The Hamiltonian Formulation of Higher Order Dynamical Systems

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Abstract

Using Dirac's approach to constrained dynamics, the Hamiltonian formulation of regular higher order Lagrangians is developed. The conventional description of such systems due to Ostrogradsky is recovered. However, unlike the latter, the present analysis yields in a transparent manner the local structure of the associated phase space and its local sympletic geometry, and is of direct application to *constrained* higher order Lagrangian systems which are beyond the scope of Ostrogradsky's approach.

1 Introduction

Canonical and path integral quantisations of systems whose dynamics is described by higher order Lagrangians—namely by Lagrangians involving time derivatives of the degrees of freedom of order at least two—is an issue which by far has not been developed to the same extent[1] as for systems whose Lagrangian only depends on the coordinates and their velocities[2]. Nevertheless, there do exist systems of physical interest described by such higher order Lagrangians, the most popular examples being perhaps higher order regularisations of quantum gauge field theories and so-called rigid strings[4, 5] or rigid particles[6]. In fact, these examples involve the additional complication that they possess local symmetries, leading therefore to constraints generating these gauge invariances on phase space.

As is well known, there does exist a generalisation of the ordinary Hamiltonian formulation in the case of higher order Lagrangians, which is due to Ostrogradsky[7]. However, on the one hand, Ostrogradsky's approach is implicitly restricted to *non constrained* systems—which in particular *do not* possess local gauge invariances—, thus rendering this approach inapplicable to most, if not all physical systems of present fundamental interest. On the other hand, in Ostrogradsky's construction the structure of phase space and in particular of its local symplectic geometry is not immediately transparent, an obvious source of possible confusion when considering canonical or path integral quantisations of such systems.

This note discusses how both problems can be resolved within the well established context of constrained systems[2] described by Lagrangians depending on coordinates and velocities only. Well known and powerful techniques become then immediately available, rendering the necessity of a separate discussion of the quantisation of higher order systems—including constrained ones, and thus in particular their BRST quantisation—void of any justification. *Any* higher order system can *always* be cast in the form of an ordinary constrained system, namely one whose Lagrangian is a function only of first order time derivatives of the degrees of freedom, but not of time derivatives of higher order.

That such a reduction of higher order Lagrangians is possible was indi-

[1]The classical analysis of such higher order regular or singular systems is available to some extent in the recent literature[1].

cated already previously[8, 9]. As should be clear, it suffices for this purpose to introduce auxiliary degrees of freedom associated to each of the successive time derivatives of the original coordinates of the system. In effect, the canonical quantisation of rigid particles has already used[6] the same idea, thus in a situation where Ostrogradsky's approach is not applicable as such.

The present note is organised as follows. In the next section, Ostrogradsky's construction is briefly considered. Sect.3 describes how any higher order Lagrangian system can be cast into the form of a constrained system whose Lagrangian involves only first order time derivatives of the degrees of freedom. The canonical Hamiltonian description of the auxiliary system is then addressed in Sect.4 while its equivalence with Ostrogradsky's formulation is established in Sect.5. Further comments are presented in the Conclusion.

2 Ostrogradsky's Construction

Let us consider a system with degrees of freedom $x_n(t)$ $(n = 1, 2, \cdots)$, t being the time evolution parameter of the system. Although the present analysis assumes that these coordinates are commuting variables, and that the index n takes a finite or an infinite number of discrete values, it should be clear that exactly the same considerations and the same conclusions as those developed hereafter are applicable to commuting and anticommuting degrees of freedom, as well as to an infinite non countable set of coordinates. The former case is that of bosonic and fermionic types of degrees of freedom, and the latter typically that of field theories. All the conclusions established in the present note are thus valid in *complete generality*, for *any* system described by some higher order Lagrangian. The restriction to a *discrete* set of *commuting* degrees of freedom is only one of ease of presentation. Moreover, in a first reading of the paper it might be useful to consider the case of only one degree of freedom[8], namely ignore the index n altogether.

These remarks having been made, let us assume that the dynamical time evolution of the system is determined from the variational principle being applied to the action functional associated to some time independent Lagrange function

$$L_0\left(x_n, \dot{x}_n, \ddot{x}_n, \cdots, x_n^{(m_n)}\right) \ . \tag{1}$$

Here, $(m_n \geq 1)$ $(n = 1, 2, \cdots)$ is the maximal order of all time derivatives of the coordinate x_n $(n = 1, 2, \cdots)$ appearing in the Lagrangian. In particular, the discussion of this paper includes the familiar case when $(m_n = 1)$ for all degrees of freedom. Throughout the analysis, it might be interesting to consider the special case $(m_n = 1)$ $(n = 1, 2, \cdots)$ to see how well-known results are recovered from the present general situation.

