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Topological Background Fields as Quantum Degrees of Freedom of Compactified Strings

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Abstract

It is shown that background fields of a topological character usually introduced as such in compactified string theories correspond to quantum degrees of freedom which parametrise the freedom in choosing a representation of the zero mode quantum algebra in the presence of non-trivial topology. One consequence would appear to be that the values of such quantum degrees of freedom, in other words of the associated topological background fields, cannot be determined by the nonperturbative string dynamics.

1 Introduction

The study of string theories compactified onto a large variety of spaces with different topologies and geometries has produced profound insights into the nonperturbative properties of these systems, ultimately leading to the discovery of D-branes and the web of dualities relating all known superstring theories to the yet to be constructed underlying M-theory (for reviews and references to the original literature, see Refs. [1, 2]). Even in the simplest framework of flat torus compactification, besides the geometry of the internal manifold, further background fields of a topological character such as an antisymmetric tensor and Wilson lines are introduced.[3] Analogous background fields also exist for more intricate compactifications with richer topology and geometry. Even though one could invoke possible dynamical mechanisms that would lead to non-vanishing expectation values of such background fields given a specific compactification topology, it may seem still somewhat unsettling to have to introduce in some ad-hoc fashion such background fields into a theory which purportedly ought to define the ultimate fundamental framework for all of matter and its quantum interactions.

In this brief note, we wish to point out that in the presence of compactifications of non-trivial topology possessing non-contractible cycles, namely possessing a non-trivial fundamental or first homotopy group, there exist specific quantum degrees of freedom to which classical string theory is oblivious. These quantum degrees of freedom play precisely a role akin to that of topological background fields, independently from the metric tensor specifying the compactified geometry of which the value is presumably determined dynamically. The values of these quantum degrees of freedom are related to a choice of representation of the Heisenberg algebra for the zero mode degrees of freedom of the string vibrating in the compactified dimensions. From that point of view, the situation is somewhat similar to that with spin for a rotationally invariant system: which representation of the rotation group algebra is to be used for the quantised system is matter of the experimental determination of the spin of the physical system under consideration. There is no known dynamical framework which would predict the spin value, say, of the electron. Likewise, values for the topological background fields in string compactifications besides the compactified geometry may possibly not be set through dynamical considerations, but could remain contingent on a choice of representation for specific algebraic structures intrinsic to quantum dynamics in the presence of non-trivial topology.

We shall not present here a detailed discussion for large classes of superstring compactifications, but restrict to the core facts we wish to emphasize by staying within the simplest context of the toroidal compactification of free oriented bosonic strings in Minkowski spacetime. As is well known, from a quantum point of view the oscillation modes of the string correspond to creation and annihilation operators, whereas the position and momentum operators describing the motion of its centre of mass satisfy the Heisenberg algebra. However, as discussed once again in a recent paper[4] even though such results have been known for a long time[5, 6] but not as widely as they would deserve to be, contrary to the situation in the case of Euclidean space, the Heisenberg algebra admits an infinity of unitarily inequivalent representations if the position operator takes its eigenvalues on a manifold with a non-trivial first homotopy group. This is the case for the string centre of mass if some spatial dimensions are compactified on a torus, for instance. The purpose of this note is to account for the existence of these inequivalent representations of the Heisenberg algebra, namely genuine quantum degrees of freedom which decouple from the classical dynamics, in torus compactifications of oriented bosonic strings.[7] Extensions to other classes of superstring theories and/or compactifications should also be of interest.

Our presentation is organised as follows. First the quantum description of the motion of a pointlike object on an arbitrary manifold, and the construction of the inequivalent representations of the corresponding Heisenberg algebra, as described in Ref. [4], are recalled in the next section. Then Sec. 3 particularises the discussion to the case of a Minkowski spacetime with some spatial dimensions compactified onto a torus. Next in Sec. 4 the analysis is applied specifically to the well known canonical quantisation of open and closed oriented bosonic strings on this compactified spacetime (see, *e.g.*, Refs. [1, 2, 8, 9, 10]). Finally Sec. 5 addresses some possible implications of our results.

2 The Heisenberg Algebra

Let us first consider a physical system describing the motion of a pointlike object¹ on an arbitrary connected oriented manifold \mathcal{M} of dimension \mathcal{N} . Units such that $\hbar = 1 = c$ are used throughout.

