1_{R,L}$.

Proposition 1.1.14 *Let $B_{tR,L}(\mathbb{A}_{\,R,L})$ be the right (resp. left) Borel subgroup of upper (resp. lower) triangular matrices of the right (resp. left) algebraic semigroup $G_{tR,L}(\mathbb{A}_{\,R,L})$. Then, the right (resp. left) Eisenstein cohomology $H^*(\partial\,\overline{S}_{K_{t_{R,L}}},\theta^1_{R,L})$ decomposes into one-dimensional eigenspaces:*

$$\begin{aligned}&H^*_{R,L}(\partial\,\overline{S}_{K_{t_{R,L}}},\,\theta^1_{R,L})\\ &\quad\simeq\bigoplus_{\omega_{R,L}}\bigoplus_{\phi_{R,L}}\mathrm{Ind}^{\pi_0(G_{R,L}(\mathbb{A}_{\,R,L})}_{\pi_0(B_{R,L}(\mathbb{A}_{\,R,L})}\ H^*_{R,L}(S^{M_{R,L}},H^*(\tilde u_{B_{tR,L}},\theta^1_{R,L})(\omega_{R,L}\cdot\lambda_{R,L}))\end{aligned}$$

where $S^{M_{R,L}}=\varinjlim_{K_{R,L}}S^{M_{R,L}}_{K^{M_{R,L}}}$.

Proof. Indeed, the cohomology $H^*(\tilde u_{B_{tR,L}},\theta^1_{R,L})$ is a right (resp. left) semimodule for the set of tori $T^1_{R,L}=\{T^1_{1R,L},\cdots,T^1_{\mu R,L},\cdots,T^1_{qR,L}\}$.

In this context, Kostant's theorem says that the cohomology decomposes into one-dimensional eigenspaces under $T^1_{R,L}$. The right (resp. left) Eisenstein cohomology then decomposes into one-dimensional eigenspaces with respect to $\omega_{R,L}$ and the type of algebraic Hecke characters $\phi_{R,L}$ according to the considered induced representation π_0 of the Borel right (resp. left) stratum of $B_{R,L}(\mathbb{A}_{\,R,L})\equiv P_{tR,L}$. ■

In correlation with Kostant's theorem, it appears necessary to develop a bit further the problem of the representation of Eisenstein cohomology into irreducible one-dimensional components. Taking into account that Eisenstein series are eigenfunctions of Hecke operators and that the decomposition of Eisenstein cohomology into irreducible submodules characterized by some weights needs a cuspical automorphic representation of the algebraic semigroups $G_{t_{R,L}}(\mathbb{A}_{\,R,L})$, we have to envisage the action of the Hecke operators in the space of cusp forms.

Note that cusp forms are directly related to the branes of "string physicists".

Definition 1.1.15 (Algebra of cusp forms) Let H denote the Poincare upper half plane in $\mathbb{C}$. Assume that f_L is a normalized eigenform, holomorphic in H and defined in $\{\mathrm{Im}(z_L)>0\}$ with respect to $z_L\in\mathbb{C}$ of $q_L=e^{2\pi iz_L}$. The normalized eigenform f_L , expanded in formal power series $f_L=\sum\limits_n a_{n_L}q_L^n$, are cusp forms of the space $S_L(N)$ and are eigenvectors of the Hecke operators T_{ℓ_L} , for $\ell\nmid N$, and U_{ℓ_L} , for $\ell\mid N$ where N is a positive integer. Then, Fourier coefficients of f_L and eigenvalues of the Hecke operator coincide: $a_1=1$ and $a_n=c(n,f_L)$ so that the $c(n,f_L)$ generate the ring of integers θ_L of the number field L^+ over $\mathbb{Q}$. The space $S_L(N)$ can then be considered as a θ_L-algebra over θ_L .

As we are concerned with the endomorphisms of the algebra of cusp forms $S_L(N)$, it is the bialgebra $S_L^e=S_L(N)\otimes_\theta S_R(N)$ which must be considered in the developments such that tensor products of Hecke operators acting on tensor products of cusp forms defined respectively in the upper and in the lower half plane will be envisaged. The coalgebra $S_R(N)$ of cusp forms is defined in the Poincare lower half plane H^* and has for elements the eigenforms $f_R=\sum\limits_n a_{n_R}q_R^n$ with $q_R^n=e^{-2\pi inz_R}$ where z_R is the complex conjugate of z_L . These eigenforms f_R are eigenfunctions of Hecke operators T_{ℓ_R} , for $\ell\nmid N$ and U_{ℓ_R} for $\ell\mid N$.

Definition 1.1.16 (Global elliptic $\mathbb{A}_{R,L}$-semimodule) In order to get an automorphic irreducible representation of the algebra of cusp forms, we shall consider that the one-dimensional semisheaf of rings $\theta^1_{R,L}$ define a global elliptic semimodule whose space is included in the space of cusp forms. Let $s_{R,L}=\Gamma(\theta^1_{R,L})$ denote the set of sections of $\theta^1_{R,L}$. For each section $s_{\mu_{R,L}}\in s_{R,L}$, let $\mathrm{End}(G_{s_{R,L}})$ be the Frobenius endomorphism of the group $G_{s_{R,L}}$ of the elements $s_{\mu_{R,L}}$ and let $q^{\pm p}\to q^{\pm(p+\mu)}\in\mathrm{End}_{\mathbb{F}_p}(G_{s_{R,L}})$ be the corresponding Frobenius substitution with $q^{\pm(p+\mu)}=e^{\pm2\pi i(p+\mu)x}$, $x\in\mathbb{R}$.

A global elliptic right (resp. left) $s_{R,L}$-semimodule $\phi_{R,L}(s_{R,L})$ in the sense of Drinfeld [Drin] is a ring homomorphism [And]: $\phi_{R,L}:s_{R,L}\to\mathrm{End}(G_{s_{R,L}})$ given by $\phi_{R,L}(s_{R,L})=\sum\limits_\mu\sum\limits_{m_\mu}\phi(s_{q_{R,L}})_{\mu,m_\mu}q^{\pm(p+\mu)}/\mathbb{Q}_{R,L}$ where $\sum\limits_\mu$ runs over the sections $T^1_{\mu_{R,L}}$ of $\theta^1_{R,L}$ having ranks n_μ and where $\sum\limits_{m_\mu}$ runs over the number of ideals of the decomposition group D_{μ^2} introduced in section 1.1.8 and corresponding to the multiplicity $m^{(\mu)}$ of the μ-th section.

Lemma 1.1.17 *The space $S_{R,L}(\phi_{R,L})$ of global elliptic $s_{R,L}$-semimodules $\phi_{R,L}(s_{R,L})$ is included into the space $S_{R,L}(N)$ of cusp forms $f_{R,L}:S_{R,L}(\phi_{R,L})\hookrightarrow S_{R,L}(N)$ such that $f_{R,L}\simeq\phi_{R,L}(s_{R,L})$.*

Definition 1.1.18 (The decomposition group) The ring of endomorphisms acting on the global elliptic $s_{R,L}$-semimodules included into weight two cusp forms is generated over $\overline{\mathbb{Z}}_{p_q}$ by the Hecke operators $T_{q_{R,L}}$ for $N\nmid q_N$ and $U_{q_{R,L}}$ for $N\mid q_N$ [M-W], [Lan3]. The coset representatives of U_{q_L} can be chosen to be upper triangular and given by integral matrices $\left(\begin{smallmatrix}1&b_N\\0&q_N\end{smallmatrix}\right)$ while the coset representatives of Uq_R are lower triangular and are given by matrices $\left(\begin{smallmatrix}1&0\\b_N&q_N\end{smallmatrix}\right)$. For general $n=a\cdot d$, we would have respectively the integral matrices $\left(\begin{smallmatrix}a&b_N\\0&d_N\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix}a_N&0\\b_N&d_N\end{smallmatrix}\right)$ of determinant $n\cdot N=ad\cdot N\equiv a_N\cdot d_N$ such that

$q_N\ =*\bmod N\simeq q\cdot N$ and $b_N=*\bmod N$.

$=q\cdot N$ (case $q_N=0\bmod N$)

But, as noticed in definition 1.1.15, we have to consider tensor products of Hecke operators. So, taking into account that the group of matrices $u(b)=\left(\begin{smallmatrix}1&b\\0&1\end{smallmatrix}\right)$ and $u(b)^t=\left(\begin{smallmatrix}1&0\\b&1\end{smallmatrix}\right)$ generate $\mathbb{F}_q$ [Lan3], the following coset representatives

$$k^D_{R\times L}(\overline{\mathbb{Z}}^2_{p_q})=\left[\begin{pmatrix}1 & b_N\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ b_N & 1\end{pmatrix}\right]\begin{pmatrix}1 & 0\\ 0 & q_N^2\end{pmatrix}$$

will be adopted for $U_{q_R}\otimes U_{q_L}$ where $\alpha_{q_N^2}=\left(\begin{smallmatrix}1 & 0\\ 0 & q_N^2\end{smallmatrix}\right)$ is the split Cartan subgroup matrix and where $D_{q_N^2,b_N}=\left(\begin{smallmatrix}1 & b_N\\ 0 & 1\end{smallmatrix}\right)\left(\begin{smallmatrix}1 & 0\\ b_N & 1\end{smallmatrix}\right)$ is the element of the decomposition group associated to $\alpha_{q_N^2}$. Indeed, the semisimplicial form $D_{q_N^2,b_N}$ is unimodular.

Proposition 1.1.19 *The eigenvalues $\lambda_\pm(q_N^2,b_N^2)$ of $k^D_{R\times L}(\overline{\mathbb{Z}}^2_{p_q})$ of $U_{q_R}\otimes U_{q_L}$ are such that:*

1) *$\lambda_+(q_N^2,b_N^2)$ being equivalent to $\lambda_-(q_N^2,b_N^2)$ is an algebraic Hecke character noted $\phi_{R,L}$ in definition 1.1.13.*

2) *they are the coefficients of the elliptic $s_{R,L}$-semimodule $\phi_{R,L}(s_{R,L})$: $\phi(s_{q_{R,L}})_{q,b}=\lambda_\pm(q_N^2,b_N^2)$.*

3) *they allow to define the radius of the torus $T^1_{q_{R,L}}$ by*

$$r(q_N^2,b_N^2)=(\lambda_+(q_N^2,b_N^2)-\lambda_-(q_N^2,b_N^2))/2\ .$$

Proof. The eigenvalues of $k^D_{R,L}(\overline{\mathbb{Z}}^2_{p_q})$ are

$$\lambda_\pm(q_N^2,b_N^2)=\frac{(1+b_N^2+q_N^2)\pm[(1+b_N^2+q_N^2)^2-4q_N^2]^{\frac12}}{2}$$

and verify

$$\begin{cases}\operatorname{trace}(k^D_{R\times L}(\overline{\mathbb{Z}}^2_{p_q}))=1+b_N^2+q_N^2\ ,\\ \det(k^D_{R\times L}(\overline{\mathbb{Z}}^2_{p_q}))=\lambda_+(q_N^2,b_N^2)\cdot\lambda_-(q_N^2,b_N^2)\ .\end{cases}$$

Assume that there exists a global elliptic $\mathbb{A}_{R\times_D L}$-(bi)semimodule

$$\phi_R(s_R)\otimes_D\phi_L(s_L)=\sum_{\mu,m_\mu}\phi(s_R)_{\mu,b}q^{-(p+\mu)}\otimes_D\sum_{\mu,m_\mu}\phi(s_L)_{\mu,b}q^{p+\mu}\ ,$$

where $q^{p+\mu}=e^{2\pi i(p+\mu)x}$, included into a diagonal tensor product of weight two cusp forms $f_R\otimes_D f_L$, then the coefficients $\phi(s_{R,L})_{\mu,b}$ are given by $\phi(s_{R,L})_{\mu,b}=\lambda_\pm(\mu_N^2,b_N^2)$ according to definition 1.1.16.

Notice that a diagonal tensor product, written $\otimes_D$, is a tensor product whose only diagonal terms with respect to a basis $\{e_{\mu,m_\mu}\otimes e_{\mu,m_\mu}\}$ are different from zero.

Let $i_R\otimes_D i_L$ be the (bi)isomorphism:

$$i_R\otimes_D i_L:\phi_R(s_R)\otimes_D\phi_L(s_L)\to\widehat{\phi}_R(s_R)\otimes_D\phi_L(s_L)$$

where

$$\widehat{\phi}_R(s_R)\otimes_D\widehat{\phi}_L(s_L)=\sum_\mu\sum_{m_\mu}r(\mu_N^2,b_N^2)q^{-(p+\mu)}\otimes_D\sum_\mu\sum_{m_\mu}r(\mu_N^2,b_N^2)q^{p+\mu}\ ,$$

which maps the eigenvalues $\lambda_\pm(\mu_N^2,b_N^2)$ to

$$r(\mu_N^2,b_N^2)=(\lambda_+(\mu_N^2,b_N^2)-\lambda_-(\mu_N^2,b_N^2))/2\ .$$

Then, $\widehat{\phi}_R(s_R)\otimes_D\widehat{\phi}_L(s_L)$ decomposes into a sum of tensor products of irreducible (semi)tori $T^1_{\mu,b_{R,L}}$ localized respectively in the upper and in the lower half space, corresponding between themselves by pairs of same ranks n_μ and same values of b_N and such that each pair of (semi)tori be characterized by a radius $r(\mu_N^2,b_N^2)$ and a center at the origin. Notice that the radius $r(\mu_N^2,b_N^2)$ is the radius of ejection $r(x_{\mu_{R,L}})$ considered in proposition 1.1.9. The isomorphism $i_{R,L}$ translates the centers of the tori from $\mathrm{cent}(\mu_N^2,b_N^2)$ $=(\mathrm{trace}(\mathrm{Frob}\,\mu_N^2))/2$ to the origin. The result is that the eigenvalues $\lambda_+(\mu_N^2,b_N^2)$ and $\lambda_-(\mu_N^2,b_N^2)$ are equivalent. ∎

Remarks 1.1.20 **1)** A cuspidal automorphic representation of Eisenstein series has thus been given in terms of global elliptic $s_{R,L}$-semimodules as developed in proposition 1.1.19: this constitutes a first step in the direction of Weil's conjectures suggesting a deep connection between the arithmetic of algebraic varieties defined over finite fields and the topology of algebraic varieties defined over $\mathbb{C}$.

2) It has thus been proved from the developments of 1.1.15 to 1.1.19 that the analytic representation of the right (resp. left) Eisenstein cohomology $H^*(\partial\overline{S}_{K_{t_{R,L}}},\theta^1_{R,L})$ is given by a global elliptic $s_{R,L}$-semimodule noted $\mathrm{ELLIP}_{R,L}(1,\mu,m_\mu)$ where " 1 " refers to the dimension: this constitutes a central challenge in the Langlands program as developed in [Pie1]. The bilinear version of the Langlands program is only really relevant and will be introduced in 1.1.23.

Proposition 1.1.21 *Each left (resp. right) exponential* $U_{v_\mu}=e^{2\pi i(p+\mu)x}$ *(resp.* $U_{\overline{v}_\mu}=e^{-2\pi i(p+\mu)x}$ *) of the* (μ,m_μ)*-term* $\phi(s_L)_{\mu,b}e^{2\pi i(p+\mu)x}$ *(resp.* $\phi(s_R)_{\mu,b}e^{-2\pi i(p+\mu)x}$ *) of* $\mathrm{ELLIP}_L(1,\mu,m_\mu)$ *(resp.* $\mathrm{ELLIP}_R(1,\mu,m_\mu)$ *) constitutes a unitary irreducible representation* $v_\mu\to U_{v_\mu}$ *(resp.* $\overline{v}_\mu\to U_{\overline{v}_\mu}$ *) associated to the left (resp. right) place* v_μ *(resp.* $\overline{v}_\mu$ *) of the algebraic extension semifield* L^+ *(resp.* L^- *) with respect to the coset representative* $k^D_{R\times L}(\overline{\mathbb{Z}}^2_{p_q})$ *of the tensor product* $T_{\mu_R}\otimes T_{\mu_L}$ *of Hecke operators.*

So, each left (resp. right) (μ,m_μ)*-term of* $\mathrm{ELLIP}_L(1,\mu,m_\mu)$ *(resp.* $\mathrm{ELLIP}_R(1,\mu,m_\mu)$ *) forms an irreducible representation of* L^+ *(resp.* L^- *) inflated from the corresponding unitary irreducible representation* U_{v_μ} *(resp.* $U_{\overline{v}_\mu}$ *) by a value* $r(\mu_N^2,b_N^2)$ *which is the radius of the considered (semi)torus* $T^1_{\mu_L}=r(\mu_N^2,b_N^2)\cdot e^{2\pi i(p+\mu)x}$ *(resp.* $T^1_{\mu_R}=r(\mu_N^2,b_N^2)\cdot e^{-2\pi i(p+\mu)x}$ *).*

As the coset representatives $k^D_{R\times L}(\overline{\mathbb{Z}}^2 p_q)=\alpha_{\mu_N^2}\cdot D_{\mu_N^2,b_N}$ of tensor products of Hecke operators have a real meaning, we are constrained to work in the context of an Eisenstein (bi)cohomology as follows:

Proposition 1.1.22 *Let the product of the semigroups* $K_R(\overline{\mathbb{Z}}_{p_q})\times K_L(\overline{\mathbb{Z}}_{p_q})$ *be given by* $K^D_{R\times L}(\overline{\mathbb{Z}}^2_{p_q})$ *. Then, the Eisenstein bicohomology* $H^*_{R\times_{(D)}L}((\partial\overline{S}_R\times_{(D)}\partial\overline{S}_L))_{K^D_{R\times L}(\overline{\mathbb{Z}}^2_{p_q})},\theta^1_R\otimes_{(D)}\theta^1_L)$ *decomposes under the decomposition group* $D_{\mu_N^2,b_N}$ *into products of pairs of one-dimensional eigenspaces:*

$$\begin{aligned}&H^*_{R\times_{(D)}L}((\partial\overline{S}_R\times_{(D)}\partial\overline{S}_L)_{K^D_{R\times L}(\overline{\mathbb{Z}}^2_{p_q})},\theta^1_R\otimes_{(D)}\theta^1_L)\\&\quad\simeq\bigoplus_\mu\bigoplus_{m_\mu}\mathrm{Ind}^{(G_{R\times_{(D)}L}(\mathbb{A}_R\times_{(D)}\mathbb{A}_L))}_{(K^D_{R\times L}(\overline{\mathbb{Z}}^2_{p_q}))}H^*_{R\times_{(D)}L}(S^{M_{R\times_{(D)}L}},H^*(\tilde{u}_{K^D_{R\times L}(\overline{\mathbb{Z}}^2_{p_q})},\theta^1_R\otimes_{(D)}\theta^1_L))\end{aligned}$$

where the sums $\bigoplus\limits_\mu\bigoplus\limits_{m_\mu}$ *run over the cosets of* $G_{t_R}(\mathbb{A}_R)\times_{(D)}G_{t_L}(\mathbb{A}_L)/K^D_{R\times L}(\overline{\mathbb{Z}}^2_{p_q})$ *having multiplicities* $m^{(\mu)}=\sup(m_\mu)$ *.*

Proof. This proposition reduced to the right or left case is clearly equivalent to proposition 1.1.14. The coefficient system given by the semisimple Lie algebra (bi)- cohomology $H^*(\tilde u_{K^D_{R\times L}(\overline{\mathbb{Z}}^2_{p_q})},\theta^1_R\otimes_{(D)}\theta^1_L)$ decomposes into sum of products of pairs of one-dimensional sections $(T^1_{\mu,b_R}\times T^1_{\mu,b_L})$ of $\theta^1_R\otimes_{(D)}\theta^1_L$ characterized by the (bi)weights $\lambda_+(\mu^2_N,b^2_N)\times\lambda_-(\mu^2_N,b^2_N)$. ■

1.1.23 Langlands bilinear global program

According to the developments from 1.1.15 to 1.1.20, the $GL_2(\mathbb{A}_{L^T_{\overline{v}}}\times\mathbb{A}_{L^T_v})$-bisemimodule $M^T_R\otimes M^T_L$ has an analytic development given by the global elliptic $s_R\otimes_D s_L$-bisemimodule $\phi_R(s_R)\otimes_D\phi_L(s_L)$ which is a product, right by left, of truncated Fourier series.

As the $\mathrm{GL}_2(\mathbb{A}_{L_{\overline{v}}}\times\mathbb{A}_{L_v})$-bisemimodule $M_R\otimes M_L$ constitutes an irreducible representation $\mathrm{Irr}^{(1)}_W(W^{ab}_{L^-}\times W^{ab}_{L^+})$ of the bilinear global Weil group $W^{ab}_{L^-}\times W^{ab}_{L^+}$ [Pie9] and as the global elliptic bisemimodule $\phi_R(s_R)\otimes_D\phi_L(s_L)$, also noted $\mathrm{ELLIP}_R(1,\cdot,\cdot)\otimes\mathrm{ELLIP}_L(1,\cdot,\cdot)$, constitutes an irreducible cuspidal representation $\mathrm{Irr\,ELLIP}(GL_2(\mathbb{A}_{L^T_{\overline{v}}}\times\mathbb{A}_{L^T_v}))$ of $GL_2(\mathbb{A}_{L_{\overline{v}}}\times\mathbb{A}_{L_v})$, we have on the Shimura bisemivariety $\partial\overline{S}_{K^D_{R\times L}}$, the Langlands irreducible global correspondence, i.e. the bijection:

$$\mathrm{Irr}^{(1)}_W(W^{ab}_{L^-}\times W^{ab}_{L^+})\ \longrightarrow\ \mathrm{Irr\,ELLIP}(GL_2(\mathbb{A}_{L^T_{\overline{v}}}\times\mathbb{A}_{L^T_v}))$$

according to [Pie9].

Let us recall that if we fix:

$$\begin{aligned}\mathrm{Gal}(L^-/K)&=\bigoplus_\mu\bigoplus_{m_\mu}(\mathrm{Gal}\,L^-_\mu/K)\ ,\\ \mathrm{Gal}(L^+/K)&=\bigoplus_\mu\bigoplus_{m_\mu}(\mathrm{Gal}\,L^+_\mu/K)\ ,\end{aligned}$$

the right (resp. left) global Weil group $W^{ab}_{L^-}$ (resp. $W^{ab}_{L^+}$) is the Galois subgroup of $\mathrm{Gal}(L^-/K)$ (resp. $\mathrm{Gal}(L^+/K)$) of the extensions L^-_μ (resp. L^+_μ) characterized by degrees:

$$\begin{aligned}n_{\mu_R}&=[L^-_\mu:K]=0\bmod N=(p+\mu)\ N\ ,\\ (\text{resp.}\quad n_{\mu_L}&=[L^+_\mu:K]=0\bmod N=(p+\mu)\ N\ )\ .\end{aligned}$$

Definition 1.1.24 The notion of quantum on the time semisheaf of rings $\theta^1_{R,L}(t)$ can be introduced as follows: let $s_{1_{R,L}}\subset\cdots\subset s_{q_{R,L}}$ be the set of sections of $\theta^1_{R,L}(t)$ and $n_1<\cdots<n_\mu<\cdots<n_q$ be the corresponding set of ranks. According to the preceding developments, it corresponds to the section $s_{\mu_{R,L}}$ a set of equivalent sections $\{s_{\mu,1_{R,L}},\cdots,s_{\mu,b_{R,L}}\}$ relative to the decomposition group $D_{\mu^2,b}$ where all the $s_{\mu,b_{R,L}}$ have the same rank n_μ .

A section $s_{\mu,b_{R,L}}$ has a rank $n_{\mu_{R,L}}=(p+\mu)\cdot N$, following 1.1.6, where N is the order of the inertia subgroup $I_{L_{\overline{v}_\mu}}$ (resp. $I_{L_{v_\mu}}$) having as representation space the $T^t_2(L_{\overline{v}^1_\mu})$-subsemimodule $M^I_{\overline{v}_\mu}$ (resp. $T_2(L_{v^1_\mu})$-subsemimodule $M^I_{v_\mu}$) which was interpreted in 1.1.6 as a right (resp. left) quantum. Thus, the section $s_{\mu,b_{R,L}}$ is composed of $(p+\mu)$ right (resp. left) time quanta, noted $\widetilde{M}^I_\mu(t)_{R,L}$, or $\widetilde{M}^T_{v^{\mu'}_{\mu,m_\mu}}=\mathrm{An\,Repsp}(T^{(t)}_2(L^T_{v^{\mu'}_{\mu,m_\mu}}))$ where $\mathrm{An\,Repsp}(T^{(t)}_2(L^T_{v^{\mu'}_{\mu,m_\mu}}))$, denoting the analytic representation space of the algebraic subgroup $T^{(t)}_2(L^T_{v^{\mu'}_{\mu,m_\mu}})$ over the irreducible subcompletion $L^T_{v^{\mu'}_{\mu,m_\mu}}$, is a "class of germ" of continuous (differentiable) function over a big point centered on $T^{(t)}_2(L^T_{v^{\mu'}_{\mu,m_\mu}})$.

Notice that the writing " $s_{\mu_{R,L}}$ " will mean, in the following developments, either a basic section $s_{\mu_{R,L}}$ alone, i.e. for a value of $b=0$, or a subset of equivalent sections $\{s_{\mu_{1_{R,L}}},\cdots,s_{\mu_{b_{R,L}}}\}$ corresponding to all the ideals of the decomposition group $D_{\mu^2,b}$, the distinction between the two cases being in general evident.

1.2 Generation of $4D$-semisheaves of rings by Eisenstein homology and $(\gamma_{t\to r}\circ E)$ morphism

Definitions 1.2.1 (1. Galois antiautomorphism) From the right (resp. left) Galois automorphic group $\Gamma_{\mu R,L}=\operatorname{Aut}_K L^{\mp}_{\mu}$, it is possible to define a Galois antiautomorphic group $\Gamma^*_{\mu R,L}=\widetilde{\operatorname{Aut}}_K L^{\mp}_{\mu}$ acting transitively on the left on the set of right (resp. left) prime ideals $b_{\mu R,L}$ of the right (resp. left) specialization semiring $B_{\mu R,L}$. We thus have a descending chain of right (resp. left) specialization ideals:

$$b_{\mu_{n\mu R,L}}\supset\cdots\supset b_{\mu_{(n_\mu-\rho_\mu)R,L}}\ ,\quad \rho<n\ ,$$

where $(n_\mu-\rho_\mu)$ is a decreasing rank.

(2. Reduced algebraic semigroups) From the right (resp. left) boundary of the compactified semispace $\partial\,\overline{S}_{K_{tR,L}}$ (see definition 1.1.7), it is possible to introduce the reduced compactified semispace:

$$\partial\,\overline{S}_{K^*_{tR,L}}=P^*_{tR,L}\setminus G^*_{tR,L}/K^*_{tR,L}\ ,$$

where

a) $G^*_{tR,L}$ is a reduced algebraic semigroup, i.e. an algebraic semigroup submitted to Galois antiautomorphisms, and having the following decomposition:

$$G^*_{t_L}(\mathbb{A}^*_L)\equiv T_2(\mathbb{A}^*_L)=D_2(\mathbb{A}^*_L)\times UT_2^{-1}(\mathbb{A}^*_L)$$

where

- $\mathbb{A}^*_L$ is a reduced adele semiring given by $\mathbb{A}^*_L\equiv\mathbb{A}^*_{L^T_v}=\prod\limits_{\nu}L^T_{v_\nu}\prod\limits_{m_\nu}L^T_{v_{\nu,m_\nu}}$, $\nu\le\mu$, and coming from $\mathbb{A}_L=\prod\limits_{\mu}L^T_{v_\mu}\prod\limits_{m_\mu}L^T_{v_{\mu,m_\mu}}$ (see 1.1.4 and 1.1.10);
- $UT_2^{-1}(\mathbb{A}^*_L)$ is the inverse of $UT_2(\mathbb{A}_L)$.

b) $P^*_{tR,L}$ is a reduced parabolic semisubgroup;

c) $K^*_{tR,L}$ is a reduced arithmetic semisubgroup of $G^*_{tR,L}$.

Proposition 1.2.2 *The right (resp. left) Eisenstein homology, defined from the action of a right (resp. left) Galois antiautomorphic group, is associated to the generation of a right (resp. left) reduced semisheaf of rings $\theta^{*1}_{R,L}$ and decomposes into [Pie3]:*

$$\begin{aligned}
&H_*(\partial\overline{S}_{K^*_{R,L}},\theta^{*1}_{R,L})\\
&\quad=\quad H_*(P^*_{tR,L}\setminus G^*_{tR,L}(\mathbb{A}^*_{R,L})/K^*_{R,L}(\overline{\mathbb{Z}}_{p_q}),\theta^{*1}_{R,L})\\
&\quad=\quad \bigoplus_{\xi_{R,L}\in\Xi_{K^*_{R,L}}} H_*(S^{M^*_{R,L}}_{K^{*M_{R,L}}(\xi_{R,L})},H_*(\tilde u^*_{R,L},\theta^{*1}_{R,L}))\ .
\end{aligned}$$

Proof. This proposition is the homological version of proposition 1.1.12. ■

Definition 1.2.3 (Reduced algebraic Hecke characters) If $\mathbb{A}^*_{R,L}$ denotes a reduced adele semiring, let

$$\lambda^*_{R,L} = (\cdots, \lambda^*_{\nu_{R,L}}, \cdots)_{\mathbb{A}_{R,L} \to \mathbb{A}^*_{R,L}}$$

be a sequence of decreasing weights.

Then, $\lambda^*_{R,L} = \{\lambda^*_{1R,L}, \cdots, \lambda^*_{\nu R,L}, \cdots, \lambda^*_{qR,L}\}$ is the sequential set of decreasing weights in $X(\theta^{*1}_{R,L})$ referring to the q set of right (resp. left) sections of the right (resp. left) reduced semisheaf of rings $\theta^{*1}_{R,L}$.

Let $\omega^*_{R,L} = \{\cdots, \omega^*_{\mu R,L}, \cdots\}$ be the set of right (resp. left) inverse actions of the Weyl groups on $\lambda^*_{R,L}$. Then, $\phi^*_{R,L} = \omega^*_{R,L} \cdot \lambda^*_{R,L}$ will denote the set of right (resp. left) reduced algebraic Hecke characters on $\theta^{*1}_{R,L}$.

Definitions 1.2.4 (1. Every **smooth endomorphism)** $E[G_{\mu R,L}]$ of the algebraic semigroup $G_{\mu R,L}$, representing the Galois subgroup $\Gamma_{\mu R,L} = \mathrm{Aut}_K L^{\mp}_{\mu}$, can decompose into the direct sum of the two nonconnected algebraic semigroups [Pie3]:

a) the reduced algebraic semigroup $G^*_{\mu R,L}$, submitted to the Galois antiautomorphic subgroup $\Gamma^*_{\mu R,L} = \widetilde{\mathrm{Aut}}_K L^{\mp}_{\mu}$;

b) the complementary algebraic semigroup $G^I_{\mu R,L}$, submitted to the complementary Galois automorphic subgroup $\Gamma^I_{\mu R,L} = \mathrm{Aut}^{(I)}_K L^{\mp}_{\mu}$, such that $G^I_{\mu R,L}$ be a semisubgroup of $G_{\mu R,L}$.

We then have

$$G_{\mu R,L} = G^*_{\mu R,L} \oplus G^I_{\mu R,L} \; .$$

Recall that the semisubgroup $K_{\mu R,L} \subset G_{\mu R,L}$ can be defined following [Har1]:

$$K_{\mu R,L} = SO(m, L^{T\mp}_{\mu}) \cdot Z^0_{\mu R,L}(L^{T\mp}_{\mu}) \; .$$

The nonconnectivity of $G^*_{\mu R,L}$ and $G^I_{\mu R,L}$ is a necessary condition to avoid triviality if the groups $SO(m, L^{T\mp}_{\mu})^* \in K^*_{\mu R,L}$ and $SO(m, L^{T\mp}_{\mu})^I \in K^I_{\mu R,L}$ had the same Witt index and the same order " m ".

(2. The **complementary Galois automorphic group)** $\Gamma^I_{\mu R,L} = \mathrm{Aut}^I_K L^{\mp}_{\mu}$ can be defined by its transitive right action on the set of prime ideals $b^I_{\mu R,L}$ of the complementary specialization semiring $B^I_{\mu R,L}$ leading to an ascending chain of complementary specialization ideals $b^I_{\mu_{1R,L}} \subset \cdots \subset b^I_{\mu_{\rho^I_\mu} R,L}$ such that the maximal rank ρ^I_μ be equal to the integer n_μ when the decreasing rank is $(n_\mu - \rho_\mu)$ (see definition 1.2.1).

(3. Complementary Eisenstein cohomology) From the compactified complementary semispace $\overline{S}_{K^I_{R,L}}$, we define its boundary by

$$\partial\, \overline{S}_{K^I_{R,L}} = P^I_{R,L} \setminus G^I_{R,L} / K^I_{R,L} \; .$$

The right (resp. left) complementary Eisenstein cohomology can then be introduced:

$$H^*_I(\partial \overline{S}_{K^I_{R,L}}, \theta^1_{I_{R,L}}) = H^*_I(P^I_{tR,L} \setminus G^I_{tR,L}(\mathbb{A}^I_{R,L}) / K^I_{R,L}(\overline{\mathbb{Z}}_{p_q}), \theta^1_{IR,L}) \; .$$

It is associated to the generation of a right (resp. left) complementary semisheaf of rings $\theta^1_{I_{R,L}}$.

Definition 1.2.5 Let $\gamma_{t\to r}$ be the **emergent morphism**, introduced in [Pie3] and mapping the complementary semisheaf of rings $\theta^1_{I_{R,L}}(t)$ from the complementary semispace $\partial\,\overline{S}_{K^I_{R,L}}(t)$ into its orthogonal complementary semispace $\partial\,\overline{S}^{\perp}_{K^I_{R,L}}(r)$ where $r=\{x,y,z\}$ is the triple of spatial variables:

$$\begin{aligned}\gamma_{t\to r}:\ \partial\,\overline{S}_{K^I_{R,L}}(t) &\ \longrightarrow\ \partial\,\overline{S}^{\perp}_{K^I_{R,L}}(r)\ ,\\ \theta^1_{I_{R,L}}(t) &\ \longrightarrow\ \theta^3_{I_{R,L}}(r)\ .\end{aligned}$$

Proposition 1.2.6 *Let $\theta^{*1}_{R,L}(t)$ be the reduced semisheaf of rings generated under the smooth endomorphism E by the right (resp. left) Eisenstein homology. Then, the morphism $(\gamma_{t\to r}\circ E)$ transforms the semisheaf of rings $\theta^1_{R,L}(t)$ into:*

$$\begin{aligned}\gamma_{t\to r}\circ E:\ \partial\,\overline{S}_{K_{R,L}}(t) &\ \longrightarrow\ \partial\,\overline{S}_{K^*_{R,L}}(t)\oplus\partial\,\overline{S}^{\perp}_{K^I_{R,L}}(r)\ ,\\ \theta^1_{R,L}(t) &\ \longrightarrow\ \theta^{*1}_{R,L}(t)\oplus\theta^3_{I_{R,L}}(r)\ ,\end{aligned}$$

such that each section $s^*_{\mu R,L}\oplus s_{I_{\mu R,L}}\in\theta^{*1}_{R,L}(t)\oplus\theta^3_{I_{R,L}}(r)$ be $T^{*1}_{\mu}(t)_{R,L}\oplus T^1_{I_\mu}(r)_{R,L}$, called a right (resp. left) elementon of space-time and noted $T^{1-1}_{\mu}(t,r)_{R,L}$, where $T^{*1}_{\mu}(t)_{R,L}$ is a set of $1D$-tori and $T^1_{I_\mu}(r)_{R,L}$ is also a set of $1D$-tori.

Proof.

a) The complementary semisheaf of rings $\theta^3_{I_{R,L}}(r)$ is three-dimensional because the groups $SO(2p,L^{T\mp})\in K^*_{R,L}$ and $SO(2p+1,L^{T\mp})\in K^{I\perp}_{R,L}$ must have the same Witt index $p=1$ in order that the endomorphism E be smooth [Pie3] but their order " m " may be different: consequently, $m=2$ for $K^*_{R,L}$ and $m'=3$ for $K^{I\perp}_{R,L}$ [Bum].

b) The fact that the section $s_{I_{\mu R,L}}$ is a set of $1D$-tori results from:

- the morphism $\gamma_{t\to r}\circ E$ where $\gamma_{t\to r}$ corresponds to the projective map :

  $$GL_2(\mathbb{A}_R\times\mathbb{A}_L)\ \longrightarrow\ P\,GL_2(\mathbb{A}_R\times\mathbb{A}_L)\hookrightarrow GL_3(\mathbb{A}_R\times\mathbb{A}_L)$$

  as developed by S. Gelbart [Gel2];

- the decomposition (or degeneration) of the representation space Repsp$(GL_3(\mathbb{A}_R\times\mathbb{A}_L))$ into one-dimensional components. ■

Definition 1.2.7 (The quantum of space) Assume that the section $T^1_{I_\mu}(r)_{R,L}\in\theta^3_{I_{R,L}}(r)$, generated under the $(\gamma_{t\to r}\circ E)$-morphism from the $1D$-section $T^1_\mu(t)$ composed in fact of a set of m_μ equivalent sections $\{T^1_{\mu,b}\}$ under the decomposition group $D_{\mu^2,b}$, is partitioned into m_μ corresponding $1D$-fibers, having each one a rank $\rho_\mu=(p+\mu)\cdot N$. Then, each $1D$ equivalent section $T^1_{I_{\mu,m_\mu}}(r)_{R,L}$ has $\mu_p=(p+\mu)$ spatial quanta, noted $\widetilde{M}^I_\mu(r)_{R,L}$, which are functions on subsemimodules of rank N . And the section $T^1_{I_\mu}(r)_{R,L}=\{T^1_{I_{\mu_1}},\cdots,T^1_{I_{\mu,m_\mu}}\}$ counts $m_\mu(p+\mu)$ space quanta.

Corollary 1.2.8 *There exists an inverse morphism $(\gamma_{r\to t}\circ E')$ transforming gradually and sequentially the $3D$-complementary semisheaf of rings $\theta^3_{I_{R,L}}(r)$ into the $1D$-semisheaf of rings $\theta^1_{R,L}(t)$.*

Proof. Let n and $(n-\rho)$ be the set of q graded ranks referring to the q sections respectively of $\theta^1_{R,L}(t)$ and $\theta^{*1}_{R,L}(t)$ according to corollary 1.1.10.

(Note that the proof is valid for the right and left cases but the indices R, L will be dropped for facility).

Then the morphism $(\gamma_{r\to t}\circ E')$ is such that

a) $E' : \theta^3_I(r)_\rho \to \theta^{*3}_I(r)_{\rho-\rho'} \oplus \theta^3_{I(I)}(r)_{\rho'}$ where $\theta^3_{I(I)}(r)_{\rho'}$ is the complementary semisheaf of $\theta^{*3}_I(r)_{\rho-\rho'}$ obtained under the smooth endomorphism E' .

b) $\gamma_{r\to t} : \theta^3_{I(I)}(r)_{\rho'} \to \theta^1(t)_{\rho'}$ where $\gamma_{r\to t}$ maps $\theta^3_{I(I)}(r)_{\rho'}$, ideal by ideal, into its $1D$-time orthogonal complementary space giving rise to $\theta^1(t)_{\rho'}$.

c) $(\gamma_{r\to t}\circ E') : \theta^{*1}(t)_{n-\rho} \oplus \theta^3_I(t)_\rho \to \theta^{*1}(t)_{n-(\rho-\rho')} + \theta^{*3}_I(r)_{(\rho-\rho')}$.

If $\rho'=\rho$, then under $(\gamma_{r\to t}\circ E')$, $\theta^3_I(r)_\rho$ has been totally transformed into $\theta^1(t)_{\rho'}$. ■

Definition 1.2.9 (Algebraic Hecke parameters) Let $\phi^*_{t;(n-\rho)_{R,L}}$ bet the set of algebraic Hecke characters referring to the generation of the reduced semisheaf of rings $\theta^{*1}_{R,L}(t)$ by Eisenstein homology and let $\phi_{r;\rho_{R,L}}$ bet the set of algebraic Hecke characters referring to the generation of the complementary semisheaf of rings $\theta^3_{I_{R,L}}(r)$ by Eisenstein cohomology.

We then have the following equality between these two sets of algebraic Hecke characters:

$$\phi^*_{t;(n-\rho)_{R,L}} = c_{t\to r}(\rho)_{R,L}\cdot\phi_{r;\rho_{R,L}}$$

where $c_{t\to r}(\rho)_{R,L} = \{c_1(\rho_1)_{R,L},\cdots,c_q(\rho_q)_{R,L}\}$ is a set of parameters referring to the q sections of the semisheaf of rings $\theta^3_{I_{R,L}}(r)$ and depending on the set of sequential ranks " ρ ".

$c_{t\to r}(\rho)_{R,L}$ can be considered as an algebraic measure giving the ratio of the generation of the complementary semisheaf of rings $\theta^3_{I_{R,L}}(r)$ with respect to the reduced semisheaf of rings $\theta^{*1}_{R,L}(t)$.

Consequently, $c_{t\to r}(\rho)$ is the most closed to the unity when $(n-\rho)=\rho$.

Proposition 1.2.10 *Each right and left 4D-elementon of space-time* $(T^{*1}_\mu(t)_{R,L}\oplus T^1_{I_\mu}(r)_{R,L}) \in \theta^{*1}_{R,L}(t)\oplus\theta^3_{I_{R,L}}(r)$, $1\le\mu\le q$, *is composed of elementary subtori* $\tau^{1-1}_\mu(t,r)_{R,L}$, *characterized by a rank* $2N$, *which are sums of a time and of a space quantum.*

Proof. (The indices R, L will be dropped in this proof). Let $(n_\mu-\rho_\mu)$ be the rank of the section $T^{*1}_\mu(t)$ and let ρ_μ be the rank of the section $T^1_{I_\mu}(r)$, taking into account that the complementary section $T^1_{I_\mu}(r)$ is generated from $T^1_\mu(t)$ by the morphism $(\gamma_{t\to r}\circ E)$.

Considering the algebraic generation of $T^1_\mu(t)$ under the action of the Galois automorphic group $\Gamma_\mu = \mathrm{Aut}_K L_\mu$ and envisaging the $(\gamma_{t\to r}\circ E)$ morphism, we then have that the elementary time prime ideal $\tau^{*1}_\mu(t)\in T^{*1}_\mu(t)$ has a rank N and the elementary space prime ideal $\tau^1_\mu(r)\in T^1_{I_\mu}(r)$ is characterized by a rank N . ■

Corollary 1.2.11 *Consider the morphism:*

$$\gamma_{t\to r}\circ E : T^{1-1}_\mu(t,r)_{R,L} \to T^1_\mu(r)_{R,L}\ , \qquad \forall\ \mu\ , \quad 1\le\mu\le q\ ,$$

such that the reduced section $T^{*1}_\mu(t)_{R,L}$ *of the semisheaf of rings* $\theta^{*1}_{R,L}(t)$ *be completely transformed into the complementary space section* $T^1_\mu(r)_{R,L}$. *Then, every elementary subtorus* $\tau^1_\mu(r)_{R,L}\in T^1_\mu(r)_{R,L}$ *is also characterized by a rank* N .

Proof. This is obvious by taking into account the proposition 1.2.10. ∎

1.2.12 Space-time structure of semiparticles

1) The mathematical and physical reasons given in the introduction and in the following developments lead us to admit that elementary particles must be composed of two symmetric objects, called a right and a left semiparticle. The basic "algebraic" space-time structure of a right and a left semiparticle (or, more exactly, of a right and a left semilepton, or semiquark, as it will be developed in section 4) will be assumed to be respectively a right and a left sequential semisheaf of rings $(\theta^{*1}_{R,L}(t)\oplus\theta^{3}_{I_{R,L}}(r))$ of which $\theta^{3}_{I_{R,L}}(r)$ can be regarded as the algebraic representation of a space physical wave packet.

The right and left $1D$-semisheaves or rings $\theta^{*1}_{R}(t)$ and $\theta^{*1}_{L}(t)$ must be viewed as the basic time structure of the right and left semiparticles while the right and left $3D$-semisheaves of rings $\theta^{3}_{I_R}(r)$ and $\theta^{3}_{I_L}(r)$ must be regarded as the basic space structure of the respective semiparticles.

2) Indeed, the fact of endowing elementary (semi)particles with an internal space-time structure from which the "mass" shell could be generated results from an attempt of the author [Pie1] to bridge the gap between general relativity and quantum field theory. The problem is that general relativity is a "classical" theory describing the mutual interaction between the geometry of space-time and the matter without explaining how matter could be generated. Now, quantum field theory asserts precisely that matter must be created from the vacuum to which the cosmological constant of the general relativity equations could correspond if it was associated to it an expanding space-time which could then constitute the fundamental structure of the vacuum of QFT. On the basis of these considerations, I have developed, in an unpublished preprint [Pie1], equations in differential geometry rather close by the equations of general relativity but referring to the quantum structure of bisemiparticles such that their most internal structures, which are space-time structures, be the fundamental structures of their own vacua from which their matter shells could be generated due to the fluctuations of these internal vacua. So, the vacuum of QFT becomes peopled to massless (bisemi)particles being potentially able to generate their mass shells due to the fluctuations of these (bisemi)particle internal vacua.

3) To the internal "space" structure of an elementary semiparticle then corresponds its linear momentum $\vec{p}$ on its "mass" shell; and, to the internal "time" structure of a semiparticle would corresponds its rest mass m_0 . The fact of considering the internal time of a semiparticle as corresponding to a topological structure can be justified by the annihilation of a pair of leptons into (pair(s)) of photons and by 2) of 1.2.10.

4) The internal space structure of a semiparticle is thus given by the semisheaf of rings $\theta^{3}_{I_{R,L}}(r)$ on a $T_3(\mathbb{A}_{\,L})$ (resp. $T_3(\mathbb{A}_{\,R})$-semimodule $M^{(T)}_{L}$ (resp. $M^{(T)}_{R}$) restricted to the upper (resp. lower) half space. Indeed, following the Langlands program briefly developed in 1.1.22, $\theta^{3}_{I_{R,L}}(r)$ has an analytic representation given by the global elliptic semimodule $\mathrm{ELLIP}_{R,L}(1,q,b)$ which corresponds to an eigenfunction of the spectral representation of an operator $T_{R,L}$ (see chapter 3) on the space structure. On the other hand, each term $\phi(s_{\mu_{R,L}})_{\mu,b}e^{\pm 2\pi i(p+\mu)x}\in\mathrm{ELLIP}_{R,L}(1,q,b)$ will be interpreted as the "space" structure of a semiphoton at $(p+\mu)$ quanta, giving then a (semi)photonic spatial structure to the semiparticle.

5) If we consider the space structure of a right (resp. left) semiparticle as given by the three-dimensional semisheaf of rings $\theta^{3}_{I_{R,L}}(r)$, then this semiparticle will be interpreted as having a wave (packet) aspect.

But, we have seen in proposition 1.2.6 that we can consider the projective map: $P_{G_{3\to 2}}: T_3(\mathbb{A}_R) \to T_2(\mathbb{A}_R)$ (resp. $T_3(\mathbb{A}_L) \to T_2(\mathbb{A}_L)$) to which corresponds the projective map: $P_{\theta_{3\to 2}}: \theta^3_{I_{R,L}}(r) \to \theta^2_{I_{R,L}}(r)$ mapping the three-dimensional semisheaf of rings $\theta^3_{I_{R,L}}(r)$ into its two-dimensional analogue $\theta^2_{I_{R,L}}(r)$, giving then to the space structure of a right (resp. left) semiparticle a "particle" aspect.

We can then formulate the first axiom referring to the generation of the "wave" space-time structure of elementary right and left semiparticles.

Axiom I 1.2.13 *The basic space-time structure of elementary right and left semiparticles is of algebraic nature.*

Proof. Indeed, the space-time structure of elementary right and left semiparticles is assumed to be given by $4D$-space-time right and left sequential semisheaves of rings $(\theta^{*1}_{R,L}(t) \oplus \theta^3_{I_{R,L}}(r))$ whose q sets of sections are " $4D$ "-elementons $T^{1-1}_\mu(t,r)_{R,L}$ generated from $1D$-symmetric splitting semifield(s) by Eisenstein cohomology and homology and by the morphism $(\gamma_{t\to r} \circ E)$. ∎

1.3 Algebraic representation of bisemiparticles by bilinear Hilbert schemes

Definition 1.3.1 (Tensor product of semisheaves of rings) The right and left semisheaves of rings $(\theta^{*1}_{R,L}(t) \oplus \theta^3_{I_{R,L}}(r))$ are defined respectively on a $G_R(\mathbb{A}_R)$-right semimodule, noted M^{ST}_R , and on a $G_L(\mathbb{A}_L)$-left semimodule, noted M^{ST}_L . The $G_R(\mathbb{A}_R)$-right semimodule M^{ST}_R and the $G_L(\mathbb{A}_L)$-left semimodule M^{ST}_L represent the basic internal space-time structures of the right and left semiparticles (essentially leptons) which act conjointly in order to form a bisemiparticle localized inside a $4D$-openball centered on the emergence point. As the $G_{R,L}(\mathbb{A}_{R,L})$-right (resp. left) semimodule $M^{ST}_{R,L}$ is also a unitary right (resp. left) $\mathbb{A}_{R,L}$-semimodule, it is an $\mathbb{A}_{R,L}$-right (resp. left) semialgebra $M^{ST}_{R,L}$. By construction, M^{ST}_R is the opposite semialgebra of M^{ST}_L . So, the tensor product $M^{ST}_R \otimes_{\mathbb{A}_R \times \mathbb{A}_L} M^{ST}_L$ will be the enveloping semialgebra of M^{ST}_L and will be assumed to constitute the space-time structure of a bisemiparticle. $(M^{ST}_R \otimes M^{ST}_L)$ will be written for $(M^{ST}_R \otimes_{\mathbb{A}_R \times \mathbb{A}_L} M^{ST}_L)$.

The space-time structure of a bisemiparticle will thus be given by the tensor product $(M^{ST}_R \otimes M^{ST}_L)$ of M^{ST}_R and M^{ST}_L such that the right semimodule M^{ST}_R be flat on the left semimodule M^{ST}_L , i.e. that for every left semimodule M'^{ST} and for every injective homomorphism $\nu : M'^{ST}_L \to M^{ST}_L$, the homomorphism $\mathbf{1}_{M^{ST}_R} \otimes \nu : M^{ST}_R \otimes M'^{ST}_L \to M^{ST}_R \otimes M^{ST}_L$ is injective [Bou1].

If the right and left semisheaves of rings $\widetilde{M}^{ST}_R$ and $\widetilde{M}^{ST}_L$ are defined respectively on the right and left semispaces M^{ST}_R and M^{ST}_L , we then get a right and a left ringed semispace $(M^{ST}_R, \widetilde{M}^{ST}_R)$ and $(M^{ST}_L, \widetilde{M}^{ST}_L)$.

Similarly, we can define the tensor product between the right and left ringed semispaces:

$$\otimes : \{(M^{ST}_R, \widetilde{M}^{ST}_R), (M^{ST}_L, \widetilde{M}^{ST}_L)\} \to (M^{ST}_R \times M^{ST}_L, \widetilde{M}^{ST}_R \otimes \widetilde{M}^{ST}_L)$$

where $M^{ST}_R \otimes M^{ST}_L$ represents the space-time structure of a bisemiparticle.

As M^{ST}_R and M^{ST}_L are semisimple, then $(M^{ST}_R \otimes M^{ST}_L)$ is also semisimple according to C. Chevalley [Che2] and J.P. Serre [Ser5], [Ser6].

The tensor product $(M^{ST}_R \otimes M^{ST}_L)$, called a bisemimodule, is characterized by a 10-dimensional noneuclidean geometry, reflecting its degree of compactness and of instability. Consequently, a blowing-up morphism will be considered in the following proposition.

Proposition 1.3.2 *There exists a blowing-up isomorphism*

$$S_L : M_R^{ST} \otimes M_L^{ST} \to (M_R^{ST} \otimes_D M_L^{ST}) \oplus (M_R^S \otimes_{magn} M_L^S) \oplus (M_R^{S-(T)} \otimes_{elec} M_L^{(S)-T})$$

transforming the bisemimodule $(M_R^{ST} \otimes M_L^{ST})$ *of dimension* 10 *(lepton case) into a set of disconnected bisemimodules which are:*

a) *the diagonal bisemimodule* $(M_R^{ST} \otimes_D M_L^{ST})$ *of dimension* 4 *characterized by a diagonal orthogonal* $4D$*-basis* $\{e^\alpha \otimes f_\alpha\}_{\alpha=0}^3$, $\forall\ e^\alpha \in M_R^{ST}$ *and* $f_\alpha \in M_L^{ST}$ *;*

b) *the magnetic bisemimodule* $(M_R^S \otimes_{magn} M_L^S)$ *characterized by a* $3D$*-nonorthogonal basis* $(e^\alpha \otimes f_\beta)_{\alpha\neq\beta=1}^3$ *, where* $M_{R,L}^S = \theta_{I_{R,L}}^3(r)$ *;*

c) *the electric bisemimodule* $(M_R^S \otimes_{elec} M_L^T)$ *or* $(M_R^T \otimes_{elec} M_L^S)$ *characterized respectively by a* $3D$*-nonorthogonal basis* $(e^\alpha \otimes f_0)_{\alpha=1}^3$ *or* $(e^0 \otimes f_\alpha)_{\alpha=1}^3$ *where* $M_{R,L}^T = \theta_{R,L}^{*1}(t)$ *.*

Proof. The blowing-up isomorphism can be understood algebraically by considering that right and left quanta are taken away respectively from the right and left semimodules M_R^{ST} and M_L^{ST} by the smooth endomorphism E , recalled [Pie3] in definition 1.2.4, in such a way that the complete bisemimodule $(M_R^{ST} \otimes M_L^{ST})$ be transformed into the diagonal bisemimodule $(M_R^{ST} \otimes_D M_L^{ST})$. Consequently, the disconnected right and left quanta will generate two off-diagonal bisemimodules having a magnetic and an electrical metric to keep a trace of the off-diagonal metric of $(M_R^{ST} \otimes M_L^{ST})$ (see also 4.3.4 and 4.3.5). The magnetic metric is given by $g_\beta^\alpha = (e^\alpha, f_\beta)_{\alpha\neq\beta=1}^3$ and the electric metric is given by $g_0^\alpha = (e^\alpha, f_0)_{\alpha=1}^3$ or $g_\alpha^0 = (e^0, f_\alpha)_{\alpha=1}^3$ where $(\cdot,\cdot)$ is a scalar product. ∎

Definition 1.3.3 (Diagonal tensor product) Let the right ringed semispace $(M_R^{ST}, \widetilde{M}_R^{ST})$ of the right semiparticle define locally the affine right semischeme S_R^{ST} and the left ringed semispace $(M_L^{ST}, \widetilde{M}_L^{ST})$ of the left semiparticle define locally the affine left semischeme S_L^{ST} [Hart].

Let $(S_R^{ST} \otimes_D S_L^{ST})$ be the diagonal tensor product between the right and left semischemes S_R^{ST} and S_L^{ST} characterized by a diagonal metric.

Consider the projective morphisms p_L and p_R :

$$\begin{aligned} p_L &: S_R^{ST} \otimes_D S_L^{ST} \to S_{R(P)/L}^{ST} , \\ p_R &: S_R^{ST} \otimes_D S_L^{ST} \to S_{L(P)/R}^{ST} , \end{aligned}$$

such that:

a) the right semischeme S_R^{ST} be projected under p_L on the left semischeme S_L^{ST} giving rise to the bisemischeme $S_{R(P)/L}^{ST}$;

b) the left semischeme S_L^{ST} be projected under p_R on the right semischeme S_R^{ST} giving rise to the bisemischeme $S_{L(P)/R}^{ST}$.

Proposition 1.3.4 *The diagonal tensor product* $S_{R(P)/L}^{ST}$ *of the right and left semischemes* S_R^{ST} *and* S_L^{ST} *such that* S_R^{ST} *is projected on* S_L^{ST} *is a covariant functor of* S_L^{ST} *representable by the bilinear Hilbert scheme* $\mathrm{Hilb}_{S_{R(P)}^{ST}/S_L^{ST}}$ *where* $S_{R(P)}^{ST}$ *is dual of* S_L^{ST} *.*

Proof. Let $S^{ST}_{R(P)}$ be a projective scheme on S^{ST}_L [G-R1].

