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Implications of an arithmetical symmetry of the commutant for modular invariants

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Abstract

We point out the existence of an arithmetical symmetry for the commutant of the modular matrices S and T. This symmetry holds for all affine simple Lie algebras at all levels and implies the equality of certain coefficients in any modular invariant. Particularizing to $\widehat{SU(3)}_k$, we classify the modular invariant partition functions when $k+3$ is an integer coprime with 6 and when it is a power of either 2 or 3. Our results imply that no detailed knowledge of the commutant is needed to undertake a classification of all modular invariants.

1. Introduction.

The classification of modular invariant partition functions remains one of the most challenging problems in two–dimensional conformal field theories. Various techniques have been set up to construct modular invariants (extensions, simple currents, automorphisms, ...), but all are lacking a completeness criterion. A conceptual understanding of the various modular invariants was neatly presented in [1], which puts the aforementioned methods in a unified perspective. However, the proof that a list of invariants is actually complete for a given theory, is a notoriously hard question, as it involves rather difficult linear and number–theoretic problems. As far as theories with an affine Lie symmetry are concerned, a complete classification is presently known only for the affine $SU(2)$ algebra at all levels [2], and is equivalent to the celebrated ADE classification. For rank two algebras, a step forward was taken in [3], where the present authors established the complete list of invariants for theories with an affine $SU(3)$ symmetry, for all levels k such that the height $k+3$ is a prime number. In these cases, it was proved that there are only four invariants at each level. The proof was based on the observation that all matrices in the commutant of S and T are invariant when their indices are simultaneously multiplied by an integer coprime with $k+3$. Because of this symmetry, the classification problem could be reduced to the validity of a crucial arithmetical lemma. The purpose of this note is two–fold. First, we point out that the symmetry alluded to above is present for any algebra at any level, and we show that it forces a series of equalities among the coefficients of any modular invariant, physical or not. This makes an arithmetical approach available in all cases and suggests a possible guideline into the full classification problem. It also provides a powerful tool to numerically investigate high rank algebras. Following this path, we then extend our previous result for $SU(3)$. As it happens, results about the relevant arithmetical problem have appeared in the mathematical litterature, which allow to classify the affine $SU(3)$ partition functions when the height is an integer coprime with 6, and when it is a power of either 2 or 3. We end with some comments about higher $SU(N)$ invariants, based on numerical results.

2. An arithmetical symmetry of the commutant.

The following discussion of the commutant holds for any (untwisted) affine simple Lie algebra, but for the sake of concreteness, we treat in detail the unitary series.

We first recall some basic facts about the representation theory for the chiral $\widehat{SU(N)}_k$ algebra [4]. We let the height of the algebra be $n = k + N$. For each integer height $n \geq N$, there are but a finite number of highest–weight unitary representations, labelled by the strictly dominant weights p of $SU(N)$ which are in the alcôve $B_n = \{p = (a_1, a_2, \ldots, a_{N-1}) \ : \ a_i \geq 1 \text{ and } a_1 + \ldots + a_{N-1} \leq n-1\}$. The number of weights in B_n is equal to $\binom{n-1}{N-1}$. We denote the corresponding (restricted) irreducible characters by $\chi_p(\tau)$. The weights in the alcôve are the "shifted" weights, so that the lowest level of the affine representation $(n; p)$ is an $SU(N)$ representation of highest weight $\lambda = p - \rho$, where $\rho = (1, 1, \ldots, 1)$ is half the sum of the positive roots.

It is well–known that the characters transform in a unitary representation of the modular group of the torus, $PSL(2, Z)$. Under the action of the two generators of this group, we have $\chi_p(\tau + 1) = \sum_{p' \in B_n} T_{p,p'} \chi_{p'}(\tau)$ and $\chi_p(\frac{-1}{\tau}) = \sum_{p' \in B_n} S_{p,p'} \chi_{p'}(\tau)$ with

$$T_{p,p'} = e\Big(\frac{p^2}{2n} - \frac{N^2-1}{24}\Big)\, \delta_{p,p'}, \tag{1.a}$$

$$S_{p,p'} = \frac{i^{N(N-1)/2}}{\sqrt{N n^{N-1}}} \sum_{w \in W} (\det w)\, e\Big(\frac{p \cdot w(p')}{n}\Big), \tag{1.b}$$

where W is the Weyl group of $SU(N)$ and $e(x)$ stands for $\exp{(2i\pi x)}$.

The partition function of a theory with a left–right $\widehat{SU(N)}_k$ symmetry takes the general form

$$Z(\tau, \tau^*) = \sum_{p,p' \in B_n} [\chi_p^*(\tau)]\, N_{p,p'}\, [\chi_{p'}(\tau)]. \tag{2}$$

Consistency of the theory on any torus requires the partition function to be modular invariant [5]. The physical interpretation of $Z(\tau, \tau^*)$ as a partition function demands in addition that the coefficients $N_{p,p'}$ be all non–negative integers, and that $N_{(1,1,\ldots,1),(1,1,\ldots,1)} = 1$. The classification problem is then to provide, for each value of n, the complete list of all functions Z (the "physical" modular invariants) which satisfy:

1. Z is modular invariant,
2. the coefficients $N_{p,p'}$ are non–negative integers,
3. Z is normalized by $N_{(1,1,\ldots,1),(1,1,\ldots,1)} = 1$.

The first condition requires the matrix N to belong to the commutant of S and T:

$$[N, S]_{p,p'} = [N, T]_{p,p'} = 0, \qquad \forall\, p, p' \in B_n. \tag{3}$$

The commutation of N with T implies the following condition:

$$p^2 \neq p'^2 \bmod 2n \qquad \Longrightarrow \qquad N_{p,p'} = 0. \tag{4}$$

The commutant of S and T can be worked out by standard techniques [2]. The affine characters χ_p, originally given for p in B_n, remain well–defined on the whole weight lattice M^*. Under the affine Weyl group, they transform as

$$\chi_{w(p)} = (\det w)\, \chi_p \qquad \text{and} \qquad \chi_{p+nM} = \chi_p, \tag{5}$$

where M is the co–root lattice. Because of the second property, all weights can be taken modulo the lattice nM. The idea is then to consider the redundant set χ_p for $p \in M^*/nM$, as if the first symmetry in (5) did not exist. Their modular transformations are described by simpler matrices $\hat{S}$ and $\hat{T}$ which read

$$\hat{T}_{p,p'} = e\Big(\frac{p^2}{2n} - \frac{N^2-1}{24}\Big)\, \delta_{p,p'} \quad \text{and} \quad \hat{S}_{p,p'} = \frac{i^{N(N-1)/2}}{\sqrt{N n^{N-1}}}\, e\Big(\frac{p \cdot p'}{n}\Big). \tag{6}$$