Note that the Lagrange function is assumed to depend on at least the first order time derivative of *each* degree of freedom x_n. Otherwise, one would have to deal with some of the equations of motion being actually constraints, a situation not considered by Ostrogradsky. Moreover, the restriction to time *independent* Lagrange functions is again for reasons of convenience rather than of principle. Time dependent higher order Lagrangians can also be analysed along the same lines as developed hereafter.

Considering the variational principle, it is clear that the Euler-Lagrange equations of motion of the system are given by

$$\sum_{k_n=0}^{m_n} (-1)^{k_n} \left(\frac{d}{dt}\right)^{k_n} \frac{\partial L_0}{\partial x_n^{(k_n)}} = 0 \ , \quad n = 1, 2, \cdots \ . \tag{2}$$

Following Ostrogradsky's lead[7] and in order to simplify the expression of these equations, let us introduce quantities p_{n,α_n} $(\alpha_n = 0, 1, \cdots, m_n - 1)$ defined recursively by

$$p_{n,i_n-1} = \frac{\partial L_0}{\partial x_n^{(i_n)}} - \frac{d}{dt} p_{n,i_n} \ , \quad i_n = 1, 2, \cdots, m_n - 1 \ , \tag{3}$$

with the initial value

$$p_{n,m_n-1} = \frac{\partial L_0}{\partial x_n^{(m_n)}} \ . \tag{4}$$

The Euler-Lagrange equations of motion in (2) take then the simpler compact form

$$\frac{\partial L_0}{\partial x_n} - \frac{d}{dt} p_{n,0} = 0 \ , \quad n = 1, 2, \cdots \ , \tag{5}$$

which are very suggestive of Hamiltonian equations of motion. Note how these expressions generalise the familiar standard definitions in the case when $(m_n = 1)$ for all degrees of freedom x_n $(n = 1, 2, \cdots)$.

In order to reveal a possible Hamitonian description in the general case when the integers m_n take arbitrary finite values, it is useful to consider the differential of the Lagrange function L_0, in which the definitions of the variables p_{n,α_n} are included. A little calculation then leads to the identity,

$$d\left(\sum_n \sum_{\alpha_n=0}^{m_n-1} x_n^{(\alpha_n+1)} p_{n,\alpha_n} - L_0\right) =$$

$$= \sum_n \sum_{\alpha_n=0}^{m_n-1}\left[x_n^{(\alpha_n+1)} dp_{n,\alpha_n} - dx_n^{(\alpha_n)}\dot{p}_{n,\alpha_n}\right] - \sum_n dx_n\left[\frac{\partial L_0}{\partial x_n} - \dot{p}_{n,0}\right] . \quad (6)$$

Note that the last sum in the r.h.s. of this expression is a combination of the Euler-Lagrange equations of motion of the system.

The meaning of this result is as follows. Consider the quantity defined by

$$H = \sum_n \sum_{\alpha_n=0}^{m_n-1} \dot{x}_n^{(\alpha_n)} p_{n,\alpha_n} - L_0 , \quad (7)$$

with the variables p_{n,α_n} determined by the recursion relations in (3) and (4). Since these variables involve the coordinates x_n and their time derivatives up to a certain order which is different for each of the coordinates x_n and each of the variables p_{n,α_n}, so would *a priori* the quantity H given in (7). However, the identity (6) establishes that this dependence is in fact rather specific, namely only through a dependence of the variables $x_n^{(\alpha_n)}$ and p_{n,α_n} $(\alpha_n = 0, 1, \cdots, m_n - 1)$ themselves,

$$H\left(x_n^{(\alpha_n)}, p_{n,\alpha_n}\right) , \quad \alpha_n = 0, 1, \cdots, m_n - 1 . \quad (8)$$

Note that this conclusion is valid irrespective of whether the Lagrangian $L_0\left(x_n, \dot{x}_n, \cdots, x_n^{(m_n)}\right)$ leads to constraints or not.

The identity (6) also shows that the Euler-Lagrange equations of motion (2) are equivalent to the set of equations

$$\dot{x}_n^{(\alpha_n)} = \frac{\partial H}{\partial p_{n,\alpha_n}} , \quad \dot{p}_{n,\alpha_n} = -\frac{\partial H}{\partial x_n^{(\alpha_n)}} , \quad \alpha_n = 0, 1, \cdots, m_n - 1 . \quad (9)$$

In other words, the system of higher order Lagrangian L_0 has been cast in Hamiltonian form, with the variables $\left(x_n^{(\alpha_n)}, p_{n,m_n}\right)$ being canonically conjugate pairs. Let us thus recapitulate.

