Noncommutative Locally Anti-de Sitter Black Holes

P. Bieliavsky

Département de Mathématique
Université Catholique de Louvain
Chemin du cyclotron, 2,
1348 Louvain-La-Neuve, Belgium
E-mail: bieliavsky@math.ucl.ac.be

S. Detournay [*] and **Ph. Spindel**

Mécanique et Gravitation
Université de Mons-Hainaut, 20 Place du Parc
7000 Mons, Belgium
E-mail: stephane.detournay@umh.ac.be,spindel@umh.ac.be

M. Rooman [†]

Service de Physique théorique
Université Libre de Bruxelles, Campus Plaine, C.P.225
Boulevard du Triomphe, B-1050 Bruxelles, Belgium
E-mail: mrooman@ulb.ac.be

January 4, 2005

Abstract

This note is based on a talk given by one of us at the workshop 'Noncommutative Geometry and Physics 2004' (Feb. 2004, Keio University, Japan). We give a review of our joint work on strict deformation of BHTZ 2+1 black holes.

This note is based on a talk given by one of us at the workshop 'Noncommutative Geometry and Physics 2004' (Feb. 2004, Keio University, Japan). We give a review of our joint work on strict deformation of BHTZ 2+1 black holes [4, 5]. However some results presented here are not published elsewhere, and an effort is made for enlightening the instrinsical aspect of the constructions. This shows in particular that the three dimensional case treated here could be generalized to an anti-de Sitter space of arbitrary dimension provided one disposes of a universal deformation formula for the actions of a parabolic subgroup of its isometry group.

1   Motivations

The universal covering space of a generic BHTZ space-time [3]—realized as some open domain in AdS_3— is, in a canonical way, the total space of a principal fibration over $\mathbb{R}$ with structure group a minimal parabolic subgroup $\mathfrak{R}$ of $G := \widetilde{SL(2,\mathbb{R})}$ [4]. The action of the structure group is isometric w.r.t. the AdS_3 metric on the total space. For spinless BHTZ black holes, the fibers are dense open subsets of twisted conjugacy classes in G associated to an exterior automorphism of G. These twisted conjugacy classes are known to be WZW branes in $G = \mathrm{AdS}_3$ [2] i.e. extremal for the DBI brane action associated to a specific 2-form B on AdS_3 (referred hereafter as the 'B-field'). In the present work, we are interested in studying deformations of BHTZ spaces which are supported on (or tangential to) the fibration fibers. We will require our deformations to be

---

[*]"Chercheur FRIA", Belgium
[†]FNRS Research Director

non-formal ('strict' in the sense of Rieffel, see below) as well as compatible with the action of the structure group. Our motivation for this is threefold. First, a deformation of the brane in the direction of the B-field is generally understood as the (non-commutative) geometrical framework for studying interactions of strings with endpoints attached to the brane [22]. Despite the fact that curved situations have been extensively studied in various non-commutative contexts, in the framework of strict deformation theory i.e. in a purely operator algebraic framework, non-commutative spaces emerging from string theory have mainly been studied in the case of constant B-fields in flat (Minkowski) backgrounds. Second, the maximal fibration preserving isometry group of (the universal cover of) a BHTZ space being the above mentioned minimal parabolic group $\mathfrak{R}$, it is natural to ask for a deformation which is invariant under the action of $\mathfrak{R}$. But there is also a deeper reason for requiring the invariance. From considerations on black hole entropy, or just by geometrical interest, one may be interested in defining higher genus locally AdS$_3$ black holes (every BHTZ black hole is topologically $S^1 \times \mathbb{R}^2$). This would of course imply implementing the action of a Fuchsian group in the BHTZ picture. At the classical level, this question was at the center of important works by Brill et al. [1]. Our point here is that one may also attend this question at the deformed level. Indeed, the symmetry group $\mathfrak{R}$ is certainly too small to contain large Fuchsian groups. However, if the deformation is already $\mathfrak{R}$-invariant one may ask wether the (classical) action of $\mathfrak{R}$ would extend to a (deformed) action of the entire G by automorphisms of the deformed algebra. If this is the case (and it is), one obtains an action of every Fuchsian group at the deformed functional level [9, 11]. The third motivation relies on Connes-Landi's and Connes-Dubois-Violette's work [16, 17] where they use Rieffel's strict defomation method for actions of tori to define spectral triples for non-commutative spherical manifolds. The main point of their construction is that the data of an *isometric* action of a torus on a spin manifold yields not only a strict deformation of its function algebra but a compatible deformation of the Dirac operator as well. Therefore, disposing of a strict deformation formula for actions of $\mathfrak{R}$ would produce, in the present situation of BHTZ spaces, examples of spectral triples for non-commutative non-compact Lorentzian manifolds with constant curvature with the additional feature that the deformation would be supported on the $\mathfrak{R}$-orbits. The difficulty here is of course that these orbits cannot be obtained as orbits of an isometric action of $\mathbb{R}^d$.