Because the alcôve B_n is essentially $\Big(M^*/nM\Big)/W$ (up to orbits of length smaller than $N!$ which label characters that are identically zero), the original S and T matrices are recovered upon the folding with the Weyl group:

$$S_{p,p'} = \sum_{w \in W} (\det w)\, \hat{S}_{p,w(p')}, \qquad \text{for} \quad p, p' \in B_n, \tag{7.a}$$

$$S_{w(p),p'} = S_{p,w(p')} = (\det w)\, S_{p,p'}, \tag{7.b}$$

and the same for T. In this way, the first property in (5) is restored. Therefore, in order to compute the most general matrix N in the commutant of S and T, one first looks for the most general matrix $\hat{N}$ in the commutant of $\hat{S}$ and $\hat{T}$ and then folds it with the Weyl group, like in (7).

The construction of the commutant of $\hat{S}$ and $\hat{T}$ has been solved in [6]. Although the commutant computed there is not well suited for practical calculations, it readily displays the symmetry we want to show. (For rank two algebras, a more explicit construction of the commutant, generalizing to all levels the simple matrices used in [3], has been given in [7].)

We start by recalling the construction of the commutant according to the reference [6]. Let $G_n = M^*/nM = Z_n^{N-2} \times Z_{nN}$ for $SU(N)$. (Z_m denotes the congruence classes modulo m. Its multiplicative group will be denoted by Z_m^*.) One considers G_n as a finite Hilbert space with orthonormal basis $|p\rangle$, $p \in G_n$. A set of operators Q^p and P^p are defined by

$$Q^p|p'\rangle = e\big(\frac{p \cdot p'}{n}\big)\,|p'\rangle, \qquad P^p|p'\rangle = |p+p'\rangle, \tag{8.a}$$

$$Q^{n\alpha} = P^{n\alpha} = 1 \quad \text{for any } \alpha \in M. \tag{8.b}$$

The operators $\hat{S}$ and $\hat{T}$ act on G_n by their matrix representations (6). Defining now the new operators $\{k,k'\} = e\big(\frac{k \cdot k'}{2n}\big)\,P^k Q^{k'}$ for pairs $(k,k') \in G_{2n} \times G_{2n}$, the key observation is that $\hat{S}$ and $\hat{T}$ generate an $SL(2,Z)$ by their adjoint action on $\{k,k'\}$

$$\hat{S}^\dagger\{k,k'\}\hat{S} = \{k',-k\} \quad \text{and} \quad \hat{T}^\dagger\{k,k'\}\hat{T} = \{k,k'-k\}. \tag{9}$$

$\hat{S}$ and $\hat{T}$ are represented on the pairs (k,k') by the right multiplication by $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 1 & -1 \\ 0 & 1 \end{smallmatrix}\right)$. Averaging the $SL(2,Z)$ action on $\{k,k'\}$ yields operators in the commutant of $\hat{S}$ and $\hat{T}$

$$I_{\mathcal{O}(k,k')} = \sum_{SL(2,Z_{2nN})} \{ak+ck', bk+dk'\}. \tag{10}$$

(From the definition of $\{k,k'\}$, only the coset $SL(2,Z_{2nN})$ acts non–trivially.) Clearly the operator (10) depends on (k,k') through its orbit $\mathcal{O}(k,k')$. Thus to each orbit of $SL(2,Z_{2nN})$ on $G_{2n} \times G_{2n}$ is associated an element of the commutant of $\hat{S}$ and $\hat{T}$. The collection of all such elements is a generating set for the commutant.

Setting $k = l \bmod G_n$ and $k' = l' \bmod G_n$, the explicit expression of $I_{\mathcal{O}(k,k')}$ depends on (l,l') only, up to the overall phase $e(\frac{k \cdot k'}{2n})$ which we omit:

$$I_{\mathcal{O}(k,k')} = \sum_{SL(2,Z_{2nN})} e\big(\frac{abl^2 + cdl'^2 + 2bcl \cdot l'}{2n}\big)\,P^{al+cl'}\,Q^{bl+dl'}. \tag{11}$$

Because of (8.b), the sum in (11) can be partially worked out and reduced to a sum over $SL(2,Z_{nN})$. The way this is done depends on whether nN is even or odd. When nN is even, we can write

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} 1+nNs & nNt \\ nNu & 1+nNs \end{pmatrix}, \quad s,t,u = 0,1, \tag{12}$$

where the first matrix on the right–hand side of (12) belongs to $SL(2,Z_{nN})$. The summation over s,t,u gives zero unless $Nl^2 = Nl'^2 = 0 \bmod 2$ (automatically satisfied if N is odd), in which case one obtains

$$I_{\mathcal{O}(k,k')} = 8 \sum_{SL(2,Z_{nN})} e\big(\frac{\alpha\beta l^2 + \gamma\delta l'^2 + 2\beta\gamma l \cdot l'}{2n}\big)\,P^{\alpha l+\gamma l'}\,Q^{\beta l+\delta l'}. \tag{13}$$

When nN is odd, we have $SL(2,Z_{2nN}) = SL(2,Z_{nN}) \times SL(2,Z_2)$ and the decomposition (12) changes accordingly. The summation over the $SL(2,Z_2)$ subgroup never vanishes in this case and yields an expression similar to (13).