Let c be the category of locally noetherian S^{ST}_L preschemes. If $T^{ST}_L \in \mathrm{obj}(c)$, consider $S_{T^{ST}_L} = S^{ST}_{R(P)} \otimes_D T^{ST}_L$ and let $F(T^{ST}_L)$ be the set of closed subpreschemes of $S_{T^{ST}_L}$ which are flat on T^{ST}_L : it is a covariant functor of T^{ST}_L representable by the Hilbert scheme $\mathrm{Hilb}_{S^{ST}_{R(P)}/S^{ST}_L}$ [Gro1], [Gro2], [Göt].

There is bilinearity on $\mathrm{Hilb}_{S^{ST}_{R(P)}/S^{ST}_L}$ with linearity on the left semischeme S^{ST}_L and antilinearity on the right semischeme $S^{ST}_{R(P)}$ if we take into account that the associated right and left ringed semispaces $(M^{ST}_R, \widetilde{M}^{ST}_R)$ and $(M^{ST}_L, \widetilde{M}^{ST}_L)$ are defined respectively on the lower half space M^{ST}_R and on the upper half space M^{ST}_L .

The projective right semischeme $S^{ST}_{R(P)}$ is flat on S^{ST}_L . Furthermore, the right semischeme $S^{ST}_{R(P)}$ is dual of the left semischeme S^{ST}_L . ∎

Corollary 1.3.5 *The Hilbert scheme* $\mathrm{Hilb}_{S^{ST}_{R(P)}/S^{ST}_L}$ *is endowed with a diagonal metric* g^α_α *of type* $(1,1)$ [Pie4].

Proof. Indeed, the components $g^\alpha_\alpha = (e^\alpha, f_\alpha)$ of the metric tensor at each point of $\mathrm{Hilb}_{S^{ST}_{R(P)}/S^{ST}_L}$ are external scalar products with respect to the basis vectors $\{(e^\alpha)^*\}^3_{\alpha=0} \in S^{ST}_{R(P)}$ and $\{f_\alpha\}^3_{\alpha=0} \in S^{ST}_L$. ∎

Proposition 1.3.6 *If we consider a bijective linear isometric map* $B_L : S^{ST}_{R(P)} \to S^{ST}_L$ *mapping each covariant element of* $S^{ST}_{R(P)}$ *into the corresponding contravariant element of* S^{ST}_L *, then the Hilbert scheme* $\mathrm{Hilb}_{S^{ST}_{R(P)}/S^{ST}_L}$ *is transformed into the internal Hilbert scheme* $\mathrm{Hilb}_{S^{ST}_{L_R}/S^{ST}_L}$ *characterized by a diagonal metric* $g_{\alpha\alpha}$ *of type* $(0,2)$.

Proof. Indeed, under the B_L map, the covariant basis vectors $\{(e^\alpha)^*\}^3_{\alpha=0}$ are transformed into the contravariant basis vectors $\{(e_\alpha)^*\}^3_{\alpha=0}$ and the components g^α_α of the metric tensor then become internal scalar products $g_{\alpha\alpha} = (e_\alpha, f_\alpha)$. ∎

Corollary 1.3.7 *The diagonal bisemischeme* $S^{ST}_{L(P)/R}$ *is a covariant functor of* S^{ST}_R *representable by the bilinear Hilbert scheme* $\mathrm{Hilb}_{S^{ST}_{L(P)}/S^{ST}_R}$ *endowed with a metric* g^α_α *of type* $(1,1)$.

Corollary 1.3.8 *By the bijective linear isometric map* $B_R : S^{ST}_{L(P)} \to S^{ST}_R$ *, the Hilbert scheme* $\mathrm{Hilb}_{S^{ST}_{L(P)}/S^{ST}_R}$ *is transformed into the internal Hilbert scheme* $\mathrm{Hilb}_{S^{ST}_{R_L}/S^{ST}_R}$ *characterized by a metric* $g^{\alpha\alpha}$ *of type* $(2,0)$.

The presentation of bilinear Hilbert schemes leads us to formulate the

Axiom II 1.3.9 *Nature is composed of bisemiparticles whose fundamental diagonal space-time structure is given locally by bilinear diagonal Hilbert schemes. This axiom is a multiplicative axiom* [Ati4].

Proof. Due to the fact that the right semischeme S_R^{ST} of the right semiparticle is topologically very close to the left semischeme S_L^{ST} of the left semiparticle such that these two semischemes S_R^{ST} and S_L^{ST} be localized in the same openball centered on the emergence point, only the left semischeme of the left semiparticle will be commonly observable in the frame of bilinear Hilbert schemes with the right semischeme of the right semiparticle generally unobservable because it is projected on the left semischeme of the left semiparticle: this corresponds to the existence of the bilinear Hilbert scheme $\mathrm{Hilb}_{S_{R(P)}^{ST}/S_L^{ST}}$. ■

Remark 1.3.10 (Twin bisemiparticles) But, there also exists a bilinear Hilbert scheme $\mathrm{Hilb}_{S_{L(P)}^{ST}/S_R^{ST}}$, as introduced in corollary 1.3.8, and resulting from the projective morphism p_R (given in definition 1.3.3) which maps the left semischeme S_L^{ST} on the right semischeme S_R^{ST} .
Thus, next to the common world in which we live and described by the bilinear Hilbert scheme $\mathrm{Hilb}_{S_{R(P)}^{ST}/S_L^{ST}}$ at the level of bisemiparticles, there is also the possibility of the existence of a twin world described by the bilinear Hilbert scheme $\mathrm{Hilb}_{S_{L(P)}^{ST}/S_R^{ST}}$ at the level of "twin bisemiparticles".

1.4 Fundamental algebraic space-time structure of semileptons, semibaryons and semiphotons

In sections 1 and 2, the basic algebraic space-time structure of the right and left semiparticles was assumed to be given by right and left semisheaves of rings $(\theta_{R,L}^{*1}(t)\oplus\theta_{I_{R,L}}^{3}(r))$ generated by Eisenstein cohomology and by the $(\gamma_{t\to r}\circ E)$ morphism. However, as it was noticed in 1.2.12, this basic space-time structure corresponds essentially to the algebraic space-time structure of the semilepton of the first family, i.e. the semielectron.

It will be seen in the first part of this section how the algebraic time structure of the semiquarks can be generated from the central algebraic time structure of a semibaryon.

Definition 1.4.1 (Smooth endomorphism E_t) 1. Instead of considering as in definitions 1.2.4 a smooth endomorphism $E[G_{\mu R,L}]$ of the algebraic semigroup $G_{\mu R,L}$ decomposing it into the direct sum of two nonconnected algebraic semigroups, we can envisage the following smooth endomorphism [Pie3]: $E_t[G_{\mu R,L}]=G^*_{(c)_{\mu R,L}}\oplus G^I_{(c)_{\mu R,L}}$ of the algebraic semigroup $G_{\mu R,L}$ decomposing it into the two connected algebraic semigroups $G^*_{(c)_{\mu R,L}}$ and $G^I_{(c)_{\mu R,L}}$ where $G^*_{(c)_{\mu R,L}}$ is the reduced algebraic semigroup, submitted to a Galois antiautomorphic subgroup, and where $G^I_{(c)_{\mu R,L}}$ is the complementary algebraic semigroup resulting from a Galois automorphic subgroup.

The smooth endomorphism E_t is such that the subgroups $SO(m_1,L_\mu^{T\mp})\in K^*_{\mu R,L}\subset G^*_{(c)_{\mu R,L}}$ and $SO(m_2,L_\mu^{T\mp})\in K^I_{\mu R,L}\subset G^I_{(c)_{\mu R,L}}$ must have the same rank but different orders, i.e. that $m_1=2t$ and $m_2=2t+1$, t being an odd integer taking the value $t=1$ here.

2. Let $\theta^1_{R,L}(t)$ be the semisheaf of rings generated by Eisenstein cohomology on the boundary of the Borel-Serre compactification $\partial\,\overline{S}_{K_{tR,L}}=P_{tR,L}\setminus G_{tR,L}/K_{tR,L}$.

Then, the smooth endomorphism E_t applied to the semisheaf $\theta^1_{R,L}(t)$ gives the following decomposition:

$$E_t[\theta^1_{R,L}(t)]=\theta^{*1}_{R,L}(t)\oplus\theta^3_{I_{R,L}}(t_1,t_2,t_3)$$

where $\theta^3_{I_{R,L}}(t_1,t_2,t_3)$ is a $3D$-complementary semisheaf of rings connected to the reduced semisheaf of rings $\theta^{*1}_{R,L}(t)$.

Lemma 1.4.2 *The* $3D$*-complementary semisheaf of rings* $\theta^3_{I_{R,L}}(t_1,t_2,t_3)$ *can only be expressed as the direct sum of 3 connected* $1D$*-time semisheaves of rings*

$$\theta^3_{I_{R,L}}(t_1,t_2,t_3)=\theta^1_{I_{R,L}}(t_1)\oplus\theta^1_{I_{R,L}}(t_2)\oplus\theta^1_{I_{R,L}}(t_3)\ .$$

Proof. Indeed, according to definition 1.4.1, the complementary semisheaf of rings $\theta^3_{I_{R,L}}(t_1,t_2,t_3)$ generated from $\theta^{*1}_{R,L}(t)$ by the smooth endomorphism E_t must be three dimensional. But, considering that:

a) $\theta^3_{I_{R,L}}(t_1,t_2,t_3)$ is defined on a $1D$ semispace $\partial\,\overline{S}_{K_{R,L}}(t)$,

b) $\theta^3_{I_{R,L}}(t_1,t_2,t_3)$ is localized in the orthogonal complement space of the $3D$-space [Sco] on which semisheaves $\theta^3_{I_{R,L}}(r_i)$ are defined and generated by the $\gamma_{t_i\to r_i}$ morphisms,

the semisheaf $\theta^3_{I_{R,L}}(t_1,t_2,t_3)$ can only be composed of three orthogonal $1D$-semisheaves of rings $\theta^1_{I_{R,L}}(t_i)$, $1\le i\le 3$ (see also [Pie9], section 4.1).

Consequently, $\theta^1_{I_{R,L}}(t_i)$ is a $1D$-time semisheaf whose sections are given by $1D$-tori $T^1_{\mu R,L}(t_i)$. ■

Proposition 1.4.3 *The algebraic time structure of a semibaryon is given by*

$$\theta^{\rm Bar}_{R,L}(t)=\theta^{*1}_{R,L}(t_c)\bigoplus_{i=1}^{3}\theta^1_{I_{R,L}}(t_i)$$

where $\theta^{*1}_{R,L}(t_c)$ *is its core time structure and where* $\theta^1_{I_{R,L}}(t_i)$ *is the time structure of a semiquark.*

Proof. The semisheaf or rings $\theta^{\rm Bar}_{R,L}(t)$ results directly from definition 1.4.1 and lemma 1.4.2 such that the reduced semisheaf or rings $\theta^{*1}_{R,L}(t_c)$ is connected to the complementary semisheaves $\theta^1_{I_{R,L}}(t_i)$.

The interpretation of $\theta^{\rm Bar}_{R,L}(t)$ as the time structure of a semibaryon is justified by the "bag" model of the baryons [C-J-J-T-W] and the confinement of the three quarks [Bjo], [C-R]. ■

Proposition 1.4.4 *The algebraic space-time structure of a semibaryon is generated from* $\theta^{\rm Bar}_{R,L}(t)$ *by* $\gamma_{t_i\to r_i}$ *morphisms following:*

$$\gamma_{t_i\to r_i}\circ E_i:\theta^{\rm Bar}_{R,L}(t)\to\theta^{\rm Bar}_{R,L}(t,r)$$

where $\theta^{\rm Bar}_{R,L}(t,r)$ *is given by*

$$\theta^{\rm Bar}_{R,L}(t,r)=\theta^{*1}_{R,L}(t_c)\bigoplus_{i=1}^{3}\theta^{1-3}_{R,L}(t_i,r_i)\ .$$

Proof.

a) The morphism $(\gamma_{t\to r}\circ E)$ does not apply on $\theta^{*1}_{R,L}(t_c)$ because it is a reduced semisheaf of rings resulting from the smooth endomorphism E_t on which $\bigoplus_{i=1}^{3}\theta^1_{I_{R,L}}(t_i)_{q_i}$ are connected.

b) The space structure of the three semiquarks is generated by considering the $(\gamma_{t_i\to r_i}\circ E_i)$ morphisms on $\theta^1_{I_{R,L}}(t_i)_{q_i}$:

$$\gamma_{t_i\to r_i}\circ E_i:\theta^1_{I_{R,L}}(t_i)_{q_i}\to\theta^{*1}_{I_{R,L}}(t_i)_{q_i}\oplus\theta^3_{I_{R,L}}(r_i)_{q_i}\quad\forall\ i\ ,\ 1\le i\le 3\ .$$ ■

Definition 1.4.5 (Constant of the strong interaction) Let n_B denote the set of q_B ranks of the q_B sections of $\theta^1_{R,L}(t_c)$ and let $(n_B-\rho_B)$ be the set of q_B decreasing ranks of $\theta^{*1}_{R,L}(t_c)$.

Then, $\phi^*_{t_c;(n_B-\rho_B)_{R,L}}$ will be the set of algebraic Hecke characters related to the generation of the reduced semisheaf or rings $\theta^{*1}_{R,L}(t_c)$ by Eisenstein homology and $\phi_{[t_1,t_2,t_3];\rho_{B_{R,L}}}$ will be the set of algebraic Hecke characters related to the generation of the complementary semisheaf or rings $\theta^3_{I_{R,L}}(t_1,t_2,t_3) = \bigoplus\limits_{i=3}^{3} \theta^1_{I_{R,L}}(t_i)_{q_i}$ by Eisenstein cohomology.

As introduced in definition 1.2.9, there is the following equality between these two sets of algebraic Heche characters:

$$\phi^*_{t_c;(n_B-\rho_B)_{R,L}} = G(\rho_B)_{t_c\to[t_1,t_2,t_3]}\phi_{[t_1,t_2,t_3];\rho_{B_{R,L}}}$$

where

$$G(\rho_B)_{t_c\to[t_1,t_2,t_3]} = \{G_1(\rho_{B_1}),\cdots,G_\mu(\rho_{B_\mu}),\cdots,G_{q_B}(\rho_{B_q})\}$$

is the set of q_B parameters measuring the generation of the complementary semisheaf $\theta^3_{I_{I_{R,L}}}(t_1,t_2,t_3)$ from $\theta^{*1}_{R,L}(t_c)$.

Proposition 1.4.6 *The parameter $\langle G(\rho_B)_{t_c\to[t_1,t_2,t_3]}\rangle = \sum\limits_{\mu=1}^{q_B} G_\mu(\rho_{B_\mu})$ must correspond to the strong constant of the strong interaction.*

Proof. Indeed, $G(\rho_B)_{t_c\to[t_1,t_2,t_3]}$ measures the generation of the time structure of the three semiquarks $\theta^3_{I_{R,L}}(t_1,t_2,t_3)$ from the core time structure $\theta^{*1}_{R,L}(t_c)$ of the envisaged semibaryon.

If $(n_B-\rho_B)\to 0$, then $\phi^*_{t_c;(n_B-\rho_B)}\to 0$ and we have asymptotic freedom [G-W], [Pol], [Wein1], corresponding to the fact that the semiquarks become free since $\bigoplus\limits_{i=1}^{3}\theta^1_{I_{R,L}}(t_i)$ are no more connected to $\theta^{*1}_{R,L}(t_c)\to 0:$ this is reflected by $\langle G(\rho_B)_{t_c\to[t_1,t_2,t_3]}\rangle\to 0$.

On the other hand, if $\phi_{[t_1,t_2,t_3]\rho_{B_{R,L}}}$ is small, then $\langle G(\rho_B)_{t_c\to[t_1,t_2,t_3]}\rangle$ will take high values. ■

Definition 1.4.7 (Algebraic Hecke characters) Consider the morphism

$$(\gamma_{t\to r}\circ E):\theta^1_{R,L}(t)\to\theta^{*1}_{R,L}(t)\oplus\theta^3_{I_{R,L}}(r)$$

transforming sequentially and gradually the $1D$-time semisheaf of rings $\theta^1_{R,L}(t)$ into the $3D$-spatial semisheaf of rings $\theta^3_{I_{R,L}}(r)$. Let $(n-\rho)_{R,L}$ be the set of ranks of the semisheaf $\theta^{*1}_{R,L}(t)$ and $\phi^*_{t;(n-\rho)_{R,L}}$ the corresponding set of algebraic Hecke characters.

Similarly, let $\rho_{R,L}$ be the set of ranks of the semisheaf $\theta^3_{R,L}(r)$ and $\phi_{r;\rho_{R,L}}$ the corresponding set of algebraic Hecke characters.

If $(n-\rho)_{R,L}\to 0$, then we have that

$$\gamma_{t\to r}\circ E:\theta^1_{R,L}(t)\xrightarrow{\sim}\theta^3_{I_{R,L}}(r)_P$$

which means that the semisheaf $\theta^1_{R,L}(t)$ has been nearly transformed into the $3D$-spatial semisheaf $\theta^3_{I_{R,L}}(r)_P$.

Let $\phi^*_{t;(n-\rho)_{R,L}} = c_{t\to r}(\rho)\phi_{r;\rho_{R,L}}$ be the equality between the corresponding sets of algebraic Hecke characters. Then,

$$c^{-1}_{t\to r}(\rho) = \{c_1^{-1}(\rho_1),\cdots,c_\mu^{-1}(\rho_\mu),\cdots,c_q^{-1}(\rho_q)\}$$

is the set of q inverse parameters measuring the generation of the semisheaf $\theta^3_{R,L}(r)_P$ from the semisheaf $\theta^{*1}_{R,L}(t)$.

Proposition 1.4.8 *If $(n-\rho)_{R,L}\to 0$, then*

1. *the average parameter $\langle c^{-1}_{t\to r}(\rho)\rangle_{\max} = \left(\sum\limits_{\mu=1}^{q} c^{-1}_{\mu}(\rho_\mu)\right)/q$ is proportional to the velocity of the light " c ".*
2. *the semisheaf of rings $\theta^3_{I_{R,L}}(r)_P$ resulting from the morphism:*

$$(\gamma_{t\to r}\circ E)_{\max} : \theta^1_{R,L}(t) \xrightarrow{\sim} \theta^3_{I_{R,L}}(r)_P$$

   gives the structure of a set of right (resp. left) semiphotons.

Proof. If $(n-\rho)_{R,L}\to 0$, the $1D$-time structure $\theta^1_{R,L}(t)$ has been nearly completely transformed into the $3D$-spatial structure $\theta^3_{I_{R,L}}(r)_P$; consequently, $\langle c^{-1}_{t\to r}(\rho)\rangle_{\max}$, giving a measure of the ratio of the spatial structure with respect to the time structure, must be proportional to the velocity of the light: indeed, as the proper time of the semiphotons tends to zero, their space(-time) structure is crudely given by $\theta^3_{I_{R,L}}(r)_P$. ■

Definition 1.4.9 (A right (resp. left) semiphoton) Considering that a photon with momentum k corresponds to a plane wave and that to each normal mode k is associated $(p+\mu)_k$ quanta, we shall assume that the internal (vacuum) space structure of a right(resp. left) semiphoton with momentum $\vec p=\hbar\,\vec k$ will be described by a spatial section $T^1_{R,L}(r_k)$ (which is a $1D$-real torus according to definition 1.2.7) having $(p+\mu)$ quanta of momenta $\vec k_\tau$ and composed of $(p+\mu)\cdot N$ prime ideals corresponding to $(p+\mu)\cdot N$ Galois automorphisms.

Then, $\vec k=(p+\mu)\vec k_\tau$ and $\vec p=\hbar_{ST}\,\vec k=(p+\mu)\hbar_{ST}\,\vec k_\tau$, where the equivalent of the Planck constant h_{ST} corresponds to the integer N in the internal (vacuum) space time unit system (see proposition 2.2.13).

Definition 1.4.10 The (vacuum) space-time structure of elementary right and left semiparticles is assumed to be given at the fundamental level by:

1. the number of sections of the space-time semisheaf of rings representing their structure;
2. the set of ranks of these sections and especially the set of parameters $c_{t\to r}(\rho)_{R,L}$ (see definition 1.2.9) measuring the generation of the complementary space semisheaf of rings $\theta^3_{I_{R,L}}(r)$ with respect to the reduced time semisheaf of rings $\theta^{*1}_{R,L}(t)$.

More precisely, the fundamental algebraic structure of:

a) semileptons will be characterized by:

   1. g_ℓ sections with $g_\ell\in I\!N^+$ (ℓ for leptons);

2. a set of g_ℓ ranks n_ℓ referring to these sections in such a way that the set of ranks $(n_\ell - \rho_\ell)$ refers to the reduced $1D$-time semisheaf of rings $\theta^{*1}_{R,L}(t)_\ell$ and ρ_ℓ refers to the complementary $3D$ semisheaf of rings $\theta^3_{I_{R,L}}(r)_\ell$. Then, the set of parameters

$$c_{t\to r}(\rho_\ell)_{R,L} = \phi^*_{t;(n_\ell-\rho_\ell)_{R,L}} \;/\; \phi_{r;(\rho_\ell)_{R,L}}$$

gives a measure of the generation of $\theta^3_{I_{R,L}}(r)_\ell$ with respect to $\theta^{*1}_{R,L}(t)_\ell$.

b) semibaryons will be characterized by:

1. q_B sections;
2. a set of q_B ranks n_B ;
3. a set of q_B parameters $G(\rho_B)_{t_c\to[t_1,t_2,t_3]}$, as introduced in definition 1.4.5 and measuring the generation of the three complementary $1D$-time semisheaves of rings $\theta^1_{I_{R,L}}(t_i)_{q_i}$ from the reduced $1D$-time semisheaf of rings $\theta^{*1}_{R,L}(t_c)$ of the baryonic core.
   Parallely, we have a set of q_B ranks $(n_B-\rho_B)$ referring to $\theta^{*1}_{R,L}(t_c)$ and a set of q_B complementary ranks ρ_B referring to $\bigoplus\limits_{i=1}^{3}\theta^1_{I_{R,L}}(t_i)_{q_i}$.
4. three sets of q_{B_i} parameters $c_{t\to r}(\rho_{q_i})$, $1\le i\le 3$, referring to the generation of the $3D$-spatial semisheaves of rings of the three semiquarks $\theta^3_{I_{R,L}}(r_i)_{q_i}$ from $\theta^1_{I_{R,L}}(t_i)_{q_i}$.

Proposition 1.4.11 *The sets of parameters $c_{t\to r}(\rho_\ell)$ of semileptons and $c_{t\to r}(\rho_{q_i})$ of the semiquarks are obstruction parameters with respect to the stability of these semiparticles.*

Proof. Indeed, these sets of parameters fix the space structures of these semiparticles with respect to their time structures according to definition 1.4.10, preventing their annihilation, i.e. the complete transformation of their time semisheaves of rings in their complementary $3D$-space semisheaves of rings by the morphisms $(\gamma_{t\to r}\circ E)$. ∎

Lemma 1.4.12 *The number of geometric points of all right (resp. left) time quanta $M^I_\mu(t)_{R,L}$ of rank N is equal.*

Proof. The sections of the $1D$-time semisheaves of rings $\theta^1_{R,L}(t)$ are generated by Eisenstein cohomology from symmetric splitting semifields having the same number of simple roots according to definitions 1.1.2 and 1.1.4. Consequently, all the sections $s_{\mu R,L}\in\theta^1_{R,L}(t)$, generated from the specialization ideals $p_{\mu R,L}$ (see definition 1.1.3), are composed of functions on right (resp. left) time quanta having the same number of geometric points [L-N]. ∎

This is true for semileptons. But, if we take into account the following proposition, it is also verified for semibaryons whose time structure originates from symmetric splitting subsemifields $L^{\mp}_\mu$ whose number is greater than for semileptons because the baryon masses are bigger than the lepton masses in a given family of elementary particles.

Proposition 1.4.13 *The right (resp. left) time quantum of a semiquark is a right (resp. left) time quantum of the baryonic core time quantum.*

Proof. Indeed, the smooth endomorphism E_t , transforming the baryonic core $1D$-time semisheaf of rings $\theta^1_{R,L}(t_c)$ into the three complementary $1D$-time semisheaves of rings $\theta^1_{I_{R,L}}(t_i)_{q_i}$ of the three semiquarks, acts on $\theta^1_{R,L}(t_c)$ prime ideal by prime ideal through the action of the Galois antiautomorphic group.

Consequently, the number of geometric points of a time quantum of the baryonic core is equal to the number of the geometric points of a time complementary quantum of a semiquark. ∎

Proposition 1.4.14 *The number of geometric points of all right (resp. left) space quanta $M^I_\mu(r)_{R,L}$ is equal.*

Proof. As by lemma 1.4.12, the number of geometric points of all time quanta $M^I_\mu(t)_{R,L}$ is equal and as the space quanta are generated from the corresponding time quanta by the morphism $(\gamma_{t\to r}\circ E)$, we reach the thesis. ∎

Proposition 1.4.15 *The number of geometric points of a space quantum of a semiquark is equal to the number of geometric points of a time quantum.*

Proof. By lemma 1.4.12 and proposition 1.4.13, we know that the number of geometric points of a time quantum of the baryonic core is equal to the number of the geometric points of a time quantum of a semiquark. Considering on the one hand the three $(\gamma_{t_i\to r_i}\circ E_i)$ morphisms responsible for the generation of the space structure $\theta^3_{I_{R,L}}(r_i)_{q_i}$ of the three semiquarks according to proposition 1.4.4 and on the other hand the $(\gamma_{t\to r}\circ E)$ morphism responsible for the generation of the space semisheaf of rings $\theta^3_{I_{R,L}}(r)_\ell$ of a semilepton, for example, we get the thesis since all time quanta have the same number of geometric points. ∎

1.4.16 The quantification rules of the space-time structure of semiparticles can then be envisaged by considering that:

a) the time structure $\theta^1_{R,L}(t)_{\ell_1}$ of a semilepton ℓ_1 can lose time quanta by the action of the smooth endomorphism E according to definition 1.2.4:

$$E:\theta^1_{R,L}(t)_{\ell_1}\to\theta^{*1}_{I_{R,L}}(t)_{\ell_1}\bigoplus_{k=1}^{m}\widetilde{M}^I_k(t)_{R,L}$$

where $\widetilde{M}^I_k(t)_{R,L}$ are disconnected time quanta (functions) of rank N .

These free right (resp. left) time quanta (functions) can then join the right (resp. left) time semisheaf of rings $\theta^1_{R,L}(t)_{\ell_2}$ of another semilepton, labeled ℓ_2 , and increase its time structure.

b) similarly, space quanta $\widetilde{M}^I_k(r)_{R,L}$ can be disconnected from the space structure $\theta^3_{R,L}(r)_{\ell_1}$ of a semilepton ℓ_1 by:

$$E:\theta^3_{R,L}(r)_{\ell_1}\to\theta^{*3}_{I_{R,L}}(r)_{\ell_1}\bigoplus_{k=1}^{m'}\widetilde{M}^I_k(r)_{R,L}$$

and increase the space structure $\theta^3_{R,L}(r)_{\ell_2}$ of another semilepton ℓ_2 .

c) the time semisheaves of rings $\theta^1_{R,L}(t_i)_{q_i}$, $1\leq i\leq 3$, of the three semiquarks q_i of a semibaryon B_1 can lose time quanta $\widetilde{M}^I_{k_i}(t_i)_{R,L}$ by means of the smooth endomorphisms

$$E_i : \theta^1_{R,L}(t_i)_{q_i} \to \theta^{*1}_{I_{R,L}}(t_i)_{q_i} \bigoplus_{k_i=1}^{m_i} \widetilde{M}^I_{k_i}(t_i)_{R,L} \ .$$

These time quanta (functions) can then increase the time semisheaves of rings of the three semiquarks of a semibaryon B_2 .

Similar conclusions are reached with space quanta.

Let us note that quantification rules with right or left quanta are not exact since only bisemiparticles have a real existence. Consequently, only quantification rules with biquanta can be considered as developed in chapter 3, section 3.

It was demonstrated in [Pie11] that the quantification rule consisting in adding time or space quanta to a semisheaf of rings corresponds to a deformation of a modular Galois representation while the quantification rule referring to the removal of quanta from a semisheaf or rings corresponds to an inverse deformation of projective type associated to an endomorphism.

More concretely, let $s_{\mu_{R,L}}$ denote a section of a semisheaf of rings having a rank $n_\mu=(p+\mu)\cdot N$.

Then, a deformation of $s_{\mu_{R,L}}$ corresponds to an equivalence class of lift:

$$\mathcal{D}^{[p+\mu]\to[p+\mu+\nu]}_{R,L} : \quad s_{\mu_{R,L}} \longrightarrow s_{\mu+\nu_{R,L}}$$

sending $s_{\mu_{R,L}}$ to a section $s_{\mu+\nu_{R,L}}$ having a rank $n_{\mu+\nu}=(p+\mu+\nu)\cdot N$ and composed of $(p+\mu+\nu)$ quanta. The deformation $\mathcal{D}^{[p+\mu]\to[p+\mu+\nu]}_{R,L}$ is associated to the exact sequence:

$$1 \longrightarrow \widetilde{M}^I_{\mu_{R,L}} \longrightarrow s_{\mu+\nu_{R,L}} \longrightarrow s_{\mu_{R,L}} \longrightarrow 1$$

whose kernel is a quantum (function) $\widetilde{M}^I_{\mu_{R,L}}$.

On the other hand, a section $s_{\mu+\nu_{R,L}}$ can be submitted to the inverse deformation

$$\mathcal{D}^{[p+\mu+\nu]\to[p+\mu]}_{R,L} : \quad s_{\mu+\nu_{R,L}} \longrightarrow s_{\mu_{R,L}}$$

which is a modular projective mapping sending a section $s_{\mu+\nu_{R,L}}$ of rank $n_{\mu+\nu}=(p+\mu+\nu)\cdot N$ to a section $s_{\mu_{R,L}}$ of rank $n_\mu=(p+\mu)\cdot N$ corresponding to an endomorphism of $s_{\mu+\nu_{R,L}}$ removing ν quanta which become “free”.

2 Deformations of the fundamental algebraic structure of semiparticles

2.1 Versal deformation and spreading-out isomorphism

External perturbations can generate singularities on the sections of the semisheaves of rings $\theta^{1-3}_{R,L}(t,r)$. This problem is analyzed in this section by considering the versal deformation of a semisheaf of germs of differentiable functions $\theta^m_{R,L}(s_{R,L})$ of dimension m having isolated singularities.

The related question consisting in the algebraic extension of the quotient algebra of the versal deformation is principally considered: it is essentially the inverse problem of the versal deformation of a sheaf of germs of differentiable functions. This problem has some analogy with the resolution [Hir], [Thoma], [Tei] of the singularities of an algebraic variety since it "reduces" the versal deformation.

Under some external perturbation, singularities [Tho1], [Lev] are assumed to be generated on the sections $s_{\mu R,L} \in \theta^m_{R,L}$. We then consider:

Definitions 2.1.1 (1. The division theorem) This theorem will be recalled for germs of differentiable functions $s_{\mu R,L}$ having an isolated singularity of corank 1 . Remark that nonisolated singularities were investigated by Siersma [Sie] and Pellikaan [Pel] who consider as starting point of their developments the group of all local isomorphisms leaving the singular locus invariant.

Let $(x_1, \cdots, x_{m-n}, w_1, \cdots, w_n)$ denote the coordinates in $(L^{\mp})^m$.

A germ $s_{\mu R,L}(w_{R,L}) \in \theta^m_{R,L}(s_{R,L})$ has a singularity of corank 1 (then, $n = 1$) and order p in $w_{R,L}$ if $s_{\mu R,L}(0, w_{R,L}) = w^p_{\mu R,L} e_\mu(w_{R,L})$ where $e_\mu(w_{R,L})$ is a differentiable unit, i.e. verifying $e_\mu(0) \neq 0$.

Let $\theta^m_{R,L}[w_{R,L}]$ be the algebra of polynomials in $w_{R,L}$ with coefficients $r_{i\mu}(x)_{R,L}$ being subfunctions of $\theta^{m-1}_{R,L}$ defined on a domain $D_{R,L} \subset B_{R,L}$ where $B_{R,L}$ is a lower (resp. upper) half open ball of radius b around $0 \in L^{m-1}$.

If $s_{\mu R,L} \in \theta^m_{R,L}(s_{R,L})$ has order p in $w_{R,L}$, then, there exist a differentiable function $q_{\mu R,L} \in \theta^m_{R,L}(q_{R,L})$ and a polynomial

$$R'_{\mu R,L} = \sum_{i=1}^{r} r_{i\mu}(x)_{R,L} w^i_{\mu R,L} \in \theta^m_{R,L}[w_{R,L}]$$

with degree $r < p$ such that

$$f_{\mu R,L} = s_{\mu R,L} \cdot q_{\mu R,L} + R'_{\mu R,L}$$

is the versal unfolding of $s_{\mu R,L}$ and corresponds to the Malgrange division theorem for the right and left cases. The Malgrange division theorem [Mal] is the differentiable version of the Weierstrass division theorem [G-R2], [G-K] valid for germs of analytic functions [Math1].

(2. The preparation theorem) Let

$$w_{\mu R,L} = w^p_{R,L} + \sum_{i=0}^{p-1} b_{i\mu}(x)_{R,L} w^i_{R,L}$$

be the Weierstrass polynomial verifying $b_{1\mu}(0) = \cdots = b_{(p-1)\mu}(0) = 0$. If $s_{\mu R,L} \in \theta^m_{R,L}(s_{R,L})$ has finite order p in $w_{R,L}$, then there exists a uniquely determined Weierstrass polynomial $w_{\mu R,L} \in \theta^m_{R,L}[w_{R,L}]$ and a unit $e_{\mu R,L} \in \theta^m_{R,L}(e)$ such that $s_{\mu R,L} = w_{\mu R,L} \cdot e_\mu(w_{R,L})$.

If $s_{\mu R,L} \in \theta^m_{R,L}[w_{R,L}]$, then $e_{\mu R,L} \in \theta^m_{R,L}[w_{R,L}]$ and we get the preparation theorem

$$f_{\mu R,L} = s_{\mu R,L} \cdot q_{\mu R,L} + R_{\mu R,L}$$

where $q_{\mu R,L} \in \theta^m_{R,L}(q)$ and

$$R_{\mu R,L} = \sum_{i=1}^{r} a_{i\mu}(x)_{R,L} w^i_{\mu R,L} \in \theta^m_{R,L}[w_{R,L}]$$

with $a_{i\mu}(x)_{R,L} \in \theta^{m-1}_{R,L}(a_i)$ and $r < p$.

Definitions 2.1.2 (1. Versal deformation) Let $\theta^{m-1}_{R,L}(a_i)$ be the semisheaf of differentiable functions $a_{i\mu}(x)_{R,L} \subset s_{\mu R,L}(x,w)$, $x=(x_1,\cdots,x_{m-n})$, and $\theta^1_{R,L}(w^i)$ be the i-th generator semisheaf of monomial functions $w^i_{\mu R,L}$. Then, $\theta_{R,L}(s_{R,L})=\{\theta^1_{R,L}(w^1_{R,L}),\cdots,\ \theta^1_{R,L}(w^i_{R,L}),\cdots,\theta^1_{R,L}(w^r_{R,L})\}$ is the right (resp. left) family of semisheaves of the right (resp. left) base $s_{R,L}$ of the versal deformation of the semisheaf $\theta^m_{R,L}(s_{R,L})$. Indeed, the versal deformation of the semisheaf $\theta^m_{R,L}(s_{R,L})$, whose sections are the differentiable functions $s_{\mu R,L}$, is given by the product [Trau1], [Trau2]:

$$\theta_{R,L}(f_{R,L})=\theta^m_{R,L}(s_{R,L})\times\theta^r_{R,L}(s_{R,L})$$

where $\theta^r_{R,L}(s_{R,L})$ is the base semisheaf such that $\theta_{R,L}(f_{R,L})$ is $s_{R,L}$-flat.

Recall succinctly that a deformation [Ill] is called versal if each deformation of $\theta_{R,L}(s_{R,L})$ is isomorphic to another deformation of $\theta_{R,L}(s_{R,L})$ induced by some transformation of the base semisheaf $\theta_{R,L}(s_{R,L})$ [Pala].

(2. Quotient algebra) The quotient algebra $\theta_{R,L}[R_{w_{R,L}}]$ of the versal deformation of the semisheaf $\theta^m_{R,L}(s_{R,L})$ is a finitely generated vector space of dimension r whose elements are the polynomials $R_{\mu R,L}=\sum\limits_{i=1}^{r} a_{i\mu}(x)_{R,L}w^i_{\mu R,L}$.

Definition 2.1.3 (Specialization semirings) Let $p(a_{i\mu})_{R,L}$ be the specialization prime ideal of the subsemiring $A_{a_{i\mu R,L}}$ referring to the generation of the function $a_{i\mu}(x)_{R,L}$. Then, $p(a_i)_{R,L}$ will denote the set of specialization prime ideals $\{p(a_{i\mu})_{R,L}\}^q_{\mu=1}$ of the semiring $A_{a_{iR,L}}$ referring to the generation of the semisheaf $\theta^{m-1}_{R,L}(a_i)$.

Similarly, let $p(w^i_{\mu R,L})$ be the specialization prime ideal of the subsemiring $A_{w^i_{\mu R,L}}$ referring to the generation of the i-th base function of the polynomial $R_{\mu R,L}$. Then, $p(w^i)_{R,L}$ will denote the set of specialization prime ideals $\{p(w^i_{\mu R,L})\}^q_{\mu=1}$ of the semiring $A_{w^i_{R,L}}$ referring to the generation of the i-th generator semisheaf $\theta^1_{R,L}(w^i_{R,L})$.

According to section 1.1.3, let $\beta(a_i)_{R,L}$ be the set of specialization ideals of the specialization semiring $B_{a_{iR,L}}$ dividing the set of specialization ideals $p(a_i)_{R,L}$. Similarly, let $\beta(w^i)_{R,L}$ be the set of specialization ideals of the specialization semiring $B_{w^i_{R,L}}$ dividing the set of specialization ideals $p(w^i)_{R,L}$.

Then, $B_{a_{iR,L}}$ is the integral closure of $A_{a_{iR,L}}$ and $\theta^{m-1}_{R,L}(a_i)$ is a semisheaf on the free $A_{a_{iR,L}}$-semimodule $B_{a_{i_{R,L}}}$.

Similarly, $\theta^1_{R,L}(w^i)$ is a semisheaf on the $A_{w^i_{R,L}}$-semimodule $B_{w^i_{R,L}}$.

Lemma 2.1.4 *The semisheaves $\theta^{m-1}_{R,L}(a_i)$, $1\le i\le r$, and $\theta^1_{R,L}(w^i)$ are characterized by the same set of ranks.*

Proof.

1) Let $n_{w^i_{R,L}}=\{n_{w^i_1},\cdots,n_{w^i_\mu},\cdots,n_{w^i_q}\}$ be the set of ranks corresponding to the set of subsemimodules $B_{w^i_{R,L}}=\{B_{w^i_1},\cdots,B_{w^i_q}\}$ and let $n_{a_{iR,L}}=\{n_{a_{i1}},\cdots,n_{a_{i\mu}},\cdots,\ n_{a_{iq}}\}$ be the set of ranks of the set of subsemimodules $B_{a_{iR,L}}=\{B_{a_{i1}},\cdots,B_{a_{iq}}\}$.

Each section $s_{\mu R,L} \in \theta^m_{R,L}(s_{R,L})$, $1 \le \mu \le q$, having a singularity of order p is $(p+1)$ determined [Math2], [Tou]. Consequently, there exist $(r-1)$ embedded and sequential subspaces of the quotient algebra $\theta_{R,L}[R_w]$ of the versal deformation $\theta_{R,L}(f_{R,L}) = \theta^m_{R,L}(s_{R,L}) \times \theta^r_{R,L}(s_{R,L})$ of the semisheaf $\theta^m_{R,L}(s_{R,L})$: this also reflects the finite determinacy [Pie5] of the quotient algebra $\theta_{R,L}[R_w]$.

Considering that the quotient algebra develops according to:

$$\theta_{R,L}[R_w] = \sum_{i=1}^{r} \theta^{m-1}_{R,L}(a_i) \times \theta^1_{R,L}(w^i) \ ,$$

each semisheaf direct product $\theta^{m-1}_{R,L}(a_i) \times \theta^1_{R,L}(w^i)$ must be finitely generated.

2) the section $a_{i\mu}(x)_{R,L} \subset s_{\mu R,L}(x,w)$ being a subfunction of $s_{\mu R,L}(x,w)$ must be characterized by a rank

$$n_{a_{i\mu}} = (h_\mu \cdot N)^{m-1}$$

where:

- the integer h_μ is a global residue degree verifying $h_\mu < \mu$ with μ being the global residue degree of the μ-th conjugacy class of $T_m(\mathbb{A}_{L_v})$ or of $T^t_m(\mathbb{A}_{L_{\overline{v}}})$ on which $s_{\mu R,L}(x,w)$ is defined.

  Note that the rank n_{s_μ} of this μ-th conjugacy class is $n_{s_\mu} = (\mu \cdot N)^m$ [Pie9].

- N is the rank of a real irreducible completion.

3) As $\theta^r_{R,L}(s_{R,L})$ is projected onto the semisheaf $\theta^{m-1}_{R,L}(a_i)$ in such a way that the semisheaf $\theta^1_{R,L}(w^i_{R,L})$, $1 \le i \le r$, be flat onto $\theta^{m-1}_{R,L}(a_i)$, the monomial function $w^i_{\mu R,L} \in \theta^1_{R,L}(w^i_{R,L})$, which is a normal crossings divisor, must have a rank $n_{w^i_\mu}$ proportional or equal to the rank $n_{a_{i\mu}}$ of $a_{i\mu}(x)_{R,L}$.

If $m-1 \le 2$, then we have that:

- $n_{w^i_\mu} = (h_\mu \cdot N)^p$ where $p \ge m-1$;
- $n_{w^i_\mu} \ge n_{a_{i\mu}}$. ■

Definition 2.1.5 (Singular ideal) The function $w^i_{\mu R,L} \in \theta^1_{R,L}(w^i)$ can have an isolated singular point in the specialization ideal $\beta_{j;w^i_\mu}$. Then, $\Delta\beta^S_{j;w^i_\mu} = \beta_{j;w^i_\mu} - \beta_{(j-1);w^i_\mu}$ will be called a singular ideal. The rank of $w^i_{\mu R,L}$ will be called the total rank, noted $n^T_{w^i_\mu}$, and will be equal to $n^T_{w^i_\mu} = (n_{w^i_\mu} - 1) + 1 \equiv n_{w^i_\mu}$ where the second term in the sum refers to the singular "rank" of the singular ideal $\Delta\beta^S_{j;w^i_\mu}$.

Lemma 2.1.6 *Let $\theta^1_{R,L}(w^i_{R,L})$ be the i-th base semisheaf of the versal deformation of $\theta^m_{R,L}(s_{R,L})$.*

Let $f^{\max}_i$ be the maximal value of its global residue degree counting the irreducible subschemes of rank N .

Then, the following smooth endomorphism

$$E_{w^i_{R,L}}[\theta^1_{R,L}(w^i_{R,L})_{f^{\max}_i}] = \theta^{*1}_{R,L}(w^i_{R,L})_{f^*_{r_i}} \oplus \theta^1_{I_{R,L}}(w^i_{R,L})_{f^I_{r_i}} \ ,$$

with $f^I_{r_i} = f^{\max}_i - f^*_{r_i} \in \mathbb{N}$, can be introduced on the semisheaf $\theta^1_{R,L}(w^i_{R,L})_{f^{\max}_i}$ in such a way that it decomposes into two non connected complementary semisheaves $\theta^{*1}_{R,L}(w^i_{R,L})_{f^*_{r_i}}$ and $\theta^1_{I_{R,L}}(w^i_{R,L})_{f^I_{r_i}}$.

Proof.

1) Referring to the rank $n_{w^i_\mu}=(h_\mu\cdot N)^p$ of the monomial function $w^i_{\mu R,L}\in\theta^1_{R,L}(w^i_{R,L})$ as given in lemma 2.1.4, we see that its unramified rank or global residue degree is given by:

$$f_{w^i_\mu}=(h_\mu)^p=n_{w^i_\mu}/N^p\ .$$

The integer $f_{w^i_\mu}$ is the number of irreducible completions of rank N on which $w^i_{\mu R,L}$ is defined. So, $f_i^{\max}$ will be in the same manner the number of irreducible completions of rank N on which $\theta^1_{R,L}(w^i_{R,L})$ is defined:

$$f_i^{\max}=\bigoplus_\mu\bigoplus_{m_\mu}(h_{\mu,m_\mu})^p\ .$$

2) The semisheaf $\theta^{*1}_{R,L}(w^i_{R,L})_{f^*_{r_i}}$ is a reduced semisheaf generated from $\theta^1_{R,L}(w^i_{R,L})_{f_i^{\max}}$ under the action of the Galois antiautomorphic group according to the endomorphism $E_{w^i_{R,L}}$ in such a way that:

- $\theta^{*1}_{R,L}(w^i_{R,L})_{f^*_{r_i}}$ is characterized by decreasing global residue degrees $f^*_{r_i}$;
- $\theta^1_{I_{R,L}}(w^i_{R,L})_{f^I_{r_i}}$ is characterized by increasing global residue $f^I_{r_i}$ verifying $f_i^{\max}=f^*_{r_i}+f^I_{r_i}$. ■

Proposition 2.1.7 *Every base semisheaf $\theta^1_{R,L}(w^i_{R,L})$ of the versal deformation of the semisheaf $\theta^m_{R,L}(s_{R,L})$ can generate under the smooth endomorphism $E_{w^i_{R,L}}$ the elements of the category $c(\theta^1_{w^i_{R,L}})$ of the (f_i-1) pairs of semisheaves of rings:*

$$\begin{aligned}c(\theta^1_{w^i_{R,L}})=\{&(\theta^{*1}_{R,L}(w^i_{R,L})_{f_i^{\max}-1}\oplus\theta^1_{I_{R,L}}(w^i_{R,L})_1),\\ &\cdots,(\theta^{*1}_{R,L}(w^i_{R,L})_{f^*_{r_i}}\oplus\theta^1_{I_{R,L}}(w^i_{R,L})_{f^I_{r_i}}),\\ &\cdots,(\theta^{*1}_{R,L}(w^i_{R,L})_1\oplus\theta^1_{I_{R,L}}(w^i_{R,L})_{f_i^{\max}-1})\}\ ,\quad 1\le f^*_{r_i}\le f_i^{\max}\ ,\end{aligned}$$

whose objects are two nonconnected semisheaves characterized by complementary global residue degrees verifying: $f_i^{\max}=f^*_{r_i}+f^I_{r_i}$.

Proof. This is a generalization of lemma 2.1.6 where $(f_i^{\max}-1)$ smooth endomorphisms $E_{w^i_{R,L}}$ are considered. ■

Corollary 2.1.8 *Let $f^*_{r_i}$ denote the global residue degree set of the i-th reduced semisheaf of rings $\theta^{*1}_{R,L}(w^i)$ and let $f^I_{r_i}$ denote the global residue degree set of the i-th complementary semisheaf of rings $\theta^1_{I_{R,L}}(w^i)$.*

Then, the smooth endomorphism $E_{w^i_{R,L}}$ applied on the semisheaf of rings $\theta^1_{R,L}(w^i)$ is maximal when $f^*_{r_i}=0$.

Proof. Indeed, if $f^*_{r_i}=0$, then $f^I_{r_i}=f_i^{\max}$, which means that the reduced semisheaf of rings $\theta^{*1}_{R,L}(w^i)$ has been completely transformed into the complementary semisheaf of rings $\theta^1_{I_{R;L}}(w^i)$. ■

Proposition 2.1.9 *Let $\theta_{R,L}(f_{R,L}) = \theta^m_{R,L}(s_{R,L}) \times \theta^r_{R,L}(s_{R,L})$ be the versal deformation of the semisheaf $\theta^m_{R,L}(s_{R,L})$ having $\theta_{R,L}[R_w] = \sum\limits_{i=1}^{r} \theta^{m-1}_{R,L}(a_i) \times \theta^1_{R,L}(w^i)$ as quotient algebra.*
Then, there exists a family of isomorphisms $\Pi_s(f^*_{r_{1_{R,L}}}, \cdots, f^*_{r_{i_{R,L}}}, \cdots, f^*_{r_{r_{R,L}}})$ given by:

$$\Pi_s(f^*_{r_{1_{R,L}}}, \cdots, f^*_{r_{i_{R,L}}}, \cdots, f^*_{r_{r_{R,L}}}) : \theta^m_{R,L}(s_{R,L}) \times \theta^r_{R,L}(s_{R,L}) \to \theta^m_{R,L}(s_{R,L}) \times \theta'_{R,L}(s_{R,L})$$
$$\bigcup \quad \{(\theta^1_{I_{R,L}}(w^1))_{f^I_{r_{1_{R,L}}}}, \cdots, (\theta^1_{I_{R,L}}(w^i))_{f^I_{r_{i_{R,L}}}}, \cdots, (\theta^1_{I_{R,L}}(w^r))_{f^I_{r_{r_{R,L}}}}\}\ , \quad 1 \leq i \leq r\ ,$$

where:

a) $(\theta^1_{I_{R,L}}(w^i))_{f^I_{r_{i_{R,L}}}}$ is the i-th complementary semisheaf having global residue degree set $f^I_{r_{i_{R,L}}}$ generated by the smooth endomorphism $E_{w^i_{R,L}}$ from the semisheaf $(\theta^1_{R,L}(w^i))_{f_i}$ having global residue degree set f_i .

b) $\theta^m_{R,L}(s_{R,L}) \times \theta'_{R,L}(s_{R,L}) = (\theta^m_{R,L}(s_{R,L}) \times \theta^r_{R,L}(s_{R,L}))-$
$$\{(\theta^1_{I_{R,L}}(w^1))_{f^I_{r_{1_{R,L}}}}, \cdots, (\theta^1_{I_{R,L}}(w^i))_{f^I_{r_{i_{R,L}}}}, \cdots, (\theta^1_{I_{R,L}}(w^r))_{f^I_{r_{r_{R,L}}}}\}\ ,\ 1 \leq i \leq r\ .$$

Proof.

1) This proposition is a generalization of proposition 2.1.7 in such a way that the smooth endomorphism $E_{w^i_{R,L}}$, generating $(f^{\max}_i - 1)$ pairs of semisheaves of the category $c(\theta^1_{w^i_{R,L}})$, is extended to all the base semisheaves $\theta^1_{R,L}(w^i_{R,L})$, $1 \leq i \leq r$, of the versal deformation.

2) The family of endomorphisms $\pi_s(f^*_{r_{1_{R,L}}}, \cdots, f^*_{r_{i_{R,L}}}, \cdots, f^*_{r_{r_{R,L}}})$ is such that:

   - $f^I_{r_{1_{R,L}}}$ irreducible subschemes of rank N are disconnected from the base semisheaf $\theta^1_{R,L}(w^1_{R,L})_{f^{\max}_1}$ on $\theta^m_{R,L}(s_{R,L})$;

   - $f^I_{r_{i_{R,L}}}$ irreducible subschemes of rank N are disconnected from the base semisheaf $\theta^1_{R,L}(w^i_{R,L})_{f^{\max}_i}$ on $\theta^m_{R,L}(s_{R,L})$;

   - and so on, $1 \leq i \leq r$.

3) The set of complementary residue degrees $(f^I_{r_{1_{R,L}}}, \cdots, f^I_{r_{i_{R,L}}}, \cdots, f^I_{r_{r_{R,L}}})$ varies in such a way that:
$$1 \leq f^I_{r_{1_{R,L}}} \leq f^{\max}_1\ ,\ \cdots\ ,\ 1 \leq f^I_{r_{R,L}} \leq f^{\max}_r$$
implying for each set $(f^I_{r_{1_{R,L}}}, \cdots, f^I_{r_{r_{R,L}}})$ a family of isomorphisms $\pi_s(f^*_{r_{1_{R,L}}}, \cdots, f^*_{r_{r_{R,L}}})$.

4) $\theta'_{R,L}(s_{R,L})$ is the residue base semisheaf resulting from the disconnection of the set $\{\theta^1_{I_{R,L}}(w^1)_{f^I_{r_{1_{R,L}}}}, \cdots, \theta^1_{R,L}(w^r)_{f^I_{r_{r_{R,L}}}}\}$. ■

Corollary 2.1.10 *The family of isomorphisms*

$$\Pi^{\max(i)}_s(f^*_{r_{1_{R,L}}}, \cdots, f^*_{r_{r_{R,L}}}) : \theta^m_{R,L}(s_{R,L}) \times \theta^r_{R,L}(s_{R,L}) \to \theta^m_{R,L}(s_{R,L}) \times \theta'_{R,L}(s_{R,L})$$
$$\bigcup \quad \{(\theta^1_{I_{R,L}}(w^1))_{f^I_{r_{1_{R,L}}}}, \cdots, (\theta^1_{I_{R,L}}(w^i))_{f^I_{r_{i_{R,L}}}}, \cdots, (\theta^1_{I_{R,L}}(w^r))_{f^I_{r_{r_{R,L}}}}\}$$

is maximal in the i-th semisheaf $(\theta^1_{R,L}(w^i)$ if $f^*_{r_{i_{R,L}}} = 0$.

Proof. If $f^{*}_{r_{i_{R,L}}} = 0$, the i-th semisheaf $\theta^1_{R,L}(w^i)$ has been completely transformed into its complementary disconnected semisheaf $\theta^1_{I_{R,L}}(w^i)$. Indeed, we have that: $f^{I}_{r_{I_{R,L}}} = f^{\max}_i$ if $f^{*}_{r_{I_{R,L}}} = 0$. ∎

Corollary 2.1.11 *The family of isomorphisms Π_s is maximal if:*

$$\Pi^{\max}_s : \theta^m_{R,L}(s_{R,L}) \times \theta^r_{R,L}(s_{R,L})) \to \theta^m_{R,L}(s_{R,L}) \bigcup \{(\theta^1_{I_{R,L}} w^1)_{f^{\max}_1}, \cdots , (\theta^1_{I_{R,L}}(w^r))_{f^{\max}_r}\} ,$$

i.e. if $f^{I}_{r_{i_{R,L}}} = f_i$, $\forall\ i$, $1 \leq i \leq r$.

Proof. Indeed, $f^{I}_{r_{i_{R,L}}} = f^{\max}_i$, if $f^{*}_{r_{i_{R,L}}} = 0$, $\forall\ i$. In this case, all the semisheaves $\theta^1_{R,L}(w^i)$, $1 \leq i \leq r$, of the quotient algebra have been disconnected from $\theta_{R,L}(f_{R,L})$. Consequently, $\theta_{R,L}(f_{R,L}) = \theta^m_{R,L}(s_{R,L}) \times \theta^r_{R,L}(s_{R,L})$ reduces to the semisheaf $\theta^m_{R,L}(s_{R,L})$. ∎

Definition 2.1.12 (Category of vertical tangent vector bundles) Let

$$T_{V_w} = \{T_{V_{w_1}}, \cdots , T_{V_{w_i}}, \cdots , T_{V_{w_r}}\}$$

denote the family of tangent vector bundles obtained by considering the projection of all complementary semisheaves $(\theta^1_{I_{R,L}}(w^i))_{f^{I}_{r_{i_{R,L}}}}$, $\forall\ i$, $1 \leq i \leq r$, in the vertical tangent spaces $T_{V_{w_i}}$ characterized by the normal vector fields $\overrightarrow{w}_i$.

Let $\tau_{V_{w_i}}$ be the proper projective map of the tangent vector bundle $T_{V_{w_i}}$:

$$\tau_{V_{w_i}} : T_{V_{w_i}}(\theta^1_{I_{R,L}}(w^i))_{f^{I}_{r_{i_{R,L}}}} \to (\theta^1_{I_{R,L}}(w^i))_{f^{I}_{r_{i_{R,L}}}}$$

so that $\tau_{V_w} = \{\tau_{V_{w_i}}\}^r_{i=1}$.
To the category $c(\theta^1_{I_{R,L}}(w^i))$ will then correspond the category $c(T_{V_{w_i}}(\theta^1_{I_{R,L}}(w^i))$ of sections of tangent vector bundles.

Proposition 2.1.13 *The extension of the quotient algebra of the versal deformation of the semisheaf $\theta^m_{R,L}(s_{R,L})$ having an isolated singularity of order p in each section $s_{\mu R,L}$ is realized by the spreading-out isomorphism $SOT = (\tau_{V_w} \circ \Pi_s)$.*

Proof. Let I_{w_i} be the kernel of the normal vector bundle $T_{V_{w_i}}$.