2 Generic BHTZ spaces revisited

A BHTZ black hole is defined as the quotient space of (an open subset of) AdS$_3$ by an isometric action of the integers $\mathbb{Z}$. Namely, if Ξ denotes a Killing vector field on AdS$_3$ and if $\{\phi_t^\Xi\}_{t\in\mathbb{R}}$ denotes its flow, the $\mathbb{Z}$-action will be of the form:

$$\mathbb{Z} \times \mathrm{AdS}_3 \longrightarrow \mathrm{AdS}_3 : (n,x) \mapsto n.x := \phi_n^\Xi(x). \tag{1}$$

Not all Killing fields Ξ give rise to black hole solutions. The relevant ones can be defined as follows. As a Lorentzian manifold, the space AdS$_3$ is identified with the universal cover, G, of the group $SL(2,\mathbb{R})$ of 2×2 real matrices of determinant equal to 1 endowed with its Killing metric. The Lie algebra of the isometry group of AdS$_3$ (i.e. the algebra of Killing fields) is then canonically isomorphic to the semisimple Lie algebra $\mathfrak{g} \oplus \mathfrak{g}$ where $\mathfrak{g} := \mathfrak{sl}(2,\mathbb{R})$ denotes the Lie algebra of G. An element $\Xi \in \mathfrak{g} \oplus \mathfrak{g}$ defines a generic (i.e. non-extremal) BHTZ black hole if and only if its adjoint orbit only intercepts split Cartan subalgebras of $\mathfrak{g} \oplus \mathfrak{g}$. The squares of the (Killing) norms of the projections of Ξ onto each ideals of $\mathfrak{g} \oplus \mathfrak{g}$ are therefore both positive. Their sum —the Killing norm of Ξ— is called the *mass* and is denoted by M. Their difference —which measures how far Ξ is from being conjugated to a diagonal element in $\mathfrak{g} \oplus \mathfrak{g}$— is called the *angular momentum* and is denoted by J. Generic spaces, as opposed to extremal spaces, correspond to non-zero values of M and J. In the present letter, we will mainly be concerned with the two following geometrical properties of generic BHTZ spaces:

(i) In a generic BHTZ space, the horizons consist in a finite union of (projected) lateral classes of minimal parabolic subgroups of G.

(ii) Every generic BHTZ space is canonically endowed with a regular Poisson structure whose characteristic distribution is (locally) generated by Killing vector fields.

We now give a detailed description of the relevant geometry in the spinless as well as in the rotating cases.

2.1 Spinless BHTZ spaces ($J = 0 \neq M$)

The global geometry in this case can be derived from the two following observations. First, the action (1) appears as the restriction to a fixed split (connected) Cartan subgroup A of G of the action of G on itself by twisted conjugation:

$$\tau : G \times G \longrightarrow G : (g, x) \mapsto g x \sigma(g^{-1}) =: \tau_g(x), \tag{2}$$

where σ denotes the unique involutive exterior automorphism of G fixing pointwise the Cartan subgroup A. Indeed, if $\mathfrak{a}$ denotes the Lie subalgebra of A and if $H \in \mathfrak{a}$ is a vector of Killing length equal to 1, one has:

$$\phi_n^\Xi = \tau_{\exp(n\sqrt{M}H)} \qquad n \in \mathbb{Z}. \tag{3}$$

Second, the action τ yields the following global decomposition of G. The map

$$\phi : A \times G/A \longrightarrow G : (a, gA) \mapsto \tau_g(a) =: \phi(a, gA) \tag{4}$$

is well-defined as a global diffeomorphism [1].

As a consequence, the space $G = \mathrm{AdS}_3$ appears as the total space of a trivial fibration over $A = SO(1,1) \simeq \mathbb{R}$ whose fibers are the τ_G-orbits i.e. the σ-twisted conjugacy classes [2]. As a homogeneous G-space every fiber is isomorphic to the affine symmetric space $G/A = \mathrm{AdS}_2$. Moreover, the $\mathbb{Z}$-action (1) which will define the BHTZ space is fiberwise [3]. The Killing metric on $G = \mathrm{AdS}_3$ turns out to be globally diagonal with respect to the twisted Iwasawa decomposition:

$$\mathrm{d}s_{\mathrm{AdS}_3}^2 = \mathrm{d}a_A^2 - \frac{1}{4}\cosh^2(a)\,\mathrm{d}s_{G/A}^2, \tag{6}$$

where $\mathrm{d}s_{G/A}^2$ denotes the canonical projected AdS_2-metric on G/A. The study of the quotient space $\mathbb{Z}\backslash G$ therefore reduces to the study of $\mathbb{Z}\backslash(G/A)$.

For a pictural visualization, we realize the space G/A as the G-equivariant universal covering space of the adjoint orbit $\mathcal{O} := \mathrm{Ad}(G)(H)$ of the element $H \in \mathfrak{g}$. Sitting in the Minkowski space $\mathfrak{g}$ endowed with its Killing form B, the orbit $\mathcal{O}$ is the sphere of radius 1—a one sheet hyperboloid. The orbits of A in $\mathcal{O}$—along which the $\mathbb{Z}$-identifications will be made— are planar: every intersection $\mathcal{O} \cap (\mathfrak{a}^\perp + \xi)$ $\xi \in \mathcal{O}$ is constituted by a union of respectively two or five A-orbits. The two planar sections $\mathcal{O} \cap (\mathfrak{a}^\perp \pm H) =: \mathcal{S}_\pm$ divide $\mathcal{O} - \mathcal{S}_\pm$ into six connected open components. In only two of them, $\mathcal{M}_\pm$, the A-orbits are timelike curves with respect to the metric on $\mathcal{O}$ induced by the Killing form B. The preimage $\mathcal{M}$ of $\mathcal{M}_+ \cup \mathcal{M}_-$ in $A \times G/A = G$ is constituted by a countable union of connected open components. Denoting by $\mathcal{M}^x$ the component of $\mathcal{M}$ containing $x \in \mathcal{M}$, one has

$$\mathcal{M} = \coprod_{z \in Z(G)} \mathcal{M}^{zJ} \quad \text{(disjoint union)}, \tag{7}$$

where $Z(G)$ denotes the center of G and where J is determined (up to sign) by

$$J \in K, \qquad J^2 = -I. \tag{8}$$

Due to the minus sign in the expression of the Killing metric (6), the open set $\mathcal{M} \subset \mathrm{AdS}_3$ is constituted by all the points $x \in \mathrm{AdS}_3$ where the Killing vector Ξ_x is spacelike. The $\mathbb{Z}$-action (1) is proper and totally discontinuous on $\mathcal{M}$, one therefore has that the quotient $\mathcal{M}^x \to \mathbb{Z}\backslash\mathcal{M}^x$ defines a metric covering