From (8.a), we find that the matrix elements of $I_{\mathcal{O}(k,k')}$ read (numerical factors neglected)

$$\langle p|I_{\mathcal{O}(k,k')}|p'\rangle = \sum_{SL(2,Z_{nN})} e\Big(\frac{\alpha\beta l^2 + \gamma\delta l'^2 + 2\beta\gamma l\cdot l' + 2(\beta l + \delta l')\cdot p'}{2n}\Big)\,\delta_{p,p'+\alpha l+\gamma l'}. \tag{14}$$

Let us now observe that, for any fixed integer ν coprime with nN, $\left(\begin{smallmatrix}\alpha & \beta\\ \gamma & \delta\end{smallmatrix}\right)$ in $SL(2,Z_{nN})$ is equivalent to $\left(\begin{smallmatrix}\nu^{-1}\alpha & \nu\beta\\ \nu^{-1}\gamma & \nu\delta\end{smallmatrix}\right)$ in $SL(2,Z_{nN})$. This change of variables allows us to replace α, β, γ, δ by $\nu^{-1}\alpha$, $\nu\beta$, $\nu^{-1}\gamma$, $\nu\delta$, which in turn, is equivalent to replacing p,p' by $\nu p,\nu p'$. We thus have

$$\langle p|I_{\mathcal{O}(k,k')}|p'\rangle = \langle \nu p|I_{\mathcal{O}(k,k')}|\nu p'\rangle, \qquad \text{for any } \nu\in Z^*_{nN}. \tag{15}$$

Equation (15) is a symmetry of any matrix $\hat{N}$ in the commutant of $\hat{S}$ and $\hat{T}$, since the operators $I_{\mathcal{O}(k,k')}$ generate it.

The symmetry (15) has a remnant at the folded level. Given two weights p and p' in B_n, we multiply them by an integer ν coprime with nN. We get two weights νp and $\nu p'$ which may or may not belong to B_n, but in any case, after reducing them modulo nM (translational part of the affine Weyl group), there will be two unique (finite) Weyl transformations w_ν and w'_ν which bring them back onto two weights of B_n, say p_ν and p'_ν. That is $p_\nu = w_\nu(\nu p)$ and $p'_\nu = w'_\nu(\nu p')$, both in B_n. (The existence of w_ν and w'_ν is guaranteed by ν being coprime with nN.) Then from the symmetry (15) and the formulae (7) for the folding, we obtain at once that any matrix $N_{p,p'}$ in the commutant of S and T satisfies

$$N_{p_\nu,p'_\nu} = (\det w_\nu)(\det w'_\nu)\,N_{p,p'}, \qquad \text{for any } \nu\in Z^*_{nN}. \tag{16}$$

This simple equation is perhaps our main result, and has far–reaching consequences if $N_{p,p'}$ is to yield a physical modular invariant. Indeed since p, p', p_ν and p'_ν are all in B_n, the two matrix elements entering (16) must be positive, and hence so must be the product of the parities of w_ν and w'_ν for *any* ν in Z^*_{nN}. If not, the matrix elements N_{p_ν,p'_ν} must vanish for all $\nu\in Z^*_{nN}$.

The above arguments, including the construction of the commutant, can be repeated verbatim for any simple Lie algebra. The main difference lies in the structure of M^*/nM as an Abelian group, which determines which factor group of $SL(2,Z)$ is to be summed over in (13). The height is in general defined by $n = k + h$, with h the dual Coxeter number. One obtains that M^*/nM is isomorphic to the following groups

$$A_k \ : \ Z_n^{k-1} \times Z_{n(k+1)}, \tag{17.a}$$

$$B_k \ : \ Z_n^{k-2} \times Z_{2n}^2 \ \ (k \text{ even}) \text{ and } \ Z_n^{k-1} \times Z_{4n} \ \ (k \text{ odd}), \tag{17.b}$$

$$C_k \ : \ Z_{2n}^k, \tag{17.c}$$

$$D_k \ : \ Z_n^{k-2} \times Z_{2n}^2 \ \ (k \text{ even}) \text{ and } \ Z_n^{k-1} \times Z_{4n} \ \ (k \text{ odd}), \tag{17.d}$$

$$E_6 \ : \ Z_n^5 \times Z_{3n} \ ; \quad E_7 \ : \ Z_n^6 \times Z_{2n} \ ; \quad E_8 \ : \ Z_n^8, \tag{17.e}$$

$$F_4 \ : \ Z_n^2 \times Z_{2n}^2 \ ; \quad G_2 \ : \ Z_n \times Z_{3n}. \tag{17.f}$$

From this follows that the summation in (13) is over $SL(2, Z_{ln})$ with the following values of l: $l = 1$ for E_8; $l = 2$ for B_{2k}, C_k, D_{2k}, E_7 and F_4; $l = 3$ for E_6 and G_2; $l = 4$ for B_{2k+1} and D_{2k+1}; $l = k + 1$ for A_k. The other minor difference is that the conditions under which $I_{\mathcal{O}(k,k')}$ in (11) vanishes change, but this does not affect the result. Therefore the relations (16) hold for any simple Lie algebra provided we let ν vary over Z_{ln}^*.

3. The parity theorem for SU(N).

We now make the above result more precise for the unitary series and first indicate how the determinant factors in (16) can be computed in those cases. Let $p = (a_1, a_2, \ldots, a_{N-1})$ be a weight of $SU(N)$. For our purpose, the following basis is actually more convenient than the Dynkin basis. Let $x_i = a_i + a_{i+1} + \ldots + a_{N-1}$ for $i = 1, 2, \ldots, N-1$. We can write p either in the Dynkin basis (round brackets) or in the x–basis (square brackets):

$$p = (a_1, a_2, \ldots, a_{N-1}) \quad \longleftrightarrow \quad \begin{cases} p = [x_1, x_2, \ldots, x_{N-1}], \\ x_i = a_i + a_{i+1} + \ldots + a_{N-1}. \end{cases} \tag{18}$$

The alcôve corresponds to

$$B_n = \{p = [x_1, \ldots, x_{N-1}] \ : \ n > x_1 > x_2 > \ldots > x_{N-1} > 0\}. \tag{19}$$

The virtue of this basis is to make the action of the Weyl group more transparent. If w_i denotes the reflector with respect to the i–th simple root, we obtain that

$$w_i[x_1, \ldots, x_i, x_{i+1}, \ldots] = [x_1, \ldots, x_{i+1}, x_i, \ldots], \quad \text{for } i = 1, \ldots, N-2, \tag{20.a}$$

$$w_{N-1}[x_1, \ldots] = [x_1 - x_{N-1}, x_2 - x_{N-1}, \ldots, x_{N-2} - x_{N-1}, -x_{N-1}]. \tag{20.b}$$

Hence the first $N-2$ reflectors generate all permutations of the x_i labels. The norm of a weight p is also much simpler: $p^2 = (x_1^2 + \ldots + x_{N-1}^2) - \frac{1}{N}(x_1 + \ldots + x_{N-1})^2$.

