Black holes in symmetric spaces : anti-de Sitter spaces

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Abstract

Using symmetric space techniques, we show that closed orbits of the Iwasawa subgroups of $SO(2, l-1)$ naturally define singularities of a black hole causal structure in anti-de Sitter spaces in $l \geq 3$ dimensions. In particular, we recover for $l = 3$ the non-rotating massive BTZ black hole. The method presented here is very simple and in principle generalizable to any semi-simple symmetric space.

Contents

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1 Introduction

Causal black holes are distinguished from metric black holes by the fact that they do not exhibit curvature singularities. They are obtained as a quotient of certain spaces under the action of a discrete isometry subgroup. To avoid closed time-like curves in the resulting space, the parts of the original space where the identifications are time-like must be cut out. In this context, the *raison d'être* of the quotient operation is to make the resulting space "causally inextensible".

The most celebrated examples, the BTZ black holes [1, 2], are built from the three dimensional anti-de Sitter space (AdS_3) by identifying points along orbits of particular Killing vectors. They represent axisymmetric and static black hole solutions of (2+1)-dimensional gravity with negative cosmological constant. Furthermore, some of them enjoy remarkable Lie group-theoretical properties, pointed out in [3, 4]. The example which will be of interest for our purposes is the *non-rotating massive* BTZ black hole. In this case, it was observed that the structure of the black hole singularities and horizons are closely linked to *minimal parabolic* (or *Iwasawa*) subgroups of $\widetilde{SL_2(\mathbb{R})} \simeq AdS_3$. We will mainly be concerned in this work to extend these observations to higher-dimensional anti-de Sitter spaces.

Higher-dimensional generalizations of the BTZ construction have been studied in the physics' literature, by classifying the one-parameter isometry subgroups of $\mathrm{Iso}(AdS_l) = SO(2, l-1)$, see [5, 6, 7, 8, 9, 10]. Nevertheless, the approach we will adopt here is conceptually different. We will first reinterpret the non-rotating BTZ black hole solution using symmetric spaces techniques and present an alternative way to express its singularities. The latter will be seen as closed orbits of Iwasawa subgroups of the isometry group. As we will show, this construction extends straightforwardly to higher dimensional cases, allowing to build a non trivial black hole on anti-de Sitter spaces of arbitrary dimension $l \geq 3$. A groupal characterization of the event horizon is also obtained. From this point of view, all anti-de Sitter spaces of dimension $l \geq 3$ appear on an equal footing. For the sake of completeness, we also analyze in some details in appendix B the two-dimensional case, for which the construction does not yield a black hole structure.

A natural question arising from this analysis is the following : *given a semi-simple symmetric space, when does the closed orbits of the Iwasawa subgroups of the isometry group seen as singularities define a non-trivial causal structure?*

We here answer this question in the case of anti-de Sitter spaces, using techniques allowing in principle for generalization to any symmetric semi-simple symmetric space.

This paper is organized as follows. Section 2 is devoted to the presentation of some aspects of the non-rotating BTZ black holes. We state some properties whose proofs are left to appendix A or to the existing literature. In the third section, we present some general elements of the theory of symmetric spaces, applicable to the study of anti-de Sitter spaces. We show how the non-rotating BTZ black holes fit in this context and how the singularities can be expressed in a way suitable for generalization to higher dimensions. In section 4, we show that the proposed definition for the singularities indeed gives rise to a black hole structure by proving the existence of an event horizon, whose characterization is provided using the Iwasawa decomposition of $\mathrm{Iso}(AdS_l)$. We leave the particular two-dimensional case to appendix B, while some explicit computation details are related in appendix C.

2 BTZ black holes and minimal parabolic subgroups

In this section, we recall for the reader's convenience the definition and construction of the non-rotating BTZ black hole [1, 2], emphasizing on some geometrical properties put forwards in [11, 12, 4, 13]. To lighten the presentation, the proofs will essentially be omitted and referred to the existing literature, or recast in appendix A.

This situation will serve us as a guideline in defining black holes in general anti-de Sitter spaces (see section 4).

Bañados, Henneaux, Teitelboim and Zanelli observed that taking the quotient of (a part of) the three-dimensional anti-de Sitter space (AdS_3) under the action of well-chosen discrete subgroups of its isometry group gives rise to solutions which correspond to axially symmetric and static black hole solutions of (2+1)-dimensional Einstein gravity with negative cosmological constant, characterized by their mass M and angular momentum J.

The space AdS_3 is defined as the (universal covering of the) simple Lie group $SL_2(\mathbb{R})$

$$AdS_3 \cong SL_2(\mathbb{R}) = \{g \in GL_2(\mathbb{R}) \,|\, \det g = 1\} := G \tag{1}$$

endowed with its Killing metric $B : \mathcal{G} \times \mathcal{G} \to \mathbb{R}$,

$$B(X,Y) = \mathrm{Tr}(\mathrm{ad}(X) \circ \mathrm{ad}(X))$$

which can be extended to the whole group by

$$B_g(X,Y) = B(dL_{g^{-1}}X, dL_{g^{-1}}Y). \tag{2}$$

Here, $\mathcal{G}$ stands for the Lie algebra of $SL_2(\mathbb{R})$:

$$\begin{aligned} \mathcal{G} &:= \mathfrak{sl}_2(\mathbb{R}) = \{X \in \mathrm{End}(2,\mathbb{R}) \,|\, \mathrm{Tr}(X) = 0\} \\ &= \{z^H H + z^E E + z^F F\}_{\{z^H, z^E, z^F \in \mathbb{R}\}}. \end{aligned} \tag{3}$$

The generators H, E and F satisfy the usual commutation relations

$$[H,E]=2E, \quad [H,F]=-2F, \quad [E,F]=H. \tag{4}$$

We define the following one-parameter subgroups of $SL_2(\mathbb{R})$:

$$A=\exp(\mathbb{R}H), \quad N=\exp(\mathbb{R}E), \quad \bar{N}=\exp(\mathbb{R}F), \quad K=\exp(\mathbb{R}T), \tag{5}$$

with $T=E-F$. They are the building blocks of the Iwasawa decomposition

$$K\times A\times N\longrightarrow SL_2(\mathbb{R}):(k,a,n)\longrightarrow kan \quad \text{or} \quad ank. \tag{6}$$

The $SL_2(\mathbb{R})$ subgroups AN and $A\bar{N}$ are called *Iwasawa subgroups*; they are minimal parabolic subgroups.

We will also use another representation of AdS_3, which can equivalently be seen as the hyperboloid

$$u^2+t^2-x^2-y^2=1 \tag{7}$$

embedded in $\mathbb{R}^{2,2}$, that is the four-dimensional flat space with metric $ds^2=-du^2-dt^2+dx^2+dy^2$.

¿From (7), the isometry group of AdS_3, denoted by Iso(G), is the four-dimensional Lorentz group $O(2,2)$. It is locally isomorphic to $G\times G$, through the action

$$(G\times G)\times G\longrightarrow G:((g_L,g_R),z)\to g_l\, z g_R^{-1}, \tag{8}$$

which corresponds to the identity component of $Iso(G)$ (from the bi-invariance of the Killing metric), and because of the Lie algebra isomorphism

$$\Phi:\mathcal{G}\times\mathcal{G}\to iso(G):(X,Y)\to\overline{X}-\underline{Y}, \tag{9}$$

where $\overline{X}$ (resp. $\underline{Y}$) denotes the right-invariant (resp. left-invariant) vector field on G associated to the element X (resp. Y) of its Lie algebra.

We have now dispose of all necessary ingredients to make the definition of BTZ black holes more precise.