---

[1]We refer to the above decomposition (4) as the *σ-twisted Iwasawa* decomposition of G with repsect to A. Indeed, if $G = KNA$ denotes an Iwasawa decomposition of G associated to the Cartan subgroup A, the homogeneous space G/A can be identified with the submanifold KN. In this setting, the decomposition (4) then reads:

$$K \times N \times A \longrightarrow G : (k, n, a) \mapsto kna\sigma(kn)^{-1},$$

which motivates our terminology.

[2]Remark that the foliation of $G = \mathrm{AdS}_3$ in τ_G-orbits (4) is the unique (up to conjugation) foliation tangent to Ξ and generated by Killing vector fields.

[3]Indeed, through the twisted Iwasawa decomposition (4) it reads:

$$n.(a, gA) = (a, \exp(n\sqrt{M}H)gA) \qquad n \in \mathbb{Z}. \tag{5}$$

of (non-complete) connected Lorentzian manifolds. These quotient spaces are all isometric to one another and homeomorphic to $S^1 \times \mathbb{R}^2$. In this picture, the black hole singularity appears at the level of G as the preimage $\mathcal{S}$ of $\mathcal{S}_+ \cup \mathcal{S}_-$ in $A \times G/A$ i.e. the boundary of $\mathcal{M}$ (on $\mathcal{S}$ the Killing field Ξ is null while it is timelike on the complement of the closure of $\mathcal{M}$, therefore yielding closed timelike curves in the quotient).

At the level of G, the horizons may be characterized as follows: a point $x \in \mathcal{M}$ belongs to an horizon if and only if the null directions from x which intersect the singularity $\mathcal{S}$ constitute a discrete set among all null directions from x. The union $\mathcal{H}$ of future and past horizons are then given by the union of the lateral classes through the element J of both minimal parabolic subgroups of G associated to A. Formally, if N and $\overline{N}$ denote the nilpotent subgroups of G normalized by A, one has

$$\mathcal{H} = (J\, Z(G)AN) \bigcup (J\, Z(G)A\overline{N}), \tag{9}$$

while the singularity is the union of the subgroups:

$$\mathcal{S} = Z(G)AN \bigcup Z(G)A\overline{N}. \tag{10}$$

Note that since the conjugation by J defines the K-adapted Cartan involution, the union $\mathcal{H}$ does not depend on the choice of right or left classes. At the level of the black hole (e.g. $\mathbb{Z}\backslash\mathcal{M}^J$), the horizons are obtained by projecting $\mathcal{H}$.

2.2 Rotating BHTZ black holes $(J \neq 0 \neq M)$

Like in the spinless case, we look for a foliation of G which is tangent to Ξ and generated by Killing vector fields. Here again there is no choice (up to conjugation) for such a foliation and it appears to underly an Iwasawa type decomposition of G. We adopt the same notations as in the preceding section and we consider $\alpha \in \mathfrak{a}^\star$ be such that $|\alpha| < 1$. Without loss of generality, one may then write the $\mathbb{Z}$-action (1) defining the rotating BHTZ space as

$$\mathbb{Z} \times G \longrightarrow G : (n, x) \mapsto n.x := \exp(nH)x\exp(n < \alpha, H > H). \tag{11}$$

We now observe that the map

$$A \times N \times K \longrightarrow G : (a, n, k) \mapsto anka^\alpha \tag{12}$$

is a global diffeomorphism [4].

We call the decomposition (12) a *modified Iwasawa* decomposition. Note that this defines a trivial fibration of G onto K whose fibers are the orbits of the following action $\overline{\tau}$ of the solvable group $\mathfrak{R} := AN$:

$$\overline{\tau} : \mathfrak{R} \times G \longrightarrow G : (an, x) \mapsto anxa^\alpha. \tag{13}$$

Consequently, every $\overline{\tau}_\mathfrak{R}$-orbit is stable under the $\mathbb{Z}$-action (1) and through the modified Iwasawa decomposition (12) it simply reads:

$$n.(r, k) = \overline{\tau}_{\exp(nH)}(r, k) = (\exp(nH)r, k) \qquad r \in \mathfrak{R}. \tag{14}$$

There are three main differences with the $J = 0$ case. First, the action $\overline{\tau}$ of $\mathfrak{R}$ does not extend to an action of G by isometries. Second, there is no global transversal which is everywhere orthogonal to the foliation. The $\mathbb{Z}$-action is everywhere proper and discontinuous so that the quotient space $\mathbb{Z}\backslash\mathrm{AdS}_3$ is globally a complete Lorentzian manifold.

The same techniques as in the spinless case nevertheless apply to obtain the causal structure in the rotating case. Again, one finds that the horizons are (projected) lateral classes of minimal parabolic subgroups.

We summarize this section by observing that to every generic BHTZ space are canonically associated two objects. First, a trivial fibration of AdS_3 whose fibers are orbits under an isometric action of a subgroup of G of dimension at least two. We call it *the action associated to the BHTZ space*. Second, a pair of conjugated minimal parabolic subgroups whose lateral classes define the horizons. We call them *the parabolic subgroups associated to the BHTZ space*.

---

[4]for $a \in A$, we set $a^\alpha := \exp(< \alpha, \log(a) > \log(a))$.