Let p be an arbitrary weight. From the two properties,

$$[x_1, \ldots, x_k + n, \ldots] = [x_1 + n, x_2, \ldots] \bmod nM, \tag{21.a}$$

$$[x_1 + nN, x_2, \ldots] = [x_1, x_2, \ldots] \bmod nM, \tag{21.b}$$

a set of representatives of M^*/nM is obtained for $x_1 \in Z_{nN}$ and $x_k \in Z_n$ for $k = 2, \ldots, N-1$. Let us write a representative as $p = [\langle x_1 \rangle + jn, \langle x_2 \rangle, \ldots, \langle x_{N-1} \rangle] \bmod nM$ where $\langle x \rangle$ is the residue of x modulo n, between 0 and $n-1$, and $0 \le j \le N-1$. There is a unique Weyl transformation w which maps p onto a weight of B_n if and only if $\langle x_i \rangle \neq \langle x_j \rangle \neq 0$ for all $i \neq j$. We define the parity of p as the determinant of w, $\mathcal{P}(p) = \det w$. The conditions on $\langle x_i \rangle$ ensure that p is not the fixed–point of an odd Weyl transformation, so that $\mathcal{P}(p)$ is well defined.

First, if $j = 0$, there is a permutation π of the reduced labels $\langle x_i \rangle$ such that $\pi(p)$ is in B_n. The permutation π is a Weyl transformation of determinant equal to $\det \pi$. Second, the Coxeter element $U = w_1 w_2 \ldots w_{N-1}$, of determinant $(-1)^{N-1}$, allows to bring j down to 0. Indeed, if $[\langle x_1 \rangle, \ldots, \langle x_{N-1} \rangle]$ is in B_n, then

$$U(p) = [jn - \langle x_{N-1} \rangle, \langle x_1 \rangle - \langle x_{N-1} \rangle, \ldots, \langle x_{N-2} \rangle - \langle x_{N-1} \rangle] \bmod nM \tag{22}$$

is in $B_n + [(j-1)n, 0, \ldots]$. By recurrence we have $U^j(B_n + [jn, 0, \ldots]) \in B_n$. Putting the two pieces together, we obtain for $p = [\langle x_1 \rangle + jn, \langle x_2 \rangle, \ldots, \langle x_{N-1} \rangle] \bmod nM$

$$\pi[\langle x_1 \rangle, \ldots, \langle x_{N-1} \rangle] \in B_n \quad \Longrightarrow \quad U^j \pi(p) \in B_n. \tag{23}$$

Equation (23) easily follows from $\pi(p) = \pi([\langle x_1 \rangle, \ldots, \langle x_{N-1} \rangle]) + [jn, 0, \ldots] \bmod nM$, a consequence of (21.$a$). Note that (23) implies $\mathcal{P}(p + [2n, 0, \ldots]) = \mathcal{P}(p)$ for any weight $p \in M^*$, and therefore also $\mathcal{P}(p + [\ldots, 0, 2n, 0, \ldots]) = \mathcal{P}(p)$ from (21.a). Since U is an even transformation when N is odd, we have the stronger invariance $\mathcal{P}(p + [\ldots, 0, n, 0, \ldots]) = \mathcal{P}(p)$ in this case.

We thus obtain the following algorithm to compute the parity of an arbitrary weight $p = [x_1, \ldots, x_{N-1}]$. First reduce the labels x_i modulo $2n$ and write $x_i = \langle x_i \rangle + \epsilon_i n \bmod 2n$, with $\epsilon_i = 0, 1$. If $\langle x_i \rangle = \langle x_j \rangle$ for some $i \neq j$ or if $\langle x_i \rangle = 0$ for some i, then the parity of p is not defined. (p would label an affine character which is identically zero.) In all other cases, find the permutation π such that $n > \langle x_{\pi(1)} \rangle > \ldots > \langle x_{\pi(N-1)} \rangle > 0$. Then the parity of p is

$$\mathcal{P}(p) = (-1)^{(\epsilon_1 + \ldots + \epsilon_{N-1})(N-1)} \det \pi. \tag{24}$$

With the parity of a weight given in (24), we obtain from (16) the following criterion to decide whether a given coefficient $N_{p,p'}$ in a physical invariant can be non–zero. From (4), we may suppose that the norms of p and p' are equal modulo $2n$. Moreover, since the parity of p depends only on its residue modulo $2n$, it is enough to take ν in Z^*_{2n} (or even in Z^*_n if N is odd). Note however that, in general, p_ν really depends on the residue of ν modulo nN (remember $p_\nu \in B_n$ is the image by an affine Weyl transformation of νp, with p itself in B_n).

Parity theorem for $SU(N)$.

Let $N_{p,p'}$ a matrix describing a physical modular invariant.
Suppose p and p' are two weights in the alcôve B_n such that $p^2 = p'^2 \bmod 2n$. Then for all integers ν coprime with nN, we have $N_{p,p'} = N_{p_\nu,p'_\nu}$. If the two parities $\mathcal{P}(\nu p)$ and $\mathcal{P}(\nu p')$ are not equal for some ν in Z^*_{2n} or Z^*_n for N even or odd respectively, then $N_{p,p'} = 0$.

Although the above condition is weaker than the commutation of N with S, which must be further checked, it is nonetheless extremely restrictive. We have defined an action of the group Z^*_{nN} on the pairs of $B_n \times B_n$, and the theorem states that the coefficients of a physical modular invariant are constant along each orbit. But it is the test on the parities which makes the hard–core of the theorem. It says when and how the symmetry *may* extend, and what the possible couplings between the characters are. Taking for example $p' = (1, 1, \ldots, 1)$ labelling the character of the identity, the list of all p passing the test enumerates the primary fields ϕ_p which may extend the affine $SU(N)$ algebra to a larger one. As we will see in the following sections, that the weights νp and $\nu p'$ have the same parity for all ν in Z^*_{nN} is something rather rare, and consequently a large number of coefficients $N_{p,p'}$ are generally required to vanish.