Then, the exact sequence

$$0 \to I_{w_i} \to T_{V_{w_i}}(\theta^1_{I_{R,L}}(w^i))_{f^{I}_{r_{i_{R,L}}}} \xrightarrow{\tau_{V_{w_i}}} (\theta^1_{I_{R,L}}(w^i))_{f^{I}_{r_{i_{R,L}}}} \to 0$$

represents an extension of the complementary semisheaf $(\theta^1_{I_{R,L}}(w^i))_{f^{I}_{r_{i_{R,L}}}}$ by the kernel I_{w_i} . ∎

Proposition 2.1.14 *The spreading-out isomorphism SOT is locally stable if the complementary semisheaves $T_{V_{w_i}}(\theta^1_{I_{R,L}}(w^i))_{f^{I}_{r_{i_{R,L}}}}$, $\forall\ i$, $1 \leq i \leq r$, generated by SOT from $\theta_{R,L}(f_{R,L})$, are locally free semisheaves.*

Proof. If $T_{V_{w_i}}(\theta^1_{I_{R,L}}(w^i))_{f^I_{r_{i_{R,L}}}}$ has a singular ideal in the sense of definition 2.1.5, then it is not stable, taking into account that a semisheaf is locally stable if it is locally free. ∎

Proposition 2.1.15 *The maximal number of complementary semisheaves* $T_{V_{w_i}}(\theta^1_{I_{R,L}}(w^i))_{f^I_{r_{i_{R,L}}}}$ *generated by the spreading-out isomorphism* SOT *is equal to the codimension of the versal deformation of the semisheaf* $\theta^m_{R,L}(s_{R,L})$.

Remark 2.1.16 Let us recall that all the singularities of generic wave fronts in spaces of dimension ≤ 7 are locally diffeomorphic to the A_{p-1} and D_{p-1} singularities [Arn1], [Arn2] whose simple genotypes in $I\!R^m \to I\!R$ have the normal forms [Mil1]:

$$\begin{aligned} A_{p-1} &\ :\ x^p + Q\ , \\ D_{p-1} &\ :\ x^2 y + y^{p-2} + Q' \end{aligned}$$

where Q and Q' are nondegenerated quadratic forms respectively of $(m-1)$ and $(m-2)$ variables.

As we are concerned in this work essentially with spaces of dimension 3, the only singularities to be considered are the corank one (i.e. with $n = 1$) singularities A_{p-1} and the corank two (i.e. with $n = 2$) singularities D_{p-1} .

Definitions 2.1.17 (Corank 2 singularities) 1. The Malgrange preparation theorem can be generalized to germs of differentiable functions $s_{\mu R,L} \in \theta^m_{R,L}(s_{R,L})$ having an isolated singularity of corank 2. Indeed, if a germ $s_{\mu R,L}$ has singularities of corank 2 and order p in the two indeterminates w_1 and w_2 , then:

a) $s_{\mu R,L}(0, w_1, w_2) = P_{\mu R,L}(w_1, w_2) \cdot e_{\mu R,L}(w_1, w_2)$ where $e_{\mu R,L}(w_1, w_2)$ is a differentiable unit and $P_{\mu R,L}(w_1, w_2)$ is a Weierstrass polynomial of degree p .

b) the quotient algebra of the Malgrange preparation theorem is a finitely generated tensoriel space of type $(0, 2)$ and dimension $r < p$.

2. The spreading-out isomorphism SOT can clearly be applied to the versal deformation of a semisheaf of germs of differentiable functions having singularities of corank 2 because the proposition 2.1.10 can be generalized to this case.

Definition 2.1.18 (Gluing-up of complementary semisheaves) The category of complementary semisheaf direct products $\{T_{V_{w_i}}(\theta^1_{I_{R,L}}(w^i))\}_{i=1}^r$ does not necessarily cover in a compact way the semisheaf $\theta^m_{R,L}(s_{R,L}) \times \theta'_{R,L}(s_{R,L})$ generated from the semisheaf $\theta^m_{R,L}(s_{R,L})$ by versal deformation and spreading-out isomorphism SOT . However, we can define a gluing-up of semisheaves $T_{V_{w_i}}(\theta^1_{I_{R,L}}(w^i))_{f^I_{r_{i_{R,L}}}}$ which are above the semisheaf $\theta^m_{R,L}(s_{R,L}) \times \theta'_{R,L}(s_{R,L})$.

As the sections of the semisheaves $T_{V_{w_i}}(\theta^1_{I_{R,L}}(w^i)_{f^I_{r_{i_{R,L}}}}$ are constituted of normal crossings divisors having a rank $n_{w^i_\mu} = (h_\mu \cdot N)^p$, with $p \geq m-1$, according to lemma 2.1.4, we can say that the dimension of these semisheaves is approximatively equal to m .

In this perspective, let us denote the i-th and the j-th complementary semisheaves by $(\theta_i^m(D(w_i))_{R,L}$ and by $(\theta_j^m(D(w_j))_{R,L}$ defined respectively on the domains $D(w_i)$ and $D(w_j)$. Consider then the gluing-up of these two complementary semisheaves in the following manner:

For each pair (i,j) , let Π_{ij} be an isomorphism from $\theta_j^m(D(w_i)\cap D(w_j))_{R,L}$ to $\theta_i^m(D(w_i)\cap D(w_j))_{R,L}$. Then, there exists a semisheaf $\theta^m(D(w_{i-j}))_{R,L}$ defined on the connected domain $D(w_{i-j})=D(w_i)\cup D(w_j))$ and an isomorphism n_i from $\theta^m(D(w_i))_{R,L}$ to $\theta_i^m(D(w_i))_{R,L}$ such that $\Pi_{ij}=n_i\circ n_j^{-1}$ in each point of $(D(w_i)\cap D(w_j))$, $\forall\ i$, $1\leq i\leq r$: this is an adapted version of a proposition of J.P. Serre [Ser1].

So, $\theta^m(D(w_{i-j}))_{R,L}$ is the semisheaf corresponding to the gluing-up of the semisheaves $\theta_i^m(D(w_i))_{R,L}$ and $\theta_j^m(D(w_j))_{R,L}$. It is then possible to envisage the gluing-up of some complementary semisheaves or of the complete category of these complementary semisheaves covering then by patches [Tho2] the semisheaf $\theta_{R,L}^m(s_{R,L})\times\theta'_{R,L}(s_{R,L})$.

Definition 2.1.19 (Sequence of spreading-out isomorphisms) Let $\theta^m_{SOT(1)_{R,L}}$ denote the family of complementary semisheaves $\{\theta_i^m(D(w_i))_{R,L}\}_{i=1}^p$, $p\leq r$, covering $\theta_{R,L}^m(s_{R,L})\times\theta'_{R,L}(s_{R,L})$ some of which can be glued together. $\theta^m_{SOT(1)_{R,L}}$ is thus generated by the spreading-out isomorphism $SOT(1)$.

According to definition 2.1.5, the germs $w^i_{\mu R,L}\in\theta^1_{R,L}(w^i)\subseteq\theta_i^m(D(w_i))_{R,L}$ can be characterized by isolated degenerated singular points.

Consequently, a versal deformation of $\theta^m_{SOT(1)_{R,L}}$ can be envisaged followed by a spreading-out isomorphism $SOT(2)$. The resulting family of complementary semisheaves will then be noted $\theta^m_{SOT(2)_{R,L}}$.

If we abbreviate $\theta_{R,L}^m(s_{R,L})\times\theta_{R,L}^r(s_{R,L})$ by $\theta_{R,L}(s,s)_{R,L}$, we have the exact sequence:

$$\begin{aligned}
\theta_{R,L}^m(s)_{R,L} \quad &\xrightarrow{Vd(1)} \quad \theta_{R,L}(s,s_1)_{R,L}\\
&\xrightarrow{SOT(1)} \quad (\theta_{R,L}(s,s_1)_{R,L})'\bigcup\theta^m_{SOT(1)_{R,L}}\\
&\xrightarrow{Vd(2)} \quad (\theta_{R,L}(s,s_1)_{R,L})'\bigcup\theta_{R,L}(s_{SOT(1)},s_2)_{R,L}\\
&\xrightarrow{SOT(2)} \quad (\theta_{R,L}(s,s_1)_{R,L})'\bigcup(\theta_{R,L}(s_{SOT(1)},s_2)_{R,L})'\bigcup\theta^m_{SOT(2)_{R,L}}
\end{aligned}$$

where

a) $Vd(1)$ and $Vd(2)$ denote the two successive versal deformations;

b) the versal deformation $Vd(2)$ of the semisheaf $\theta^m_{SOT(1)_{R,L}}$ gives the semisheaf $\theta_{R,L}(s_{SOT(1)},s_2)_{R,L}$ in such a way that the dimension of its quotient algebra q verifies $q<r$ where r is the dimension of the quotient algebra of $Vd(1)$.

Proposition 2.1.20 *A sequence of maximum two successive spreading-out isomorphisms can be envisaged from a given semisheaf of germs of differentiable functions $\theta_{R,L}^m(s)_{R,L}$ where $m\leq 3$.*

Proof. As $m\leq 3$, the corank " *ck* " of the degenerated singularities on $\theta_{R,L}^m(s)_{R,L}$ is $ck\leq 2$ according to the remarks 2.1.16 and the codimension " *cd* " of the versal deformation of $\theta_{R,l}^m(s)_{R,L}$ is $cd\leq 3$. Consequently, the possible degenerated singularities on the family of complementary semisheaves $\theta^m_{SOT(1)_{R,L}}$, obtained from $\theta_{R,L}^m(s)_{R,L}$ by the $(SOT(1)\circ Vd(1))$ morphism, have a codimension $cd\leq 2$. Thus, one and only one supplementary $(SOT(2)\circ Vd(2))$ morphism can be envisaged from the semisheaf $\theta^m_{SOT(1)_{R,L}}$. ∎

2.2 The three embedded structures of semiparticles

The aim of this section is to prove that the algebraic structure of semiparticles is composed of three embedded semisheaves of rings whose most internal is the space-time semisheaf of rings studied in chapter 1.

If singularities are generated on this space-time semisheaf of rings, then a sequence of maximum two successive spreading-out isomorphisms consecutive to versal deformations can be considered leading to the generation of two embedded semisheaves of rings covering the fundamental space-time semisheaf of rings.

The developments will be made for semileptons because they are easier to handle but they are also valid for semibaryons.

Definition 2.2.1 (Extension of the quotient algebra) Consider the $4D$-space-time semisheaf of rings $(\theta^{*1}_{R,L}(t)\oplus\theta^3_{I_{R,L}}(r))_{ST}$, noted " ST " for space-time, whose q sections are differential functions isomorphic to

$$(T^{*1}_\mu(t)_{R,L}\oplus T^1_{I_\mu}(r)_{R,L})\;,\qquad 1\le\mu\le q\;.$$

Assume that under some external perturbations:

a) all the sections $s^{*1}_\mu(t)_{R,L}\in\theta^{*1}_\mu(t)$ are endowed with the same isolated singularities of corank 1 and codimension $r\le 3$.

b) or/and that all the complementary $3D$-sections $s_{I_\mu}(r)_{R,L}\in\theta^3_{I_{R,L}}(r)$ have the same isolated singularities of corank $ck\le 2$ and codimension $r\le 3$.

According to proposition 2.1.9, the versal deformation of the q sections $s^{*1}_\mu(t)_{R,L}\in\theta^{*1}_{R,L}(t)$ defines the quotient algebra

$$\theta_{R,L}[R_w]=\sum_{i=1}^r\theta^0_{R,L}(a_i)\times\theta^1_{R,L}(w^i)$$

where $\theta^0_{R,L}(a_i)$ is a constant semisheaf and where $\theta^1_{R,L}(w^i)$ is the i-th generator semisheaf of the versal unfolding, when the versal deformation of the semisheaf of rings $\theta^3_{I_{R,L}}(r)$ having singularities of corank 1 gives rise to the quotient algebra

$$\theta^2_{R,L}[R_w]=\sum_{i=1}^r\theta^2_{R,L}(b_i)\times\theta^1_{R,L}(w^i)\;.$$

But, if the singularities are of corank 2 on $\theta^3_{I_{R,L}}(r)$, the quotient algebra of the versal deformation will be

$$\theta^1_{R,L}[R_{w_1,w_2}]=\sum_{i,j=1}^r\theta^1_{R,L}(c_i)\times\theta^2_{R,L}(w_{1i},w_{2j})\;.$$

Then, the extension of the quotient algebra of the versal deformation realized by the spreading-out isomorphism generates:

a) for the semisheaf of rings $\theta^{*1}_{R,L}(t)_{ST}$ the category $c(T_{w_i}(\theta^1_{R,L}(w^i)))$ of sections of vertical tangent bundles.

b) for the semisheaf of rings $\theta^3_{I_{R,L}}(r)_{ST}$ having singularities of corank 1 the category $c(T_{w_i}(\theta^1_{R,L}(w^i))))$ of sections of vertical tangent bundles and for the semisheaf of rings $\theta^3_{I_{R,L}}(r)_{ST}$ having singularities of corank 2 the category $c(T_{w_{ij}}(\theta^2_{R,L}(w_{1i},w_{2j})))$ of sections of vertical tangent bundles.

Definition 2.2.2 (Covering the spreading-out isomorphism) Let $\{\theta^1_i(D(w_i))_{R,L}\}^p_{i=1}$ (resp. $\{\theta^3_i(D(w_i))_{R,L}\}^p_{i=1}$) denote a family, i.e. with $p \leq r$, of the category $c(T_{w_i}(\cdots))$ (resp. $c(T_{w_i}(\cdots))$ of vertical tangent bundle sections which are complementary semisheaves.

Assume that $\theta^1_{R,L}(t_{res})_{ST}$ (resp. $\theta^{3'}_{R,L}(r_{res})_{ST}$), being the residue semisheaf after the versal deformation and the spreading-out isomorphism $SOT(1) \circ Vd(1)$ (resp. $SOT(1') \circ Vd(1')$) (see definition 2.1.19) of $\theta^{*1}_{R,L}(t)_{ST}$ (resp. $\theta^{3'}_{I_{R,L}}(r)_{ST}$), is partially covered [Ful] by the semisheaf $\theta^1_{R,L}(t)_{MG}$ (resp. $\theta^{3'}_{R,L}(r)_{MG}$) denoting one family $\{\theta^1_i(D(w_i))_{R,L}\}^p_{i=1}$ (resp. $\{\theta^3_i(D(w_i))_{R,L}\}^p_{i=1}$) or several families

$$\{\{\theta^1_i(D(w_i))_{R,L}\}^p_{i=1},\cdots,\{\theta^1_k(D(w_k))_{R,L}\}^s_{k=1}\}$$

$$(\text{resp.}\quad \{\{\theta^3_i(D(w_i))_{R,L}\}^p_{i=1},\cdots,\{\theta^3_k(D(w_k))_{R,L}\}^s_{k=1}\}\ )$$

where p and s are inferior or equal to the respective dimensions of the versal deformations.

The covering by several families of semisheaves must be considered because every section $s^{*1}_{\mu}(t)_{R,L} \in \theta^{*1}_{R,L}(t)_{ST}$ (resp. $s'_{I_\mu}(r)_{R,L} \in \theta^{3'}_{I_{R,L}}(r)_{ST}$) can have several isolated degenerated singularities. If there is a covering by one or several families of semisheaves, some of these semisheaves can be glued together according to definition 2.1.18.

Definition 2.2.3 (Embedded semisheaves of rings) Let $\theta^1_{R,L}(t)_{MG}$ (resp. $\theta^{3'}_{R,L}(r)_{MG}$) be the semisheaf covering partially $\theta^1_{R,L}(t_{res})_{ST}$ (resp. $\theta^{3'}_{R,L}(r_{res})_{ST}$) where " MG " is the abbreviated form for "middle ground". According to proposition 2.1.20, if the codimension of the degenerated singularities on the semisheaves $\theta^{*1}_{R,L}(t)_{ST}$ (resp. $\theta^{3'}_{I_{R,L}}(r)_{ST}$) is superior or equal to 3 , a versal deformation and a spreading-out isomorphism $(SOT(2)\circ Vd(2))$ (resp. $(SOT(2')\circ Vd(2'))$) can be envisaged from $\theta^1_{R,L}(t)_{MG}$ (resp. $\theta^{3'}_{R,L}(r)_{MG}$) leading to a semisheaf $\theta^1_{R,L}(t)_M$ (resp. $\theta^{3'}_{R,L}(r)_M$) covering partially the residue semisheaf $\theta^1_{R,L}(t_{res})_{MG}$ (resp. $\theta^{3'}_{R,L}(r_{res})_{MG}$), where " M " is the abbreviated form for "mass".

If the semisheaves $\theta^1_{R,L}(t_{res})_{MG}$ and $\theta^1_{R,L}(t)_M$ (resp. $\theta^{3'}_{R,L}(r_{res})_{MG}$ and $\theta^{3'}_{R,L}(r)_M$) are generated by versal deformation and spreading-out isomorphism, then the corresponding complementary semisheaves $\theta^3_{I_{R,L}}(r)_{MG}$ and $\theta^3_{I_{R,L}}(r)_M$ (resp. $\theta^{1'}_{I_{R,L}}(t)_{MG}$ and $\theta^{1'}_{I_{R,L}}(t)_M$) can be generated respectively from $\theta^1_{R,L}(t_{res})_{MG}$ and $\theta^1_{R,L}(t)_M$ (resp. $\theta^{3'}_{R,L}(r_{res})_{MG}$ and $\theta^{3'}_{R,L}(r)_M$) by a $(\gamma_{t\to r}\circ E)$ morphism (resp. a $(\gamma_{r\to t}\circ E')$ morphism according to proposition 1.2.6 and corollary 1.2.8.

The sequence of the two successive versal deformations and spreading-out isomorphisms from the $1D$-semisheaf of rings $\theta^1_{R,L}(t)_{ST}$ are summarized in the two following diagrams A) and B):

A)

$$\begin{array}{ccccc}
\theta^1_{R,L}(t)_{ST} & \xrightarrow{SOT(1)\circ Vd(1)} & \theta^1_{R,L}(t_{res})_{ST}\cup\theta^1_{R,L}(t)_{MG} & \xrightarrow{SOT(2)\circ Vd(2)} & \theta^1_{R,L}(t_{res})_{ST}\cup\theta^1_{R,L}(t_{res})_{MG}\cup\theta^1_{R,L}(t)_M \\
\Big\downarrow{\scriptstyle \gamma^{ST}_{t\to r}\circ E} & & \Big\downarrow{\scriptstyle \gamma^{MG}_{t\to r}\circ E} & & \Big\downarrow{\scriptstyle \gamma^{M}_{t\to r}\circ E} \\
\theta^3_{I_{R,L}}(r)_{ST} & & \theta^3_{I_{R,L}}(r_{res})_{ST}\cup\theta^3_{I_{R,L}}(r)_{MG} & & \theta^3_{I_{R,L}}(r_{res})_{ST}\cup\theta^3_{I_{R,L}}(r_{res})_{MG}\cup\theta^3_{I_{R,L}}(r)_M
\end{array}$$

B)

$$\begin{tikzcd}
\theta^1_{R,L}(t)_{ST} \arrow[d, "\gamma^{ST}_{t\to r}\circ E'"] & \theta^{1'}_{I_{R,L}}(t_{res})_{ST} \cup \theta^{1'}_{I_{R,L}}(t)_{MG} & \theta^{1'}_{I_{R,L}}(t_{res})_{ST} \cup \theta^{1'}_{I_{R,L}}(t_{res})_{MG} \cup \theta^{1'}_{I_{R,L}}(t)_{M} \\
\theta^{3'}_{I_{R,L}}(r)_{ST} \arrow[r, "SOT(1')\circ Vd(1')"] & \theta^{3'}_{R,L}(r_{res})_{ST} \cup \theta^{3'}_{R,L}(r)_{MG} \arrow[u, "\gamma^{MG}_{r\to t}\circ E'"] \arrow[r, "SOT(2')\circ Vd(2')"] & \theta^{3'}_{R,L}(r_{res})_{ST} \cup \theta^{3'}_{R,L}(r_{res})_{MG} \cup \theta^{3'}_{R,L}(r)_{M} \arrow[u, "\gamma^{M}_{r\to t}\circ E'"]
\end{tikzcd}$$

leading to the two sets of three embedded semisheaves of rings for the diagram A):

$$\theta^1_{R,L}(t_{res})_{ST} \subset \theta^1_{R,L}(t_{res})_{MG} \subset \theta^1_{R,L}(t)_{M}$$
$$\theta^3_{I_{R,L}}(r_{res})_{ST} \subset \theta^3_{I_{R,L}}(r_{res})_{MG} \subset \theta^3_{I_{R,L}}(r)_{M}$$

in the sense that there is a topological embedding for all their q sections, $1 \leq \mu \leq q$, i.e.:

$$\begin{array}{ccccc}
s_{\mu R,L}(t_{res})_{ST} & \subset & s_{\mu R,L}(t_{res})_{MG} & \subset & s_{\mu R,L}(t)_{M} \\
s_{\mu R,L}(r_{res})_{ST} & \subset & s_{\mu R,L}(r_{res})_{MG} & \subset & s_{\mu R,L}(r)_{M}
\end{array}$$

where $s_{\mu R,L}(t_{res})_{ST} \in \theta^1_{R,L}(t_{res})_{ST}$ and so on.

With respect to the diagram B), we have the corresponding embedding of semisheaves of rings:

$$\theta^{1'}_{I_{R,L}}(t_{res})_{ST} \subset \theta^{1'}_{I_{R,L}}(t_{res})_{MG} \subset \theta^{1'}_{I_{R,L}}(t)_{M}$$
$$\theta^{3'}_{R,L}(r_{res})_{ST} \subset \theta^{3'}_{R,L}(r_{res})_{MG} \subset \theta^{3'}_{R,L}(r)_{M}$$

Proposition 2.2.4 *The semisheaves of rings $\theta^3_{I_{R,L}}(r_{res})_{ST} \cup \theta^3_{I_{R,L}}(r)_{MG}$, generated by the morphism $\gamma^{MG}_{t\to r} \circ E \circ SOT(1) \circ Vd(1))$ from $\theta^1_{R,L}(t)_{ST}$, may be isomorphic to the semisheaves of rings $\theta^{3'}_{R,L}(r_{res})_{ST} \cup \theta^{3'}_{R,L}(r)_{MG}$, generated by the morphism $(SOT(1') \circ Vd(1') \circ \gamma^{ST}_{t\to r} \circ E')$ from $\theta^1_{R,L}(t)_{ST}$ if and only if:*

a) singularities of corank 1 and of the same codimension are at the origin of the versal deformations $Vd(1)$ and $Vd(1')$;

b) the singularities on the semisheaves $\theta^1_{R,L}(t_{res})_{ST} \cup \theta^1_{R,L}(t)_{MG}$ are conserved under the morphism $\gamma^{MG}_{t\to r} \circ E$.

Proof. By hypothesis, only singularities of corank 1 and of the same codimension are taken into account in the versal deformations $Vd(1)$ and $Vd(1')$: this is justified physically by the fact that the same kind of perturbation must be envisaged on the semisheaf $\theta^1_{R,L}(t)_{ST}$ for the versal deformation $Vd(1)$ and on the complementary semisheaf $\theta^{3'}_{I_{R,L}}(t)_{ST}$ for the versal deformation $Vd(1')$.

The quotient algebra of the versal deformation $Vd(1)$ is

$$\theta_{R,L}[R_w] = \sum_{i=1}^{r} \theta^0_{R,L}(a_i) \times \theta^1_{R,L}(w^i)$$

while it is

$$\theta^2_{R,L}[R_w] = \sum_{i=1}^{r} \theta^2_{R,L}(b_i) \times \theta^1_{R,L}(w^i)$$

for the versal deformation $Vd(1')$ according to definition 2.2.1.

Now, under the hypothesis of the proposition, the generator semisheaves $\theta^1_{R,L}(w^i)$ in $\theta_{R,L}[R_w]$ and in $\theta^2_{R,L}[R_w]$ are composed of time quanta, i.e. time prime ideals of rank N . These time quanta are composed of the same number of geometric points as the time quanta of the semisheaf $\theta^1_{R,L}(t)_{ST}$ because the semisheaves $\theta^1_{R,L}(w^i)$ and $\theta^1_{R,L}(t)_{ST}$ are supposed to be generated by Eisenstein cohomology from $1D$-time symmetric splitting semifields (see definition 1.1.2).

Considering that the semisheaf $\theta^2_{R,L}(b_i) \in \theta^2_{R,L}[R_w]$ is a stratum semisheaf of $\theta^{3'}_{I_{R,L}}(r)_{ST}$, we can admit that we reach the thesis since the $\gamma_{t\to r}$ morphism is a morphism essentially based on the inverse Kronecker's specialization [Lan1] such that a ring of irreducible polynomials in n variables can be extended to a ring of irreducible polynomials in m variables, where $n<m$.

Or, more directly, we have seen that the $3D$ semisheaves of rings $\theta^{3'}_{R,L}(r_{\rm res})_{ST}$ and $\theta^{3'}_{R,L}(r_{\rm res})_{MG}$ degenerate into $1D$-semisheaves of rings according to proposition 1.2.6. ∎

Corollary 2.2.5 *The semisheaves of rings $\theta^3_{I_{R,L}}(r_{res})_{MG} \cup \theta^3_{I_{R,L}}(r)_M$ may be isomorphic to the semisheaves of rings $\theta^{3'}_{R,L}(r_{res})_{MG} \cup \theta^{3'}_{R,L}(r)_M$ if and only if*

a) singularities of corank 1 and of the same codimension are at the origin of the versal deformations $Vd(2)$ and $Vd(2')$;

b) the singularities on the semisheaves $\theta^1_{R,L}(t_{res})_{MG} \cup \theta^1_{R,L}(t)_M$ are conserved under the mosphism $\gamma^M_{t\to r} \circ E$.

Proposition 2.2.6 *Let $M^I_{R,L}(t)_{ST} \in \theta^1_{R,L}(t_{res})_{ST}$, $M^I_{R,L}(t)_{MG} \in \theta^1_{R,L}(t_{res})_{MG}$ and $M^I_{R,L}(t)_M \in \theta^1_{R,L}(t)_M$ be the time quanta on which the corresponding semisheaves of rings ST , MG and M are defined. Then, these time quanta have the same number of geometric points.*

Proof. A time quantum is a time submodule having a rank N . As the MG-time semisheaf $\theta^1_{R,L}(t)_{MG}$ is generated from the ST-$1D$-time semisheaf $\theta^1_{R,L}(t)_{ST}$ by the $(SOT(1)\circ Vd(1))$ morphism, it results that the MG-time quanta $M^I_{R,L}(t)_{MG}$ are submodules of the category $c(T_{aw_i}(\theta^0_{R,L}(a_i)\times \theta^1_{R,L}(w^i))$ according to definition 2.2.1.

Now, as the semisheaf $\theta^1_{R,L}(w^i)$ is generated on and from irreducible completions $L_{v^1_\mu}$, i.e. quanta according to section 1.1.4, we have that $M^I_{R,L}(t)_{ST}$ has the same number of geometric points as $M^I_{R,L}(t)_{MG}$.

Similarly, we can prove that $M^I_{R,L}(t)_{MG}$ has the same number of geometric points as $M^I_{R,L}(t)_M$. ∎

Corollary 2.2.7 *If $M^I_{R,L}(r)_{ST} \in \theta^3_{R,L}(r)_{ST}$, $M^I_{R,L}(r)_{MG} \in \theta^3_{R,L}(r)_{MG}$ and $M^I_{R,L}(r)_M \in \theta^3_{R,L}(r)_M$ are space quanta, then they have the same number of geometric points.*

Proof. As the space quanta are generated from the time quanta by the $\gamma_{t\to r}\circ E$ morphism and as the time quanta have the same number of geometric points, we have the thesis. ∎

Proposition 2.2.8 *The semisheaves of rings $\theta^1_{R,L}(t_{res})_{MG}$ and $\theta^1_{R,L}(t)_M$ as well as the semisheaves of rings $\theta^3_{R,L}(r_{res})_{MG}$ and $\theta^3_{R,L}(r)_M$ are not necessarily compact and Zariski dense in such a way that their sections are open strings.*

Proof. Indeed, by construction (see definition 2.2.2), these semisheaves $\theta^1_{R,L}(t_{\rm res})_{MG}$ and $\theta^1_{R,L}(t)_M$ (idem for $\theta^3_{R,L}(t_{\rm res})_{MG}$ and $\theta^3_{R,L}(t)_M$) cover partially by patches the semisheaf $\theta^1_{R,L}(t_{res})_{ST}$.

As the sections of $\theta^1_{R,L}(t_{\rm res})_{MG}$ and of $\theta^1_{R,L}(t)_M$ (resp. of $\theta^3_{R,L}(r_{\rm res})_{MG}$ and $\theta^3_{R,L}(r)_M$) are one-dimensional and cover partially the sections of $\theta^1_{R,L}(t_{\rm res})_{ST}$ (resp. $\theta^3_{R,L}(t_{\rm res})_{ST}$), they are open strings.

More precisely, as the coefficients $a_{i_\mu}(x)$ of the quotient algebra of the versal deformation of germs $s_{\mu_{R,L}}(w_{R,L})$ are (germs of) functions defined on domains $D_{\mu_{R,L}}$ included into half open balls $B_{\mu_{R,L}}$ whose radii increase in function of the global residue degrees of the sections $s_{\mu_{R,L}}$, the numbers of quanta $\widetilde{M}^I_{R,L}(t)_{MG}$ and $\widetilde{M}^I_{R,L}(t)_M$, covering the sections $s_{\mu_{R,L}}(t)_{ST}$ of the semisheaf $\theta^1_{R,L}(t_{\rm res})_{ST}$, increase according to the global residue degrees $f_{\mu_{ST}}$ of $s_{\mu_{R,L}}(t)_{ST}$. Thus, if $n_{M^I(t)_{MG}}(s_\mu)=f_{\mu_{MG}}$ and $n_{M^I(t)_{MG}}(s_{\mu+1})=f_{(\mu+1)_{MG}}$ denote the numbers of quanta, i.e. the corresponding global residue degrees, respectively of the μ-th and $(\mu+1)$-th sections of the semisheaf $\theta^1_{R,L}(t_{\rm res})_{MG}$ covering the corresponding sections of the semisheaf $\theta^1_{R,L}(t_{\rm res})_{ST}$, then $n_{M^I(t)_{MG}}(s_\mu)<n_{M^I(t)_{MG}}(s_{\mu+1})$.

And, $n_{M^I(t)_{ST}}(s_\mu)\approx n_{M^I(t)_{MG}}(s_\mu)$, i.e. that the number of quanta $n_{M^I(t)_{MG}}(s_\mu)$ of the μ-th section of $\theta^1_{R,L}(t_{\rm res})_{MG}$ is approximately equal to the number of quanta $n_{M^I(t)_{ST}}(s_\mu)$ of the μ-th section of $\theta^1_{R,L}(t_{\rm res})_{ST}$. ∎

Proposition 2.2.9 *Every semisheaf of rings $\theta^1_{R,L}(t_{res})_{ST}$, $\theta^1_{R,L}(t_{res})_{MG}$, $\theta^1_{R,L}(t)_M$, $\theta^3_{R,L}(r_{res})_{ST}$, $\theta^3_{R,L}(r_{res})_{MG}$ or $\theta^3_{R,L}(r)_M$ which is locally free corresponds to a Stein space.*

Proof. A sheaf of rings θ_P , defined on a closed subset P of a topological space X , is locally free if it has no degenerated singularity. Consequently, it cannot be submitted to a versal deformation and must satisfy the condition $H^q(P,\theta_P)=0$, $\forall\ q\geq 1$, [G-R3]. If this is the case, the sheaf of rings θ_P corresponds to a Stein space and is locally free. ∎

Definition 2.2.10 (Semialgebras on $\mathbb{A}_{R,L}$-semimodules) Let $\theta^{1-3}_{R,L}(t,r)=\theta^1_{R,L}(t)\oplus\theta^3_{R,L}(r)$ denote the direct sum of the $1D$- and $3D$-semisheaves of rings. As $\theta^{1-3}_{R,L}(t,r)_{ST}$, $\theta^{1-3}_{R,L}(t,r)_{MG}$ and $\theta^{1-3}_{R,L}(t,r)_M$ are semisheaves on semimodules over $\mathbb{A}_{R,L}$, they are semialgebras [F-D].

This leads us to the following proposition:

Proposition 2.2.11 *The semialgebras $\theta^{1-3}_{R,L}(t,r)_{ST}$, $\theta^{1-3}_{R,L}(t,r)_{MG}$ and $\theta^{1-3}_{R,L}(t,r)_M$ are commutative while the semialgebras*

$$\theta^{1-3}_{R,L}(t,r)_{ST}\oplus\theta^{1-3}_{R,L}(t,r)_{MG}\ ,$$

$$\theta^{1-3}_{R,L}(t,r)_{ST}\oplus\theta^{1-3}_{R,L}(t,r)_{MG}\oplus\theta^{1-3}_{R,L}(t,r)_M\ ,$$

$$\text{and}\quad \theta^{1-3}_{R,L}(t,r)_{MG}\oplus\theta^{1-3}_{R,L}(t,r)_M$$

extended from $\theta^{1-3}_{R,L}(t,r)_{ST}$ by versal deformation(s) and spreading-out isomorphism(s) are noncommutative.

Proof. 1. The semialgebra $\theta^{1-3}_{R,L}(t,r)_{ST}$ is commutative by construction (see chapter 1) since $\theta^1_{R,L}(t)_{ST}$ is generated by right (resp. left) Eisenstein cohomology from the $1D$-symmetric splitting semifields $L^\mp_\mu$.

The semialgebra $\theta^3_{R,L}(r)_{ST}$ is also commutative since it is generated by the $(\gamma_{t\to r}\circ E)$ morphism from $\theta^1_{R,L}(t)_{ST}$ according to proposition 1.2.6.

In fact, the semialgebras $\theta^{1-3}_{R,L}(t,r)_{ST}$, $\theta^{1-3}_{R,L}(t,r)_{MG}$ and $\theta^{1-3}_{R,L}(t,r)_{M}$ are commutative because it is possible for each one of these to define a unique centralizator.

2. A semialgebra extended by versal deformation and spreading-out isomorphism, for example $\theta^{1-3}_{R,L}(t,r)_{ST}\oplus\theta^{1-3}_{R,L}(t,r)_{MG}$, is noncommutative because it is impossible to define for it a unique centralizator. Indeed, the generator semisheaves $\theta_{R,L}(w^i)$ of the versal unfolding leading to the generation of the $1D$-extended semisheaf $\theta^1_{R,L}(t)_{MG}$ originate from the specialization prime ideals $p(w^i_{\mu R,L})$ (see definition 2.1.3) while the $1D$-semisheaf $\theta^1_{R,L}(t)_{ST}$ originates from specialization prime ideals $p_{\mu R,L}$ (see definition 1.1.3). As these specialization ideals $p(w^i_{\mu R,L})$ and $p_{\mu R,L}$ are not equal, we reach the thesis. ■

Definition 2.2.12 The emission quantification of the space-time, middle ground and mass structures of semiparticles can be envisaged by considering that these three embedded structures are constituted by the three embedded time semisheaves of rings $\theta^1_{R,L}(t_{res})_{ST}\cup\theta^1_{R,L}(t_{res})_{MG}\cup\theta^1_{R,L}(t)_{M}$, noted in abbreviated form $\theta^1_{R,L}(t)_{ST-MG-M}$, and by the three embedded space semisheaves $\theta^3_{R,L}(r)_{ST-MG-M}\equiv\theta^3_{R,L}(r_{res})_{ST}\cup\theta^3_{R,L}(r_{res})_{MG}\cup\theta^3_{R,L}(r)_{M}$.

Taking into account that the middle ground and mass semisheaves of rings are above the space-time semisheaves of rings, the middle ground and mass quanta will be above the space-time quanta. Consequently, a smooth endomorphism $E_{ST-MG-M}$, acting simultaneously on the three embedded semisheaves ST , MG and M , can be defined by:

$$E_{ST-MG-M}:\theta^1_{R,L}(t)_{ST-MG-M}\to\theta^{*1}_{I_{R,L}}(t)_{ST-MG-M}\bigoplus_{k=1}^{m}\widetilde{M}^I_{k_{R,L}}(t)_{ST-MG-M}$$

where $\widetilde{M}^I_{k_{R,L}}(t)_{ST-MG-M}=\widetilde{M}^I_{k_{R,L}}(t)_{ST}\cup\widetilde{M}^I_{k_{R,L}}(t)_{MG}\cup M^I_{k_{R,L}}(t)_{M}$ are three "disconnected" functions on time quanta from $\theta^1_{R,L}(t)_{ST-MG-M}$ so that $\widetilde{M}^I_{k_{R,L}}(t)_{MG}$ is above $\widetilde{M}^I_{k_{R,L}}(t)_{ST}$ and $\widetilde{M}^I_{k_{R,L}}(t)_{M}$ is above $\widetilde{M}^I_{k_{R,L}}(t)_{MG}$.

This smooth endomorphism $E_{ST-MG-M}$ then represents a three stratum time quantification of emission of semiparticles.

Similarly, a three stratum space quantification of emission would be introduced by applying a smooth endomorphism $E_{ST-MG-M}$ on $\theta^3_{R,L}(r)_{ST-MG-M}$ disconnecting space quanta $\widetilde{M}^I_{R,L}(r)_{ST-MG-M}$ from this space semisheaf.

Proposition 2.2.13 1. *The standard quanta of quantum field theory are the spatial left quanta* $M^I_L(r)_M\in\theta^3_L(r)_M$.

2. *The Planck constant h corresponds to the value of the integer N in the mass unit system, where N refers to the order of the global inertia subgroup.*

Proof. The quanta of quantum theories are the spatial left quanta $M^I_L(r)_M$ because only the mass structure of elementary particles is presently observable and corresponds to the left semisheaf of rings $\theta^3_L(r)_M$ of left semiparticles. However, right (resp. left) quanta are in fact spatial quanta $M^I_{k_{R,L}}(r)_{ST-MG-M}\in\theta^3_{R,L}(r)_{ST-MG-M}$.

On the other hand, according to axiom II 1.3.9, a "real" spatial quantum of an algebraic quantum theory must be a biquantum given by the product of a right and a left spatial quantum: $M^I_R(r)_{ST,MG,M}\times M^I_L(r)_{ST,MG,L}$.

2) The Planck constant, introduced by Planck in Physics to take into account the discontinued behavior of matter, must then correspond to the value of the integer N in the system of units of the algebraic mass semisheaf of rings. ▪

Definition 2.2.14 (Vertical tangent semibundles) The $1D$-time semisheaves of rings $\theta^1_{R,L}(t_{res})_{MG}$ and $\theta^1_{R,L}(t)_M$ as well as the $3D$-space semisheaves of rings $\theta^3_{R,L}(r_{res})_{MG}$ and $\theta^3_{R,L}(r)_M$, generated from the semisheaf $\theta^1_{R,L}(t)_{ST}$ by versal deformations, spreading-out isomorphisms and $(\gamma_{t\rightleftarrows r}\circ E)$ morphisms according to definition 2.2.3, are total spaces respectively of the $1D$-middle ground (resp. mass) vertical tangent bundle

$$\begin{aligned}
& T^{(1)}_{MG_{R,L}}(\theta^1_{R,L}(t_{res})_{MG},\theta^1_{R,L}(t_{res})^B_{MG},\tau^{(1)}_{V_{MG}}) \\
(\text{resp.}\quad & T^{(1)}_{M_{R,L}}(\theta^1_{R,L}(t)_M,\theta^1_{R,L}(t)^B_M,\tau^{(1)}_{V_M})\ )
\end{aligned}$$

and of the $3D$-middle ground (resp. mass) vertical tangent bundle

$$\begin{aligned}
& T^{(3)}_{MG_{R,L}}(\theta^3_{R,L}(r_{res})_{MG},\theta^3_{R,L}(r_{res})^B_{MG},\tau^{(3)}_{V_{MG}}) \\
(\text{resp.}\quad & T^{(3)}_{M_{R,L}}(\theta^3_{R,L}(r)_M,\theta^3_{R,L}(r)^B_M,\tau^{(3)}_{V_M})\ ),
\end{aligned}$$

where

a) $\theta^1_{R,L}(t_{res})^B_{MG}$ (resp. $\theta^3_{R,L}(r_{res})^B_{MG}$) is the basis of the vertical tangent bundle as resulting globally from the isomorphism Π_s (see proposition 2.1.10).

b) $\theta^1_{R,L}(t_{res})_{MG}$ (resp. $\theta^3_{R,L}(r_{res})_{MG}$) is the total space of the vertical tangent bundle

$$\begin{aligned}
& T^{(1)}_{MG_{R,L}}(\theta^1_{R,L}(t_{res})_{MG},\theta^1_{R,L}(t_{res})^B_{MG},\tau^{(1)}_{V_{MG}}) \\
(\text{resp.}\quad & T^{(3)}_{MG_{R,L}}(\theta^3_{R,L}(r_{res})_{MG},\theta^3_{R,L}(r_{res})^B_{MG},\tau^{(3)}_{V_{MG}})\ )
\end{aligned}$$

obtained by considering the projection of the complementary semisheaf direct products (i.e. $\theta^1_{R,L}(t_{res})^B_{MG}$ (resp. $\theta^3_{R,L}(r_{res})^B_{MG}$) in the vertical tangent space, according to definition 2.1.13.

c) $\tau^{(1)}_{V_{MG}}$ (resp. $\tau^{(3)}_{V_{MG}}$) is the projective map.

Definition 2.2.15 The generators of the $1D$- and $3D$-translation groups of the vertical tangent semibundles $T^{(1)}_{MG_{R,L}}$, $T^{(3)}_{MG_{R,L}}$, $T^{(1)}_{M_{R,L}}$ and $T^{(3)}_{M_{R,L}}$ are respectively given by the following elliptic differential operators:

$$\begin{aligned}
m_{0R,L;MG} &= \pm i\hbar_{MG}\,\frac{\partial}{\partial t_0}\ , \\
p_{R,L;MG} &= \left\{\pm i\frac{\hbar_{MG}}{c_{t\to r;MG}}\,\frac{\partial}{\partial x},\pm i\frac{\hbar_{MG}}{c_{t\to r;MG}}\,\frac{\partial}{\partial y},\pm i\frac{\hbar_{MG}}{c_{t\to r;MG}}\,\frac{\partial}{\partial z}\right\}\ ,
\end{aligned}$$

$$m_{0R,L;M} = \pm i\hbar_M \frac{\partial}{\partial t_0} ,$$

$$p_{R,L;M} = \left\{ \pm i \frac{\hbar_M}{c_{t\to r;M}} \frac{\partial}{\partial x}, \pm i \frac{\hbar_M}{c_{t\to r;M}} \frac{\partial}{\partial y}, \pm i \frac{\hbar_M}{c_{t\to r;M}} \frac{\partial}{\partial z} \right\} ,$$

where $\hbar_{MG}$ and $\hbar_M$ are constants corresponding to the integer N in the " MG " and " M " unit systems. $\hbar_M$ is the Planck constant $\hbar$: it recalls that the total spaces $\theta^1_{R,L}(t)_M$ and $\theta^3_{R,L}(r)$ respectively of the vertical tangent semibundles $T^{(1)}_{M_{R,L}}$ and $T^{(3)}_{M_{R,L}}$ are quantified since they are composed of mass quanta.

The constant $c^{-1}_{t\to r;M} = \langle c^{-1}_{t\to r}(\rho)\rangle$ refers to an average value of the quotient between algebraic Hecke characters according to propositions 1.4.8 and 1.4.11 and gives a measure of the transformation of the semisheaf $\theta^3_{R,L}(r)_M$ from the semisheaf $\theta^1_{R,L}(t)_M$. $c^{-1}_{t\to r;M} \approx c^{-1}$ where c is the light velocity.

An equivalent interpretation can be given to the constants $\hbar_{MG}$ and $c^{-1}_{t\to r;MG}$.

Definition 2.2.16 The elliptic differential operators $m_{0R,L}$ and $p_{R,L}$ can be directional gradients, Lie derivatives or covariant derivatives [B-G], [Kob]. The covariant derivative $\Delta_{\vec{V}_M}$ of a semisheaf, for example $\theta^3_{R,L}(r)_M$, along a vector field $\vec{V}_M$ is such that this semisheaf if parallely transported along a family of geodesics orthogonal to it with tangent vectors $\vec{V}_M$ [Del1], [H-E].

2.3 Phase spaces associated to the vibrations of the three embedded structures and the vacuum of Quantum Field Theory

Proposition 2.3.1 *To each $1D$- and $3D$-space-time (ST), middle ground (MG) and mass (M) semisheaf of rings corresponds a phase space which is homeomorphic to $\mathbb{R}^1 \times \mathbb{R}^1$ or $\mathbb{R}^3 \times \mathbb{R}^3$ and which has the structure of a F-Steenrod bundle whose basis is given by the considered semisheaf of rings.*

Proof. Let, for example, $\theta^3_{R,L}(r)_M$ be the $3D$-space-mass semisheaf of rings. Then, its associated F-Steenrod bundle is given by $(\theta^3_{R,L}(r,p)_M, \theta^3_{R,L}(r)_M, pr^{(3)}_M)$ where $\theta^3_{R,L}(r,p)_M$ is the total space and whose topological group is $GL(3,\mathbb{R})$. $\theta^3_{R,L}(r,p)_M = \theta^3_{R,L}(r)_M \times \theta^3_{R,L}(p)_M$ where the fiber $\theta^3_{R,L}(p)_M$ has a F-structure where $F = \mathbb{R}^3$. This $\mathbb{R}^3$-structure is given by a set of homeomorphisms $\mathbb{R}^3 \to \theta^3_{R,L}(p)_M$ so that each homeomorphism sends the action of the group $G = GL(3,\mathbb{R})$ from $F = \mathbb{R}^3$ to $\theta^3_{R,L}(p)_M$. ∎

Definition 2.3.2 (F-equivalent fibers of a F-Steenrod bundle) Two fibers $\theta^3_{R,L}(p)^1_M$ and $\theta^3_{R,L}(p)^2_M$ will be said to be F-equivalent if they are homotopic, i.e. if there exists a continuous mapping from the one to the other.

Proposition 2.3.3 *To each $1D$- or $3D$- " ST ", " MG " or " M " F-Steenrod bundle corresponds a set of F-equivalent sections above a given basis related to a given frequency of vibration of this basis.*

Proof. At a given basis of a F-Steenrod bundle corresponds a set of F-equivalent fibers according to the definition 2.3.2 and thus a set of F-equivalent sections.

Each set of F-equivalent sections of a F-Steenrod bundle is then interpreted as corresponding to all the possible vibrations of the basis at a given frequency. ∎

Proposition 2.3.4 *The frequencies of vibration of the $1D$ and $3D$ space-time (ST), middle ground (MG) and mass (M) semisheaves of rings are quantified.*

Proof. The semisheaves of rings " ST ", " MG " and " M " are assumed to be defined on quanta, i.e. submodules of rank N . Thus, the semisheaves of rings $\theta^1_{R,L}(t)_{ST,MG,M}$ and $\theta^3_{R,L}(r)_{ST,MG,M}$ are quantified. As they are the basis of F-Steenrod bundles and as a given frequency is associated to each basis of an F-Steenrod bundle according to the preceding proposition, we reach the thesis. ■

Definition 2.3.5 The mass frequency of an elementary semiparticle is an average measure of the vibration of all the points of the semisheaf of rings $\theta^3_{R,L}(r)_M$. From the preceding developments, it becomes clear that there exists a correspondence between the ranks of sections (i.e. classes of degrees of Galois extensions) and the integer numbers of the quantum mechanics referring to vibrations. Indeed, these integer numbers n_μ refer to the numbers of quanta of μ-th substates of a (semi)particle.

Proposition 2.3.6 *The [semi]wave-[semi]particle duality of quantum theory results from the quantification of the vibration frequency(ies) of the (ST , MG and) M [semi]sheaf(ves) of rings.*

Proof. Indeed, this duality is essentially traduced by the relations $E = h\nu$, $\vec{p} = \hbar\vec{k}$ between the dynamical variables related to the mass structure of the semiparticles and the frequencies of the associated semiwaves [Mes], [deBro]. ■

Remark 2.3.7 The vacuum in this algebraic quantum model is not external to elementary semiparticles but is composed of their $4D$-" ST " and " MG " semisheaves of rings which presently are unobservable and whose spatial extension is of the order of the Planck length $\simeq 10^{-33}$ cm. The mass of a semiparticle is given by the $4D$-" M " semisheaf of rings $\theta^{1-3}_{R,L}(t,r)_M = \theta^1_{R,L}(t)_M \cup \theta^3_{R,L}(r)_M$ which is generated from the corresponding $4D$-" MG " semisheaf of rings $\theta^{1-3}_{R,L}(t,r)_{MG} = \theta^1_{R,L}(t)_{MG} \cup \theta^3_{R,L}(r)_{MG}$ by versal deformation $Vd(2)$ or $Vd(2')$, spreading-out isomorphism $SOT(2)$ or $SOT(2')$ and $\gamma^M_{t\to r} \circ E$ or $\gamma^M_{r\to t} \circ E'$ morphism.

Consequently, the composition of morphisms:

$$\gamma^M_{t\to r} \circ E \circ SOT(2) \circ Vd(2) : \theta^{1-3}_{R,L}(t,r)_{MG} \to \theta^{1-3}_{R,L}(t_{res}, r_{res})_{MG} \cup \theta^{1-3}_{R,L}(t,r)_M$$

corresponds to the creation operator of Quantum Field Theory.

Proposition 2.3.8 *The $4D$-" M " semisheaf of rings of a semiparticle is observable while the $4D$-" ST " and " MG " semisheaves of rings are unobservable because the vibration frequencies of the " M " semisheaf is inferior to the vibration frequencies of the " ST " and " MG " semisheaves.*

Proof. The " M " semisheaf of rings, being generated by versal deformation and spreading-out isomorphism from the " MG " semisheaf of rings, is characterized by a set of ranks n_{θ_M} inferior or equal to the set of ranks $n_{\theta_{MG}}$ of the " MG " semisheaf of rings since the codimension of the singularities on the " MG " semisheaf of rings is inferior to the codimension of the singularities on the " ST " semisheaf of rings.

According to proposition 2.3.4, the vibration frequency of the " M " semisheaf of rings must thus be inferior to the vibration frequency of the " MG " semisheaf of rings since:

- the frequency vibrations of the " MG " and " M " semisheaves are quantified;

- the sections of the " M " semisheaf are open strings covering partially from outside the open strings of the " MG " semisheaf. ∎

Remark 2.3.9 (Dark energy) If the semiparticles are composed of the $4D$-" ST " semisheaves of rings $\theta^{1-3}_{R,L}(t_{res},r_{res})_{ST}$ or of the $4D$-" ST " and " MG " semisheaves of rings $\theta^{1-3}_{R,L}(t_{res},r_{res})_{ST}\cup\theta^{1-3}_{R,L}(t,r)_{MG}$, noted $\theta^{1-3}_{R,L}(t,r)_{ST-MG}$, they are massless and unobservable and could contribute to the dark energy of the Universe.

2.4 The electric charge and the existence of three families of semiparticles

Let $\theta^{1-3}_{R,L}(t,r)_{ST-MG-M}$ denote the three embedded $4D$-semishaves of rings.
Consider that an external perturbation generates on each section of the semisheaves of rings $\theta^{1-3}_{R,L}(t_{res},r_{res})_{ST}$, $\theta^{1-3}_{R,L}(t_{res},r_{res})_{MG}$ and $\theta^{1-3}_{R,L}(t,r)_{M}$ an isolated degenerated singularity of corank 1.

Then, it will be seen that singularities of codimension 1 on $\theta^{1}_{R,L}(t)_{ST-MG-M}$ may be interpreted as being at the origin of the time structure of the electric charge and that singularities of codimension 2 and 3 on $\theta^{3}_{R,L}(r)_{ST-MG-M}$ or on $\theta^{1}_{R,L}(t)_{ST-MG-M}$ are at the origin of the generation of the second and of the third family of elementary semiparticles.

Note that the time structure of the electrical charge is supposed to be generated by versal deformation and spreading-out isomorphism because it must have a permanent structure on the contrary of the magnetic moment of a semiparticle which is generated only on the basis of the smooth endomorphism " E " as it will be seen in the following.

Definition 2.4.1 (The time structure of the electric charge) Let $\theta^{1}_{R,L}(t)_{ST-MG-M}\equiv\theta^{1}_{R,L}(t_{res})_{ST}$ $\cup\,\theta^{1}_{R,L}(t_{res})_{MG}\cup\theta^{1}_{R,L}(t)_{M}$ denote the $1D$-time " ST ", " MG " and " M " semisheaves of rings of a semilepton or of a semiquark. Consider that each section of these semisheaves is endowed with an isolated degenerated singularity of codimension one due to an external perturbation.
Then, the versal deformation and spreading-out isomorphism, applied to $\theta^{1}_{R,L}(t)_{ST-MG-M}$ gives:

$$SOT(e)\circ Vd(e):\theta^{1}_{R,L}(t)_{ST-MG-M}\to\theta^{1}_{R,L}(t_{res})_{ST-MG-M}\cup\theta^{1}_{R,L}(t)^{(e)}_{ST-MG-M}$$

where $\theta^{1}_{R,L}(t)^{(e)}_{ST-MG-M}$ is interpreted as the time structure of the electric charge of a semilepton or of a semiquark; its $3D$-spatial structure is given by a $3D$-semisheaf of rings $\theta^{3}_{L,R}(r)^{(e)}_{ST-MG-M}$ composed of $3D$-left (resp. right) quanta generated by the smooth endomorphism $E_{ST-MG-M}$ (see definition 2.2.12) acting simultaneously on the $3D$-semisheaves of rings $\theta^{3}_{L,R}(r)_{ST-MG-M}$ of its associated semiparticle.

If $\theta^{1}_{R,L}(t)^{(e)}_{ST-MG-M}$ represents the time structure of the electric charge of a semiquark, then the ranks of the " ST ", " MG " and " M " semisheaves $\theta^{1}_{R,L}(t)^{(e)}_{ST-MG-M}$ are equal to $\frac{1}{3}$ or $\frac{2}{3}$ [L-P-F] of the ranks of the corresponding $1D$-electric semisheaves of a semilepton because the electric charge must be conserved.

Proposition 2.4.2 *Only three families of elementary semiparticles can exist in the above-mentioned mathematical frame.*

Proof. Let $\theta^3_{R,L}(r)^{(A)}_{ST-MG-M} = \theta^3_{R,L}(r_{res})_{ST} \cup \theta^3_{R,L}(r_{res})_{MG} \cup \theta^3_{R,L}(r)_M$ be the three embedded $3D$-semisheaves of rings " ST ", " MG " and " M " of a semilepton or of a semiquark of the first family A , i.e. a semielectron or a semiquark "up".

Under some strong external perturbation, each section of the $3D$-semisheaves or rings " ST ", " MG " and " M " is assumed to have one or a set of degenerated singularities of corank 1 and of the same codimension $cd = 2$.

Then, the versal deformation and the spreading-out isomorphism of the three embedded semisheaves of rings of a semiparticle of the first family " A ", $\theta^3_{R,L}(t,r)^{(A)}_{St-MG-M}$ generate the three embedded semisheaves of rings of a semiparticle of the second family " B " according to:

$$SOT(A) \circ Vd(A) : \theta^3_{R,L}(r)^{(A)}_{ST-MG-M} \to \theta^3_{R,L}(r_{res})^{(A)}_{ST-MG-M} \cup \theta^3_{R,L}(r)^{(B)}_{ST-MG-M}$$

where $\theta^3_{R,L}(r_{res})^{(A)}_{ST-MG-M} \cup \theta^3_{R,L}(r)^{(B)}_{ST-MG-M}$ represents the three embedded structures of this semiparticle " B ".

But, if the singularities on the sections of $\theta^3_{R,L}(r)^{(A)}_{ST-MG-M}$ are of corank 1 and codimension 3, then the three embedded semisheaves of rings of a semiparticle of the third family " C " can be generated by versal deformation and spreading-out isomorphism from the remaining degenerated singularities of corank 1 and codimension 1 on the sections of $\theta^3_{R,L}(r)^{(B)}_{ST-MG-M}$. We then have:

$$SOT(B) \circ Vd(B) : \theta^3_{R,L}(r)^{(B)}_{ST-MG-M} \to \theta^3_{R,L}(r_{res})^{(B)}_{ST-MG-M} \cup \theta^3_{R,L}(r)^{(C)}_{ST-MG-M}$$

where $\theta^3_{R,L}(r_{res})^{(A)}_{ST-MG-M} \cup \theta^3_{R,L}(r_{res})^{(B)}_{ST-MG-M} \cup \theta^3_{R,L}(r)^{(C)}_{ST-MG-M}$ represents the three embedded spatial structures of a semiparticle of the third family C .

