Kramers-Wannier dualities via symmetries

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Kramers-Wannier dualities in lattice models are intimately connected with symmetries. We show that they can be found directly and explicitly from the symmetry transformations of the boundary states in the underlying conformal field theory. Intriguingly the only models with a self-duality transformation turn out to be those with an auto-orbifold property.

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Universal properties of two-dimensional critical phenomena, when conformally invariant, are described by conformal field theories. Virtually all their aspects can be understood within an appropriate conformal theory. Duality transformations, which have played a prominent role in the history of critical phenomena, have however so far largely remained outside the conformal description.

The notion of duality is inspired by the duality discovered by Kramers-Wannier in the Ising model [1], and more specifically its interpretation in terms of disorder lines given by Kadanoff and Ceva [2]: a duality exchanges order with disorder fields, where the disorder fields are associated with defects lines, non-local perturbations concentrated on lines. The Kramers-Wannier duality had the surprising and important feature of relating the high and low temperature regimes. Dualities were subsequently discovered in many other models, and also generalized to relate different models [3].

Recently the dualities in self-dual models have been reconsidered from a conformal theory point of view [4]. It was shown there that the information about the dualities is in fact contained in the fusion algebra of special fields associated with conformal defects.

In this letter, we provide an alternative point of view by connecting the dualities directly with the symmetries of the underlying conformal theory. We give a simple and completely explicit way to find out when there is a duality, and to compute the duality transformations. Although the method is general, we will mostly deal with the unitary Virasoro minimal theories. We will conclude that only six of them have a non trivial duality.

The exact way the present work relates to [4] is not clear to us. The dualities discussed in [4] are generated by specific duality defects which are not group-like. In contrast, our approach is based on the transformation laws of the boundary states under internal symmetries. The results nonetheless seem to fully agree, which indicates that the two approaches are complementary.

The dualities discussed here are derived in the conformal setting, namely at criticality. They however extend at least to the perturbative neighbourhood of the critical point, where the conformal picture holds. When the critical point is self-dual, the duality is a true symmetry which establishes relations among correlators and operator product coefficients. The duality is also useful to connect different relevant perturbations. If the perturbing term is $\lambda \int \phi(x)$ for a duality odd field, the duality establishes a correspondence between the $\lambda > 0$ and the $\lambda < 0$ regimes, and between the corresponding renormalization group flows, generalizing the duality between the low and high temperature phases of the Ising model.

The Ising model. - It is instructive to reconsider the Ising model at zero magnetic field. On an $L \times T$ cylindrical lattice with free boundary conditions along the bottom and top boundaries, the contour representation of the partition functions is

$$Z_{P,ff}^{L\times T}(\beta) = 2^{LT}\,(\cosh\beta)^{2LT-L} \sum_{C} (\tanh\beta)^{|C|}, \qquad (1)$$

where the sum runs over closed contours. A contour is a set of loops drawn on the lattice bonds, where each bond is used at most once ($|C|$ is the number of bonds of C).

A contour C defines the variations of a dual spin configuration s^* on the dual lattice [5]. The dual spins are constant inside the loops of C, and change sign when one crosses a contour line. If the dual spin at a single dual site is fixed, the whole dual configuration is unambiguously fixed by C. By relating the energy of the dual configuration to the value of $|C|$, one can write the above partition function in terms of a partition function for the dual spins. This dual partition function is evaluated at the dual temperature β^* defined by $\tanh\beta = \mathrm{e}^{-2\beta^*}$, while the dual spins are subjected to dual boundary conditions, usually distinct from the original ones. In the present case, the relation reads

$$Z_{P,ff}^{L\times T}(\beta) = 2^{L/2}\,(\sinh 2\beta)^{LT-L/2} \times \Big[Z_{P,++}^{L\times(T+1)}(\beta^*) + Z_{P,+-}^{L\times(T+1)}(\beta^*)\Big], \qquad (2)$$

where $Z_{P,++}$ (resp. $Z_{P,+-}$) is the cylinder partition function for fixed and equal (resp. opposite) spins on the two boundaries.

