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LINEARISATION OF UNIVERSAL FIELD EQUATIONS

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Abstract

The Universal Field Equations, recently constructed as examples of higher dimensional dynamical systems which admit an infinity of inequivalent Lagrangians are shown to be linearised by a Legendre transformation. This establishes the conjecture that these equations describe integrable systems. While this construction is implicit in general, there exists a large class of solutions for which an explicit form may be written.

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1. Nonlinear Equations with an Infinity of Conservation Laws.

In a recent series of papers[1,2,3] an investigation of potentially integrable systems in higher dimensions was initiated. The characteristic property of the equations exhibited in those papers is that they arise as the Euler variational equations of an infinite number of inequivalent Lagrangians, which, since they do not involve the fields explicitly, could be construed as providing an infinite number of conservation laws. In the simplest case of just one field, the equation of motion possesses the property of covariance, i.e. any function of a solution is also a solution.

One of the most remarkable properties of this equation which led us to describe it as 'Universal' is that it arises from an arbitrary function $\mathcal{F}(\phi_i)$, homogeneous of degree one in the first derivatives $\phi_i = \frac{\partial\phi}{\partial x_i}$ of a scalar field $\phi(x_i)$ over a manifold of dimension d by an iterative procedure of the following nature. Denote by $\mathcal{E}$ the Euler differential operator

$$\mathcal{E} = -\frac{\partial}{\partial\phi} + \partial_i\frac{\partial}{\partial\phi_i} - \partial_i\partial_j\frac{\partial}{\partial\phi_{ij}}\dots \tag{1.1}$$

(In principle the expansion continues indefinitely but it is sufficient here to terminate at the stage of second derivatives ϕ_{ij}).

Now consider the sequence of iterations;

$$\begin{aligned} &\mathcal{E}\mathcal{F}, \\ &\mathcal{E}\mathcal{F}\mathcal{E}\mathcal{F}, \\ &\mathcal{E}\mathcal{F}\mathcal{E}\mathcal{F}\mathcal{E}\mathcal{F} \quad \text{etc.} \end{aligned} \tag{1.2}$$

This sequence terminates after d iterations by vanishing identically. At the penultimate step the resulting expression set to zero may be regarded as a universal equation of motion; i.e. it is independent of the details of $\mathcal{F}$, and is in fact the equation of motion for the Lagrangian

$$\mathcal{L} = \mathcal{F}\mathcal{E}\mathcal{F}\cdots\mathcal{E}\mathcal{F} \quad (d-1 \text{ factors}). \tag{1.3}$$

This result is subject to the provision that $\mathcal{F}$ is generic, i.e that the Hessian $M_{ij} = \frac{\partial^2\mathcal{F}(\phi_k)}{\partial\phi_i\partial\phi_j}$ is of maximal rank, namely $d-1$. In fact as it is shown in [3] it is even possible to choose different $\mathcal{F}$'s in each factor in (1.3) without affecting the universality of the resulting equation, which takes the form

$$\det\begin{pmatrix} 0 & \phi_1 & \phi_2 & \dots & \phi_d \\ \phi_1 & \phi_{11} & \phi_{12} & \dots & \phi_{1d} \\ \phi_2 & \phi_{12} & \phi_{22} & \dots & \phi_{2d} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \phi_d & \phi_{1d} & \phi_{2d} & \dots & \phi_{dd} \end{pmatrix} = 0. \tag{1.4}$$

Since it arises as the Euler variation of an infinite number of Lagrangians (1.3), it possesses an infinite number of conservation laws, a property which led us to speculate that this equation might be completely integrable. A more symmetric form of this equation, suitable for further generalisation to the multifield case can be obtained by introducing an additional variable x_0, and re-defining $\phi(x_0, x_i)$ as $x_0\phi(x_i)$. Then (1.4) is equivalent to

$$\det\left(\frac{\partial^2\phi}{\partial x_i \partial x_j}\right) = 0 \quad i, j = 0, 1, \ldots, d. \tag{1.5}$$

The primary objective of this paper is to demonstrate that this is indeed the case, by exhibiting a linearisation of (1.4) using the Legendre Transform.[4] This is a transformation which replaces a description of a hypersurface in terms of points by a description in terms of parameters of tangent hyperplanes, and as such is a version of Penrose's well known twistor transform.[5] It has been used in this capacity to linearise the Plebanski equation[6] and also to construct Hyperkähler manifolds.[7]

