Positronium Spectroscopy
in
a Magnetic Field

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Abstract

Hyperfine spectroscopy of positronium formed in the presence of a static magnetic field is considered. Generalising the situation hitherto developed in the literature, the magnetic field is not assumed to be parallel to the momentum of incoming polarised positrons, while the possibility of electron polarisation is also included in the analysis. The results are of application to high sensitivity positron polarimeters used in current β decay experiments.

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1 Introduction

Hyperfine structure in positronium is a basic physical fact with wide ranging implications. The energy difference between singlet and triplet states, and their lifetimes dominated by disintegrations into two and three photons, respectively, are observables providing crucial tests for quantum electrodynamics (QED)[1] and the Standard Model[2] of electroweak and strong interactions. On a more practical level, positronium physics is also essential in a series of technological developments both in the design of detectors for particle physics experiments and in applications to problems of solid state physics (for a detailed review with references to the original literature, see Refs.[1, 3]).

In particular, hyperfine structure of positronium has been put to use since over thirty years[4] in measurements of positron polarisations. One instance where such experiments come immediately to bear on the structure of fundamental interactions is in the case of positrons emitted in the β decay of radioactive nuclei. Specifically, any deviation from purely right-handed polarisation of positrons emitted in the β decay of polarised nuclei would point to physics beyond the Standard Model in its electroweak sector. Actually, experiments with this purpose in mind and using such a positronium based positron polarisation measurement technique, have been pursued at our Institute[5, 6, 7]. Results are promising[5, 6], and compete well with limits on physics beyond the Standard Model obtained from experiments at much higher energies.

Typically in such β decay experiments, one is interested in measuring the longitudinal polarisation of positrons emitted parallel or antiparallel to the direction in which the decaying nucleus is polarised. The emitted positron is stopped in some specific medium where positronium is formed. The medium being placed in a strong magnetic field[1], the decay spectrum of positronium formed provides information[4] on the polarisation of the incoming positron. The analysis of actual experimental positronium spectra usually assumes that, at the location where positronium is formed, the applied magnetic field is *exactly* parallel to the direction in which the positron is emitted. Since, strictly speaking, such an assumption can never be correct in practice, the more general situation has to be considered, namely when the direction of the magnetic field is arbitrary with respect to the positron momentum. However, the present author has been unable to find in the literature[3, 4, 9, 10] a detailed discussion of this point, hence this note addressing the problem specifically.

In sect.2, a brief review of positronium hyperfine structure in the absence of any external electromagnetic field is presented, with also the purpose of specifying our notation. Sect.3 develops the discussion of the positronium ground state and its hyperfine structure in the presence of an arbitrary static magnetic field. Sects. 4 and 5 consider lifetimes and time evolution of hyperfine populations, respectively, while sect.6 ends with some conclusions.

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[1]Since positrons and electrons have opposite electric charges, an external *electric* field does not affect positronium states nor their *intrinsic* decay rates (however, "pick-up" processes of external electrons or other "quenching" effects can affect these rates[3]). Nevertheless, an external electric field can indeed improve[8] the positronium formation rate in a given medium.

2 Hyperfine Structure of Positronium

As is well known, the $1S$ positronium ground state is in fact split into two hyperfine levels of total spin $S = 0$ and $S = 1$. Their energy difference is due to spin-spin interactions between the positron and the electron, to relativistic corrections to their kinetic energies and to virtual pair annihilation in the $S = 1$ channel[2].

A non relativistic representation of the associated wave functions is sufficient for our purposes, as well as being justified. Accordingly, wave functions separate into space and spin components, with the space component being identical for both hyperfine states. Namely, the $S = 0$ singlet or parapositronium state is given by a wave function of the form[3]

$$\mid 0, 0 >= \psi(r)\ \frac{1}{\sqrt{2}}\,[\ |+>|->-|->|+>\ ]\ . \tag{1}$$

Here, $\psi(r)$ is the space wave function of the $1S$ ground state, while the second factor in the r.h.s. of this expression is the spin wave function. The convention used throughout is that in the product of two spin ket vectors, the first always stands for the electron spin component whereas the second stands for the positron spin component. These components are taken with respect, say to the z axis of some reference frame. Later on, this axis will of course correspond to the direction in which the β decay positron is emitted. It is also assumed that the space wave function $\psi(r)$ is properly normalised,

$$4\pi\ \int_0^\infty dr\ r^2\ \mid \psi(r)\mid^2 = 1\ , \tag{2}$$

and that the basis vectors $|+>$ and $|->$ in spin space both for the electron and for the positron are normalised in the usual way, namely

$$<+|+> = 1 = <-|->\ ,\quad <+|-> = 0 = <-|+>\ \ . \tag{3}$$

Consequently, the $\mid 0, 0 >$ state in (1) is also of norm 1.

Similarly, the three components $m = \pm 1, 0$ of the $S = 1$ triplet orthopositronium state are given by the normalised state vectors

$$\begin{aligned} \mid 1, m = 1 >&= \psi(r)\ |+>|+>\ , \\ \mid 1, m = -1 >&= \psi(r)\ |->|->\ , \\ \mid 1, m = 0 >&= \psi(r)\ \frac{1}{\sqrt{2}}\,[\ |+>|->+|->|+>\ ]\ . \end{aligned} \tag{4}$$

The hyperfine states in (1) and (4) are eigenstates of the total Hamiltonian H_0 of the system in the absence of an external magnetic field. In the non relativistic limit, this Hamiltonian is comprised of the ordinary Schrödinger type Hamiltonian $H_{(C)}$—which, apart from the usual kinetic term, only includes the Coulomb interaction between the

[2]The latter effect is absent in the $S = 0$ channel due to charge conjugation. A photon has $C = -1$, whereas the $S = 0$ and $S = 1$ states have $C = +1$ and $C = -1$, respectively.