Definition 1. *The one-parameter subgroup of $Iso(G)$ defined by*

$$\psi_t(g)=\exp(t\,a\,H)\,g\exp(-t\,a\,H), \quad a\in\mathbb{R}_0, \quad g\in G \tag{10}$$

*is called the **BHTZ subgroup**. Its generator $\Xi=a(H,H)$ is called the **identification vector**. The **BHTZ action** associated to Ξ is $\psi_{\mathbb{Z}}:G\to G$.*

Definition 2. *A **safe region** in AdS_3 is defined as an open an connected domain*

$$||\Xi||^2:=\beta_z(\Xi,\Xi)>0 \quad . \tag{11}$$

Definition 3. *A **non-rotating massive BTZ black hole** is obtained as the quotient of a safe region in AdS_3 under the BHTZ action.*

This definition deserves some comments. First, the restriction to a safe region in AdS_3 ensures that the resulting quotient space be free of closed time-like curves. This means that the other parts of AdS_3 have to be “cut out” from the original space. Furthermore, due to the identifications, one may restrict to a fundamental domain of the BHTZ action. Secondly, the **black hole singularities** $\mathscr{S}$ are defined as the surfaces where the identification vector becomes light-like :

$$\mathscr{S} = \{z \in AdS_3 \,|\, \beta_z(\Xi, \Xi) = 0\}. \tag{12}$$

Thus, the BTZ black hole singularities represent singularities in the causal structure, not curvature ones. The resulting space is causally inextensible, i.e. trying to extend it would produce closed time-like curves. Finally, the BTZ space-time exhibits all characteristic features of a black hole. Namely, it has *event horizons*, that is, surfaces hiding a region (*the interior region*, see hereafter) causally disconnected from spatial infinity.

Note that it is the choice of identification vector which dictates the nature (rotating, extremal, vacuum or non-rotating massive) of the resulting black hole. Moreover, not all choices give rise to black holes.

The reason why we here focus on the non-rotating massive case lies in the peculiar geometrical properties of its horizons and singularities. To define the horizons properly, we will need the concept of light-rays and light-cones issued from a point.

Definition 4. *A **light-ray** starting from a point g in a safe region is a curve*

$$l_g^k(s) = \exp(-sAd(k)E)g \tag{13}$$

*for a given $k \in K$. The **future** and **past light-cones** at g are given by*

$$C_g^{\pm} = \{l_g^k(s)\}_{\substack{k\in K \\ s\in\mathbb{R}^{\pm}}}. \tag{14}$$

We are now ready to define the horizons.

Definition 5. *A point g will be said to lie in the **future interior region**, denoted by $\mathcal{M}^{int,+}$, if all future-directed light-rays issued from g necessarily fall into the black hole singularity, that is*

$$g \in \mathcal{M}^{int,+} \Leftrightarrow \forall k \in K, \exists s \in \mathbb{R}^+ s.t. \|\underline{H} - \overline{H}\|^2_{l_g^k(s)} = 0. \tag{15}$$

*The **future horizon** $\mathscr{H}^+$ is defined as the boundary of $\mathcal{M}^{int,+}$.*

Equation (15) simply expresses that any future-directed causal signal necessarily falls into the black hole singularity and cannot escape it. The **past interior region** and **past horizon** are defined in a similar way.

Using the embedding (7) of AdS_3 into $\mathbb{R}^{2,2}$, one finds, from (12) and (15), that

$$\mathscr{S} \equiv t^2 - y^2 = 0 \quad \text{and} \quad \mathscr{H} \equiv u^2 - x^2 = 0, \tag{16}$$

where $\mathscr{H} = \mathscr{H}^+ \cup \mathscr{H}^-$.

These results can be stated more intrinsically as follows:

Proposition 6. *In $G = AdS_3$, the non-rotating BTZ black hole singularities are given by a union of minimal parabolic subgroups of G:*

$$\mathscr{S} = Z(G)AN \cup Z(G)A\overline{N}, \tag{17}$$

where $Z(G) = \{e, -e\}$ denotes the center of $G = SL_2(\mathbb{R})$.

Proposition 7. *In $G = AdS_3$, the non-rotating BTZ black hole horizons correspond to a union of lateral classes of minimal parabolic subgroups of G :*

$$\mathscr{H} = Z(G)ANJ \cup Z(G)A\overline{N}J \quad , \tag{18}$$

where $J = \exp(\frac{3\pi}{2}T) \in K$ satisfies $J^2 = e$.

These two propositions actually follows directly from (16), by using the parametrization $g = \begin{pmatrix} u+x & y+t \\ y-t & u-x \end{pmatrix}$. They show that the black hole structure is closely related to the minimal parabolic subgroup of $SL_2(\mathbb{R})$. Of course, this construction cannot be generalized in a straightforward way to higher-dimensional anti-de Sitter spaces, because of the peculiar nature of the three-dimensional case, being the only to enjoy a group manifold structure. Rather, we will reconsider in the next section the case treated here in a more general framework, putting on an equal footing all anti-de Sitter spaces. Again, a minimal parabolic subgroup will reveal crucial in the construction.

3 Symmetric space structure on anti-de Sitter

Most of the material of this section can be found in a general framework in [14, 15, 16, 17].

3.1 Basic facts

As physical space, AdS_l is the set of points $(u, t, x_1, \ldots, x_{l-1}) \in \mathbb{R}^{2,l-1}$ such that $u^2 + t^2 - x_1^2 - \ldots - x_{l-1}^2 = 1$. The transitive (an isometric) action of $SO(2, l-1)$ on AdS_l yields an homogeneous space structure. Let's parameterize the matrix representation of the groups in such a way that $SO(1, l-1)$ –seen as a subgroup of $SO(2, l-1)$– leaves unchanged the vector $(1, 0, \ldots, 0)$. In this case we have an homogeneous space isomorphism

$$AdS_l = \frac{SO(2,n)}{SO(1,n)}$$

with $n = l-1$. The isomorphism is explicitly given by

$$[g] \to g \cdot \begin{pmatrix} 1 \\ 0 \\ \vdots \end{pmatrix} \tag{19}$$

where the dot denotes the action matrix times vector of the representant $g \in [g]$ in the defining representation of $SO(2, n)$. The classes are taken from the right : $[g] = \{gh \,|\, h \in H\}$.

From now we set $G := SO(2, n)$ and $H := SO(1, n)$; the symbols $\mathcal{G}$ and $\mathcal{H}$ denote their respective Lie algebras. We also write $\vartheta = [e]$ and $M = G/H = AdS_l$. We consider a Cartan involution $\theta : \mathcal{G} \to \mathcal{G}$ which gives a Cartan decomposition

$$\mathcal{G} = \mathcal{K} \oplus \mathcal{P},$$

and an involutive automorphism $\sigma = \mathrm{id}\,|_{\mathcal{H}} \oplus (-\,\mathrm{id})|_{\mathcal{Q}}$ which gives a reductive symmetric space decomposition

$$\mathcal{G} = \mathcal{H} \oplus \mathcal{Q}$$

with

$$[\mathcal{H}, \mathcal{H}] \subset \mathcal{H}, \quad [\mathcal{H}, \mathcal{Q}] \subset \mathcal{Q}, \quad [\mathcal{Q}, \mathcal{Q}] \subset \mathcal{H}. \tag{20}$$

One can choose them in such a manner that $[\sigma, \theta] = 0$.

The space $\mathcal{Q}$ can be identified with the tangent space $T_{[e]}M$. We can extend this identification by defining $\mathcal{Q}_g = dL_g\mathcal{Q}$. In this case $d\pi : \mathcal{Q}_g \to T_{[g]}M$ is a vector space isomorphism.

The last point is to find Iwasawa decompositions $\mathcal{H} = \mathcal{A}_{\mathcal{H}} \oplus \mathcal{N}_{\mathcal{H}} \oplus \mathcal{K}_{\mathcal{H}}$ and $\mathcal{G} = \mathcal{A} \oplus \mathcal{N} \oplus \mathcal{K}$ with $\mathcal{A}_{\mathcal{H}} \subset \mathcal{A}$ and $\mathcal{N}_{\mathcal{H}} \subset \mathcal{N}$. We denote by A, N and K the exponentials of $\mathcal{A}$, $\mathcal{N}$ and $\mathcal{K}$; and $\overline{N} = \theta(N)$.