Classically, within the Hamiltonian setting, the state of the system is described by local configuration space coordinates q^n and their associated canonical conjugate momenta p_n , with the Poisson brackets

$$\{q^n, q^m\} = 0, \quad \{p_n, p_m\} = 0, \quad \{q^n, p_m\} = \delta_m^n, \quad n, m = 1, 2, \dots, \mathcal{N}. \quad (1)$$

Following the usual rules of canonical quantisation, one associates to these classical degrees of freedom linear self-adjoint operators \hat{q}^n and \hat{p}_n acting on the ‘‘Hilbert’’ space of quantum states, satisfying the Heisenberg algebra

$$[\hat{q}^n, \hat{q}^m] = 0, \quad [\hat{p}_n, \hat{p}_m] = 0, \quad [\hat{q}^n, \hat{p}_m] = i\delta_m^n, \quad n, m = 1, 2, \dots, \mathcal{N}. \quad (2)$$

In order to represent the Heisenberg algebra on the Hilbert space, let us assume there exists a basis of eigenvectors $|q\rangle$ of the configuration space or position operators \hat{q}^n ,

$$\hat{q}^n |q\rangle = q^n |q\rangle, \quad (3)$$

in one-to-one correspondence with the points of the manifold \mathcal{M} . Assuming implicitly a positive definite hermitian inner product on the space of quantum states, the normalisation of the position eigenstates,

$$\langle q|q'\rangle = \frac{1}{\sqrt{g(q)}} \delta(q - q'), \quad (4)$$

may always be expressed in terms of a volume \mathcal{N} -form

$$\Omega(q) = \sqrt{g(q)} dq^1 \wedge \dots \wedge dq^{\mathcal{N}}. \quad (5)$$

Position operators act on a quantum state $|\psi\rangle$ according to

$$\langle q|\hat{q}^n|\psi\rangle = q^n \langle q|\psi\rangle, \quad (6)$$

while it may be shown, based on the abstract Heisenberg algebra in (2), that the action of the momentum operators may be parametrised according to

$$\langle q|\hat{p}_n|\psi\rangle = \frac{-i}{g^{1/4}(q)} \left[\frac{\partial}{\partial q^n} + iA_n(q) \right] g^{1/4}(q) \langle q|\psi\rangle \quad (7)$$

¹The possible generalisation of similar considerations to extended objects is an open issue.

in terms of a real closed 1-form

$$A(q) = A_n(q) dq^n, \quad dA(q) = 0. \quad (8)$$

The real forms $\Omega(q)$ and $A(q)$ characterise the considered representation of the Heisenberg algebra.

Under a general change of position eigenstate basis in Hilbert space of the following form,

$$|q\rangle^{(2)} = R(q) e^{i\theta(q)} |q\rangle^{(1)}, \quad (9)$$

namely through a simple change of phase and normalisation both left unspecified by the above considerations, and which thus relates, up to their normalisation, unitarily equivalent representations² of the Heisenberg algebra, the two relevant forms characteristic of any such representation transform as

$$\Omega^{(2)}(q) = \frac{1}{R^2(q)} \Omega^{(1)}(q), \quad A^{(2)}(q) = A^{(1)}(q) + d\theta(q). \quad (10)$$

Hence unitarily inequivalent representations of the Heisenberg algebra are in one-to-one correspondence with the equivalence classes of closed 1-forms differing by exact 1-forms. These classes constitute the 1-cohomology of the manifold \mathcal{M} , which is non-trivial if the manifold is not simply connected. In the case of the simply connected Euclidean configuration space manifold, one recovers the well known Stone–von Neumann theorem stating that for Euclidean spaces, up to arbitrary unitary transformations there exists a single representation of the Heisenberg algebra given by the usual plane wave realisations.

3 The Heisenberg Algebra and Torus Compactification

Let us now particularise our discussion to the motion of a pointlike object on a Minkowski spacetime of total dimension $D = D_1 + D_2$, where D_2 spatial dimensions are compactified onto a Euclidean torus.