The same calculation can be repeated for antiperiodic boundary condition along the perimeter of the cylinder.

In this case, the energy is computed as before except that two neighbouring spin columns are coupled antiferromagnetically. The duality relation reads in this case

$$Z^{L\times T}_{A,ff}(\beta) = 2^{L/2}\,(\sinh 2\beta)^{LT-L/2}\times \Big[Z^{L\times(T+1)}_{P,++}(\beta^*) - Z^{L\times(T+1)}_{P,+-}(\beta^*)\Big]. \qquad (3)$$

At the critical, self-dual point, specified by $\sinh 2\beta = 1$, the relations (2) and (3) imply the following identities among universal partition functions,

$$Z_{f|f} = Z_{+|+} + Z_{+|-}, \qquad Z^A_{f|f} = Z_{+|+} - Z_{+|-}. \qquad (4)$$

They not only show that the fixed boundary conditions are dual to the free condition, but also emphasize the role of the symmetry group.

A toric lattice can also be considered. The contours break up into four subclasses, according to the number, even or odd, of non contractible loops running in the two directions. Interestingly the duality relation takes the form

$$Z_{PP}(\beta) = \frac{1}{2}\,(\sinh 2\beta)^{LT}\times \Big[Z_{PP}(\beta^*) + Z_{PA}(\beta^*) + Z_{AP}(\beta^*) + Z_{AA}(\beta^*)\Big]. \qquad (5)$$

At the critical point, these relations show that the Ising model is its own orbifold. (When a conformal theory has a symmetry group, the orbifold construction produces another conformal theory by rearranging the fields of the various sectors [6]. For the minimal models, this construction has been worked out in their restricted solid-on-solid lattice realization, and off criticality in [7].)

Duality in the conformal theory. - The conformal description of a system with a boundary starts with conformally invariant boundary conditions. These can be written as $|\alpha\rangle = \sum_{i\in\mathcal{E}} c^i_\alpha\,|i\rangle\!\rangle$, where $|i\rangle\!\rangle$ are unphysical boundary states called Ishibashi states and the c^i_α are numerical coefficients [8]. The set $\mathcal{E} = \mathcal{E}_e \cup \mathcal{E}_g \cup \ldots$ is a set of labels for all the scalar bulk fields of the theory, *including those contained in sectors twisted by elements of the symmetry group* [9, 10]. We will see that the Ishibashi states from the twisted sectors play a central role in our derivation of the dualities.

Again the case of the Ising model is particularly instructive. There are three physical conformal boundary conditions, the free condition $|f\rangle$, and the fixed ones $|+\rangle$ and $|-\rangle$. The periodic sector contains three scalar fields, the identity, the spin field σ and the energy density ε, of chiral conformal weight 0, $\frac{1}{16}$ and $\frac{1}{2}$ respectively. They lead to three Ishibashi states $|0\rangle\!\rangle_P$, $|\frac{1}{16}\rangle\!\rangle_P$ and $|\frac{1}{2}\rangle\!\rangle_P$. The second, antiperiodic sector contains a single scalar field, the disorder field μ, with the same conformal weight $\frac{1}{16}$ as the spin field, and gives rise to one Ishibashi state $|\frac{1}{16}\rangle\!\rangle_A$.