Some explicit matrix choices are given in appendix C. Since the Killing form B is an Ad_H-invariant product on $\mathcal{Q}$, we can define

$$B_g(X, Y) = B_g(dL_{g^{-1}}X, dL_{g^{-1}}Y) \tag{21}$$

which descent to an homogeneous metric on $T_{[g]}M$:

$$B_{[g]}(d\pi X, d\pi Y) = B_g(\mathtt{pr}\,X, \mathtt{pr}\,Y) \tag{22}$$

where $\mathtt{pr} : T_gG \to dL_g\mathcal{Q}$ is the canonical projection. Properties of this product are given in [18].

3.2 Causal structure on anti-de Sitter space

Let us start this section by computing the closed orbits of the action of AN and $A\overline{N}$ on AdS_l. In order to see if $x = [g] \in M$ lies in a closed orbit of AN, we "compare" the basis $\{d\pi dL_g q_i\}$ of T_xM and the space spanned by the fundamental vectors of the action. If these two spaces are the same then x belongs to an open orbit (because a submanifold is open if and only if it has same dimension as the main manifold). This idea is precisely contained in the following proposition.

Proposition 8. *If R is a subgroup of G with Lie algebra $\mathcal{R}$, then the orbit $R\cdot\vartheta$ is open in G/H if and only if the projection $\mathtt{pr} : \mathcal{R} \to \mathcal{Q}$ is surjective.*

In order to check the openness of the R-orbit of $[g]$, we look at the openness of the $\mathbf{Ad}(g^{-1})R$-orbit of ϑ using the proposition.

A great simplification is possible. The AN-orbits are trivially AN-invariant. So the K part of $[g] = ank$ alone fix the orbit in which $[g]$ belongs. In the explicit parametrization of K, we know that the $SO(n)$ part is "killed" by the quotient with respect to $SO(1, n)$. In definitive, we are left with *at most* one AN-orbit for each element in $SO(2)$. Computations using proposition 8 show that the closed orbits are given by

$$\mathscr{S} = \{\pm[AN], \pm[A\overline{N}]\}. \tag{23}$$

We are now in position to make a link with the non-rotating BTZ black hole we discussed in the previous section, through the following

Proposition 9. *The singularities of the non-rotating BTZ black hole, given in* (16)*, coincide with the closed orbits of the action of the subgroups AN and $A\overline{N}$ of $SO(2,2)$ on AdS_3.*

This may be checked by computing the fundamental vector fields of the actions of AN and $A\overline{N}$, then by determining the loci where they span a space of dimension less than 3, and finally observing that this actually corresponds to the equation for $\mathscr{S}$ in (16).

The advantage of this reinterpretation is that it allows, this time, for a straightforward generalization to higher-dimensional anti-de Sitter spaces. Proposition 9 motivates the following

Definition 10. *A point in AdS_l is **singular** if it belongs to a closed orbit of the Iwasawa group AN or $A\overline{N}$.*

This definition finds its origin in the next proposition, which we will mainly concerned with in the next section :

Proposition 11. *In AdS_l, for $l \geq 3$, defining singularities as the closed orbits of the Iwasawa subgroups AN and $A\overline{N}$ of $SO(2,l-1)$ gives rise to a black-hole structure, in the sense that there exists a non empty event horizon.*

Let us make this more precise. As in the three dimensional case, we need to define the notion of light-cone in AdS_l.

General theory about symmetric spaces says that if E is nilpotent in $\mathcal{Q}$, then $\{\mathrm{Ad}(k)E\}_{k\in K_H}$ is the set of all the light-like vectors in $T_{[\vartheta]}AdS_l \simeq \mathcal{Q}$. So the future light cone of ϑ is given by

$$C^+_{[\vartheta]} = \{\pi(e^{-t\,\mathrm{Ad}(k)E})\}_{\substack{t\in\mathbb{R}^+\\ k\in K_H}}$$

and the one of a general element $[g] \in AdS_l$ is obtained by the (isometric) action of g thereon :

$$C^+_{\pi(g)} = \{\pi(ge^{-t\,\mathrm{Ad}(k)E})\}_{\substack{t\in\mathbb{R}^+\\ k\in K_H}}\,. \tag{24}$$

It should be noted that this definition is independent of the choice of the representant g in the class $\pi(g)$ because, for any $h \in H$, $\pi(ghe^{-t\,\mathrm{Ad}(k)E}) = \pi(ghe^{-t\,\mathrm{Ad}(k)E}h^{-1})$ which is simply a reparametrization in K_H.

We are now able to define the causality as follows. A point $[g] \in AdS_l$ belongs to the **interior region** if for all direction $k \in K_H$, the future light ray through $[g]$ intersects the singularity within a *finite* time. In other words, it is interior when the whole light cone ends up in the singularity. A point is **exterior** when it is not interior. A particularly important set of point is the **event horizon**, or simply *horizon*, defined as the boundary of the interior. When a space contains a non trivial causal structure (i.e. when there exists a non empty horizon), we say that the definition of singularities gives rise to a **black hole**.

4 Black hole structure on anti-de Sitter spaces

4.1 General method for computing the singularities

First, let us give an alternative to proposition 8 to study the openness of an AN-orbit. We denote by $\mathscr{S}_{AN}$ the closed orbits of AN and by $\mathscr{S}_{A\overline{N}}$ the ones of $A\overline{N}$. We explain the method for $\mathscr{S}_{AN}$, but the same with trivial adaptations is true for $\mathscr{S}_{A\overline{N}}$.

If $x \in M$ belongs to $\mathscr{S}_{AN}$, the tangent space of his AN-orbit has lower dimension that the tangent space of M. In this case the volume spanned by the fundamental vectors at x is zero. The idea is simple : we build the volume form ν_x of T_xM and we apply it on a basis of the fundamental fields. If the result is zero, then x belongs to the $\mathscr{S}_{AN}$. The action is given by

$$\begin{aligned} \tau \,:\, AN \times M &\to M \\ (an, [g]) &\to [ang]. \end{aligned} \tag{25}$$

If $X \in \mathcal{A} \oplus \mathcal{N}$ and $[g] \in M$, then

$$X^*_{[g]} = -d(\pi \circ R_g)X. \tag{26}$$

As said before, if $\{q_i\}$ is a basis of $\mathcal{Q}$ then a basis of $T_{[g]}M$ is given by $\{d\pi dL_g q_i\}$. We define

$$\nu = q_0^\flat \wedge q_1^\flat \wedge \ldots \wedge q_{l-1}^\flat$$

where $q^\flat_{i[g]} = B_{[g]}(d\pi dL_g q_i, \cdot)$. The condition for $[g]$ to belongs to $\mathscr{S}_{AN}$ reads

$$\nu_{[g]}(N^*_{1\,[g]}, N^*_{2\,[g]}, \ldots, N^*_{l\,[g]}) = 0 \tag{27}$$

for all choices of N_j in a basis of $\mathcal{A} \oplus \mathcal{N}$. It corresponds to the vanishing of $l \times l$ determinants. Our purpose is now to compute the products

$$\begin{aligned} B_{[g]}(d\pi dL_g q_i, N^*_{j\,[h]}) &= -B_g(dL_g q_i, dR_g N_j) \\ &= -B_e(q_i, \mathrm{Ad}(g^{-1})N_j). \end{aligned}$$

We note

$$\Delta_{ij}([g]) = B(q_i, \mathrm{Ad}(g^{-1})N_j)$$

where N_j runs over a basis of $\mathcal{A} \oplus \mathcal{N}$ and q_i a one of $\mathcal{Q}$. Our problem of light cone (see explanations around expression (24)) leads us to compute

$$\Delta_{ij}(\pi(ge^{-tk\cdot E})) = B(\mathrm{Ad}(e^{-tk\cdot E})q_i, \mathrm{Ad}(g^{-1})N_j) \tag{28}$$

where $k \cdot E$ is a notation for $\mathrm{Ad}(k)E$.