To represent this direct product compactified spacetime, first, one has a Minkowski spacetime M^{D_1} of dimension D_1 with coordinates x^μ and metric

$$\eta_{\mu\nu} = \text{diag}(-1, 1, \dots, 1), \quad \mu, \nu = 0, 1, \dots, D_1 - 1, \quad (11)$$

and second, one has a Euclidean space E^{D_2} of dimension D_2 , with cartesian coordinates y^I and Euclidean metric

$$\delta_{IJ} = \text{diag}(1, \dots, 1), \quad I, J = 1, 2, \dots, D_2. \quad (12)$$

On the space E^{D_2} and its dual E^{D_2*} , let us then introduce dual bases $\{e_a^I\}$ and $\{e_I^{*a}\}$, with $e_I^{*a} e_b^I = \delta_b^a$, generating the lattices

$$\Lambda = \{l^I = l^a e_a^I : l^a \in \mathbb{Z}\}, \quad \Lambda^* = \{k_I = k_a e_I^{*a} : k_a \in \mathbb{Z}\}. \quad (13)$$

The torus T^{D_2} is then constructed by identifying points of E^{D_2} of the form $y^I + 2\pi l^I$, with l^I in Λ . Finally the D -dimensional compactified spacetime is the direct product $M^{D_1} \times T^{D_2}$.

²Strictly speaking, a change of normalisation does not define a unitary transformation. However since the normalisation of position eigenstates, parametrised through the volume form, is of no physical relevance, only local phase redefinitions of these eigenstates associated to any 0-form $\theta(q)$ on \mathcal{M} are considered to define unitary equivalent realisations of the Heisenberg algebra.

The compactified spacetime $M^{D_1} \times T^{D_2}$ is not simply connected, and its 1-cohomology is thus non-trivial. Consequently the Heisenberg algebra of the position, \hat{x}^μ, \hat{y}^I , and momentum, $\hat{p}_{x\mu}, \hat{p}_{yI}$, operators admits inequivalent representations on the space of quantum states. Each of these representations may univocally be characterised, on the one hand, say by the canonical unit D -form associated to the Minkowskian geometry

$$\Omega = dx^1 \wedge \dots \wedge dx^{D_1} \wedge dy^1 \wedge \dots \wedge dy^{D_2}, \quad (14)$$

and on the other hand, by a real closed 1-form

$$A = A_I dy^I \quad (15)$$

associated to a constant vector A_I in the unit cell of the dual lattice Λ^* . Introducing a basis of eigenvectors $|x, y\rangle$ of the position operators \hat{x}^μ, \hat{y}^I , with the normalisation

$$\langle x, y | x', y' \rangle = \delta(x - x') \delta(y - y'), \quad (16)$$

the action of the position and momentum operators on a quantum state $|\psi\rangle$ may be expressed as

$$\langle x, y | \hat{x}^\mu | \psi \rangle = x^\mu \langle x, y | \psi \rangle, \quad \langle x, y | \hat{p}_{x\mu} | \psi \rangle = -i \frac{\partial}{\partial x^\mu} \langle x, y | \psi \rangle, \quad (17)$$

and

$$\langle x, y | \hat{y}^I | \psi \rangle = y^I \langle x, y | \psi \rangle, \quad \langle x, y | \hat{p}_{yI} | \psi \rangle = -i \left(\frac{\partial}{\partial y^I} + i A_I \right) \langle x, y | \psi \rangle. \quad (18)$$

Finally it is useful to identify in the Hilbert space a basis of vectors $|p_x, p_y\rangle$ diagonalising the momentum operators $\hat{p}_{x\mu}, \hat{p}_{yI}$. Their configuration space wave functions obey the differential equations

$$-i \frac{\partial}{\partial x^\mu} \langle x, y | p_x, p_y \rangle = p_{x\mu} \langle x, y | p_x, p_y \rangle \quad (19)$$

and

$$-i \left(\frac{\partial}{\partial y^I} + i A_I \right) \langle x, y | p_x, p_y \rangle = p_{yI} \langle x, y | p_x, p_y \rangle, \quad (20)$$

with the general solution

$$\langle x, y | p_x, p_y \rangle = C \exp i(p_x \cdot x + (p_y - A) \cdot y), \quad (21)$$