3 BHTZ domains and extensions

We consider a generic BHTZ space with corresponding $\mathbb{Z}$-action (1) on AdS_3. We define a *BHTZ domain* of AdS_3 as a $\mathbb{Z}$-stable connected simply connected open subset $\mathcal{U}$ of AdS_3 maximal for the following two properties.

  (i) the restricted action $\mathbb{Z} \times \mathcal{U} \to \mathcal{U}$ is proper and totally discontinuous;

  (ii) the Killing vector field Ξ is everywhere spacelike on $\mathcal{U}$.

    Roughly speaking, a BHTZ domain is thus a realization in AdS_3 of the universal covering space of a generic BHTZ space-time.

Similarly, we define a *BHTZ extension domain* as a $\mathbb{Z}$-stable connected simply connected open subset $\mathcal{E}$ of AdS_3 maximal only for property (i) above. We call the corresponding quotient manifold $\mathbb{Z}\backslash\mathcal{E}$ a *BHTZ extension* of the BHTZ space-time at hand. The latter therefore lies in every of its maximal extensions.

Remark that for rotating generic BHTZ spaces, the whole AdS_3 constitutes an extension domain. While for spinless BHTZ spaces, we have seen that any extension domain never entirely contains a τ_G-orbit.

The rest of this section will be devoted to the following assertion: every BHTZ domain admits a unique (up to conjugation) extension domain foliated by the orbits of the restriction of its associated action to the neutral component of one of its associated minimal parabolic subgroup.

The above statement is obvious in the rotating case. While for the spinless case, it is just worth observing that at the level of the adjoint orbit $\mathcal{O} \subset \mathfrak{g}$ (cf. Subsection 2.1), the subgroup AN (resp. $A\overline{N}$) has exactly two open orbits: the connected components of the complement of the planar intersection $\mathcal{O} \cap \mathfrak{n}^\perp$ (resp. $\overline{\mathfrak{n}}^\perp$) where $\mathfrak{n}$ (resp. $\overline{\mathfrak{n}}$) denotes the Lie algebra of N (resp. $\overline{N}$). Each of them being identified with the subgroup itself (the stabilizer groups are all trivial), the restricted action of $\exp(\mathbb{Z}\sqrt{M}H)$ is proper and discontinuous. One then readily verifies that any connected component of the preimage in $A \times G/A = G$ of one of these open orbits defines a BHTZ extension domain.

4 Poisson structures

We have seen that the orbits for the associated action of a non-spinning BHTZ space are the σ-twisted conjugacy classes in $G = AdS_3$ (σ is an involutive exterior automorphism of G). This remark naturally leads to consider a particular Poisson structure on AdS_3 whose symplectic leaves are the τ_G-orbits. Indeed, on the one hand, these submanifolds are known to be WZW 1- branes. That is, extremals of the DBI action associated with the data of a 2-form B on AdS_3 such that its exterior derivative dB is, up to a constant factor, equal to the metric volume ν on AdS_3. On the second hand, each orbit is, as a G-homogeneous space, isomorphic to the symmetric space G/A which is endowed with a unique (up to a constant factor) G-invariant symplectic structure $\omega^{G/A}$ [12]. It is then natural to ask wether the restriction of the B-field to the tangent distribution of the orbit foliation yields on each orbit (a multiple of) its canonical G-invariant symplectic structure. It actually does it, and in a canonical way. Indeed, the twisted Iwasawa decomposition (4) $\phi : A \times G/A \to G$ yields a global transversal (A) to the orbit foliation. By extending by 0 on A the 2-form $\omega^{G/A}$ to $A \times G/A$ one gets a closed 2-form on $G = AdS_3$: $\omega := (\phi^{-1})^\star(0 \oplus \omega^{G/A})$. Every τ-invariant 2-form on G is then of the form $f\omega$ where f is leafwise constant i.e. $f = f(a)$ $a \in A$. Such a form therefore defines an admissible B-field if and only if $\nu = f' \, da \wedge \omega$. This last condition determines a unique (up to an additive constant) function f. Indeed, both ν and ω being τ-invariant, it is enough to check the condition on the transversal, which defines f'[5].

    Now for a given generic BHTZ space, the above remark leads us to consider the particular class of Poisson structures on AdS_3 constituted by leafwise symplectic structures which are invariant under the associated action. Since the group $\mathfrak{R}$ has a unique (up to a constant factor) left-invariant symplectic structure, such Poisson structures exist in the rotating case as well.

---

[5]a computation at the level of the fundamental vector fields yields

$$f(a) = \tanh(\frac{a}{2}) + \text{const.}$$

5 Strict deformations for solvable Lie group actions

5.1 Connes spectral triples for Abelian Lie group actions

In this subsection we very roughly recall Rieffel strict deformation theory for actions of $\mathbb{R}^d$ [21] and how Connes-Landi and Connes-Dubois-Violette used it to define non-commutative spaces [16, 17].