Given the Lagrange function $L_0\left(x_n, \dot{x}_n, \cdots, x_n^{(m_n)}\right)$, first one introduces the conjugate momenta p_{n,m_n-1} defined by

$$p_{n,m_n-1} = \frac{\partial L_0}{\partial x_n^{(m_n)}}\left(x_n, \dot{x}_n, \cdots, x_n^{(m_n)}\right) \ . \tag{10}$$

Regarding the variables $\left(x_n, \dot{x}_n, \cdots, x_n^{(m_n-1)}\right)$ as independent, the relations (10) may be inverted to give

$$x_n^{(m_n)} = \dot{x}_n^{(m_n-1)} = \dot{x}_n^{(m_n-1)}\left(x_n, \dot{x}_n, \cdots, x_n^{(m_n-1)}, p_{n,m_n}\right) \ . \tag{11}$$

The remaining conjugate momenta p_{n,i_n-1} ($i_n = 1, 2, \cdots, m_n - 1$) are then determined by the recursion relations in (3). However, these relations are *not* used in order to express the variables $\dot{x}_n^{(i_n-1)}$ ($i_n = 1, 2, \cdots, m_n - 1$) in terms of $(x_n, \dot{x}_n, \cdots, x_n^{(i_n-1)})$, the conjugate momenta $(p_{n,i_n}, \cdots, p_{n,m_n-1})$ and time derivatives of the latter. Indeed in Ostrogradsky's construction, the variables $(x_n, \dot{x}_n, \cdots, x_n^{(m_n-1)})$ have to be considered as being *independent*. We shall come back to this point shortly.

Once the expressions for the quantities $\dot{x}_n^{(m_n-1)}$ determined as in (11), the canonical Hamiltonian of the system is defined by (7), or equivalently,

$$H(x_n^{(\alpha_n)}, p_{n,\alpha_n}) =$$

$$= \dot{x}_n^{(m_n-1)} p_{n,m_n-1} - L_0\left(x_n, \dot{x}_n, \cdots, \dot{x}_n^{(m_n-1)}\right) + \sum_n \sum_{i_n=1}^{m_n-1} x_n^{(i_n)} p_{n,i_n-1} \ , \tag{12}$$

in which only the substitutions for the variables $\dot{x}_n^{(m_n-1)}$ are performed, while the degrees of freedom $x_n^{(\alpha_n)}$ and p_{n,α_n} ($\alpha_n = 0, 1, \cdots, m_n - 1$) are considered as being independent.

Finally, the Hamiltonian equations of motion are given by (9). Introducing the fundamental Poisson brackets,

$$\left\{x_n^{(\alpha_n)}, x_m^{(\alpha_m)}\right\} = 0 \ , \ \left\{x_n^{(\alpha_n)}, p_{m,\alpha_m}\right\} = \delta_{nm}\delta_{\alpha_n\alpha_m} \ , \ \{p_{n,\alpha_n}, p_{m,\alpha_m}\} = 0 \ , \tag{13}$$

with ($\alpha_n = 0, 1, \cdots, m_n - 1$) and ($\alpha_m = 0, 1, \cdots, m_m - 1$) ($n, m = 1, 2, \cdots$), the equations (9) take the Hamiltonian form

$$\dot{x}_n^{(\alpha_n)} = \left\{x_n^{(\alpha_n)}, H\right\} \ , \ \ \dot{p}_{n,\alpha_n} = \{p_{n,\alpha_n}, H\} \ , \ \ \alpha_n = 0, 1, \cdots, m_n - 1 \ . \tag{14}$$

In other words, the variables $\left(x_n^{(\alpha_n)}, p_{n,\alpha_n}\right)$ $(\alpha_n = 0, 1, \cdots, m_n - 1)$ are pairs of conjugate degrees of freedom, thus defining the phase space of the system and its local symplectic structure.

Obviously, certain comments are in order. It is clear that Ostrogradsky's construction is applicable only to those higher order Lagrangians for which the inversions required in the determination of the quantities $\dot{x}_n^{(m_n-1)}$ are non degenerate. Namely, the Lagrangian L_0 *cannot* lead to constraints of any kind. *Constrained* higher order Lagrangians are beyond the scope of Ostrogradsky's approach.