Finally, the corresponding $1D$-time semisheaves of rings are obtained from the $3D$-semisheaves of rings by the morphisms $(\gamma_{r\to t} \circ E)$ (see definition 2.2.3). ■

Remark 2.4.3 As the three embedded semisheaves of rings of semiparticles of the second and of the third family are supercompact by construction, they are highly distorted. Consequently, the three embedded structures of semiparticles of these families B and C are highly unstable which explains their rapid decays.

Proposition 2.4.4 *The heavy semiquark of a given family can be obtained from the lighter semiquark of the same family by versal deformation and spreading-out isomorphism of the singularities of corank 1 and codimension 1 on the sections of the three embedded semisheaves of rings of this lighter semiquark.*

Proof. Let $\theta^3_{R,L}(r)^{(Li)}_{ST-MG-M}$ be the three embedded $3D$-spatial semisheaves of rings of a light semiquark of a given family " A ", " B " or " C ". Assume that the sections of $\theta^3_{R,L}(r)^{(Li)}_{ST-MG-M}$ are endowed with singularities of corank 1 and codimension 1 under some external perturbation. Then, under versal deformation and spreading-out isomorphism, $\theta^3_{R,L}(r)^{(Li)}_{ST-MG-M}$ is transformed according to:

$$SOT(Li) \circ Vd(Li) : \theta^3_{R,L}(r)^{(Li)}_{ST-MG-M} \to \theta^3_{R,L}(r_{res})^{(Li)}_{ST-MG-M} \cup \theta^3_{R,L}(r)^{(He)}_{ST-MG-M}$$

where $\theta^3_{R,L}(r_{res})^{(Li)}_{ST-MG-M} \cup \theta^3_{R,L}(r)^{(He)}_{ST-MG-M}$ represents the three embedded $3D$-semisheaves of rings of the heavier semiquark. The corresponding $1D$-semisheaves of rings are obtained by the $(\gamma_{r\to t} \circ E)$ morphism. ■

Remark 2.4.5 As the middle ground (MG) and mass (M) structures of semiparticles are generated from the space-time (ST) structure by versal deformation and spreading-out isomorphism, we shall not consider that the creation of these " MG " and " M " structures correspond to an axiom which, otherwise, would have been an homotopy axiom according to M. Atiyah [Ati3].

3 Bialgebras of von Neumann, probability calculus and quantification rules

The main purpose of this chapter is to introduce the bialgebras of von Neumann and to restore in this manner the classical probability calculus in quantum theories dealing thus with the sixth problem of Hilbert which consists in the ontological meaning of the theory of probabilities.

In this context, the spectral representation of a (bi)operator is explicitly given as:

- corresponding to the representation of the general bilinear semigroup $GL_{2(n)}(\mathbb{A}_R \times \mathbb{A}_L)$ in the $G_R(\mathbb{A}_R) \times G_L(\mathbb{A}_L)$ bisemimodule $(M_R \otimes M_L)$ where $M_{R,L}$ is $3(n)$-dimensional;

- resulting from the representation of the Lie algebra $\mathfrak{g}\ell_{2(n)}(\mathbb{A}_R \times \mathbb{A}_L)$ of the general bilinear semigroup $GL_{2(n)}(\mathbb{A}_R \times \mathbb{A}_L)$ in the shifted $G_R(\mathbb{A}_R) \times G_L(\mathbb{A}_L)$-bisemimodule $(M^a_R \otimes M^a_L)$ which is a perverse bisemisheaf.

As our objective is the study of the space-time structure of elementary particles which become bisemiparticles in this mathematical frame and as a massive bisemiparticle is composed of a left and a right semiparticle whose structure is given by the three embedded structures " ST ", " MG " and " M ", we shall have to consider a bialgebra of von Neumann on each of these three structures.

3.1 Hilbert, magnetic and electric bilinear spaces

We thus begin this section by introducing the structure of a massive bisemiparticle and the space on which it is defined.

Definition 3.1.1 (Structure of a massive right and left semiparticle) The three embedded $4D$-structures of a right and a left semiparticle, i.e. essentially of a semilepton or of a semiquark, is given respectively by the three embedded right $4D$-semisheaves of rings

$$\theta^{1-3}_R(t,r)_{ST-MG-M} = \theta^{1-3}_R(t,r)_{ST} \cup \theta^{1-3}_R(t,r)_{MG} \cup \theta^{1-3}_R(t,r)_M$$

and by the three embedded left $4D$-semisheaves of rings

$$\theta^{1-3}_L(t,r)_{ST-MG-M} = \theta^{1-3}_L(t,r)_{ST} \cup \theta^{1-3}_L(t,r)_{MG} \cup \theta^{1-3}_L(t,r)_M$$

as developed in chapter 2, section 2.

As $\theta^{1-3}_{R,L}(t,r)_{ST} \cap \theta^{1-3}_{R,L}(t,r)_{MG} = \emptyset$, $\theta^{1-3}_{R,L}(t,r)_{ST} \cap \theta^{1-3}_{R,L}(t,r)_M = \emptyset$ and $\theta^{1-3}_{R,L}(t,r)_{MG} \cap \theta^{1-3}_{R,L}(t,r)_M = \emptyset$, we shall envisage the direct sum of the three embedded semisheaves " ST ", " MG " and " M ":

$$\Theta_{R,L} : \theta^{1-3}_{R,L}(t,r)_{ST-MG-M} \to \theta^{1-3}_{R,L}(t,r)_{ST} \oplus \theta^{1-3}_{R,L}(t,r)_{MG} \oplus \theta^{1-3}_{R,L}(t,r)_M$$

noted $\theta^{1-3}_{R,L}(t,r)_{ST\oplus MG\oplus M}$.

$\theta^{1-3}_{R,L}(t,r)_{ST\oplus MG\oplus M}$ is then defined on 3 embedded topological spaces of dimension 4.

Definition 3.1.2 (Structure of a massive bisemiparticle) Massive elementary stable objects of Nature are in fact biobjects, i.e. bisemiparticles according to axiom II 1.3.9. Their space-time structure is crudely given by the tensor product between the three embedded right and left $4D$-semisheaves of rings:

$$\theta^{1-3}_R(t,r)_{ST\oplus MG\oplus M}\otimes\theta^{1-3}_L(t,r)_{ST\oplus MG\oplus M}$$

which allows to generate interactions between the right and left structures, i.e. between the right and left " ST ", " MG " and " M " semisheaves of rings.

Consider the condensed notation $\theta^4_{R;ST}$ for $\theta^{1-3}_R(t,r)_{ST}$.

This tensor product then develops according to:

$$\begin{aligned}
&\theta^{1-3}_R(t,r)_{ST\oplus MG\oplus M}\otimes\theta^{1-3}_L(t,r)_{ST\oplus MG\oplus M}\equiv\theta^4_{R;ST\oplus MG\oplus M}\otimes\theta^4_{L;ST\oplus MG\oplus M}\\
&\quad=\quad(\theta^4_{R;ST}\oplus\theta^4_{R;MG}\oplus\theta^4_{R;M})\otimes(\theta^4_{L;ST}\oplus\theta^4_{L;MG}\oplus\theta^4_{L;M})\\
&\quad=\quad(\theta^4_{R;ST}\otimes\theta^4_{L;ST})\oplus(\theta^4_{R;MG}\otimes\theta^4_{L;MG})\oplus(\theta^4_{R;M}\otimes\theta^4_{L;M})\\
&\qquad\qquad\oplus(\theta^4_{R;ST}\otimes\theta^4_{L;MG})\oplus(\theta^4_{R;MG}\otimes\theta^4_{L;ST})\oplus(\theta^4_{R;ST}\otimes\theta^4_{L;M})\\
&\qquad\qquad\oplus(\theta^4_{R;M}\otimes\theta^4_{L;ST})\oplus(\theta^4_{R;MG}\otimes\theta^4_{L;M})\oplus(\theta^4_{R;M}\otimes\theta^4_{L;MG})
\end{aligned}$$

where the three first tensor products refer to the " ST ", " MG " and " M " structures of the considered bisemiparticle while the six other tensor products refer to the interactions between the right and left " ST ", " MG " and " M " structures.

Definition 3.1.3 (Duality of semisheaves) Let $\widetilde M_R$ and $\widetilde M_L$ denote a right semisheaf $\theta^4_{R;ST}$, $\theta^4_{R;MG}$ or $\theta^4_{R;M}$ and a left semisheaf $\theta^4_{L;ST}$, $\theta^4_{L;MG}$ or $\theta^4_{L;M}$.

Their tensor product is given by the bisemimodule $(\widetilde M_R\otimes\widetilde M_L)$ which decomposes under the blowing-up isomorphism S_L (see proposition 1.3.2) into the direct sum of

a) the diagonal bisemisheaf $(\widetilde M_R\otimes_D\widetilde M_L)$,

b) the magnetic bisemisheaf $(\widetilde M^S_R\otimes_{\text{magn}}\widetilde M^S_L)$,

c) the electric bisemisheaf $(\widetilde M^{T-(S)}_R\otimes_{\text{elec}}\widetilde M^{S-(T)}_L)$,

where $\widetilde M^S_{R,L}$ is a $3D$-spatial subsemisheaf and where $\widetilde M^T_{R,L}$ is a $1D$-time subsemisheaf.

For the facility of notations, $(\widetilde M^S_R\otimes_{\text{magn}}\widetilde M^S_L)$ will be written $(\widetilde M_R\otimes_m\widetilde M_L)$ and
$(\widetilde M^{T-(S)}_R\otimes_{\text{elec}}\widetilde M^{S-(T)}_L)$ will be written $(\widetilde M_R\otimes_e\widetilde M_L)$.

If we consider the projective linear map:

$$p_L:\widetilde M_R\otimes_{D,m,e}\widetilde M_L\to\widetilde M_{R(P)/_{D,m,e}L}$$

of the right semisheaf $\widetilde M_R$ onto the left semisheaf $\widetilde M_L$ with respect to the diagonal, magnetic or electric metric, then $\widetilde M_{R(P)}$ is the dual semisheaf of $\widetilde M_L$ and is called a coleft semisheaf whose elements are coleft differential functions.

But, if we take into account the projective linear map

$$p_R:\widetilde M_R\otimes_{D,m,e}\widetilde M_L\to\widetilde M_{L(P)/_{D,m,e}R}$$

projecting the left semisheaf $\widetilde M_L$ onto the right semisheaf $\widetilde M_R$, then $\widetilde M_{L(P)}$ is the dual semisheaf of $\widetilde M_R$ and will be called a coright semisheaf whose elements are coright functions.

Remarks 3.1.4 **1.** The " ST ", " MG " or " M " diagonal (bi)structure of a bisemiparticle is thus given by the diagonal bisemisheaf $\widetilde{M}_{R(P)/_D L}$ (resp. $\widetilde{M}_{L(P)/_D R}$) constituted by the diagonal tensor product between the left (resp. right) semisheaf $\widetilde{M}_L$ (resp. $\widetilde{M}_R$) of the left (resp. right) semiparticle and the projected right (resp. left) semisheaf $\widetilde{M}_{R(P)}$ (resp. $\widetilde{M}_{L(P)}$) of the projected right (resp. left) semiparticle.

The projected right (resp. left) semisheaf $\widetilde{M}_{R(P)}$ (resp. $\widetilde{M}_{L(P)}$) is thus called a coleft (resp. coright) semisheaf and the projected right (resp. left) semiparticle is then called a coleft (resp. coright) semiparticle.

2. The following developments about bilinear Hilbert spaces concern the bisemisheaves $(\widetilde{M}_R \otimes \widetilde{M}_L)$ as well as the $G_{R\times L}(\mathbb{A}_R \times \mathbb{A}_L)$-bisemimodules $(M_R \otimes M_L)$ on which they are defined.

Definition 3.1.5 (Algebraic external Hilbert, magnetic and electric bilinear spaces) **1.** By the projective linear map p_L (resp. p_R), the diagonal bisemisheaf $(\widetilde{M}_R \otimes_D \widetilde{M}_L)$ is transformed into $\widetilde{M}_{R(P)/_D L}$ (resp. $\widetilde{M}_{L(P)/_D R}$). If we endow $\widetilde{M}_{R(P)/_D L}$ (resp. $\widetilde{M}_{L(P)/_D R}$) with an external scalar product characterized by an euclidian metric δ^α_β of type $(1,1)$, $0 \le \alpha,\beta \le 3$, then we get a left (resp. right) external bilinear Hilbert space noted $\mathcal{H}^a_L$ (resp. $\mathcal{H}^a_R$) [Pie4], which is of algebraic nature.

2. Similarly, the projective linear map p_L (resp. p_R) transforms the magnetic bisemisheaf $(\widetilde{M}_R \otimes_m \widetilde{M}_L)$ into a left (resp. right) external bilinear magnetic bisemisheaf $\widetilde{M}_{R(P)/_m L}$ (resp. $\widetilde{M}_{L(P)/_m R}$) which becomes a left (resp. right) external bilinear magnetic space, noted $V^{m;a}_L$ (resp. $V^{m;a}_R$), if it is endowed with an external magnetic product $\langle \phi_{R(P)},\phi_L\rangle_m$ (resp. $\langle \phi_{L(P)},\phi_R\rangle_m$) defined from $(\widetilde{M}_{R(P)} \times_m \widetilde{M}_L)$ to $\mathbb{C}$ (resp. from $(\widetilde{M}_{L(P)} \times_m \widetilde{M}_R)$ to $\mathbb{C}$) and characterized by a noneuclidian magnetic metric g^α_β , $\forall\ \alpha \neq \beta$, $1 \le \alpha,\beta \le 3$, of type $(1,1)$.

3. The electric bisemisheaf $(\widetilde{M}_R \otimes_e \widetilde{M}_L)$ is transformed by the projective linear map p_L (resp. p_R) into the left (resp. right) external bilinear electric bisemisheaf $\widetilde{M}_{R(P)/_e L}$ (resp. $\widetilde{M}_{L(P)/_e R}$) which becomes a left (resp. right) external bilinear electric space, noted $V^{e;a}_L$ (resp. $V^{e;a}_R$), if it is endowed with an external electric product $\langle \phi_{R(P)},\phi_L\rangle_e$ (resp. $\langle \phi_{L(P)},\phi_R\rangle_e$) defined from $(\widetilde{M}_{R(P)} \times_e \widetilde{M}_L)$ to $\mathbb{C}$ (resp. from $(\widetilde{M}_{L(P)} \times_e \widetilde{M}_R)$ to $\mathbb{C}$) and characterized by a noneuclidian electric metric g^α_β of type $(1,1)$ with $\alpha = 0$ and $1 \le \beta \le 3$ or with $1 \le \alpha \le 3$ and $\beta = 0$.

Proposition 3.1.6 *The left and right external bilinear Hilbert spaces $\mathcal{H}^a_L$ and $\mathcal{H}^a_R$ are characterized by bilinear orthogonal $4D$-basis while the left and right external bilinear electric and magnetic spaces are characterized by $3D$-basis.*

Proof. 1. The bilinear Hilbert spaces $\mathcal{H}^a_L$ and $\mathcal{H}^a_R$ are characterized by $4D$-orthogonal bilinear basis since they result from the diagonal bisemisheaf $(\widetilde{M}_R \otimes_D \widetilde{M}_L)$.

2. The electric basis is three-dimensional and not six-dimensional because the set of electric basis bivectors $\{e^0 \otimes f_\beta\}^3_{\beta=1}$ are orthogonal to the electric basis bivectors $\{e^\beta \otimes f_0\}^3_{\beta=1}$; indeed, we have that $\langle (e^0)^* \otimes f_\beta, (e^\beta)^* \otimes f_0)\rangle = 0$ implying $\langle (e^0)^*, f_0\rangle \langle (e^\beta)^*, f_\beta\rangle = 0$ since $\langle (e^0)^*, f_0\rangle = \langle (e^\beta)^*, f_\beta)\rangle = 0$, with $1 \le \beta \le 3$, by hypothesis on the electric metric.

3. Similar conclusions are obtained for the external bilinear magnetic spaces $V^{m;a}_L$ and $V^{m;a}_R$. ∎

Definition 3.1.7 (Algebraic internal Hilbert, magnetic and electric bilinear spaces) Let $B_L : M_{R(P)} \to M_L$ (resp. $B_R : M_{L(P)} \to M_R$) be the bijective linear isometric map from $M_{R(P)}$ (resp. $M_{L(P)}$) to M_L (resp. M_R) mapping each covariant element of $M_{R(P)}$ (resp. $M_{L(P)}$) into a contravariant element of M_L , noted M_{L_R} (resp. of M_R , noted M_{R_L}) as introduced in proposition 1.3.6.

Then, B_L (resp. B_R) transforms:

1. the left (resp. right) external bilinear Hilbert space $\mathcal{H}_L^a$ (resp. $\mathcal{H}_R^a$) into the left (resp. right) internal bilinear Hilbert space $\mathcal{H}_a^+$ (resp. $\mathcal{H}_a^-$) in such a way that:
   a) the bielements of $\mathcal{H}_a^+$ (resp. $\mathcal{H}_a^-$) are bivectors, i.e. two confounded vectors;
   b) each external scalar product of $\mathcal{H}_L^a$ (resp. $\mathcal{H}_R^a$) is transformed into an internal scalar product defined from $M_{L_R} \times_D M_L$ (resp. $M_{R_L} \times_D M_R$) to $\mathbb{C}$.

2. the left (resp. right) external bilinear magnetic space $V_L^{m;a}$ (resp. $V_R^{m;a}$) into the left (resp. right) internal bilinear magnetic space $V_{m;a}^+$ (resp. $V_{m;a}^-$) in such a way that the external magnetic product of $V_L^{m;a}$ (resp. $V_R^{m;a}$) be transformed into an internal magnetic product defined from $M_{L_R} \times_m M_L$ (resp. $M_{R_L} \times_m M_R$) to $\mathbb{C}$. This internal magnetic space $V_{m;a}^+$ (resp. $V_{m;a}^-$) is characterized by a noneuclidian metric $g_{\alpha\beta}$ of type $(0,2)$, $\forall\ \alpha \neq \beta$, $1 \leq \alpha, \beta \leq 3$.

3. the left (resp. right) external bilinear electric space $V_L^{e;a}$ (resp. $V_R^{e;a}$) into the left (resp. right) internal bilinear electric space $V_{e;a}^+$ (resp. $V_{e;a}^-$) such that the external electric product of $V_L^{e;a}$ (resp. $V_R^{e;a}$) be transformed into an internal electric product defined from $M_{L_R} \times_e M_L$ (resp. $M_{R_L} \times_e M_R$) to $\mathbb{C}$.

Definitions 3.1.8 (1. Algebraic extended external bilinear Hilbert spaces H_L^a and H_R^a) Let $\widetilde{M}_R$ and $\widetilde{M}_L$ denote respectively the $4D$-right semisheaf and the $4D$-left semisheaf. Then, we consider on the noneuclidian bisemisheaf $\widetilde{M}_R \otimes \widetilde{M}_L$ the projective linear map:

$$\begin{aligned} p_L : \widetilde{M}_R \otimes \widetilde{M}_L &\to \widetilde{M}_{R(P)/_cL} \qquad (\text{“} c \text{''}:\ \text{for complete}) \\ \text{or} \quad p_R : \widetilde{M}_R \otimes \widetilde{M}_L &\to \widetilde{M}_{L(P)/_cR} \end{aligned}$$

of the right (resp. left) semisheaf $\widetilde{M}_R$ (resp. $\widetilde{M}_L$) on the left (resp. right) semisheaf $\widetilde{M}_L$ (resp. $\widetilde{M}_R$).

If we endow the bisemisheaf $\widetilde{M}_{R(P)/_cL}$ (resp. $\widetilde{M}_{L(P)/_cR}$) with a complete external bilinear form defined from $\widetilde{M}_{R(P)} \times \widetilde{M}_L$ (resp. $\widetilde{M}_{L(P)} \times \widetilde{M}_R$) to $\mathbb{C}$, we get a left (resp. right) extended external bilinear Hilbert space H_L^a (resp. H_R^a) characterized by a nonorthogonal basis.

(2. Algebraic extended internal bilinear Hilbert spaces H_a^+ and H_a^-) The left (resp. right) extended external bilinear Hilbert space H_L^a (resp. H_R^a) is transformed into the left (resp. right) extended internal bilinear Hilbert space H_a^+ (resp. H_a^-) by means of a bijective bilinear isometric map B_L (resp. B_R) from $\widetilde{M}_{R(P)}$ (resp. $\widetilde{M}_{L(P)}$) to $\widetilde{M}_L$ (resp. $\widetilde{M}_R$).

The complete external bilinear form of H_L^a (resp. H_R^a) is then transformed into a complete internal bilinear form of H_a^+ (resp. H_a^-).

Definition 3.1.9 (Analytic Hilbert, magnetic and electric bilinear spaces) Let $X_{R,L}^s$ be the analytic semivariety associated to the semispace $\partial\,\overline{S}_{K_{R,L}}$ and let $\widetilde{M}_{R,L}^s$ be an analytic semisheaf on $X_{R,L}^s$.

From the complete, diagonal, magnetic or electric tensor product between the right and left semisheaves $\widetilde{M}^s_R$ and $\widetilde{M}^s_L$, we can construct by application of the composition of maps $B_L \circ p_L$ (resp. $B_R \circ p_R$) on the bisemisheaves $\widetilde{M}^s_R \otimes \widetilde{M}^s_L$, $\widetilde{M}^s_R \otimes_D \widetilde{M}^s_L$, $\widetilde{M}^s_R \otimes_m \widetilde{M}^s_L$ or $\widetilde{M}^s_R \otimes_e \widetilde{M}^s_L$ an analytic left (resp. right) internal bilinear extended Hilbert space H^+_h (resp. H^-_h):

$$\begin{tikzcd} \{\widetilde{M}^s_R, \widetilde{M}^s_L\} \arrow[r] & \widetilde{M}^s_R \otimes \widetilde{M}^s_L \arrow[r, "B_L\circ p_L"] \arrow[dr, "B_R\circ p_R"'] & \widetilde{M}^s_{L_R} \otimes \widetilde{M}^s_L \subset H^+_h \\ & & \widetilde{M}^s_{R_L} \otimes \widetilde{M}^s_R \subset H^-_h \end{tikzcd}$$

an analytic left (resp. right) internal bilinear (diagonal) Hilbert space $\mathcal{H}^+_h$ (resp. $\mathcal{H}^-_h$):

$$\begin{tikzcd} \{\widetilde{M}^s_R, \widetilde{M}^s_L\} \arrow[r] & \widetilde{M}^s_R \otimes_D \widetilde{M}^s_L \arrow[r, "B_L\circ p_L"] \arrow[dr, "B_R\circ p_R"'] & \widetilde{M}^s_{L_R} \otimes_D \widetilde{M}^s_L \subset \mathcal{H}^+_h \\ & & \widetilde{M}^s_{R_L} \otimes_D \widetilde{M}^s_R \subset \mathcal{H}^-_h \end{tikzcd}$$

an analytic left (resp. right) internal bilinear magnetic space $V^+_{m;h}$ (resp. $V^-_{m;h}$):

$$\begin{tikzcd} \{\widetilde{M}^s_R, \widetilde{M}^s_L\} \arrow[r] & \widetilde{M}^s_R \otimes_m \widetilde{M}^s_L \arrow[r, "B_L\circ p_L"] \arrow[dr, "B_R\circ p_R"'] & \widetilde{M}^s_{L_R} \otimes_m \widetilde{M}^s_L \subset V^+_{m;h} \\ & & \widetilde{M}^s_{R_L} \otimes_m \widetilde{M}^s_R \subset V^-_{m;h} \end{tikzcd}$$

an analytic left (resp. right) internal bilinear electric space $V^+_{e;h}$ (resp. $V^-_{e;h}$):

$$\begin{tikzcd} \{\widetilde{M}^s_R, \widetilde{M}^s_L\} \arrow[r] & \widetilde{M}^s_R \otimes_e \widetilde{M}^s_L \arrow[r, "B_L\circ p_L"] \arrow[dr, "B_R\circ p_R"'] & \widetilde{M}^s_{L_R} \otimes_e \widetilde{M}^s_L \subset V^+_{e;h} \\ & & \widetilde{M}^s_{R_L} \otimes_e \widetilde{M}^s_R \subset V^-_{e;h} \end{tikzcd}$$

All these internal bilinear spaces are endowed with the corresponding internal bilinear forms in complete analogy with which was developed in definition 3.1.7.

Definition 3.1.10 (Diagonal, complete, magnetic and electric products of right and left Eisenstein cohomologies) In chapter 1, section 1, right and left Eisenstein cohomologies $H^*_{R,L}(\partial\, \overline{S}_{K_{R,L}}, \widetilde{M}_{R,L})$ defined on the right and left semispaces $\partial\, \overline{S}_{K_{R,L}}$ and associated to the generation of the right and left semisheaves $\widetilde{M}_{R,L}$ were studied.

This allows to generate a diagonal, complete, magnetic or electric bisemisheaf $\widetilde{M}_R \otimes_{(D),m,e} \widetilde{M}_L$ on $\partial\, \overline{S}_{K_R} \times_{(D),m,e} \partial\, \overline{S}_{K_L}$ by the diagonal, complete, magnetic or electric product of right and left Eisenstein cohomology groups:

$$\begin{aligned} H^E_{R\times_{(D),m,e}L} \quad : \quad & H^*_R(\partial\, \overline{S}_{K_R}, \widetilde{M}_R) \times_{(D),m,e} H^*_L(\partial\, \overline{S}_{K_L}, \widetilde{M}_L) \\ & \qquad \to H^*_{R\times_{(D),m,e}L}(\partial\, \overline{S}_{K_R} \times_{(D),m,e} \partial\, \overline{S}_{K_L}, \widetilde{M}_R \otimes_{(D),m,e} \widetilde{M}_L)\ . \end{aligned}$$

Proposition 3.1.11 *The bilinear Eisenstein cohomology*

$$H^*_{R\times_{(D),m,e}L}\ (\partial\, \overline{S}_{K_R} \times_{(D),m,e} \partial\, \overline{S}_{K_L}, \widetilde{M}_R \otimes_{(D),m,e} \widetilde{M}_L)$$

associated to the coefficient system $\widetilde{M}_R \otimes_{(D),m,e} \widetilde{M}_L$ decomposes into sum of products of one-dimensional eigenspaces according to:

$$H^*_{R\times_{(D),m,e}L}\ (\partial\,\overline{S}_{K_R}\times_{(D),m,e}\partial\,\overline{S}_{K_L},\widetilde{M}_R\otimes_{(D),m,e}\widetilde{M}_L)$$

$$\begin{aligned}&=\bigoplus_{\mu}\bigoplus_{m_\mu}\bigoplus_{1_\ell}\operatorname{Ind}^{G_{R\times_{(D),m,e}L}(\mathbb{A}_R\times\mathbb{A}_L)}_{K^D_{R\times_{(D),m,e}L}(\overline{\mathbb{Z}}^2_{pq})}\\&\quad\times H^{1_\ell,1_\ell}_{R\times_{(D),m,e}L}(S^{M_{R\times_{(D),m,e}L}},H^{1_\ell,1_\ell}(\tilde{u}_{K^D_{R\times_{(D),m,e}L}},\widetilde{M}^{1_\ell}_R(\mu,m_\mu)\otimes_{(D),m,e}\widetilde{M}^{1_\ell}(\mu,m_\mu)))\end{aligned}$$

where the sum over 1_ℓ , $1\le\ell\le n$, refers to the decomposition of the $n^{(2)}$-dimensional bisemisheaf $\widetilde{M}_R\otimes\widetilde{M}_L$ into products of 1-dimensional subsemisheaves $\widetilde{M}^{1_\ell}_R\otimes\widetilde{M}^{1_\ell}_L$ on the representation of $GL_{2_\ell}(\mathbb{A}_R\times\mathbb{A}_L)$ [Pie9].

Proof. This immediately results from proposition 1.1.22 and from the reducible Langlands program developed in [Pie9] in such a way that

$$\operatorname{Rep}(GL_{2n=2_1+\cdots+2_\ell+\cdots+2_n}(\mathbb{A}_R\times\mathbb{A}_L))=\boxplus_{\ell=1}^{n}\operatorname{Rep}GL_{2_\ell}(\mathbb{A}_R\times\mathbb{A}_L)\ .$$

■

Definition 3.1.12 (The analytic de Rham cohomology) As in the algebraic case, the analytic cohomology $H^*(X^s_{R,L},\widetilde{M}^s_{R,L})$ can be computed through the analytic de Rham complex.

We can also define a diagonal, complete, magnetic or electric product of right and left analytic cohomology groups:

$$\begin{aligned}H^h_{R\times_{(D),m,e}L}\quad:\quad&H^*(X^s_R,\widetilde{M}^s_R)\times_{(D),m,e}H^*(X^s_L,\widetilde{M}^s_L)\\&\qquad\to H^*_{R\times_{(D),m,e}L}(X^s_R\times_{(D),m,e}X^s_L,\widetilde{M}^s_R\otimes_{(D),m,e}\widetilde{M}^s_L)\end{aligned}$$

with coefficients in the respective product $\widetilde{M}^s_R\otimes_{(D),m,e}\widetilde{M}^s_L$ of the analytic semisheaves $\widetilde{M}^s_R$ and $\widetilde{M}^s_L$.

Proposition 3.1.13 *There is an isomorphism:*

$$\begin{aligned}&i_{H^*(\partial\,\overline{S}_{K_{R\times_{(D),m,e}L}})-H^*(X^s_{R\times_{(D),m,e}L})}\ :\\&\quad H^*_{R\times_{(D),m,e}L}(X^s_R\times_{(D),m,e}X^s_L,\widetilde{M}^s_R\otimes_{(D),m,e}\widetilde{M}^s_L)\\&\qquad\to H^*_{R\times_{(D),m,e}L}(\partial\,\overline{S}_{K_R}\times_{(D),m,e}\ \partial\,\overline{S}_{K_L},\widetilde{M}_R\otimes_{(D),m,e}\widetilde{M}_L)\end{aligned}$$

between products of Eisenstein cohomologies and analytic de Rham cohomologies.

Proof. According to Grothendick [Gro3], there is an isomorphism between the de Rham cohomologies of Ω^*-smooth differential forms with respect to $\partial\,\overline{S}_{K_{R,L}}$ and $X^s_{R,L}$:

$$H^*(\Omega^*_{\partial\,\overline{S}_{K_{R,L}}})\simeq H^*(\Omega^*_{X^s_{R,L}})$$

leading to the following isomorphism

$$H^*(\partial\,\overline{S}_{K_{R,L}},\widetilde{M}_{R,L})\simeq H^*(X^s_{R,L},\widetilde{M}^s_{R,L})$$

and thus to the thesis. ■

3.2 Bialgebras of Von Neumann

Definition 3.2.1 (Diagonal, complete, magnetic and electric products of operators) **1.** Let $(\widetilde M_{L_R}\otimes_{(D),m,e}\widetilde M_L)$ (resp. $(\widetilde M_{R_L}\otimes_{(D),m,e}\widetilde M_R)$) be the algebraic diagonal, complete, magnetic or electric bisemisheaf respectively of the left (resp. right) algebraic internal diagonal Hilbert, extended Hilbert, magnetic or electric bilinear space $\mathcal{H}^\pm_a$, $H^\pm_a$, $V^\pm_{m;a}$ or $V^\pm_{e;a}$.

Similarly, let $(\widetilde M^s_{L_R}\otimes_{(D),m,e}\widetilde M^s_L)$ (resp. $(\widetilde M^s_{R_L}\otimes_{(D),m,e}\widetilde M^s_R)$ be the corresponding analytic bisemisheaves of the analytic bilinear spaces $\mathcal{H}^\pm_h$, $H^\pm_h$, $V^\pm_{m;h}$ or $V^\pm_{e;h}$.

2. Consider the diagonal, complete, magnetic or electric tensor product between a right and a left elliptic (linear differential) operator D_R and D_L acting respectively on a right and a left algebraic or analytic semisheaf $\widetilde M^{(s)}_{L_R}$ or $\widetilde M^{(s)}_R$ and $\widetilde M^{(s)}_L$ or $\widetilde M^{(s)}_{R_L}$ of $\mathcal{H}^\pm_{h,a}$, $H^\pm_{h,a}$, $V^\pm_{m;h,a}$ or $V^\pm_{e;h,a}$:

$$\begin{aligned}
\{D_R,D_L\}\{\widetilde M^{(s)}_{L_R},\widetilde M^{(s)}_L\} &\to (D_R\otimes_{(D),m,e}D_L)(\widetilde M^{(s)}_{L_R}\otimes_{(D),m,e}\widetilde M^{(s)}_L)\\
\{D_R,D_L\}\{\widetilde M^{(s)}_{R_L},\widetilde M^{(s)}_R\} &\to (D_R\otimes_{(D),m,e}D_L)(\widetilde M^{(s)}_{R_L}\otimes_{(D),m,e}\widetilde M^{(s)}_R)\ .
\end{aligned}$$

The index [Ati3] of a diagonal, complete, magnetic or electric product of a right and a left elliptic operators is given by:

$$\gamma_{R\times L}(D_R\otimes_{(D),m,e}D_L)=\gamma_R(D_R)\times\gamma_L(D_L)$$

taking into account that $\gamma_{R,L}(D_{R,L})$ is the index of a right (resp. left) operator.

Furthermore, we have that:

$$\gamma_{R,L}(D_R\otimes_{(D),m,e}D_L)=\dim\operatorname{Ker}(D_R\otimes_{(D),m,e}D_L)-\dim\operatorname{co}\operatorname{Ker}(D_R\otimes_{(D),m,e}D_L)\ .$$

3. If the complete, diagonal, magnetic or electric tensor product between a right and a left operator is bounded and has a finite-dimensional kernel and cokernel, then it is a complete, diagonal, magnetic or electric Fredholm bioperator, noted $(T_{F_R}\otimes_{(D),m,e}T_{F_L})$.

4. Let $\mathcal{L}^B_{R,L}(\widetilde M^{(s)}_{R,L})$ denote the algebra of right (resp. left) bounded operators $T_{R,L}$ acting respectively on the right or left semisheaf $\widetilde M^{(s)}_R$ or $\widetilde M^{(s)}_L$.

Then, the algebra of right (resp. left) self-adjoint bounded operators $T_{R,L}$ acting on $\mathcal{H}^\pm_{a,h}$, $H^\pm_{a,h}$, $V^\pm_{m;a,h}$ or $V^\pm_{e;a,h}$ will be noted $\mathcal{L}^B_{R,L}(\mathcal{H}^\pm_{a,h})$, $\mathcal{L}^B_{R,L}(H^\pm_{a,h})$, $\mathcal{L}^B_{R,L}(V^\pm_{m;a,h})$ or $\mathcal{L}^B_{R,L}(V^\pm_{e;a,h})$ while the bialgebra of diagonal, complete, magnetic or electric tensor product of a right and a left bounded operators $(T_R\otimes_{(D),m,e}T_L)$ acting on the corresponding bilinear spaces will be noted $\mathcal{L}^B_{R\times_D L}(\mathcal{H}^\pm_{a,h})\equiv\mathcal{L}^B_R\otimes_D\mathcal{L}^B_L(\mathcal{H}^\pm_{a,h})$, $\mathcal{L}^B_{R\times L}(H^\pm_{a,h})$, $\mathcal{L}^B_{R\times_m L}(V^\pm_{m;a,h})$ or $\mathcal{L}^B_{R\times_e L}(V^\pm_{e;a,h})$.

Lemma 3.2.2 *The bialgebra $\mathcal{L}^B_{R\times_D L}(\mathcal{H}^\pm_a)$ is abelian.*

Proof. Considering that $\mathcal{H}^\pm_a$ is characterized by a diagonal metric, the bialgebra $\mathcal{L}^B_{R\times_D L}(\mathcal{H}^\pm_a)$ must then be abelian. ∎

Definitions 3.2.3 (1. Self-adjointness) Consider that the right and left bounded operators T_R and T_L are self-adjoint, i.e. that we have $T_R\equiv T^\dagger_L=T_L$.

A left and a right involutions are then defined by:

$$\begin{aligned} i_L &: \quad T_L \to T_L^\dagger \equiv T_R \ , \\ i_R &: \quad T_R \to T_R^\dagger \equiv T_L \ . \end{aligned}$$

The physical interpretation of the self-adjointness consists in the fact that the action of the self-adjoint operator T_R on the co-left semisheaf $\widetilde M_{L_R}$ is equal to its antiunitary involutary action on the left semisheaf $\widetilde M_L$.

The mathematical origin of the self-adjointness results from the fact that the centralizer of the co-left semimodule M_{L_R} is $Z_0(L^-)$ while the centralizer of the left semimodule M_L is $Z_0(L^+)$ according to [Pie4].

(2. Complete, diagonal, magnetic or electric norm topologies) The complete, diagonal, magnetic or electric norm topology on $(T_R \otimes_{(D),m,e} T_L)$ will be given by:

$$\|T_R \otimes_{(D),m,e} T_L\| = \sup \ \frac{(T_R\psi_{L_R}, T_L\psi_L)_{(D),m,e}}{(\psi_{L_R}, \psi_L)_{(D),m,e}} \quad \forall\ \psi_{L_R} \in \widetilde M_{L_R} \ , \ \forall\ \psi_L \in \widetilde M_L$$

where $(\cdot,\cdot)_{(D),m,e}$ is respectively a complete, diagonal, magnetic or electric internal bilinear form as introduced in definition 3.1.7 and characterized by a complete, diagonal, magnetic or electric metric.

(3.) A **weight** on the algebra $\mathcal{L}^B_{R,L}(H^+_a)$ is given by the positive bilinear form $(T_R\psi_{L_R}, \psi_L)$ or $(\psi_{L_R}, T_L\psi_L)$ which is a map from $\mathcal{L}^B_{R,L}(M_{L_R} \times M_L)$ into $\mathbb{C}$ for every section $\psi_{L_R} \in \widetilde M_{L_R}$ and $\psi_L \in \widetilde M_L$.

Similarly, a weight on the bialgebra $(\mathcal{L}^B_R \otimes \mathcal{L}^B_L)(H^+_a)$ is given by the positive bilinear form $(T_R\psi_{L_R}, T_L\psi_L)$ which is a map from $(\mathcal{L}^B_R(M_{L_R}) \otimes \mathcal{L}^B_L(M_L))$ into $\mathbb{C}$ for all $T_{R,L} \in \mathcal{L}^B_{R,L}$.

Proposition 3.2.4 *The extended bilinear Hilbert spaces $H^\pm_{a,h}$ are the natural representation spaces for the algebras and the bialgebras of bounded operators.*

Proof. The representation of a group G in a linear Hilbert space h is an application such that to each element g of G corresponds a linear operator $T(g)$. In the finite-dimensional case, $T(g)$ is defined by a matrix of $M_n(K)$.

On the other hand, the enveloping algebra M^e of the semimodule $M^{(s)}_{R,L}$ is given by

$$M^e_{(s)} = M^{(s)}_R \otimes_{\mathbb{A}_R \times \mathbb{A}_L} M^{(s)}_L$$

where $M^{(s)}_R$ (resp. $M^{(s)}_L$) must be considered as the opposite algebra of $M^{(s)}_L$ (resp. $M^{(s)}_R$).

If $M^{(s)}_{R,L}$ is a projective right (resp. left) semimodule of dimension n , then $M^{(s)}_{R,L} \simeq \mathbb{A}^n_{R,L}$ and we have:

$$M^e_{(s)} \simeq \mathrm{End}_{\mathbb{A}_R \times \mathbb{A}_L}(M^{(s)}_{R,L}) \simeq \mathrm{End}_{\mathbb{A}_R \times \mathbb{A}_L}(((\mathbb{A}_R \times \mathbb{A}_L)^n) = M_n(\mathbb{A}_R \times \mathbb{A}_L)$$

where $M_n(\mathbb{A}_R \times \mathbb{A}_L)$ is the ring of matrices of order n over $\mathbb{A}_R \times \mathbb{A}_L$.

The homomorphism $E_{M^e} : M^{(s)}_{R,L} \to M_n(\mathbb{A}_{R,L})$ is the n-dimensional representation of $M^{(s)}_{R,L}$.

As the extended bilinear Hilbert space $H^\pm_{a,h}$ is composed of a bisemisheaf defined on a bisemimodule $(M^{(s)}_{L_R} \otimes M^{(s)}_L)$ or $(M^{(s)}_{R_L} \otimes M^{(s)}_R)$ which is an enveloping algebra isomorphic to $M_n(\mathbb{A}_R \times \mathbb{A}_L)$,we have that $H^\pm_{a,h}$ is the natural representation space for the algebras and the bialgebras of bounded operators acting on the above defined semisheaves or bisemisheaves. ∎

Definition 3.2.5 (Algebras and bialgebras of von Neumann on extended bilinear Hilbert spaces) **1)** A right (resp. left) algebra of von Neumann $\mathbb{M}^{a,h}_{R,L}(H^{\pm}_{a,h})$ in the representation algebraic or analytic extended bilinear Hilbert space $H^{\pm}_{a,h}$ is an involutive subalgebra of $\mathcal{L}^{B}_{R,L}(H^{\pm}_{a,h})$ having a closed norm topology.

2) A bialgebra of von Neumann $\mathbb{M}^{a,h}_{R\times L}(H^{+}_{a,h})$ in the representation space $H^{\pm}_{a,h}$ is an involutive subalgebra of $\mathcal{L}^{B}_{R\times L}(H^{\pm}_{a,h})$ having a closed norm topology.

3) A bialgebra of von Neumann $\mathbb{M}^{a,h}_{R\times_D L}(\mathcal{H}^{\pm}_{a,h})$, $\mathbb{M}^{a,h}_{R\times_m L}(V^{\pm}_{m;a,h})$ or $\mathbb{M}^{a,h}_{R\times_e L}(V^{\pm}_{e;a,h})$ is an involutive subalgebra of respectively $\mathcal{L}^{B}_{R\times_D L}(\mathcal{H}^{\pm}_{a,h})$, $\mathcal{L}^{B}_{R\times_m L}(V^{\pm}_{m;a,h})$ or $\mathcal{L}^{B}_{R\times_e L}(V^{\pm}_{e;a,h})$ having a closed norm topology.

Proposition 3.2.6 *Between the algebraic and analytic von Neumann algebras and bialgebras, we have the following isomorphisms:*

$$\begin{aligned}
i_{\mathbb{M}^a_{R,L}-\mathbb{M}^h_{R,L}} &: \mathbb{M}^{a}_{R,L}(H^{\pm}_{a}) \to \mathbb{M}^{h}_{R,L}(H^{\pm}_{h}) \;, \\
i_{\mathbb{M}^a_{R\times L}-\mathbb{M}^h_{R\times L}} &: \mathbb{M}^{a}_{R\times L}(H^{\pm}_{a}) \to \mathbb{M}^{h}_{R\times L}(H^{\pm}_{h}) \;, \\
i_{\mathbb{M}^a_{R\times_D L}-\mathbb{M}^h_{R\times_D L}} &: \mathbb{M}^{a}_{R\times_D L}(\mathcal{H}^{\pm}_{a}) \to \mathbb{M}^{h}_{R\times_D L}(\mathcal{H}^{\pm}_{h}) \;, \\
i_{\mathbb{M}^a_{R\times_m L}-\mathbb{M}^h_{R\times_m L}} &: \mathbb{M}^{a}_{R\times_m L}(V^{\pm}_{m;a}) \to \mathbb{M}^{h}_{R\times_m L}(V^{\pm}_{m;h}) \;, \\
i_{\mathbb{M}^a_{R\times_e L}-\mathbb{M}^h_{R\times_e L}} &: \mathbb{M}^{a}_{R\times_e L}(V^{\pm}_{e;a}) \to \mathbb{M}^{h}_{R\times_e L}(V^{\pm}_{e;h}) \;.
\end{aligned}$$

Proof. This results immediately from the isomorphisms between the Eisenstein and the analytic de Rham cohomologies according to proposition 3.1.13. ∎

Definitions 3.2.7 (Shifted actions of (bi)operators on the functional representations of (bi)linear semigroups) **1)** Let $T_{R,L}\in\mathbb{M}^{h}_{R,L}(H^{\pm}_{h})$ be a right (resp. left) bounded linear operator of the algebra of von Neumann $\mathbb{M}^{h}_{R,L}(H^{\pm}_{h})$. It can be assumed that this operator $T_{R,L}$ is a differential operator of the form $T_{R,L}=\sum\limits_{n}\sum\limits_{r}c_{mr}U^{m}_{r}$ where U is the unitary translation operator. This operator is supposed to be a regular representation of the discrete compact triangular semigroup $T^{t}_{m}(\mathbb{C}\,)$ (resp. $T_{m}(\mathbb{C}\,)$) in the extended bilinear Hilbert space $H^{\pm}_{h}$ such that $T^{t}_{m}(\mathbb{C}\,)$ (resp. $T_{m}(\mathbb{C}\,)$) acts on the right (resp. left) n-dimensional semisheaf $\widetilde{M}^{(s)}_{R,L}$ of $H^{\pm}_{h}$ with $m\le n$.

Similarly, let $(T_R\otimes T_L)$ be the tensor product of a right and a left bounded linear operators acting on the bisemisheaf of the extended bilinear Hilbert space $H^{\pm}_{h}$. So, $(T_R\otimes T_L)$ belongs to the bialgebra of von Neumann $\mathbb{M}^{h}_{R\times L}(H^{\pm}_{h})$. This bioperator $(T_R\otimes T_L)$ is supposed to be the regular representation of the product $GL_m(\mathbb{C}\times\mathbb{C}\,)=T^{t}_{M}(\mathbb{C}\,)\times T_m(\mathbb{C}\,)$ of the compact semigroups $T^{t}_{m}(\mathbb{C}\,)$ and $T_m(\mathbb{C}\,)$.

2) More concretely, a differential bioperator $(T_R\otimes T_L)$, being the regular representation of $GL_m(\mathbb{C}\times\mathbb{C}\,)$ in a bisemisheaf $\widetilde{M}_R\otimes\widetilde{M}_L$ on a $GL_n(\mathbb{A}_R\times\mathbb{A}_L)$-bisemimodule $M_R\otimes M_L$, has a representation in the bilinear Lie algebra $\mathfrak{g}\ell_m(\mathbb{C}\times\mathbb{C}\,)$ of the bilinear Lie semigroup $GL_m(\mathbb{C}\times\mathbb{C}\,)$. Then, the action of the differential bioperator $(T_R\otimes T_L)$ on the bisemisheaf $\widetilde{M}_R\otimes\widetilde{M}_L$ is equivalent to:

a) consider a shift in $(m\times m)$-dimensions of the bisemisheaf $\widetilde{M}_R\otimes\widetilde{M}_L$ constituting a functional representation of the bilinear Lie semigroup $GL_n(\mathbb{A}_R\times\mathbb{A}_L)$ leading to the homomorphism of the functional representation of the bilinear semigroup:

$$\begin{aligned}T_R\otimes T_L:\quad & \widetilde M_R\otimes\widetilde F_L=\mathrm{FRep}(GL_n(\mathbb{A}_R\times\mathbb{A}_L))\\ &\qquad\longrightarrow \widetilde M^a_{R_{n[m]}}\otimes\widetilde M^a_{L_{n[m]}}=\mathrm{FRep}(GL_{n[m]}((\mathbb{A}_R\otimes\mathbb{C})\times(\mathbb{A}_L\otimes\mathbb{C})))\end{aligned}$$

where $\mathrm{FRep}(GL_{n[m]}((\mathbb{A}_R\otimes\mathbb{C})\times(\mathbb{A}_L\otimes\mathbb{C}))$ denotes the functional representation of the bilinear semigroup $GL_n(\mathbb{A}_R\times\mathbb{A}_L)$ shifted in $(m\times m)$ dimensions.

b) map $\widetilde M_R\otimes\widetilde M_L$ in the bisemisheaf $\widetilde M^a_{Rn[m]}\otimes\widetilde M^a_{Ln[m]}$ shifted in $(m\times m)$ dimensions such that $\widetilde M^a_{Rn[m]}\otimes\widetilde M^a_{Ln[m]}$ be a perverse bisemisheaf, i.e. an object of the derived category $D(\widetilde M_R\otimes\widetilde M_L,\mathbb{C})$ [Pie12]. $\widetilde M^a_{Rn[m]}\otimes\widetilde M^a_{Ln[m]}$ will be written in condensed form $\widetilde M^a_R\otimes\widetilde M^a_L$.

3) Similarly, we have on the bilinear subsemigroup $K^D_{R\times L;n}(\overline{\mathbb{Z}}^2_{pq})$ the following shifted action resulting from the action of the differential bioperator:

$$\begin{aligned}T_R\otimes T_L:\quad & K^D_{R\times L;n}(\overline{\mathbb{Z}}^2_{pq})=D_n(\overline{\mathbb{Z}}^2_{pq})\times[UT^t_n(\overline{\mathbb{Z}}_{pq})\times UT_n(\mathbb{Z}_{pq})]\\ &\longrightarrow K^D_{R\times L;n[m]}(\overline{\mathbb{Z}}^2_{pq}\otimes\mathbb{C}^2)\\ &\qquad=D_{n[m]}(\overline{\mathbb{Z}}^2_{pq}\otimes\mathbb{C}^2)\times[UT^t_{n[m]}\overline{\mathbb{Z}}_{pq}\otimes\mathbb{C})\times UT_{n[m]}(\overline{\mathbb{Z}}_{pq}\otimes\mathbb{C})]\end{aligned}$$

where

- $D_{n[m]}(\overline{\mathbb{Z}}^2_{pq}\otimes\mathbb{C})=D_n(\overline{\mathbb{Z}}^2_{pq})\times D_{n[m]}(\mathbb{C}^2)$

  such that $D_{n[m]}(\mathbb{C}^2)$ is the subgroup of diagonal matrices of order n shifted in m dimensions, i.e. whose elements $d_{n[m]}(\mathbb{C}^2)$ are

$$\begin{array}{cl}\left(\begin{matrix}\times & & & & \\ & \times & & 0 & \\ & & 1 & & \\ & 0 & & 1 & \\ & & & & 1\end{matrix}\right) & \begin{matrix}\left.\vphantom{\begin{matrix}a\\b\end{matrix}}\right] m\\ \left.\vphantom{\begin{matrix}a\\b\\c\end{matrix}}\right] n-m\end{matrix}\end{array}$$

  The m shifts of $d_{n[m]}(\mathbb{C}^2)$ are the squares of the infinitesimal generators of the Lie algebra of the diagonal subgroup $D_m(\mathbb{C})$ of order m .

- $UT_{n[m]}(\overline{\mathbb{Z}}_{pq}\otimes\mathbb{C})=UT_n(\overline{\mathbb{Z}}_{pq})\times UT_{n[m]}(\mathbb{C})$

  such that the shifts in m dimensions of $UT_{n[m]}(\mathbb{C})$ correspond to the generators of the nilpotent Lie algebra.

4) Under the action of $(T_R\otimes T_L)$, the functional representation of the bilinear parabolic subgroup

$P_n(\mathbb{A}_{L^T_{\overline{v}1}} \times \mathbb{A}_{L^T_{v1}})$ is shifted in $(m \times m)$ dimensions according to:

$$\begin{aligned} T_R \otimes T_L :\ & \mathrm{FRep}(P_n(\mathbb{A}_{L^T_{\overline{v}_1}} \times \mathbb{A}_{L^T_{v_1}})) = \mathrm{FRep}(D_n(\mathbb{A}_{L^T_{\overline{v}1}} \times \mathbb{A}_{L^T_{v1}}) \times [UT^t_n(\mathbb{A}_{L^T_{\overline{v}1}}) \times UT_n(\mathbb{A}_{L^T_{v1}})]) \\ & \longrightarrow \mathrm{FRep}(P_{n[m]}((\mathbb{A}_{L^T_{\overline{v}1}} \otimes \mathbb{C}) \times (\mathbb{A}_{L^T_{v1}} \otimes \mathbb{C}))) \\ & \quad = \mathrm{FRep}(D_{n[m]}((\mathbb{A}_{L^T_{\overline{v}1}} \otimes \mathbb{C}) \times (\mathbb{A}_{L^T_{v1}} \otimes \mathbb{C}))[UT^t_{n[m]}(\mathbb{A}_{L^T_{\overline{v}1}} \otimes \mathbb{C}) \times UT_{n[m]}(\mathbb{A}_{L^T_{v1}} \otimes \mathbb{C})]) \end{aligned}$$

where:

$$D_{n[m]}(\mathbb{A}_{L^T_{v1}} \otimes \mathbb{C}) = D_n(\mathbb{A}_{L^T_{\overline{v}1}}) \times D_{n[m]}(\mathbb{C}) .$$

3.2.8 Shifted Shimura bisemivariety

Under the action of the differential bioperator $(T_R \otimes T_L)$, the functional representation of the Shimura bisemivariety $\partial \overline{S}_{K^D_{R\times L,n}}$ given by the bisemisheaf $\widetilde{M}_R \otimes \widetilde{M}_L = \mathrm{FRep}(\partial \overline{S}_{K^D_{R\times L,n}}) = \mathrm{FRep}(P_n(\mathbb{A}_{L^T_{\overline{v}1}} \times \mathbb{A}_{L^T_{v1}}) \setminus GL_n(\mathbb{A}_R \times \mathbb{A}_L) \Big/ K^D_{R\times L;n}(\overline{\mathbb{Z}}^2_{pq}))$ is shifted in $(m \times m)$ dimensions according to:

$$T_R \otimes T_L : \quad \widetilde{M}_R \otimes \widetilde{M}_L = \mathrm{FRep}(\partial \overline{S}_{K^D_{R\times L;n}}) \longrightarrow \widetilde{M}^a_R \otimes \widetilde{M}^a_L = \mathrm{FRep}(\partial \overline{S}_{K^D_{R\times L;n[m]}})$$

where the shifted Shimura bisemivariety $\partial \overline{S}_{K^D_{R\times L;n[m]}}$ is given by:

$$\begin{aligned} \partial \overline{S}_{K^D_{R\times L;n[m]}} = & P_{n[m]}(\mathbb{A}_{L^T_{\overline{v}1}} \otimes \mathbb{C}) \times (\mathbb{A}_{L^T_{v1}} \otimes \mathbb{C})) \setminus \\ & GL_n((\mathbb{A}_R \otimes \mathbb{C}) \times (\mathbb{A}_L \otimes \mathbb{C})) \Big/ K^D_{R\times L;n[m]}(\overline{\mathbb{Z}}^2_{pq} \otimes \mathbb{C}^2) . \end{aligned}$$

Proposition 3.2.9 *The semimodules $M_{L_{R,L}}$, $M^a_{L_{R,L}}$ and $M^s_{L_{R,L}}$ have a basis of dimension $i = t$ corresponding to the upper degree of the Galois extensions.*

Proof. Under the automorphisms $\sigma_{R,L}$ of the algebraic semigroup $T^t_n(\mathbb{A}_R)$ (resp. $T_n(\mathbb{A}_L)$), the semimodule $M_{L_{R,L}}$ decomposes into:

$$M_{L_{R,L}} = \bigoplus_{\sigma_{R,L}=1}^{t} M_{L_{R,L}}(\sigma_{R,L})$$

where the number t of automorphisms is the degree of the Galois extension.

Now, under the cross action of $T^t_{n[m]}(\mathbb{A}_R \otimes \mathbb{C})$ (resp. $T_{n[m]}(\mathbb{A}_L \otimes \mathbb{C})$), the semimodule $M^a_{L_{R,L}}$ decomposes into:

$$M^a_{L_{R,L}} = \bigoplus_{\sigma_{R,L}} M^a_{L_{R,L}}(\sigma_{R,L})$$

where the number of cross automorphisms is also t , corresponding to the same upper degree of Galois extension as for the semimodule $M_{L_{R,L}}$.

So, the semimodules $M_{L_{R,L}}$ and $M^a_{L_{R,L}}$ have a basis with the same dimension $i = t$.