5.1.1 Rieffel's machinery

Given an action τ of the Abelian group $V := \mathbb{R}^d$ on a manifold $\mathcal{X}$ Rieffel's machinery produces a one parameter deformation $\{\mathbb{A}_\theta\}_{\theta \in \mathbb{R}}$ of the commutative algebra $\mathbb{A}_0 := C(\mathcal{X})$ of complex valued functions on X vanishing at infinity [6]. The algebras $\mathbb{A}_\theta$'s involved in the deformation are in fact $C^\star$-algebras [7] and the family $\{\mathbb{A}_\theta\}_{\theta \in \mathbb{R}}$ is a continuous field of $C^\star$-algebras. One starts with a fixed anti-symmetric matrix $\mathcal{J} \in \mathfrak{so}(\mathbb{R}^d)$. Note that, via the action τ, it determines a Poisson structure on $C^\infty(X)$: $\{u, v\} := \mathcal{J}^{ij} X_i^\star.u\, X_j^\star.v$ where $u, v \in C^\infty(\mathcal{X})$ and where $\{X_i^\star\}$ are the fundamental vector fields w.r.t. the action τ associated to a basis $\{X_i\}$ of (the Lie algebra of) $\mathbb{R}^d$. Denote by $\alpha : V \times C(\mathcal{X}) \to C(\mathcal{X})$ the action on functions induced by τ and by $\mathbb{A}_0^\infty$ the space of smooth vectors of α. One then defines the deformed product at the level of the smooth vectors by the following oscillatory integral formula:

$$a \star_\theta b := \int_{V \times V} e^{ix.y} \alpha_x(a) \alpha_{\theta \mathcal{J} y}(b) \, \mathrm{d}x \mathrm{d}y \tag{15}$$

where $a, b \in \mathbb{A}_0^\infty$ and where $x.y$ denotes the Euclidean dot product on $V = \mathbb{R}^d$. The pair $(A_0^\infty, \star_\theta)$ then turns out to be a pre-$C^\star$-algebra for a suitable choice of involution and norm. Its $C^\star$-completion is denoted by $\mathbb{A}_\theta$. Note that for $\theta = 0$, the RHS of Formula (15) reduces to the usual commutative product of functions ab. Also, the first order expansion term in θ of the oscillatory integral (15) is $\frac{\theta}{2i}\{a, b\}$. The product $a \star_\theta b$ is thus an associative deformation of ab in the direction of the Poisson bracket coming from the action τ.

An important example obtained from Rieffel's machinery is the so-called *quantum torus* $C(\mathbb{T}^d)_\theta$. It is defined as the deformation of the d-torus $\mathbb{T}^d$ obtained from the above procedure when applied to the natural action of $\mathbb{R}^d$ on $\mathbb{T}^d$.

5.1.2 Deformed spectral triples for compact spin $\mathbb{T}^d$-manifolds

Assume M is a compact spin manifold which the torus $\mathbb{T}^d$ acts on by isometries. The point is that Rieffel's machinery applied to M only produces a deformation of the *topological* space M. One would want to deform the metric structure as well. For doing this, Connes introduced the notion of spectral triple. The typical commutative example of a spectral triple is the one associated to a compact spin manifold; it roughly goes as follows. The basic idea is that the metric aspect of the Riemannian manifold M is encoded, in the operator algebra framework, by the Dirac operator D[15]. The spectral triple associated to M is then defined as the triple $(\mathbb{A}_0, \mathcal{H}, D)$ where $\mathbb{A}_0 = C(M)$ and where $\mathcal{H}$ is the Hilbert space of L^2-spinor fields on which the Dirac operator D acts as well as the $C^\star$-algebra $\mathbb{A}_0$ (simply by multiplication). Generally, a (not necessarily commutative) spectral triple is defined as a triple $(\mathbb{A}, \mathcal{H}, D)$ where $\mathbb{A}$ is a $C^\star$-algebra, $\mathcal{H}$ is a Hilbert space representation of $\mathbb{A}$ by bounded operators and D is a self-adjoint unbounded operator on $\mathcal{H}$ such that, among other things, $[D, a]$ is bounded for all a in $\mathbb{A}$[8]. To produce a deformed spectral triple from an action of $\mathbb{T}^d$ on M, one first observes that the deformation $C^\infty(M)_\theta$ obtained from Rieffel's machinery can equivalently be described as the closed subalgebra $(C^\infty(M) \otimes C(\mathbb{T}^d)_\theta)^{\tau \times L^{-1}}$ of $C^\infty(M) \otimes C(\mathbb{T}^d)_\theta$ constituted by the elements invariant under the natural action of $\mathbb{T}^d$ on the latter tensor product algebra (L denotes the regular action). Essentially, this is the co-action $C^\infty(M) \to C^\infty(M) \otimes C(\mathbb{T}^d)$ which yields the identification with $C^\infty(M)_\theta$. Second, one uses the previous identification to deform the $C^\infty(M)$-module of sections $\Gamma(M, S)$ of the spinor bundle $S \to M$. Since the action τ is isometric, up to a double covering of $\mathbb{T}^d$, the spinor bundle is $\mathbb{T}^d$-equivariant. The space of invariants $\Gamma(M, S)_\theta := (\Gamma(M, S) \otimes C(\mathbb{T}^d)_\theta)^{\tau \times L^{-1}}$ is therefore a $C^\infty(M)_\theta$-module stable by the operator $D \otimes I$. The restriction of the latter on $\Gamma(M, S)_\theta$ then defines the deformed Dirac operator D_θ.

---

[6]if X is compact, $C(\mathcal{X})$ is the whole algebra of continuous functions.

[7]the $C^\star$-norm on $C(\mathcal{X})$ is the sup-norm.

[8]this is the essential property needed to define the metric distance in operator terms.

We end this subsection by stressing the fact that the way Connes and Dubois-Violette define a deformation of the Dirac operator via an action of $\mathbb{T}^d$ essentially relies on two crucial properties. First, the invariance of Rieffel's deformed product on $C(\mathbb{T}^d)_\theta$ under the (left or right) regular representation (this allows to define the algebra (or module) structure on the space of invariant elements in the tensor product). Second, the fact that the action τ of $\mathbb{T}^d$ on M is isometric.