C being some arbitrary integration constant. The operators $\hat{p}_{x\mu}$ admit real eigenvalues $p_{x\mu} = p_\mu$, while single-valuedness or periodicity conditions on the torus

$$\langle x, y + 2\pi l | p_x, p_y \rangle = \langle x, y | p_x, p_y \rangle, \quad l^I \in \Lambda, \quad (22)$$

restrict the eigenvalues of the operators \hat{p}_{yI} to the quantised spectrum $p_{yI} = k_I + A_I$, where k_I is a vector of the dual lattice Λ^* . Imposing the normalisation conditions

$$\langle p, k + A | p', k' + A \rangle = \delta(p - p') \delta_{kk'} \quad (23)$$

and fixing the arbitrary phase to unity, the configuration space wave functions read

$$\langle x, y | p, k + A \rangle = (2\pi)^{-D/2} V^{-1/2} \exp i(p \cdot x + k \cdot y), \quad (24)$$

where V is the volume of the torus.

4 Torus Compactification of Bosonic Strings

We are now ready to address open and closed oriented bosonic strings on the toroidally compactified spacetime, focusing on the motion of the centre of mass in light of the previous discussion.

Consider a free relativistic oriented bosonic string. Its world-sheet, parametrised by coordinates σ^α ($\alpha = 0, 1$) with $\sigma^0 \equiv \tau$ in $[\tau_1, \tau_2]$ and $\sigma^1 \equiv \sigma$ in $[0, \pi]$, is equipped with an intrinsic metric $\gamma_{\alpha\beta}(\tau, \sigma)$ of signature $(-, +)$, and embedded through functions $x^\mu(\tau, \sigma)$ and $y^I(\tau, \sigma)$ into the compactified spacetime $M^{D_1} \times T^{D_2}$, with total dimension D restricted to the critical value, $D = 26$, for the usual reasons of quantum consistency. The dynamics of the system is determined by the linear Polyakov action

$$S[x, y, \gamma] = -\frac{1}{4\pi\alpha'} \int_{\tau_1}^{\tau_2} d\tau \int_0^\pi d\sigma \sqrt{-\det \gamma} \gamma^{\alpha\beta} [\eta_{\mu\nu} \partial_\alpha x^\mu \partial_\beta x^\nu + \delta_{IJ} \partial_\alpha y^I \partial_\beta y^J], \quad (25)$$

where α' is the Regge slope.

The usual Hamiltonian analysis and canonical quantisation of the system is readily achieved. Local gauge symmetries associated to world-sheet diffeomorphisms and Weyl transformations are partly fixed by choosing the conformal gauge. Results remain identical of course when working in the light-cone gauge.³

4.1 Open strings

Let us first concentrate on the case of open oriented strings. In the conformal gauge, the equations of motion

$$\partial_\tau^2 x^\mu - \partial_\sigma^2 x^\mu = 0, \quad \partial_\tau^2 y^I - \partial_\sigma^2 y^I = 0, \quad (26)$$

together with the Neumann boundary conditions

$$\partial_\sigma x^\mu(\tau, 0) = \partial_\sigma x^\mu(\tau, \pi) = 0, \quad \partial_\sigma y^I(\tau, 0) = \partial_\sigma y^I(\tau, \pi) = 0, \quad (27)$$

lead to the classical solutions

$$x^\mu(\tau, \sigma) = x^\mu + 2\alpha' p_x^\mu \tau + i\sqrt{2\alpha'} \sum_{n \in \mathbb{Z}_0} \frac{1}{n} \alpha_n^\mu e^{-in\tau} \cos n\sigma, \quad (28)$$

$$y^I(\tau, \sigma) = y^I + 2\alpha' p_y^I \tau + i\sqrt{2\alpha'} \sum_{n \in \mathbb{Z}_0} \frac{1}{n} \alpha_n^I e^{-in\tau} \cos n\sigma. \quad (29)$$

At the quantum level, the zero modes \hat{x}^μ, \hat{y}^I and $\hat{p}_{x\mu}, \hat{p}_{yI}$ satisfy a Heisenberg algebra, while the oscillator modes $\hat{\alpha}_m^\mu, \hat{\alpha}_m^I$ correspond to creation or annihilation operators according to whether the integer mode index m is negative or positive, respectively. The space of quantum states is obtained as the tensor product of Heisenberg and Fock representation spaces.