Another issue with the present construction is the risk of confusion which arises when dealing with the variables $x_n^{(\alpha_n)}$ and their first order time derivatives $\dot{x}_n^{(\alpha_n)}$ $(\alpha_n = 0, 1, \cdots, m_n - 1)$, a situation which becomes even the more acute when considering canonical or path integral quantisations of such systems. As emphasized above, only the first order time derivatives of the variables $x_n^{(m_n-1)}$ are to be solved for in terms of the conjugate momenta p_{n,m_n-1} and the variables $x_n^{(\alpha_n)}$ $(\alpha_n = 0, 1, \cdots, m_n - 1)$, the latter considered to be independent of one another rather than being simply time derivatives of order α_n of the coordinates x_n. It is in this manner only that the canonical Hamiltonian defined in (12) can be made a function of the pairs of conjugate degrees of freedom $(x_n^{(\alpha_n)}, p_{n,\alpha_n})$. To illustrate the possible confusion which might arise when this point is not fully appreciated, the reader is invited to consider a simple example in the case of a single degree of freedom $x(t)$, such as,

$$L_0(x, \dot{x}, \ddot{x}) = \frac{1}{2} a x \ddot{x}^2 - \frac{1}{2} b x \dot{x}^2 \ , \tag{15}$$

with a and b being arbitrary constant parameters. If one attempts solving both for $\ddot{x}$ and for $\dot{x}$ in terms of x and p_0 and p_1, there appear in the canonical Hamiltonian time derivative terms of the conjugate momentum p_1! It is thus important to develop Ostrogradsky's construction precisely in the manner emphasized above, keeping the variables x and $\dot{x}$ as independent, and inverting only for $\ddot{x}$ in terms of x, $\dot{x}$ and p_1.

Nevertheless, when solving the Hamiltonian equations of motion (14) for the degrees of freedom $x_n(t)$, it becomes necessary, *after having computed the Poisson brackets*, to impose the condition that the variables $x_n^{(i_n)}$ are time derivatives of order i_n $(i_n = 1, 2, \cdots, m_n - 1)$ of the coordinates $x_n(t)$.

It is clear that both issues are solved at once by emphasizing *explicitly* the fact that in the Hamiltonian approach—hence also when considering canonical and path integral quantisations of such systems—, all variables $x_n^{(\alpha_n)}$ are to be considered as being *independent*. This is readily achieved by introducing *independent auxiliary* degrees of freedom, each corresponding to a time derivative of a given order of one of the original degrees of freedom. The *same* system can then be described in terms of an extended Lagrangian including a dependence on the auxiliary degrees of freedom, such that time derivatives of first order only are involved. In this manner, one is brought back[8, 9] into the realm of the usual type of dynamical systems for which most powerful techniques are available, with the additional advantage that *constrained* higher order Lagrangians do not need to be considered on a separate basis any longer.

3 The Auxiliary Lagrangian

Given a system of degrees of freedom $x_n(t)$ $(n = 1, 2, \cdots)$ with Lagrange function $L_0(x_n, \dot{x}_n, \cdots, x_n^{(m_n)})$ $(m_n \geq 1)$—*be it regular or not*—, let us introduce new *independent* variables $q_{n,\alpha_n}(t)$ $(\alpha_n = 0, 1, \cdots, m_n - 1)$ such that the following recursion relations would hold,

$$q_{n,i_n} = \dot{q}_{n,i_n-1} \ , \quad i_n = 1, 2, \cdots, m_n - 1 \ , \tag{16}$$

with the initial value

$$q_{n,0} = x_n \ . \tag{17}$$

Clearly, the variables q_{n,i_n} $(i_n = 1, 2, \cdots, m_n - 1)$ would then correspond to the time derivatives $x_n^{(i_n)}$ of order i_n of the coordinates x_n, the latter being identical to the coordinates $q_{n,0}$.

In order to inforce the relations (16) and (17) for the *independent* variables q_{n,α_n}, additional Lagrange multipliers $\mu_{n,i_n}(t)$ $(i_n = 1, 2, \cdots, m_n - 1)$ are introduced. The variables $(q_{n,\alpha_n}, \mu_{n,i_n})$ thus determine the set of *independent* degrees of freedom of the extended Lagrangian system, with (q_{n,i_n}, μ_{n,i_n}) $(i_n = 1, 2, \cdots, m_n - 1)$ being auxiliary degrees of freedom as compared to the original coordinates $(x_n(t) = q_{n,0}(t))$. The auxiliary Lagrange function of this extended description of the system is given by

$$L(q_{n,\alpha_n}, \dot{q}_{n,\alpha_n}, \mu_{n,i_n}) =$$
$$= L_0(q_{n,0}, q_{n,1}, \cdots, q_{n,m_n-1}, \dot{q}_{n,m_n-1}) + \sum_n \sum_{i_n=1}^{m_n-1} (q_{n,i_n} - \dot{q}_{n,i_n-1})\, \mu_{n,i_n} \ . \tag{18}$$

Note that as advertised, the auxiliary Lagrangian L involves only *first* order time derivatives of the extended set of degrees of freedom. Obviously, due to the presence of the Lagrange multipliers μ_{n,i_n}, the Lagrange function L in (18) defines a constrained system, to which the usual analysis[10, 2] of constrained dynamics is applicable.