5.2 Symmetric spaces and universal deformation formulae

In this section we recall a consruction of a universal (strict) deformation formula for the actions of a non-Abelian Lie group. This formula was obtained via geometric methods based on symplectic symmetric spaces. We begin by recalling the notion of symplectic symmetric space, we then pass to the specific example which will be relevant here.

A **symplectic symmetric space** [7, 8] is triple (M, ω, s) where (M, ω) a symplectic manifold and $s : M \times M \to M$ is a smooth map such that for each point $x \in M$ the partial map $s_x : M \to M : y \mapsto s_x(y) := s(x, y)$ is an involutive symplectic diffeomorphism which admits x as isolated fixed point. Under these hypotheses, the following formula defines a symplectic torsionfree affine connection ∇ on M:

$$\omega_x(\nabla_X Y, Z) := \frac{1}{2} X_x.\omega(Y + s_{x\star}Y, Z); \tag{16}$$

where X, Y and Z are smooth vector fields on M. This connection is the only one being invariant under the *symmetries* $\{s_x\}_{x \in M}$ which turns out to be the geodesic symmetries. The group $G(M)$ generated by the products $s_x \circ s_y$ is a Lie group of transformations of M which acts transitively on M. It is called the **transvection** group of M.

The particular example we will be concerned with is the one where the transvection group is the Poincaré group $P_{1,1} := SO(1,1) \times \mathbb{R}^2$. It is a three dimensional solvable Lie group whose generic (two dimensional) coadjoint orbits are symplectic symmetric spaces admitting $P_{1,1}$ as tranvection group. They are all equivalent to oneanother ($P_{1,1}/\mathbb{R}$ as homogeneous spaces) and topologically $\mathbb{R}^2$.

Let (M, ω) be such an orbit and let ∇ be tis canonical connection. Such a space turns out to be strictly geodesically convex in the sense that between two points in M there is a unique geodesic. It has moreover the property that given any three points x, y and z in M, there's a unique fourth point t in M satisfying

$$s_x s_y s_z t = t. \tag{17}$$

This allows to define the **three point phase** of M as the three point function $S \in C^\infty(M \times M \times M, \mathbb{R})$ [23] given by

$$S(x, y, z) := \int_{\Delta(t, s_z t, s_y s_z t)} \omega \tag{18}$$

where $\Delta(x, y, z)$ denotes the geodesic triangle with vertices x, y and z. The three point **amplitude** $A \in C^\infty(M \times M \times M, \mathbb{R})$ [20, 6] is defined as the ratio

$$A(x, y, z) := \left(\frac{\int_{\Delta(t, s_z t, s_y s_z t)} \omega}{\int_{\Delta(x, y, z)} \omega} \right)^{\frac{1}{2}}. \tag{19}$$

Both S and A are invariant under the diagonal action of the symmetries. For compactly supported $u, v \in C_c^\infty(M)$, one sets

$$u \star_\theta v(x) := \frac{1}{\theta^2} \int_{M \times M} A(x, y, z) \, e^{\frac{i}{\theta} S(x, y, z)} u(y) \, v(z) \, dy \, dz, \text{ or shortly } u \star_\theta v = \frac{1}{\theta^2} \int_{M \times M} A \, e^{\frac{i}{\theta} S} u \otimes v; \tag{20}$$

where dx denotes the symplectic measure on (M, ω). This formula defines a symmetry invariant associative (non-formal) deformation of M led by the Poisson structure associated to ω on some topological function space $\mathcal{E}_\theta(M) \subset C(M)$ []. For every real value of θ the space $\mathcal{E}_\theta(M)$ contains the test functions. Moreover, the deformed product is *strongly closed* in the sense that, one has the trace property:

$$\int_M u \star_\theta v = \int_M uv \tag{21}$$

for all compactly supported smooth functions u and v.

We now explain how this is relevant to our situation of BHTZ spaces. The transvection group $P_{1,1}$ contains a copy of the group $\mathfrak{R}$ which acts on maximal extensions of BHTZ spaces as discussed above. Moreover the restricted action $\mathfrak{R} \times M \to M$ turns out to be strictly transitive, so that one has the identification $M \simeq \mathfrak{R}$. Transporting the above structure to the group manifold $\mathfrak{R}$, one gets a left-invariant deformation, again denoted by $\star_\theta$, on the symplectic group $(\mathfrak{R}, \omega^\mathfrak{R})$. Let us denote by $K_\theta^\mathfrak{R}$ the associated (oscillating) kernel:

$$u \star_\theta v = \int_{\mathfrak{R} \times \mathfrak{R}} K^\mathfrak{R}{}_\theta \, u \otimes v \quad u, v \in \mathcal{E}_\theta(\mathfrak{R}), \tag{22}$$

where the integration is taken w.r.t. a left invariant Haar measure on $\mathfrak{R}$. Now, in a similar spirit as Rieffel's machinery, this defines a universal deformation formula for actions of $\mathfrak{R}$. Indeed, if $\mathcal{X}$ is a space the group $\mathfrak{R}$ acts on via an action τ, one defines for $a \in C(\mathcal{X})$ and $x \in \mathcal{X}$ the function $\alpha^x : \mathfrak{R} \to \mathbb{C} : g \mapsto a(\tau_{g^{-1}}x)$. Setting

$$\mathcal{E}_\theta(\mathcal{X}) := \{a \in C(\mathcal{X}) : \alpha^x a \in \mathcal{E}_\theta(\mathfrak{R}) \, \forall x \in \mathcal{X}\}, \tag{23}$$

the following formula yields an associative product on the space $\mathcal{E}_\theta(\mathfrak{R})$:

$$(a \star_\theta b)(x) := \int_{\mathfrak{R} \times \mathfrak{R}} K^\mathfrak{R}{}_\theta(e, g, h) \, \alpha^x a(g) \, \alpha^x b(h) \, dg \, dh; \tag{24}$$

or shortly: $a \star_\theta b =: \int_{\mathfrak{R} \times \mathfrak{R}} K^\mathfrak{R}{}_\theta \, \alpha a \otimes \alpha b$. Of course, such a product enjoys trace properties similar to (21).