[3]Note that a compactified notation for state vectors is used throughout, whereby their isotropic radial dependence is not displayed explicitly.

two oppositely charged particles—to which diverse relativistic corrections are added. The latter contributions correspond to spin-spin interactions between the electron and the positron, to relativistic corrections to their kinetic energies, to virtual pair annihilation effects for orthopositronium states and to further QED radiative corrections. Restricting for a moment the discussion to the non relativistic purely Coulomb Hamiltonian, the singlet and triplet states above are degenerate eigenstates of $H_{(C)}$ with energy[4]

$$E_{(C)} = -\frac{1}{4}\alpha^2 mc^2 = -6.803 \text{ eV} \ . \tag{5}$$

In fact, in this purely Coulomb limit, the space wave function is simply

$$\psi_{(C)}(r) = \left(\pi a^3\right)^{-1/2} \ e^{-r/a} \ , \tag{6}$$

with the positronium radius

$$a = \frac{2\hbar}{\alpha mc} = 1.0584 \text{ Å} \ . \tag{7}$$

Even though the *complete* wave function $\psi(r)$ in (1) departs from the simple radial dependence in (6), the parameter a in (7) gives a measure of the spatial extension of the positronium ground state.

Relativistic effects just mentioned lift the singlet-triplet degeneracy[5] according to the eigenvalues

$$H_0 \mid 0,0 >= E_0 \mid 0,0 > \ , \quad H_0 \mid 1, m=\pm 1, 0 >= E_1 \mid 1, m=\pm 1, 0 > \ , \tag{8}$$

with energies[3]

$$E_0 = \left\{ -\frac{1}{4} + \alpha^2 \left[-\frac{1}{4} - \frac{5}{64} \right] + \mathcal{O}(\alpha^3, \alpha^2 \ln \alpha^{-1}) \right\} \ \alpha^2 mc^2 \ , \tag{9}$$

$$E_1 = \left\{ -\frac{1}{4} + \alpha^2 \left[\frac{1}{12} - \frac{5}{64} + \frac{1}{4} \right] + \mathcal{O}(\alpha^3, \alpha^2 \ln \alpha^{-1}) \right\} \ \alpha^2 mc^2 \ , \tag{10}$$

and the hyperfine difference[13, 14]

$$\begin{aligned} \Delta E = E_1 - E_0 \ &= \left[\tfrac{7}{3} - \tfrac{\alpha}{\pi} \left(\tfrac{32}{9} + 2 \ln 2 \right) + \tfrac{5}{6} \alpha^2 \ln \alpha^{-1} + \mathcal{O}(\alpha^2) \right] \ \tfrac{1}{4} \alpha^4 mc^2 \\ &= 8.41 \times 10^{-4} \text{ eV} \ . \end{aligned} \tag{11}$$

Except for their first term corresponding to the purely Coulomb contribution (5), the different contributions of order α^4 in (9) and (10) are as follows. Both in E_0 and in E_1, the first such contribution is that of the spin-spin interaction energy, while the

[4]Throughout, α, m and c of course stand for the fine structure constant, the electron and positron mass and the speed of light, respectively. Numerical values for these parameters are from Ref.[11].

[5]In the purely Coulomb situation of (5), the singlet-triplet degeneracy is due[12] to a dynamical $SO(4)$ symmetry explicitly broken by effects now considered. Nevertheless, the $S = 1$ triplet states $m = \pm 1, 0$ remain degenerate since, in the absence of external electromagnetic fields, the total Hamiltonian H_0 is invariant under rotations in space.

second—common to both expressions—is that due to relativistic corrections to the total kinetic energy. Finally, the third term of order α^4 in E_1 follows from virtual pair annihilation in the orthopositronium triplet state.

To conclude, let us consider positronium lifetimes. Due to charge conjugation properties, the singlet state can only decay into an even number of photons and the triplet states into an odd number (beginning of course with three photons). Therefore in very good approximation, parapositronium decays predominantly into two photons and orthopositronium into three photons, since compared to each of these processes, the rate for any further photon pair emission is suppressed each time by an additional power of α^2. Hence, only 2γ and 3γ decay processes are considered in this note, and these two channels are assumed to encompass all possible decay modes of positronium.

The associated lifetimes, including radiative corrections, have been computed within QED[15, 16]. For the singlet state, one has the 2γ decay rate

$$\begin{aligned}\lambda_S &= \tfrac{1}{2}\alpha^5\tfrac{mc^2}{\hbar}\ \left[1-\tfrac{\alpha}{\pi}\left(5-\tfrac{\pi^2}{4}\right)+\tfrac{2}{3}\alpha^2\ln\alpha^{-1}+\mathcal{O}(\alpha^2)\right] \\ &= (0.125209\times10^{-9}\ \mathrm{s})^{-1} = 7.98665\times10^{9}\ \mathrm{s}^{-1}\ .\end{aligned} \tag{12}$$

Similarly for triplet states, their 3γ decay rate is

$$\begin{aligned}\lambda_T &= \tfrac{2}{9\pi}\alpha^6\tfrac{mc^2}{\hbar}(\pi^2-9)\ \left[1-(10.266\pm0.008)\tfrac{\alpha}{\pi}-\tfrac{1}{3}\alpha^2\ln\alpha^{-1}+\mathcal{O}(\alpha^2)\right] \\ &= (142.074\times10^{-9}\ \mathrm{s})^{-1} = 7.03859\times10^{6}\ \mathrm{s}^{-1}\ .\end{aligned} \tag{13}$$

Note the rather large ratio

$$\frac{\lambda_S}{\lambda_T}\ =\ 1134.695\ . \tag{14}$$

3 Coupling to a Magnetic Field

Let us now consider positronium states formed in the presence of some external static magnetic field $\vec{B}=(B_x,B_y,B_z)$. For all practical purposes, certainly always realised in actual experimental conditions, it will be assumed that whenever $\vec{B}(\vec{x})$ might have a non vanishing gradient, this gradient is nevertheless negligible on the scale of the spatial extension of the positronium state, namely

$$a\,\frac{\mid\vec{\nabla}B_i\mid}{\mid\vec{B}\mid}\ll 1\ ,\quad \text{for all } i=x,y,z\ . \tag{15}$$

Here, a is the positronium radius of (7). Effectively, one may then consider the magnetic field $\vec{B}$ not only to be static but also to be constant, which is thus the situation to be assumed in the analysis developed in this note. Furthermore, the field $\vec{B}$ is not taken to be necessarily parallel to the z axis with respect to which spin eigenstates were defined in the previous section, since in practical applications, the latter axis is often defined by the positron momentum instead.