2. The Legendre Transformation

Suppose a scalar field $\phi(x_i)$ in d dimensional space-time obeys the dynamical equation $f(\phi(x_i), \phi_j, \phi_{jk}) = 0$, where subscripts denote partial derivatives as above. This equation can sometimes be simplified by the use of the Legendre transformation,[4] which involves the introduction of a set of variables $\xi_i,\ w(\xi_i)$ dual to $x_i,\ \phi(x_i)$, in the following sense. Consider for simplicity the case $d = 2$. Then $z = \phi(x, y)$ determines a surface with tangent plane at the point $x_0, y_0, z_0 = \phi(x_0, y_0)$ given by the equation

$$z - z_0 - (x - x_0)\phi_x(x_0, y_0) - (y - y_0)\phi_y(x_0, y_0) = 0. \tag{2.1}$$

Now the general equation of a plane may be specified by three parameters w, ξ, η as follows

$$z - \xi x - \eta y + w = 0. \tag{2.2}$$

Comparing (2.1) and (2.2) it is evident that the conditions such that (2.2) is a tangent plane to the surface at the point x_0, y_0, z_0 are

$$\xi = \phi_{x_0}, \quad \eta = \phi_{y_0}, \quad w = x_0\phi_{x_0} + y_0\phi_{y_0} - z_0. \tag{2.3}$$

Now the surface $z = \phi(x, y)$ is also determined if w is given as a function of ξ, η by which the two parameter family of tangent planes is characterized. So, since x_0, y_0, z_0 is a generic point on the surface, we can drop the subscript and write the conditions as

$$\begin{aligned} \phi(x, y) + w(\xi, \eta) &= x\xi + y\eta, \\ \xi = \phi_x, \qquad & \eta = \phi_y, \\ x = w_\xi, \qquad & y = w_\eta. \end{aligned} \tag{2.4}$$

The last two relations may be obtained by partial differentiation of the first equation with the aid of the second two. This set then demonstrates a duality in the alternative descriptions of the geometry of the situation in terms of point and plane coordinates and (2.4) assigns to every surface element $x, y, \phi, \phi_x, \phi_y$ a surface element $\xi, \eta, w, w_\xi, w_\eta$. This transform, which is clearly involutive has the flavour, as was remarked earlier of a twistor transform. The generalisation to an arbitrary number of independent variables is immediate;

$$\begin{aligned}\phi(x_1, x_2, \ldots, x_d) + w(\xi_1, \xi_2, \ldots, \xi_d) =& x_1\xi_1 + x_2\xi_2 + \ldots, x_d\xi_d. \\ \xi_i = \frac{\partial \phi}{\partial x_j}, \quad x_i =& \frac{\partial w}{\partial \xi_i}, \quad \forall i.\end{aligned} \tag{2.5}$$

To evaluate the second derivatives ϕ_{ij} in terms of derivatives of w it is convenient to introduce two Hessian matrices; $\Phi,\ W$ with matrix elements ϕ_{ij} and $w_{\xi_i\xi_j} = w_{ij}$ respectively. Then assuming that Φ is invertible, $\Phi W = 1\!\!1$ and

$$\frac{\partial^2 \phi}{\partial x_i \partial x_j} = (W^{-1})_{ij}, \quad \frac{\partial^2 w}{\partial \xi_i \partial \xi_j} = (\Phi^{-1})_{ij}. \tag{2.6}$$

Since $\det \Phi \neq 0$, equation (1.4) can be written as

$$\sum_{i,j} \phi_i (\Phi^{-1})_{ij} \phi_j = 0. \tag{2.7}$$

The effect of the Legendre transformation is immediate; in the new variables the equation becomes simply

$$\sum_{i,j} \xi_i \xi_j \frac{\partial^2 w}{\partial \xi_i \partial \xi_j} = 0, \tag{2.8}$$

a linear second order equation for the function w! Indeed, in terms of variables $y_j = \log(\xi_j)$ the equation is linear with constant coefficients, and is thus completely understood. The general solution to this equation is simply

$$w(\xi_i) = v_0(\xi_i) + v_1(\xi_i), \tag{2.9}$$

where $v_0,\ v_1$ are two arbitrary functions, homogeneous of degree zero and one respectively in ξ_i. The solution of the original problem is given implicitly by elimination of ξ_i from the equations

$$\frac{\partial w}{\partial \xi_i} = x_i, \quad \phi(x_k) = \sum_j x_j \xi_j - w(\xi_k) = -v_0(\xi_k). \tag{2.10}$$