Analoguous universal strict deformation formulae for more general solvable groups have been given in [].

5.3 Spectral triples from non-Abelian Lie group actions

We now give a noncommutative spectral triple deforming the commutative one canonically associated to a maximally extended non-rotating BHTZ-space. We start, more generally, with the data of a pseudo-Riemannian spin manifold M of signature (m, n). We set D for the Dirac operator on M acting on the $C^\infty(M)$-bi-module of smooth sections $\underline{S}$ of the spinor bundle $S \to M$ over M[9]. We note that the fact that, for all a in $C^\infty(M)$, $[D, a]$ is continuous with respect to any reasonable topology on the spinor bi-module relies on the property that D is a derivation of the module structure. In the case of Connes and Landi's construction, this property of the (undeformed) Dirac operator remains through the deformation because the torus is an Abelian group. One difficulty in the case of a non-Abelian Lie group action is therefore to force the derivation property of the Dirac operator with respect to the deformed module structure.

5.3.1 Deformations of the spinor module and module endomorphisms

Assume our deformation at the level of functions on M comes from an action α of a symplectic solvable group $\mathfrak{R}$ in such a way that the spinor module is $\mathfrak{R}$-equivariant. This means that there exists an action $\alpha^S : \mathfrak{R} \times \underline{S} \to \underline{S}$ such that $\alpha_g(a.\Psi) = \alpha_g(a).\alpha_g^S(\Psi)$ for all $a \in C^\infty(M)$. Denoting by K_θ the convolution kernel which defines the deformation, the following formula yields, at least formally, a deformation of the (right) module structure:

$$\Psi \star_\theta a := \int_{\mathfrak{R} \times \mathfrak{R}} K_\theta(e, g, h) \, \alpha_h(a).\alpha_g^S(\Psi) \, dg \, dh; \tag{25}$$

where e denotes the identity in $\mathfrak{R}$. One writes a similar formula $a \star_\theta \Psi$ for a deformation of the left module structure. In the same way, one deforms the module structure attached to any vector bundle associated to the spin structure over M. Note that, for the particular case of the endomorphism bundle, one formally has, for all $\Psi \in \underline{S}$, $\gamma \in \underline{S^\star \otimes S}$ and $a \in C^\infty M$,

$$(\gamma \star_\theta \Psi) \star_\theta a = \gamma \star_\theta (\Psi \star_\theta a). \tag{26}$$

In other words, the left deformed module of endomorphisms acts on the right deformed spinor module.

---

[9]More generally, given a vector bundle $E \to M$ over M, we shall denote its space of smooth sections by $\underline{E}$.

5.3.2 Deforming the Dirac operator: case of trivial fibrations

First consider the case of a differential operator $\Theta : \underline{S} \to \underline{S}$ of the form

$$\Theta(\Psi) = \gamma\left(\nabla_X \Psi\right), \tag{27}$$

where ∇ denotes the spin connection, where X is a vector field on M and where $\gamma \in \underline{S^\star \otimes S}$ is a section of the endomorphism bundle. Note that Θ acts as a derivation of the (undeformed) module structure.
If the vector field X commute with the actions α and α^S, i.e. $X.\alpha_g a = \alpha_g(X.a)$ and $\nabla_X \alpha_g^S \Psi = \alpha_g^S\left(\nabla_X \Psi\right)$, then it automatically acts as derivation of the deformed multiplications. Indeed, one formally has: $X.(a \star_\theta b) = \int K_\theta\, X.(\alpha a \otimes \alpha b) = \int K_\theta\left(\alpha(X.a) \otimes \alpha b + \alpha a \otimes \alpha(X.b)\right) = (X.a) \star_\theta b + a \star_\theta (X.b)$, and similarly for $\nabla_X(\Psi \star_\theta a)$. In particular, in the case the element γ is constant, the operator Θ is then a derivation of the deformed module structure.

Now, we turn back to our (extended) BHTZ spaces. In the preceeding sections, we have seen that their maximal extension domains are total spaces of trivial principal fibrations over $\mathbb{R}$ with structure group $\mathfrak{R}$. In particular, this implies that the space $\mathcal{E}_\theta(\mathcal{U})$, resulting from the application of the deformation formula to the left action τ of $\mathfrak{R}$ by isometries on $\mathcal{U}$, is non-trivial. Moreover, the (non-isometric) *right* action of $\mathfrak{R}$ on itself— and hence on a maximal extension domain $\mathcal{U}$— defines for any element X in the Lie algebra of the group $\mathfrak{R}$ a fundamental vector field again denoted by X which commutes with the actions α and α^S of $\mathfrak{R}$. Observing that $\tau_\star X = X$, one immediately concludes in the case of the non-rotating case $J = 0$ [10] that the Dirac operator D on $\mathcal{U}$ acts as a derivation of the (right) deformed module structure. This yields the formula $[D, a].\Psi = \Psi \star_\theta (Da)$, showing that for every a the commutator $[D, a]$ is a bounded operator in the space $\mathcal{H}$ of L^2 spinors. The triple $(\mathcal{E}_\theta(\mathcal{U}), \mathcal{H}, D)$ is thus a spectral triple in the sense of Connes[11]. A distance formula for such Lorentzian spectral triples has been formulated by V. Moretti in []. In a future work, we plan to investigate the question of understanding how causality is encoded at the operator algebraic level.