A way to proceed is to express all our elements of $SO(2,n)$ in the root space decomposition

$$\mathcal{G} = \mathcal{G}_{(0,0)} \bigoplus_{\lambda \in \Sigma} \mathcal{G}_\lambda.$$

The purpose of this resides in the fact that the Killing form $B(X,Y)$ is most easy to compute when X and Y are in some root spaces. In order to be more synthetic in the text, most of explicit decompositions are given in appendix C.

An important computational remark is the fact that E is nilpotent, so $\mathrm{Ad}(k)E$ also is and $\mathrm{Ad}(e^{-t\,\mathrm{Ad}(k)E})X = e^{-t\,\mathrm{ad}(k)E}X$ only gives second order expressions with respect to t. These computations are nevertheless heavy, but can fortunately be circumvented by a simple counting of dimensions, as we describe in the next subsection.

4.2 AdS_l-adapted method for computing the singularities

We here explicitly use the description of AdS_l in terms of the embedding coordinates $(u, t, x_1, \ldots, x_{l-1}) \in \mathbb{R}^{2,l-1}$ (see subsection 3.1), and the choices of generators related in appendix C .

Proposition 12. *In term of the embedding of AdS_l in $\mathbb{R}^{2,l-1}$, the closed orbits of $AN \subset SO(2, l-1)$ are located at $y - t = 0$. Similarly, the closed orbits of $A\overline{N}$ correspond to $y + t = 0$.*

In other words, the equation

$$t^2 - y^2 = 0 \tag{29}$$

describes the singularity $\mathscr{S} = \mathscr{S}_{AN} \cup \mathscr{S}_{A\overline{N}}$.

Proof. The different fundamental vector fields of the AN actions can be computed by $X^*_{[g]} = -Xg \cdot \vartheta$. For example, in AdS_3,

$$\begin{aligned} M^*_{[g]} &= \begin{pmatrix} 0 & -1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & -1 & 0 & 1 \\ 1 & 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} u \\ t \\ x \\ y \end{pmatrix} = \begin{pmatrix} -t+y \\ u-x \\ -t+y \\ u-x \end{pmatrix} \\ &= (y-t)\partial_u + (u-x)\partial_t + (y-t)\partial_x + (u-x)\partial_y . \end{aligned}$$

Full results are

$$J_1^* = -y\partial_t - t\partial_y \tag{30a}$$
$$J_2^* = -x\partial_u - u\partial_x \tag{30b}$$
$$M^* = (y-t)\partial_u + (u-x)\partial_t + (y-t)\partial_x + (u-x)\partial_y \tag{30c}$$
$$L^* = (y-t)\partial_u + (u+x)\partial_t + (t-y)\partial_x + (u+x)\partial_y \tag{30d}$$
$$W_i^* = -x_i\partial_t - x_i\partial_y + (y-t)\partial_i \tag{30e}$$
$$V_j^* = -x_j\partial_u - x_j\partial_x + (x-u)\partial_j \quad , \quad i, j = 3, \ldots, l-1 \tag{30f}$$

First consider points satisfying $t - y = 0$. It is clear that, at these points, the l vectors J_1^*, M^*, L^* and W_i^* are linearly dependent. Then, there are at most $l-1$ linearly independent vectors amongst the $2(l-1)$ vectors (30), thus the points belong to a closed orbit.

We now show that a point with $t - y \neq 0$ belongs to an open orbit of AN. It is easy to see that J_1^*, L^* and M^* are three linearly independent vectors. The vectors V_i^* gives us $l-3$ more. Then they span a l-dimensional space.

The same can be done with the closed orbits of $A\overline{N}$. The result is that a points belongs to a closed orbit of $A\overline{N}$ if and only if $t + y = 0$. □

Corollary 13. *The singularities coincide with the set of points in AdS_l where $\|J_1^*\|^2 = 0$.*

This generalizes proposition 9 to any dimension. Hence, a discrete quotient of AdS_l along orbits of J_1^* gives a direct higher-dimensional generalization of the non-rotating BTZ black hole.

4.3 Existence of an horizon

Let us show that the definition 10 gives rise to a black hole causal structure, namely that it leads to the existence of horizons, as defined in subsection 3.2.

We first consider points of the form $K \cdot \vartheta$, which are parameterized by an angle μ. Up to choice of this parametrization, a light-like geodesic trough μ is given by

$$K \cdot \mathrm{e}^{-s\,\mathrm{Ad}(k)E} \cdot \vartheta \tag{31}$$

with $k \in SO(l-1)$ and $s \in \mathbb{R}$.

This geodesic reaches $\mathscr{S}_{AN}$ and $\mathscr{S}_{A\overline{N}}$ for values s_{AN} and $s_{A\overline{N}}$ of the affine parameter, given by

$$s_{AN} = \frac{\sin\mu}{\cos\mu - \cos\alpha}, \quad \text{and} \quad s_{A\overline{N}} = \frac{\sin\mu}{\cos\mu + \cos\alpha} \tag{32}$$

where $\cos\alpha$ is the second component of the first column of k, see appendix C and equation (90).

Because the part $\sin\mu = 0$ is $\mathscr{S}_{AN}$, we may restrict ourselves to the open connected domain of AdS_l given by $\sin\mu > 0$. More precisely, $\sin\mu = 0$ is the equation of $\mathscr{S}_{AN}$ is the ANK decomposition. In the same way, $\mathscr{S}_{A\overline{N}}$ is given by $\sin\mu' = 0$ in the $A\overline{N}K$ decomposition. In order to escape the singularity, the point μ needs $s_{AN}, s_{A\overline{N}} < 0$. It is only possible to find directions (i.e. an angle α) which gives it when $\cos u < 0$. So the point $\cos u = 0$ is one point of the horizon.

This proves proposition 11. Remark that the two-dimensional case here appears as degenerate. Therefore, it is treated in appendix B, where we show that *no black hole arises from this construction in* AdS_2.

4.4 A characterization of the horizon

Let $D[g]$ be the set of the light-like directions (vectors in $SO(n)$) for which the point $[g]$ falls into $\mathscr{S}_{AN}$. Similarly, the set $\overline{D}[g]$ is the one of directions which fall into $\mathscr{S}_{A\overline{N}}$. A great result is the fact that it is possible to express $\overline{D}$ in terms of D. Indeed

$$\begin{aligned} k \in \overline{D}[g] &\text{ iff } \pi(ge^{tk\cdot E}) \in \mathscr{S}_{A\overline{N}} \\ &\text{ iff } \pi(\theta(g)\theta(e^{tk\cdot E_1})) \in \mathscr{S}_{AN} \\ &\text{ iff } \theta(k) \in D(\theta[g]) \\ &\text{ iff } k \in (D(\theta[g]))_\theta. \end{aligned} \tag{33}$$

So

$$\overline{D}[g] = (D\theta[g])_\theta \tag{34}$$

where the definition of k_θ is

$$\theta(\mathrm{Ad}(k)E) = \mathrm{Ad}(k_\theta)E.$$

This definition is possible because θ is an inner automorphism.

It is easy to see that θ changes the sign of the spatial part of k, i.e. changes $w_i \to -w_i$.

How to express the condition $g \in \mathscr{H}$ in terms of $D[g]$? The condition to be in the black hole is $D[g] \cup \overline{D}[g] = SO(n)$. If the complementary of $D[g] \cup \overline{D}[g]$ has an interior (i.e. if it contains an open subset), then by continuity the complementary $D[g'] \cup \overline{D}[g']$ has also an interior for all $[g']$ near $[g]$. In this case, $[g]$ cannot belong to the horizon. So a characterization of $\mathscr{H}$ is the fact that the boundary of $D[g]$ and $\overline{D}[g]$ coincide. Equation (34) shows that $\mathscr{H}$ is θ-invariant.