The freedom available in the choice of Heisenberg representation for the compactified zero modes and parametrised by a vector A_I in the fundamental cell of the dual lattice Λ^* , implies that one could consider different types or sectors of such open strings, each characterised by a different value of $A_I(\alpha)$ distinguished by the label α . This situation is reminiscent of that for spinning strings for which, depending on the boundary conditions for the world-sheet spinors, one

³The latter has to be properly defined in the presence of compactified spacelike dimensions, namely the gauge fixed light-cone coordinate must not involve any of the compactified components.

obtains Ramond or Neveu–Schwarz sectors distinguished by the representations of the fermionic Fock algebra, and in particular that of its zero mode sector.

More generally still, by adding Chan–Paton factors to the ends of open strings, we seem to be free to choose independently in each sector (i, j) ($1 \leq i, j \leq N$), possibly further distinguished by the label α , an arbitrary representation of the Heisenberg algebra, characterised by a vector $A_I^{ij}(\alpha)$ in the unit cell of the dual lattice Λ^* .

However a consistent description of string interactions requires charge and momentum conservation. These conservation laws lead, for the annihilation of two, three or more strings, to relations such as

$$A_I^{ij}(\alpha) + A_I^{ji}(\beta) = 0 \pmod{\Lambda^*}, \quad A_I^{ij}(\alpha) + A_I^{jk}(\beta) + A_I^{ki}(\gamma) = 0 \pmod{\Lambda^*}, \quad \dots \quad (30)$$

where the values of the sector labels α , β and γ must be correlated in a fashion to be specified, in relation to the properties of the string interactions being considered.

A general analysis of the consequences of the freedom, of a topological origin, in the choice of quantum degrees of freedom $A_I^{ij}(\alpha)$ for the compactified bosonic zero mode algebra would be of interest. In this note we restrict solely to the simpler setting, namely that in which there is only a single type of open string in each Chan–Paton sector, the label α above distinguishing the choices $A_I^{ij}(\alpha)$ then taking a single value. In that case, the solutions for the vectors A_I^{ij} are of the form

$$A_I^{ij} = A_I^i - A_I^j, \quad i, j = 1, 2, \dots, N, \quad (31)$$

with A_I^i in the fundamental cell of the dual lattice Λ^* .

Physical states are defined by the usual annihilation Viraroso constraints. In particular, the zero mode generator inclusive of the quantum subtraction constant required by conformal invariance,

$$\hat{L}_0 = \alpha' \hat{p}_x^2 + \alpha' \hat{p}_y^2 + \hat{N} - 1, \quad (32)$$

where \hat{N} is the total string level excitation operator, leads to the following mass spectrum

$$\alpha' M^2 = \alpha' (k + A^i - A^j)^2 + N - 1, \quad (33)$$

where k_I is a vector of the dual lattice Λ^* , and N a positive integer specifying the \hat{N} eigenvalue.

4.2 Closed strings

Let us now turn to the case of closed oriented strings. In the conformal gauge, the equations of motion

$$\partial_\tau^2 x^\mu - \partial_\sigma^2 x^\mu = 0, \quad \partial_\tau^2 y^I - \partial_\sigma^2 y^I = 0, \quad (34)$$

together with the periodicity conditions

$$x^\mu(\tau, \pi) = x^\mu(\tau, 0), \quad y^I(\tau, \pi) = y^I(\tau, 0) + 2\pi l^I, \quad (35)$$

where the vector l^I in the lattice Λ parametrises the torus winding sector of the string configuration, possess the following classical solutions

$$x^\mu(\tau, \sigma) = x^\mu + 2\alpha' p_x^\mu \tau + \frac{i}{2} \sqrt{2\alpha'} \sum_{n \in \mathbb{Z}_0} \frac{1}{n} \left[\alpha_n^\mu e^{-2in(\tau-\sigma)} + \bar{\alpha}_n^\mu e^{-2in(\tau+\sigma)} \right], \quad (36)$$

$$y^I(\tau, \sigma) = y^I + 2\alpha' p_y^I \tau + 2l^I \sigma + \frac{i}{2} \sqrt{2\alpha'} \sum_{n \in \mathbb{Z}_0} \frac{1}{n} \left[\alpha_n^I e^{-2in(\tau-\sigma)} + \bar{\alpha}_n^I e^{-2in(\tau+\sigma)} \right]. \quad (37)$$

At the quantum level, in a given winding sector l^I , the zero modes \hat{x}^μ, \hat{y}^I and $\hat{p}_{x\mu}, \hat{p}_{yI}$ satisfy a Heisenberg algebra, while the oscillator modes $\hat{\alpha}_m^\mu, \hat{\alpha}_m^I$ and $\hat{\tilde{\alpha}}_m^\mu, \hat{\tilde{\alpha}}_m^I$ correspond to creation or annihilation operators. The space of quantum states is constructed as the tensor product of Heisenberg and Fock representation spaces.