Remark 5.1 (i) In the more general case of a non-constant element γ, the deformed operator Θ_θ defined as

$$\Theta_\theta(\Psi) := (\nabla_X \Psi) \star_\theta \gamma, \tag{28}$$

acts as a derivation of the left deformed module structure (as a immediate consequence of the fact that the right deformed module of endomorphisms acts on the left deformed spinor module).

(ii) Another situation of interest is when the vector field X is Hamiltonian with respect to the Poisson structure induced by the action α. Assume moreover that the spinor bundle $S \to M$ is trivial—as it is the case for AdS_3 or BHTZ-spaces. In this case, one has a globally defined connection form $\Gamma \in \Omega^1(S^\star \otimes S)$ such that

$$\nabla_X \Psi = X.\Psi + \Gamma(X)\Psi. \tag{29}$$

Denoting by $\lambda_X \in C^\infty(M)$ the hamiltonian function associated to X, one deforms the covariant derivative in the direction of X as

$$(\nabla_X)^\theta \Psi := \lambda_X \star_\theta \Psi - \Psi \star_\theta \lambda_X + \Psi \star_\theta \Gamma(X). \tag{30}$$

Note that $(\nabla_X)^\theta$ provides a derivation of the left deformed module structure in the sense that $(\nabla_X)^\theta(a \star_\theta \Psi) = [\lambda_X, a]_\theta \star_\theta \Psi + a \star_\theta (\nabla_X)^\theta \Psi$ where $[a, b]_\theta := a \star_\theta b - b \star_\theta a$.

These remarks have been used in [] to define strict deformations of BHTZ spaces themselves rather than their extension domains.

References

[1] S. Aminneborg, I. Bengtsson, D. Brill, S. Holst, P. Peldan, Class.Quant.Grav. 15(1998) 627 ;D. Brill, Annalen Phys. 9(2000) 217

---

[10] In the rotating case, the lack of a transversal everywhere orthogonal to the foliation makes the argument more involved.
[11] More precisely, because of noncompactess, in the sense of [].

[2] C. Bachas, M. Petropoulos, JHEP 0102 (2001) 025

[3] M. Bañados, M. Henneaux, C. Teitelboim, J. Zanelli, Phys. Rev. **D48** (1993) 1506, gr-qc/9302012

[4] P. Bieliavsky, M. Rooman, Ph. Spindel, Nucl.Phys. B645 (2002) 349-364

[5] P. Bieliavsky, S. Detournay, M. Herquet, M. Rooman, Ph. Spindel, Phys.Lett. B570 (2003) 231-236

[6] P. Bieliavsky, S. Detournay, M. Rooman, Ph. Spindel, *Star products on extended massive nonrotating BTZ black holes*, JHEP06(2004)031.

[7] P. Bieliavsky, *Espaces symétriques symplectiques.* Ph. D. thesis, Université Libre de Bruxelles, 1995.

[8] P. Bieliavsky, M. Cahen and S. Gutt, *Symmetric symplectic manifolds and deformation quantization*, Modern group theoretical methods in Physics, Math. Studies, **18** (1995) 63–75.

[9] P. Bieliavsky, *Symmetric spaces and star representations.* Advances in geometry, 71–82, Progr. Math., 172, Birkhuser Boston, Boston, MA, 1999.

[10] P. Bieliavsky, Journal of Symplectic Geometry, Vol.1 N.2 (2002) 269, math.QA/0010004

[11] P. Bieliavsky; M. Pevzner, *Symmetric spaces and star representations. III. The Poincar disc.* Noncommutative harmonic analysis, 61–77, Progr. Math., 220, Birkhuser Boston, Boston, MA, 2004.

[12] P. Bieliavsky, *Semisimple symplectic symmetric spaces.* Geom. Dedicata 73 (1998), no. 3, 245–273.

[13] P. Bieliavsky and Y. Maeda, *Convergent star product algebras on "ax + b".* Lett. Math. Phys. 62 (2002), no. 3, 233–243.

[14] P. Bieliavsky and M. Massar, *Oscillatory integral formulae for left-invariant star products on a class of Lie groups.* Lett. Math. Phys. 58 (2001), no. 2, 115–128.

[15] A. Connes, *Noncommutative geometry*, Academic Press, San Diego (1994)

[16] A. Connes and G. Landi, *Noncommutative manifolds, the instanton algebra and isospectral deformations.* Comm. Math. Phys. 221 (2001), no. 1, 141–159.

[17] A. Connes, M. Dubois-Violette, *Noncommutative finite-dimensional manifolds. I. Spherical manifolds and related examples.* Comm. Math. Phys. 230 (2002), no. 3, 539–579.

[18] V. Gayral, J.M. Gracia-Bonda, B. Iochum, T. Schcker, J.C. Vrilly, *Moyal planes are spectral triples.* Comm. Math. Phys. 246 (2004), no. 3, 569–623.

[19] V. Moretti, Rev.Math.Phys. Vol.15 (2003) 1-46, gr-qc/0203095

[20] Z. Qian, *Groupoids, Midpoints and Quantizations.* Ph.D. Thesis, University of California, Berkeley 1997.

[21] M. A. Rieffel, *Deformation quantization for actions of* $\mathbf{R}^d$. Mem. Amer. Math. Soc. 106 (1993), no. 506.

[22] N. Seiberg, E. Witten, JHEP 9909:032, 1999, hep-th/9908142

[23] A. Weinstein. *Traces and triangles in symmetric symplectic spaces.* Symplectic geometry and quantization (Sanda and Yokohama, 1993), Contemp. Math. 179 (1994), 261–270.