Note that the property of covariance of the solution ϕ is a reflection of the arbitrariness of $v_0(\xi_k)$. An explicit solution will not be possible in general, though in particular cases it might be feasible. Note that equations (2.9,2.10) imply

$$\sum_j x_j \frac{\partial \phi(x_i)}{\partial x_j} = v_1(\xi_i). \tag{2.11}$$

Now suppose $v_1(\xi_i) = 0$. This imposes the restriction that $\phi(x_i)$ is a function, homogeneous of degree zero in the variables x_i, but is otherwise arbitrary. That such an explicit function satisfies (1.4) may be verified directly, an observation which already has been recorded in [1]. The Legendre Transform method fails for the choice $v_0(\xi_i) = 0$.

As an illustrative example the case of $d = 2$, the so called Bateman equation will now be considered. In terms of the original variables, this equation takes the form

$$\phi_y^2\phi_{xx} - 2\phi_x\phi_y\phi_{xy} + \phi_x^2\phi_{yy} = 0. \tag{2.12}$$

Under the Legendre transformation (2.4) this equation becomes

$$\xi^2 w_{\xi\xi} + 2\xi\eta w_{\xi\eta} + \eta^2 w_{\eta\eta} = 0. \tag{2.13}$$

This linear equation admits the general solution

$$w = f(\frac{\xi}{\eta}) + (\xi + \eta)g(\frac{\xi}{\eta}), \tag{2.14}$$

where $f,\ g$ are arbitrary functions. Differentiation with respect to $\xi,\ \eta$ yields

$$\begin{aligned} x = w_\xi =& \frac{1}{\eta}(f' + (\xi + \eta)g') + g, \\ y = w_\eta =& -\frac{\xi}{\eta^2}(f' + (\xi + \eta)g') + g. \end{aligned} \tag{2.15}$$

Thus

$$x\xi + y\eta = (\xi + \eta)g = w + \phi, \tag{2.16}$$

giving

$$\phi = -f(\frac{\xi}{\eta}). \tag{2.17}$$

Thus $\frac{\xi}{\eta}$ is an arbitrary function of ϕ. Division of the first relation of (2.16) by η, followed by redefinition, gives the standard construction of a general implicit solution to the Bateman equation which runs as follows: Constrain two arbitrary functions $f_1(\phi), f_2(\phi)$ by the relation

$$xf_1(\phi) + yf_2(\phi) = c \quad \text{(constant)}, \tag{2.18}$$

and solve for ϕ. Then this ϕ solves the Bateman equation. For general d it will not be possible to carry out the explicit elimininination of the auxiliary variables ξ_i except in very special circumstances. The reason that this works here is that the equation (2.19) is parabolic.

This method of solution fails when $\det \Phi = 0$. A large class of evident solutions to (1.4) fall into this category, for example those for which ϕ is a function of all x_i except one. A less trivial example consists of those for which ϕ is given by an extension of (2.17);

$$\sum_{i=1}^{i=d} x_i f_i(\phi) = c \quad \text{(constant)}, \tag{2.18}$$

an implicit functional relation for ϕ in terms of arbitrary functions $f_i(\phi)$.[1] This equation implies the following structure for the second derivatives;

$$\phi_{ij} = \alpha_i \phi_j + \alpha_j \phi_j. \tag{2.19}$$

The precise form of the functions α_i, α_j is not revelant, but is easily found from (2.18). The important point is that (2.19) implies that $\det \Phi = 0$ for $d > 2$.

3. Other Transformable Equations.

It is clear that many other examples of integrable nonlinear equations of second order in field derivatives may now be constructed by reversing the Legendre transformation on a linear equation. Whether these equations also enjoy similar properties to those exhibited in [1,2,3] is a matter for speculation; however it is instructive to consider the case of the equation

$$\phi_t^2 \phi_{xx} - \phi_x^2 \phi_{tt} = 0, \tag{3.1}$$

which results from the substitution of $u(x,t) = -\frac{\phi_t}{\phi_x}$ into the first order differential equation describing nonlinear waves