The relations implied by the theorem are clearly satisfied by the diagonal invariants, but it takes a little check to show that the parity conditions are met for the complementary invariants [8,9]. These are constructed from the outer automorphism μ of $\widehat{SU(N)}$, defined in the Dynkin basis by

$$\mu(a_1, \ldots, a_{N-1}) = (n - \sum a_i, a_1, a_2, \ldots, a_{N-2}) = (n, 0, \ldots, 0) + U(p), \tag{25}$$

and which generates a cyclic group of order N, $\mu^N = 1$. The complementary invariants typically couple p and $\mu^k(p)$ for some k. From (24), one easily checks that

$$\mathcal{P}(\nu\mu^k(p)) = (-1)^{k(\nu+1)(N-1)}\, \mathcal{P}(\nu p). \tag{26}$$

Therefore the coupling $N_{p,\mu^k(p)} \neq 0$ is compatible with the parity test if $k(\nu{+}1)(N{-}1)$ is always even, which it is if N is odd. If N is even, ν is always odd since it must be coprime with nN.

Another example where the relations among the $N_{p,p'}$ can be checked is the exceptional invariant for $SU(3)$, at height $n = 24$ [10]

$$\begin{aligned}E_{24}(\tau,\tau^*) = {}&|\chi_{(1,1)} + \chi_{(5,5)} + \chi_{(7,7)} + \chi_{(11,11)} + \chi_{(1,22)} + \chi_{(22,1)} + \chi_{(5,14)} + \chi_{(14,5)} \\ &+ \chi_{(7,10)} + \chi_{(10,7)} + \chi_{(2,11)} + \chi_{(11,2)}|^2 + |\chi_{(1,11)} + \chi_{(5,7)} + \chi_{(12,1)} + \chi_{(11,12)} \\ &+ \chi_{(12,5)} + \chi_{(7,12)} + \chi_{(11,1)} + \chi_{(7,5)} + \chi_{(1,12)} + \chi_{(12,11)} + \chi_{(5,12)} + \chi_{(12,7)}|^2.\end{aligned} \tag{27}$$

Finally note that for each $\nu \in Z^*_{nN}$, the map $M_\nu : p \to p_\nu$ is invertible on B_n and so defines a permutation of it. However the matrix $(M_\nu)_{p,p'}$ does not qualify to describe a physical invariant unless $\nu = -1$. Indeed the map M_{-1} is just the charge conjugation C acting by $C(a_1, a_2, \ldots, a_{N-1}) = (a_{N-1}, a_{N-2}, \ldots, a_1)$.

4. The case of SU(3) when n is coprime with 6.

The equation (24) gives the parity of any weight in the x–basis. For $SU(3)$ however, the expression is just as easy in the Dynkin basis. If $p = (\langle a\rangle, \langle b\rangle) \bmod n$, then

$$\mathcal{P}(p) = \begin{cases} +1 & \text{if } \langle a\rangle + \langle b\rangle < n, \\ -1 & \text{if } \langle a\rangle + \langle b\rangle > n. \end{cases} \tag{28}$$

If we use the affine Dynkin basis, writing $\tau = (a, b, n-a-b)$, then the equation (28) is equivalent to $\mathcal{P}(\tau) = +1$ or -1 according to whether $\langle a\rangle + \langle b\rangle + \langle n-a-b\rangle = n$ or $2n$. This makes it clear that the parity is invariant under any permutation of the affine Dynkin labels. For this reason, it is better to use the affine weights, that we generically denote by τ. τ is in B_n if its three labels are integers between 1 and $n-1$.

The parity theorem of Section 3 is precisely what was used in [3] to classify the modular invariants of $SU(3)$ when n is a prime number. Choosing $\tau = (1, 1, n-2)$, it was proved that for no weight τ' in B_n are the parities $\mathcal{P}(\nu\tau)$ and $\mathcal{P}(\nu\tau')$ equal for all ν, except for the trivial solutions, namely τ' is a permutation of τ. (The equality of the norms of the two weights was not imposed.) Although in a totally different context, the parity theorem for $SU(3)$, in its full power, was in fact investigated by Koblitz and Rohrlich some fifteen years ago [11]. Their main result is as follows.

Theorem [11]. Let n an integer coprime with 6. Let $\tau = (a, b, n-a-b)$ and

$\tau' = (a', b', n - a' - b')$ two weights of B_n. Then the parities of $\nu\tau$ and $\nu\tau'$ are equal for every $\nu \in Z_n^*$ if and only if τ' is a permutation of τ.

This remarkable result allows the classification for the corresponding heights without looking any further into the details of the commutant.

From the above theorem, the character of the identity representation $\chi_{(1,1)}$ can only couple to itself, $\chi_{(1,n-2)}$ and $\chi_{(n-2,1)}$. The last two possibilities are readily excluded because the norms of the corresponding weights are not equal modulo $2n$ to that of $(1,1)$. Therefore, any partition function looks like $Z(\tau, \tau^*) = |\chi_{(1,1)}|^2 + \ldots$, which shows that the affine $SU(3)$ symmetry does not extend. As to the other weights, the above theorem and the norm condition imply that a weight p can only couple to one of the following possibilities: p itself, $C(p)$, $\sigma(p)$ or to $C\sigma(p)$, where $\sigma(p) = \mu^{nt(p)}(p)$, C is the charge conjugation, μ is the outer automorphism (25) and $t(p)$ is the triality of p. This is a local result, valid for each weight p taken separately, but it is not difficult to show that these couplings must be global as well (see reference [12] for more details). We thus obtain the final result that there are only four modular invariant partition functions, the diagonal and the complementary found in [8], given by

$$N_{p,p'} = \delta_{p',p}, \quad \text{and} \quad N_{p,p'} = \delta_{p',\sigma(p)}, \tag{29}$$

and their C–twisted version, obtained by replacing $N_{p,p'}$ by $N_{p,Cp'}$. For $n = 5$, the second invariant of (29) is identical to the C–conjugate of the first one.

5. Powers of 2 and 3 for SU(3).

Koblitz and Rohrlich also examined the parity theorem when n is a power of 2 or 3. We start with the powers of 2, which is the simplest case. As in the theorem we used in Section 4, they do not impose any condition on the norms of τ, τ'. When n is a power of 2, we do impose such conditions in order to make the statements simpler. We say that the norms of τ and τ' do not match modulo $2n$ if $p^2 \neq p'^2 \bmod 2n$ for any choice of 2–label (*i.e.* non–affine) weights p and p' obtained respectively from τ and τ'.