Inserting the values of the coefficients c^i_α [8, 9], we obtain the following suggestive expansions,

$$|+\rangle = \frac{1}{\sqrt{2}}\Big[|0\rangle\!\rangle_P + \sqrt[4]{2}\,|\tfrac{1}{16}\rangle\!\rangle_P + |\tfrac{1}{2}\rangle\!\rangle_P\Big], \qquad (6)$$

$$|f\rangle = \Big[|0\rangle\!\rangle_P + \sqrt[4]{2}\,|\tfrac{1}{16}\rangle\!\rangle_A - |\tfrac{1}{2}\rangle\!\rangle_P\Big], \qquad (7)$$

$$|-\rangle = \frac{1}{\sqrt{2}}\Big[|0\rangle\!\rangle_P - \sqrt[4]{2}\,|\tfrac{1}{16}\rangle\!\rangle_P + |\tfrac{1}{2}\rangle\!\rangle_P\Big]. \qquad (8)$$

These equations reflect the transformation laws under the symmetry group Z_2, but also reveal the duality transformations. The duality exchanges σ and μ, and therefore the states $|\frac{1}{16}\rangle\!\rangle_P$ and $|\frac{1}{16}\rangle\!\rangle_A$ which are built on them. This results in the exchange of $|f\rangle$ and $|+\rangle$ or $|-\rangle$ ($|\frac{1}{16}\rangle\!\rangle_A$ is defined up to a phase, a sign in particular) provided the energy density is odd (it defines the thermal perturbation, and so changes sign under the low-high temperature duality). In fact, the relations (4) suggest to write the duality relations as $|f\rangle_P \leftrightarrow \frac{|+\rangle+|-\rangle}{\sqrt{2}}$ and $|f\rangle_A \leftrightarrow \frac{|+\rangle-|-\rangle}{\sqrt{2}}$, where $|f\rangle_{P,A}$ denote the projections of $|f\rangle$ on the P and A sectors. These two dualities are consistent with the symmetry. Together they imply that $|f\rangle = |f\rangle_P + |f\rangle_A$ is exchanged with $|+\rangle$ (or indeed $|-\rangle$).

We now proceed to generalize these observations. The unitary minimal Virasoro models are classified by pairs $(\mathcal{A}_{2m}, \mathcal{G})$ with $\mathcal{G}$ a simply-laced Lie algebra with Coxeter number $q = 2m$ or $2m+2$, and $m \geq 1$ an integer [11]. All the required data are encoded in the Dynkin diagrams of $\mathcal{A}_{2m}$ and $\mathcal{G}$ [8, 9, 10].

The boundary states $|\alpha\rangle = |(a,b)\rangle = \sum_{i\in\mathcal{E}} c^i_\alpha\,|i\rangle\!\rangle$ can be labelled by a node $a = 1, 2, \ldots, m$ of the diagram $\mathcal{T}_m = \mathcal{A}_{2m}/Z_2$ (tadpole diagram with m nodes), and a node b in the diagram of $\mathcal{G}$. The model $(\mathcal{A}_{2m}, \mathcal{G})$ possesses an internal symmetry group G equal to the automorphism group of $\mathcal{G}$, and moreover the action of G on the boundary states coincides with its action on the diagram $\mathcal{G}$, *i.e.* ${}^g|(a,b)\rangle = |(a, g(b))\rangle$. In terms of the Ishibashi states $|i\rangle\!\rangle$, this action is induced by the action of G on the scalar fields $i \in \mathcal{E}$ of all sectors. The set $\mathcal{E}$ itself decomposes as $\mathcal{E} = \cup_{g\in G}\,\mathcal{E}_g$, where $\mathcal{E}_e$ corresponds to the periodic sector. Each subset $\mathcal{E}_g$ labels the scalar fields contained in the sector twisted by g, and equals the set of Kac labels (r,s) where r and s run over the exponents of $\mathcal{A}_{2m}$ and of $\mathcal{G}^g$ respectively (up to the symmetry of the Kac table). Here $\mathcal{G}^g$ is the part of the diagram of $\mathcal{G}$ that is left fixed by g, and is itself a Dynkin diagram. Finally the coefficients c^i_α are explicitly known in terms of the eigendata of the fused adjacency matrices of $\mathcal{A}_{2m}\times\mathcal{G}^g$. In particular, and this will be crucial for what follows, if $i \in \mathcal{E}_g$ refers to a scalar field in the sector twisted by g, then $c^i_\alpha = 0$ if α is not invariant under g. F.i. in the Ising model $(\mathcal{A}_2, \mathcal{A}_3)$, $|f\rangle = |(1,2)\rangle$ (first node of $\mathcal{T}_1$, second of $\mathcal{A}_3$) is the only Z_2 invariant boundary state, and thus the only one to have a projection on $|\frac{1}{16}\rangle\!\rangle_A$.