We can explicitly express $D[u]$ for $u \in SO(2)$ by examining equation (32). Let us write w_2 instead of $\cos\alpha$. The set $D[u]$ is the set of $w_2 \in [-1,1]$ such that $\cos u - w_2 > 0$:

$$D[u] = [-1, \cos\mu[. \tag{35}$$

So in order for $[u]$ to belong to $\mathscr{H}$, it must satisfy

$$D[\theta]_\theta = [-1, \cos\mu'[_\theta =] - \cos\mu', 1].$$

Consequently, if u is the K component of g in the ANK decomposition and u' the one of θu, then we can describe the horizon by

$$\cos u = -\cos u' \tag{36}$$

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A Horizons of the non-rotating BTZ black holes

A.1 Global description of the black hole

In this appendix, we use results and techniques of [11, 3, 4, 13] to derive the equation of the non-rotating BTZ black holes horizons. We will begin by stating some results which will be useful in describing the global geometry of the black hole.

Proposition 14. *Let σ be the unique exterior automorphism of G fixing point-wise the Cartan subgroup A and consider the following* twisted *action of G on itself :*

$$\tau : G \times G \longrightarrow G : (g, x) \rightarrow \tau_g(x) := g\, x\, \sigma(g^{-1}). \tag{37}$$

Then, the BHTZ action (see definition 1) can be rewritten as

$$\psi_n = \tau_{\exp(n\sqrt{M}H)}, \quad n \in \mathbb{Z}. \tag{38}$$

The proof follows from the fact that σ fixes the generator H. Using the action τ, one finds the following global decomposition of G :

Proposition 15. *The map*

$$\phi : A \times G/A \longrightarrow G : (a, [g]) \rightarrow \phi(a, [g]) := \tau_g(a) \tag{39}$$

is well-defined as a global diffeomorphism.

This follows from the observation that the application

$$\phi : K \times A \times N \to G : (k, a, n) \to \phi(k, a, n) = \tau_{kn}(a) \tag{40}$$

is a global diffeomorphism on G ("twisted Iwasawa decomposition"). As a consequence, the space G appears as the total space of a trivial fibration over $A = SO(1,1) \simeq \mathbb{R}$ whose fibers are the τ_G-orbits, i.e. the $\sigma-$twisted conjugacy classes. As a homogeneous $G-$space, every fiber is isomorphic to $G/A = AdS_2$. Moreover, the BHTZ action is fiberwise, because

$$\tau_h(\phi(a, [g])) = \phi(a, h.[g]) = \phi(a, [hg]). \tag{41}$$

The Killing metric on G turns out to be globally diagonal with respect to the twisted Iwasawa decomposition [11] :

$$ds^2_G = da^2_A - \frac{1}{4}\cosh^2(a) ds^2_{G/A}, \tag{42}$$

where $ds^2_{G/A}$ denotes the canonical projected AdS_2-metric on G/A. The study of the quotient space $G/\mathbb{Z}$ therefore reduces to the study of $(G/A)/\mathbb{Z}$.

The space G/A can be realized as the $G-$equivariant universal covering space of the adjoint orbit $\mathcal{O} := \mathrm{Ad}(G)H$ in $\mathfrak{sl}_2(\mathbb{R})$, where it corresponds to a one sheet hyperboloid. In this picture, we may identify the part of the hyperboloid corresponding to a safe region (see definition 2) in G.

Lemma 16. *In $\mathcal{O}$, a connected region where the orbits of the BHTZ action are space-like is given by*

$$\{X = x^H H + x^E E + x^F F \in \mathcal{O} \mid -1 < x^H < 1\}. \tag{43}$$

Furthermore, it can be parameterized as

$$X = Ad\left(\exp(\frac{\theta}{2}H)\exp(-\frac{\tau}{2}T)\right) H,\, 0 < \tau < \pi\,,\, -\infty < \theta < +\infty. \tag{44}$$

This has been proven in [11]. From this and the preceding proposition, we find a global description of a safe region in G well adapted to the BHTZ identifications, through the

Proposition 17. *A global description of a safe region in G is given by*

$$z(\rho, \theta, \tau) = \tau_{\exp(\frac{\theta}{2}H)\ \exp(-\frac{\tau}{2}T)}(\exp(\rho H)). \tag{45}$$

Furthermore, the action of the BHTZ subgroup reads in these coordinates

$$(\tau, \rho, \theta) \to (\tau, \rho, \theta + 2na). \tag{46}$$

A.2 Derivation of the horizons

We now have to study the equation of (15). Using the bi-invariance of the Killing metric and the Ad-invariance of the Killing form, it reduces to

$$B(H,H) - B(H, Ad(\mathrm{e}^{-sAd(k)E})Ad(x)H) = 0. \tag{47}$$

Lemma 18. *$\mathcal{M}^{int,+}$ is A bi-invariant.*

Proof. This equation is clearly invariant under $x \to x \cdot a$, $a \in A$. In order to see the invariance under $x \to a \cdot x$, one uses the cyclicity of the trace to bring the second term to

$$B(H, Ad(Ad(a^{-1})\mathrm{e}^{uAd(k)E})Ad(x)H).$$

But $Ad(a^{-1})\mathrm{e}^{-sAd(k)E} = \mathrm{e}^{-\tilde{s}Ad(\tilde{k})E}$, with $\tilde{s} = s(\mathrm{e}^{-2a}\cos^2\theta + \mathrm{e}^{2a}\sin^2\theta)$ and $\cot t = \mathrm{e}^{-2a}\cot\theta$, where $k = \mathrm{e}^{\theta T}$ and $\tilde{k} = \mathrm{e}^{tT}$. The net result is thus simply a relabelling of the parameters (note that s and $\tilde{s}$ have the same signs!) $\square$

Let us now consider a light-ray (definition 4) starting from a safe region in G. Because of (45) and lemma 18, we may restrict our study to

$$\begin{aligned} z &= \mathrm{e}^{-\tau/2T}\mathrm{e}^{\rho H}\sigma(\mathrm{e}^{\tau/2T}) \qquad (48)\\ &= \mathrm{e}^{-\tau/2T}\mathrm{e}^{\rho H}\mathrm{e}^{-\tau/2T}. \qquad (49)\end{aligned}$$

The equation to study reduces to

$$B(H,H) - B(H, Ad(\mathrm{e}^{-sAd(k)E})Ad(\mathrm{e}^{-\tau/2T}\mathrm{e}^{\rho H}\mathrm{e}^{-\tau/2T})H) = 0 \tag{50}$$

with $\tau \in]0,\pi[$ and $\rho \in \mathbb{R}$.

Let us focus on the points in $Ad(G)H$ corresponding to

$$\mathcal{B} := Ad(\mathrm{e}^{-\tau/2T}\mathrm{e}^{\rho H}\mathrm{e}^{-\tau/2T})H,$$

with $\tau \in]0,\pi[$, $\rho \in \mathbb{R}$. First note that $Ad(\mathrm{e}^{\rho H}\mathrm{e}^{-\tau/2H})H$ precisely corresponds to a safe region on the hyperboloid . Thus $\mathcal{B}$ is the region swept out by the a safe region when rotating it counterclockwise around the T-axis with an angle π.

It can be seen that the domain $\mathcal{B}$ can be decomposed into three regions :

$$\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2 \cup \mathcal{B}_3, \tag{51}$$

with

$$\begin{aligned} \mathcal{B}_1 &= Ad(A)Ad(\mathrm{e}^{-\beta/2T})H & \beta \in]0,2\pi[, \qquad (52a)\\ \mathcal{B}_2 &= Ad(A)Ad(\mathrm{e}^{t(E+F)})H & t \in \mathbb{R}, \qquad (52b)\\ \mathcal{B}_3 &= Ad(A)(-H \pm E) \text{ or } Ad(A)(-H \pm F). & \qquad (52c)\end{aligned}$$

Thanks to the A bi-invariance, we may forget about the $Ad(A)$ in the above equations. We are thus led to analyze the existence of solutions of (50) with $X \in \mathcal{B}$ of the form $X_1 = Ad(\mathrm{e}^{-\beta/2T})H$, $X_2 = Ad(\mathrm{e}^{t(E+F)})H$ and $X_3 = -H \pm E$, $-H \pm F$.