Set $n = 2^m \geq 16$. The following result has been proved in [11]. Let $\tau = (a, b, n - a - b)$ and $\tau' = (a', b', n - a' - b')$ two weights of B_n with matching norms. Suppose in addition that $\gcd(\tau, \tau') = 1$. Then the parities of $\nu\tau$ and $\nu\tau'$ are equal for every $\nu \in Z_{2m}^*$ if and only if τ and τ' are either permutations of each other, or else permutations, modulo n, of $u(1,1,n-2)$ and $u(\frac{n}{2}-1,\frac{n}{2}-1,2)$ for some $u \in Z_{2m}^*$.

We proceed as follows. First this theorem shows that $(1,1)$ can only couple to itself and to $(\frac{n}{2}-1,\frac{n}{2}-1)$, which signals a possible extension of the symmetry by the field $\phi_{(\frac{n}{2}-1,\frac{n}{2}-1)}$. If $(1,1)$ does not couple to $(\frac{n}{2}-1,\frac{n}{2}-1)$, *i.e.* $N_{(1,1),(n/2-1,n/2-1)}=0$ hence $N_{u(1,1),u(n/2-1,n/2-1)}=0$ by the parity theorem, then the $\widehat{SU}(3)$ symmetry does not extend and, using the same argument as in Section 4, the only invariants are the diagonal and the complementary (same as in (29)), plus their C–conjugates. On the other hand, if $(1,1)$ does couple to $(\frac{n}{2}-1,\frac{n}{2}-1)$, then the invariant is necessarily of the form $Z(\tau,\tau^*)=|\chi_{(1,1)}+\chi_{(n/2-1,n/2-1)}|^2+\ldots$ (because of (16) with $\nu=\frac{n}{2}-1$), and thus involves an extension of the symmetry. We show that this is not compatible with modular invariance unless $n=8$.

In order to do this, we look at another part of N. From the above theorem, the weight $(\epsilon,n-3)$, with $\epsilon=n \bmod 3$, only couples to itself and its conjugate. The corresponding 2×2 blocks of N and S must commute, which yields

$$N_{p,p'}=\begin{pmatrix}\alpha & \beta\\ \beta & \alpha\end{pmatrix}, \qquad \text{for} \quad p,p'\in\{(\epsilon,n-3),\,(n-3,\epsilon)\}, \tag{30}$$

which is also a consequence of (16). If we now enforce the commutation $[N,S]_{p,p'}=0$ for p in $\{(1,1),(\frac{n}{2}-1,\frac{n}{2}-1)\}$ and p' in $\{(\epsilon,n-3),(n-3,\epsilon)\}$, we find that there is no solution for α,β unless the equation $\sin\frac{2\pi}{n}-\sin\frac{6\pi}{n}=0$ holds, which it does for $n=8$ only. Hence for $n\geq 16$, there is no extension of the symmetry and the only physical invariants are the diagonal and the complementary. For $n=8$, it is easy enough to check by hand that there is one exceptional invariant, given by

$$\begin{aligned}E_8(\tau,\tau^*)=&|\chi_{(1,1)}+\chi_{(3,3)}|^2+|\chi_{(1,3)}+\chi_{(4,3)}|^2+|\chi_{(3,1)}+\chi_{(3,4)}|^2\\&+|\chi_{(1,4)}+\chi_{(4,1)}|^2+|\chi_{(2,3)}+\chi_{(6,1)}|^2+|\chi_{(3,2)}+\chi_{(1,6)}|^2.\end{aligned} \tag{31}$$

The invariant (31) is the reduction of the diagonal invariant of $\widehat{SU}(6)$ level 1, into which $\widehat{SU}(3)$ level 5 is conformally embedded [10]. Thus for all heights $n=2^m\geq 16$, there are four modular invariants in terms of the unrestricted characters, and they are given in (29). There are six invariants for $n=8$, and only two for $n=4$.

Finally, we consider the powers of 3, $n=3^m\geq 9$. This case is slightly more difficult since the symmetry always extends. We start by giving the relevant result from [11].

Let $\tau = (a, b, n - a - b)$ and $\tau' = (a', b', n - a' - b')$ two weights of B_n. Then the parities of $\nu\tau$ and $\nu\tau'$ are equal for every $\nu \in Z^*_{3m}$ if and only if τ and τ' are either permutations of each other, or else they are permutations, modulo n, of $u(3^k, \frac{n}{3} - 2 \cdot 3^k, \frac{2n}{3} + 3^k)$ and $u(3^{k+1}, \frac{n}{3} - 2 \cdot 3^k, \frac{2n}{3} - 3^k)$ for some k between 0 and $m - 2$ and some $u \in Z^*_{3^{m-k}}$.

From this result and the equality of the norms, we obtain that the three weights $(1, 1)$, $(n - 2, 1)$, $(1, n - 2)$ can only be coupled among themselves, and the same is true for the two weights (1,2) and (2,1). Imposing $[N, S]_{p,p'} = 0$ for p, p' running over these five weights leads to the following two situations. Either the symmetry does not extend and the only invariants are the diagonal and its C–twisted version [12], or else the symmetry extends and the partition function looks like $Z(\tau, \tau^*) = |\chi_{(1,1)} + \chi_{(n-2,1)} + \chi_{(1,n-2)}|^2 + \ldots$. In the second case, $Z(\tau, \tau^*)$ must be expressible in terms of the combinations $\tilde{\chi}_{(a,b)} = \chi_{(a,b)} + \chi_{(n-a-b,a)} + \chi_{(b,n-a-b)}$ and $\chi_{(\frac{n}{3},\frac{n}{3})}$. Moreover the three weights labelling the characters in $\tilde{\chi}_{(a,b)}$ must have the same norm modulo $2n$, implying that $\tilde{\chi}_{(a,b)}$ cannot appear if (a, b) is a weight with a non–zero triality.