We will impose three basic requirements on the dualities: (i) a duality must exchange bulk fields from different sectors, since otherwise the transformation would be called a symmetry rather than a duality; (ii) it must be invertible; and (iii) it has to be consistent with both the chiral algebra and the internal symmetry (necessarily non-trivial).

Unitary minimal conformal models and relatives. - The diagonal theories $(\mathcal{A}_{2m}, \mathcal{A}_{q-1})$ all have a Z_2 symmetry. The boundary conditions are parametrized by a in $\{1,2,\ldots,m\}$ and b in $\{1,2,\ldots,q-1\}$, among which those with $b=\frac{q}{2}$ are Z_2 invariant. We choose $\mathcal{E}_e = \{(r,s) \in [1,2m]\times[1,q-1] \;:\; r \text{ odd}\}$ to label the scalar fields of the periodic sector. For $i=(r,s)$ in $\mathcal{E}_e$, the coefficients are $c^i_{(a,b)} \sim \sin\frac{\pi q a r}{2m+1}\,\sin\frac{\pi(2m+1)bs}{q}$ up to a non-zero factor.

There are two sectors, periodic and antiperiodic. Requirement (i) means that some $|i\rangle\rangle_P$ are exchanged with some $|j\rangle\rangle_A$, and therefore implies that a boundary state $|\alpha\rangle$ which is not invariant under the Z_2 symmetry and which has a non-zero projection on all the states $|i\rangle\rangle_P$, is necessarily exchanged with a boundary state $|\alpha^*\rangle$ which expands on Ishibashi states from the A sector, that is, with a Z_2 invariant boundary state. Consistency with the symmetry Z_2 actually requires that $|\alpha^*\rangle$ be dual to the combination $(|\alpha\rangle + {}^g|\alpha\rangle)/\sqrt{2}$.

Since the duality must be invertible, the number of such pairs $|\alpha\rangle$, ${}^g|\alpha\rangle$ must be smaller or equal to the number of invariant boundary states, equal to m. From the formula given above, the coefficients $c^i_{(a,b)}$ are different from zero for all i in the periodic sector, if and only if a is coprime with $2m+1$ and b is coprime with q. The total number of pairs $|\alpha\rangle$, ${}^g|\alpha\rangle$ is thus equal to $\frac{1}{4}\phi(q)\phi(2m+1)$, with $\phi(n)$ the Euler totient function, so that the aforementioned inequality reads $\phi(q)\phi(2m+1) \leq 4m$. The crude lower bound $\phi(n) \geq n^{3/5}$ if n is odd, and $\phi(n) \geq (\frac{n}{2})^{3/5}$ if n is even, is enough to show that the inequality is violated for all $m > 128$. Checking the finite number of remaining cases leaves only four cases: $(m,q) = (1,4)$, $(2,4)$, $(2,6)$ and (3,6).

By looking at the expansions of the boundary conditions in terms of the Ishibashi states, the two cases $(m,q) = (2,6), (3,6)$ are easily ruled out for not having a consistent duality. The last two cases $(m,q) = (1,4)$ and (2,4) correspond to the Ising model, discussed above, and the tricritical Ising model, both self-dual at the critical point.