Referring to the isomorphism between the Eisenstein cohomology and the analytic de Rham cohomology, it appears that the semisheaf $\widetilde{M}^s_{L_{R,L}}$ must have a basis $\{e^i_{R,L(s)}\}^t_{i=1}$ with the same dimension $i = t$ as the algebraic basis $\{e^i_{R,L(a)}\}^t_{i=1}$ of the semisheaf $\widetilde{M}^a_{L_{R,L}}$. ∎

3.2.10 Shift of the Eisenstein bicohomology

Let $(T_R\otimes T_L)$ be the tensor product of bounded differential operators of the von Neumann bialgebra $\mathbb{M}^a_{R\times L}(H^\pm_a)$. Its shifted action on the bilinear Eisenstein cohomology will be:

$$T_R\otimes T_L: \quad H^*_{R\times L}(\partial\overline{S}_{K^D_{R\times L;n}},\widetilde{M}_R\otimes\widetilde{M}_L) \longrightarrow H^*_{R\times L}(\partial\overline{S}_{K^D_{R\times L;[n[m]}},\widetilde{M}^a_R\otimes\widetilde{M}^a_L)$$

such that $H^*_{R\times L}(\cdot,\cdot)$ decomposes into the double sum $\bigoplus\limits_\mu\bigoplus\limits_{m_\mu}$, associated to the places μ with multiplicities m_μ of the semifield L^T_v (or $L^T_{\overline{v}}$), according to:

$$\begin{aligned}&H^*_{R\times L}(\partial\overline{S}_{K^D_{R\times L;[n[m]}},\widetilde{M}^a_R\otimes\widetilde{M}^a_L)\\ &\quad=\bigoplus_\mu\bigoplus_{m_\mu}\mathrm{Ind}^{GL_{n[m]}((\mathbb{A}_R\otimes\mathbb{C})\times(\mathbb{A}_L\otimes\mathbb{C}))}_{K^D_{R\times L;n[m]}(\mathbb{Z}^2_{pq}\otimes\mathbb{C}^2)}\ H^*_{R\times L}(S^{M(\mathbb{A}_R\otimes\mathbb{C})\times M(\mathbb{A}_L\otimes\mathbb{C})},H^*(\tilde{u}_{K^D_{R\times L;n[m]}},\widetilde{M}^a_R\otimes\widetilde{M}^a_L))\end{aligned}$$

where

$$\begin{aligned}&S^{M(\mathbb{A}_R\otimes\mathbb{C})\times M(\mathbb{A}_L\otimes\mathbb{C})}\\ &\quad=D_{n[m]}((\mathbb{A}_{L^T_{\overline{v}^1}}\otimes\mathbb{C})\times(\mathbb{A}_{L^T_{v^1}}\otimes\mathbb{C}))\setminus D_{n[m]}((\mathbb{A}_R\otimes\mathbb{C})\times(\mathbb{A}_L\otimes\mathbb{C}))\Big/D_{n[m]}(\overline{\mathbb{Z}}^2_{pq}\otimes\mathbb{C}^2)\end{aligned}$$

following the notations of definition 1.1.11.

The coefficient system given by the Lie algebra cohomology $H^*(\tilde{u}_{K^D_{R\times L;n[m]}},\widetilde{M}^a_R\otimes\widetilde{M}^a_L)$ decomposes according to the cosets of $GL_{n[m]}((\mathbb{A}_R\otimes\mathbb{C})\times(\mathbb{A}_L\otimes\mathbb{C}))\big/K^D_{R\times L;n[m]}(\overline{\mathbb{Z}}^2_{pq}\otimes\mathbb{C}^2)$ generating the set of subrepresentatives $\{\widetilde{M}^a_{\overline{v}_{\mu,m_\mu}}\otimes\widetilde{M}^a_{v_{\mu,m_\mu}}\}^q_{\mu=1}$ on $GL_{n[m]}((\mathbb{A}_R\otimes\mathbb{C})\times(\mathbb{A}_L\otimes\mathbb{C}))$.

Note that it was proved in [Pie12] that the shifted bilinear Eisenstein cohomology $H^*_{R\times L}(\partial\overline{S}_{K^D_{R\times L;[n[m]}},\widetilde{M}^a_R\otimes\widetilde{M}^a_L)$ is isomorphic to the adjoint functional representation Ad FRep$(GL_n(\mathbb{A}_R\times\mathbb{A}_L))$ which corresponds to FRep$(GL_{n[m]}((\mathbb{A}_R\otimes\mathbb{C})\times(\mathbb{A}_L\otimes\mathbb{C}))$ where FRep$(\cdot)$ denotes the functional representation of the considered bilinear semigroup.

Proposition 3.2.11 *Applying the Kostant's theorem, we can decompose the bilinear Eisenstein cohomology $H^*_{R\times L}(\partial\overline{S}_{K^D_{R\times L;[n[m]}},\widetilde{M}^a_R\otimes\widetilde{M}^a_L)$ into sums of products of pairs of one-dimensional eigenspaces following:*

$$\begin{aligned}&H^*_{R\times L}(\partial\overline{S}_{K^D_{R\times L;[n[m]}},\widetilde{M}^a_R\otimes\widetilde{M}^a_L)\\ &\quad=\bigoplus_\mu\bigoplus_{m_\mu}\bigoplus_{1_\ell}\mathrm{Ind}^{GL_{n[m]}((\mathbb{A}_R\otimes\mathbb{C})\times(\mathbb{A}_L\otimes\mathbb{C}))}_{K^D_{R\times L;n[m]}}\\ &\qquad\qquad\times H^{1_\ell,1_\ell}_{R\times L}(S^{M(\mathbb{A}_R\otimes\mathbb{C})\times M(\mathbb{A}_L\otimes\mathbb{C})},H^{1_\ell,1_\ell}(\tilde{u}_{K^D_{R\times L;n[m]}},\widetilde{M}^{a_{1_\ell}}_R(\mu,m_\mu)\otimes\widetilde{M}^{a_{1_\ell}}_L(\mu,m_\mu)))\ .\end{aligned}$$

Then, the decomposition of the Lie algebra cohomology $H^{1_\ell,1_\ell}(\tilde{u}_{K^D_{R\times L;n[m]}},\widetilde{M}^{a_{1_\ell}}_R(\mu,m_\mu)\otimes\widetilde{M}^{a_{1_\ell}}_L(\mu,m_\mu))$ into sums of products of pairs of one-dimensional eigenspaces involves a decomposition of the bilinear Hilbert space $H^\pm_h$ into a tower of embedded bilinear Hilbert subspaces $H^\pm_h\{\mu\}^q_1$ decomposing into pairs of one-dimensional subspaces.

Proof. 1. The decomposition of the shifted bilinear Eisenstein cohomology into sums of products of pairs of one-dimensional eigenspaces results from proposition 3.1.11.

2. The embedded representation subspaces $H^\pm_a\{\mu\}$ of $H^\pm_a\simeq\widetilde{M}_{R(P)/_cL}$ forms a Jordan-Hölder serie for the homomorphism

$$\Pi_{H-\mathfrak{g}\ell}:\quad H^*_{R\times L}(\tilde{u}_{K^D_{R\times L;n[m]}},\widetilde{M}^{a_{1_\ell}}_R\otimes\widetilde{M}^{a_{1_\ell}}_L)\longrightarrow\mathfrak{g}\ell(H^\pm_a)$$

of the Lie algebra $H^*_{R\times L}(\cdot,\cdot)$ into the Lie algebra $\mathfrak{g}\ell(H^{\pm}_a)$ of the automorphisms of $H^{\pm}_a$ isomorphic to $H^{\pm}_h$. We thus have a sequence of embedded bilinear Hilbert subspaces: $H^{\pm}_h\{1\}\subset\cdots\subset H^{\pm}_h\{\mu\}\subset\cdots\subset H^{\pm}_h\{q\}$ where

$$H^{\pm}_h\{\mu\}=\bigoplus_{\nu=1}^{\mu}H^{\pm}_h(\nu)$$

with $H^+_h(\nu)$ the extended bilinear Hilbert subspace constituted by the ν-th subbisemisheaf $\widetilde{M}^s_{L_R}(\nu)\otimes\widetilde{M}^s_L(\nu)$ corresponding to the ν-th biplace of $L^T_{\overline{v}}\times L^T_v$. ■

Remarks 3.2.12 In order to include the above-mentioned cases in a uniform presentation, we shall admit until the end of Section 3.2 that the integer “ i ” refers to:

a) a Galois extension degree related to the dimension of the basis of the semimodule $M^a_{L_{R,L}}$;

or b) a class of degrees of Galois extensions which corresponds to the global class residue degree f_{v_i} (see 1.1.4) labelling the i-th coset of $GL_{n[m]}((\mathbb{A}_R\otimes\mathbb{C})\times(\mathbb{A}_L\otimes\mathbb{C}))/K^D_{R\times L;,[m]}(\overline{\mathbb{Z}}^2_{pq}\otimes\mathbb{C}^2)$ (in this case, $i=\mu$).

Definition 3.2.13 (Random bioperators on analytic bilinear Hilbert spaces) Let $T_R\otimes T_L$ be the tensor product of a right and a left bounded linear operators being the regular representation of $GL_m(\mathbb{C}\times\mathbb{C})$ in $(\widetilde{M}_R\otimes\widetilde{M}_L)$.

$GL_m(\mathbb{C}\times\mathbb{C})$ has for bilinear semigroup of inner automorphisms [Kac] $\operatorname{Int}\Gamma^h_R\times\operatorname{Int}\Gamma^h_L$ (see definition 3.2.7) and has the inner conjugacy biclasses noted $g^h_R\times g^h_L$ if the fixed bielement is of dimension 1 with respect to the basis of $(\widetilde{M}^a_R\otimes\widetilde{M}^a_L)$ in the case a) of 3.2.12.

$GL_m(\mathbb{C}\times\mathbb{C})$ has $\Gamma^h_R\times\Gamma^h_L$ for bilinear semigroup of modular automorphisms and has the modular conjugacy biclasses $\gamma^h_R\times\gamma^h_L$ if the fixed bielement, which is a normal bilinear subsemigroup, is of dimension N^2 with respect to the (algebraic) basis of $(M^a_R\otimes M^a_L)$ in the case b) of 3.2.12.

The right (resp. left) bounded linear operator $T_{R,L}(\Gamma^h_{R,L})$ is a random operator if it decomposes into a set of right (resp. left) bounded linear operators $\{T_{R,L}(g^h_{R,L}(i))\}$, $\forall\ g^h_{R,L}(i)\in\operatorname{Int}(\Gamma^h_{R,L})$ or $\{T_{R,L}(\gamma^h_{R,L}(i))\}$, $\forall\ \gamma^h_{R,L}(i)\in\Gamma^h_{R,L}$.

So, the tensor product $(T_R(\Gamma^h_R)\otimes T_L(\Gamma^h_L))$ of a right and a left bounded linear operators is a random bioperator if it decomposes into a tensor product of a set of right and left bounded linear operators:

$$\begin{aligned}
&T_R(\Gamma^h_R)\otimes T_L(\Gamma^h_L)=\{T_R(g^h_R(i))\otimes T_L(g^h_L(i'))\}^t_{i,i'=1}\\
\text{(resp.}\quad &T_R(\Gamma^h_R)\otimes T_L(\Gamma^h_L)=\{T_R(\gamma^h_R(i))\otimes T_L(\gamma^h_L(i'))\}^q_{i,i'=1}\ ),\qquad t\geq q\ .
\end{aligned}$$

Let

$$\begin{aligned}
&g^h_{R,L}\{i\}=\bigoplus_{j=1}^{i}g^h_{R,L}(j)\ ,\qquad 1\leq i\leq t\\
\text{(resp.}\quad &\gamma^h_{R,L}\{i\}=\bigoplus_{\nu=1}^{i}g^h_{R,L}(\nu)\ ,\qquad 1\leq\nu\leq q\ ),
\end{aligned}$$

denote the sum of inner (resp. modular) conjugacy classes of $\Gamma^h_{R,L}$. This leads to define a sum of inner (resp. modular) random operators by:

$$T^D_{R,L}(g^h_{R,L}\{i\}) = \bigoplus_{j=1}^{i} T^D_{R,L}(g^h_{R,L}(j))$$

$$(\text{resp.} \qquad T^D_{R,L}(\gamma^h_{R,L}\{i\}) = \bigoplus_{\nu=1}^{i} T^D_{R,L}(g^h_{R,L}(\nu))\ ),$$

such that

$$T^D_{R,L}(g^h_{R,L}\{i\}) \in \mathbb{M}^{h(\mathrm{in})}_{R,L}(H^{\mp}_h\{i\})$$

$$(\text{resp.} \qquad T^D_{R,L}(\gamma^h_{R,L}\{i\}) \in \mathbb{M}^{h(\mathrm{mod})}_{R,L}(H^{\mp}_h\{i\})\ ),$$

where $\mathbb{M}^{h(\mathrm{in})}_{R,L}(H^{\mp}_h\{i\})$ (resp. $\mathbb{M}^{h(\mathrm{mod})}_{R,L}(H^{\mp}_h\{i\})$) is an inner (resp. modular) von Neumann subalgebra referring to the i-th sum of inner (resp. modular) random operators.

So, a tower of inner (resp. modular) von Neumann subalgebras can be intoduced by:

$$\mathbb{M}^{h(\mathrm{in})}_{R,L}(H^{\mp}_h\{1\}) \subset \cdots \subset \mathbb{M}^{h(\mathrm{in})}_{R,L}(H^{\mp}_h\{i\}) \subset \cdots \subset \mathbb{M}^{h(\mathrm{in})}_{R,L}(H^{\mp}_h\{t\})$$

$$(\text{resp.} \quad \mathbb{M}^{h(\mathrm{mod})}_{R,L}(H^{\mp}_h\{1\}) \subset \cdots \subset \mathbb{M}^{h(\mathrm{mod})}_{R,L}(H^{\mp}_h\{i\}) \subset \cdots \subset \mathbb{M}^{h(\mathrm{mod})}_{R,L}(H^{\mp}_h\{q\})\ ),$$

such that:

$$\mathbb{M}^{h(\mathrm{in})}_{R,L}(H^{\mp}_h\{i\}) = \bigoplus_{j=1}^{i} \mathbb{M}^{h(\mathrm{in})}_{R,L}(H^{\mp}_h(j))$$

$$(\text{resp.} \quad \mathbb{M}^{h(\mathrm{mod})}_{R,L}(H^{\mp}_h\{i\}) = \bigoplus_{\nu=1}^{i} \mathbb{M}^{h(\mathrm{mod})}_{R,L}(H^{\mp}_h(\nu))\ ).$$

Proposition 3.2.14 *Let $T_{R,L}(g^h_{R,L}(t))$ and $T_{R,L}(g^h_{R,L}(r))\}$ be two right or left inner random operators such that $t < r$.*

Then, the random bioperator $T_R(g^h_R(r)) \otimes T_L(g^h_L(r))\}$ is an extension of the random bioperator $T_R(g^h_R(t)) \otimes T_L(g^h_L(t))$ corresponding to a Galois extension of degree $(r-t)$.

Let $T_{R,L}(\gamma^h_{R,L}(q))$ and $T_{R,L}(\gamma^h_{R,L}(s))$ bet two right or left modular random operators such that $q < s$.

Then, the random bioperator $T_R(\gamma^h_R(s)) \otimes T_L(\gamma^h_L(s))$ is an extension of the random bioperator $T_R(\gamma^h_R(q)) \otimes T_L(\gamma^h_L(q))$ corresponding to a Galois extension of class of degree $(s-q)$.

Proof. Indeed, $g^h_{R,L}(i)$ (resp. $\gamma^h_{R,L}(i)$) is a inner (resp. modular) conjugacy class of the discrete semigroup $T^{(t)}_m(\mathbb{C})$ whose representation semispace $M^s_{R,L}$ has a basis of dimension t whose entire number t (resp. q) corresponds to a Galois extension of degree t or a class of Galois extension degrees q . ∎

Proposition 3.2.15 *Let $\mathbb{M}^h_{R\times L}(H^{\pm}_h)$ be the von Neumann bialgebra of bounded self-adjoint bioperators on the analytic extended bilinear Hilbert space $H^{\pm}_h$.*

Let $\mathbb{M}^h_{R\times L}(H^{\pm}_h\{i\})$ be the von Neumann bialgebra of random bioperators on the analytic extended bilinear subspace $H^{\pm}_h\{i\}$ and let $\mathbb{M}^h_{R\times L}(\mathcal{H}^{\pm}_h\{i\})$ be the corresponding von Neumann bialgebra on the analytic internal bilinear subspace $\mathcal{H}^{\pm}_h\{i\}$.

Then, the discrete (diagonal) spectrum $\sigma_D(T_R\otimes T_L)$ of a bioperator $T_R\otimes T_L\in\mathbb{M}^h_{R\times L}(H^\pm_h)$ is obtained by the isomorphism:

$$\begin{aligned} i^h_{\{i\}^D_{R\times L}}\circ i^h_{\{i\}_{R\times L}}\ :\ \mathbb{M}^h_{R\times L}(H^\pm_h) &\ \to\ \{\mathbb{M}^h_{R\times L}(\mathcal{H}^\pm_h\{i\})\}_i\ ,\\ T_R\otimes T_L &\ \to\ \sigma_D(T_R\otimes T_L)\ , \end{aligned}$$

where the isomorphisms $i^h_{\{i\}_{R\times L}}$ and $i^h_{\{i\}^D_{R\times L}}$ are defined by

$$\begin{aligned} i^h_{\{i\}_{R\times L}} &\ :\ \mathbb{M}^h_{R\times L}(H^\pm_h)\ \to\ \{\mathbb{M}^h_{R\times L}(H^\pm_h\{i\})\}_i\ ,\\ i^h_{\{i\}^D_{R\times L}} &\ :\ \{\mathbb{M}^h_{R\times L}(H^\pm_h\{i\})\}_i\ \to\ \{\mathbb{M}^h_{R\times L}(\mathcal{H}^\pm_h\{i\})\}_i\ . \end{aligned}$$

Proof. The isomorphism $i^h_{\{i\}_{R\times L}}$ is an isomorphism transforming the bounded bioperator $(T_R(\Gamma^h_R)\times T_L(\Gamma^h_L))$ into the set of bounded bioperators $\{T_R(g^h_R\{i\})\otimes T_L(g^h_L\{i'\})\}$ (resp. $\{T_R(\gamma^h_R\{i\})\otimes T_L(\gamma^h_L\{i'\})\}$).

On the other hand, the isomorphism $i^h_{\{i\}^D_{R\times L}}$ is an isomorphism transforming the nonabelian von Neumann subbialgebras $\{\mathbb{M}^h_{R\times L}(H^\pm_h\{i\})\}_i$ into the abelian or diagonal von Neumann subbialgebras $\{\mathbb{M}^h_{R\times L}(\mathcal{H}^\pm_h\{i\})\}_i$ of random bioperators acting on the "diagonal" enveloping algebra $(\mathcal{H}^\pm_h\{i\})$.
$\{\mathbb{M}^h_{R\times L}(\mathcal{H}^\pm_h\{i\})\}_i$ is thus the spectral algebra of the bounded bioperator $(T_R\otimes T_L)$. ∎

Corollary 3.2.16 *Let $\mathbb{M}^h_{R\times_D L}(H^\pm_h)$ be the diagonal bialgebra of von Neumann on the analytic extended bilinear Hilbert space $H^\pm_h$.*

Then, the discrete spectrum $\sigma_D(T_R\otimes_D T_L)$ of the bioperator $T_R\otimes_D T_L\in\mathbb{M}^h_{R\times_D L}(H^\pm_h)$ is obtained by the isomorphism:

$$i^h_{\{i\}^D_{R\times_D L}}\circ i^h_{\{i\}_{R\times_D L}}\ :\ \mathbb{M}^h_{R\times_D L}(H^\pm_h)\to\{\mathbb{M}^h_{R\times_D L}(\mathcal{H}^\pm_h\{i\})\}_i\ .$$

Proof. This proposition is a generalization of the preceding one to the von Neumann bialgebra $\mathbb{M}^h_{R\times_D L}(H^\pm_h)$.

The corresponding spectrum is then defined on the von Neumann bialgebra $\{\mathbb{M}^h_{R\times_D L}(\mathcal{H}^\pm_h\{i\})\}_i$ with a spectrum characterized by a diagonal metric. ∎

Proposition 3.2.17 *There exists a set of spectral bimeasures $\{\mu_R(i)\times_D\mu_L(i)\}$ on the spectrum $\sigma_D(T_R\otimes_D T_L)$ such that every bivector of the space $\mathcal{H}^\pm_h\{i\}$ of the von Neumann bialgebra $\mathbb{M}^h_{R\times_D L}(\mathcal{H}^\pm_h\{i\})$ be an eigenbivector of the bioperator $(T_R\times_D T_L)$ where i is a degree of Galois extension or a class of degrees of Galois extensions.*

Proof. The existence of spectral bimeasures $\{\mu_R(i)\times_D\mu_L(i)\}$ on the spectrum $\sigma_D(T_R\otimes_D T_L)$ is a consequence of the isomorphisms $i^h_{\{i\}^D_{R\times_D L}}\circ i^h_{\{i\}_{R\times_D L}}$ introduced in proposition 3.2.15. ∎

Results concerning the von Neumann (bi)algebras 3.2.18 **1)** If the integer " i " refers to a class of Galois extension degrees related to a coset of $GL_{n[m]}((\mathbb{A}_R\otimes\mathbb{C})\times(\mathbb{A}_L\otimes\mathbb{C}))/K^D_{R\times L;n[m]}(\overline{\mathbb{Z}}^2_{pq}\otimes\mathbb{C}^2)$, then the algebra of von Neumann $\mathbb{M}^h_{R\times L}(H^\pm_h)$ decomposes into $\mathbb{M}^h_{R\times L}(h^\pm_h)=\bigoplus\limits_i\mathbb{M}^h_{R\times L}(\mathcal{H}^\pm_h\{i\})$.

The spectrum $\sigma_D(T_R\otimes T_L)$ is degenerated if there is an action of the decomposition group in the sense of section 3.2.10.

2) If the integer " i " refers to a class of Galois extension degrees related to one-dimensional cosets of $GL_{n[m]}((\mathbb{A}_R \otimes \mathbb{C}) \times (\mathbb{A}_L \otimes \mathbb{C}))/K^D_{R\times L;n[m]}(\overline{\mathbb{Z}}^2_{pq} \otimes \mathbb{C}^2)$, then the algebra of von Neumann $\mathbb{M}^h_{R\times L}(H^{\pm}_h)$ decomposes into a direct sum of factors such that the set of integers $\{1,\cdots,i,\cdots,q\}$ are the entire dimensions of a von Neumann (bi)algebra of type I_q . The multiplicity of the spectrum of $(T_R \otimes T_L)$ results from the action of the decomposition group as introduced in proposition 3.2.11.

3) The spectrum of the operator $T_R \otimes T_L \in \mathbb{M}^h_{R\times L}(H^{\pm}_h)$ is obtained through the isomorphism:

$$i^h_{\{i\}^D_{R\times L}} \circ i^h_{\{i\}_{R\times L}} : \mathbb{M}^h_{R\times L}(H^{\pm}_h) \to \{\mathbb{M}^h_{R\times L}(\mathcal{H}^{\pm}_h\{i\})\}_i$$

such that we have the embedding of the $\mathcal{H}^{\pm}_h\{i\}$:

$$\mathcal{H}^{\pm}_h\{1\} \subset \cdots \subset \mathcal{H}^{\pm}_h\{i\} \subset \cdots \subset \mathcal{H}^{\pm}_h\{q\} ,$$

and the development of the i-th eigenbifunction $\psi_{L_R}(i) \otimes_D \psi_L(i) \in \mathcal{H}^{\pm}_h\{i\}$ following:

$$\psi_{L_R}(i) \otimes_D \psi_L(i) = \sum_i \sum_{m_i} c_{i_R}\phi_{L_R}(i) \otimes_D c_{i_L}\phi_L(i)$$

where

- $\psi_{L_R}(1) \otimes_D \psi_L(1) \equiv \phi_{L_R}(1) \otimes_D \phi_L(1)$ is the first eigenfunction in $\mathcal{H}^{\pm}_h\{1\}$;
- $\psi_{L_R}(i) \otimes_D \psi_L(i)$ is (isomorphic to) a n-dimensional truncated global elliptic bisemimodule;
- $\phi_{L_R}(i) \otimes_D \phi_L(i)$ is a section of $\widetilde{M}^s_{L_R} \otimes_D \widetilde{M}^s_L \in \mathcal{H}^+_h$ (see section 3.1.9).

Indeed, the bioperator $(T_R \otimes T_L)$ maps the bisemisheaf $(\widetilde{M}_R \otimes \widetilde{M}_L)$ over the $GL_n(\mathbb{A}_R \otimes \mathbb{A}_L)$-bisemimodule $(M_R \otimes M_L)$ into the perverse bisemisheaf $(\widetilde{M}^a_R \otimes \widetilde{M}^a_L)$ over the shifted $GL_{n[m]}((\mathbb{A}_R \otimes \mathbb{C}) \times (\mathbb{A}_L \otimes \mathbb{C}))$-bisemimodule $(M^a_R \times M^a_L)$ decomposed into sums over the conjugacy classes i with multiplicities m_i according to section 3.2.7. Now, the Langlands program [Pie9], [Pie12], succinctly introduced in 1.1.23, sets up bijections between:

- $(M_R \otimes M_L)$ and the n-dimensional global elliptic bisemimodule $\phi_R(s_R) \otimes_D \phi_L(s_L)$ (see proposition 1.1.19);
- $(M^a_R \otimes M^a_L)$ and the n-dimensional shifted global elliptic bisemimodule $\phi^a_R(s_R) \otimes_D \phi^a_L(s_L)$.

This leads to the following proposition:

Proposition 3.2.19 *Let $\phi_R(s_R) \otimes_D \phi_L(s_L)$ be a n-dimensional global elliptic bisemimodule constituting an analytic representation of the $GL_n(\mathbb{A}_R \times \mathbb{A}_L)$-bisemimodule $M_R \otimes M_L$.*

Let $\phi^a_R(s_R) \otimes_D \phi^a_L(s_L)$ denote the corresponding n-dimensional shifted global elliptic bisemimodule constituting the analytic representation of the perverse bisemimodule $M^a_R \otimes M^a_L$.

Then, the action of the bioperator $(T_R \otimes T_L)$ is such that:

$$(T_R \otimes T_L) : \quad \phi_R(s_R) \otimes_D \phi_L(s_L) \longrightarrow \phi^a_R(s_R) \otimes_D \phi^a_L(s_L) .$$

The shifted global elliptic bisemimodule $\phi^a_R(s_R) \otimes_D \phi^a_L(s_L)$ gives rise to the eigenbivalue equation:

$$\phi^a_R(s_R) \otimes_D \phi^a_L(s_L) = \lambda_R(n,i) \cdot \lambda_L(n,i)(\phi_R(s_R) \otimes_D \phi_L(s_L))$$

rewritten following:

$$(T_R \otimes_D T_L)(\phi_R(s_R) \otimes_D \phi_L(s_L)) = \lambda_R(n,i) \cdot \lambda_L(n,i)(\phi_R(s_R) \otimes_D \phi_L(s_L))$$

where the right (resp. left) eigenvalue $\lambda_{R,L}(n,i)$ was interpreted in [Pie12] as a set of shifts in m dimensions of Hecke characters, i.e. infinitesimal generators of the considered Lie algebra.

Proof. As the bialgebra of von Neumann $\mathbb{M}_{R\times L}(H_h^{\pm})$ can be considered as a solvable bialgebra, i.e. implying a sequence of embedded subalgebras:

$$\mathbb{M}_{R\times L}(H_h^{\pm}\{1\}) \subset \cdots \subset \mathbb{M}_{R\times L}(H_h^{\pm}\{i\}) \subset \cdots \subset \mathbb{M}_{R\times L}(H_h^{\pm}\{q\}) \ ,$$

the set of eigenvalues of $(T_R \otimes T_L)$ forms an embedded sequence:

$$\lambda_R(n,1)\cdot\lambda_L(n,1) \subset \cdots \subset \lambda_R(n,i)\cdot\lambda_L(n,i) \subset \cdots \subset \lambda_R(n,q)\cdot\lambda_L(n,q)$$

in one-to-one correspondence with the set of embedded eigenbifunctions:

$$\psi_{L_R}(1) \otimes_D \psi_L(1) \subset \cdots \subset \psi_{L_R}(i) \otimes_D \psi_L(i) \subset \cdots \subset \psi_{L_R}(q) \otimes_D \psi_L(q) \ ,$$

where $\psi_{L_R}(q) \otimes_D \psi_L(q)$ is isomorphic to a n-dimensional truncated global elliptic bisemimodule given by

$$\phi_R(s_R) \otimes_D \phi_L(s_L) = \sum_{i,m_i=1}^{q} \phi(s_R)_{i,m_i}\ e^{-i\pi i(p+i)z} \otimes_D \sum_{i,m_i=1}^{q} \phi(s_L)_{i,m_i}\ e^{i\pi i(p+i)z} \ , \quad z \in \mathbb{R}^n \ . \qquad \blacksquare$$

Proposition 3.2.20 *Let $\mathbb{M}^{a}_{R\times_{(D)}L}(H_a^{\pm})$ be the complete (resp. diagonal) von Neumann bialgebra of bounded self-adjoint bioperators on the algebraic extended bilinear Hilbert space $H_a^{\pm}$.*

Let $\mathbb{M}^{a}_{R\times L}(\mathcal{H}_a^{\pm}\{i\})$ and $\mathbb{M}^{a}_{R\times_D L}(\mathcal{H}_a^{\pm}\{i\})$ be the complete and diagonal subbialgebras of von Neumann on the closed algebraic internal bilinear subspaces $\mathcal{H}_a^{\pm}\{\sigma\}$. Then, the discrete spectrum of the bioperator $T_R \otimes_{(D)} T_L \in \mathbb{M}^{a}_{R\times_{(D)}L}(H_a^{\pm})$ is obtained through the isomorphism(s):

$$\begin{aligned}
i^a_{\{\sigma\}^D_{R\times L}} \circ i^a_{\{\sigma\}_{R\times L}} &: \mathbb{M}^{a}_{R\times L}(H_a^{\pm}) \ \longrightarrow\ \{\mathbb{M}^{a}_{R\times L}(\mathcal{H}_a^{\pm}\{i\})\}_i \ ,\\
i^a_{\{\sigma\}^D_{R\times_D L}} \circ i^a_{\{\sigma\}_{R\times_D L}} &: \mathbb{M}^{a}_{R\times_D L}(H_a^{\pm}) \ \longrightarrow\ \{\mathbb{M}^{a}_{R\times_D L}(\mathcal{H}_a^{\pm}\{i\})\}_i \ .
\end{aligned}$$

Proof. This proposition is the algebraic correspondent of proposition 3.2.15 and corollary 3.2.16 and results from the isomorphisms between analytic and algebraic von Neumann bialgebras as developed in proposition 3.2.6. $\blacksquare$

3.3 Quantification rules, probability calculus, spin, *PCT* map and relativity invariants

As the entire dimensions of the von Neumann bialgebras can correspond to classes of degrees of Galois extensions, biquanta $\widetilde{M}^I_{k_R} \otimes_{D,m,e} \widetilde{M}^I_{k_L}$, i.e. $1D$-irreducible closed subschemes of rank N^2 , can be emitted from (or absorbed by) the algebraic bisemisheaf $(\theta^{1-3}_{R;ST,MG,M} \otimes_{D,m,e} \theta^{1-3}_{L;ST,MG,M})$.

Definition 3.3.1 (“ ST ”, “ MG ” and “ M ” bistructures) Referring to the structure of a massive bisemiparticle as described in definitions 3.1.2 and 3.1.3, we recall that:

Each ST , MG and M structure $(\theta^4_{R;ST}\otimes\theta^4_{L;ST})$, $(\theta^4_{R;MG}\otimes\theta^4_{L;MG})$ and $(\theta^4_{R;M}\otimes\theta^4_{L;M})$ of a bisemiparticle decomposes under the blowing-up isomorphism into:

a) a diagonal bisemisheaf $(\theta^4_{R;ST,MG,M}\otimes_D\theta^4_{L;ST,MG,M})$,

b) a magnetic bisemisheaf $(\theta^3_{R;ST,MG,M}\otimes_m\theta^3_{L;ST,MG,M})$,

c) an electric bisemisheaf $(\theta^{1-(3)}_{R;ST,MG,M}\otimes_e\theta^{3-(1)}_{L;ST,MG,M})$,

where ST , MG , M means “ ST ”, “ MG ” or “ M ”.

Proposition 3.3.2 *The quantification rules of emission of biquanta on the $ST\oplus MG\oplus M$ bistructure of a bisemiparticle are obtained by considering the diagonal, magnetic or electric products of the right and left smooth endomorphisms $(E_{R;ST\oplus MG\oplus M}\times_{D,M,E}E_{L;ST\oplus MG\oplus M})$ applied on $(\theta^{1-3}_{R;ST\oplus MG\oplus M}\otimes_{D,m,e}\theta^{1-3}_{L;ST\oplus MG\oplus M})$ until the fundamental rank sets $n^0_{D,m,e;ST,MG,M}$ are reached.*

Proof. This proposition is an adaptation of the emission quantification rules introduced in 1.4.16 and in definition 2.2.12 to the “ $ST\oplus MG\oplus M$ ” bistructure of a bisemiparticle. We then have

$$\begin{aligned}&E_{R;ST\oplus MG\oplus M}\times_{D,m,e}E_{L;ST\oplus MG\oplus M}:\\&\quad\theta^{1-3}_{R;ST\oplus MG\oplus M}\otimes_{D,m,e}\theta^{1-3}_{L;ST\oplus MG\oplus M}\\&\qquad\to(\theta^{*1-(3)}_{R;ST\oplus MG\oplus M}\otimes_{D,m,e}\theta^{*1-(3)}_{L;ST\oplus MG\oplus M})\bigoplus_k(\widetilde M^I_{k_{R;ST\oplus MG\oplus M}}\otimes_{D,m,e}\widetilde M^I_{k_{L;ST\oplus MG\oplus M}})\end{aligned}$$

where $(\widetilde M^I_{k_{R;ST\oplus MG\oplus M}}\otimes_{D,m,e}\widetilde M^I_{k_{L;ST\oplus MG\oplus M}})$ are $1D$-time or $1D$-space diagonal, magnetic or electric biquanta on the three bistructures $ST\oplus MG\oplus M$. ∎

Remarks 3.3.3 1. The standard quantification rules of quantum (field) theory would be obtained by considering the smooth endomorphism

$$E_{L;M}:\theta^3_{L;M}\to\theta^{*3}_{L;M}\bigoplus_{k=1}^{m}\widetilde M^I_{k_{L;M}}$$

applied on the mass (“ M ”) left-$3D$-semisheaf of rings $\theta^3_{L;M}$.

2. The quantum theories work essentially with analytic functions. Due to the hypothesis considered in this work, namely that the quantum nature is algebraic, algebraic (semi-) sheaves of rings have been essentially taken into account.

According to the preceding section, algebraic semisheaves of rings were considered as isomorphic to analytic semisheaves: this is among others a consequence of the (iso)morphism of J.P. Serre [Ser7]. Thus, if we want to reach the mathematical objects of quantum theories, we have to consider bijections between algebraic semisheaves of rings and analytic global elliptic semimodules following the Langlands program.

Definition 3.3.4 (Bispectrum of Fredholm diagonal bioperators) Let $(T_R \otimes_D T_L)$ be a diagonal Fredholm bioperator acting from $\mathcal{H}^+_{h;M}$ to $\mathcal{H}^+_{h;M}$.

Let $\{\psi_{L_R(i)} \otimes_D \psi_{L(i)}\}_{i=1}^q$, $q \leq \infty$, be the set of eigenbivectors of $(T_R \otimes_D T_L)$ and let $\{\lambda_{R(i)} \times \lambda_{L(i)}\}_{i=1}^q = \{\lambda_i^2\}_{i=1}^q$ be the corresponding set of eigenbivalues occurring with probability measures $\{P_{\lambda_i} = \mu_{R(i)} \times_D \mu_{L(i)}\}$.

Now, the probability measure P_{λ_i} can be written in Dirac terminology [Dir4] following:

$$P_{\lambda_i} = \langle \psi_{L_R}(i) \mid \Psi \rangle \langle \Psi \mid \psi_L(i) \rangle$$

where $\langle \cdot \mid \cdot \rangle$ is an internal scalar product between a bra eigenvector $\langle \psi_{L_R}(i) \mid$ and the total left wave function.

As we are working in the frame of an orthogonal geometry with a diagonal metric, abbreviated by " $\times_D$ ", the eigenbivectors are orthogonal between themselves [Pie4].

Proposition 3.3.5 *The semisheaf $\widetilde{M}_L$ (resp. $\widetilde{M}_R$) on the $G_L(\mathbb{A}_L)$-left semimodule M_L (resp. the $G_R(\mathbb{A}_R)$-right semimodule M_{L_R}) constitutes an algebraic representation of a left (resp. right) wave function which is a ket (resp. bra) vector in the terminology of Dirac. The wave function has then an algebraic structural interpretation in terms of algebraic eigenvectors and a statistical interpretation as given classically in the quantum theories.*

Proof. In the terminology introduced in definition 3.1.3, a ket vector is a left vector and a bra vector is a coleft vector. The bisemisheaf $\widetilde{M}_{L_R} \otimes_D \widetilde{M}_L$ "at the mass level" has an automorphic irreducible representation in terms of global elliptic bisemimodule as developed in 1.1.14 to 1.1.20 and in 3.2.10 to 3.2.20.

Let $\{\psi_{L_R}(i) \otimes \psi_L(i)\}_i$ be the set of eigenbivectors of an operator $T_R \otimes T_L$ as defined in results 3.2.18 and in definition 3.3.4. Then, the semisheaf $\widetilde{M}_L$ (resp. $\widetilde{M}_{L_R}$) has for spectral representation the left (resp. right) wave function $\mid \Psi \rangle$ (resp. $\langle \Psi \mid$) developed following:

$$\mid \Psi \rangle = \sum_i d_i \mid \psi_L(i) \rangle \qquad (\text{resp. } \langle \Psi \mid = \sum_i d_i^* \langle \psi_{L_R}(i) \mid)$$

if we refer to definition 3.3.4 where d_i (resp. d_i^*) is given by

$$d_i = \langle \psi_{L_R}(i) \mid \Psi \rangle \qquad (\text{resp. } d_i^* = \langle \Psi \mid \psi_L(i) \rangle \text{).}$$

■

Remark 3.3.6 Referring to proposition 3.3.5, we notice that the coefficients d_i and d_i^* are probability measures. We shall then see that the traditional calculus of probability amplitudes of quantum theories [B-vonN] is replaced in this context by the probability calculus with intensities.

Proposition 3.3.7 *The traditional calculus with the amplitudes of probability [Fey1], [Dir5] of quantum (field) theory is replaced by a calculus with intensities of probability in the context of this algebraic quantum theory.*

Proof. If we realize on a bisystem, an elementary bisemiparticle for example, an observation " A ", corresponding to the Fredholm bioperator $T_R \otimes_D T_L$ defined from $\mathcal{H}^+_{h;M}$ to $\mathcal{H}^+_{h;M}$, we shall obtain the eigenbivalue λ_a^2 (or more exactly $+\sqrt{\lambda_a^2}$ since the associated coleft particle is unobservable) with probability $P_{\lambda_a} = \langle \psi_{L_R}(a) \mid \Psi\rangle\langle\Psi \mid \psi_L(a)\rangle$. An observation " B " on the same bisystem will give the eigenbivalue λ_b^2 with probability $P_{\lambda_b} = \langle \psi_{L_R}(b) \mid \Psi\rangle\langle\Psi \mid \psi_L(b)\rangle$.

Thus, $P_{\lambda_b} \cdot P_{\lambda_a} = P_{\lambda_b\cdot\lambda_a} = \langle \psi_{L_R}(b) \mid \Psi\rangle\langle\Psi \mid \psi_L(b)\rangle\langle \psi_{L_R}(a) \mid \Psi\rangle\langle\Psi \mid \psi_L(a)\rangle$ will correspond to the probability of two successive measurements " A " and " B " on a bisystem.

This differs from the ordinary calculus with amplitudes of probability [Fey1] $\psi_{\lambda_b\cdot\lambda_a} = \langle \psi_{L_R}(b) \mid \Psi\rangle\langle\Psi \mid \psi_L(a)\rangle$ of quantum theory dealing with elementary particles and not with elementary bisemiparticles as considered here.

As we have $P_{\lambda_b\cdot\lambda_a} = \psi^*_{\lambda_b\cdot\lambda_a} \cdot \psi_{\lambda_b\lambda_a}$, we see that the classical probability calculus with intensities is restored in quantum theory if bisystems throughout bisemiparticles are taken into account. ■

Definition 3.3.8 The *PCT* map of quantum field theory ([B-D], [Lüd], [Wig1], [W-W-W]) transforms the fields of particles into the fields of antiparticles and vice-versa. Its equivalent in this AQT model is the following set of maps:

a) $B_R \circ p_R : \widetilde{M}_L \to \widetilde{M}_{R_L}$,
transforming the left semisheaf $\widetilde{M}_L$ of the left semiparticle into the (involuted) coright-semisheaf $\widetilde{M}_{R_L}$) ;

b) $p_L^{-1} \circ B_L^{-1} : \widetilde{M}_{L_R} \to \widetilde{M}_R$,
transforming the coleft semisheaf $\widetilde{M}_{L_R}$ of the right semiparticle into the (involuted) right semisheaf $\widetilde{M}_R$.

The maps $p_{R,L}$ and $B_{R,L}$ are described in definitions 3.1.5 and 3.1.7.

Then, the left bisemisheaf $\widetilde{M}_{L_R} \otimes_D \widetilde{M}_L$ of a left bisemiparticle, associated to the left internal bilinear Hilbert space $\mathcal{H}^+_a$, is transformed into the right bisemisheaf $\widetilde{M}_{R_L} \otimes_D \widetilde{M}_R$ of a right bisemiparticle, associated to the right internal bilinear Hilbert space $\mathcal{H}^-_a$, according to:

$$(p_L^{-1} \circ B_L^{-1}) \otimes_D (B_R \circ p_R) : \widetilde{M}_{L_R} \otimes_D \widetilde{M}_L \to \widetilde{M}_{R_L} \otimes_D \widetilde{M}_R \ .$$

$(p_L^{-1} \circ B_L^{-1}) \otimes_D (B_R \circ p_R)$ is thus a parity time bimap whose physical meaning is given in 1.3.10.

$\widetilde{M}_{L_R} \otimes_D \widetilde{M}_L$ is the "physical field" of the left (bisemi)particle and $\widetilde{M}_{R_L} \otimes_D \widetilde{M}_R$ is the " physical field" of the right (bisemi)particle according to section 1.1.6.

If the bisemiparticle is electrically charged, a supplementary set of maps $(p_L^{-1} \circ B_L^{-1}) \otimes_{m,e} (B_R \circ p_R)$ must be applied on the magnetic and electric bisemisheaves reversing then the electric charge and the magnetic moment; this will thus correspond to a charge conjugation.

Note that the intrinsic parity-time of a $4D$-semisheaf corresponds to its orientation: this results from its generation by Eisenstein cohomology from the symmetric splitting semifield $L^{\mp}$ (see definition 1.1.2).

Definitions 3.3.9 (1. Right and left $4D$-elliptic operators) As the cohomology $H^*(\Gamma)$ of an arithmetic subgroup Γ may be identified with the cohomology of Γ-invariant smooth differential forms of de

Rham [Bor1], [Gro3], we shall assume that the $4D$- differential operator

$$
\begin{aligned}
& T_{R,L;ST,MG,M} \\
= & \left\{\pm i\hbar_{ST,MG,M}\ dt_0, \pm i\frac{\hbar_{ST,MG,M}}{c_{t\to r;ST,MG,M}}\ dx, \pm i\frac{\hbar_{ST,MG,M}}{c_{t\to r;ST,MG,M}}\ dy, \pm i\frac{\hbar_{ST,MG,M}}{c_{t\to r;ST,MG,M}}\ dz\right\}
\end{aligned}
$$

can apply on the $4D$-semisheaf of rings $\theta^{1-3}_{R,L}(t,r)_{ST}$, $\theta^{1-3}_{R,L}(t,r)_{MG}$ or $\theta^{1-3}_{R,L}(t,r)_{M}$,
where

a) the " + " or " − " sign is a convention depending on the sense of rotation of the considered semisheaf of rings;

b) $c^{-1}_{t\to r;ST,MG,M}$ is an average parameter equal to the ratio of algebraic Hecke characters (see definition 1.4.10).

On the other hand, as the semisheaves of rings $\theta^{1-3}_{R,L}(t,r)_{MG}$ and $\theta^{1-3}_{R,L}(t,r)_{M}$ are the basis of the vertical tangent semibundles $T^{(1-3)}_{MG_{R,L}}$ and $T^{(1-3)}_{M_{R,L}}$ according to definition 2.2.14, their projective maps are given by the elliptic operators:

$$
\begin{aligned}
& DT_{R,L;MG,M} \\
= & \left\{\pm i\hbar_{MG,M}\ \frac{\partial}{\partial t_0}, \pm i\frac{\hbar_{MG,M}}{c_{t\to r;MG,M}}\ \frac{\partial}{\partial x}, \pm i\frac{\hbar_{MG,M}}{c_{t\to r;MG,M}}\ \frac{\partial}{\partial y}, \pm i\frac{\hbar_{MG,M}}{c_{t\to r;MG,M}}\ \frac{\partial}{\partial z}\right\}
\end{aligned}
$$

where $\hbar_{MG,M}$ corresponds to the order of the global inertia subgroup respectively in the " MG " or " M " system of units. In particular, $\hbar_M \equiv \hbar$, i.e. the Planck's constant.

(2. Tensor products of right and left elliptic operators) Let, for example, the bioperators $(T_{R;ST}\otimes T_{L;ST})$, $(DT_{R;MG}\otimes DT_{L;MG})$ and $(DT_{R;M}\otimes DT_{L;M})$ act respectively on the bisemisheaves $(\theta^{1-3}_{R}(t,r)_{ST}\otimes \theta^{1-3}_{L}(t,r)_{ST})$, $(\theta^{1-3}_{R}(t,r)_{MG}\otimes \theta^{1-3}_{L}(t,r)_{MG})$ and $(\theta^{1-3}_{R}(t,r)_{M}\otimes \theta^{1-3}_{L}(t,r)_{M})$.

Then, the bioperator $(T_{R;ST}\oplus DT_{R;MG}\oplus DT_{R;M})\otimes(T_{L;ST}\oplus DT_{L;MG}\oplus DT_{L;M})$ will act on the bisemisheaf:

$$
\begin{aligned}
& \theta^{1-3}_{R}(t,r)_{ST-MG-M}\otimes\theta^{1-3}_{L}(t,r)_{ST-MG-M} \\
= & \ (\theta^{1-3}_{R}(t,r)_{ST}\oplus\theta^{1-3}_{R}(t,r)_{MG}\oplus\theta^{1-3}_{R}(t,r)_{M}) \\
& \quad \otimes(\theta^{1-3}_{L}(t,r)_{ST}\oplus\theta^{1-3}_{L}(t,r)_{MG}\oplus\theta^{1-3}_{L}(t,r)_{M})
\end{aligned}
$$

representing the complete massive structure of a bisemiparticle.

According to the development of the bisemisheaf $(\theta^{1-3}_{R}(t,r)_{ST-MG-M}\otimes \theta^{1-3}_{L}(t,r)_{ST-MG-M})$ in direct sums of bisemisheaves as given in definition 3.1.2, the bioperator $(T_{R;ST}\oplus DT_{R;MG}\oplus DT_{R;M})\otimes(T_{L;ST}\oplus DT_{L;MG}\oplus DT_{L;M})$ will decompose into:

$$
\begin{aligned}
& (T_{R;ST}\oplus DT_{R;MG}\oplus DT_{R;M})\otimes(T_{L;ST}\oplus DT_{L;MG}\oplus DT_{L;M}) \\
= & \ (T_{R;ST}\otimes T_{L;ST})+(DT_{R;MG}\otimes DT_{L;MG})+(DT_{R;M}\otimes DT_{L;M}) \\
& \quad +(T_{R;ST}\otimes DT_{L;MG})+(DT_{R;MG}\otimes T_{L;ST})+(T_{R;ST}\otimes DT_{L;M}) \\
& \quad +(DT_{R;M}\otimes T_{L;ST})+(DT_{R;MG}\otimes DT_{L;M})+(DT_{R;M}\otimes DT_{L;MG})\ .
\end{aligned}
$$

$(T_{R;ST} \otimes DT_{L;MG})$ has for $3D$-spatial off-diagonal components

$$L_L(k) = -i(dr_i p^j_{MG} + dr_j p^i_{MG}) \ , \qquad 1 \leq i,j \leq 3 \ ,$$

and $(DT_{R;MG} \otimes T_{L;ST})$ has for $3D$-spatial off-diagonal components

$$L_R(k) = +i(p_{i;MG} dr^j + p_{j;MG} dr^i) \ ,$$

where

a) $dr_i = \dfrac{\hbar_{ST}}{c_{t\to r;ST}}\, dx_i$ so that, if $i=1\,,\quad x_i \equiv x\,,$

$i=2\,,\quad x_i \equiv y\,,$

$i=3\,,\quad x_i \equiv z\,;$

b) $p_{i;MG} = DT_{(i)_{R;MG}} = +i \dfrac{\hbar_{MG}}{c_{t\to r;MG}} \, \dfrac{\partial}{\partial x_i}$;

c) $p^i_{MG} = DT_{(i)_{L;MG}} = -i \dfrac{\hbar_{MG}}{c_{t\to r;MG}} \, \dfrac{\partial}{\partial x^i}$.

Definition 3.3.10 (Right and left internal angular momenta) $L_{L;MG(ST)}(k)$ is interpreted as the components of the angular momentum vector $\vec{L}_{L;MG(ST)}$ of the left middle-ground structure of the left semiparticle and $L_{R;MG(ST)}(k)$ is interpreted as the components of the angular momentum vector $\vec{L}_{R;MG(ST)}$ of the right middle-ground structure of the right semiparticle. Thus, $\vec{L}_{R,L;MG(ST)}$ represents the "angular momentum" of all the sections of the right (resp. left) semisheaf of rings $\theta^3_{R,L}(r)_{MG}$ with respect to the left (resp. right) semisheaf of rings $\theta^3_{L,R}(t)_{ST}$.

Similarly, the $3D$-spatial off-diagonal components of $(T_{R;ST} \otimes DT_{L;M})$ (resp. of $(DT_{R;M} \otimes T_{L;ST})$) will be $L_{L;M(ST)}(k)$ (resp. $L_{R;M(ST)}(k)$). $L_{L;M(ST)}(k)$ (resp. $L_{R;M(ST)}(k)$) are thus the components of the angular momentum vector $\vec{L}_{L;M(ST)}$ (resp. $\vec{L}_{R;M(ST)}$) interpreted as the angular momentum of the "mass" structure $\theta^3_{L;R}(r)_M$ of the left (resp. right) semiparticle with respect to the right (resp. left) space-time structure $\theta^3_{R,L}(r)_{ST}$ of the right (resp. left) semiparticle.

To each right (resp. left) " ST ", " MG " or " M " semisheaf or rings corresponds a right (resp. left) internal angular momentum vector $\vec{L}_{R;ST,MG,M}$ (resp. $\vec{L}_{L;ST,MG,M}$) which indicates its angular velocity and its sense of rotation with respect to its associated corresponding left (resp. right) semisheaf of rings. The right (resp. left) internal momentum vector $\vec{L}_{R,L;ST,MG,M}$ then corresponds to the spin concept [Pau], [Dir3] of quantum (field) theory.

Proposition 3.3.11 *A right and a left semiparticle rotate in opposite senses and have only two possible spin states.*

Proof. In definitions 3.3.9 and 3.3.10, we have defined the right and the left internal angular momentum components of the right and the left " MG " semisheaf of rings $\theta^3_{R,L}(r)_{MG}$ by

$$L_R(k) \quad = \quad +i(p_{i;MG} dr^j + p_{j;MG} dr^i)$$

and by

$$L_L(k) \quad = \quad -i(dr_i p^j_{MG} + dr_j p^i_{MG}) \; .$$

It is then evident that $L_R(k) = -L_L(k)$ which proves that:

a) a right and a left associated semiparticle have opposite rotation senses;

b) two senses of rotation can only exist for a right and a left semiparticle and also for a bisemiparticle since only the left semiparticle is observable in a bisemiparticle and thus only its own left internal angular momentum.

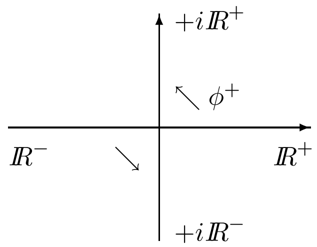
Indeed, it can be remarked that the sign of $L_{R,L}(k)$ depends on the sign of dr^i , $1 \le i \le 3$, which is reflected by the mapping:

$$\begin{aligned} \phi_\pm : \quad \mathbb{R}^\pm \quad &\to \quad \mathbb{R}^\pm(\pm i) \; , \\ \pm dr^i \quad &\rightsquigarrow \quad \pm dr^i(\pm i) \; . \end{aligned}$$

of the positive (resp. negative) reals in the positive or negative pure imaginary reals.

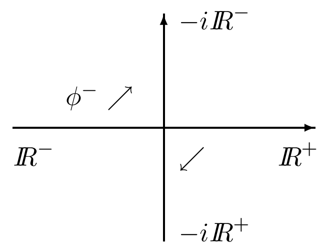
A left handled rotation corresponds to the mapping:

$$\phi_+ : (\mathbb{R}^+, \mathbb{R}^-) \to (\mathbb{R}^+(+i), \mathbb{R}^-(+i)) \; ,$$



and a right handled rotation corresponds to the mapping:

$$\phi_- : (\mathbb{R}^+, \mathbb{R}^-) \to (\mathbb{R}^+(-i), \mathbb{R}^-(-i)) \; .$$



■

3.3.12 Spin-statistics and supersymmetry

According to proposition 3.3.11, each elementary (semi)particle has two senses of rotation: this is the case for:

- the elementary leptons: e^- , μ^- , τ^- and their neutrinos ν_{e^-} , ν_{μ^-} , ν_{τ^-} ;
- the quarks: u^+ , d^- , s^- , c^+ , b^- , t^+ ;
- the photons.

So, this algebraic quantum (field) theory, which does not refer to a (non)abelian gauge theory, takes up the spin concept differently from quantum field theories. However, it seems evident that elementary (semi)fermions must always obey the Fermi-Dirac statistics while the photons behave in accordance with the Bose-Einstein statistics since they can increase their quanta number as developed in section 1.4.16.

Consequently, the supersymmetry, whose aim is the transformation of half integer spin particles into interger spin particles, does not seem essential in the present context and will not be taken into account.

Remark 3.3.13 (Interpretation of special relativity invariants) Let $T_{R,L;ST,MG,M}$ be the right or left $4D$-differential operator acting on the " ST ", " MG " or " M " $4D$-semisheaf of rings $\theta^{1-3}_{R,L}(t,r)_{ST,MG,M}$ and let

$$dt^2=(T_{R;ST,MG,M},T_{L;ST,MG,M})=\hbar^2_{ST,MG,M}(dt_0^2+c^{-2}_{t\to r}\ dx^2+c^{-2}_{t\to r}\ dy^2+c^{-2}_{t\to r}\ dz^2)$$

be their internal scalar product which is an additional structure of the corresponding " ST ", " MG " or " M " internal bilinear Hilbert space $\mathcal{H}^{\mp}_{ST,MG,M}$.

The corresponding Minkowsky space-time differential form of special relativity [Ein2] is

$$dt_0^2=dt^2-c^{-2}\ dx^2-c^{-2}\ dy^2-c^{-2}\ dz^2\ .$$

It is an invariant whose meaning in view of the developments of this paper can be interpreted as follows: if we remember that the Eisenstein cohomology classes are represented by differential forms in bijection with Eisenstein series, we can deduce from it that every increasing or decreasing of dt^2 , i.e. finally of $(\theta^{1-3}_R(t,r)_{ST,MG,M}\otimes_D\theta^{1-3}_L(t,r)_{ST,MG,M})$, happens by external capture or loosing of biquanta $(\widetilde M^I_R(r)_{ST,MG,M}\otimes_D\widetilde M^I_L(r)_{ST,MG,M})$ throughout the smooth biendomorphism $(E_{R;ST,MG,M}\times_D E_{L;ST,MG,M})$ according to proposition 3.3.2: indeed, this corresponds to the increasing or to the decreasing of $c^{-2}(dx^2+dy^2+dz^2)$.

On the other hand, the euclidian invariant

$$dt^2=\hbar^2_{ST,MG,M}(dt_0^2+c^{-2}_{t\to r}(dx^2+dy^2+dz^2))\ ,$$

valid for a closed system and essentially envisaged in this work, can be interpreted in function of the internal morphism $((\gamma_{t\rightleftarrows r}\circ E_{R;ST,MG,M})\times_D(\gamma_{t\rightleftarrows r}\circ E_{L;ST,MG,M}))$ applied on $(\theta^{1-3}_R(t,r)_{ST,MG,M}\otimes_D\theta^{1-3}_L(t,r)_{ST,MG,M})$. Indeed, if dt^2 is invariant, then time biquanta can be transformed into $3D$-spatial biquanta and vice-versa.

4 Second order differential bilinear equations

This chapter is devoted to the study of the differential equations relative to the bisemiparticles. It is thus necessary to classify the bisemiparticles with respect to the presently observed elementary particles and in function of their general structure as developed in the preceding chapters: this is the object of this first section. We shall take for reference the traditional statistical classification of fermions and bosons.