Before turning to that important issue however, let us first establish the equivalence of the auxiliary Lagrangian with the original formulation of the system determined by the Lagrangian L_0. Applied to L in (18), the variational principle leads to the following equations of motion for the Lagrange multipliers μ_{n,i_n},

$$q_{n,i_n} = \dot{q}_{n,i_n-1} \ , \quad i_n = 1,2,\cdots,m_n-1 \ , \tag{19}$$

while for the degrees of freedom q_{n,i_n} ($i_n = 1,2,\cdots,m_n-1$), one obtains,

$$\mu_{n,j_n} = -\frac{\partial L_0}{\partial q_{n,j_n}} - \frac{d}{dt}\mu_{n,j_n+1} \ , \quad j_n = 1,2,\cdots,m_n-2 \ , \tag{20}$$

and

$$\mu_{n,m_n-1} = -\frac{\partial L_0}{\partial q_{n,m_n-1}} + \frac{d}{dt}\frac{\partial L_0}{\partial \dot{q}_{n,m_n-1}} \ . \tag{21}$$

Finally, the equations of motion for $q_{n,0}$ are

$$\frac{\partial L_0}{\partial q_{n,0}} + \dot{\mu}_{n,1} = 0 \ . \tag{22}$$

Note that the latter equations are in fact the actual equations of motion of the system. Indeed, all the other equations for q_{n,i_n} and μ_{n,i_n} are constraint equations which determine the auxiliary degrees of freedom q_{n,i_n} in terms of successive time derivatives of the original coordinates ($q_{n,0} = x_n$), as well as the Lagrange multipliers μ_{n,i_n} in terms of successive partial derivatives of the Lagrange function L_0. By substitution in (22) of the successive definitions of the Lagrange multipliers μ_{n,i_n}, the equations of motion for $q_{n,0}$ reduce to

$$\sum_{\alpha_n=0}^{m_n-1}(-1)^{\alpha_n}\left(\frac{d}{dt}\right)^{\alpha_n}\frac{\partial L_0}{\partial q_{n,\alpha_n}} + (-1)^{m_n}\left(\frac{d}{dt}\right)^{m_n}\frac{\partial L_0}{\partial \dot{q}_{n,m_n-1}} = 0 \ . \tag{23}$$

Upon the substitution of the recursion relations (19), one then indeed recovers the original Euler-Lagrange equations of motion in (2). Hence, the complete equivalence between the auxiliary formulation of the system and the original one based on the higher order Lagrange function $L_0\left(x_n, \dot{x}_n, \cdots, x_n^{(m_n)}\right)$ is established.

4 The Hamiltonian Formulation

Given the auxiliary Lagrangian formulation of higher order systems of the previous section, let us apply to it the ordinary analysis[2] of constraints in order to develop its Hamiltonian description. The momenta canonically conjugate to the degrees of freedom q_{n,α_n} ($\alpha_n = 0, 1, \cdots, m_n - 1$) and μ_{n,i_n} ($i_n = 1, , 2 \cdots, m_n - 1$) are of course defined by, respectively,

$$p_{n,\alpha_n} = \frac{\partial L}{\partial \dot{q}_{n,\alpha_n}} \ , \quad \pi_{n,i_n} = \frac{\partial L}{\partial \dot{\mu}_{n,i_n}} \ . \tag{24}$$

However, the phase space degrees of freedom $(q_{n,\alpha_n}, p_{n,\alpha_n}; \mu_{n,i_n}, \pi_{n,i_n})$ are not all independent. In fact, the system possesses the following primary constraints,

$$\Phi_{n,i_n} = 0 \ , \quad \pi_{n,i_n} = 0 \ , \quad i_n = 1, 2, \cdots, m_n - 1 \ , \tag{25}$$

where

$$\Phi_{n,i_n} \equiv p_{n,i_n-1} + \mu_{n,i_n} \ , \quad i_n = 1, 2, \cdots, m_n - 1 \ . \tag{26}$$

Both sets of primary constraints follow from the particular way in which the auxiliary degrees of freedom are introduced in the definition of the extended Lagrange function L in (18). The primary constraints obey the algebra of Poisson brackets

$$\{\Phi_{n,i_n}, \Phi_{m,i_m}\} = 0 \ , \quad \{\Phi_{n,i_n}, \pi_{m,i_m}\} = \delta_{nm}\delta_{i_n i_m} \ , \quad \{\pi_{n,i_n}, \pi_{m,i_m}\} = 0 \ , \tag{27}$$

with ($i_n = 1, 2, \cdots, m_n - 1$) and ($i_m = 1, 2, \cdots, m_m - 1$) ($n, m = 1, 2, \cdots$), showing therefore already at this stage that the primary constraints are certainly also second class constraints.