The presence of the magnetic field $\vec{B}$ induces an additional interaction term in the total Hamiltonian for the positronium system. The total Hamiltonian H now includes the previous Hamiltonian H_0 with its lowest energy eigenstates in (1) and (4), to which the magnetic coupling to magnetic moments is added, namely

$$H_B = -\vec{\mu}_- \cdot \vec{B} - \vec{\mu}_+ \cdot \vec{B} \ . \tag{16}$$

Here, $\vec{\mu}_-$ and $\vec{\mu}_+$ are the electron and positron magnetic moments, respectively. In terms of their spin operators $\vec{\sigma}_-/2$ and $\vec{\sigma}_+/2$, respectively, we have[6]

$$\vec{\mu}_\pm = \mp\mu\frac{\vec{\sigma}_\pm}{2} \ . \tag{17}$$

The magnetic dipole moment μ is given by

$$\mu = g\ \frac{e\hbar}{2m} \ , \tag{18}$$

with the gyromagnetic factor[17, 18, 19, 20, 1]

$$g = 2\left\{1 + \frac{\alpha}{2\pi} + \left[\frac{3}{4}\zeta(3) - 3\zeta(2)\ln 2 + \frac{1}{2}\zeta(2) + \frac{197}{144}\right]\left(\frac{\alpha}{\pi}\right)^2 + \mathcal{O}\left(\left(\frac{\alpha}{\pi}\right)^3\right)\right\} \ . \tag{19}$$

Therefore, the magnetic energy contribution to the total Hamiltonian reads

$$H_B = \frac{1}{2}\mu(\vec{\sigma}_- - \vec{\sigma}_+) \cdot \vec{B} \ . \tag{20}$$

To complete this list of notations, it turns out that the parameter setting the physical scale of magnetic fields in the present system, is the combination

$$\frac{2\mu}{\Delta E} = \frac{1}{3.628575\ \text{Tesla}} \ , \tag{21}$$

leading to the definition of the *positive* quantity

$$x = \frac{2\mu}{\Delta E} \mid \vec{B} \mid = \frac{\mid \vec{B} \mid}{3.63\ \text{Tesla}} \ . \tag{22}$$

In addition, it proves convenient[7] to introduce the following combinations of the B_x and B_y components of the magnetic field $\vec{B}$,

$$B_+ = \frac{B_x + iB_y}{\sqrt{2}} \ , \quad B_- = \frac{B_x - iB_y}{\sqrt{2}} \ . \tag{23}$$

Given the total Hamiltonian

$$H = H_0 + H_B \ , \tag{24}$$

[6]Here, $\vec{\sigma}$ are the usual Pauli matrices defining the spin 1/2 representation of the (double covering $SU(2)$) of the rotation group $SO(3)$ in three dimensions.

[7]The actual reason why these definitions are convenient is the fact that it is the spin 1/2 representation of the three dimensional rotation group which appears throughout.

it is now a simple matter to proceed diagonalising it for its lowest energy states with spherical symmetry, namely in the sector of positronium $1S$ hyperfine states of the previous section. First, one easily finds

$$H \mid 0,0 >= E_0 \mid 0,0 > +\mu B_z \mid 1,0 > -\mu B_- \mid 1,1 > +\mu B_+ \mid 1,-1 > , \tag{25}$$

$$H \mid 1,0 >= E_1 \mid 1,0 > +\mu B_z \mid 0,0 > , \tag{26}$$

$$H \mid 1,1 >= E_1 \mid 1,1 > -\mu B_+ \mid 0,0 > , \tag{27}$$

$$H \mid 1,-1 >= E_1 \mid 1,-1 > +\mu B_- \mid 0,0 > \ . \tag{28}$$

Given these results, eigenstates of H and their eigenvalues can be derived after some work. The corresponding four eigenstates are denoted $\mid \psi_{S'} >$, $\mid \psi_{T'} >$ and $\mid \psi_\pm >$. In the limit of vanishing magnetic field $\vec{B}$, these states reduce—possibly up to some phase—to the singlet $\mid 0,0 >$, the triplet $\mid 1,0 >$ and the triplet $\mid 1,\pm 1 >$ states, respectively, hence the notation. In particular, the $\mid \psi_{S'} >$ and $\mid \psi_{T'} >$ states are referred to as the "pseudo-singlet" and "pseudo-triplet" states, respectively.

The pseudo-singlet state is given by

$$\begin{aligned} \mid \psi_{S'} >= & \ \tfrac{1}{\sqrt{2}}\,(1+x^2)^{-1/4}(\sqrt{1+x^2}+1)^{-1/2} \times \Big\{ (\sqrt{1+x^2}+1) \mid 0,0 > - \\ & -\tfrac{2\mu}{\Delta E} B_z \mid 1,0 > + \tfrac{2\mu}{\Delta E} B_- \mid 1,1 > - \tfrac{2\mu}{\Delta E} B_+ \mid 1,-1 > \Big\} \ , \end{aligned} \tag{29}$$

with the eigenvalue

$$E_{S'} = -\frac{1}{2}\Delta E \ \left[\sqrt{1+x^2}-1\right] \ + E_0 \ . \tag{30}$$

The pseudo-triplet state is

$$\begin{aligned} \mid \psi_{T'} >= & \ \tfrac{1}{\sqrt{2}}\,(1+x^2)^{-1/4}(\sqrt{1+x^2}-1)^{-1/2} \times \Big\{ (\sqrt{1+x^2}-1) \mid 0,0 > + \\ & +\tfrac{2\mu}{\Delta E} B_z \mid 1,0 > - \tfrac{2\mu}{\Delta E} B_- \mid 1,1 > + \tfrac{2\mu}{\Delta E} B_+ \mid 1,-1 > \Big\} \ , \end{aligned} \tag{31}$$

with the eigenvalue

$$E_{T'} = +\frac{1}{2}\Delta E \ \left[\sqrt{1+x^2}+1\right] \ + E_0 \ . \tag{32}$$