Consider the first case. With $Ad(\mathrm{e}^{-\tau/2T}\mathrm{e}^{\rho H}\mathrm{e}^{-\tau/2T})H$ of the form $Ad(\mathrm{e}^{-\beta/2T})H$, (50) becomes the following equation :

$$\frac{1}{4}s^2(\cos\beta-\cos(\beta+4\theta))+s\sin\beta+2\sin^2\beta=0. \tag{53}$$

We are looking for the values of β for which this equation admits a solution for $s>0$, for all $\theta\in\ [0,\pi]$ –this range for θ originates from the fact that G/A is a $\mathbb{Z}_2$ covering of $\mathrm{Ad}(G)H$. By considering the particular case $\theta=0$, we find $s=-\tan\frac{\beta}{2}$, thus the allowed values of β have to lie in the range $]\pi,2\pi[$. Let us look at the constrains imposed by other values of θ. We denote by s_1 and s_2 the two roots of (53). We have

$$s_1\cdot s_2 = \frac{4\sin^2\beta/2}{\sin 2\theta\sin(\beta+2\theta)}, \tag{54}$$

$$s_1+s_2 = \frac{-2\sin\beta}{\sin 2\theta\sin(\beta+2\theta)}. \tag{55}$$

First note that, $\forall\beta\in]0,2\pi[$, $\sin 2\theta\sin(\beta+2\theta)$ may be positive or negative as θ varies in the range $[0,\pi]$. If $\sin\beta<0$, then there are two positive roots when $\sin 2\theta\sin(\beta+2\theta)>0$, and one positive and one negative when $\sin 2\theta\sin(\beta+2\theta)<0$. Thus there always exist a positive solution for u, for any θ. If $\sin\beta>0$, there are two negative roots when $\sin 2\theta\sin(\beta+2\theta)>0$. Consequently, the interior region will correspond to points $X_1=Ad(\mathrm{e}^{-\beta/2T})H\ ,\ \beta\in]\pi,2\pi[$ on the adjoint orbit.

For the second case, $X_2=Ad(\mathrm{e}^{t(E+F)})H$, the equation we get is

$$\frac{1}{4}s^2(\cosh 2t-\cos 4\theta\cosh 2t+2\sin 2\theta\sinh 2t)+s\cos 2\theta\sinh 2t+(1-\cosh 2t)=0. \tag{56}$$

By considering two special cases, it is easy to see that this equation does not admit a positive solution in u for all θ. Indeed, for $\theta=\pi/2$, one finds $s=-\tanh t$, while for $\theta=0$, one gets $s=\tanh t$. Thus there is no $t\neq 0$ satisfying both conditions. The last case yields no positive solution for all θ neither.

As a conclusion we find that the interior region is given by

$$x\in\mathcal{M}^{\mathrm{int},+}\Leftrightarrow Ad(x)H=Ad(A)Ad(\mathrm{e}^{-\beta/2T})H,\quad \text{with } \beta\in]\pi,2\pi[. \tag{57}$$

The boundaries of the corresponding region in $Ad(G)H$ are given by $-H+r^2E$ and $-H+r^2F$ or

$$Ad(N^-)(-H)\cup Ad(\bar{N}^+)(-H), \tag{58}$$

with $N^-=\{\mathrm{e}^{tE}\}_{t\leq 0}$ and $\bar{N}^+=\{\mathrm{e}^{tF}\}_{t\geq 0}$.

The horizons can be deduced as

$$x\in\mathcal{H}^+\Leftrightarrow Ad(x)H=Ad(N^-)(-H)\text{ or }Ad(x)H=Ad(\bar{N}^+)(-H). \tag{59}$$

Because of the A-invariance, we may write $x=\tau_{\mathrm{e}^{-\frac{\tau}{2}T}}(\mathrm{e}^{\rho H})$ and look for the relation between τ and ρ such that

$$Ad(\tau_{\mathrm{e}^{-\tau/2T}}(\mathrm{e}^{\rho H}))H=Ad(N^-)Ad(\mathrm{e}^{\pi/2T})H. \tag{60}$$

This amounts to require that

$$\left(\mathrm{e}^{-\tau/2T}\mathrm{e}^{\rho H}\mathrm{e}^{-\tau/2T}\right)^{-1}(\mathrm{e}^{-t^2E}\mathrm{e}^{\pi/2T})\in A\cup Z(G). \tag{61}$$

This condition gives $\cos\tau=\tanh\rho$, $\rho<0$, $\tau\in]\pi/2,\pi[$. By replacing e^{-t^2E} with e^{t^2F}, one gets $\cos\tau=-\tanh\rho$, $\rho>0$, $\tau\in]\pi/2,\pi[$.

The domain $\mathcal{M}^{\text{int},-}$ is of course defined as

$$x\in\mathcal{M}^{\text{int},-}\Leftrightarrow\forall k\in K,\exists u\in\mathbb{R}^-\text{s.t.}\|\underline{H}-\overline{H}\|^2_{l^k_x(u)}=0. \tag{62}$$

The *past horizon* $\mathcal{H}^-$ is defined as the boundary of $\mathcal{M}^{\text{int},-}$. By proceeding the same way, we find that

$$x\in\mathcal{M}^{\text{int},-}\Leftrightarrow Ad(x)H=Ad(A)Ad(\mathrm{e}^{-\beta/2T})H,\quad\text{with }\beta\in]\pi,2\pi[, \tag{63}$$

and

$$x\in\mathcal{H}^-\Leftrightarrow Ad(x)H=Ad(N^+)(-H)\text{ or }Ad(x)H=Ad(\bar{N}^-)(-H), \tag{64}$$

or in coordinates : $\tau\ \in]0,\pi/2[$, $\cos\tau=\tanh\rho$ for $\rho>0$ and $\cos\tau=-\tanh\rho$ for $\rho<0$.

We thus established the following

Proposition 19. *In a safe region in G parameterized by*

$$z(\rho,\theta,\tau)=\tau_{\exp(\frac{\theta}{2}H)\ \exp(-\frac{\tau}{2}T)}(\exp(\rho H)),$$

the horizons $\mathcal{H}:=\mathcal{H}^+\cup\mathcal{H}^-$ of the non-rotating BTZ black hole are given by

$$\cos\tau=\pm\tanh\rho. \tag{65}$$

As a direct consequence, we have the

Corollary 20. *In terms of the embedding coordinates* (7) *of G in $\mathbb{R}^{2,2}$, the horizons of the non-rotating BTZ black hole are*

$$\mathcal{H}\equiv u^2-x^2=0. \tag{66}$$

B The two dimensional case

B.1 Singularity and physical space

The two dimensional case is very special because it doesn't present a black hole structure. The particular structure directly appears in the groupal formalism[1]. Here $G=SL_2(\mathbb{R})$ and, as homogeneous space, up to a double covering,

$$AdS_2=G/A=\mathrm{Ad}(G)H \tag{67}$$

[1]See section 2 for notations related to $SL_2(\mathbb{R})$.

where $A = e^{\mathbb{R}H}$ is the abelian part of G with respect to the Iwasawa decomposition. In the basis $\{H, E, F\}$ of $SL_2(\mathbb{R})$, the matrix of the Killing form is given by

$$B = \begin{pmatrix} 8 & & \\ & & 4 \\ & 4 & \end{pmatrix} \tag{68}$$

while the basis $\{H, E+F, E-F\}$ gives

$$B = \begin{pmatrix} 8 & & \\ & 8 & \\ & & -8 \end{pmatrix},$$

so that we have the following isometry, $(\mathfrak{sl}_2(\mathbb{R}), B) \sim (\mathbb{R}^3, \eta_{1,2})$. It will be convenient to see AdS_2 as an hyperboloid in $\mathbb{R}^3$. We will use the Cartan involution $\theta(X) = -X^t$.