$$\frac{\partial u}{\partial t} = u \frac{\partial u}{\partial x}. \tag{3.2}$$

(Substitution of $u(x,t) = +\frac{\phi_t}{\phi_x}$ yields the Bateman equation.) This equation can be derived from the Lagrangian $\mathcal{L} = \log \frac{\phi_t}{\phi_x}$, and admits an infinite set of conservation laws of the form

$$\frac{\partial}{\partial t} \frac{\partial}{\partial \phi_t} F(\phi_t \phi_x) - \frac{\partial}{\partial x} \frac{\partial}{\partial \phi_x} F(\phi_t \phi_x) = 0, \tag{3.3}$$

where F is an arbitrary differentiable function of the product $\phi_t \phi_x$ and $\phi(t,x)$ satisfies (3.1). Since (3.2) possesses an infinite number of conservation laws of the form

$$\frac{\partial}{\partial t}(u^n) = \frac{\partial}{\partial x}(\frac{n}{n+1} u^{n+1}), \tag{3.4}$$

where n is arbitrary, there are also independent conservation laws of the form

$$(n+1) \frac{\partial}{\partial t} (\frac{\phi_t}{\phi_x})^n + n \frac{\partial}{\partial x} (\frac{\phi_t}{\phi_x})^{n+1} = 0. \tag{3.5}$$

In fact, this last equation can be written in terms of an arbitrary function $G(\frac{\phi_t}{\phi_x})$ which admits a power series expansion as

$$\frac{\partial}{\partial t} \frac{\partial}{\partial \phi_t} (G(\frac{\phi_t}{\phi_x}) \phi_x) - \frac{\partial}{\partial x} \frac{\partial}{\partial \phi_x} (G(\frac{\phi_t}{\phi_x}) \phi_x) = 0. \tag{3.6}$$

It is curious that both (3.4) and (3.6) both have the form of an Euler variation of a Lagrangian, except for the introduction of a 'Lorentz metric' into the Euler operator which is here

$$\mathcal{E}' = \frac{\partial}{\partial t} \frac{\partial}{\partial \phi_t} - \frac{\partial}{\partial x} \frac{\partial}{\partial \phi_x} \tag{3.7}$$

and $G(\frac{\phi_t}{\phi_x})\phi_x$ is homogeneous of degree one in derivatives of ϕ, just like the Bateman Lagrangian. The equation of motion can then be written in the form

$$\mathcal{E}'\mathcal{L} = \frac{\partial}{\partial t}\frac{\partial}{\partial \phi_t}\mathcal{L} - \frac{\partial}{\partial x}\frac{\partial}{\partial \phi_x}\mathcal{L} = 0, \tag{3.8}$$

with a 'Lagrangian' written as $\mathcal{L} = F(\phi_t\phi_x) + G(\frac{\phi_t}{\phi_x})\phi_x$. The application of the Legendre transform to (3.1) produces the equation

$$\xi^2 w_{\xi\xi} - \eta^2 w_{\eta\eta} = 0, \tag{3.9}$$

with general solution

$$w = f(\xi\eta) + \eta g(\frac{\xi}{\eta}), \tag{3.10}$$

where $f,\ g$ are arbitrary functions of one variable. Note the appearance of a similar functional dependence to that in the conservation laws. The general solution of (3.1) is then obtained from the elimination of $\xi,\ \eta$ from the equations

$$\begin{aligned}\frac{\partial w}{\partial \xi} &= x = \eta f'(\xi\eta) + g'(\frac{\xi}{\eta}),\\ \frac{\partial w}{\partial \eta} &= t = \xi f'(\xi\eta) - \frac{\xi}{\eta}g'(\frac{\xi}{\eta}) + g,\\ \phi &= 2\xi\eta f'(\xi\eta) - f(\xi\eta),\end{aligned} \tag{3.11}$$

The last equation implies that the product $\xi\eta$ is an arbitrary function of ϕ, which might as well be taken as ϕ itself since equation (3.1) possesses the same covariance property as the Universal equation, and any function of a solution is also a solution. From the first pair of equations (3.11) one may deduce that $\frac{\xi}{\eta}x - t$ is an arbitrary function of the ratio $\frac{\xi}{\eta}$, but nothing more without making a specific choice of g.