Finally, the remaining two states are

$$\begin{aligned} \mid \psi_\pm >= & \ \tfrac{1}{\sqrt{2}}\,\tfrac{\sqrt{B_x^2+B_y^2}}{|\vec{B}|} \ \mid 1,0 > + \\ & + \tfrac{1}{\sqrt{2}}\,\tfrac{B_z \pm |\vec{B}|}{|\vec{B}|}\,\tfrac{B_-}{\sqrt{B_x^2+B_y^2}} \ \mid 1,1 > + \tfrac{1}{\sqrt{2}}\,\tfrac{-B_z \pm |\vec{B}|}{|\vec{B}|}\,\tfrac{B_+}{\sqrt{B_x^2+B_y^2}} \mid 1,-1 > , \end{aligned} \tag{33}$$

with the degenerate eigenvalue

$$E_\pm = \Delta E + E_0 = E_1 \ . \tag{34}$$

Here, upper (resp. lower) signs in the r.h.s. of (33) correspond to the state $\mid \psi_+ >$ (resp. $\mid \psi_- >$).

By construction, the four states just given not only diagonalise the total Hamiltonian H, but are also orthonormalised. Namely, these states are orthogonal by pairs and are normalised to 1. Of course, orthogonality is automatic for non degenerate states but not for degenerate ones; by construction, the states $\mid \psi_+ >$ and $\mid \psi_- >$ given above do indeed have a vanishing inner product.

Before commenting on these expressions, it is useful to consider the limit in which the magnetic field $\vec{B}$ is parallel to the z axis, namely when $B_x = 0$ and $B_y = 0$. Of course, in this limit, the eigenvalues of the four states above remain unchanged, since their energies only depend on the variable x, *i.e.* the norm of the magnetic field. However, the states specified above, which diagonalise the total Hamiltonian H, then reduce to

$$\mid \psi_{S'} >= \cos\theta \mid 0,0 > - \sin\theta \mid 1,0 > \ , \tag{35}$$

$$\mid \psi_{T'} >= \eta \ \{ \ \sin\theta \mid 0,0 > + \cos\theta \mid 1,0 > \} \ , \tag{36}$$

$$\mid \psi_{\pm} >= \pm e^{\pm i\eta\omega} \ \mid 1, \pm\eta > \ . \tag{37}$$

In these expressions, $\eta = \text{sign}(B_z)$ is the sign of B_z, ω is some arbitrary phase whose value is dependent on the manner in which the limit $B_x = 0, \ B_y = 0$ is taken[8], and the mixing angle θ is defined by

$$\cos\theta = \frac{1}{\sqrt{2}} \sqrt{1 + \frac{1}{\sqrt{1+x^2}}} \ , \quad \sin\theta = \eta \ \frac{1}{\sqrt{2}} \sqrt{1 - \frac{1}{\sqrt{1+x^2}}} \ . \tag{38}$$

Of course, these expressions coincide with those usually found in the literature[4, 10, 3], in which case it is customary to take $\mid \psi_{\pm} >=\mid 1, \pm 1 >$ and the z axis along the magnetic field $\vec{B}$, namely $\eta = +1$. In particular, note that in the limit of an *infinite* magnetic field $\vec{B}$, the pseudo-singlet and pseudo-triplet states in (35) and (36) further reduce to (in the notation of (1))

$$\mid \psi_{S'} >= -\eta \ \psi(r) \mid -\eta >\mid +\eta > \ , \quad \mid \psi_{T'} >= +\eta \ \psi(r) \mid +\eta >\mid -\eta > \ . \tag{39}$$

In order to comment on the physical significance of these results, let us first consider the case where the magnetic field is *parallel* to the z axis. Even though the vector $\vec{B}$ then explicitly breaks rotational invariance of the positronium system in vacuum, there still remains the symmetry of arbitrary rotations around the z axis. Consequently, the spin projection m on that axis still defines a good quantum number for positronium states (of vanishing angular momentum.). Therefore, the states $\mid 1,1 >$ and $\mid 1,-1 >$ must remain eigenstates of the total Hamiltonian, whereas the other two states with $m = 0$, namely $\mid 0,0 >$ and $\mid 1,0 >$, are now allowed to mix. Moreover, for the former two states with $m = \pm 1$, since the electron and positron have their spins then aligned and since their magnetic moments are equal in norm but opposite, the magnetic coupling energy H_B vanishes identically, leading for these two states to the same energy eigenvalue E_1 as in the absence of any field $\vec{B}$. Finally, for the remaining two states

[8]Indeed, the coefficients of the states $\mid 1,1 >$ and $\mid 1,-1 >$ in (33) are non analytic functions of B_x and B_y.

with $m = 0$ diagonalising the total Hamiltonian H, in the limit[9] where the magnetic coupling H_B is much larger than all other contributions to H, namely for magnetic fields whose magnitude is much larger than $(\Delta E/(2\mu) = 3.63$ Tesla), the state of lowest (resp. highest) energy, *i.e.* $\mid \psi_{S'} >$ (resp. $\mid \psi_{T'} >$), is the one for which the electron and positron magnetic moments are both aligned parallel (resp. antiparallel) to the magnetic field $\vec{B}$, or equivalently the one for which the electron spin is aligned antiparallel (resp. parallel) to $\vec{B}$ and the positron spin is parallel (resp. antiparallel) to $\vec{B}$.

These properties—indeed, solely expected on physical grounds independently of any explicit calculation—are beautifully confirmed by the results above in the case where $B_x = 0$ and $B_y = 0$. At this stage, it thus appears that an explicit calculation serves the purpose only of determining the mixing angle θ in (38) between the two $m = 0$ states, and of deriving the energy eigenvalues $E_{S'}$ and $E_{T'}$ in (30) and (32), as functions of the magnetic field $\vec{B} = (0, 0, B_z)$.