Among all conjugate momenta, p_{n,m_n-1} certainly play a distinguished role since on the one hand, they are the only ones not involved in any of the primary constraints above, and on the other hand, their conjugate coordinates q_{n,m_n-1} are the only variables whose first order time derivatives do appear in the original Lagrange function L_0. Indeed, we have

$$p_{n,m_n-1} = \frac{\partial L_0}{\partial \dot{q}_{n,m_n-1}} (q_{n,0}, q_{n,1}, \cdots, q_{n,m_n-1}, \dot{q}_{n,m_n-1}) \ . \tag{28}$$

As in Ostrogradsky's approach, let us then assume that for *fixed* values of q_{n,α_n} ($\alpha = 0, 1, \cdots, m_n - 1$), these relations are invertible, leading therefore to the velocities,

$$\dot{q}_{n,m_n-1} = \dot{q}_{n,m_n-1}\left(q_{n,\alpha_n}, p_{n,m_n-1}\right) \ . \tag{29}$$

In other words, given fixed values for q_{n,i_n-1} ($i_n = 1, 2, \cdots, m_n - 1$), the dynamical system of degrees of freedom q_{n,m_n-1} with Lagrange function $L_0\left(q_{n,0}, q_{n,1}, \cdots, q_{n,m_n-1}, \dot{q}_{n,m_n-1}\right)$ is assumed to be a *regular* system, namely *not leading* to any constraints for the conjugate momenta p_{n,m_n-1}. Consequently, the constraints in (25) determine the full set of primary contraints in the extended formalism of the higher order Lagrangian $L_0\left(x_n, \dot{x}_n, \cdots, x_n^{(m_n)}\right)$.

The distinguished role of the conjugate variables $(q_{n,m_n-1}, p_{n,m_n-1})$ justifies the definition of the *restricted Legendre transform* of $L_0\left(q_{n,\alpha_n}, \dot{q}_{n,m_n-1}\right)$, leading to the *restricted canonical Hamiltonian*,

$$\overline{H}_0(q_{n,\alpha_n}, p_{n,m_n-1}) = \sum_n \dot{q}_{n,m_n-1} p_{n,m_n-1} - L_0\left(q_{n,\alpha_n}, \dot{q}_{n,m_n-1}\right) \ . \tag{30}$$

In the same way as was established for the Hamiltonian H in (7), note that the restricted Hamiltonian $\overline{H}_0$ is a function of the variables $(q_{n,\alpha_n}, p_{n,m_n-1})$ only, *irrespective of whether the relations (28) are invertible or not*, namely irrespective of whether $L_0\left(q_{n,0}, q_{n,1}, \cdots, q_{n,m_n-1}, \dot{q}_{n,m_n-1}\right)$ defines a regular system in the coordinates q_{n,m_n-1} or not[3]. In the present discussion, the assumption of regularity is necessary only in order that no further primary constraints beyond those in (25) appear in the analysis.

In terms of the definitions and the primary constraints above, the canonical Hamiltonian of the extended system,

$$H_0 = \sum_n \sum_{\alpha_n}^{m_n-1} \dot{q}_{n,\alpha_n} p_{n,\alpha_n} + \sum_n \sum_{i_n=1}^{m_n-1} \dot{\mu}_{n,i_n} \pi_{n,i_n} - L \ , \tag{31}$$

is readily found to be given by

$$H_0\left(q_{n,\alpha_n}, p_{n,\alpha_n}; \mu_{n,i_n}\right) = \overline{H}_0\left(q_{n,\alpha_n}, p_{n,m_n-1}\right) - \sum_n \sum_{i_n=1}^{m_n-1} \mu_{n,i_n} q_{n,i_n} \ . \tag{32}$$