4.1 Classification of bisemiparticles

Definition 4.1.1 (Bisemifermions and bisemibosons) Let $(\theta^{1-3}_R(t,r)_{ST-MG-M}\otimes\theta^{1-3}_L(t,r)_{ST-MG-M})$ denote the three embedded structures of a massive bisemiparticle as developed in definition 3.1.2. According to definition 3.1.1, this bistructure corresponds essentially to a bisemilepton or to a bisemiquark, i.e. to an elementary massive bisemifermion. In definition 3.1.2, this tensor product has been decomposed into the direct sum of three tensor products referring to the " ST ", " MG " and " M " bistructures and of six other tensor products referring to the interactions between the right and the left " ST ", " MG " and " M " structures. Taking into account the general structure of a bisemiparticle as given above, we can classify the bisemiparticles in the following four categories:

1) **An elementary massive bisemifermion** has a bistructure given by $(\theta_R^{1-3}(t,r)_{ST-MG-M}\otimes\theta_L^{1-3}(t,r)_{ST-MG-M})$ whose each of the nine constitutive tensor products: $(\theta_R^{1-3}(t,r)_{ST}\otimes\theta_L^{1-3}(t,r)_{ST})$, $\cdots$, $(\theta_R^{1-3}(t,r)_M\otimes\theta_L^{1-3}(t,r)_{MG})$, (noted in a general abbreviated form $(\theta_R^{1-3}(t,r)\otimes\theta_L^{1-3}(t,r))$), (see definition 3.1.2) decomposes under the blowing-up isomorphism S_L into:

   a) a diagonal bisemisheaf $(\theta_R^{1-3}(t,r)\otimes_D\theta_L^{1-3}(t,r))$ giving in the case of $(\theta_R^{1-3}(t,r)_{ST,MG,M}\otimes_D\theta_L^{1-3}(t,r)_{ST,MG,M})$ the diagonal central bistructure of the three embedded bisemisheaves of rings “ ST ”, “ MG ” or “ M ”;

   b) a magnetic bisemisheaf $(\theta_R^3(r)^{(m)}\otimes_m\theta_L^3(r)^{(m)})$ which is composed in the case of $(\theta_R^3(r)^{(m)}_{ST,MG,M}\otimes_m\theta_L^3(r)^{(m)}_{ST,MG,M})$ of nonorthogonal “ ST ”, “ MG ” or “ M ” magnetic biquanta $(\widetilde M^{I(3)}_{k_{R;ST,MG,M}}\otimes_m\widetilde M^{I(3)}_{k_{L;ST,MG,M}})$.
   In fact, these magnetic biquanta are generated by the magnetic smooth biendomorphism according to proposition 3.3.2:

$$\begin{aligned}E_{R;ST,MG,M}\times_m E_{L;ST,MG,M}:\theta_R^{1-3}(t,r)_{ST,MG,M}\otimes\theta_L^{1-3}(t,r)_{ST,MG,M}\\ \to\quad(\theta_R^{1-(3)^*}(t,r)_{ST,MG,M}\otimes\theta_L^{1-(3)^*}(t,r)_{ST,MG,M})\\ \bigoplus_{k=1}^{m}(\widetilde M^{I(3)}_{k_{R;ST,MG,M}}\otimes_m\widetilde M^{I(3)}_{k_{L;ST,MG,M}})\end{aligned}$$

   and constitute the magnetic moment of the considered bisemifermion;

   c) an electric bisemisheaf $(\theta_R^{1-(3)}(t,(r))^{(e)}\otimes_e\theta_L^{3-(1)}((t),r)^{(e)})$ which is composed in the case of $(\theta_R^{1-(3)}(t,(r))^{(e)}_{ST,MG,M}\otimes_e\theta_L^{3-(1)}((t),r)^{(e)}_{ST,MG,M})$ of “ ST ”, “ MG ” or “ M ” electric time-space biquanta $(\widetilde M^{I(1)}_{k_{R;ST,MG,M}}\otimes_e\widetilde M^{I(3)}_{k_{L;ST,MG,M}})$ or space-time biquanta $(\widetilde M^{I(3)}_{k_{R;ST,MG,M}}\otimes_e\widetilde M^{I(1)}_{k_{L;ST,MG,M}})$ such that the right (resp. left) time quanta are generated by versal deformation and spreading-out isomorphism $SO(e)\circ Vd(e)$ according to definition 2.4.1 while the left (resp. right) space quanta are generated by smooth endomorphism $E_{L,R;ST,MG,M}$. The electric bisemisheaf $(\theta_R^{1-(3)}(t,(r))^{(e)}_{ST-MG-M}\otimes_e\theta_L^{3-(1)}((t),r)^{(e)}_{ST-MG-M})$ constitutes the electric charge of the considered massive bisemifermion.

2) **A bisemiphoton** has a spatial structure given by the tensor product $(T_R^1(r_k)_{ST-MG-M}\otimes T_L^1(r_k)_{ST-MG-M})$ of a right semiphoton by a left semiphoton which can split under the blowing-up isomorphism S_L into:

$$\begin{aligned}&S_L:(T_R^1(r_k)_{ST-MG-M}\otimes T_L^1(r_k)_{ST-MG-M})\\ &\to\ (T_R^1(r_k)_{ST-MG-M}\otimes_D T_L^1(r_k)_{ST-MG-M})\oplus(T_R^1(r_k)^{(m)}_{ST-MG-M}\otimes_m T_L^1(r_k)^{(m)}_{ST-MG-M})\end{aligned}$$

   where $(T_R^1(r_k)_{ST-MG-M}\otimes_D T_L^1(r_k)_{ST-MG-M})$ refers to the three embedded diagonal bisections “ ST ”, “ MG ” and “ M ” representing its central space bistructure and where $(T_R^1(r_k)^{(m)}_{ST-MG-M}\otimes_m T_L^1(r_k)^{(m)}_{ST-MG-M})$ refers to the three embedded magnetic bisections of rings “ ST ”, “ MG ” and “ M ” representing its magnetic structure composed of magnetic space biquanta.

3) **A bisemiboson of magnetic structure** is an electrically neutral meson which will be proved in chapter 5 to be generated by a magnetic biendomorphism from a bisemiquark. In this category, we may also include the magnetic biquanta whose structure is given by the magnetic bisemisheaf $(\theta^3_R(r)^{(m)}_{ST-MG-M}\otimes_m\theta^3_L(r)^{(m)}_{ST-MG-M})$.

4) **An electrically charged bisemiboson** is an electrically charged meson generated from a bisemiquark as it will be seen in chapter 5. In this category, we may also include the electric charge whose structure is given by the electric bisemisheaf: $(\theta^{1-(3)}_R(t,(r))^{(e)}_{ST-MG-M}\otimes_e\theta^{3-(1)}_L((t),r)^{(e)}_{ST-MG-M})$ where $\theta^{1-(3)}$ means a $1D$-time or $3D$-space semisheaf of rings.

Definition 4.1.2 (Annihilation of a semilepton pair) Let $(\theta^{1-3}_R(t,r)_{ST-MG-M}\otimes_D\theta^{1-3}_L(t,r)_{ST-MG-M})\oplus(\theta^3_R(r)^{(m)}_{ST-MG-M}\otimes_m\theta^3_L(r)^{(m)}_{ST-MG-M})\oplus(\theta^{1-(3)}_R(t,(r))^{(e)}_{ST-MG-M}\otimes_e\theta^{3-(1)}_L(t,r)^{(e)}_{ST-MG-M})$ be the " $ST\oplus MG\oplus M$ "-semisheaves of rings constituting the massive structure of a bisemilepton. Under some external perturbation, a breaking of the diagonal bisemisheaves $(\theta^{1-3}_R(t,r)_{ST-MG-M}\otimes_D\theta^{1-3}_L(t,r)_{ST-MG-M})$ can occur such that the right and left semisheaves are no more localized in some open ball of radius R where R is the radius of the topological domain on which the constitutive bisemisheaf $(\theta^{1-3}_R(t,r)_{ST-MG-M}\otimes\theta^{1-3}_L(t,r)_{ST-MG-M})$ is defined.

If the right and left semisheaves $\theta^{1-3}_R(t,r)_{ST-MG-M}$ and $\theta^{1-3}_L(t,r)_{ST-MG-M}$ are no more localized in the same open ball, a new electric and magnetic " $ST-MG-M$ " bisemisheaf can be generated. Indeed, a right and a left electric and magnetic bisemisheaves, corresponding to a positive and a negative electric charges and magnetic moments, can exist simultaneously because they are no more orthogonal according to proposition 3.1.6. The result is that a pair of semileptons is generated such that each semilepton is endowed with an electric charge and a magnetic moment of opposite signs.

But this pair of semileptons can annihilate. Indeed, by electromagnetic attraction, this pair will be again concentrated in a same open ball which involves that the two electric bisemisheaves, representing their electric charges, become orthogonal. Consequently, they cannot conserve the same structure as remarked above: they must then transform themselves into magnetic bisemisheaves as follows:

$$\begin{aligned}\gamma_{t\to r}\circ E:(&\theta^{1-(3)}_R(t,(r))^{(e)}_{ST-MG-M}\otimes_e\theta^{3-(1)}_L((t),r)^{(e)}_{ST-MG-M})\\ &\to\ (\theta^{3-(3)}_{I_R}(r,(r))^{(m)}_{ST-MG-M}\otimes_m\theta^{3-(3)}_L((r),r)^{(m)}_{ST-MG-M})\ .\end{aligned}$$

The resulting magnetic bisemisheaves can be transformed later in diagonal bisemisheaves.

On the other hand, as the time structures of the electric charge are generated by the morphisms $SOT(e)\circ Vd(e)$ from the right and the left semisheaves $\theta^{1-3}_{R,L}(t,r)_{ST-MG-M}$ of the considered semileptons according to definition 2.4.1 and as the $1D$-time semisheaves of the electric charge must be transformed into their complementary $3D$-space semisheaves by the fact of the collision, it is reasonable to admit that the external perturbation provoking a $(\gamma_{t\to r}\circ E)$ morphism on the $1D$-time structures of the electric charge will also provoke a $(\gamma_{t\to r}\circ E)$ morphism on the two semisheaves $\theta^{1-3}_{R,L}(t,r)_{ST-MG-M}$ constituting the central massive structure of the two semileptons. We then will have:

$$(\gamma_{t\to r}\circ E_{R,L}):\theta^{1-3}_{R,L}(t,r)_{ST-MG-M}\to\theta^3_{R,L}(r)_{ST-MG-M}\simeq T^1_{R,L}(r_k)_{ST-MG-M}$$

transforming the central $4D$-structures of the pair of semileptons into $3D$-structures of semiphoton(s).

Indeed, we have finally that the pair of semileptons annihilate into a pair of semiphotons according to:

$$
\begin{aligned}
&[(\theta_R^{1-3}(t,r)_{ST-MG-M})\oplus(\theta_R^1(t)^{(e)}_{ST-MG-M}\otimes_e\theta_L^3(r)^{(e)}_{ST-MG-M})\\
&\oplus\theta_R^3(r)^{(m)}_{ST-MG-M}\otimes_m\theta_L^3(r)^{(m)}_{ST-MG-M})]\\
&\cup[(\theta_L^{1-3}(t,r)_{ST-MG-M})\oplus(\theta_R^3(r)^{(e)}_{ST-MG-M}\otimes_e\theta_L^1(t)^{(e)}_{ST-MG-M})\\
&\oplus(\theta_R^3(r)^{(m)}_{ST-MG-M}\otimes_m\theta_L^3(r)^{(m)}_{ST-MG-M})]\\
&\quad\to\quad[(T^1_{I_R}(r_k)_{ST-MG-M})\oplus(\theta_R^3(r)^{(m)}_{ST-MG-M}\otimes_m\theta_L^3(r)^{(m)}_{ST-MG-M})]\\
&\qquad\qquad\cup[(T^1_{I_L}(r_k)_{ST-MG-M})\oplus(\theta_R^3(r)^{(m)}_{ST-MG-M}\otimes_m\theta_L^3(r)^{(m)}_{ST-MG-M})]\;.
\end{aligned}
$$

Remark 4.1.3 (Hypothesis concerning the structure of semineutrinos) Consider that a pair of semileptons, endowed each one with its electric charge and magnetic moment, comes into collision in such a way that almost all the "mass" quanta of the semisheaves of rings $\theta^{1-3}_{R,L}(t,r)_{ST-MG-M}$ blow up by an endomorphism as described in section 1.2 such that $\theta^{1-3}_{R,L}(t,r)_{ST-MG-M}$ be reduced to $\theta^{1-3}_{R,L}(t,r)_{ST-MG-(M\to0)}$. Then, the (semi)lepton central structure $\theta^{1-3}_{R,L}(t,r)_{ST-MG-M}$ has been transformed into a (semi)neutrino central structure $\theta^{1-3}_{R,L}(t,r)_{ST-MG-(M\to0)}$.

4.2 Second order elliptic bilinear equations on extended bilinear Hilbert spaces

For the facility of manipulations and notations, the elliptic differential bilinear equations will be considered for the mass (" M ") structure of the lightest massive bisemilepton, i.e. the bisemielectron or classical electron, considering that the elliptic differential bilinear equations relative to the other bisemiparticles and to the other structures " ST " and " MG " are exactly of the same type.

Definition 4.2.1 (Bisections of bisemisheaves) Let $(\theta^{1-3}_{L_R}(t,r)_M\otimes\theta^{1-3}_L(t,r)_M)$ be the left-bisemisheaf defined on the left extended internal bilinear Hilbert space H^+_a and representing the mass structure of a bisemielectron. This left-bisemisheaf then results from the bisemisheaf $(\theta^{1-3}_R(t,r)_M\otimes\theta^{1-3}_L(t,r)_M)$ by application of the composition of maps $B_L\circ p_L$ according to:

$$B_L\circ p_L:(\theta^{1-3}_R(t,r)_M\otimes\theta^{1-3}_L(t,r)_M)\to(\theta^{1-3}_{L_R}(t,r)_M\otimes\theta^{1-3}_L(t,r)_M)\;.$$

The right (resp. left) semisheaf $\theta^{1-3}_{L_R}(t,r)_M$ (resp. $\theta^{1-3}_L(t,r)_M$) is composed of q sections which are (isomorphic to) differentiable right (resp. left) functions $\phi_{L_{R\mu}}(t,r)$ (resp. $\phi_{L_\mu}(t,r)$), $1\le\mu\le q$, defined on a compact domain homeomorphic to a compact domain D_{e^+} of $(I\!R^+)^1\times(I\!R^+)^3$ centered in the upper half plane with respect to the emergence point.

The right (resp. left) function $\phi_{L_{R\mu}}(t,r)$ (resp. $\phi_{L_\mu}(t,r)$) can be decomposed following:

$$
\begin{aligned}
\phi_{L_{R\mu}}(t,r)&=\phi_{L_{R\mu}}(t)\oplus\phi_{L_{R\mu}}(r)\\
(\text{resp. }\phi_{L_\mu}(t,r)&=\phi_{L_\mu}(t)\oplus\phi_{L_\mu}(r)\ )
\end{aligned}
$$

where " r " denotes the triple of spatial variables $\{x_1,x_2,x_3\}$.

This is a consequence of the generation of the three-dimensional semisheaf of rings $\theta^3_{L_R,L}(r)$ from the one-dimensional semisheaf of rings $\theta^1_{L_R,L}(t)$ according to Section 1.2.

The left bisemisheaf $(\theta^{1-3}_{L_R}(t,r)_M\otimes\theta^{1-3}_{L}(t,r)_M)$ is composed of algebraic bifunctions isomorphic to $(\phi_{L_{R_\mu}}(t,r)\otimes\phi_{L_\mu}(t,r))$ defined on the left extended internal bilinear Hilbert space H^+_h .

These bifunctions are defined on a curved space-time domain homeomorphic to a domain $D_{e^+}\times D_{e^+}$ of $(I\!R^+)^4\times(I\!R^+)^4\simeq(I\!R^+)^{10}$ into $I\!R$.

Definition 4.2.2 (4D-elliptic differential operator) As we are dealing with the vertical tangent semibundles $T^{(1)}_{M_{R,L}}$ and $T^{(3)}_{M_{R,L}}$ according to definition 2.2.15, the elliptic differential right (resp. left) operator to be considered is the following:

$$\mathbb{M}_{R,L}=\{m_{0_{R,L;M}},\vec p_{R,L;M}\}$$

whose explicit development is given by:

$$\mathbb{M}_{R,L}=\left\{\pm i\hbar_M\ \frac{\partial}{\partial t_0},\pm i\ \frac{\hbar_M}{c_{t\to r;M}}\ \frac{\partial}{\partial x_1},\cdots,\pm i\ \frac{\hbar_M}{c_{t\to r;M}}\ \frac{\partial}{\partial x_3}\right\}\ .$$

But we must take into account the spin of the bisemielectron, i.e. the rotation of the sections of $\theta^{1-3}_{L_R}(t,r)_M$ and $\theta^{1-3}_{L}(t,r)_M$ (see definition 3.3.11). This can be achieved by considering that the elliptic differential operators $m_{0_{R,L;M}}$ and $\vec p_{R,L;M}$ are respectively 1D- and 3D-directional gradients $s_{0_{R,L}}\nabla_{0_{R,L}}$ and $\vec s_{R,L}\vec{\nabla}_{R,L}$ where $s_{0_{R,L}}$ and $\vec s_{R,L}$ are 1D- and 3D-unit vectors referring to the spin with direction cosines $\{s_{0_{R,L}}\}$ and $\{s_{1_{R,L}},s_{2_{R,L}},s_{3_{R,L}}\}$.

Indeed, the spin cannot be introduced judiciously by γ or σ matrices because bisemifermions are characterized by complete bilinear forms of H^+_a .

The 4D-elliptic self-adjoint differential mass operator will thus be written as follows:

$$\mathbb{M}_{R,L}=\left\{\pm i\hbar s_0\ \frac{\partial}{\partial t_0},\pm i\ \frac{\hbar}{c}s_1\ \frac{\partial}{\partial x_1},\pm i\ \frac{\hbar}{c}s_2\ \frac{\partial}{\partial x_2},\pm i\ \frac{\hbar}{c}s_3\ \frac{\partial}{\partial x_3}\right\}$$

where $\hbar$ and c are abbreviated notations respectively for $\hbar_M$ and $c_{t\to r;M}$.

Definition 4.2.3 (Second order differential bilinear equation) The differential mass bioperator will be $(\mathbb{M}_{R\mu}\otimes\mathbb{M}_{L\mu})$ acting on the differentiable bifunction $(\phi_{L_{R_\mu}}(t,r)\otimes\phi_{L\mu}(t,r))$ such that the μ-th mass second order elliptic differential bilinear equation to be considered is:

$$(\mathbb{M}^2_\mu-E^2_\mu)(\phi_{L_{R\mu}}\cdot\phi_{L\mu})=0$$

if we take into account the self-adjointness of the right (resp. left) elliptic operator $\mathbb{M}_{R\mu}$ (resp. $\mathbb{M}_{L\mu}$).

This differential bilinear equation can be explicitly written according to:

$$\begin{aligned}&\sum_{i,j=1}^{3}A^{ij}_\mu\ \frac{\partial^2(\phi_{L_{R\mu}}(r)\cdot\phi_{L\mu}(r))}{\partial x_i\ \partial x_j}\ +\ \sum_{i=1}^{3}A^{0i}_\mu\ \frac{\partial\phi_{L_{R\mu}}(t)}{\partial t}\cdot\ \frac{\partial\phi_{L\mu}(r)}{\partial x_i}\\ &+\ \sum_{i=1}^{3}A^{0i}_\mu\ \frac{\partial\phi_{L_{R\mu}}(r)}{\partial x_i}\cdot\ \frac{\partial\phi_{L\mu}(t)}{\partial t}\ +\ A^{00}_\mu\ \frac{\partial^2(\phi_{L_{R\mu}}(t)\cdot\phi_{L\mu}(t))}{\partial t^2}\\ &\qquad\qquad-\ E^2_\mu(\phi_{L_{R\mu}}(t,r)\cdot\phi_{L\mu}(t,r))\ =\ 0\end{aligned}$$

where

$$A^{ij}_\mu=-\ \frac{\hbar^2}{c^2}s^i_\mu s^j_\mu\ ,\qquad A^{00}_\mu=-\hbar^2s^0_\mu s^0_\mu\ ,\qquad A^{0i}_\mu=-\ \frac{\hbar^2}{c}s^0_\mu s^i_\mu\ .$$

It is a second order elliptic differential bilinear equation which is degenerated because the bilinear form $(p_{R\mu}\phi_{L_{R\mu}}(r), p_{L\mu}\phi_{L_\mu}(r))_c$ is degenerated.

$p_{R\mu,L_\mu}$ stands for the $3D$-linear momentum operator given by

$$p_{R\mu,L\mu} = \left\{ -i\ \frac{\hbar}{c} s_{1\mu}\ \frac{\partial}{\partial x_1}, -i\ \frac{\hbar}{c} s_{2\mu}\ \frac{\partial}{\partial x_2}, -i\ \frac{\hbar}{c} s_{3\mu}\ \frac{\partial}{\partial x_3} \right\}$$

(for the literature on the second order elliptic differential linear equation, see [K-N1], [K-N2], [G-I-L]).

Proposition 4.2.4 *A right and a left isomorphisms transform the second order degenerated elliptic differential bilinear equation:*

$$\begin{aligned}
\sum_{i,j=1}^{3} A_\mu^{ij}\ \tfrac{\partial^2(\phi_{L_{R\mu}}(r)\cdot\phi_{L\mu}(r)}{\partial x_i\ \partial x_j}\ &+\ \sum_{i=1}^{3} A_\mu^{0i}\ \tfrac{\partial\phi_{L_{R\mu}}(t)}{\partial t}\cdot\ \tfrac{\partial\phi_{L\mu}(r)}{\partial x_i} \\
+\ \sum_{i=1}^{3} A_\mu^{0i}\ \tfrac{\partial\phi_{L_{R\mu}}(r)}{\partial x_i}\cdot\ \tfrac{\partial\phi_{L\mu}(t)}{\partial t}\ &+\ A_\mu^{00}\ \tfrac{\partial^2(\phi_{L_{R\mu}}(t)\cdot\phi_{L\mu}(t))}{\partial t^2} \\
&-\ E_\mu^2(\phi_{L_{R\mu}}(t,r)\cdot\phi_{L\mu}(t,r))\quad=\quad 0
\end{aligned}$$

into the canonical second order elliptic-parabolic differential bilinear equation:

$$\begin{aligned}
\bar{A}_\mu^{33}\ \frac{\partial^2(\phi_{L_{R\mu}}(z)\cdot\phi_{L\mu}(z)}{\partial z^2}\ &+\ \bar{A}_\mu^{03}\ \frac{\partial\phi_{L_{R\mu}}(t)}{\partial t}\cdot\ \frac{\partial\phi_{L\mu}(z)}{\partial z} \\
+\ \bar{A}_\mu^{03}\ \frac{\partial\phi_{L_{R\mu}}(z)}{\partial z}\cdot\ \frac{\partial\phi_{L\mu}(t)}{\partial t}\ &+\ \bar{A}_\mu^{00}\ \frac{\partial^2(\phi_{L_{R\mu}}(t)\cdot\phi_{L\mu}(t))}{\partial t^2} \\
&-\ E_\mu^2(\phi_{L_{R\mu}}(t,z)\cdot\phi_{L\mu}(t,z))\quad=\quad 0
\end{aligned}$$

where

$$\bar{A}_\mu^{33} = \frac{-\hbar^2}{c^2}(s_1^2+s_2^2+s_2^3)\ \equiv\ -\ \frac{-\hbar^2}{c^2}s_\mu^2\ , \qquad \bar{A}_\mu^{03} = \frac{\hbar^2}{c}s_\mu\ .$$

Proof. The canonical form of the second order degenerated elliptic differential bilinear equation is obtained for a fixed " μ " throughout the following change of variables [K-S-G]:

a) $\xi_k = \sum\limits_{i=1}^{3} x_i\beta_k^i$ such that

$$\xi_k\xi_\ell = \sum_{i,j=1}^{3} \beta_k^i x_i x_j \beta_\ell^j = \beta_L x x^T \beta_R^\dagger\ ,$$

with $\beta \in SO(3)$, x a $3D$-vector, x^T its transposed and such that $B^{ij}x_ix_j$ be transformed into $\bar{B}^{k\ell}\xi_k\xi_\ell$, where

$$B^{ij} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \qquad \text{and} \qquad \bar{B}^{k\ell} = \beta_L B^{ij}\beta_R^\dagger\ ;$$

b) $\dfrac{\partial}{\partial x_i} = \sum\limits_{k=1}^{3} \frac{\partial}{\partial \xi_k} \cdot \frac{\partial \xi_k}{\partial x_i} = \sum\limits_{k=1}^{3} \frac{\partial}{\partial \xi_k} \alpha_i^k$ such that $A^{ij} \dfrac{\partial^2}{\partial x_i \, \partial x_j}$ be transformed into $\bar{A}^{k\ell} \dfrac{\partial}{\partial \xi_k} \cdot \dfrac{\partial}{\partial \xi_\ell}$ where

$$\begin{aligned}
\bar{A}^{k\ell} &= \sum_{i,j=1}^{3} \alpha_i^k A^{ij} \alpha_j^\ell = \alpha_L \; A \alpha_R^\dagger \,, \\
\bar{A}^{0k} &= \sum_{i=1}^{3} A^{0i} \alpha_i^k \,.
\end{aligned}$$

The transformed equation then becomes:

$$\begin{aligned}
&\sum_{k,\ell=1}^{3} \bar{A}^{k\ell} \, \tfrac{\partial^2 (\phi_{L_R}(\rho) \cdot \phi_L(\rho))}{\partial \xi_k \, \partial \xi_\ell} \; + \; \sum_{k=1}^{3} \bar{A}^{0k} \, \tfrac{\partial \phi_{L_R}(t)}{\partial t} \cdot \, \tfrac{\partial \phi_L(\rho)}{\partial \xi_k} \\
&+ \; \sum_{k=1}^{3} \bar{A}^{k0} \, \tfrac{\partial \phi_{L_R}(\rho)}{\partial \xi_k} \cdot \, \tfrac{\partial \phi_L(t)}{\partial t} \; + \; \bar{A}^{00} \, \tfrac{\partial^2 (\phi_{L_R}(t) \cdot \phi_L(t))}{\partial t^2} \\
&\qquad - \; E^2 (\phi_{L_R}(t, \rho) \cdot \phi_L(t, \rho)) \;\; = \;\; 0
\end{aligned}$$

where $\rho = \{\xi_1, \xi_2, \xi_3\}$ stands for the triple of ξ-spatial variables. The canonical second order elliptic parabolic differential bilinear equation reduces if we remark that:

1) $\bar{B}^{k\ell} = \beta_L B^{ij} \beta_R^\dagger = E$ corresponds to a left and a right unitary transformations, such that the eigenvalue diagonal matrix λ has for unique eigenvalue $e = \sqrt{3}$ (this point was outlined to me by G. Raseev). Consequently, $\phi_{L_R}(\rho) \cdot \phi_L(\rho)$ becomes $\phi_{L_R}(\xi_3) \cdot \phi_L(\xi_3)$ rewritten $\psi(z) = \phi_{L_R}(z) \cdot \phi_L(z)$ if $z \equiv \xi_3$.

2) $\bar{A}^{k\ell} = \alpha_L \; A \alpha_R^\dagger = D$ also corresponds to a left and a right unitary transformations such that D is the diagonal matrix whose unique eigenvalue different from zero is:

$$a = - \tfrac{\hbar^2}{c^2} (s_1^2 + s_2^2 + s_3^2) = s^2 \cdot \tfrac{\hbar^2}{c^2} \,. \qquad \blacksquare$$

Notice that the differential bilinear equation over the set of four variables $\{t, x_1, x_2, x_3\}$ reduces to a canonical differential bilinear equation over the set of two variables $\{t, z\}$. Indeed, the three-dimensional spatial section $\phi_{L_{R_\mu}}(r)$ (resp. $\phi_{L_\mu}(r)$) degenerates into the one-dimensional function $\phi_{L_R}(z)$ (resp. $\phi_L(z)$) justifying the decomposition of the shifted Eisenstein bicohomology into pairs of one-dimensional eigenspaces following proposition 3.2.11.

Definition 4.2.5 (Differential bilinear equation of the bisemielectron)
In a first step, we shall suppose that the time variable is constant. Consequently, the canonical second order elliptic-parabolic differential bilinear equation, relative to a "mass" irreducible section of the bisemielectron, will become [Pie2]:

$$- \frac{\hbar^2}{c^2} S^2 \; \frac{\partial^2 \psi(z)}{\partial z^2} - 2i \; \frac{\hbar}{c} m_0 S \; \frac{\partial \psi(z)}{\partial z} + (m_0^2 - E^2) \psi(z) = 0 \,.$$

This equation is 1-dimensional, and is thus defined on the left internal bilinear Hilbert space $\mathcal{H}_h^+$.

Proposition 4.2.6 *The $1D$-mass equation for one irreducible bisection of the bisemielectron*

$$-\frac{\hbar^2}{c^2}S^2\,\frac{\partial^2\psi}{\partial z^2}-2i\,\frac{\hbar}{c}m_0S\,\frac{\partial\psi}{\partial z}+(m_0^2-E^2)\psi=0$$

is the equation of a damped harmonic oscillator whose general solution consists in the superposition of two damped waves in phase opposition with frequencies given by

$$E=+\,\frac{\hbar}{c}\nu S$$

and whose general motion corresponds to a damped sinusoidal motion whose dephasage is proportional to the linear momentum $\vec{p}_{\mu_{R,L}}$ of the considered section of the right or left semielectron.

The energy E of a section $\phi_{L_\mu}(r)$ at $(p+\mu)$ quanta is equal to $(p+\mu)$-times the quantum energy E^I_μ which can be found from the corresponding nontrivial zero of the Riemann zeta function.

Proof. This equation was already worked out elsewhere in [Pie2]. However, we shall briefly give the following elements of the proof:

1) Rewrite the $1D$-mass biwave equation of the bisemielectron in the form:

$$\frac{d^2\psi}{dz^2}+k\,\frac{d\psi}{dz}+\omega^2\psi=0$$

where

$$k=2i\,\frac{c}{\hbar}S^{-1}m_0\;,\qquad \omega^2=\frac{c^2}{\hbar^2}S^{-2}(E^2-m_0^2)\;.$$

This equation is the one of a damped harmonic oscillator. The nature of the solution depends on the characteristic roots [Stru]:

$$\lambda=-\,\frac{k}{2}\pm\left[\left(\frac{k}{2}\right)^2-\omega^2\right]^{1/2}\;.$$

If $\omega^2>\left(\frac{k}{2}\right)^2$, i.e. if $E^2>0$, then $\omega_1=\left[\omega^2-\left(\frac{k}{2}\right)^2\right]^{1/2}$ has to be considered such that $\lambda=-\,\frac{k}{2}\pm i\omega_1$.

Using

$$\omega^2=\frac{c^2}{\hbar^2}S^{-2}(E^2-m_0^2)\;,$$

we find that $\omega_1=\frac{c}{\hbar}S^{-1}E$ or that $E=\frac{h}{c}\nu S$ if $\omega_1=2\pi\nu$.

We then recover the famous relation of the Broglie except for the spin factor S . Note that this formula concerns the total energy " E " of a section of a left or right semielectron which can then be interpreted as follows:

If this section is the μ-th section $s_{\mu_{R,L}}$ having $\mu_p=p+\mu$ quanta (see definition 1.2.17), then the energy E_μ of this section will be $E_\mu=\mu_pE^I_\mu$ where E^I_μ is the energy of one quantum in $s_{\mu_{R,L}}$. Now, $E^I_\mu=h\nu_{\tau_\mu}$ where h is the Planck constant associated with the degree of Galois extension N and where ν_{τ_μ} is assumed to be the frequency of a prime ideal corresponding to one Galois automorphism. One then has $E_\mu=\mu_pE^I_\mu=\mu_ph\nu_{\tau_\mu}=h\nu_\mu$ where $\nu_\mu=\mu_p\nu_{\tau_\mu}$ is the frequency of μ_p prime ideals.

Remark that it was proved in [Pie10] that the energy E^I_μ of a quantum M^I_μ in a section $s_{\mu_{R,L}}$ can be obtained from the corresponding nontrivial zero $\lambda(4\nu^2,i^2,E^I_{4\nu^2})$ of the Riemann zeta function

$\zeta(s)=\sum\limits_n n^{-s}$. Indeed, the trivial zeros of $\zeta(s)$ are the negative integers -2 , -4 , $\ldots$, -2ν , $\ldots$, -2η such that the even integer $\mu_p=2\nu$ be the global class residue degree of the section $s_{\mu_{R,L}}$ (see 1.1.4). Now, the nontrivial zeros can be obtained from the corresponding trivial ones by the action of the Lie algebra of the decomposition group, whose coset representative is:

$$D_{4\nu^2,i^2}=\left[\begin{pmatrix}1 & i\\ 0 & 1\end{pmatrix}\ \begin{pmatrix}1 & 0\\ i & 1\end{pmatrix}\right]\ ,$$

such that the eigenvalues of:

$$D_{4\nu^2,i^2}\cdot\varepsilon_{4\nu^2}\cdot\alpha_{4\nu^2}=\left[\begin{pmatrix}1 & i\\ 0 & 1\end{pmatrix}\ \begin{pmatrix}1 & 0\\ i & 1\end{pmatrix}\right]\begin{pmatrix}E^I_{4\nu^2} & 0\\ 0 & 1\end{pmatrix}\ \begin{pmatrix}4\nu^2 & 0\\ 0 & 1\end{pmatrix}$$

be the nontrivial zeros:

$$\lambda_\pm(4\nu^2,i^2,E^I_{4\nu^2})=\frac{1\pm i\sqrt{16\nu^2\cdot E^I_{4\nu^2}-1}}{2}=\frac{1}{2}\pm i\gamma_{\mu_p}\ ,$$

where $\gamma_{\mu_p}=(16\nu^2(E^I_{2\nu})^2-1)^{\frac12}/2$ with $E^I_{4\nu^2}\equiv(E^I_{2\nu})^2$.

We thus have that:

$$\lambda_+(4\nu^2,i^2,E^I_{4\nu^2})\cdot\lambda_-(4\nu^2,i^2,E^I_{4\nu^2})=4\nu^2(E^I_{2\nu})^2=\frac{1}{4}+\gamma^2_{\mu_p}$$

leading to:

$$2\nu E^I_{2\nu}\equiv\mu_pE^I_{2\nu}=\left(\frac{1}{4}+\gamma^2_{\mu_p}\right)^{\frac12}\simeq\gamma\quad\text{for}\quad\gamma^2_{\mu_p}\gg\frac{1}{4}\ .$$

So, $\mu_pE^I_\nu$, which is the energy of the μ-th section s_μ or of a photon at μ_p quanta, is approximatively equal to the imaginary part γ_{μ_p} of the nontrivial zero $\lambda_\pm(4\nu^2ni^2,E^I_{4\nu^2})$.

2) The solution of the $1D$-mass biwave equation of the bisemielectron is:

$$\begin{aligned}\psi(z)\quad=\quad&\frac{\psi_0}{2}\exp\left(-i\ \frac{c}{\hbar}S^{-1}m_0z\right)\\&\left[\left(1+\ \frac{m_0}{E}\right)\exp\left(i\ \frac{c}{\hbar}S^{-1}Ez\right)+\left(1-\ \frac{m_0}{E}\right)\exp\left(-i\ \frac{c}{\hbar}S^{-1}Ez\right)\right]\end{aligned}$$

where

- the Cauchy initial conditions are at $z=0$, $\psi=\psi_0$ and $\frac{d\psi}{dz}=0$;
- $\exp\left(-i\ \frac{c}{\hbar}S^{-1}m_0z\right)$ is a damping factor depending on the rest mass m_0 .

The solution $\psi(z)$ is thus the sum of two damped waves in phase opposition: the positive frequency wave refers to the left semielectron and the negative frequency wave refers to the right semielectron.

This solution $\psi(z)$ can be written in the following form:

$$\psi(z)=\psi_0\left(1-\ \frac{m_0^2}{E^2}\right)\exp\left(-i\ \frac{c}{\hbar}S^{-1}m_0z\right)\sin\left(\ \frac{c}{\hbar}S^{-1}Ez+\tan^{-1}\ \frac{E}{im_0}\right)$$

which corresponds to a damped sinusoidal motion of period

$$T = \frac{2\pi\hbar S^{+1}}{cE}$$

and dephasage

$$\Delta = \tan^{-1} \frac{E}{im_0} \simeq \tan^{-1} p \ .$$

The mass structure of a one-dimensional bisection of the bisemielectron behaves thus globally like a damped harmonic oscillator having a dephasage Δ proportional to the linear momentum $\vec{p}$ of the μ-th section of the right or left semielectron. ■

Proposition 4.2.7 *The total energy " $E_{eR,L}$ " of a right (resp. left) semielectron is given by*

$$E_{eR,L} = \sum_{\mu=1}^{q} \sum_{m_\mu} E_{\mu,m_{\mu eR,L}}$$

where $E_{\mu,m_{\mu eR,L}}$, noted E above, is the energy of a one-dimensional irreducible subsection of the μ-th section. The total linear momentum $p_{eR,L}$ is similarly given by

$$p_{eR,L} = \sum_{\mu=1}^{q} \sum_{m_\mu} p_{\mu,m_{\mu eR,L}} \ .$$

Definition 4.2.8 (Generic biconnexion) By a generic connexion of a fiber bundle $T_{M_{R;L}}$, we mean a distribution which admits in each point P of the total space of the fibration of $T_{M_{R;L}}$ an horizontal direction tangent to this total space, and which is transversal to the fiber in P .

We shall consider a generic connexion associated to the mass vertical tangent semibundle $T^{(1-3)}_{M_{R;L}}$ and generated by the $4D$-multiplicative operator:

$$\partial A(t,r)_{R,L} = \left\{ +\hbar Ze \ \frac{1}{t_0}, + \ \frac{\hbar}{c} Ze \ \frac{1}{x_1}, + \ \frac{\hbar}{c} Ze \ \frac{1}{x_2}, + \ \frac{\hbar}{c} Ze \ \frac{1}{x_3} \right\}$$

corresponding to the action of a strong external (super) heavy nuclear system of charge Ze .

If the time variable is supposed to be constant, the mass bioperator of the bisemielectron $(\mathbb{M}_R, \mathbb{M}_L)_C$, endowed with the generic biconnexion $(\partial A(r)_R, \partial A(r)_L)_C$, will be:

$$\begin{aligned}
&(\mathbb{M}_R + \partial A_R, \mathbb{M}_L + \partial A_L)_C \\
&= \left(\left\{ m_0, -i \ \frac{\hbar}{c} s_1 \ \frac{\partial}{\partial x_1} + \ \frac{\hbar}{c} Ze \ \frac{1}{x_1}, -i \ \frac{\hbar}{c} s_2 \ \frac{\partial}{\partial x_2} + \ \frac{\hbar}{c} Ze \ \frac{1}{x_2}, -i \ \frac{\hbar}{c} s_3 \ \frac{\partial}{\partial x_3} + \ \frac{\hbar}{c} Ze \ \frac{1}{x_3} \right\}, \right. \\
&\qquad \left. \left\{ m_0, -i \ \frac{\hbar}{c} s_1 \ \frac{\partial}{\partial x_1} + \ \frac{\hbar}{c} Ze \ \frac{1}{x_1}, -i \ \frac{\hbar}{c} s_2 \ \frac{\partial}{\partial x_2} + \ \frac{\hbar}{c} Ze \ \frac{1}{x_2}, -i \ \frac{\hbar}{c} s_3 \ \frac{\partial}{\partial x_3} + \ \frac{\hbar}{c} Ze \ \frac{1}{x_3} \right\} \right)_C
\end{aligned}$$

where $(\cdot,\cdot)_C$ is a complete bilinear form of the left extended internal bilinear Hilbert space H_h^+ .

Corollary 4.2.9 *The mass equation of the bisemielectron endowed with the "strong" biconnexion $(\partial A_R, \partial A_L)_C$ is a second order degenerated elliptic differential bi(linear) equation whose canonical form has a set of particular solutions obtained with the condition*

$$m_0^2 - \frac{1}{2^n}(m_0^2 - E^2) = 0 \; , \qquad n \in I\!N \; ,$$

or $E = \sqrt{2^n - 1}m_0$ which allows to find the energy levels of the right or left semielectron in the strongly perturbated confining phase.

Proof. The treatment of the mass equation of the bisemielectron endowed with the biconnexion $(\partial A_R, \partial A_L)_C$ was developed in [Pie2].

Let us note that the first calculated energy levels of the (semi)electron given by the formula

$$E_{eR,L} = \sqrt{2^n - 1}\; m_{0_{eR,L}}$$

corresponds quite well to the observed values. ■

Definition 4.2.10 (S_L-isomorphism) Instead of considering that the bisemielectron mass structure is defined on the left extended internal Hilbert space H_a^+ as done since definition 4.2.1, we could consider that it is described by the sum of the three bisemisheaves, according to definition 4.1.1:

$$\left(\theta_R^{1-3}(t,r)_M \otimes_D \theta_L^{1-3}(t,r)_M\right) \oplus \left(\theta_R^3(r)_M^{(m)} \otimes_m \theta_L^3(r)_M^{(m)}\right) \oplus \left(\theta_R^{1-(3)}(t,(r))_M^{(e)} \otimes_e \theta_L^{3-(1)}((t),r)_M^{(e)}\right)$$

obtained by application of the S_L-isomorphism on the complete tensor product $\theta_R^{1-3}(t,r)_M \otimes \theta_L^{1-3}(t,r)_M$ between the right semielectron semisheaf $\theta_R^{1-3}(t,r)_M$ and the left semielectron semisheaf $\theta_L^{1-3}(t,r)_M$.

According to definitions 3.1.5 and 3.1.7, this sum of three bisemisheaves can be transformed by means of the $B_L \circ p_L$ map into:

$$\left(\theta_{L_R}^{1-3}(t,r)_M \otimes_D \theta_L^{1-3}(t,r)_M\right) \oplus \left(\theta_{L_R}^3(r)_M^{(m)} \otimes_m \theta_L^3(r)_M^{(m)}\right) \oplus \left(\theta_{L_R}^{1-(3)}(t,(r))_M^{(e)} \otimes_e \theta_L^{3-(1)}((t),r)_M^{(e)}\right)$$

such that $\left(\theta_{L_R}^{1-3}(t,r)_M \otimes_D \theta_L^{1-3}(t,r)_M\right)$ be defined on the left internal bilinear Hilbert space $\mathcal{H}_a^+$, $\left(\theta_{L_R}^3(r)_M^{(m)} \otimes_m \theta_L^3(r)_M^{(m)}\right)$ on the left internal bilinear magnetic space $V_{m;a}^+$ and $\left(\theta_{L_R}^{1-(3)}(t,(r))_M^{(e)} \otimes_e \theta_L^{3-(1)}((t),r)_M^{(e)}\right)$ on the left internal bilinear electric space $V_{e;a}^+$.

Definition 4.2.11 (Bilinear diagonal, magnetic and electric wave equations)
We shall then obtain a set of three second order elliptic differential bilinear equations:

a) a central mass biwave equation:

$$\sum_{i=0}^{3} A^{ii} \; \frac{\partial^2 \psi(t,r)}{\partial x_i^2} - E_D^2 \psi(t,r) = 0$$

where

- $A^{ii} = -\dfrac{\hbar^2}{c^2} s^i s^i \; , \; 1 \leq i \leq 3 \; ; \quad A^{00} = -\hbar^2 s^0 s^0 \; ;$

- $\psi(t,r)=\phi_{L_R}(t,r)\otimes_D\phi_L(t,r)\in\theta^{1-3}_{L_R}(t,r)_M\otimes_D\theta^{1-3}_L(t,r)_M$ is a diagonal bisection such that $\psi(t,r)$ be defined on a compact euclidian domain of $I\!R^4$ of the left internal Hilbert space $\mathcal{H}^+_h$.

This equation corresponds to the Klein-Gordon equation except that the metric δ^i_i is euclidian and not pseudo-euclidian or of Minkowsky type.

b) a bilinear magnetic mass biwave equation:

$$\sum_{\substack{i,j=1\\ i\neq j,i>j}}^{3} A^{ij}\ \frac{\partial\phi_{L_R}(r)^{(m)}}{\partial x_i}\cdot\ \frac{\partial\phi_L(r)^{(m)}}{\partial x_j}-\mu(\phi_{L_R}(r)^{(m)}\cdot\phi_L(r)^{(m)})=0$$

where

- $\phi_L(r)^{(m)}$ (resp. $\phi_{L_R}(r)^{(m)}$) refers to a left (resp. right) magnetic section of $\theta^3_L(r)^{(m)}_M$ (resp. of $\theta^3_{L_R}(r)^{(m)}_M$);
- $A^{ij}=-\dfrac{\hbar^2}{c^2}s^is^j$, $i\neq j$;
- μ refers to the magnetic moment of the bisemielectron.

c) a bilinear electric mass biwave equation:

$$\sum_{i=1}^{3} A^{i0}\ \frac{\partial\phi_{L_R}(r)^{(e)}}{\partial x_i}\cdot\ \frac{\partial\phi_L(t)^{(e)}}{\partial t}-e(\phi_{L_R}(r)^{(e)}\cdot\phi_L(t)^{(e)})=0$$

where

- $\phi_L(t)^{(e)}$ (resp. $\phi_{L_R}(r)^{(e)}$) refers to a left (resp. right) electric time (resp. space) section of $\theta^1_L(t)^{(e)}_M$ (resp. of $\theta^3_{L_R}(r)^{(e)}_M$);
- $A^{i0}=-\dfrac{\hbar^2}{c}s^is^0$;
- e corresponds to the electric charge of the bisemielectron.

Proposition 4.2.12 *If the elliptic differential mass biwave equation of the bisemielectron as given in definition 4.2.3 splits into the set of three elliptic differential equations:*

a) a central mass biwave equation;

b) a bilinear magnetic mass biwave equation;

c) a bilinear electric mass biwave equation,

the general bilinear solution of the elliptic differential mass biwave equation is given by the sum of the solutions of the three split elliptic differential equations, i.e. by:

$$\phi_{L_R}(t,r)\cdot\phi_L(t,r)=\psi(t,r)+\phi_{L_R}(r)^{(m)}\cdot\phi_L(r)^{(m)}+\phi_{L_R}(r)^{(e)}\cdot\phi_L(t)^{(e)}$$

where

- *$\psi(t,r)$ is defined on the left internal bilinear Hilbert space $\mathcal{H}_h^+$;*
- *$\phi_{L_R}(r)^{(m)} \cdot \phi_L(r)^{(m)}$ is defined on the left internal bilinear magnetic space $V_{m;a}^+$;*
- *$\phi_{L_R}(r)^{(e)} \cdot \phi_L(t)^{(e)}$ is defined on the left internal bilinear electric space $V_{e;a}^+$.*

Outline of the proof : The fact that the general bilinear solution of the elliptic differential mass biwave equation of the bisemielectron (see definition 4.2.3)

$$(\mathbb{M}^2 - E^2)\phi_{L_R}(t,r) \cdot \phi_L(t,r) = 0$$

can be developed as the sum of the solutions of the three split bilinear elliptic differential equations results from the S_L-isomorphism according to definition 4.1.1. ■

Definition 4.2.13 (Bisemiphoton wave equation) As the bisemiphoton mass structure is given on the left extended internal bilinear Hilbert space H_a^+ by a bisection isomorphic to $T^1_{L_R}(r)_M \otimes T^1_L(r)_M$ according to definition 4.1.1, the elliptic differential mass biwave equation of the mass structure of a bisemiphoton at $(p+\mu)$ biquanta $\widetilde{M}^I_\mu(r)_R \otimes \widetilde{M}^I_\mu(r)_L$ will be:

$$\sum_{i,j=1}^{3} A^{ij} \; \frac{\partial^2(\phi_{L_R}(r)\cdot\phi_L(r))}{\partial x_i \; \partial x_j} - E^2_\mu(\phi_{L_R}(r)\cdot\phi_L(r)) = 0$$

where

- $\phi_{L_R}(r)\cdot\phi_L(r) \simeq T^1_{L_R,\mu}(r)_M \times T^1_{L,\mu}(r)_M$,
- $A^{ij} = -\dfrac{\hbar^2}{c^2} s^i s^j$,

so that s^i is the i-th component of a $3D$ unit vector of polarization of the semiphoton referring to the two possible different rotations of its sections.

This equation is a second order differential elliptic-parabolic bilinear equation which is degenerated.

Definition 4.2.14 (Canonical wave equation of the bisemiphoton) A right and a left unitary inner automorphisms transform the degenerated second order differential elliptic bilinear equation of the bisemiphoton into the $1D$-canonical second order elliptic differential equation:

$$\bar{A}^{33} \; \frac{\partial^2(\phi_{L_R}(z)\cdot\phi_L(z))}{\partial z^2} - E^2_\mu(\phi_{L_R}(z)\cdot\phi_L(z)) = 0$$

where

$$A^{33} = -\frac{\hbar^2}{c^2}(s_1^2 + s_2^2 + s_3^2) = -\frac{\hbar^2}{c^2} S^2 \; , \qquad z \equiv x_3 \; .$$

This is the equation of an harmonic oscillator:

$$\frac{\partial^2\psi(z)}{\partial z^2} + \omega^2_\mu \psi(z) = 0$$

where

$$\omega^2_\mu = \frac{c^2}{\hbar^2} S^{-2} E^2_\mu \qquad \text{and} \qquad \psi(z) = \phi_{L_R}(z)\cdot\phi_L(z) \; .$$

The general solution of the harmonic oscillator equation of the bisemiphoton consists in the superposition of two waves in phase opposition having frequencies given by

$$\omega_\mu = \frac{c}{\hbar} S^{-1} E_\mu$$

leading to the well-known relation of Einstein:

$$E_\mu = \frac{h}{c} \nu_\mu S \, ,$$

excepting the factor S , where ν_μ is the frequency of μ_p prime ideals as introduced in proposition 4.2.6.

This general solution $\psi(z)$ is explicitly given by:

$$\begin{aligned} \psi(z) &= c_1 \exp\left(i \, \frac{c}{\hbar} S^{-1} E_\mu z\right) + c_2 \exp\left(-i \, \frac{c}{\hbar} S^{-1} E_\mu z\right) \\ &= A \sin\left(\frac{c}{\hbar} S^{-1} E_\mu z + \delta\right) = A \sin(\omega_\mu z + \delta) \end{aligned}$$

corresponding to a sinusoidal motion.

Definition 4.2.15 The solution $\psi(z)$ is a linear combination of two one-dimensional waves corresponding to the one-dimensional irreducible components of the bisemiphoton. The coefficients c_1 and c_2 allow to define the radii of the tori $\exp\left(\pm i \, \frac{c}{\hbar} S^{-1} E_\mu z\right)$ according to proposition 1.1.18.

Assume that $\psi(z)$ has $p_\mu = (p+\mu)$ biquanta. Then, the limit condition gives in $z \equiv a =$ radius:

$$\psi(z) \simeq A \sin(\omega_\mu z) = 0$$

whose solution can be obtained only if $\omega_\mu a = \mu\pi$, i.e. if $\omega_\mu = \frac{\mu\pi}{a}$.

Definition 4.2.16 (Kinetic energy of a bisemilepton) Consider now the central mass diagonal bi-operator $(\mathbb{M}_{L_R}, \mathbb{M}_L) = \mathbb{M}_D^2$ of the central mass biwave equation of a bisemilepton defined on the internal Hilbert space $\mathcal{H}_h^+$ (see definition 4.2.11). It is:

$$\mathbb{M}_D^2 = m_0^2 + \sum_{i=1}^{3} p_i^2$$

if we do not take the spin vector into account.

The norm of the central mass M_D of a bisemilepton is then:

$$\|\mathbb{M}_D\| = \|E^2\| = m_0 \left(1 + \frac{1}{m_0^2} \sum_{i=1}^{3} p_i^2\right)^{1/2} \simeq m_0 + \omega$$

if $\omega \ll m_0$. We thus have that:

$$E^2 = m_0^2 + \sum_{i=1}^{3} p_i^2 \simeq m_0^2 + 2m_0\omega + \omega^2$$

which implies that:

$$\omega = \frac{1}{2m_0} \sum_{i=1}^{3} p_i^2 \, .$$

This term is the kinetic energy of the semilepton and corresponds to the harmiltonian of the Schrödinger equation in a zero potential.

The following proposition results from this.

Proposition 4.2.17 *The kinetic energy of a right or left semilepton is equal to the normed (by the factor $\frac{1}{2m_0}$) inner product between the linear momenta of the right and left semileptons.*

Proof. Indeed, the inner product (p_R, p_L) between the linear momenta of the right and left semileptons is $\sum\limits_{i=1}^{3} p_i^2$. Thus, this inner product (p_R, p_L) normed by the factor $\frac{1}{2m_0}$ corresponds to the kinetic energy of a semilepton $\omega = \frac{1}{2m_0}(p_R, p_L)$. ■

Definition 4.2.18 (The concept of field in algebraic quantum theory) The nature of the field in quantum field theory proceeds from the treatment of the harmonic oscillator in classical mechanisms. It is considered that an infinite number of harmonic oscillators brings us to a field theory with the field at each point of space considered as independent generalized coordinate and that the field quantization results from the quantization of an infinite assemblage of harmonic oscillators [B-D].

In this model of algebraic quantum theory, each pair of right and left one-dimensional sections, isomorphic to one-dimensional tori, can be considered as a (damped) harmonic oscillator according to propositions 4.2.6, 4.2.12 and 4.2.14. Let us recall that each elementary bisemiparticle (bisemifermion or bisemiboson) has a central (i.e. diagonal) "mass" spatial structure composed of pairs of right and left one-dimensional sections (which are in fact one-dimensional waves or strings) of the mass bisemisheaf $(\theta^3_{L_R}(r)_M \otimes_D \theta^3_L(r)_M)$, the electric charge and the magnetic moment being also composed of pairs of one-dimensional sections characterized by an electric and a magnetic metric as developed in proposition 3.1.6.

Thus, every elementary bisemiparticle has a central "mass" spatial structure given by a corresponding field $\theta^3_{L_R}(r)_M \otimes_D \theta^3_L(r)_M$ behaving like a sum of independent harmonic oscillators if we refer for example to the definition of the wave (bi)function of an elementary particle having a spectral decomposition of algebraic type as introduced in proposition 3.3.5. By this way, we recover the classical concept of field of the quantum theories [Wein2].

However, let us note that the electron field is given in quantum field theory by

$$\psi(x) = \sum_k u_k(\vec{x}) e^{-i\omega_k t} a_k$$

where $\{u_k(\vec{x})e^{-i\omega_k t}\}$ is a set of orthonormal plane-wave solutions of Dirac equation and where the a_k are annihilation operators. The annihilation and creation operators were defined in this algebraic quantum model as morphisms of type $\gamma^M_{t\to r} \circ E \circ SOT(2) \circ Vd(2)$ (see remark 2.3.7) generating the "mass" semisheaves of rings from the vacuum composed of the internal semisheaves of rings " ST " and " MG ".

In quantum field theory, one is dealing with a system of an infinite number of degrees of freedom, leading to a nonseparable Hilbert space [Wigh1]. This refers to the old problem of quantum mechanics consisting in the difference between its discontinued space " Z " ($=1,2,\cdots$) of discrete values of the index μ and its configuration space " Ω " which is continued with k dimensions where k is the number of degrees of freedom of the system [V.Neu1].

If we consider the algebraic spectral decomposition of the wave bifunction in terms of pairs of one-dimensional sections which correspond to the degrees of freedom of the envisaged system, it appears that the spaces " Z " and " Ω " are in one-to-one correspondence since:

- the global class residue degree f_μ of an irreducible one-dimensional section at μ_p quanta is a degree of freedom of the system;

- the index μ of the configuration space " Ω " is the integer μ labelling the μ-th eigenbifunction $\psi_{L_R}(\mu) \otimes \psi_L(\mu) \in \mathbb{M}^{\,h}_{R\times L}(\mathcal{H}^+_h\{\mu\})$ (see 3.2.18 and 3.2.19).

This brings a new light on the Hilbert spaces of infinite dimensions of quantum field theory since the spectral decomposition of a wave (bi)function can have an infinite dimension.

Definition 4.2.19 (Invariance of elements of bisemimodules) We shall now envisage the invariance of the space-time structure of bisemifermions and more particularly of bisemileptons. Referring to definition 4.1.1, the space-time left bisemisheaf of a bisemilepton defined on the left extended internal bilinear Hilbert space H^+_a splits under the S_L isomorphism into the set of the three disconnected left bisemisheaves:

1) the $4D$-space-time diagonal left bisemisheaf $\theta^{1-3}_{L_R}(t,r) \otimes_D \theta^{1-3}_L(t,r)$ whose elements are the diagonal bielements $\phi_{L_R} \otimes_D \phi_L \in \mathcal{H}^+_a$ characterized by a $4D$(-euclidian) metric of type δ^α_α , $0 \leq \alpha \leq 3$.