The tricritical Ising model $(\mathcal{A}_4, \mathcal{A}_3)$ has six conformal boundary conditions. Four of them expand as

$$|(1,1)\rangle = C\Big[|0\rangle\rangle_P + \eta|\tfrac{1}{10}\rangle\rangle_P + \eta|\tfrac{3}{5}\rangle\rangle_P + |\tfrac{3}{2}\rangle\rangle_P + \sqrt[4]{2}|\tfrac{7}{16}\rangle\rangle_P + \sqrt[4]{2}\eta|\tfrac{3}{80}\rangle\rangle_P\Big], \qquad (9)$$

$$|(1,2)\rangle = \sqrt{2}C\Big[|0\rangle\rangle_P - \eta|\tfrac{1}{10}\rangle\rangle_P + \eta|\tfrac{3}{5}\rangle\rangle_P - |\tfrac{3}{2}\rangle\rangle_P + \sqrt[4]{2}|\tfrac{7}{16}\rangle\rangle_A + \sqrt[4]{2}\eta|\tfrac{3}{80}\rangle\rangle_A\Big], \qquad (10)$$

$$|(2,1)\rangle = C\Big[\eta^2|0\rangle\rangle_P - \eta^{-1}|\tfrac{1}{10}\rangle\rangle_P - \eta^{-1}|\tfrac{3}{5}\rangle\rangle_P + \eta^2|\tfrac{3}{2}\rangle\rangle_P - \sqrt[4]{2}\eta^2|\tfrac{7}{16}\rangle\rangle_P + \sqrt[4]{2}\eta^{-1}|\tfrac{3}{80}\rangle\rangle_P\Big], \qquad (11)$$

$$|(2,2)\rangle = \sqrt{2}C\Big[\eta^2|0\rangle\rangle_P + \eta^{-1}|\tfrac{1}{10}\rangle\rangle_P - \eta^{-1}|\tfrac{3}{5}\rangle\rangle_P - \eta^2|\tfrac{3}{2}\rangle\rangle_P - \sqrt[4]{2}\eta^2|\tfrac{7}{16}\rangle\rangle_A + \sqrt[4]{2}\eta^{-1}|\tfrac{3}{80}\rangle\rangle_A\Big], \qquad (12)$$

where $C = \sqrt{\frac{\sqrt{5}-1}{8}}$ and $\eta = \sqrt{\frac{\sqrt{5}+1}{2}}$. The last two, $|(1,3)\rangle$ and $|(2,3)\rangle$, are the Z_2 transforms of $|(1,1)\rangle$ and $|(2,1)\rangle$ respectively, and are simply obtained from them by changing the sign of the coefficients of $|\frac{7}{16}\rangle\rangle_P$ and $|\frac{3}{80}\rangle\rangle_P$, since the corresponding bulk fields are odd under the Z_2.

The duality transformations [12] are easily read off from these expansions. The two bulk fields $\varepsilon = (\frac{1}{10},\frac{1}{10})$ and $\varepsilon'' = (\frac{3}{2},\frac{3}{2})$ are odd, whereas $\varepsilon' = (\frac{3}{5},\frac{3}{5})$ is even; $\sigma = (\frac{3}{80},\frac{3}{80})_P$ is exchanged with $\mu = (\frac{3}{80},\frac{3}{80})_A$, and $\sigma' = (\frac{7}{16},\frac{7}{16})_P$ with $\mu' = (\frac{7}{16},\frac{7}{16})_A$; finally the duality between the boundary conditions reads

$$|(1,2)\rangle_{P,A} \;\leftrightarrow\; \tfrac{1}{\sqrt{2}}[|(1,1)\rangle \pm |(1,3)\rangle], \qquad (13)$$

$$|(2,2)\rangle_{P,A} \;\leftrightarrow\; \tfrac{1}{\sqrt{2}}[|(2,1)\rangle \pm |(2,3)\rangle]. \qquad (14)$$

The complementary models $(\mathcal{A}_{2m}, \mathcal{D}_{\frac{q}{2}+1})$ have a Z_2 internal symmetry for $q \geq 8$, and an S_3 symmetry if $q=6$.