Consider now the general situation when the direction of the magnetic field $\vec{B}$ is arbitrary with respect to the z axis. In fact, the results just discussed can be used in order to understand the general case as well. Indeed, the only difference between the two configurations is that the axis with respect to which the spin part of state vectors is expanded, is different. Hence, by an appropriate change of basis *in the spin sector*, effected through a rotation in the spin 1/2 representation, the eigenstates of the total Hamiltonian H in the arbitrary case $(B_x, B_y) \neq (0, 0)$ can in principle be constructed from the expressions of these states when $(B_x, B_y) = (0, 0)$. Therefore, since under such a rotation in spin space representations of spin 0 and of spin 1 are invariant, only the states $\mid 1, m = 0, \pm 1 >$ can mix among themselves. Consequently, in the general case, both the pseudo-singlet and pseudo-triplet states should be given as some superposition of all *four* states $\mid 0, 0 >$ and $\mid 1, m = 0, \pm 1 >$, with in particular the coefficient of the $\mid 0, 0 >$ component *independent* of the *components* of the magnetic field but only dependent on its norm $\mid \vec{B} \mid$, whereas the remaining two states $\mid \psi_\pm >$ can only involve the three states $\mid 1, m = 0, \pm 1 >$. In addition, for all four eigenstates, the coefficients of the states $\mid 1, m = 0, \pm 1 >$ must depend on *all three components* B_x, B_y *and* B_z of the magnetic field.

Indeed, these are features of the results in (29), (31) and (33), which therefore find their origin in the fact that spin 0 and spin 1 representations are invariant under space rotations. However, only an explicit calculation—either along the lines just sketched or by direct diagonalisation of the Hamiltonian as done in this note—can determine the specific mixing coefficients defining each of the eigenstates of the total Hamiltonian H. Incidentally, note that the same argument of invariance under space rotations explains why eigenvalues of H remain unchanged when the magnetic field $\vec{B}$ is no longer parallel to the z axis, *i.e.* why these eigenvalues only depend on the norm $\mid \vec{B} \mid$ of the magnetic field.

[9]This situation was pointed out to the author by J. Deutsch.

4 Positronium Lifetimes

As a first application of results so far, let us compute decay rates for all four eigenstates of the total Hamiltonian H in the presence of a magnetic field $\vec{B}$. Actually, such a calculation is rather straightforward, given the decay rates λ_S and λ_T of the singlet and triplet states $\mid 0,0>$ and $\mid 1, m=0,\pm 1>$, respectively.

First, consider decay rates into two photons. Due to charge conjugation, only the $\mid 0,0>$ state has 2γ decays. Therefore, the 2γ decay rate of the pseudo-singlet state $\mid \psi_{S'}>$ is

$$\lambda_{S'}^{(2\gamma)} = \frac{1}{2}\left[1+\frac{1}{\sqrt{1+x^2}}\right]\lambda_S \ = \ \lambda_S \cos^2\theta \ . \tag{40}$$

For the pseudo-triplet state $\mid \psi_{T'}>$, we have

$$\lambda_{T'}^{(2\gamma)} = \frac{1}{2}\left[1-\frac{1}{\sqrt{1+x^2}}\right]\lambda_S \ = \ \lambda_S \sin^2\theta \ . \tag{41}$$

Finally, the 2γ decay rate of the remaining two states $\mid \psi_\pm >$ is

$$\lambda_{\pm}^{(2\gamma)} = 0 \ . \tag{42}$$

Similarly, for the 3γ decay rates of these states in the same order, one finds

$$\lambda_{S'}^{(3\gamma)} = \frac{1}{2}\left[1-\frac{1}{\sqrt{1+x^2}}\right]\lambda_T \ = \ \lambda_T \sin^2\theta \ , \tag{43}$$

$$\lambda_{T'}^{(3\gamma)} = \frac{1}{2}\left[1+\frac{1}{\sqrt{1+x^2}}\right]\lambda_T \ = \ \lambda_T \cos^2\theta \ , \tag{44}$$

$$\lambda_{\pm}^{(3\gamma)} = \lambda_T \ . \tag{45}$$

In these expressions, the angle θ is defined in (38).

Finally, total decay rates—ignoring the much suppressed rates into four or more photons—are simply

$$\lambda_{S'} = \frac{1}{2}(\lambda_S+\lambda_T)+\frac{1}{2}\frac{1}{\sqrt{1+x^2}}(\lambda_S-\lambda_T) \ = \ \lambda_S\cos^2\theta+\lambda_T\sin^2\theta \ , \tag{46}$$

$$\lambda_{T'} = \frac{1}{2}(\lambda_S+\lambda_T)-\frac{1}{2}\frac{1}{\sqrt{1+x^2}}(\lambda_S-\lambda_T) \ = \ \lambda_S\sin^2\theta+\lambda_T\cos^2\theta \ , \tag{47}$$

$$\lambda_{\pm} = \lambda_T \ . \tag{48}$$

As ought to be expected, these expressions only depend on the norm of the magnetic field $\vec{B}$, but not on its direction. Indeed, since the choice of axis with respect to which spin states are expanded does not affect the calculation of decay rates[10], that axis can

[10] Indeed, positronium states can "remember" the direction and polarisation of the incoming positron only through the *populations* of the four Hamiltonian eigenstates (this is the topic of the next section). Decay rates are *intrinsic* properties of each of these states, and as such, are thus independent of any variable possibly affecting positronium formation.

always be taken along the magnetic field, in which case decay rates can only depend on $\mid B_z \mid$, namely the norm of the magnetic field. Incidentally, note that *total* 2γ and 3γ decay rates—λ_S and λ_T, respectively—are independent of $\vec{B}$—a consequence of unitarity.

It is also easy to check that the statistical decay rates into two and three photons, *i.e.* the average of each of these rates over the four Hamiltonian eigenstates, are independent of the magnetic field. These statistical rates thus coincide with their values when $\vec{B} = \vec{0}$, namely $\lambda_S/4$ and $3\lambda_T/4$ for two and three photon decays, respectively.