However, as is well known[2], due to the presence of constraints, the Hamiltonian generating the genuine time evolution of the system under which the constraints are preserved, is in general given by the canonical Hamiltonian H_0 and a linear combination of the constraints. Thus in the present case, the would-be Hamiltonian is of the form,

$$H_* = H_0 + \sum_n \sum_{i_n=1}^{m_n-1} \left[\lambda^{(1)}_{n,i_n}\Phi_{n,i_n} + \lambda^{(2)}_{n,i_n}\pi_{n,i_n}\right] \ , \qquad (33)$$

with $\lambda^{(1)}_{n,i_n}$ and $\lambda^{(2)}_{n,i_n}$ being Lagrange multipliers for the constraints. Consistent time evolution of the primary constraints Φ_{n,i_n} and π_{n,i_n} then imposes the relations

$$\lambda^{(1)}_{n,i_n} = q_{n,i_n} \ , \quad i_n = 1, 2, \cdots, m_n - 1 \ , \qquad (34)$$

as well as

$$\lambda^{(2)}_{n,1} = \frac{\partial \overline{H}_0}{\partial q_{n,0}} \ , \quad \lambda^{(2)}_{n,j_n} = \frac{\partial \overline{H}_0}{\partial q_{n,j_n-1}} - \mu_{n,j_n-1} \ , \quad j_n = 2, 3, \cdots, m_n - 1 \ . \qquad (35)$$

Consequently, the extended formulation of the system does not possess secondary constraints, while its extended Hamiltonian reduces to

$$H_* = \overline{H}_0 + \sum_n \sum_{i_n=1}^{m_n-1} q_{n,i_n} p_{n,i_n-1} + $$
$$+ \sum_n \pi_{n,1}\frac{\partial \overline{H}_0}{\partial q_{n,0}} + \sum_n \sum_{j_n=2}^{m_n-1} \pi_{n,j_n}\left[\frac{\partial \overline{H}_0}{\partial q_{n,j_n-1}} - \mu_{n,j_n-1}\right] \ . \qquad (36)$$

However, as already pointed out previously, all primary constraints Φ_{n,i_n} and π_{n,i_n} are second class, and may thus be solved for explicitly provided the canonical Poisson brackets are traded for appropriate Dirac brackets[10, 2]. Choosing to solve the constraints in terms of

$$\mu_{n,i_n} = -p_{n,i_n-1} \ , \quad \pi_{n,i_n} = 0 \ , \quad i_n = 1, 2, \cdots, m_n - 1 \ , \qquad (37)$$

the reduced phase space degrees of freedom are then simply $(q_{n,\alpha_n}, p_{n,\alpha_n})$ $(\alpha_n = 0, 1, \cdots, m_n - 1)$. On the other hand, given the algebra (27) of constraints, the Dirac brackets of the reduced Hamiltonian description are easily seen to remain canonical $(\alpha_n = 0, 1, \cdots, m_n - 1;\ \alpha_m = 0, 1, \cdots, m_m - 1)$,

$$\{q_{n,\alpha_n}, q_{m,\alpha_m}\}_D = 0 \ , \quad \{q_{n,\alpha_n}, p_{m,\alpha_m}\}_D = \delta_{nm}\delta_{\alpha_n\alpha_m} \ , \quad \{p_{n,\alpha_n}, p_{m,\alpha_m}\}_D = 0 \ . \qquad (38)$$

The constrained description of higher order Lagrangian systems has thus lead to the following Hamiltonian formulation. The local phase space coordinates are the canonically conjugate pairs of degrees of freedom $(q_{n,\alpha_n}, p_{n,\alpha_n})$ $(\alpha_n = 0, 1, \cdots, m_n - 1)$, with the local symplectic structure determined by the Dirac brackets in (38). Time evolution in phase space is specified through these brackets by the extended Hamiltonian,

$$H_E\left(q_{n,\alpha_n}, p_{n,\alpha_n}\right) = \overline{H}_0\left(q_{n,\alpha_n}, p_{n,m_n-1}\right) + \sum_n \sum_{i_n=1}^{m_n-1} q_{n,i_n} p_{n,i_n-1} \ . \tag{39}$$

In particular, the fundamental equations of motion are $(i_n = 1, 2, \cdots, m_n - 1)$,

$$\dot{q}_{n,i_n-1} = q_{n,i_n} \ , \quad \dot{q}_{n,m_n-1} = \frac{\partial \overline{H}_0}{\partial p_{n,m_n-1}} \ , \tag{40}$$

as well as

$$\dot{p}_{n,0} = -\frac{\partial \overline{H}_0}{\partial q_{n,0}} \ , \quad \dot{p}_{n,i_n} = -\frac{\partial \overline{H}_0}{\partial q_{n,i_n}} - p_{n,i_n-1} \ . \tag{41}$$