Another example involves a slight generalisation of (1.4) to the case where the zero in the top left corner is replaced by $q\phi$, where q is a numerical factor. Then the equation (2.7) becomes

$$q\phi - \sum_{i,j}\phi_i(\Phi^{-1})_{ij}\phi_j = 0, \tag{3.12}$$

which translates into

$$(\sum_j \xi_j\frac{\partial}{\partial \xi_j})^2 w - (q+1)\sum_j \xi_j\frac{\partial}{\partial \xi_j}w + qw = 0, \tag{3.13}$$

with general solution of exactly the same form as (2.9)

$$w(\xi_i) = v_q(\xi_i) + v_1(\xi_i), \tag{3.14}$$

where v_q, v_1 are two arbitrary functions, homogeneous of degree q and one respectively in ξ_i, provided $q \neq 1$. The solution of the original equation proceeds in principle by elimination of ξ_i, using (3.13), from

$$\frac{\partial w}{\partial \xi_i} = x_i, \quad \phi(x_k) = \sum_j x_j \xi_j - w(\xi_k) = (q-1) v_q(\xi_k). \tag{3.15}$$

If $q = 1$, then the solution (3.14) requires modification, with attendant consequences for (3.15).

4. Multicomponent Field Generalisation.

In paper [3], a generalisation which was already conjectured in [1], of the Universal Field Equation (1.4) to an arbitrary number of fields, (but fewer than the number of space-time dimensions), was proved. Essentially the trick is to augment the number of space co-ordinates by an additional set u_a equal to the number k of fields $f^a(x_j)$ and write the Universal Equation (1.4) in terms of a master field

$$\phi(u_a, x_j) = \sum_a^k u_a f^a(x_j) \tag{4.1}$$

Then the equation may be writtten

$$\det \begin{pmatrix} 0 & \phi_{u_1 u_2} & \cdots & \phi_{u_1 u_k} & \phi_{u_1 x_1} & \cdots & \phi_{u_1 x_d} \\ \phi_{u_2 u_1} & \phi_{u_2 u_2} & \cdots & \phi_{u_k u_k} & \phi_{u_2 x_1} & \cdots & \phi_{u_2 x_d} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \phi_{u_k u_1} & \phi_{u_k u_2} & \cdots & \phi_{u_k u_k} & \phi_{u_k x_1} & \cdots & \phi_{u_k x_d} \\ \phi_{x_1 u_1} & \phi_{x_1 u_2} & \cdots & \phi_{x_1 u_k} & \phi_{x_1 x_1} & \cdots & \phi_{x_1 x_d} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \phi_{x_d u_1} & \phi_{x_d u_2} & \cdots & \phi_{x_d u_k} & \phi_{x_d x_1} & \cdots & \phi_{x_d x_d} \end{pmatrix} = 0. \tag{4.2}$$

In fact, though the equations have been expressed in this way to emphasise that they are effectively just a particular case of (1.5), when the linear dependence of ϕ on u_a is invoked, the property $\phi_{u_a u_b} = 0, \ \forall a, b$ implies that the the leading $k \times k$ submatrix in (4.2) vanishes, and the first row and column may be re-expressed after re-organising the determinant as $(0, \cdots, 0, \frac{\partial \phi}{\partial x_1}, \cdots, \frac{\partial \phi}{\partial x_d})$. The coefficients of monomials of total degree $d+k$ in the variables u_a in the expansion of the left hand side of (4.2), set individually to zero form an overdetermined set of equations for the description of the fields $f^a(x_j)$. This set of equations is generally covariant, i.e. any set of functions of the solution set is also a solution set.

It is now comparatively easy to adapt the method of section (2) to the solution of this equation. All that is necessary is to introduce a set of $k-1$ variables λ_a, conjugate to the

$u_a,\ a = 2, \ldots, k$ in the same way as the ξ_j are to the x_j. u_1 plays a special role, and is not conjugated. The Legendre Transform becomes

$$
\begin{aligned}
\phi(u_a,\ x_j) + w(\lambda_a,\ \xi_j) &= \sum_2^k u_a \lambda_a + \sum_1^d x_j \xi_j, \\
\lambda_a = \frac{\partial \phi}{\partial u_a}, &\quad u_a = \frac{\partial w}{\partial \lambda_a}, \quad \forall a \neq 1, \\
\xi_i = \frac{\partial \phi}{\partial x_j}, &\quad x_i = \frac{\partial w}{\partial \xi_i}, \quad \forall i.
\end{aligned}
\tag{4.3}
$$

The transform of equation (4.2) is simply (2.8) with modified solution

$$
w(\lambda_a, \xi_i) = v_0(\lambda_a, \xi_i) + v_1(\lambda_a, \xi_i), \tag{4.4}
$$

where $v_0,\ v_1$ are two arbitrary functions, homogeneous of degrees zero and one respectively in ξ_i, but with thus far unrestricted dependence on λ_a. The arbitrariness in dependence on λ is a reflection of the general covariance of the solution for f^a. The information that ϕ is a linear form in the variables u_a must now be imposed upon the implicit solution of the functional relationships (4.3).