Finally, let us remark that the difference between the total decay rates for the pseudo-triplet and $\mid \psi_\pm >$ states,

$$\lambda_{T'} - \lambda_T = \frac{1}{2}(\lambda_S - \lambda_T) \left[1 - \frac{1}{\sqrt{1+x^2}}\right] = (\lambda_S - \lambda_T)\sin^2\theta \ , \tag{49}$$

is a quantity always positive for all values of the magnetic field.

5 Positronium Populations

Let us now address the specific topic of this note; the formation of positronium in a medium placed in a magnetic field. In view of applications, the incoming positron is assumed to have a polarisation P_+ $(-1 \le P_+ \le +1)$. This polarisation P_+ is the expectation value of the positron spin projected onto its momentum, the latter vector thus also defining the z axis for spin quantisation from now on. Therefore, up to a physically irrelevant overall phase, the spin component of the incoming positron wave function is given by

$$\frac{1}{\sqrt{2}}\sqrt{1+P_+} \mid + > + \, e^{i\phi_+}\frac{1}{\sqrt{2}}\sqrt{1-P_+} \mid - > \ . \tag{50}$$

Here, ϕ_+ is an arbitrary *phase difference*—thus possibly leading to observable physical effects—between the two spin components defining a positron state of polarisation P_+.

Similarly, it will be assumed[11] that the positron capturing electron has a polarisation P_- $(-1 \le P_- \le +1)$ along the same z axis. Though in most practical applications, the positronium formation medium is at temperatures such that electrons are effectively not polarised, some experiments at very low temperatures are being planned, for which an investigation of possible effects due to electron polarisation in the applied magnetic field might therefore be found useful. Consequently, again up to a physically irrelevant overall phase, the spin part of the electron wave function is also of the form

$$\frac{1}{\sqrt{2}}\sqrt{1+P_-} \mid + > + \, e^{i\phi_-}\frac{1}{\sqrt{2}}\sqrt{1-P_-} \mid - > \ , \tag{51}$$

where ϕ_- is another arbitrary phase shift.

Hence, at the moment of positronium formation, it is assumed that the spin component of the positronium state vector $\mid \psi, t = 0 >$ is simply given by the tensor

[11] The interest of this possibility was pointed out to the author by F. Gimeno-Nogues.

product[12] of the electron and positron spin vectors in (50) and (51), while the space part of the state vector is of course the wave function $\psi(r)$ in (1). In order to obtain the time evolution of the associated state vector, and thus also the time dependence of its decay products, the resulting wave function $\mid \psi, t = 0 >$ of formed positronium has to be expanded in the basis of eigenstates of the total Hamiltonian H in the presence of the magnetic field $\vec{B}$. This change of basis thus defines coefficients $C_{S'}$, $C_{T'}$ and $C_\pm$ such that

$$\mid \psi, t = 0 >= \sum_{a=S',T',+,-} C_a \ \mid \psi_a > \ . \tag{52}$$

Explicit expressions for these coefficients are given in Appendix A. Time evolution of the positronium state formed is then given by

$$\mid \psi, t >= \sum_{a=S',T',+,-} C_a \ \mid \psi_a > \ \exp\left(-\frac{i}{\hbar}E_a t - \frac{1}{2}\lambda_a t\right) \ , \tag{53}$$

with the quantities E_a and λ_a $(a = S', T', +, -)$ defined in (30), (32), (34) and (46), (47) and (48).

Given these expressions, time evolution of each of the populations associated to each of the four states $\mid \psi_a >$ $(a = S', T', +, -)$ is simply obtained as

$$\mid C_a \mid^2 \ e^{-\lambda_a t} \ , \ \ a = S', T', +, - \ . \tag{54}$$

Expressions for all observables of interest are then easily written down. For example, 2γ and 3γ production rates are[13], respectively,

$$R_{(2\gamma)}(t) = \sum_{a=S',T',+,-} \lambda_a^{(2\gamma)} \mid C_a \mid^2 \ e^{-\lambda_a t} \ , \tag{55}$$

$$R_{(3\gamma)}(t) = \sum_{a=S',T',+,-} \lambda_a^{(3\gamma)} \mid C_a \mid^2 \ e^{-\lambda_a t} \ , \tag{56}$$

while the total photon production rate—simply the sum of the latter two rates—is itself

$$R(t) = \sum_{a=S',T',+,-} \lambda_a \mid C_a \mid^2 \ e^{-\lambda_a t} \ . \tag{57}$$

The rather lengthy expressions for the populations at $t = 0$, namely the coefficients $\mid C_a \mid^2$ $(a = S', T', +, -)$, are given in Appendix B. Results probably more relevant at this point are the same coefficients *averaged*[14] over the phase shifts ϕ_- and ϕ_+. Indeed, one ought to expect that for most sources where the β decay process responsible

[12]Note that it is always possible to "rotate away" *one* of the two phases ϕ_- or ϕ_+—*but not both*— by an appropriate rotation around the z axis. However, dependence on the cancelled phase then reappears through the B_x and B_y components of the rotated magnetic field.

[13]Note that when actual experimental data are considered, the 2γ production rate in (55) should also include a "fast" component due to *direct* pair annihilation of incoming positrons with electrons of the positronium forming medium.