   These bielements are invariant under a (bi)representation of $SO(4,I\!\!R) \times SO(4,I\!\!R)$.

   Note that $SO(n,I\!\!R)$ is the orthogonal unimodular group of order n acting linearly on a left or right n-dimensional semimodule.

2) the $3D$-space magnetic left bisemisheaf $\theta^3_{L_R}(r)^{(m)} \otimes_m \theta^3_L(r)^{(m)}$ whose bielements are characterized by a $3D$ metric of type $g_{\alpha\beta}$, $1 \leq \alpha$, $\beta \leq 3$, $\alpha \geq \beta$, $\alpha \neq \beta$.

   The magnetic bielements are then invariant under a representation of $SL(3,I\!\!R) \times SL(3,I\!\!R)$ where $SL(3,I\!\!R)$ is the unimodular special linear group of order 3 .

   The group of left or right magnetic invariance is $SL(3,I\!\!R)$ because, if $g_m \in SL(3,I\!\!R)$, then the magnetic invariance condition is $(g^T_m)_R(g_m)_L = h_m$ where

$$h_m = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \qquad \text{or} \qquad h_m = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

   has determinant equal to one.

3) the $3D$-space-time electric left bisemisheaf

$$\theta^{1-(3)}_{L_R}(t,(r))^{(e)} \otimes_e \theta^{(1)-3}_L((t),r)^{(e)}$$

   whose bielements are characterized by a $3D$ metric of type $g_{0\alpha}$ or $g_{\alpha 0}$, $1 \leq \alpha \leq 3$.

   The electric bielements are then invariant under a representation of $SL(1,I\!\!R) \times SL(3,I\!\!R)$ corresponding to a $1D$-right (resp. $3D$-left) translation on the right (resp. left) semisheaf and to a $3D$-left (resp. $1D$-right) translation on the left (resp. right) semisheaf.

Note that the space-structure of a bisemiphoton defined on H_a^+ is given by the complete bisection $T^1_{L_R}(r) \otimes T^1_L(r)$ which splits under the S_L-isomorphism into the sum of the two disconnected bisections:

1) the $3D$-space diagonal bisection $T^1_{L_R}(r) \otimes_D T^1_L(r)$ whose diagonal bielements are invariant under a representation of $SO(3, I\!R) \times SO(3, I\!R)$;

2) the $3D$-space magnetic bisection $T^1_{L_R}(r)^{(m)} \otimes_m T^1_L(r)^{(m)}$ whose bielements are invariant under a representation of $SL(3, I\!R) \times SL(3, I\!R)$.

Proposition 4.2.20 *The $4D$ (resp. $3D$) diagonal bielements of a $4D$ (resp. $3D$) diagonal left bisemisheaf defined on a bilinear internal Hilbert space $\mathcal{H}_a^+$ are invariant under a right and a left action of $SO(4, I\!R) \times SO(4, I\!R)$ (resp. of $SO(3, I\!R) \times SO(3, I\!R)$) which correspond to a right and a left inner automorphism.*

The $3D$ magnetic bielements of a $3D$ magnetic bisemisheaf are invariant under a right and a left action of $SL(3, I\!R) \times SL(3, I\!R)$.

The $3D$ electric bielements of a $3D$ electric bisemisheaf are invariant under a right (resp. left) and a left (resp. right) action of $SL(1, I\!R) \times SL(3, I\!R)$ (resp. of $SL(3, I\!R) \times SL(1, I\!R)$).

4.3 The bidynamics of bisemiparticles

Definition 4.3.1 (Bidynamics and bisemiflow) Let $\theta_R^{1-3}(t,r) \otimes_{D,m,e} \theta_L^{1-3}(t,r)$ be the diagonal, magnetic or electric tensor product between the right semisheaf $\theta_R^{1-3}(t,r)$, referring to a right semiparticle, and the left semisheaf $\theta_L^{1-3}(t,r)$, referring to a left semiparticle.

The dynamics of this bisemisheaf is a bidynamics corresponding to right and left diffeomorphisms at the one parameter time " t " applied respectively to the envisaged right and left semisheaves. We are then led to define a local bisemisheaf as follows:

Let $\Gamma(\theta^1_{R,L}(t))$ and $\Gamma(\theta^3_{R,L}(r))$ be the right or left $1D$ and $3D$ sections of $\theta^1_{R,L}(t)$ and $\theta^3_{R,L}(r)$ above respectively $1D$-time and $3D$-space domains.

By a local bisemiflow for a densely defined self-adjoint bioperator, for example,

$$p_R \otimes_D p_L = \left\{ +i\ \frac{\hbar}{c} s_1\ \frac{\partial}{\partial x_1}, \cdots, +i\ \frac{\hbar}{c} s_3\ \frac{\partial}{\partial x_3} \right\} \otimes_D \left\{ -i\ \frac{\hbar}{c} s_1\ \frac{\partial}{\partial x_1}, \cdots, -i\ \frac{\hbar}{c} s_3\ \frac{\partial}{\partial x_3} \right\}$$

acting on $\Gamma(\theta^3_R(r)) \otimes_D \Gamma(\theta^3_L(r))$, we mean a bijective bilinear map:

$$F(p_R)_{-t} \otimes_D F(p_L)_{+t} : \Gamma(\theta^3_R(r)) \otimes_D \Gamma(\theta^3_L(r)) \to \Gamma(\theta^3_R(r)) \otimes_D \Gamma(\theta^3_L(r))$$

such that

$$F(p_R)_{-(t_1+t_2)} \otimes_D F(p_L)_{+(t_1+t_2)} = F(p_R)_{-t_1} \cdot F(p_R)_{-t_2} \otimes_D F(p_L)_{+t_1} \cdot F(p_L)_{+t_2}$$

be a geodesic bisemiflow corresponding to a "time" translation on the right and the left $1D$-sections of $\theta^3_R(r)$ and $\theta^3_L(r)$ [Lan2], [Sma1].

A bivector field with domain of $\Gamma(\theta^3_R(r)) \otimes_D \Gamma(\theta^3_L(r))$ is a bimap:

$$p_R \otimes_D p_L : \Gamma(\theta^3_R(r)) \otimes_D \Gamma(\theta^3_L(r)) \to \Gamma(T^{(3)}_R(\theta^3_R(r))) \otimes_D \Gamma(T^{(3)}_L(\theta^3_L(r)))$$

into the diagonal tangent bibundles $(T^{(3)}_R \otimes_D T^{(3)}_L)$.

We can thus generalize the Stone theorem [C-M] in the case of a local bisemiflow.

Proposition 4.3.2 *The local bisemiflow $F(O_R)_{-t}\otimes_{D,m,e}F(O_L)_{+t}$ for a self-adjoint bioperator $O_R\otimes_{D,m,e} O_L$ is given by the right and left actions $U_R(-t)\otimes_{D,m,e}U_L(+t)$ of the continuous one-parameter unitary Lie group such that:*

$$U_R(-t)\otimes_{D,m,e}U_L(+t)=e^{-itO_R}\otimes_{D,m,e}e^{+itO_L}\qquad \textit{for } t\in I\!R^+ \ .$$

Proof. This proposition generalizes the Stone theorem in the case of a bisemiflow. Remark that O_R is a right operator semibounded from above and O_L is a left operator semibounded from below.

The bigenerator $O_R\otimes_{D,m,e}O_L$ of the local bisemiflow $F(O_R)_{-t}\otimes_{D,m,e}F(O_L)_{+t}$ is given by:

$$O_R\otimes_{D,m,e}O_L=\left.\frac{d}{dt}F(O_R)_{-t}\right|_{t=0}\otimes_{D,m,e}\left.\frac{d}{dt}F(O_L)_{+t}\right|_{t=0} .$$

We then have for the bisemimodule $\theta_R(-t)\otimes_{D,m,e}\theta_L(+t)$:

$$\theta_R(-t)\otimes_{D,m,e}\theta_L(+t)=F(O_R)_{-t}\theta_R(0)\otimes_{D,m,e}F(O_L)_{+t}\theta_L(0) .$$

■

Remarks 4.3.3 Evolution of bisemiparticles and the classical symplectic structure

1) This bidynamics is a bidiffeomorphism with respect to the time variable describing the (bi)evolution of bisemiparticles from an initial event localized at the time $T=0$. This initial event may be the big-bang of physics.

   Consequently, the bisemiflow should take into account this delayed (bi)evolution by a parameter (or a constant following the traditional terminology) depending on time, called $B(\pm t)$ and related likely to the Hubble constant $H(\pm t)$, such that:

$$U(-t)\otimes_{D,m,e}U(+t)=e^{-itB(-t)O_R}\otimes_{D,m,e}e^{+itB(+t)O_L} .$$

2) This bidynamics must be envisaged simultaneously on the three embedded structures "space-time", "middle-ground" and "mass" of the bisemiparticles giving their irreversible evolution in time.

   It can happen that, under some external perturbation during a small interval of time dt , there are fixed points on the bisemiparticle structures " ST , " MG " or " M " with respect to the bidiffeomorphism at the one parameter time group. These fixed points then correspond to degenerated singularities of genotype attractors, problem which was developed elsewhere [Pie8].

3) It is natural to regard the complex Hilbert space as the analog of the cotangent bundle in a classical system [Sim]. Endowed with a symplectic form which is the imaginary part of the inner product, we get a symplectic structure [God], [Sha], [Lich], from which the classical (and quantum) Lagrangian and hamiltonian dynamics proceed.

   Let us note that the (bi)dynamics, developed in this paper, does not result from a Lagrangian or an hamiltonian method which presents the following difficulties:

   a) The Lagrangian action is not well understood in mathematics [C-M] likely because the " q " and " p " variables are inextricably mixed [Tho3]. This could be explained by the fact that classical and quantum theories "work" on a single structure or level while, in the present AQT , three embedded structures have been taken into account leading to a noncommutative algebra [Gui], [Con1].

b) The automorphisms of the symplectic structure generate a space of infinite dimension [Kib] while the automorphisms of the euclidian structure leads to a space of finite dimension.

c) The second order elliptic linear Laplace equation $\frac{\partial^2 u}{\partial t^2}+\Delta u=0$ leads to no really decent Banach space on which this equation generates a flow and an energy of definite sign [C-M], which is not the case here.

4) It is commonly admitted that the classical mechanics is the limit case of the quantum mechanics providing that the Planck constant satisfies $\hbar \to 0$. Now, quantum theories deal with the discontinued behavior of the matter while the classical theories are only concerned with continuous objects. As the discontinued behavior of matter is described here algebraically, it becomes evident that the structure of the quantum theories must be described by coherent algebraic sheaves of rings while the structure of classical theories could refer to coherent analytic sheaves, i.e. ring of (germs of) holomorphic functions. It then results that the isomorphism from quantum theories to classical theories given by the condition $\hbar \to 0$ corresponds likely to the isomorphism of J.P. Serre between coherent algebraic sheaves and coherent analytic sheaves [Ser7].

   Indeed, according to definition 1.1.24, a quantum $\widetilde{M}^I_{\mu_{R,L}}$ is a continuous subfunction of the algebraic semisheaf of rings $\theta^1_{R,L}$ over a big point centered on $M^I_{\mu_{R,L}}$. So, when $\hbar \to 0$, the "big point" on which is centered $M^I_{\mu_{R,L}}$ and to which the Zariski topology corresponds tends to an "ordinary point" associated with the ordinary finer topology. Consequently, the algebraic semisheaf of rings $\theta^1_{R,L}$ is transformed into an analytic sheaf.

Definition 4.3.4 (Physical internal machinery of a bisemiparticle) The internal machinery of a bisemiparticle allows to justify the absorption and the emission of right and left quanta from the space-time, middle-ground or mass bisemisheaves of rings, noted in abbreviated form $\theta^{1-3}_R \otimes \theta^{1-3}_L$. According to definition 2.2.12, the emission of right or left quanta occurs by means of the smooth endomorphism $E_{R,L}$.

The considered bisemiparticle will be a bisemilepton for simplicity and the following developments will be envisaged for the spatial bistructure $\theta^3_R \otimes \theta^3_L$.

It was proved in proposition 3.3.11 that the right and left semiparticles rotate in opposite senses. This means that each spatial bisection of the " ST , " MG " or " M " bisemisheaf of rings $\theta^3_R \otimes_D \theta^3_L$, and thus that the " ST , " MG " or " M " spatial bistructure, behaves globally like two adjacent gyroscopes having opposite torques τ_R and τ_L so that the right and left torques are defined at the points $P_{R,L} \in \theta^3_{R,L}$ by $\tau_{R,L} = \frac{dL_{R,L}}{dt}$ where $L_{R,L}$ is the right (resp. left) angular momentum.

Now, it is well-known that the centripetal force

$$F_{p_{R,L}} = -\frac{mv_J^2}{r}$$

acting on a point, having a linear momentum

$$p_{R,L} = (mv)_{R,L} = (m\omega r)_{R,L} \ ,$$

can decompose into the sum of the three forces [F-L-S]:

$$F_{p_{R,L}} = -\frac{mv_J^2}{r} = -\frac{mv_M^2}{r} - 2mv_M\omega - m\omega^2 r$$

where

a) the velocity v_J of the rotating point $P_{R,L}$ is the sum of the rotational velocity v_M and of an additional angular velocity ωr :

$$v_J = v_M + \omega r \; ;$$

b) r is the distance from the point $P_{R,L}$ to the emergence point of the bisemiparticle;

c) $F_D = -\frac{mv_M^2}{r}$ is the "diagonal" centripetal force which is in fact independent of the rotation. $F_M = -2mv_M\omega$ is the Coriolis force responsible for the torque $\tau = \frac{dL}{dt} = F_M \times r$ in action in the gyroscope.

   $F_E = -m\omega^2 r$ is the centripetal force acting on points $P_{R,L}$ even still in $\theta^3_{R,L}$.

Proposition 4.3.5 *1) The space of diagonal biquanta $\widetilde{M}^I_R \otimes_D \widetilde{M}^I_L$ are generated from the bisemisheaf $\theta^3_R \otimes_D \theta^3_L$ under the action of the diagonal centripetal biforce $F_{D_R} \otimes_D F_{D_L}$, where $F_D = -\frac{mv_M^2}{r}$, responsible for the smooth biendomorphism $E_R \otimes_D E_L$ acting on $\theta^3_R \otimes_D \theta^3_L$.*

2) The magnetic space biquanta $\widetilde{M}^I_R \otimes_m \widetilde{M}^I_L$ are generated from the bisemisheaf $\theta^3_R \otimes_D \theta^3_L$ under the action of the Coriolis biforce $F_{M_R} \otimes F_{M_L}$, where $F_M = -2mv_M\omega$, responsible for the smooth biendomorphism $E_R \otimes_m E_L$ acting on $\theta^3_R \otimes_D \theta^3_L$.

3) The emission (and the subsequent reabsorption) of left and right magnetic quanta by the left and right semisheaves θ^3_L and θ^3_R of a left and right semiparticles having different magnitudes of rotational velocities results from the differences between right and left torques and generates by reaction a global movement of translation of the bisemiparticle.

Proof.

1) The emission of diagonal biquanta $\widetilde{M}^I_R \otimes_D \widetilde{M}^I_L$ must result from the diagonal centripetal force $F_{D_R} \otimes_D F_{D_L}$. Indeed, a diagonal centripetal force F_D must provoke Galois antiautomorphisms on an algebraic semigroup leading to the smooth endomorphism $E_{R,L}$.

2) Knowing that the Coriolis force is a force acting sidewise, it seems natural to attribute to this force the cause of the emission of magnetic quanta. Indeed, magnetic quanta would be emitted in order to balance any variation of the rotational kinetic energy between a left and a right semisheaves because the work done against the centrifugal force ought to agree with the difference in rotational energy. If it was not the case, the centrifugal force would not be equilibrated and would run out.

   Note also in this context that the magnetic moment of the electron is equal to $\mu = \frac{e}{2mc} \cdot \ell$ and is thus proportional to its angular momentum ℓ .

3) Consider that a left and a right $3D$-space semisheaves do not have the same magnitude of rotational velocity. Then we can assign to every point $P_R \in \theta^3_R$ a torque τ_R whose length and direction are different from those of a torque τ_L corresponding to a point $P_L \in \theta^3_L$.

   We thus have a resulting torque $\delta\tau = \tau_R - \tau_L$ at the "bipoint" $P_R \times P_L$. It is reasonable to admit that the action of the resulting torque $\delta\tau$ will be at the origin of a smooth endomorphism $E_{R,L}$.

   As a consequence, a left or a right magnetic quantum will be emitted which will provoke by reaction a movement of translation of the bisemiparticle.

   Thus, the Coriolis force which is in fact apparent becomes here effective by the emission of magnetic quanta. The magnetic quanta, emitted towards the emergence point of the bisemiparticle, are reabsorbed later on. Thus, a process of emission-reabsorption of magnetic left and right quanta generates a global movement of translation of the bisemiparticle. ∎

Remarks 4.3.6 1) It is likely that the centripetal force $F_e = -m\omega^2 r$ is responsible for the emission of $3D$-space electric quanta.

2) The assertions of proposition 4.3.5 are valid for bisemiphotons, which explains why a set of bisemiphotons generates a magnetic field.

5 The gravito-electro-magnetic interactions between bisemiparticles

It was seen in chapters 3 and 4 that the structure of bisemiparticles is given by bisemisheaves so that an action-reaction process is generated in a bisemiparticle by the interactions between the right-semisheaves of the right semiparticle and the left-semisheaves of the left semiparticle.

Generalizing this concept to a set of bisemiparticles, one easily demonstrates that the interactions between a set of bisemiparticles result from the interactions between the right and the left semisheaves belonging to different bisemiparticles leading to a set of mixed action-reaction processes of bilinear nature.

Mathematically, if we have a set of N (distinguishable) bisemiparticles, their $3D$ spatial structure is given by the completely reducible representation $\operatorname{Rep}(GL_{2N}(\mathbb{A}_R \times \mathbb{A}_L))$ of the bilinear general semigroup $GL_{2N}(\mathbb{A}_R \times \mathbb{A}_L)$. Indeed, given a partition $2N = 2_1 + 2_2 + \cdots + 2_N$ of $2N$, the tensor product $\operatorname{Rep}(GL_{2_1}(\mathbb{A}_R \times \mathbb{A}_L)) \otimes \cdots \otimes \operatorname{Rep}(GL_{2_N}(\mathbb{A}_R \times \mathbb{A}_L))$ has an irreducible quotient given by the formal sum

$$\begin{aligned}
&\operatorname{Rep}(GL_{2N=2_1+\cdots+2_i\cdots+2_N}(\mathbb{A}_R \times \mathbb{A}_L)) \\
&\quad = \operatorname{Rep}(GL_{2_1}(\mathbb{A}_R \times \mathbb{A}_L)) \boxplus \cdots \\
&\qquad\quad \boxplus \operatorname{Rep}(GL_{2_i}(\mathbb{A}_R \times \mathbb{A}_L)) \boxplus \cdots \boxplus \operatorname{Rep}(GL_{2_N}(\mathbb{A}_R \times \mathbb{A}_L)) \;, \qquad 1 \le i \le N \;,
\end{aligned}$$

which constitutes a completely reducible orthogonal representation of $GL_{2_N}(\mathbb{A}_R \times \mathbb{A}_L)$.

The nonorthogonal completely reducible representation of $GL_{2_N}(\mathbb{A}_R \times \mathbb{A}_L)$ is reached if we add to $\operatorname{Rep}(GL_{2N=2_1+\cdots+2_i\cdots+2_N}(\mathbb{A}_R \times \mathbb{A}_L))$ the direct sum of off-diagonal irreducible bilinear representations $\operatorname{Rep}(T^t_{2_{i_R}}(\mathbb{A}_R) \times T_{2_{j_L}}(\mathbb{A}_L))$ for all pairs of semiparticle indices, $i \neq j$.

These mathematical considerations allow to develop Langlands global bilinear correspondences for reducible representations of $GL(2N)$ [Pie9] and to introduce the general mathematical frame of the interactions between bisemiparticles as studied in section 5.1.

5.1 Interactions between bisemiparticles

Definition 5.1.1 (Interacting bisemiparticles) Let $(\theta_{R_i}\otimes\theta_{L_i})$ be a bisemisheaf " ST ", " MG " or " M " of a bisemiparticle " i ". The total right (resp. left) semisheaf Θ_{R_N} (resp. Θ_{L_N}) of a set of N right (resp. left) semiparticles on $GL_{2N}(\mathbb{A}_R)$ (resp. $GL_{2N}(\mathbb{A}_L)$) is the union of the disconnected right (resp. left) semisheaves:

$$\Theta_{R,L_N}=\theta_{R,L_1}\cup\cdots\cup\theta_{R,L_i}\cup\cdots\cup\theta_{R,L_N}$$

given by the direct sum [Art] of all right (resp. left) semisheaves $\Theta_{R,L_N}=\bigoplus\limits_{i=1}^{N}\theta_{R,L_i}$ if and only if $\theta_{R,L_i}\cap\theta_{R,L_{i+1}}=\emptyset$.

Then, the bisemisheaf $(\Theta_{R_N}\otimes\Theta_{L_N})$ of a set of N interacting bisemiparticles on $GL_{2N}(\mathbb{A}_R\times\mathbb{A}_L)$ will be:

$$\Theta_{R_N}\otimes\Theta_{L_N}=\left(\bigoplus_{i=1}^{N}\theta_{R_i}\right)\otimes\left(\bigoplus_{j=1}^{N}\theta_{L_j}\right)=\bigoplus_{i,j=1}^{N}\left(\theta_{R_i}\otimes\theta_{L_j}\right)$$

which can be decomposed following:

$$\Theta_{R_N}\otimes\Theta_{L_N}=\bigoplus_{i=1}^{N}(\theta_{R_i}\otimes\theta_{L_i})\bigoplus_{\substack{i,j=1\\ i\neq j}}^{N}\left(\theta_{R_i}\otimes\theta_{L_j}\right)$$

where

a) the direct sum $\bigoplus\limits_{i=1}^{N}(\theta_{R_i}\otimes\theta_{L_i})$ refers to the total bisemisheaf of N-noninteracting (i.e. free) bisemiparticles verifying the condition of nonconnectivity between the bisemisheaves " i " and " j ": $(\theta_{R_i}\otimes\theta_{L_i})\cap(\theta_{R_j}\otimes\theta_{L_j})=\emptyset$.

b) the "mixed" direct sum $\bigoplus\limits_{i,j=1}^{N}(\theta_{R_i}\otimes\theta_{L_j})$ refers to the bilinear interactions between the right semisheaves θ_{R_i} of the N right semiparticles and the left semisheaves θ_{L_j} of the N left semiparticles. The bisemisheaf of a "mixed" direct sum is thus an interference bisemisheaf between the N interacting bisemiparticles.

Definition 5.1.2 (Interaction bisemisheaves of interacting bisemileptons) The $(i-j)$-th interaction bisemisheaf $(\theta_{R_i}\otimes\theta_{L_j})$ of the "mixed" direct sum of the total bisemisheaf $(\Theta_{R_N}\otimes\Theta_{L_N})$ of a set of N interacting bisemileptons represents the interactions between the right semisheaf θ_{R_i} of the i-th bisemilepton and the left semisheaf θ_{L_j} of the j-th bisemilepton and can be developed following:

$$\begin{aligned}\theta_{R_i}\otimes\theta_{L_j}&\equiv\theta^{1-3}_{R_i}(t_i,r_i)\otimes\theta^{1-3}_{L_j}(t_j,r_j)\\ &\xrightarrow{S_L}\quad(\theta^{1-3}_{R_i}(t_i,r_i)\otimes_D\theta^{1-3}_{L_j}(t_j,r_j))\\ &\qquad\bigoplus(\theta^{3}_{R_i}(r_i)^{(m)}\otimes_m\theta^{3}_{L_j}(r_j)^{(m)})\bigoplus(\theta^{1-(3)}_{R_i}(t_i,(r_i))^{(e)}\otimes_e\theta^{(1)-3}_{L_j}((t_j),r_j)^{(e)}\end{aligned}$$

if we take into account the S_L isomorphism (see definition 4.1.1).

Note that

1) $\theta^{1-3}_{R_i}(t_i,r_i)\otimes_D\theta^{1-3}_{L_j}(t_j,r_j)$ is a "mixed" diagonal $4D$-space-time bisemisheaf composed of:

a) "mixed" orthogonal space biquanta $\widetilde{M}^I_{R_i}(r_i) \otimes_D \widetilde{M}^I_{L_j}(r_j)$ generated by a spatial smooth biendomorphism $E_R \otimes_D E_L$ (see proposition 4.3.5);

b) "mixed" time biquanta $\widetilde{M}^I_{R_i}(t_i) \otimes \widetilde{M}^I_{L_j}(t_j)$ generated by a time smooth biendomorphism $E_R \otimes E_L$.

2) $\theta^3_{R_i}(r_i)^{(m)} \otimes_m \theta^3_{L_j}(r_j)^{(m)}$ is a "mixed" $3D$ magnetic bisemisheaf composed of "mixed" magnetic biquanta $\widetilde{M}^I_{R_i}(r_i)^{(m)} \otimes_m \widetilde{M}^I_{L_j}(r_j)^{(m)}$ of a magnetic field.

3) $\theta^{1-(3)}_{R_i}(t_i(,r_i))^{(e)} \otimes_e \theta^{(1)-3}_{L_j}((t_j),r_j)^{(e)}$ is a "mixed" electric bisemisheaf composed of electric biquanta $(\widetilde{M}^I_{R_i}(t_i)^{(e)} \otimes_e \widetilde{M}^I_{L_j}(r_j)^{(e)})$ or $(\widetilde{M}^I_{R_i}(r_i)^{(e)} \otimes_e \widetilde{M}^I_{L_j}(t_j)^{(e)})$ of an electric field.

Remark that all these "mixed" biquanta of interaction are localized between the bisemileptons labelled " i " and " j ".

Definition 5.1.3 (Interaction bisemisheaves of interacting bisemiphotons) Similarly, the $(i-j)$-th interaction bisection $(T_{R_i} \otimes T_{L_j})$ of the "mixed" direct sum of the total bisemisheaf $(T_{R_M} \otimes T_{L_M})$ of a set of M interacting bisemiphotons is given by:

$$\begin{aligned}
&T_{R_i} \otimes T_{L_j} \equiv T^1_{R_i}(r_i) \otimes T^1_{L_j}(r_j) \\
&\xrightarrow{S_L} (T^1_{R_i}(r_i) \otimes_D T^1_{L_j}(r_j)) \bigoplus (T^1_{R_i}(r_i)^{(m)} \otimes_m T^1_{L_j}(r_j)^{(m)})
\end{aligned}$$

where

1) $T^1_{R_i}(r_i) \otimes_D T^1_{L_j}(r_j)$ is a "mixed" diagonal space bisection composed of "mixed" orthogonal space biquanta $\widetilde{M}^I_{R_i}(r_i) \otimes_D \widetilde{M}^I_{L_j}(r_j)$.

2) $T^1_{R_i}(r_i)^{(m)} \otimes_m T^1_{L_j}(r_j)^{(m)}$ is a "mixed" magnetic bisection composed of magnetic biquanta of the magnetic field of the interacting bisemiphotons.

Definition 5.1.4 (Bisemileptons interacting with bisemiphotons) The total bisemisheaf $\Theta_{R_{N-M}} \otimes \Theta_{L_{N-M}}$ of a set of N interacting bisemileptons interacting with a set of M interacting bisemiphotons is given by

$$\begin{aligned}
\Theta_{R_{N-M}} \bigotimes \Theta_{L_{N-M}} &= \left(\bigoplus_{i=1}^N \theta_{R_i} \bigoplus_{k=1}^M T_{R_k} \right) \bigotimes \left(\bigoplus_{j=1}^N \theta_{L_j} \bigoplus_{h=1}^M T_{L_h} \right) \\
&\equiv \left(\bigoplus_{i=1}^N \theta^{1-3}_{R_i}(t_i,r_i) \bigoplus_{k=1}^M T^1_{R_k}(r_k) \right) \bigotimes \left(\bigoplus_{j=1}^N \theta^{1-3}_{L_j}(t_j,r_j) \bigoplus_{h=1}^M T^1_{L_h}(r_j) \right) \\
&= \bigoplus_{i,j=1}^N \left(\theta^{1-3}_{R_i}(t_i,r_i) \otimes \theta^{1-3}_{L_j}(t_j,r_j) \right) \bigoplus_{k,h=1}^M \left(T^1_{R_k}(r_k) \otimes T^1_{L_h}(r_h) \right) \\
&\qquad \bigoplus_{i,h=1}^{N,M} \left(\theta^{1-3}_{R_i}(t_i,r_i) \otimes T^1_{L_h}(r_h) \right) \bigoplus_{k,j=1}^{N,M} \left(T^1_{R_k}(r_k) \otimes \theta^{1-3}_{L_j}(t_j,r_j) \right)
\end{aligned}$$

where

a) the direct sum $\bigoplus\limits_{i,j=1}^{N}$ refers to the total bisemisheaf of N interacting bisemileptons;

b) $\bigoplus\limits_{k,h=1}^{M}$ refers to the total bisemisheaf of M interacting bisemiphotons;

c) the mixed direct sum $\bigoplus\limits_{i,h=1}^{N,M}$ refers to the bilinear interactions between the right semisheaves $\theta^{1-3}_{R_i}(t_i,r_i)$ of the N right semileptons and the left semisheaves $T^1_{L_h}(r_h)$ of the M left semiphotons.

The $(i-h)$-th interaction bisemisheaf of this direct sum $\bigoplus\limits_{i,h=1}^{N,M}$ decomposes under the S_L-isomorphism into the bisemisheaves:

$$\begin{aligned} S_L &: \theta^{1-3}_{R_i}(t_i,r_i)\otimes T^1_{L_h}(r_h) \\ &\to \quad \left(\theta^3_{R_i}(r_i)\otimes_D T^1_{L_h}(r_h)\right)\bigoplus\left(\theta^3_{R_i}(r_i)^{(m)}\otimes_m T^1_{L_h}(r_h)^{(m)}\right)\bigoplus\left(\theta^1_{R_i}(t_i)^{(e)}\otimes_e T^1_{L_h}(r_h)^{(e)}\right)\end{aligned}$$

where

1. $\theta^3_{R_i}(r_i)\otimes_D T^1_{L_h}(r_h)$ is a mixed diagonal space bisemisheaf composed of mixed orthogonal space biquanta $\widetilde{M}^I_{R_i}(r)\otimes_D \widetilde{M}^I_{L_h}(r)$;
2. $\theta^3_{R_i}(r_i)^{(m)}\otimes_m T^1_{L_h}(r_h)^{(m)}$ is a mixed $3D$ magnetic bisemisheaf generating a magnetic field;
3. $\theta^1_{R_i}(t_i)^{(e)}\otimes_e T^1_{L_h}(r_h)^{(e)}$ is a mixed $3D$ electric bisemisheaf generating an electric field.

d) the mixed sum $\bigoplus\limits_{k,j=1}^{M,N}$ refers to the bilinear interactions between the right sections $T^1_{R_k}(r_k)$ of the M right semiphotons and the left semisheaves $\theta^{1-3}_{L_j}(t_j,r_j)$ of the N left semileptons.

The $(k-j)$-th interaction bisemimodule of this direct sum $\bigoplus\limits_{k,j=1}^{M,N}$ can be handled under the S_L-isomophism similarly as in c).

Lemma 5.1.5 *Let $\theta^3_{R_i}(r_i)\otimes_D\theta^3_{L_j}(r_j)$ be the mixed $3D$ diagonal space bisemisheaf of interaction between the $3D$ space right semisheaf $\theta^3_{R_i}(r_i)$ of a right semiparticle (semifermion or semiphoton) and the $3D$ space left semisheaf $\theta^3_{L_j}(r_j)$ of a left semiparticle (semifermion or semiphoton).*

Then, the mixed orthogonal space biquanta:

$$\widetilde{M}^I_{R_i}(r_i)\otimes_D\widetilde{M}^I_{L_j}(r_j)\in\theta^3_{R_i}(r_i)\otimes_D\theta^3_{L_j}(r_j)$$

are gravitational biquanta of the gravitational field between the i-th right semiparticle and the j-th left semiparticle.

Respectively, the mixed $1D$ time biquanta $\widetilde{M}^I_{R_i}(t_i)\otimes\widetilde{M}^I_{L_j}(t_j)$ will be assumed to generate a scalar gravitational field between the i-th right semifermion and the j-th left semifermion.

Proof. The gravitational biquanta $\widetilde{M}^I_{R_i}(r_i)\otimes_D \widetilde{M}^I_{L_j}(r_j)$ are elements of the gravitational field because they belong to the bisections of the tangent bibundles $T^{(3)}_{R_i;ST,MG,M}\otimes_D T^{(3)}_{L_j;ST,MG,M}$ whose inverse projective bimaps $(p_{R_i}\otimes_D p_{L_j})$ are the $3D$ diagonal momentum bioperators (see definition 4.2.2).

Now, these bioperators $(p_{R_i}\otimes_D p_{L_j})$ can be considered as operators of "mixed" acceleration if the tangent bibundles $T^{(3)}_{L_{R_i};ST,MG,M}\otimes_D T^{(3)}_{L_j;ST,MG,M}$ are defined on the internal bilinear Hilbert spaces $\mathcal{H}^+_a$ (see definition 3.1.7).

Knowing that the intensity of the gravitational field is proportional to an acceleration, we have the thesis. ∎

We are thus led to the following proposition:

Proposition 5.1.6 *A set of bisemileptons interact by means of a gravito-electro- magnetic field.*

A set of bisemiphotons interact by means of a gravito-magnetic field.

A set of bisemileptons and of bisemiphotons interact by means of a gravito-electro-magnetic field.

Proof. This proposition results from the definitions 5.1.2 and 5.1.4 and from the lemma 5.1.5. However, let us remak that:

a) the mixed diagonal interactions generate the gravitational field while the mixed off-diagonal interactions generate the electro-magnetic field.

b) if we work in the context of a bilinear quantum theory, then the bisemiphotons interact between themselves while in the standard linear quantum theory, the interactions are generated by gauge theories (see for example [A-L], [Langa]) as in the $U(1)$ abelian gauge theory, excluding any interaction between photons. ∎

Proposition 5.1.7 *Let us adopt the convention that the structure of a negative electric charge is given by an electric bisemisheaf of type $\theta^1_R(t)^{(e)}\otimes_e \theta^3_L(r)^{(e)}$ and that the structure of a positive electric charge is characterized by an electric bisemisheaf of type $\theta^3_R(r)^{(e)}\otimes_e \theta^1_L(t)^{(e)}$.*

Then, we have that:

a) a set of N electric charges of the same sign of N interacting bisemifermions interact by means of an electric field;

b) a set of N electric charges of opposite sign of N interacting bisemifermions interact by means of a magnetic field and by means of a time scalar gravitational field;

c) a set of N magnetic moments of N interacting bisemifermions interact by means of a magnetic field.

Proof.

a) Let $\theta^1_{R_i}(t)^{(e)}\otimes_e \theta^3_{L_i}(r)^{(e)}$ and $\theta^1_{R_j}(t)^{(e)}\otimes_e \theta^3_{L_j}(r)^{(e)}$ be two bisemisheaves characterizing two electric charges of the same sign (here negative) of two interacting bisemifermions labelled " i " and " j . These two negative electric bisemisheaves interact between themselves by means of the two interaction bisemisheaves: $\theta^1_{R_i}(t)^{(e)}\otimes_e \theta^3_{L_j}(r)^{(e)}$ and $\theta^1_{R_j}(t)^{(e)}\otimes_e \theta^3_{L_i}(r)^{(e)}$ according to definition 5.1.1.

Clearly, these two interaction bisemisheaves are of electric nature composed of electric biquanta which are elements of an electric field. As these two interaction bisemisheaves are of the same nature, they will be of repulsive nature.

b) The two electric bisemisheaves $\theta^1_{R_i}(t)^{(e)} \otimes_e \theta^3_{L_i}(r)^{(e)}$ and $\theta^3_{R_j}(r)^{(e)} \otimes_e \theta^1_{L_j}(t)^{(e)}$ characterizing two electric charges of opposite sign of two interacting bisemifermions " i " and " j " interact by means of the two interaction bisemisheaves:

1. $\theta^3_{R_j}(r)^{(e)} \otimes_e \theta^3_{L_i}(r)^{(e)}$ which is a $3D$ space bisemisheaf, i.e. a magnetic field; we thus have the identity:
$$\theta^3_{R_j}(r)^{(e)} \otimes_e \theta^3_{L_i}(r)^{(e)} \equiv \theta^3_{R_j}(r)^{(m)} \otimes_m \theta^3_{L_i}(r)^{(m)}$$
so that the magnetic bisemisheaf $\theta^3_{R_j}(r)^{(m)} \otimes_m \theta^3_{L_i}(r)^{(m)}$ is composed of magnetic biquanta of a magnetic field;
2. $\theta^1_{R_i}(t)^{(e)} \otimes_e \theta^1_{L_i}(t)^{(e)}$ which is a mixed time bisemisheaf composed of scalar gravitational time biquanta that are of attractive nature.

c) From the definitions 1.1.5 and 5.1.2, it appears that a set of N magnetic moments of N interacting bisemifermions interact by means of a magnetic field. ∎

Definition 5.1.8 (Internal bilinear mixed Hilbert space) According to definition 5.1.1, the total bisemisheaf $(\Theta_{R_N} \otimes \Theta_{L_N})$ of a set of N interacting bisemifermions or bisemiphotons is given by:
$$\Theta_{R_N} \otimes \Theta_{L_N} = \bigoplus_{i=1}^{N} (T_{R_i} \otimes T_{L_i}) \bigoplus_{\substack{i,j=1\\ i\neq j}}^{N} (T_{R_i} \otimes T_{L_j}) \; .$$

Let p_L be a projective linear map, mapping the right semisheaf Θ_{R_N} onto the left semisheaf Θ_{L_N} and let B_L be a bijective linear isometric map from the projected right semisheaf $\Theta_{R_{N(P)}}$ to Θ_{L_N} so that the bisemisheaf $\Theta_{R_N} \otimes \Theta_{L_N}$ be transformed as follows [Pie4]:
$$(B_L \circ p_L) : \Theta_{R_N} \otimes \Theta_{L_N} \to \Theta_{L_{R_N}} \otimes \Theta_{L_N}$$
leading to
$$\Theta_{L_{R_N}} \otimes \Theta_{L_N} = \bigoplus_{i=1}^{N} (T_{L_{R_i}} \otimes T_{L_i}) \bigoplus_{\substack{i,j=1\\ i\neq j}}^{N} (T_{L_{R_i}} \otimes T_{L_j}) \; .$$

Then, the diagonal bisections $\{T_{L_{R_i}} \otimes_D T_{L_i}\}_{i=1}^N$ and $\{T_{L_{R_i}} \otimes_D T_{L_j}\}_{i,j=1}^N$, obtained under the S_L-isomorphism, are defined respectively on the internal bilinear left Hilbert space $\mathcal{H}_i^+$ and on the internal bilinear mixed left Hilbert space $\mathcal{H}^+_{(i-j)}$.

Definition 5.1.9 (What differentiate bisemifermions from bisemiphotons) The bisemisheaves $\{\theta_{L_{R_i}} \otimes \theta_{L_i}\}_{i=1}^N$ of the bisemifermions differ from the corresponding bisections of the bisemiphotons by the existence of the electric bisemisheaves at the origin of the electric charges of the bisemifermions. Similarly, the bisemifermion interaction bisemisheaves $\{\theta_{L_{R_i}} \otimes \theta_{L_j}\}_{\substack{i,j=1\\ i\neq j}}^N$ differ from the corresponding bisemiphoton interaction bisections by the existence of electric bisemisheaves at the origin of an electric field.

This way of handling interactions between bisemiparticles differs from the standard description, in linear quantum theory, of a set of N fermions given by an antisymmetric Fock space over the Hilbert space while a set of N photons is described by a symmetric Fock space over the Hilbert space [Foc], [VN-Mu].

Let us be more explicit by working out the standard Hilbert space at two particles and the present bilinear Hilbert space at two bisemiparticles. For the purpose of simplicity, we shall exclude the magnetic and electric bisemisheaves of the bisemifermions and the magnetic bisemisheaves of the bisemiphotons. We then have:

Proposition 5.1.10 (Bilinear Hilbert space) *The standard Hilbert space at two particles is given by $\mathcal{H}_2 = \mathcal{H}_1 \otimes \mathcal{H}_2$ while the internal bilinear left Hilbert space at two bisemiparticles is:*

$$\mathcal{H}_2^+ = \mathcal{H}_{(1)}^+ \oplus \mathcal{H}_{(2)}^+ \oplus [\mathcal{H}_{(1-2)}^+ \oplus \mathcal{H}_{(2-1)}^+]_{\rm int} \; .$$

$\mathcal{H}_2$ is thus a Hilbert space at two free particles while $\mathcal{H}_2^+$ leads naturally to a bilinear Hilbert space at two interacting bisemiparticles.

Proof. The structure of the Hilbert space $\mathcal{H}_2^+$ at two bisemiparticles results directly from definitions 5.1.1 and 5.1.8.

Let us point out the difference between $\mathcal{H}_2$ and $\mathcal{H}_2^+$ by developing the additional structures of these two Hilbert spaces.

a) If $\{f_\gamma\}_{\gamma=1}^{n_1}$ and $\{f_\delta\}_{\delta=1}^{n_2}$ are the orthonormal basis of $\mathcal{H}_1$ and $\mathcal{H}_2$, then a vector ϕ of $\mathcal{H}_2 = \mathcal{H}_1 \otimes \mathcal{H}_2$ will write:

$$\phi = \sum_{\gamma,\delta=1}^{n_1,n_2} c_{\gamma\delta}(f_\gamma \otimes f_\delta)$$

and the scalar product (ϕ,ϕ) defined on $\mathcal{H}_2 \times \mathcal{H}_2$ will be given by [R-S]:

$$\begin{aligned}(\phi,\phi) &= (\textstyle\sum c_{\gamma\delta}(f_\gamma\otimes f_\delta), \sum c_{\mu\nu}(f_\mu \otimes f_\nu))\\ &= \textstyle\sum \overline{c}_{\gamma\delta} c_{\mu\nu}(f_\gamma, f_\mu)\cdot (f_\delta, f_\nu)\;;\end{aligned}$$

b) on the other hand, a bivector $\phi_{L_{R_2}} \otimes_D \psi_{L_2}$ of

$$\mathcal{H}_2^+ \simeq \theta_{L_{R_2}} \otimes_D \theta_{L_2} = (\theta_{L_{R(1)}} \oplus \theta_{L_{R(2)}}) \otimes_D (\theta_{L(1)} \oplus \theta_{L(2)})$$

(see definition 5.1.1) will write:

$$\begin{aligned}\phi_{L_{R_2}} \otimes_D \psi_{L_2} &= (\phi_{L_{R(1)}} \oplus \phi_{L_{R(2)}}) \otimes_D (\psi_{L(1)} \oplus \psi_{L(2)})\\ &= \left(\sum_\alpha \overline{b}_\alpha (e^\alpha)^* \oplus \sum_\beta \overline{c}_\beta (f^\beta)^*\right) \otimes_D \left(\sum_\gamma b_\gamma(e_\gamma) \oplus \sum_\delta c_\delta(f_\delta)\right)\end{aligned}$$

if $\{(e^\alpha)^*\}_{\alpha=1}^{n_1}$ and $\{(f^\beta)^*\}_{\beta=1}^{n_2}$ are orthonormal basis respectively of the right semimodules $\theta_{L_{R(1)}}$ and $\theta_{L_{R(2)}}$ while $\{e_\gamma\}_{\gamma=1}^{n_1}$ and $\{f_\delta\}_{\delta=1}^{n_2}$ are orthonormal basis respectively of the left semimodules $\theta_{L(1)}$ and $\theta_{L(2)}$.

Then, the internal scalar product on $\theta_{L_{R_2}} \otimes_D \theta_{L_2}$ will be given by:

$$\begin{aligned}(\phi_{L_{R_2}}, \psi_{L_2}) &= \sum_{\alpha=\gamma} \overline{b}_\alpha b_\gamma (e^\alpha, e_\gamma) + \sum_{\beta=\delta} \overline{c}_\beta c_\delta (f^\beta, f_\delta)\\ &\quad + \sum_{\alpha=\delta} \overline{b}_\alpha c_\delta (e^\alpha, f_\delta) + \sum_{\beta=\gamma} \overline{c}_\beta b_\gamma (f_\beta, e_\gamma)\; .\end{aligned}$$

Remark that

$$\begin{aligned}\phi_{L_{R_2}} \otimes_D \psi_{L_2} &= (\phi_{L_{R(1)}} \otimes_D \psi_{L(1)}) \oplus (\phi_{L_{R(2)}} \otimes_D \psi_{L(2)}) \\ &\bigoplus (\phi_{L_{R(1)}} \otimes_D \psi_{L(2)}) \oplus (\phi_{L_{R(2)}} \otimes_D \psi_{L(1)})\end{aligned}$$

where $(\phi_{L_{R(1)}} \otimes_D \psi_{L(2)}) \in \mathcal{H}^+_{(1-2)}$ and $(\phi_{L_{R(2)}} \otimes_D \psi_{L(1)}) \in \mathcal{H}^+_{(2-1)}$.

$[\mathcal{H}^+_{(1-2)} \oplus \mathcal{H}^+_{(2-1)}]_{\text{int}}$ is the direct sum of the interaction bilinear Hilbert spaces between the two bisemiparticles (1) and (2) .

Then we have that

1. if $[\mathcal{H}^+_{(1-2)} \oplus \mathcal{H}^+_{(2-1)}]_{\text{int}} = 0$, the two bisemiparticles do not interact;
2. If $[\mathcal{H}^+_{(1-2)} \oplus \mathcal{H}^+_{(2-1)}]_{\text{int}} \neq 0$, the two bisemiparticles interact by means of a gravitational field (see lemma 5.1.5) generated by:
   - gravitational biquanta from the $3D$ mixed bilinear Hilbert spaces
   $$[\mathcal{H}^{\text{space}}_{(1-2)} \oplus \mathcal{H}^{\text{space}}_{(2-1)}] \subset \mathcal{H}^+_{(1-2)} \oplus \mathcal{H}^+_{(2-1)} \ ;$$
   - time mixed biquanta of a scalar gravitational field from $1D$ mixed bilinear Hilbert spaces
   $$[\mathcal{H}^{\text{time}}_{(1-2)} \oplus \mathcal{H}^{\text{time}}_{(2-1)}] \subset \mathcal{H}^+_{(1-2)} \oplus \mathcal{H}^+_{(2-1)} \ .$$

■

Definition 5.1.11 (Wave equations of interacting bisemiparticles) We shall now introduce the "wave" equation of N interacting bisemiparticles.

Referring to definition 5.1.1, the total bisemisheaf $(\Theta_{R_N} \otimes \Theta_{L_N})$ of a set of N interacting bisemiparticles is given by:

$$\Theta_{R_N} \otimes \Theta_{L_N} = \bigoplus_{i,j=1}^{N} (\theta_{R_i} \otimes \theta_{L_j}) \ .$$

According to definition 3.1.8, this bisemisheaf $(\Theta_{R_N} \otimes \Theta_{L_N})$ will generate the extended internal bilinear Hilbert space H^+_N if we apply to it the $(B_L \circ p_L)$ map transforming it into the bisemisheaf $\Theta_{L_{R_N}} \otimes \Theta_{L_N}$.

If we take into account the section 2 of chapter 4 where the "mass" second order elliptic differential bilinear equation was introduced, we can state that the "mass" biwave equation of N interacting bisemiparticles can be developed following:

$$\sum_{\substack{i,j=1\\ \mu,\nu}}^{N} [(\mathbb{M}_{R_{i\mu}} \otimes \mathbb{M}_{L_{j\nu}}) - (E_{R_{i\mu}} \otimes E_{L_{j\nu}})][\phi_{L_{R_{i\mu}}}(t,r) \otimes \phi_{L_{j\nu}}(t,r)] = 0$$

where

a) $\mathbb{M}_{R_{i\mu}}$ (resp. $\mathbb{M}_{L_{j\nu}}$) is the mass differential right (resp. left) operator, given explicitly in definition 4.2.2, acting on the μ-th right (resp. ν-th left) section $\phi_{L_{R_{i\mu}}}(t,r)$ (resp. $\phi_{L_{j\nu}}(t,r)$) of the right (resp. left) semisheaf $\theta_{L_{R_i}}$ (resp. θ_{L_j}) defined on the bilinear Hilbert space H^+_N ;

b) $(E_{R_{i\mu}} \otimes E_{L_{j\nu}})$ is the corresponding eigenbivalue.

We then have the following proposition:

Proposition 5.1.12 *The biwave equation of N interacting bisemiparticles separates automatically into N_q biwave equations of the N_q bisections of N bisemiparticles and into $((N_q)^2 - N_q)$ biwave equations referring to the interactions between the right and left sections of these N bisemiparticles.*

Proof. The mass biwave equation of N interacting bisemiparticles on the extended internal bilinear Hilbert space H_N^+ :

$$\sum_{\substack{i,j=1\\ \mu,\nu}}^{N} [(\mathbb{M}_{R_{i\mu}} \phi_{L_{R_{i\mu}}}(t,r) \otimes \mathbb{M}_{L_{j\nu}} \phi_{L_{j\nu}}(t,r)) - (E_{R_{i\mu}} \otimes E_{L_{j\nu}})(\phi_{L_{R_{i\mu}}}(t,r) \otimes \phi_{L_{j\nu}}(t,r))] = 0$$

decomposes into:

$$\sum_{i,\mu} [(\mathbb{M}_{R_{i\mu}} \phi_{L_{R_{i\mu}}}(t,r) \otimes \mathbb{M}_{L_{i\mu}} \phi_{L_{i\mu}}(t,r)) - (E_{R_{i\mu}} \otimes E_{L_{i\mu}})(\phi_{L_{R_{i\mu}}}(t,r) \otimes \phi_{L_{i\mu}}(t,r))]$$

$$+ \sum_{\substack{i,j,\mu,\nu\\ i\neq j,\mu\neq\nu}} [(\mathbb{M}_{R_{i\mu}} \phi_{L_{R_{i\mu}}}(t,r) \otimes \mathbb{M}_{L_{j\nu}} \phi_{L_{j\nu}}(t,r)) - (E_{R_{i\mu}} \otimes E_{L_{j\nu}})(\phi_{L_{R_{i\mu}}}(t,r) \otimes \phi_{L_{j\nu}}(t,r))] = 0$$

where

a) $\sum\limits_{i,\mu}$ refers to the sum of N_q biwave equations relative to the N_q bisections of the N free bisemiparticles;

b) $\sum\limits_{i,j,\mu,\nu}$ refers to the sum of $[(N_q)^2 - (N_q)]$ biwave equations relative to the interactions between the right and left sections of the N bisemiparticles. ■

Definition 5.1.13 (Biwave equation between two different sections) The biwave equation relative to a bisection of a free bisemiparticle was already handled in section 2 of chapter 4. We shall now develop the biwave equation between two different right and left sections, i.e. when $\mu \neq \nu$: this corresponds to a term b) of the proof of proposition 5.1.12:

$$[(\mathbb{M}_{R_{i\mu}} \phi_{L_{R_{i\mu}}} \otimes \mathbb{M}_{L_{j\nu}} \phi_{L_{j\nu}}) - (E_{R_{i\mu}} \otimes E_{L_{j\nu}})(\phi_{L_{R_{i\mu}}} \otimes \phi_{L_{j\nu}})] = 0 \ .$$

Proceeding as in definition 4.2.3, this biwave equation becomes:

$$\begin{aligned}
&\Bigg[\sum_{p,q=1}^{3} A_{\mu\nu}^{pq} \frac{\partial^2(\phi_{L_{R_{i\mu}}}(r_{i\mu})\phi_{L_{j\nu}}(r_{j\nu}))}{\partial x_{i\mu_p}\ \partial x_{j\nu_q}} + \sum_{p=1}^{3} A_{\mu\nu}^{p0} \frac{\partial \phi_{L_{R_{i\mu}}}(r_{i\mu})}{\partial x_{i\mu_p}} \cdot \frac{\partial \phi_{L_{j\nu}}(t_{j\nu})}{\partial t_{j\nu}} \\
&\quad + \sum_{q=1}^{3} A_{\mu\nu}^{0q} \frac{\partial \phi_{L_{R_{i\mu}}}(t_{i\mu})}{\partial t_{i\mu}} \cdot \frac{\partial \phi_{L_{j\nu}}(r_{j\nu})}{\partial x_{j\nu_q}} + A_{\mu\nu}^{00} \frac{\partial^2(\phi_{L_{R_{i\mu}}}(t_{i\mu})\phi_{L_{j\nu}}(t_{j\nu}))}{\partial t_{i\mu}\ \partial t_{j\nu}} \Bigg] \\
&\qquad - [(E_{R_{i\mu}} \cdot E_{L_{j\nu}})(\phi_{L_{R_{i\mu}}}(t_{i\mu}, r_{i\mu}) \cdot \phi_{L_{j\nu}}(t_{j\nu}, r_{j\nu}))] \quad = \quad 0
\end{aligned}$$

where

$$A_{\mu\nu}^{pq} = -\frac{\hbar^2}{c^2}\ s_\mu^p\ s_\nu^q \ , \qquad A_{\mu\nu}^{00} = -\hbar^2\ s_\mu^0\ s_\nu^0 \ ,$$

$$r_{i\mu} = \{x_{i\mu_1}, x_{i\mu_2}, x_{i\mu_3}\} \ .$$

It is a second order degenerated elliptic differential bilinear equation which can be solved by separation of variables. This equation allows to find the interaction energy, which is of gravitational, electric or magnetic nature according to proposition 5.1.6.

Remarks 5.1.14 1) In [Pie1], Green's propagators for bisemiparticles, i.e. Green's bipropagators, were evaluated which allowed to develop the S-matrix for bisemiparticles.

It was then demonstrated that the traditional Feynman graphs [B-D], like the electron self-mass and the vacuum polarisation, "open" and split giving rise to new Feynman "bigraphs" in the context of bisemiparticles: the result is that the Feynman graphs, which are divergent in the context of quantum field theory [Schwe], are transformed into corresponding bisemiparticle bigraphs which were proved not to be more divergent.

2) This way of handling the interactions between N bisemiparticles, and more particularly between N bisemielectrons, clears up the problem of the electronic correlation between N electrons (see for example [Löw]).

5.2 The gravito-electro-magnetism

Instead of considering the interactions between a set of N well defined and localized bisemiparticles as done until now, it is possible to envisage the interaction of a given bisemiparticle with an external field representing the global influence of the set of $(N-1)$ remaining bisemiparticles.

Definition 5.2.1 (The tensor of the gravito-bifield) This external field will be given by the generic biconnexion $(A_R(t,r)\otimes A_L(t,r))$ such that $A_{R,L}(t,r)$ be a right (resp. left) connexion, i.e. a right (resp. left) distribution at the considered point $P_{R,L}(t,r)$ (see definition 4.2.8).

$$A_{R,L}(t,r)=\{A^t_{R,L},A^x_{R,L},A^y_{R,L},A^z_{R,L}\}$$

is a four-vectorial distribution whose components $A^m_{R,L}$, $m=x,y,z$, are given for example by [B-D]:

$$A^m_{R,L}(t,r)=\int d^3k_{R,L}A_{R,L}(\vec{k},\vec{S})e^{\pm i\vec{k}\vec{r}}\varepsilon(k_{R,L},\lambda)$$

where $\varepsilon(k_{R,L},\lambda)$ is the polarization unit vector depending on the integer $\lambda=1,2$ referring to the two transverse polarization modes of the semiphotons.

The mass bisemisheaf of a bisemiparticle in an external field on the left extended internal bilinear Hilbert space H^+_a will then be written:

$$[(\mathbb{M}_R+eA_R(t,r))\otimes(\mathbb{M}_L+eA_L(t,r))](\theta_{L_R}\otimes\theta_L)$$

where " e " is the classical charge parameter modulating the connexion $A_{R,L}(t,r)$ in order to have an interaction between two proportional charges throughout an infinitesimal right or left connexion.