When accounting for all possible winding sectors of the string, one may apparently freely choose independently in each sector l^I an arbitrary representation of the Heisenberg algebra characterised by a vector $A_I(l)$ in the unit cell of the dual lattice Λ^* .⁴ However once again winding and momentum conservation laws lead, for the annihilation of two, three or more strings, to restrictions such as

$$\begin{aligned} A_I(l_1) + A_I(l_2) &= 0 \pmod{\Lambda^*}, & l_1 + l_2 &= 0, \\ A_I(l_1) + A_I(l_2) + A_I(l_3) &= 0 \pmod{\Lambda^*}, & l_1 + l_2 + l_3 &= 0, \quad \dots \end{aligned} \quad (38)$$

which suggest a general solution of the linear form

$$A_I(l) = B_{IJ} l^J, \quad (39)$$

where the real coefficients B_{IJ} may be viewed as defining some 2-index covariant tensor of the compactified Euclidean space. However, since $A_I(l)$ is defined modulo a vector of the dual lattice Λ^* , B_{IJ} itself is defined modulo a 2-tensor of Λ^* , namely B_{IJ} belongs only to the “fundamental cell” of such 2-tensors. Furthermore B_{IJ} may be decomposed into a symmetric and an antisymmetric part in the two indices I and J .

Among the Virasoro constraints defining physical states, the zero mode generators

$$\hat{L}_0 = \frac{\alpha'}{4} \hat{p}_x^2 + \frac{\alpha'}{4} (\hat{p}_y - \alpha'^{-1} \hat{l})^2 + \hat{N} - 1, \quad \hat{\tilde{L}}_0 = \frac{\alpha'}{4} \hat{p}_x^2 + \frac{\alpha'}{4} (\hat{p}_y + \alpha'^{-1} \hat{l})^2 + \hat{\tilde{N}} - 1, \quad (40)$$

with \hat{N} and $\hat{\tilde{N}}$ the total string level excitation operators for each of the world-sheet chiral sectors, lead to the following mass spectrum

$$\alpha' M^2 = \alpha' (k + Bl)^2 + \alpha'^{-1} l^2 + 2N + 2\bar{N} - 4 \quad (41)$$

as well as the level matching condition

$$N - \bar{N} = k_I l^I + B_{IJ} l^I l^J, \quad (42)$$

where l^I is a winding vector of the lattice Λ , k_I a momentum vector of the dual lattice Λ^* , and N, \bar{N} positive integers.

In order that whatever the winding sector the set of physical states be non-empty, we must further require that the symmetric part of B_{IJ} is a 2-tensor of the dual lattice Λ^* , while since $A_I(l)$ is defined up to vectors in the dual lattice anyway, one may simply restrict B_{IJ} to be purely antisymmetric and thus be lying in the fundamental cell of antisymmetric 2-tensors of Λ^* . Consequently, the level matching condition finally reads

$$N - \bar{N} = k_I l^I. \quad (43)$$

Furthermore, it may easily be shown that such a choice of representation of the Heisenberg algebra, characterised by an antisymmetric tensor B_{IJ} defined modulo the dual lattice Λ^* , is also

⁴Here again we ignore the possibility of still more generality, which would be afforded by including different string sectors $A_I(l; \alpha)$ in the manner discussed previously for the open string case.

consistent with one-loop modular invariance of the partition function of the closed string. Given that our results coincide, as indicated hereafter, with those of closed string torus compactification in presence of a constant background antisymmetric field B_{IJ} for which modular invariance is well established,[3] the same conclusion readily follows in our setting. Any other form chosen for $A_I(l)$ does not seem capable of meeting the requirement of modular invariance necessary for the quantum consistency of interacting strings.[7]

5 Discussion and Perspectives

In this note, general results relevant to the Heisenberg algebra on an arbitrary manifold of non-trivial topology have been applied to the canonical quantisation of open and closed oriented bosonic strings on a toroidally compactified Minkowski spacetime. Taking into account inequivalent representations of the algebra describing the centre of mass of the strings, which then arise from the non-trivial spacetime topology, results of interest have been established, in particular through the physical mass spectra of such string models.