The question of the introduction of this constraint complicates the issue as to whether this is a genuinely linearisable problem since the conditions $\phi_{u_a u_b} = 0$ translate into highly nonlinear restrictions on W, namely that the corresponding matrix elements of W^{-1} vanish.

The class of explicit solutions in section 2, may however be trivially extended to the multifield case. All that is necessary is to observe that the choice of $f^a(x_i)$ as a set of arbitrary functions, homogeneous of degree zero in their arguments automatically satisfies (4.2)!

5. Conclusion.

This analysis has demonstrated that the Universal Field Equations proposed in [1,2] which are covariant in the field, or reparametrisation invariant in the base space are linearisable by a Legendre Transform, and thus may be added to the dossier of examples of integrable systems linearisable by a transform method. It thus justifies the hopes for integrability presented in those papers, based upon the existence of an infinite number of conservation laws. (There was a flurry of activity in the mid 60's when such conservation laws were written down for linear systems)[8,9,10,11]

The more general class of overdetermined equations, conjectured in [1,2] and demonstrated to be simply a particular case of the single field in $d + k - 1$ dimensions, and thus potentially integrable, still require a further technical trick before the assertion of integrability can be justified, on account of the difficulty in implementing the requirement of a linear decomposition of ϕ into its component fields. The most interesting case is that for $d = 4,\ k = 2$. Then the equations describe a new reparametrisation invariant string in 4 dimensions, whose world sheet is specified by the intersection of the hypersurfaces $f(x, y, z, t) = 0$ and $g(x, y, z, t) = 0$. More explicitly this set of equations is given by requiring

$$\det\begin{pmatrix} 0 & 0 & f_x & f_y & f_z & f_t \\ 0 & 0 & g_x & g_y & g_z & g_t \\ f_x & g_x & u_1 f_{xx}+u_2 g_{xx} & u_1 f_{xy}+u_2 g_{xy} & u_1 f_{xz}+u_2 g_{xz} & u_1 f_{xt}+u_2 g_{xt} \\ f_y & g_y & u_1 f_{xy}+u_2 g_{xy} & u_1 f_{yy}+u_2 g_{yy} & u_1 f_{yz}+u_2 g_{yz} & u_1 f_{yt}+u_2 g_{yt} \\ f_z & g_z & u_1 f_{xz}+u_2 g_{xz} & u_1 f_{yz}+u_2 g_{yz} & u_1 f_{zz}+u_2 g_{zz} & u_1 f_{zt}+u_2 g_{zt} \\ f_t & g_t & u_1 f_{xt}+u_2 g_{xt} & u_1 f_{yt}+u_2 g_{yt} & u_1 f_{zt}+u_2 g_{zt} & u_1 f_{tt}+u_2 g_{tt} \end{pmatrix} = 0 \tag{5.1}$$

for all choices of $u_1,\ u_2$. It will be interesting to examine solutions either based upon the techniques of this paper, or otherwise. Note the particularly simple class of solutions $f(x,y,z,t),\ g(x,y,z,t)$ as arbitrary homogeneous functions of degree zero in x,y,z,t. The corresponding hypersurfaces mentioned above are now generalised cones.

For a large class of known solutions (2.18), the Legendre Transform method technically fails, because Φ is singular for these solutions. Furthermore, although these equations are linearisable by the Legendre Transform, this does not mean that they are tractable, as the solution so obtained is only implicit. This is by no means unusual in dealing with nonlinear systems; for example the well known ADHM construction[12] of $SU(2)$ self dual Yang Mills instantons is not explicitly solvable in the general case, nor is the corresponding Multi-monopole construction.[13]

The intriguing feature of the Universal Field Equations is that they are derivable by a process of iteration of the Euler variation, for which there is not so far a geometrical motivation. By applying the converse transformation to other linear systems, such as those exemplified in section 3, more understanding of the nature of the higher dimensional integrable systems of this type may be gained.

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