The mass bioperator $(\mathbb{M}_R\otimes\mathbb{M}_L)$ of a bisemiparticle endowed with the infinitesimal biconnexion $(eA_R(t,r)\otimes eA_L(t,r))$, noted $(A_R\otimes A_L)$, will develop according to:

$$(\mathbb{M}_R+A_R)\otimes(\mathbb{M}_L+A_L)=(\mathbb{M}_R\otimes\mathbb{M}_L)+(A_R\otimes A_L)+((\mathbb{M}_R\otimes A_L)+(A_R\otimes\mathbb{M}_L))$$

where $((\mathbb{M}_R \otimes A_L) + (A_R \otimes \mathbb{M}_L))$ represents the interaction bioperator between the mass bioperator of a bisemiparticle and the infinitesimal biconnexion giving the global influence of an external bifield. This interaction bioperator is the sum of two tensors of the same type and will be noted $\mathbb{M}A_{mn}$.

The interaction tensor $\mathbb{M}A_{mn}$ is a tensor whose components are:

$$\mathbb{M}A_{mn} = \mathbb{M}_m A_n + A_m \mathbb{M}_n \ ,$$

with $\mathbb{M}_m = \{m_0, p_x, p_y, p_z\}$ and $A_m = \{A_t, A_x, A_y, A_z\}$.

This interaction tensor $\mathbb{M}A_{mn}$, called the gravito-electro-magnetic tensor or GEM tensor, can be explicitly written as follows:

$$\mathbb{M}A_{mn} = \begin{bmatrix} G_t & E_x^- & E_y^- & E_z^- \\ E_x^+ & G_x & B_z^- & B_y^+ \\ E_y^+ & B_z^+ & G_y & B_x^- \\ E_z^+ & B_y^- & B_x^+ & G_z \end{bmatrix}$$

where

a) $\vec{E}^{\pm} = \{E_x^{\pm}, E_y^{\pm}, E_z^{\pm}\}$ is a $3D$-positively (resp. negatively) charged electric field vector;

b) $\vec{B}^{\pm} = \{B_x^{\pm}, B_y^{\pm}, B_z^{\pm}\}$ is a $3D$-positive (resp. negative) magnetic field vector;

c) $G = \{G_x, G_y, G_z\}$ is a $3D$ gravitational field diagonal tensor and G_t is a scalar gravitational field.

Proposition 5.2.2 *The interaction tensor $\mathbb{M}A_{mn}$ is transformed into the antisymmetric tensor F_{mn} of electromagnetism if $\mathbb{M}A_{mn}$ is submitted to the bijective antisymmetric map $C : \mathbb{M}A_{mn} \to F_{mn}$ transforming the right components of A_m into their corresponding left components and the left components of $\mathbb{M}_m$ into their corresponding right components, which corresponds to a map transforming a symplectic metric into an orthogonal metric.*

Proof.

a) The off-diagonal left electric components of the interaction tensor $\mathbb{M}A_{mn}$ are:

$$E_i^- \equiv E_{L_i} = m_0 A_i + A_t p_i \simeq +i\hbar \ \frac{\partial}{\partial t} \ A_i - A_t \cdot i \ \frac{\hbar}{c} \ \frac{\partial}{\partial i} \ , \quad i = x, y, z \ .$$

The " C " map defined as

$$m_0 A_i + A_t p_i \to m_0 A_i + p_i A_t$$

transforms E_{L_i} into $-E_i$:

$$C : E_i^- \equiv E_{L_i} \to -E_i$$

where

$$-E_i = +i \left(\frac{\partial A_i}{\partial t} - \frac{\partial A_t}{\partial i} \right)$$

in the $c = \hbar = 1$ systems of units.

It then appears that $\{-E_i\}_{i=x,y,z}$ are the components similar at a sign of the $3D$ negatively charged left electric field vector:

$$-\vec{E} = +\vec{\nabla} A_t + \frac{\partial \vec{A}}{\partial t}$$

of classical electromagnetism.

Similarly,

$$E_i^+ \equiv E_{R_i} = p_i A_t + A_i m_0 \simeq +i \frac{\hbar}{c} \frac{\partial}{\partial i} \ A_t - i\hbar A_i \ \frac{\partial}{\partial t}$$

can be transformed by the " C " map into:

$$C : E_i^+ \equiv E_{R_i} \to +E_i$$

where

$$E_i = +i \left(\frac{\partial A_t}{\partial i} - \frac{\partial A_i}{\partial t} \right) .$$

Clearly, $\{+E_i\}_{i=x,y,z}$ are very closed at a sign to the components of the $3D$ positively charged right electric field vector

$$\vec{E} = -\vec{\nabla} A_t - \frac{\partial \vec{A}}{\partial t}$$

of electromagnetism.

b) In a similar way, the off-diagonal left magnetic components of the GEM interaction tensor $\mathbb{M}\, A_{mn}$ are:

$$B_k^- \equiv B_{L_k} = p_i A_j + A_i p_j \simeq +i \ \frac{\hbar}{c} \ \frac{\partial}{\partial i} \ A_j - i \ \frac{\hbar}{c} \ A_i \ \frac{\partial}{\partial j} \ , \quad \{i,j,k\} \Leftrightarrow \{x,y,z\} \ .$$

The " C " map transforms $B_k^- \equiv B_{L_k}$ into

$$-B_k = +i \left(\frac{\partial A_j}{\partial i} - \frac{\partial A_i}{\partial j} \right)$$

in the $c = \hbar = 1$ system of units.

$\{-B_k\}_{k=x,y,z}$ are the components of the $3D$ left negative magnetic field vector $-\vec{B} = -\vec{\nabla} \times \vec{A}$ of electromagnetism.

Similarly,

$$B_k^+ \equiv B_{R_k} = p_j A_i + A_j p_i \simeq +i \ \frac{\hbar}{c} \ \frac{\partial}{\partial j} \ A_i - i \ \frac{\hbar}{c} \ A_j \ \frac{\partial}{\partial i} \ .$$

The " C " map transforms $B_k^+ \equiv B_{R_k}$ into

$$+B_k = +i \left(\frac{\partial A_i}{\partial j} - \frac{\partial A_j}{\partial i} \right)$$

where $\{B_k\}_{k=x,y,z}$ are the components of the $3D$ right positive magnetic field vector $+\vec{B} = \vec{\nabla} \times \vec{A}$.

c) Finally, the diagonal components $G_t = m_0 A_t + A_t m_0$ and $G_i = p_i A_i + A_i p_i$, $i = x,y,z$, of the interaction tensor $\mathbb{M}\, A_{mn}$ are the components respectively of a scalar gravitational field G_t and of a $3D$ gravitational field diagonal tensor G because G_t and G are "mixed" $1D$ and $3D$ diagonal bioperators acting respectively on "mixed" $1D$ and $3D$ space orthogonal bisemisheaves which are gravitational bisemisheaves according to lemma 5.1.5.

The " C " map transforms $G_t = m_0 A_t + A_t m_0$ and $G_i = p_i A_i + A_i p_i$ respectively into

$$\underset{\sim t}{G} = +i\hbar \left(\frac{\partial A_t}{\partial t} - \frac{\partial A_t}{\partial t} \right)$$

and into

$$\underset{\sim i}{G} = +i\ \frac{\hbar}{c}\ \left(\frac{\partial A_i}{\partial i} - \frac{\partial A_i}{\partial i} \right)\ ;$$

so, if the right components of A_m anticommute with the left components of $\mathbb{M}_m$, then

$$\underset{\sim t}{G} = \underset{\sim i}{G} = 0\ .$$

This explains why the gravitational field is so hardly observable and why it does not appear in the tensor F_{mn} of electromagnetism as we should expect it.

Summarizing, we have:

$$C: \quad \mathbb{M}\, A_{mn} \rightarrow F_{mn}$$

where

$$\mathbb{M}\, A_{mn} \quad = \quad \begin{bmatrix} G_t & E_x^- & E_y^- & E_z^- \\ E_x^+ & G_x & B_z^- & B_y^+ \\ E_y^+ & B_z^+ & G_y & B_x^- \\ E_z^+ & B_y^- & B_x^+ & G_z \end{bmatrix}$$

and

$$F_{mn} \quad = \quad \begin{bmatrix} 0 & -E_x & -E_y & -E_z \\ +E_x & 0 & -B_z & +B_y \\ +E_y & +B_z & 0 & -B_x \\ +E_z & -B_y & +B_x & 0 \end{bmatrix}\ .$$

■

Proposition 5.2.3 *The "GEM" gravito-electro-magnetic tensor* $\mathbb{M}\, A_{mn}$ *is reduced to the "GM" gravito-magnetic tensor* $\mathbb{M}\, A^p_{ij}$ *,* $i, j = x, y, z$ *, in the case of bisemiphotons, i.e. when a bisemiphoton interacts with an external field.*

Proof. According to proposition 5.1.6, bisemiphotons interact by means of a gravito-magnetic field. Consequently, the tensor $\mathbb{M}\, A_{mn}$ reduces to the tensor

$$\mathbb{M}\, A^p_{ij} = \begin{bmatrix} G_x & B_z^- & B_y^+ \\ B_z^+ & G_y & B_x^- \\ B_y^- & B_x^+ & G_z \end{bmatrix}$$

which is transformed into the tensor

$$F^p_{ij} = \begin{bmatrix} 0 & -B_z & +B_y \\ +B_z & 0 & -B_x \\ -B_y & +B_x & 0 \end{bmatrix}$$

under the action of the " C " map. ∎

Remark 5.2.4 The bisemifermions interact by means of gravitational, electric and magnetic biquanta which could be interpreted as virtual photons [Fey2], [Fey3] like in quantum electrodynamics.

Indeed, the external field is given by the generic biconnexion $A_R(t,r) \otimes A_L(t,r)$ which could be interpreted as a "bi"semiphoton gauge field as in quantum electrodynamics.

Thus, the existence of a local bilinear gauge transformation on a physical field is equivalent to consider a deformation of this field following 1.4.16 and [Pie11].

Proposition 5.2.5 *The condition* $4D$*-nul divergence:* $\partial^n \mathbb{M}\, A_{mn} = 0$ *, i.e.*

$$(1 \otimes \delta_L)[(\mathbb{M}_R \otimes A_L) + (A_R \otimes \mathbb{M}_L)] = 0 \;,$$

applied to the GEM tensor $\mathbb{M}\, A_{mn}$ *leads to a set of formal differential equations:*

$$\begin{cases} \vec{\nabla} \cdot \vec{E} &= \dfrac{\partial G_t}{\partial t} \;, \\[2ex] \vec{\nabla} \times \vec{B} &= \vec{\nabla} \cdot G + \dfrac{d\vec{E}}{dt} \;. \end{cases}$$

analog to the second set of Maxwell equations: $\partial^n F_{mn} = 4\Pi j_m$ *, or*

$$\begin{cases} \vec{\nabla} \cdot \vec{E} &= \rho \;, \\[2ex] \vec{\nabla} \times \vec{B} &= \vec{j} + \dfrac{d\vec{E}}{dt} \;, \end{cases}$$

where $j_m = \{\rho, j_x, j_y, j_z\}$ *.*

Proof. As $A_{R,L}$ is a right (resp. left) connexion in contrast with the vector potential $A_m(r,t) = \{\phi, A_x, A_y, A_z\}$ of electro-magnetism given classically by [F-L-S]:

$$\begin{aligned} \phi(1,t) &= \int \frac{\rho(2, t - r_{12/c})}{4\Pi\varepsilon_0 r_{12}}\, dv_2 \;, \\ \vec{A}(1,t) &= \int \frac{\vec{j}(2, t - r_{12/c})}{4\Pi\varepsilon_0 c^2 r_{12}}\, dv_2 \;, \end{aligned}$$

i.e. defined respectively from the charge density $\rho(2,\cdots)$ and from the current density $j(2,\cdots)$, the $1D$-divergence $\frac{\partial G_t}{dt}$ and the $3D$-"divergence"

$$\vec{\nabla} \cdot G = \frac{\partial G_x}{\partial x} + \frac{\partial G_y}{\partial y} + \frac{\partial G_z}{\partial z}$$

of the gravitational field G_m appears formally in the set of equations:

$$\begin{cases} \vec{\nabla}\cdot\vec{E} & = & \dfrac{\partial G_t}{\partial t}\ , \\[2ex] \vec{\nabla}\times\vec{B} & = & \vec{\nabla}\cdot G + \dfrac{d\vec{E}}{dt}\ . \end{cases}$$

The conditions of $4D$-nul divergence $\partial^n \mathbb{M}\, A_{mn} = 0$ of the tensor $\mathbb{M}\, A_{mn}$ leads to the conditions $(\delta_L, A_L) = (\delta_L, \mathbb{M}_L) = 0$, where δ_L is a $4D$-left divergence and $(\cdot,\cdot)$ is a scalar product. Now, $(\delta_L, A_L) = 0$ corresponds to the radiation gauge or to the Lorentz condition [B-D] of electromagnetism while $(\delta_L, \mathbb{M}_L) = 0$ is a condition of conservation of the left mass of the reference left semiparticle or is a condition of nonaccelerated (i.e. uniform) motion.

This set of differential equations gives the possibility of generating a $1D$ and a $3D$ gravitational field respectively from an electric field and from an electromagnetic field.

The transformation

$$\vec{\nabla}\cdot\vec{E} = \frac{\partial G_t}{\partial t}$$

is realized through a $(\gamma_{r\to t}\circ E)$ morphism applied to $\vec{\nabla}\cdot\vec{E}$. ■

Definition 5.2.6 ($4D$-external current) If the tensor $\mathbb{M}\, A_{mn}$ is no more conserved, i.e. if $(\delta_L, A_L) \neq 0$ and if $(\delta_L, M_L) \neq 0$, then we have:

$$(1\otimes\delta_L)[(\mathbb{M}_R\otimes A_L) + (A_R\otimes\mathbb{M}_L)] = J_R$$

or

$$\partial^n \mathbb{M}\, A_{mn} = J_m$$

where $J_m = \{J_t, J_x, J_y, J_z\}$ is a $4D$-external perturbating right current.

This condition $\partial^n \mathbb{M}\, A_{mn} = J_m$ leads to the set of differential equations:

$$\begin{cases} J_t + \vec{\nabla}\cdot\vec{E} & = & \dfrac{\partial G_t}{\partial t}\ , \\[2ex] \vec{J} + \vec{\nabla}\times\vec{B} & = & \vec{\nabla}\cdot G + \dfrac{\partial\vec{E}}{\partial t}\ , \end{cases}$$

with $\vec{J} = \{J_x, J_y, J_z\}$.

Definition 5.2.7 ($3D$ external current) The condition of $3D$ nul divergence:

$$\nabla^j \mathbb{M}\, A^p_{ij} = 0\ , \qquad i,j = x,y,z\ ,$$

applied to the GM tensor $\mathbb{M}\, A^p_{ij}$ of bisemiphotons leads to the set of differential equations:

$$\vec{\nabla}\times\vec{B} = \vec{\nabla}\cdot G$$

which gives the possibility of generating a gravitational field G from a magnetic field $\vec{B}$.

If the tensor $\mathbb{M}\, A^p_{ij}$ is no more conserved, i.e. if $(\nabla_L, A_L) \neq 0$ and if $(\nabla_L, p_L) \neq 0$ where ∇_L is a $3D$ divergence, then we have:

$$\nabla^j \mathbb{M}\, A^p_{ij} = J^p_i$$

where $\vec{J}^p = \{J_x^p, J_y^p, J_z^p\}$ is a $3D$ external right current.

The condition $\nabla^j \mathbb{M}\, A_{ij}^p = J_i^p$ leads to the set of differential equations

$$\vec{J}^p + \vec{\nabla} \times \vec{B} = \vec{\nabla} G \ .$$

Remark 5.2.8 It is commonly assumed that light waves are electromagnetic waves. However, considering the preceding developments, it appears that isolated light waves, i.e. bisemiphotons, generate only a magnetic field. It is only when light waves interact with bisemifermions that an electromagnetic field of interaction is produced according to proposition 5.1.6.

On the other hand, bisemiphotons could not have a proper mass (i.e. components depending on their proper time) strictly equal to zero following proposition 1.4.8, because, otherwise, the velocity of light would be infinite. But, the proper mass of the bisemiphotons is too tiny to generate an internal electric field.

Proposition 5.2.9 *The gravitational field is of attractive nature while the electromagnetic field is of repulsive and/or of attractive nature.*

Proof. This results from the fact that the gravitational field consists of diagonal biquanta while the electromagnetic field is composed of pairs of off-diagonal magnetic and electric biquanta, generating a positive or a negative field following the sense of rotation of the sections of the magnetic and electric bisemisheaves of the corresponding magnetic and electric fields. ■

Remark 5.2.10 (The gravitational field and the theory of general relativity) Let us finally make two remarks concerning the gravitational field.

1) In this algebraic quantum model (AQT), the gravitation results directly from the diagonal interactions between bisemiparticles. The question is now to find some connexion between the way by which gravitation has been introduced in AQT and the way by which it was described by A. Einstein in general relativity [Ein3], [Ein4].

   The solution is not immediate. Indeed, it appears that there are two fundamental tools in general relativity:

   a) the metric tensor $g_{\mu\nu}$, interpreted as a gravitational potential leads to a description of gravitation in terms of curvature of space-time throughout the Ricci tensor $R_{\mu\nu}$ without really attending to the cause of gravitation;

   b) the equation of Poisson: $\Delta\phi = 4\Pi\kappa\rho$ having been used as a guiding principle for deriving the equations of general relativity:

   $$R_{\mu\nu} - \frac{1}{2}\, g_{\mu\nu} R = -\kappa T_{\mu\nu} \ .$$

   Now it appears that the concept of interaction between (bi)objects at the basis of the generation of the gravitational field could be related to the basic metric tensor $g_{\mu\nu}$. On the other hand, Poisson's equation, describing the "dynamics" of the production of the density of matter ρ from the gravitational field, likely refers in AQT to the transformation of $3D$ graviphotons into $3D$ bisemiphotons.

2) It was demonstrated that every massive elementary left or right semiparticle is constituted by three embedded semisheaves of rings:

$$\theta^{1-3}_{L,R}(t,r)_{ST} \subset \theta^{1-3}_{L,R}(t,r)_{MG} \subset \theta^{1-3}_{L,R}(t,r)_{M}$$

such that the middle-ground and mass semisheaves of rings be generated from the space-time semisheaf of rings. This idea was initially developed in order to get a bridge between quantum field theory and general relativity. Let us recall the pioneer work of L. Parker [Par] in this field. Indeed, Sakharov [Sak] suggested that gravitation could be some quantized phenomenon due to the vacuum energy. The aim pursued in [Pie1] was then to consider that:

   a) the two internal structures of an elementary semiparticle, i.e. the space-time and middle-ground structures, could correspond to its unobservable vacuum from which the mass structure could be generated. The middle-ground structure was then interpreted as being of gravitational nature, which was promptly realized to be incorrect.

   b) the equations of general relativity can be lightly modified [Pie1] by relating nicely the Ricci tensor to the matter stress tensor in order to take into account the creation of matter from gravitational energy.

5.3 The strong interactions

Definition 5.3.1 (The space-time structure of a semibaryon) The strong interactions are widely believed to be generated by a nonabelian $SU(3)$ gauge theory of colored quarks and gluons which are permanently confined in color singlet hadronic bound states: this is quantum chromodynamics [M-P].

This theory is principally justified by the beautiful discovery that nonabelian gauge theories are asymptotically free [G-W], [Pol], [Wein1], but unfortunately, quantum chromodynamics does not give a simple qualitative and dynamical understanding of confinement [C-J-J-T-W].

Now, in this algebraic quantum model, the strong interactions and the nature of the confinement of the semiquarks result directly from the space-time structure of the semibaryons.

Indeed, it was proved in proposition 1.4.3 that the algebraic time structure of a semibaryon is given by:

$$\theta^{\mathrm{Bar}}_{R,L}(t) = \theta^{*1}_{R,L}(t_c) \bigoplus_{i=1}^{3} \theta^{1}_{I_{R,L_i}}(t_i)$$

where $\theta^{*1}_{R,L}(t_c)$ is the core time structure of the semibaryon and where $\theta^{1}_{I_{R,L_i}}(t_i)$ is the time structure of a semiquark.

The algebraic space-time structure of a semibaryon is generated from the semisheaf of rings $\theta^{\mathrm{Bar}}_{R,L}(t)$ by $\gamma_{t_i\to r_i} \circ E_i$ morphisms:

$$\gamma_{t_i\to r_i} \circ E_i : \theta^{\mathrm{Bar}}_{R,L}(t) \to \theta^{\mathrm{Bar}}_{R,L}(t,r) = \theta^{*1}_{R,L}(t_c) \bigoplus_{i=1}^{3} \theta^{1-3}_{R,L_i}(t_i,r_i)$$

according to proposition 1.4.4.

As in QCD (i.e. quantum chromodynamics), the color is related to a quark state [Kok] and corresponds to one of the indices “ i ” of the semiquark semimodule $\theta^{1-3}_{R,L_i}(t_i,r_i)$, $1 \le 1 \le 3$.

$$\text{We would thus have the equivalences:}\quad \begin{aligned} i=1 &\quad\sim\quad \text{red color,}\\ i=2 &\quad\sim\quad \text{blue color,}\\ i=3 &\quad\sim\quad \text{yellow color.}\end{aligned}$$

Recall that the set of parameters

$$G(\rho_B)_{t_c\to[t_1,t_2,t_3]}=\{G_1(\rho_{B_1}),\cdots,G_\mu(\rho_{B_\mu}),\cdots,G_q(\rho_{B_q})\}\ ,$$

noted in abbreviated form $G(\rho_B)$ and defined by (see definition 1.4.5):

$$\phi^*_{t_c;(n_B-\rho_B)_{R,L}}=G(\rho_B)_{t_c\to[t_1,t_2,t_3]}\cdot\phi_{[t_1,t_2,t_3];\rho_{B_{R,L}}}$$

where

a) $\phi^*_{t_c;(n_B-\rho_B)_{R,L}}$ is the set of algebraic Hecke characters related to the generation of the reduced semisheaf $\theta^{*1}_{R,L}(t_c)$ by Eisenstein homology,

b) $\phi_{[t_1,t_2,t_3];\rho_{B_{R,L}}}$ is the set of algebraic Hecke characters related to the generation of the complementary semisheaf

$$\theta^3_{I_{R,L}}(t_1,t_2,t_3)=\bigoplus_{i=1}^{3}\theta^1_{I_{R,L}}(t_i)\ ,$$

leads to the definition of the strong constant of the strong interaction:

$$\langle G(\rho_B)_{t_c\to[t_1,t_2,t_3]}\rangle=\left(\sum_{\mu=1}^{q_B}G_\mu(\rho_{B_\mu})\right)$$

noted $G(\rho)$.

Now, we can state the proposition:

Proposition 5.3.2 *1) The confinement of the 3 semiquarks originates naturally from the generation of the 3 semiquarks from the core time semisheaf of rings of the semibaryon by the smooth endomorphism* E_t *.*

2) The asymptotic freedom of the 3 semiquarks could result from a complete transformation of the core time semisheaf of rings of the semibaryon $\theta^{*1}_{R,L}(t_c)$ *into the complementary time semisheaves of rings of the 3 semiquarks under the conditions that:*

a) $\theta^{*1}_{R,L}(t_c)\simeq 0$ *;*

b) $G(\rho_B)_{t_c\to[t_1,t_2,t_3]}\simeq 0$ *;*

c) $(n_B-\rho_B)_{R,L}\to 0$ *.*

Outline of proof: Asymptotic freedom, which is a consequence of Bjorken scaling [Bjo] at high momentum transfer [D-D-T], occurs if the semiquarks are free, i.e. if $\theta^{*1}_{R,L}(t_c) \simeq 0$.

This is realized when the rank set $\rho_{B_{R,L}}$ of the complementary semisheaf of rings of the 3 semiquarks $\theta^3_{I_{R,L}}(t_1,t_2,t_3)$ is equal to the rank set of the core time semisheaf of rings $\theta^1_{R,L}(t_c)$, i.e. if $(n_B-\rho_B)_{R,L} \to 0$.

And, from the definition of the set of parameters $G(\rho_B)_{t_c\to[t_1,t_2,t_3]}$, it appears that asymptotic freedom is reached when $G(\rho_B)_{t_c\to[t_1,t_2,t_3]} \simeq 0$. ∎

Definition 5.3.3 (Mass-operator of a semibaryon) Referring to the space-time structure of a semibaryon as given in definition 5.3.1, it is immediate that the elliptic self-adjoint differential operator corresponding to the mass operator of a right (resp. left) semibaryon is

$$\begin{aligned}
\mathbb{M}^{\rm Bar}_{R,L} \quad = \quad & \{\mp i\hbar s_0 \, \frac{\partial}{\partial t_{c0}}, \{\pm i\hbar G(\rho) s^R_0 \, \frac{\partial}{\partial t_0}, \pm i \, \frac{\hbar}{c_R} G(\rho) s^R_x \, \frac{\partial}{\partial x}, \cdots, \pm i \, \frac{\hbar}{c_R} G(\rho) s^R_z \, \frac{\partial}{\partial z}\},\\
& \{\pm i\hbar G(\rho) s^B_0 \, \frac{\partial}{\partial t_0}, \cdots, \pm i \, \frac{\hbar}{c_B} G(\rho) s^B_z \, \frac{\partial}{\partial z}\},\\
& \{\pm i\hbar G(\rho) s^Y_0 \, \frac{\partial}{\partial t_0}, \cdots, \pm i \, \frac{\hbar}{c_Y} G(\rho) s^Y_z \, \frac{\partial}{\partial z}\}\}
\end{aligned}$$

where

a) the indices R, B, Y refer to the colors;

b) $\{s^R_i\}_{i=x,y,z}$ are the components of a $3D$ unit vector referring to the spin of the red semiquark;

c) $G(\rho)$ is the strong constant defined in definition 5.3.1;

d) $c_{R,B,Y}$ is the abbreviated notation for the parameter $c_{t\to r}(\rho_{q_{R,B,Y}})_{R,L}$ referring to the generation of the $3D$ spatial semisheaf of rings of the semiquark R , B or Y from its corresponding $1D$ time semisheaf of rings.

Definition 5.3.4 (The space-time structure of a bisemibaryon) According to the axiom II 1.3.9 and definition 3.1.2, we have to consider bisemibaryons whose " ST ", " MG " or " M " structure is given by the bisemisheaves
$\theta^{\rm Bar}_R(t,r)_{ST,MG,M} \otimes \theta^{\rm Bar}_L(t,r)_{ST,MG,M}$ characterized by the tensor products between the right semisheaf $\theta^{\rm Bar}_R(t,r)_{ST,MG,M}$, referring to a right semibaryon, and a left semisheaf $\theta^{\rm Bar}_L(t,r)_{ST,MG,M}$ referring to a left semibaryon. On the "mass" structure, we will have:

$$\begin{aligned}
& \theta^{\rm Bar}_R(t,r)_M \otimes \theta^{\rm Bar}_L(t,r)_M \\
& \quad = \quad (\theta^{*1}_R(t_c) \oplus (\theta^{1-3}_R(t_R,r_R) \oplus \theta^{1-3}_R(t_B,r_B) \oplus \theta^{1-3}_R(t_Y,r_Y)) \\
& \qquad \bigotimes (\theta^{*1}_L(t_c) \oplus (\theta^{1-3}_L(t_R,r_R) \oplus \theta^{1-3}_L(t_B,r_B) \oplus \theta^{1-3}_L(t_Y,r_Y))) \, .
\end{aligned}$$

Under the S_L-isomorphism, this baryonic bisemisheaf splits into the following set of bisemisheaves:

$$\begin{aligned}\theta_R^{\rm Bar}(t,r)_M\otimes\theta_L^{\rm Bar}(t,r)_M \quad\to\quad & (\theta_R^{*1}(t_c)\otimes\theta_R^{*1}(t_c))\\ & \bigoplus_{i=1}^{3}(\theta_{R_i}^{1-3}(t_i,r_i)\otimes\theta_{L_i}^{1-3}(t_i,r_i))\bigoplus_{\substack{i,j=1\\ i\neq j}}^{3}(\theta_{R_i}^{1-3}(t_i,r_i)\otimes\theta_{L_j}^{1-3}(t_j,r_j))\\ & \bigoplus_{i=1}^{3}(\theta_R^{*1}(t_c)\otimes\theta_{L_i}^{1-3}(t_i,r_i))\bigoplus_{i=1}^{3}(\theta_{R_i}^{1-3}(t_i,r_i)\otimes\theta_L^{*1}(t_c))\end{aligned}$$

where

a) the bisemisheaf $(\theta_R^{*1}(t_c)\otimes\theta_L^{*1}(t_c))$ refers to the core central time structure of the bisemibaryon;

b) the bisemisheaf $(\theta_{R_i}^{1-3}(t_i,r_i)\otimes\theta_{L_i}^{1-3}(t_i,r_i))$ refers to the $10D$-space-time structure of the bisemiquark " i " ($i=$ red, blue, yellow).

This bisemisheaf $(\theta_{R_i}^{1-3}(t_i,r_i)\otimes\theta_{L_i}^{1-3}(t_i,r_i))$ splits under the S_L-isomorphism into the direct sum of the three bisemisheaves:

- the $4D$-diagonal space-time bisemisheaf $(\theta_{R_i}^{1-3}(t_i,r_i)\otimes_D\theta_{L_i}^{1-3}(t_i,r_i))$,
- the $3D$ magnetic bisemisheaf $\theta_{R_i}^{3}(r_i)^{(m)}\otimes_m\theta_{L_i}^{3}(r_i)^{(m)}$, and
- the $3D$ electric bisemisheaf $\theta_{R_i}^{1}(t_i)^{(e)}\otimes_e\theta_{L_i}^{3}(r_i)^{(e)}$ or $\theta_{R_i}^{3}(r_i)^{(e)}\otimes_e\theta_{L_i}^{1}(t_i)^{(e)}$.

Remark that the $4D$-space-time diagonal bisemisheaf $\theta_{R_i}^{1-3}(t_i,r_i)\otimes_D\theta_{L_i}^{1-3}(t_i,r_i)$ of the i-th bisemiquark is at the origin of "biquanta" $\widetilde{M}_{R_i}^{I}(r_i)\otimes_D\widetilde{M}_{L_i}^{I}(r_i)$ which are generated from the $3D$ space orthogonal bisemisheaf $\theta_{R_i}^{3}(r_i)\otimes_D\theta_{L_i}^{3}(r_i)$ by a smooth biendomorphism $E_R\otimes_D E_L$.

The electric bisemisheaf of a bisemiquark is at the origin of the electric charge of this bisemiquark whose absolute value is $\left|\frac{1}{3}\right|e$ or $\left|\frac{2}{3}\right|e$. Indeed, following that the electric biquanta of the electric bisemisheaf of a bisemiquark are invariant under an electric subgroup of $SL(1,I\!\!R)\times SL(3,I\!\!R)$ (or of $SL(3,I\!\!R)\times SL(1,I\!\!R)$) (see definition 4.2.19) at one or two bigenerators, the eigenbivalues (in " e " units) of the electric charge of a bisemiquark will be $\left|\frac{1}{3}\right|$ or $\left|\frac{2}{3}\right|$.

c) the "mixed" direct sum $\bigoplus\limits_{\substack{i,j=1\\ i\neq j}}^{3}(\theta_{R_i}^{1-3}(t_i,r_i)\otimes\theta_{L_j}^{1-3}(t_j,r_j))$ refers to the bilinear interactions between the right semisheaves $\theta_{R_i}^{1-3}(t_i,r_i)$ of the right semiquarks and the left semisheaves $\theta_{L_j}^{1-3}(t_j,r_j)$ of the left semiquarks.

The $(i-j)$-th interaction bisemisheaf $(\theta_{R_i}^{1-3}(t_i,r_i)\otimes\theta_{L_j}^{1-3}(t_j,r_j))$ between the i-th right semiquark and the j-th left semiquark splits into

1. a mixed diagonal $4D$- space-time bisemisheaf:

$$(\theta_{R_i}^{1-3}(t_i,r_i)\otimes_D\theta_{L_j}^{1-3}(t_j,r_j))$$

composed of mixed $1D$ time biquanta and mixed $3D$ orthogonal space biquanta which are biquanta of the gravitational field between the i-th and the j-th bisemiquark,

2. a mixed $3D$ magnetic bisemisheaf
$$(\theta^3_{R_i}(r_i)^{(m)} \otimes_m \theta^3_{L_j}(r_j)^{(m)})$$
composed of magnetic biquanta of the magnetic field between the i-th and the j-th bisemiquark,
3. a mixed $3D$ electric bisemisheaf
$$(\theta^1_{R_i}(t_i)^{(e)} \otimes_e \theta^3_{L_j}(r_j)^{(e)}) \quad \text{or} \quad (\theta^3_{R_i}(r_i)^{(e)} \otimes_e \theta^1_{L_j}(t_j)^{(e)})$$
composed of electric biquanta of the electric field between the i-th and the j-th bisemiquark.

d) the mixed direct sum of the bisemisheaves $\bigoplus\limits_{i=1}^{3}(\theta^{*1}_R(t_c) \otimes \theta^{1-3}_{L_i}(t_i,r_i))$ and $\bigoplus\limits_{i=1}^{3}(\theta^{1-3}_{R_i}(t_i,r_i) \otimes \theta^{*1}_L(t_c))$ refer respectively to the bilinear interactions between the right central time semisheaf of rings $\theta^{*1}_R(t_c)$ of the right semibaryon and the left semisheaves $\theta^{1-3}_{L_i}(t_i,r_i)$ of the left semiquarks and to the bilinear interactions between the right semisheaves $\theta^{1-3}_{R_i}(t_i,r_i)$ of the right semiquarks and the left central time semisheaf of rings $\theta^{*1}_L(t_c)$ of the left semibaryon.

The i-th interaction bisemisheaf $(\theta^{*1}_R(t_c) \otimes \theta^{1-3}_{L_i}(t_i,r_i))$ splits into:

1. a mixed $1D$ time bisemisheaf $(\theta^{*1}_R(t_c) \otimes \theta^1_{L_i}(t_i))$ referring to the interaction between the right central time semisheaf of rings of the right semibaryon and the i-th left time semisheaf of rings of the i-th left semiquark.

   This bisemisheaf is composed of mixed $1D$ time biquanta $\widetilde{M}^I_R(t_c) \otimes \widetilde{M}^I_{L_i}(t_i)$ which are of gravitational nature according to lemma 5.1.5.
2. a mixed $3D$ electric bisemisheaf $(\theta^{*1}_R(t_c)^{(e)} \otimes_e \theta^3_{L_i}(r_i)^{(e)})$ composed of electric biquanta $(\widetilde{M}^I_R(t_c)^{(e)} \otimes_e \widetilde{M}^I_{L_i}(r_i)^{(e)})$ which must be of "strong" nature and responsible for a "strong" force between the central core right semisheaf of the right semibaryon and the $3D$ space semisheaf of the i-th left semiquark.

   These strong electric biquanta $(\widetilde{M}^I_R(t_c)^{(e)} \otimes_e \widetilde{M}^I_{L_i}(r_i)^{(e)})$ are likely rather massive.

   To each of the three electric strong bisemisheaves $(\theta^{*1}_R(t_c)^{(e)} \otimes_e \theta^3_{L_i}(r_i)^{(e)})$, we can associate a "blue", "yellow" or "red" color in the sense that the localization of such a bisemisheaf on a bisemiquark gives to it the corresponding color.

We have a similar splitting of the i-th interaction bisemisheaf $(\theta^{1-3}_{R_i}(t_i,r_i) \otimes \theta^{*1}_L(t_c))$.

We are thus led to the following proposition:

Proposition 5.3.5 *A right and a left semibaryon of a given bisemibaryon interact by means of:*

1) the electric charges and the magnetic moments of the 3 bisemiquarks;

2) a gravito-electro-magnetic field resulting from the bilinear interactions between the right and the left semiquarks of different bisemiquarks;

3) a strong gravitational and electric field resulting from the bilinear interactions between the central core structures of the left and right semibaryons and the right and left semiquarks.

Proof. The assertions of this proposition result from the developments of definition 5.3.4∎

Corollary 5.3.6 *In the lightest, stable (nonradioactive) nuclei, the number of protons is equal to the number of neutrons in such a way that the strong positive electric field of the up bisemiquarks equilibrates the strong negative electric field of the down bisemiquarks. As a result, the strong force between up and down bisemiquarks is attractive.*

Proof. According to proposition 3.1.6, the electric basis must be three-dimensional. Consequently, for a given bisemiquark, the electric strong field must refer to the strong electric negative bisemisheaf $(\theta^{*1}_R(t_c)^{(e)}\otimes_e\theta^3_{L_i}(r_i)^{(e)})$ or to the strong electric positive bisemisheaf $(\theta^3_{R_i}(r_i)^{(e)}\otimes_e\theta^{*1}_L(t_c))$.

As the quark composition of the proton is u,u,d and u,d,d for the neutron, we see that the mixed $3D$ strong electric positive bisemisheaves $(\theta^3_{R_i}(r_i)^{(e)}\otimes_e\theta^{*1}_L(t_c))$ of the right up semiquarks will compensate the mixed $3D$ strong electric negative bisemisheaves $(\theta^{*1}_R(t_c)^{(e)}\otimes_e\theta^3_{L_i}(r_i)^{(e)})$ of the left down semiquarks: in other terms, the strong electric field between up and down bisemiquarks will be of attractive nature. ∎

Proposition 5.3.7 *A set of bisemibaryons interact by means of:*

1) *a gravito-electro-magnetic field resulting from the bilinear interactions between the right and the left semiquarks belonging to different bisemibaryons;*
2) *a strong gravitational and electric field resulting from the bilinear interactions between the central core structures of the semibaryons and the semiquarks belonging to different bisemibaryons.*

Proof. According to definition 5.1.1, the general bisemisheaf of a set of N interacting bisemibaryons is given by:

$$\Theta^{\rm Bar}_{R_N}\otimes\Theta^{\rm Bar}_{L_N}=\bigoplus_{i=1}^N(\theta^{\rm Bar}_{R_i}\otimes\theta^{\rm Bar}_{L_i})\bigoplus_{\substack{i,j=1\\ i\neq j}}^N(\theta^{\rm Bar}_{R_i}\otimes\theta^{\rm Bar}_{L_j})\ .$$

The $(i-j)$-th interaction bisemisheaf $(\theta^{\rm Bar}_{R_i}\otimes\theta^{\rm Bar}_{L_j})$ decomposes under the S_L-isomorphism into:

$$\begin{aligned}\theta^{\rm Bar}_{R_i}\otimes\theta^{\rm Bar}_{L_j}\quad\longrightarrow\quad&(\theta^{*1}_{R_i}(t_{c_i})\otimes\theta^{*1}_{L_j}(t_{c_j}))\\ &\bigoplus_{\alpha,\beta=1}^3(\theta^{1-3}_{R_{i\alpha}}(t_{i\alpha},r_{i\alpha})\otimes\theta^{1-3}_{L_{j\beta}}(t_{j\beta},r_{j\beta}))\bigoplus_{\alpha=1}^3(\theta^{*1}_{R_i}(t_{c_i})\otimes\theta^{1-3}_{L_{j\alpha}}(t_{j\alpha},r_{j\alpha}))\\ &\bigoplus_{\alpha=1}^3(\theta^{1-3}_{R_{i\alpha}}(t_{i\alpha},r_{i\alpha})\otimes\theta^{*1}_{L_j}(t_{c_j}))\end{aligned}$$

where

a) the mixed direct sum $\bigoplus\limits_{\alpha,\beta=1}^3(\theta^{1-3}_{R_{i\alpha}}(t_{i\alpha},r_{i\alpha})\otimes\theta^{1-3}_{L_{j\beta}}(t_{j\beta},r_{j\beta}))$ refers to the bilinear interactions between the right semisheaves $\theta^{1-3}_{R_{i\alpha}}(t_{i\alpha},r_{i\alpha})$ of the right semiquarks and the left semisheaves $\theta^{1-3}_{L_{j\beta}}(t_{j\beta},r_{j\beta}))$ of the left semiquarks.

   According to proposition 5.3.5, the $(i\alpha-j\beta)$-th interaction bisemisheaf generates a gravito-electro-magnetic field.

b) the mixed direct sum $\bigoplus\limits_{\alpha=1}^{3}(\theta^{*1}_{R_i}(t_{c_i})\otimes\theta^{1-3}_{L_{j\alpha}}(t_{j\alpha},r_{j\alpha}))$ refers to the bilinear interactions between the central core right semisheaf $\theta^{*1}_{R_i}(t_{c_i})$ of the i-th right semibaryon and the left semisheaves $\theta^{1-3}_{L_{j\alpha}}(t_{j\alpha},r_{j\alpha}))$ of the left semiquarks of the j-th left semibaryon.

According to proposition 5.3.5, this $(i-j\alpha)$-th interaction bisemisheaf generates a strong gravitational and electric field. ∎

Remark 5.3.8 The classical charge parameter " e " is the coupling constant modulating the connexion $A_{R,L}(t,r)$ tied up with a right (resp. left) semilepton as envisaged in definition 5.2.1. It would then be natural to choose the parameter $\frac{e}{3}G(\rho)$ [Pie1] as the coupling constant modulating the connexion $A_{q_{R,L}}(t,r)$ tied up with a right (resp. left) semiquark.

5.4 The decays of bisemiparticles

Definition 5.4.1 (Main decays of bisemiparticles) In definition 2.4.2, it was demonstrated that the second and the third families of elementary right and left semiparticles are generated from the first (resp. the second) family by a $SO(\cdot)\circ Vd(\cdot)$ morphism where:

a) $Vd(\cdot)$ denotes the versal deformation;

b) $SO(\cdot)$ is the spreading-out isomorphism.

As the second and the third families of elementary bisemiparticles are unstable, they decay into lighter bisemiparticles, i.e. finally into bisemiparticles of the first family.

The decays of bisemibaryons are of two types:

1) leptonic decays which are of general form:

$$A\to B+\ell+\nu_1$$

where

a) A and B are bisemibaryons so that the bisemibaryon A has a bisemiquark composition of higher mass than that of the bisemibaryon B ;

b) ℓ is a bisemilepton and ν_1 is a bisemineutrino;

2) nonleptonic decays which are of general form:

$$A\to B+mes$$

where

a) the bisemibaryon A has a bisemiquark composition of higher mass than that of the bisemibaryon B ;

b) mes denotes a meson having a bisemiquark composition q_Rq_L such that the right semiquark q_R has generally a different flavor from the left semiquark q_L .

The decay of bisemileptons are of general type:

$$\ell_a \to \ell_b + \nu_b + \overline{\nu}$$

where the bisemilepton ℓ_a is of higher family than the bisemilepton ℓ_b and where ν_b and $\overline{\nu}$ are bisemineutrinos.

Proposition 5.4.2 *The leptonic decay $A \to B + \ell + \nu_1$ of a bisemibaryon A results from the "diagonal" emission of a bisemilepton " ℓ " by a bisemiquark q_i of A throughout the biendomorphism $(E_R \otimes_D E_L)$ applied to the 10D-space-time bisemisheaf $(\theta^{1-3}_{R_i}(t_i,r_i) \otimes \theta^{1-3}_{L_i}(t_i,r_i))$ of the bisemiquark q_i .*

As a consequence, the bisemiquark q_i is transformed into a bisemiquark q'_i of lighter mass and a bisemineutrino ν_1 is emitted to take into account the bilinear interaction between the bisemiquark q'_i and the bisemilepton ℓ .

Summarizing, we have:

$$q_i \to q'_i + \ell + \nu_1 \ , \qquad q_i \in A \ , \quad q'_i \in B \ .$$

Proof. Let $(\theta^{1-3}_{R_i}(t_i,r_i) \otimes \theta^{1-3}_{L_i}(t_i,r_i))$ be the ST , MG or M bisemisheaf of the i-th bisemiquark q_i belonging to the bisemibaryon A . This bisemiquark q_i is supposed to be of the family B or C (see definition 2.4.2).

Let then $(E_R \otimes_D E_L)$ be the diagonal smooth biendomorphism applied to this bisemisheaf $(\theta^{1-3}_{R_i}(t_i,r_i) \otimes \theta^{1-3}_{L_i}(t_i,r_i))$:

$$\begin{aligned} E_R \otimes_D E_L : & \ \theta^{1-3}_{R_i}(t_i,r_i) \otimes \theta^{1-3}_{L_i}(t_i,r_i) \\ \to \quad & (\theta^{1-3}_{R_{i'}}(t_{i'},r_{i'}) \otimes \theta^{1-3}_{L_{i'}}(t_{i'},r_{i'})) + (\theta^{1-3}_{R_1}(t,r) \otimes \theta^{1-3}_{L_1}(t,r)) \\ & + (\theta^{1-3}_{R_{i'}}(t_{i'},r_{i'}) \otimes \theta^{1-3}_{L_1}(t,r)) + (\theta^{1-3}_{R_1}(t,r) \otimes \theta^{1-3}_{L_{i'}}(t_{i'},r_{i'})) \end{aligned}$$

so that

a) $(\theta^{1-3}_{R_{i'}}(t_{i'},r_{i'}) \otimes \theta^{1-3}_{L_{i'}}(t_{i'},r_{i'}))$ is the bisemisheaf of the i-th bisemiquark having decreased to a lighter family;

b) $(\theta^{1-3}_{R_1}(t,r) \otimes \theta^{1-3}_{L_1}(t,r))$ is the bisemisheaf of the generated bisemilepton ℓ such that the left and right semisheaves $\theta^{1-3}_{R_\ell,L_\ell}(t,r)_{\rho_\ell}$ of the bisemilepton ℓ have ranks ρ_ℓ equal to the difference between the ranks n_i and $(n-\rho)_i$, of the semisheaves $\theta^{1-3}_{R_i,L_i}(t_i,r_i)_{n_i}$ and $\theta^{1-3}_{R_{i'},L_{i'}}(t_{i'},r_{i'})_{(n-\rho)_i}$ of the left and right semiquarks q_{R,L_i} and $q_{R,L_{i'}}$.
We thus have that:
$$\rho_\ell = n_i - (n-\rho)_i \ ;$$

c) the leptonic bisemisheaf $(\theta^{1-3}_{R_\ell}(t,r) \otimes \theta^{1-3}_{L_\ell}(t,r))$ is disconnected from the i'-th bisemiquark bisemisheaf $(\theta^{1-3}_{R_{i'}}(t_{i'},r_{i'}) \otimes \theta^{1-3}_{L_{i'}}(t_{i'},r_{i'}))$;

d) the sum of the two bisemisheaves $(\theta^{1-3}_{R_{i'}}(t_{i'},r_{i'}) \otimes \theta^{1-3}_{L_1}(t,r))$ and $(\theta^{1-3}_{R_1}(t,r) \otimes \theta^{1-3}_{L_{i'}}(t_{i'},r_{i'}))$ is the interaction bisemisheaf between the bisemilepton ℓ and the generated bisemiquark $q_{i'}$ and is allowed to generate a new bisemifermion, under the circumstances a bisemineutrino ν_ℓ . ∎

Proposition 5.4.3 *The nonleptonic decay $A \to B + mes$ of a bisemibaryon A results from the "off-diagonal" emission of a meson " mes " by a bisemiquark q_i of A throughout the nonorthogonal biendomorphism $(E_R \otimes_{m,e} E_L)$ (i.e. a magnetic or electric biendomorphism) applied to the space-time bisemisheaf $(\theta^{1-3}_{R_i}(t_i,r_i)\otimes\theta^{1-3}_{L_i}(t_i,r_i))$ of the bisemiquark q_i .*

As a consequence, the bisemiquark q_i is transformed into a bisemiquark $q_{i'}$ of a lighter mass.

Proof. Let $(\theta^{1-3}_{R_i}(t_i,r_i)\otimes\theta^{1-3}_{L_i}(t_i,r_i))$ be the ST , MG or M bisemisheaf of the i-th bisemiquark q_i of the bisemibaryon A .

Let $(E_R \otimes_m E_L)$ be the "magnetic" smooth biendomorphism applied to the $3D$ space bisemisheaf $\theta^3_{R_i}(r_i)\otimes_D\theta^3_{L_i}(r_i)\subset\theta^{1-3}_{R_i}(t_i,r_i)\otimes\theta^{1-3}_{L_i}(t_i,r_i)$ so that the $3D$ space magnetic biquanta are emitted, i.e. are disconnected from $\theta^3_{R_i}(r_i)\otimes_D\theta^3_{L_i}(r_i)$.

The set of emitted magnetic biquanta then generate the magnetic bisemisheaf $(\theta^3_{R_{mes}}\otimes_m\theta^3_{L_{mes}})$ which can get a "mass" throughout the bimorphism $(\gamma_{r\to t}\circ E)_R\otimes_m(\gamma_{r\to t}\circ E)_L$. We then have that

$$(\gamma_{r\to t}\circ E)_R\otimes_m(\gamma_{r\to t}\circ E)_L : (\theta^3_{R_{mes}}\otimes_m\theta^3_{L_{mes}}) \to (\theta^{1-3}_{R_{mes}}\otimes_{\text{nonorth}}\theta^{1-3}_{L_{mes}})$$

where the nonorthogonal bisemisheaf $(\theta^{1-3}_{R_{mes}}\otimes_{\text{nonorth}}\theta^{1-3}_{L_{mes}})$ corresponds to the generated meson from the i-th bisemiquark q_i . This bisemisheaf $(\theta^{1-3}_{R_{mes}}\otimes_{\text{nonorth}}\theta^{1-3}_{L_{mes}})$ is characterized by a metric $g_{\alpha\beta}$ so that

$$\begin{cases} g_{\alpha\beta}=0 & \text{if } \alpha=\beta \ , \\ g_{\alpha\beta}\neq 0 & \text{if } \alpha\neq\beta \ . \end{cases}$$

Similarly, we can envisage an "electric" smooth biendomorphism $(E_R\otimes_e E_L)$ applied to the bisemisheaf $(\theta^{1-3}_{R_i}(t_i,r_i)\otimes\theta^{1-3}_{L_i}(t_i,r_i))$ of the bisemiquark q_i so that electric biquanta are emitted, i.e. are disconnected from $(\theta^{1-3}_{R_i}(t_i,r_i)\otimes\theta^{1-3}_{L_i}(t_i,r_i))$.

The set of emitted electric biquanta then generate the electric bisemisheaf $(\theta^1_{R_{mes}}\otimes_e\theta^3_{L_{mes}})$ which can get a mass throughout the bimorphism $(\gamma_{t\to r}\circ E)_R\otimes_e(\gamma_{r\to t}\circ E)_L$. We then have that:

$$(\gamma_{t\to r}\circ E)_R\otimes_e(\gamma_{r\to t}\circ E)_L : (\theta^1_{R_{mes}}\otimes_e\theta^3_{L_{mes}}) \to (\theta^{1-3}_{R_{mes}}\otimes_{\text{nonorth}}\theta^{1-3}_{L_{mes}}) \ .$$

Thus, in the case of a "magnetic" or an "electric" smooth biendomorphism, a massive meson $(\theta^{1-3}_{R_{mes}}\otimes_{\text{nonorth}}\theta^{1-3}_{L_{mes}})$ can be generated from the bisemiquark i .
If this massive meson develops a morphism $(SO\circ Vd)$ on its time structure, it will be endowed with an electric charge (see definition 2.4.1). The consequence of the generation of the meson " mes'' " from the bisemiquark q_i of " A " is the transformation of this bisemiquark into a bisemiquark $q_{i'}$ of different flavor and with a mass lighter than this of the bisemiquark q_i . ■

Proposition 5.4.4 *The space-time structure of a meson is given by the nonorthogonal space-time bisemisheaf $ST-MG-M$: $(\theta^{1-3}_{R_{mes}}\otimes_{\text{nonorth}}\theta^{1-3}_{L_{mes}})_{ST-MG-M}$ characterized by a nonorthogonal metric $g_{\alpha\beta}\neq 0$ if $\alpha\neq\beta$.*

Proposition 5.4.5 *The decay $\ell_a\to\ell_b+\nu_b+\overline{\nu}$ of a bisemilepton ℓ_a results from the diagonal emission of a bisemineutrino ν_b throughout the biendomorphism $(E_R\otimes_D E_L)$ applied to the space-time bisemisheaf $(\theta^{1-3}_{R_{\ell_a}}(t,r)\otimes\theta^{1-3}_{L_{\ell_a}}(t,r))$ of the bisemilepton ℓ_a .*

As a consequence, the bisemilepton ℓ_a is transformed into a bisemilepton ℓ_b of lighter family than ℓ_a and a bisemineutrino $\overline{\nu}$ of different helicity from ν_b is emitted to take into account the bilinear interaction between the bisemilepton ℓ_b and the bisemineutrino ν_b .

Proof. Let $(\theta^{1-3}_{R_{\ell_a}}(t,r)\otimes\theta^{1-3}_{L_{\ell_a}}(t,r))$ be the ST , MG or M bisemisheaf of the bisemilepton ℓ_a . Let then $(E_R\otimes_D E_L)$ be the diagonal smooth biendomorphism applied to this bisemisheaf:

$$\begin{aligned}(E_R\otimes_D E_L):\ & \theta^{1-3}_{R_{\ell_a}}(t,r)\otimes\theta^{1-3}_{L_{\ell_a}}(t,r)\\ \to\quad & (\theta^{1-3}_{R_{\ell_b}}(t,r)\otimes\theta^{1-3}_{L_{\ell_b}}(t,r))+(\theta^{1-3}_{R_{\nu_b}}(t,r)\otimes\theta^{1-3}_{L_{\nu_b}}(t,r))\\ & +(\theta^{1-3}_{R_{\ell_b}}(t,r)\otimes\theta^{1-3}_{L_{\nu_b}}(t,r))+(\theta^{1-3}_{R_{\nu_b}}(t,r)\otimes\theta^{1-3}_{L_{\ell_b}}(t,r))\end{aligned}$$

where

a) $(\theta^{1-3}_{R_{\ell_b}}(t,r)\otimes\theta^{1-3}_{L_{\ell_b}}(t,r))$ refers to the bisemisheaf of the bisemilepton ℓ_b resulting from the decay of the bisemilepton ℓ_a ;

b) $(\theta^{1-3}_{R_{\nu_b}}(t,r)\otimes\theta^{1-3}_{L_{\nu_b}}(t,r))$ is the bisemisheaf of the bisemineutrino ν_b emitted by the bisemilepton ℓ_b ;

c) $[(\theta^{1-3}_{R_{\ell_b}}(t,r)\otimes\theta^{1-3}_{L_{\nu_b}}(t,r))+(\theta^{1-3}_{R_{\nu_b}}(t,r)\otimes\theta^{1-3}_{L_{\ell_b}}(t,r))]$ refers to the interaction between the bisemilepton ℓ_b and the bisemineutrino ν_b and is allowed to generate a new bisemineutrino, under the circumstances a bisemineutrino $\overline{\nu}$ of which the left semineutrino $\overline{\nu}_L$ differs by its helicity (which is right) from the helicity (left) of the left semineutrino ν_{b_L} of the bisemineutrino ν_b .

Thus, in the terminology, a bisemineutrino $\overline{\nu}$ whose left semineutrino has right helicity is the "antineutrino" of the bisemineutrino ν whose left semineutrino has left helicity. ∎

5.5 The EPR paradox

Let us recall that the famous EPR paradox raises two kinds of questions [E-P-R], [Bel]:

1) Does the wave function describe the objective reality of an elementary particle?

2) How is it possible that two elementary particles, having interacted in the past, can still interfere in the future, even instantaneously, although the Hilbert space (representing the mathematical frame of quantum mechanics) only deals with tensor products of one-particle Hilbert spaces, excluding interactions between elementary particles?

We shall prove in the next proposition that this new algebraic quantum model gives a response to the EPR paradox and that the two types of questions raised by this paradox are in fact intimely interconnected.

Proposition 5.5.1 *1) The wave function of quantum mechanics, defined on the linear Hilbert space $\mathcal{H}$, is replaced in AQT by a wave "bi"function referring to the state of a bisemiparticle and defined on a bilinear Hilbert space $H^\pm$.*

2) Two elementary bisemiparticles can interact:

a) through the space by means of a gravito-magneto-(electric) field;

b) through the time by means of a $1D$ time gravitational field.

Proof.

1) Point 1) was already developed especially in definition 4.2.1. Thus, the linear wave function of linear quantum mechanics does not describe the objective reality of an elementary (bisemi)particle. Only, the wave "bi"function referring to a state of a bisemiparticle describes the objective reality of a bisemiparticle.

2)
   a) Two bisemiparticles interact in an interval of time dt through the space by means of a gravito-electro-magnetic field according to proposition 5.1.6. These two bisemiparticles can interact "nonlocally" through the internal time only by means of a $1D$ time gravitational field according to lemma 5.1.5. The internal time can then be considered as an hidden variable.
   b) the structure of the bilinear Hilbert space $H^{\pm}$ at two bisemiparticles having interacted in the past makes possible their possible interaction in the future at the condition that a gravito-electro-magnetic field might be generated between these two bisemiparticles. ∎

It can be concluded that the description of the interferences between two bisemiparticles having interacted in the past is only possible by the consideration of biobjects.

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