The remaining characters, $\tilde{\chi}_{(a,b)}$ for (a, b) a root and $\chi_{(\frac{n}{3},\frac{n}{3})}$, are the reduced characters of an extended theory $\mathcal{T}$, possessing a larger symmetry than $\widehat{SU(3)}$. ($\mathcal{T}$ contains three different characters $\tilde{\chi}^i_{(\frac{n}{3},\frac{n}{3})}$, $i = 1, 2, 3$, which all reduce to the same affine character.) According to the results of [1], every partition function of $\mathcal{T}$ originates from an automorphism σ of the fusion rules of $\mathcal{T}$ and is necessarily of the form (the sum also includes the $\tilde{\chi}^i_{(\frac{n}{3},\frac{n}{3})}$)

$$Z(\tau, \tau^*) = \sum_{(a,b) \in B_n \cap M} [\tilde{\chi}^*_{(a,b)}(\tau)]\ [\tilde{\chi}_{\sigma(a,b)}(\tau)]. \tag{32}$$

Furthermore σ is a permutation that has to commute with the matrix $\tilde{S}$ of the extended theory.

Let us show that the extra couplings, namely those for which the affine weights τ and τ' are not permutations of each other, must be excluded. Suppose the contrary, namely that the partition function is $Z(\tau, \tau^*) = \ldots + \tilde{\chi}^*_{u(3^k,n/3-2\cdot 3^k)}\ \tilde{\chi}_{u(3^{k+1},n/3-2\cdot 3^k)} + \ldots$ for some k and some u. (The only other possibility is the same coupling with one of the two weights conjugated, but clearly these two cases are both compatible with modular invariance or none of them is. Note that instead of $u(3^k, \frac{n}{3} - 2 \cdot 3^k)$ and $u(3^{k+1}, \frac{n}{3} - 2 \cdot 3^k)$, we should take their residues modulo n, themselves in B_n. It makes no difference in the following.) From the parity theorem, the above coupling cannot depend on u, so we may take $u=1$. Because Z must be of the form (32), the automorphism σ which exchanges $p_1=(3^k,\frac{n}{3}-2\cdot 3^k)$ and $p_2=(3^{k+1},\frac{n}{3}-2\cdot 3^k)$ must leave the extended $\tilde{S}$ matrix invariant. In particular, one must have $\tilde{S}_{(1,1),p_1}=\tilde{S}_{(1,1),p_2}$. But $(1,1)$, p_1 and p_2 are roots (different from $(\frac{n}{3},\frac{n}{3})$) and thus these matrix elements of $\tilde{S}$ are just three times the same matrix elements of S. So we obtain the same condition on the original S matrix, $S_{(1,1),p_1}=S_{(1,1),p_2}$, which explicitly reads

$$\sin\frac{2\pi(\frac{n}{3}-3^k)}{n}-\sin\frac{2\pi 3^k}{n}=\sin\frac{2\pi(\frac{n}{3}+3^k)}{n}-\sin\frac{2\pi 3^{k+1}}{n}. \tag{33}$$

By using the identity $\sin\left(\frac{2\pi}{3}+x\right)+\sin x=\sin\left(\frac{2\pi}{3}-x\right)$, the equation (33) simplifies to $\sin\frac{2\pi 3^{k+1}}{n}=0$, which is impossible on account of the inequalities $0\leq k\leq m-2$. Therefore all these 'exceptional' couplings are ruled out.

At this stage, we have that each extended character $\tilde{\chi}_{(a,b)}$ can couple to itself or to $\tilde{\chi}_{(b,a)}$. This means that, in (32), the automorphism σ maps (a,b) onto itself or onto (b,a), for each root (a,b) separately. However, if it is to commute with S, the charge conjugation C can only act globally, even when it is, like here, restricted to act on the roots of B_n only. (To obtain this, one shows that, for $n=3^m$, the matrix element $S_{(1,4),(a,b)}$ is real if and only if $a=b$, $b=n-2a$ or $a=n-2b$.) Thus the automorphism σ is either the identity or C, leading to two and only two invariants with an extension of the symmetry (they were already found in [9]):

$$Z(\tau,\tau^*)=\sum_{\substack{(a,b)\in B_n\cap M\\ (a,b)\neq(\frac{n}{3},\frac{n}{3})}}|\tilde{\chi}_{(a,b)}(\tau)|^2\;+\;3\,|\chi_{(\frac{n}{3},\frac{n}{3})}|^2 \tag{34}$$

and its C–conjugate. Putting everything together, we obtain, for $n=3^m\geq 9$, four modular invariant partition functions in terms of the unrestricted affine characters. Clearly, for $n=3$, there is only one invariant.

6. Numerical investigation for higher ranks.

In this last section, we report on some numerical results concerning higher $SU(N)$ algebras. The parity theorem of Section 3 is particularly well suited for numerical studies as the computations are at all times performed on a few integer variables (n, ν and the Dynkin labels). It requires no large memory, unlike a systematic search of all physical invariants. We emphasize that the list of couplings $N_{p,p'}$ allowed by the parity theorem does not classify the modular invariants. In particular, the commutation with S must be separately checked. Despite this, its real power is to reveal the heights where something special may be expected.

It is straightforward to run a computer program that examines the parity theorem. It only needs the formulae for the norm and the parity of a weight, both easier in the x–basis (see Section 3). To test the parities, we note that we can let ν range from 1 to n or to $\frac{n}{2}$ for N even or odd respectively, on account of the following property of the parity: $\mathcal{P}((2n-\nu)p) = (-1)^{N(N-1)/2}\,\mathcal{P}(\nu p)$ for N even, $0 < \nu < n$, and $\mathcal{P}((n-\nu)p) = (-1)^{N(N-1)/2}\,\mathcal{P}(\nu p)$ for N odd, $0 < \nu < \frac{n}{2}$.

By choosing $p' = \rho = (1, 1, \ldots, 1)$ in the parity theorem, we have looked for heights n at which an extension of the symmetry can be expected. For $SU(3)$ ($n \leq 500$), $SU(4)$ ($n \leq 150$) and $SU(5)$ ($n \leq 100$) we have listed the weights p in the alcôve such that $p^2 = \rho^2 = \frac{N(N^2-1)}{12} \bmod 2n$ and such that the parities of the ν–multiples of ρ and p coincide for all ν. The results are more conveniently expressed in terms of the orbits of the weights under the automorphism μ of equation (25) and the charge conjugation C. Such orbits are in general of length $2N$. We will say that two orbits are allowed by the parity theorem to couple if each orbit contains one weight allowed to couple with at least one weight in the other orbit. So we look for the orbits which can couple to the orbit of the identity ρ.