¿From Definition 10, the singularities are here the closed orbits of AN and $A\overline{N}$ for the adjoint action on $AdS_2 = \mathrm{Ad}(G)H$. A basis of the Lie algebra $\mathcal{A} \oplus \mathcal{N}$ is given by $\{E, H\}$. So x will belong to a closed orbit if and only if $E^*_x \wedge H^*_x = 0$. If we put $x = x_H H + x_E E + x_F F$, the computation is

$$\begin{aligned} E^*_x \wedge H^*_x &= [E, x] \wedge [H, x] \\ &= 4x_H x_F E \wedge F + 2x_E x_F H \wedge E - 2x_F^2 H \wedge F. \end{aligned}$$

It is zero if and only if $x_F = 0$. The closed orbit of $A\overline{N}$ is given by the same computation with $H^*_x \wedge F^*_x$. The part of these orbits contained in AdS_2 is the one with norm 8 :

$$B(x, x) = 8(x_H^2 + x_E x_F).$$

In both cases, it gives $x_H = \pm 1$, and the closed orbits in AdS_2 are given by

$$\pm H + \lambda F \tag{69a}$$

$$\pm H + \lambda E, \tag{69b}$$

Proposition 21. *The singularities can equivalently be defined as*

$$\mathscr{S} = \{x \in \mathrm{Ad}(G)H \mid \|H^*_x\| = 0\} \tag{70}$$

where H^* is the fundamental field associated to the vector H:

$$H^*_x = \frac{d}{dt}\Big[x \cdot e^{-tH}\Big]_{t=0} = \frac{d}{dt}\Big[\mathrm{Ad}(e^{-tH})x\Big]_{t=0} = -[H, x]. \tag{71}$$

Proof. The condition (70) for x to belong to the singularity is

$$B([H, x], [H, x]) = 0. \tag{72}$$

The most general[2] element x in $\mathfrak{sl}_2(\mathbb{R})$ is $x = aH + bE + cF$. It is easy to see that $[x, H] = -2bE + 2cF$, so that the condition (72) becomes $bc = 0$. Then the

[2] It is actually *more* than the most general element to be considered because our space is $\mathrm{Ad}(G)H$, and not the whole $\mathfrak{sl}_2(\mathbb{R})$.

two possibilities are $x = aH + bE$ and $x = aH + cF$. The singularities in $\mathfrak{sl}_2(\mathbb{R})$ are the planes (H, F) and (H, E). The intersection between the plane (H, F) and the hyperboloid is given by the equation

$$B(aH + bF, aH + bF) = 8$$

whose solutions are $a = \pm 1$. The same is also true for the plane (H, E). So we find back the fact that the singularities are given by the four lines

$$\pm H + \lambda E \text{ and } \pm H + \lambda F. \tag{73}$$

□

Another way to express the singularities is

$$\mathrm{Ad}(e^{nE})(\pm H) \text{ and } \mathrm{Ad}(e^{fF})(\pm H), \tag{74}$$

which clearly shows that these are orbits of AN and $A\overline{N}$. Indeed, as $\mathrm{Ad}(a)$ fixes H, we can write $\mathrm{Ad}(an)H = \mathrm{Ad}(ana^{-1})H$. Using the CBH formula we find

$$ana^{-1} = e^{nE + 2anE + \ldots} = e^{ne^{2a}E} = n' \in N.$$

The same can be done with f. So $\mathrm{Ad}(an)H = \mathrm{Ad}(n')H$ and $\mathrm{Ad}(af)H = \mathrm{Ad}(f')H$. This shows that for all $n \in N$ and $a \in A$, there exists a $n' \in N$ such that

$$\mathrm{Ad}(an)H = \mathrm{Ad}(n')H \tag{75a}$$

The same is true with f :

$$\mathrm{Ad}(af)H = \mathrm{Ad}(f')H. \tag{75b}$$

In the basis E, F, H the singularities are four lines with angle=45° trough H and $-H$. They divide the space AdS_2 into four pieces. We define the **physical space** as the part of AdS_2 contained between $H + \lambda E$ and $-H + \lambda E$. The K part of $SL_2(\mathbb{R})$ gives a double covering of this curve. The part contained between the singularities $H + \lambda F$ and $-H + \lambda F$ should be another choice of physical space.

The following proposition gives an useful characterization of the physical space.

Proposition 22. *Any point in the physical space can be written as* $\mathrm{Ad}(ak)H$, *with* $k \in]0, \pi/2[$.

Proof. The physical space contains the curve $\cos\beta H + \sin\beta(E + F)$ with $\beta \in]0, \pi[$, which is exactly $\mathrm{Ad}(k)H$ for $k \in]0, \pi/2[$. It is also the intersection of AdS_2 and the part of $\mathfrak{sl}_2(\mathbb{R})$ between the planes (E, H) and (F, H). If we use the coordinates x, y, z on $\mathfrak{sl}_2(\mathbb{R})$ (i.e. $\overline{r} = xH + yE + zF$), our physical space is given by the inequations

$$\begin{cases} x^2 + yz = 1 \\ y > 0 \\ z > 0. \end{cases}$$

The first equation gives a β such that $x = \cos\beta$, $yz = \sin^2\beta$. It is always possible to define a $a \in \mathbb{R}$ such that $y = e^{2a}\sin\beta$ and $z = e^{-2a}\sin\beta$. Finally, the physical space is parameterized by

$$\overline{r} = \cos\beta H + \sin\beta(e^{2a}E + e^{-2a}F). \tag{76}$$

On the other hand, from commutation relations in $\mathfrak{sl}_2(\mathbb{R})$, one finds

$$\mathrm{Ad}(e^{aH})E = e^{2a}E, \tag{77a}$$
$$\mathrm{Ad}(e^{aH})F = e^{-2a}F. \tag{77b}$$
$$\tag{77c}$$

Then

$$\begin{aligned} \mathrm{Ad}(ak)H &= \mathrm{Ad}(e^{aH})(\cos\beta H + \sin\beta(E+F)) \\ &= \cos\beta H + \sin\beta(e^{2a}E + e^{-2a}F). \end{aligned} \tag{78}$$

□

B.2 Light cone

The light-like vectors of $\mathfrak{sl}_2(\mathbb{R})$ are E and F, so at $\mathrm{Ad}(g)H$, the light-cone consists in two parts :

$$\mathrm{Ad}(g)\,\mathrm{Ad}(e^{tE})H \text{ and } \mathrm{Ad}(g)\,\mathrm{Ad}(e^{tF})H.$$

It is best rewritten in the compact form

$$C^+_{\mathrm{Ad}(g)H} = \{\mathrm{Ad}(g)\,\mathrm{Ad}(e^{t\epsilon E})H\}_{\substack{t>0 \\ \epsilon=\mathrm{id},\theta}} \tag{79}$$

where ϵ is the identity or the Cartan involution.