Assume first $q \geq 8$. There are again two sectors, P and A, and correspondingly two sets of exponents,

$$\mathcal{E}_e = \{(r,s) \in [1,2m]\times[1,q-1] \;:\; r,s \text{ odd}\} \cup \{(r,\tfrac{q}{2}) \;:\; r \text{ odd} \in [1,2m]\}, \qquad (15)$$

$$\mathcal{E}_g = \{(r,s) \in [1,2m]\times[1,q-1] \;:\; r \text{ odd, } s \text{ even}\}. \qquad (16)$$

As before an essential requirement is that the duality exchanges Ishibashi states from the two sectors, $|i\rangle\rangle_P \leftrightarrow |j\rangle\rangle_A$. As it must also preserve the scaling dimensions, this actually requires $i=j$ and so $\mathcal{E}_e \cap \mathcal{E}_g$ must be non-empty. That alone excludes the cases $q = 2 \bmod 4$.

In addition, the inclusion $\mathcal{E}_g \subset \mathcal{E}_e$ should hold for a duality to exist. Assume the contrary. The states $|j\rangle\rangle_A$ for those j which are not in $\mathcal{E}_e$ should be left fixed by the duality, and so the Z_2 invariant physical boundary states which have a non-zero projection on them would be permuted among themselves. The values of the coefficients show that this is the case of all invariant states. The duality would then mix the Z_2 invariant states among themselves, and being invertible, would also mix the non-invariant states among themselves, contradicting the exchange of some $|i\rangle\rangle_P$ with some $|j\rangle\rangle_A$.

A simple look at the sets (15) and (16) shows that no value of $q \geq 8$ satisfies the condition $\mathcal{E}_g \subset \mathcal{E}_e$. The remaining two models $q=6$, namely the critical and tricritical 3-Potts models, are the only ones to qualify for a non-trivial duality, and easily checked to have one. The nature of these dualities is somewhat different from that in the Ising model, as they are induced by the Z_3 subgroup of a larger symmetry group S_3.

In order to treat the two models simultaneously, and also to lighten the expressions, we will compute their duality in terms of their parent theory through the coset construction, namely the $\mathcal{D}_4$ model with affine symmetry $\widehat{su}(2)$ at level $k=4$. The way this model descends onto the two 3-Potts models $(\mathcal{A}_4, \mathcal{D}_4)$ (critical) and $(\mathcal{A}_6, \mathcal{D}_4)$ (tricritical) is standard and yields the results in a straightforward fashion.

The $\mathcal{D}_4$ model has an S_3 internal symmetry, for which all twisted torus and cylinder partition functions have been computed in [10]. It has four physical boundary conditions, labelled by the nodes of the $\mathcal{D}_4$ Dynkin diagram, with the state $|4\rangle$ being fully invariant under S_3. The other three are cyclically rotated by the Z_3 subgroup, and each one is left invariant by a Z_2 subgroup. We call r the generator of the Z_3 subgroup, and g, g', g'' the generators of the three conjugate Z_2 subgroups.

The full expansions in terms of the affine Ishibashi states read [10]

$$|1\rangle = \frac{1}{\sqrt[4]{3}}\Big\{|1\rangle\!\rangle_e + |3\rangle\!\rangle_e + |3'\rangle\!\rangle_e + |5\rangle\!\rangle_e + \sqrt[4]{3}(|2\rangle\!\rangle_g + |4\rangle\!\rangle_g)\Big\}, \tag{17}$$

$$|2\rangle = \frac{1}{\sqrt[4]{3}}\Big\{|1\rangle\!\rangle_e + \omega\,|3\rangle\!\rangle_e + \omega^2\,|3'\rangle\!\rangle_e + |5\rangle\!\rangle_e + \sqrt[4]{3}(|2\rangle\!\rangle_{g'} + |4\rangle\!\rangle_{g'})\Big\}, \tag{18}$$