In fact, these spectra are well known in the literature,[1, 2, 3, 8, 9, 10] but they are then obtained by quantising strings interacting with constant external topological fields in the compactified spatial dimensions (whether an antisymmetric tensor or Wilson lines) in addition to the metric background, while choosing the trivial representation for the algebra describing their centre of mass. As a matter of fact, the open string spectrum (33) can be reproduced by coupling the ends of an open string to a constant diagonal $U(N)$ gauge field⁵ $A_I = \text{diag}(A_I^1, \dots, A_I^N)$

$$S_A[x, y, \gamma, g_0, g_\pi] = S[x, y, \gamma] + i \int_{\tau_1}^{\tau_2} d\tau \text{Tr} [K g^{-1} (\partial_\tau + i A_I \partial_\tau y^I) g] \Big|_{\sigma=0}^{\sigma=\pi}. \quad (44)$$

Under T-duality to the D-brane picture, the values for the parameters A_I^i are known to correspond to positions of the D-branes on the compactified torus.[1, 2] In the present setting, these positions are seen to correspond, through T-duality, to quantum degrees of freedom arising from the non-trivial representation theory of the bosonic zero mode Heisenberg algebra in the presence of non-trivial topology. In the same way, the closed string spectrum (41), together with the level matching condition (43), can be reproduced by coupling a closed string to a constant antisymmetric tensor field B_{IJ} [3]

$$S_B[x, y, \gamma] = S[x, y, \gamma] - \frac{1}{4\pi} \int_{\tau_1}^{\tau_2} d\tau \int_0^\pi d\sigma \epsilon^{\alpha\beta} B_{IJ} \partial_\alpha y^I \partial_\beta y^J. \quad (45)$$

Even though reproducing well known results, our approach provides thus a new way to look at the interaction of bosonic strings with these topological background fields (this does not include the background metric field describing the compactified geometry). While these couplings are usually imposed by hand as external constraints with values to be determined presumably through nonperturbative dynamics, here they are rather understood to correspond to new intrinsic quantum degrees of freedom arising from a complete quantisation of the strings, when proper account is given of the freedom in choosing representations for the abstract algebraic

⁵In the literature, the spectrum of an open string in a constant diagonal $U(N)$ gauge field is determined[1, 2] by analogy with that of a particle in a constant $U(1)$ gauge field. In fact, this spectrum follows from the action (44), where K is the highest weight of the fundamental representation of $U(N)$, and $g_0(\tau)$ and $g_\pi(\tau)$ in $U(N)$ describe the Chan–Paton degrees of freedom attached to both ends of the string. See Ref. [11] for a detailed discussion.

structures defining a quantised dynamics. From that point of view, this result also suggests that values for such topological background fields are not to be determined through the dynamics.

Our discussion could be extended in many ways. Here it was presented in the simplest of possible contexts, namely that of open and closed oriented bosonic strings compactified on torii. The generalization to superstrings may appear to be obvious at least for the bosonic zero mode Heisenberg algebra, but a closer look at the issue of fermionic zero modes with regards to the situation in the bosonic sector and manifest world-sheet supersymmetry would seem warranted nevertheless. More general settings than the simple example discussed here, involving different interacting string sectors, could also be investigated thoroughly, along the lines mentioned above. It is to be expected that more general solutions to the conditions in (30) would translate, in the T-dual D-brane picture, to constructions based on orbifold procedures. And finally, extensions to other classes of non-flat compactifications of greater relevance to string phenomenology would also deserve a detailed study, including the case of non-oriented strings.

In parallel to such issues, it could also be of interest to investigate, in the case of open strings, how these quantum degrees of freedom associated to zero modes of non-trivial topology, would translate into quantum degrees of freedom for the closed string channel of one-loop open string amplitudes, and *vice versa*. A preliminary study[7] of that issue has shown that the quantum degrees of freedom discussed in this note for both the open and the closed string sectors are not in direct relationship through this open-closed string duality.

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