[14]This average *does not* amount to setting $e^{i\phi_\pm} = 0$ in the original expressions for the coefficients C_a, and can only be applied once the *complete* expressions of Appendix B for the coefficients $\mid C_a \mid^2$ have been obtained.

for positron production is taking place, positron states with all possible phase shifts ϕ_+ are statistically populated, thereby justifying an average of the final positronium populations over ϕ_+. Similarly, in the positronium forming medium, one should also expect that all electron states with different phase shifts ϕ_- are statistically populated, again justifying an average over the phase ϕ_-. Under these assumptions[15], the averaged populations are given by

$$\overline{\mid C_{S',T'} \mid^2} = \frac{1}{4}\left[1 - P_- P_+\right] \mp \frac{1}{4}\frac{1}{\sqrt{1+x^2}}\frac{2\mu}{\Delta E}B_z\left[P_- - P_+\right] + \\ + \frac{1}{2}\left[1 \mp \frac{1}{\sqrt{1+x^2}}\right]\frac{B_- B_+}{\mid \vec{B} \mid^2}P_- P_+ \ , \qquad (58)$$

and

$$\overline{\mid C_\pm \mid^2} = \frac{1}{4}\left[1 \pm \frac{B_z}{\mid \vec{B} \mid}P_-\right]\left[1 \pm \frac{B_z}{\mid \vec{B} \mid}P_+\right] \ . \qquad (59)$$

In the r.h.s. of these two expressions, the upper (resp. lower) sign refers to $\overline{\mid C_{S'} \mid^2}$ (resp. $\overline{\mid C_{T'} \mid^2}$) and $\overline{\mid C_+ \mid^2}$ (resp. $\overline{\mid C_- \mid^2}$), respectively. Note that the sum of the four phase averaged populations does indeed reduce to 1, as ought be the case since, by definition, the state $\mid \psi, t = 0 >$ is normalised to 1.

In turn, phase averaged photon production rates $\overline{R}_{(2\gamma)}(t)$, $\overline{R}_{(3\gamma)}(t)$ and $\overline{R}(t)$ are defined as in (55), (56) and (57), of course involving now the phase averaged coefficients $\overline{\mid C_a \mid^2}$ ($a = S', T', +, -$) just given. Clearly, these averaged photon rates are more readily amenable to experimental measurement than are the non phase averaged rates considered previously.

6 Conclusions

This note reports on the calculation of hyperfine positronium populations formed in the presence of an arbitrary magnetic field—whose gradient is assumed to be vanishingly small over the spatial extension of the positronium bound state—for arbitrary positron and electron polarisations. The analysis generalises results available[4, 10, 3] in the literature in two respects. On the one hand, the magnetic field is *not* assumed to be necessarily aligned along the momentum of the incoming positron. On the other hand, allowing for possible electron polarisation effects enables the present results to be of application to positron polarimeters operated at very low temperatures.

Expressions derived in this note provide the basic information required in any experimental analysis of results obtained using positron polarimeters based on either time or energy distributions of positronium decay photons. As a simple but explicit illustration of relevance to current β decay experiments[5, 6, 7], let us consider for

[15] Note that taking such averages is even more justifiable in the instance—often realised in practice—that the magnetic fields present in an experimental set-up possess an axial symmetry along the axis of incoming positrons. Indeed, as was noticed previously, either of the phases ϕ_- or ϕ_+ can always be "rotated away" by an appropriate rotation around the z axis, then also a symmetry transformation of the magnetic fields.

example the phase averaged photon production time spectrum $\overline{R}(t)$ when electrons in the positronium formation medium are not polarised ($P_- = 0$). Expressions derived in the previous section then lead to

$$\overline{R}(t) = \frac{1}{2}\lambda_T e^{-\lambda_T t} + \frac{1}{4}\lambda_{T'}\left[1 - \frac{1}{\sqrt{1+x^2}}\frac{2\mu}{\Delta E}B_z P_+\right]e^{-\lambda_{T'}t} + + \frac{1}{4}\lambda_{S'}\left[1 + \frac{1}{\sqrt{1+x^2}}\frac{2\mu}{\Delta E}B_z P_+\right]e^{-\lambda_{S'}t} \ . \quad (60)$$

However, since the pseudo-singlet decay rate $\lambda_{S'}$ is much larger than the pseudo-triplet one $\lambda_{T'}$, the pseudo-singlet contribution in the r.h.s. of (60) becomes effectively negligible after a nanosecond or so, leaving only the first two terms. Hence in effect, the photon time spectrum $\overline{R}(t)$

$$\overline{R}(t \geq 1 \text{ ns}) \approx \frac{1}{2}\lambda_T e^{-\lambda_T t} + \frac{1}{4}\lambda_{T'}\left[1 - \epsilon P_+\right]e^{-\lambda_{T'}t} \ , \quad (61)$$

with the analysing power

$$\epsilon = \frac{1}{\sqrt{1+x^2}}\frac{2\mu}{\Delta E}B_z \ , \quad (62)$$

provides the means of measuring positron polarisations. Note that, when compared to the usual result obtained for a magnetic field $\vec{B}$ assumed to be parallel to the incoming positron momentum, the sole effect of non vanishing components B_x and B_y in this simple example is to *decrease* the effective analysing power ϵ multiplying the positron polarisation P_+. Nevertheless, there certainly exist other instances where the effects of non vanishing components B_x and B_y have to be properly accounted for when analysing actual experimental data. The results of this note provide the basis for such investigations.

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Appendix A

This Appendix gives the coefficients of the linear combination of the total Hamiltonian eigenstates defining the state vector of positronium formed in the presence of a magnetic field $\vec{B}$ (see (53)).

For the pseudo-singlet and pseudo-triplet states, one has

$$
\begin{aligned}
C_{S',T'} = \; & \frac{1}{\sqrt{2}} \, (1+x^2)^{-1/4} \, (\sqrt{1+x^2} \pm 1)^{-1/2} \times \\
\times \Bigg\{ \; & \frac{1}{2\sqrt{2}} \left[\sqrt{1+P_-} \, \sqrt{1-P_+} \, e^{i\phi_+} - \sqrt{1-P_-} \, \sqrt{1+P_+} \, e^{i\phi_-} \right] (\sqrt{1+x^2} \pm 1) \mp \\
& \mp \frac{1}{2\sqrt{2}} \left[\sqrt{1+P_-} \, \sqrt{1-P_+} \, e^{i\phi_+} + \sqrt{1-P_-} \, \sqrt{1+P_+} \, e^{i\phi_-} \right] \frac{2\mu}{\Delta E} B_z \pm \\
& \pm \frac{1}{2} \sqrt{1+P_-} \, \sqrt{1+P_+} \, \frac{2\mu}{\Delta E} B_+ \mp \\
& \mp \frac{1}{2} \sqrt{1-P_-} \, \sqrt{1-P_+} \, e^{i(\phi_- + \phi_+)} \, \frac{2\mu}{\Delta E} B_- \Bigg\} \; ,
\end{aligned}
\tag{63}
$$

where the upper (resp. lower) sign refers to the coefficient $C_{S'}$ (resp. $C_{T'}$).