$$|3\rangle = \frac{1}{\sqrt[4]{3}}\Big\{|1\rangle\!\rangle_e + \omega^2\,|3\rangle\!\rangle_e + \omega\,|3'\rangle\!\rangle_e + |5\rangle\!\rangle_e + \sqrt[4]{3}(|2\rangle\!\rangle_{g''} + |4\rangle\!\rangle_{g''})\Big\}, \tag{19}$$

$$|4\rangle = \sqrt[4]{3}\,\Big\{|1\rangle\!\rangle_e + |3\rangle\!\rangle_r + |3\rangle\!\rangle_{r^2} - |5\rangle\!\rangle_e + \frac{1}{\sqrt[4]{3}}(|2\rangle\!\rangle_g + |2\rangle\!\rangle_{g'} + |2\rangle\!\rangle_{g''} - |4\rangle\!\rangle_g - |4\rangle\!\rangle_{g'} - |4\rangle\!\rangle_{g''})\Big\}, \tag{20}$$

where $\omega \neq 1$ is a third root of unity.

One deduces the following duality transformations: the periodic fields 3_e and $3'_e$ are exchanged with the twist fields 3_r and 3_{r^2}, whereas 5_e is odd; in the three pairs $(2_g, 4_g)$, $(2_{g'}, 4_{g'})$ and $(2_{g''}, 4_{g''})$, the fields with label 2 are even and those with label 4 are odd; for the boundary conditions, one has

$$|4\rangle_e \;\leftrightarrow\; (|1\rangle_e + |2\rangle_e + |3\rangle_e)/\sqrt{3}, \quad |4\rangle_g \leftrightarrow |1\rangle_g, \tag{21}$$

$$|4\rangle_r \;\leftrightarrow\; (|1\rangle_e + \omega|2\rangle_e + \omega^2|3\rangle_e)/\sqrt{3}, \tag{22}$$

up to symmetries.

The critical Potts model is simply obtained by juxtaposing an $r = 1, 2$ index to the above $su(2)$ labels to form a Kac label. The known duality transformations are then recovered [13]. The duality of the tricritical 3-Potts model is similar.

The last unitary Virasoro minimal models with a symmetry (Z_2) are the two models $(\mathcal{A}_{10}, \mathcal{E}_6)$ and $(\mathcal{A}_{12}, \mathcal{E}_6)$. They both possess a non trivial duality which comes directly from the duality of the $\mathcal{E}_6$ model with affine symmetry $\widehat{su}(2)$ at level $k = 10$.

To conclude, only six unitary minimal models have a duality, and they are precisely the only six models to be their own orbifold. They all inherit their duality from that of the parent theories, the $\mathcal{A}_3, \mathcal{D}_4$ and $\mathcal{E}_6$ models with an $\widehat{su}(2)$ affine symmetry, which again are the only three $\widehat{su}(2)$ affine theories to be self-orbifold.

The reference [7], which generalizes the torus identity (5) to all unitary minimal models, strongly points to a general relation between a duality and the orbifold construction. The above observations can be repeated for pairs of models which are orbifold of each other, and lead to duality relations. For instance, the orbifold of the $\mathcal{D}_4$ model with affine symmetry $\widehat{su}(2)_4$ under one of its Z_2 subgroup is the $\mathcal{A}_5$ model (same level). One finds the duality relations

$$\frac{1}{\sqrt{2}}(|1\rangle_{\mathcal{A}} + |5\rangle_{\mathcal{A}}) \leftrightarrow |1\rangle_{\mathcal{D}}\,, \quad \frac{1}{\sqrt{2}}(|2\rangle_{\mathcal{A}} + |4\rangle_{\mathcal{A}}) \leftrightarrow |4\rangle_{\mathcal{D}}\,, \quad |3\rangle_{\mathcal{A}} \leftrightarrow \frac{1}{\sqrt{2}}(|2\rangle_{\mathcal{D}} + |3\rangle_{\mathcal{D}})\,, \tag{23}$$

between the boundary conditions of the two models.

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