In addition to its self–coupling, the orbit of the identity for $SU(3)$ can also couple to that of $(\frac{n}{2}-1, \frac{n}{2}-1)$ when n is divisible by 4. It is easy to check from (24) or (28) that the parity test is satisfied, while the equality of the norms requires $n = 0 \bmod 4$. The surprising outcome of our numerical computation is that these 'regular' couplings are the only ones allowed by the parity theorem for n up to 500, except in two cases, $n = 24$ and $n = 60$.

For $n = 24$, the identity can couple to (the orbits of) (1,1), (5,5), (7,7) and (11,11). All these couplings remain allowed by the commutation with S, and lead to the exceptional invariant of equation (27). It was numerically checked in [13] that there is no other exceptional invariant at that height. The case $n = 60$ is in some respects similar to $n = 24$. We find that the identity can couple to (1,1), (11,11), (19,19) and to (29,29). However here none except the self–coupling survives the commutation with S, so that there is no exceptional extension of the symmetry.

In all other cases, namely $n \neq 24, 60$, the only new extension of the symmetry can only come from $(\frac{n}{2}-1, \frac{n}{2}-1)$ when n is divisible by 4. Using the numerical fact that $(1,4)$ can only couple to its own orbit (not true for $n=24$), we have checked that the extension by $(\frac{n}{2}-1, \frac{n}{2}-1)$ is not compatible with the commutation with S unless $n=8$ or $n=12$, where there is indeed an exceptional invariant E_8 (given in (31)) and E_{12} [10].

From this, we conclude that for $n \leq 500$, the only new modular invariants can only be of the kind found in [1] at $n=12$, *i.e.* associated to automorphisms of the theory extended by the primary fields $\phi_{(1,n-2)}$ and $\phi_{(n-2,1)}$ at height $n = 0 \bmod 3$. Combined to the results of [12], our numerical calculations prove that the list of invariants of [10] is complete for n not divisible by 3 and smaller or equal to 500.

Before leaving $SU(3)$, we would like to make the following remark. It may be desirable to study the parity theorem without imposing the norm requirement, as was done in [11]. However, numerical evidence shows that the resulting classification of couplings (even to (1,1) only) is bound be something rather complicated when n is even. For example, at $n=42$, no less than 12 orbits can couple to the identity if the norm matching condition is dropped, whereas only one remains (the identity orbit with itself) if that condition is reinstalled. Strangely, the norm condition seems to play almost no rôle when n is odd. This pattern strengthens for higher ranks.

Our next example is $SU(4)$ with $n \leq 150$. Here too we find one series of regular couplings when n is even: the orbit of the identity can couple to that of $(\frac{n}{2}-2, 1, \frac{n}{2}-2)$ provided $n \neq 4 \bmod 8$ (for norm reasons). These couplings appear in the exceptional invariants coming from conformal embeddings, at $n = 8, 10$ [14] and $n = 12$ [15]. Apart from this regular series, the situation much depends on whether n is even or odd, as indicated above. When n is odd, we find just one case where the identity can couple to another orbit: at $n = 15$, with the orbit of (1,3,4). If n is even, there are additional couplings to the identity for most even heights up to 90. (They are not many though: their maximal number is 4, attained at $n = 30$.) In the range from 92 to 150 and presumably onwards, the only allowed couplings are those of the above regular series. We have not investigated the question as to whether they yield acceptable extensions.

Finally in the case of $SU(5)$, there are three series of regular couplings, which again appear when n is even: $(1, \frac{n}{2}-2, \frac{n}{2}-2, 1)$ (no condition on n from the norm), $(\frac{n}{2}-3, 1, 1, \frac{n}{2}-3)$ (for $n = 0 \bmod 4$) and $(\frac{n}{2}-3, 2, 2, \frac{n}{2}-3)$ (also for $n = 0 \bmod 4$). (It is straightforward although tedious to check that they all pass the parity test for any n.) As for $SU(3)$ and $SU(4)$, the couplings belonging to these regular series are involved in the exceptional invariants coming from conformal embeddings, in this case at $n = 8$, $n = 10$ [14] and $n = 12$ [15]. As to the other couplings, the situation is very much like in $SU(4)$. When n is odd, there are just two heights with additional couplings: at $n = 15$, the identity can couple to (1,3,4,4), and at $n = 17$, it can couple to (3,3,5,4). For n even, there are generally (many) more couplings (16 more couplings for $n = 42$).

In conclusion, we see that the combined requirements from the norm condition and especially the parity test put severe constraints on the way the characters can be coupled to form a physical modular invariant. This is even more true when the height is an odd integer. The numerical results show that for the $SU(N)$ series, there seems to be a considerable difference between the even and odd heights, as already well illustrated by $SU(2)$ and $SU(3)$. We note that all conformal embeddings of $\widehat{SU(N)}_k$ occur at even heights [16], and that the conjectured list of exceptional invariants due to automorphisms of an extended algebra also contains even heights only [17]. We are not aware of the existence of an exceptional invariant at an odd height, and our results indicate that nothing exceptional is to be expected there.

7. Conclusion.

We have shown that, for any affine simple Lie algebra at any level, the (unfolded) commutant of the modular matrices $\hat{S}$ and $\hat{T}$ possesses an arithmetical symmetry. Any matrix in the commutant is invariant when its two indices are simultaneously multiplied by an integer coprime with ln, n being the height (level plus dual Coxeter number) and l is an algebra dependent integer. At the folded level, this symmetry results in a parity theorem, which says that certain coefficients of a modular invariant must be equal, and gives a condition under which these coefficients must vanish.

This result has two main virtues. It first shows that a precise account of the commutant is not really needed. The parity theorem precisely embodies what we believe is its most important property. Instead, the classification of modular invariants is reduced to the study of the parity theorem, which is a purely arithmetical problem. Secondly, it opens the possibility of a much wider numerical investigation, allowing to study high rank algebras at large levels, a thing which was so far beyond computational feasability.

Using available results relevant to the parity theorem, we classified the $SU(3)$

partitions functions when the height is coprime with 6, and when it is a power of 3 or 2.

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