Similarly, the coefficients of the remaining two states $\mid \psi_\pm >$ are

$$
\begin{aligned}
C_\pm = \; & \frac{1}{4} \left[\sqrt{1+P_-} \, \sqrt{1-P_+} \, e^{i\phi_+} + \sqrt{1-P_-} \sqrt{1+P_+} \, e^{i\phi_-} \right] \frac{\sqrt{B_x^2 + B_y^2}}{\mid \vec{B} \mid} + \\
& + \frac{1}{2\sqrt{2}} \sqrt{1+P_-} \, \sqrt{1+P_+} \, \frac{B_z \pm \mid \vec{B} \mid}{\mid \vec{B} \mid} \frac{B_+}{\sqrt{B_x^2 + B_y^2}} + \\
& + \frac{1}{2\sqrt{2}} \sqrt{1-P_-} \, \sqrt{1-P_+} \, e^{i(\phi_- + \phi_+)} \, \frac{-B_z \pm \mid \vec{B} \mid}{\mid \vec{B} \mid} \frac{B_-}{\sqrt{B_x^2 + B_y^2}} \; .
\end{aligned}
\tag{64}
$$

Here again, the upper (resp. lower) sign refers to the coefficient C_+ (resp. C_-).

Appendix B

This Appendix gives the populations of positronium states formed in the presence of a magnetic field, for electrons and positrons of initial polarisation P_- and P_+, respectively. With the same conventions as to upper and lower signs as in Appendix A, the results are as follows.

The $\mid \psi_{S',T'} >$ populations are

$$\begin{aligned} \mid C_{S',T'} \mid^2 = & \ \frac{1}{2} \ (1+x^2)^{-1/2} \ (\sqrt{1+x^2} \pm 1)^{-1} \times \\ & \times \Big\{ \ \frac{1}{2}(1+x^2)^{1/2}(\sqrt{1+x^2} \pm 1) \, [1 - P_- P_+] \ \mp \\ & \mp \frac{1}{2}(\sqrt{1+x^2} \pm 1)\frac{2\mu}{\Delta E} B_z \, [P_- - P_+] + \left(\frac{2\mu}{\Delta E}\right)^2 (B_- B_+) P_- P_+ \ \mp \\ & \mp \ \frac{1}{2}(\sqrt{1+x^2} \pm 1)\sqrt{1-P_-^2} \ \sqrt{1-P_+^2} \ \cos(\phi_- - \phi_+) \ - \\ & - \frac{1}{4}\left(\frac{2\mu}{\Delta E}\right)^2 \sqrt{1-P_-^2} \ \sqrt{1-P_+^2} \left[B_- e^{i\phi_+} + B_+ e^{-i\phi_+}\right] \left[B_- e^{i\phi_-} + B_+ e^{-i\phi_-}\right] \ \pm \\ & \pm \frac{1}{2\sqrt{2}} \left(\frac{2\mu}{\Delta E}\right) \sqrt{1-P_+^2} \left[B_- e^{i\phi_+} + B_+ e^{-i\phi_+}\right] \left[\sqrt{1+x^2} \pm 1) \mp \frac{2\mu}{\Delta E} \ B_z \ P_-\right] \ \mp \\ & \mp \frac{1}{2\sqrt{2}} \left(\frac{2\mu}{\Delta E}\right) \sqrt{1-P_-^2} \left[B_- e^{i\phi_-} + B_+ e^{-i\phi_-}\right] \left[(\sqrt{1+x^2} \pm 1) \pm \frac{2\mu}{\Delta E} \ B_z \ P_+\right]\Big\} \ . \end{aligned} \tag{65}$$

Similarly, the $\mid \psi_\pm >$ populations are

$$\begin{aligned} \mid C_\pm \mid^2 = & \ \frac{1}{4}\frac{B_z^2}{\mid \vec{B} \mid^2} [1 + P_- P_+] \pm \ \frac{1}{4}\frac{B_z}{\mid \vec{B} \mid} [P_- + P_+] + \ \frac{1}{2}\frac{B_- B_+}{\mid \vec{B} \mid^2} \ \pm \\ & \pm \ \frac{1}{4\sqrt{2}}\sqrt{1-P_+^2} \left[1 \pm \frac{B_z}{\mid \vec{B} \mid} P_-\right] \left[\frac{B_-}{\mid \vec{B} \mid} e^{i\phi_+} + \frac{B_+}{\mid \vec{B} \mid} e^{-i\phi_+}\right] \ \pm \\ & \pm \ \frac{1}{4\sqrt{2}}\sqrt{1-P_-^2} \left[1 \pm \frac{B_z}{\mid \vec{B} \mid} P_+\right] \left[\frac{B_-}{\mid \vec{B} \mid} e^{i\phi_-} + \frac{B_+}{\mid \vec{B} \mid} e^{-i\phi_-}\right] \ + \\ & + \frac{1}{8}\sqrt{1-P_-^2} \ \sqrt{1-P_+^2} \left[\frac{B_-}{\mid \vec{B} \mid} e^{i\phi_+} + \frac{B_+}{\mid \vec{B} \mid} e^{-i\phi_+}\right] \left[\frac{B_-}{\mid \vec{B} \mid} e^{i\phi_-} + \frac{B_+}{\mid \vec{B} \mid} e^{-i\phi_-}\right] \ . \end{aligned} \tag{66}$$

Note that in each of these two equations, the last three terms could be combined further into the product of two terms. However, results in the form given here are more amenable to the phase average discussed in the main text. Incidentally, it is straightforward to check that the sum of all four populations does indeed reduce to 1, since, by construction, the state $\mid \psi, t = 0 >$ is normalised to 1.

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