It is somewhat easy to remark that for all X, Y in a Lie algebra and all automorphism φ, the formula $\varphi(\mathrm{Ad}(e^X)Y) = \mathrm{Ad}(e^{\varphi X})(\varphi Y)$ holds. Then

$$\mathrm{Ad}(e^{t\epsilon E})H = s(\epsilon)\epsilon(\mathrm{Ad}(e^{tE})H) \tag{80}$$

with

$$s(\epsilon) = \begin{cases} 1 & \text{if } \epsilon = \mathrm{id}, \\ -1 & \text{if } \epsilon = \theta. \end{cases}$$

Since $H^*_x = -[H, x]$, the intersection of the light-cone with the singularity is expressed, using Proposition 21, as

$$\|[H, \mathrm{Ad}(g)\,\mathrm{Ad}(e^{t\epsilon E})H]\|^2 = 0. \tag{81}$$

B.3 No black hole

The light cone of the point $\mathrm{Ad}(ak)H$ –which is a general point of the physical space– is given by $\mathrm{Ad}(ak)s(\epsilon)\epsilon(\mathrm{Ad}(e^{tE})H)$. The computation of $\mathrm{Ad}(ak)(H - 2tE)$ and $-\mathrm{Ad}(ak)(-H + 2tF)$ gives

$$(\cos(2k) - t\sin(2k))H - e^{2a}(\sin(2k) + 2t\cos^2 k)E - e^{-2a}(\sin(2k) - 2t\sin^2 k)F \tag{82a}$$

and

$$(\cos(2k) - t\sin(2k))H - e^{2a}(\sin(2k) - 2t\sin^2 k)E - e^{-2a}(\sin(2k) + 2t\cos^2 k)F \tag{82b}$$

With respect to t, these are two straight lines, so they are the intersection of AdS_2 and the tangent plane to AdS_2 at $\mathrm{Ad}(ak)H$.

This is important because it allows us immediately to infer the non-existence of a black-hole structure for this choice of singularity. The light cone at $x \in AdS_2$ is given by the tangent plane C of AdS_2 at x. The part of the singularity passing by H is given by a vertical plane S. The intersection of these two planes is a line, and the intersection of a line with AdS_2 is two points. Then each of the two lines of $C \cap AdS_2$ intersect one of the two lines of $S \cap AdS_2$. The same is true for the other part of the singularity.

The conclusion is that both two lines of the light cone intersect the singularity passing by H *and* the one passing by $-H$. So any point comes from the singularity and returns to the singularity; no point is connected to the infinity.

C Explicit matrix choices

The first choice is to parameterize $SO(2,n)$ and $SO(1,n)$ in such a way the latter leaves unchanged the vector $(1, 0, 0, \ldots)$. Then

$$\mathcal{H} = \mathfrak{so}(1,n) \rightsquigarrow \begin{pmatrix} \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \begin{pmatrix} \cdots 0 \cdots \\ \leftarrow v^t \rightarrow \end{pmatrix} \\ \begin{pmatrix} \vdots & \uparrow \\ 0 & v \\ \vdots & \downarrow \end{pmatrix} & B \end{pmatrix}. \tag{83}$$

where v is $n \times 1$ and B is skew symmetric $n \times n$. When we speak about $\mathfrak{so}(n)$, we usually refer to B. A complementary space $\mathcal{Q}$ such that $[\mathcal{H}, \mathcal{Q}] \subset \mathcal{Q}$ is given by

$$\mathcal{Q} \rightsquigarrow \begin{pmatrix} \begin{matrix} 0 & a \\ -a & 0 \end{matrix} & \begin{pmatrix} \leftarrow w^t \rightarrow \\ \cdots 0 \cdots \end{pmatrix} \\ \begin{pmatrix} \uparrow & \vdots \\ w & 0 \\ \downarrow & \vdots \end{pmatrix} & 0 \end{pmatrix} \tag{84}$$

We consider the involutive automorphism $\sigma = \mathrm{id}_{\mathcal{H}} \oplus (-\,\mathrm{id})_{\mathcal{Q}}$ and the corresponding symmetric space structure on $\mathcal{G}$. As basis of $\mathcal{Q}$, we choice q_0 as the 2×2 antisymmetric upper-left square and as q_i, the one obtained with w full of zero apart a 1 on the ith component. Next we choice the Cartan involution $\theta(X) = -X^t$ which gives rise to a Cartan decomposition

$$\mathcal{G} = \mathcal{K} \oplus \mathcal{P}.$$

The latter choice is made in such a way that $[\sigma, \theta] = 0$. It can be computed, but it is not astonishing that the compact part $\mathcal{K}$ is made of "true" rotations while $\mathcal{P}$ contains the boost. So

$$\mathcal{K} = \begin{pmatrix} \mathfrak{so}(2) & \\ & \mathfrak{so}(n) \end{pmatrix}.$$

In order to build an Iwasawa decomposition, one has to choose a maximal abelian subalgebra $\mathcal{A}$ of $\mathcal{P}$. Since rotations are in $\mathcal{K}$, they must be boosts and the fact that there are only two time-like directions restricts $\mathcal{A}$ to a two dimensional algebra. Up to reparametrization, it is thus generated by $t\partial_x + x\partial_t$ and $u\partial_y + y\partial_t$. Our matrix choices are

$$J_1 = \begin{pmatrix} & 0 & & \\ 0 & 0 & 0 & 1 \\ & 0 & & \\ & 1 & & \end{pmatrix} \in \mathcal{H}, \text{ and } J_2 = q_1 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & & & \\ 1 & & & \\ 0 & & & \end{pmatrix} \in \mathcal{Q}.$$

From here, we have to build root spaces. There still remains a lot of arbitrary choices –among them, the positivity notion on the dual space $\mathcal{A}^*$. An elements X in $\mathcal{G}_{(a,b)}$ fulfill $\mathrm{ad}(X)J_1 = aJ_1$ and $\mathrm{ad}(X)J_2 = bJ_2$. The symbol E_{ij} denote the matrix full of zeros with a 1 on the component ij. Results are

$$\mathcal{G}_{(0,0)} \rightsquigarrow \begin{pmatrix} & & x & 0 & \\ & & 0 & y & \\ x & 0 & & & \\ 0 & y & & & \\ & & & & D \end{pmatrix}, \tag{85}$$

where $D \in M_{(n-2)\times(n-2)}$ is skew-symmetric,

$$\mathcal{G}_{(1,0)} \rightsquigarrow W_i = E_{2i} + E_{4i} + E_{i2} - E_{i4}, \tag{86a}$$
$$\mathcal{G}_{(-1,0)} \rightsquigarrow Y_i = -E_{2i} + E_{4i} - E_{i2} - E_{i4}, \tag{86b}$$
$$\mathcal{G}_{(0,1)} \rightsquigarrow V_i = E_{1i} + E_{3i} + E_{i1} - E_{i3}, \tag{86c}$$
$$\mathcal{G}_{(0,-1)} \rightsquigarrow X_i = -E_{1i} + E_{3i} - E_{i1} - E_{i3} \tag{86d}$$

with $i\,:\,5 \to n+2$ and

$$\mathcal{G}_{(1,1)} \rightsquigarrow M = \begin{pmatrix} 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \end{pmatrix}, \quad \mathcal{G}_{(1,-1)} \rightsquigarrow L = \begin{pmatrix} 0 & 1 & 0 & -1 \\ -1 & 0 & -1 & 0 \\ 0 & -1 & 0 & 1 \\ -1 & 0 & -1 & 0 \end{pmatrix}, \tag{87}$$

$$\mathcal{G}_{(-1,1)} \rightsquigarrow N = \begin{pmatrix} 0 & 1 & 0 & 1 \\ -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \end{pmatrix}, \quad \mathcal{G}_{(-1,-1)} \rightsquigarrow F = \begin{pmatrix} 0 & 1 & 0 & 1 \\ -1 & 0 & -1 & 0 \\ 0 & -1 & 0 & -1 \\ 1 & 0 & 1 & 0 \end{pmatrix}. \tag{88}$$

The choice of positivity is

$$\mathcal{N} = \{V_i, W_j, M, L\}. \tag{89}$$

The following result is important in the computation of the light cones : if $k \in SO(n)$, then the choice $E = q_0 + q_2$ of nilpotent element in $\mathcal{Q}$ gives

$$\mathrm{Ad}(k)E = \begin{pmatrix} 0 & 1 & w_1 & w_2 & \ldots \\ -1 & & & & \\ w_1 & & & & \\ w_2 & & & & \\ \vdots & & & & \end{pmatrix} \tag{90}$$

where the vector w is the first column of k, whose components satisfy $\sum_{i=1}^{l-1} w_i^2 = 1$.

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