However, the restricted canonical Hamiltonian $\overline{H}_0$ is such that

$$\frac{\partial \overline{H}_0}{\partial q_{n,\alpha_n}}\left(q_{n,\alpha_n}, p_{n,m_n-1}\right) = -\frac{\partial L_0}{\partial q_{n,\alpha_n}}\left(q_{n,\alpha_n}, \dot{q}_{n,m_n-1}(q_{n,\alpha_n}, p_{n,m_n-1})\right) \ , \tag{42}$$

so that the equations (41) for p_{n,α_n} are equivalent to

$$\frac{\partial L_0}{\partial q_{n,0}} - \frac{d}{dt} p_{n,0} = 0 \ , \tag{43}$$

with

$$p_{n,i_n-1} = \frac{\partial L_0}{\partial q_{n,i_n}} - \frac{d}{dt} p_{n,i_n} \ , \quad i_n = 1, 2, \cdots, m_n - 1 \ . \tag{44}$$

Expressed in this manner, it is clear how these Hamiltonian equations of motion are indeed equivalent to the Euler-Lagrange equations (19) to (22) of Sect.3, when the constraints $(\mu_{n,i_n} = -p_{n,i_n-1})$ are accounted for. Indeed, (43) determine the actual equations of motion of the system and are equivalent to (22). The conjugate momenta $p_{n,0}$ are defined recursively through the equations (44), which are equivalent to (20) and (21), given the momenta p_{n,m_n-1}. Finally, the latter quantities, which are absent of course in the Lagrangian equations of motion except through their definition as

$$p_{n,m_n-1} = \frac{\partial L_0}{\partial \dot{q}_{n,m_n-1}} \left(q_{n,\alpha_n}, \dot{q}_{n,m_n-1}\right) \ , \tag{45}$$

are determined implicitly by the Hamiltonian equations of motion,

$$\dot{q}_{n,m_n-1} = \frac{\partial \overline{H}_0}{\partial p_{n,m_n-1}} \left(q_{n,\alpha_n}, p_{n,m_n-1}\right) \ . \tag{46}$$

Since the Lagrangian $L_0\left(q_{n,\alpha_n}, \dot{q}_{n,m_n-1}\right)$ is assumed to be regular in the coordinates q_{n,m_n-1}, this latter relation is indeed invertible, leading back to the relation (45). It is in this way that the present Hamiltonian equations of motion are equivalent to the Euler-Lagrange equations under the Lagrangian reduction, namely the reduction of conjugate momenta p_{n,α_n} in terms of the coordinates q_{n,α_n} and their velocities $\dot{q}_{n,\alpha_n}$. Therefore, since the auxiliary Lagrangian formulation was shown to reproduce the Euler-Lagrange equations of the higher order Lagrangian, the present Hamiltonian construction is established to be equivalent to the original description of the system as well.

5 Ostrogradsky's Approach Revisited

The equivalence of the results obtained through the analysis of constraints applied to the auxiliary formulation of *regular* higher order Lagrangian systems with Ostrogradsky's construction is now obvious.

In the latter approach at the Hamiltonian level, the successive time derivatives $x_n^{(\alpha_n)}$ $(\alpha_n = 0, 1, \cdots, m_n - 1)$ of the degrees of freedom x_n *have to be considered as being independent*. In the constrained formulation, these variables correspond to the *independent* auxiliary coordinates q_{n,α_n}, with in particular $(q_{n,0} = x_n)$. In addition, the fundamental brackets (13) in Ostrogradsky's formulation are identical to the canonical Dirac brackets(38) of the reduced phase space degrees of freedom $(q_{n,\alpha_n}, p_{n,\alpha_n})$.

Finally, it is clear that the extended Hamiltonian H_E in (39) is *identical* to Ostrogradsky's canonical Hamiltonian H in (12). In particular, note how in the definition of the latter quantity, the restricted canonical Hamiltonian

$\overline{H}_0$ defined in (30) appears naturally, indeed emphasizing once again the distinguished role played by the time derivatives $x_n^{(m_n)}$ of maximal order of the degrees of freedom $x_n(t)$.

Therefore, the analysis of the previous section, based on the auxiliary formulation of regular higher order Lagrangian systems and Dirac's analysis of constraints, has recovered precisely Ostrogradsky's Hamiltonian description of such systems.

6 Conclusion

This note has established the equivalence of the Hamiltonian formulation of *regular* higher order Lagrangian systems due to Ostrogradsky[7], with a constrained auxiliary description[8, 9] of such systems in which time derivatives of degrees of freedom of at most first order only are involved. The latter approach offers the following advantages, however.

In Ostrogradsky's construction, time derivatives of the coordinates of different order have to be considered as being *independent*. Such a situation is a possible source of confusion, especially at the quantum level when translating Poisson brackets into (anti)commutation relations for the fundamental quantum operators. Indeed, it is not always clear when to consider a time derivative of given order as an independent variable or as the first order time derivative of some other variable in Ostrogradsky's phase space. In the auxiliary approach, this issue is avoided altogether *ab initio*, since *independent* auxiliary degrees of freedom are introduced explicitly, each being associated with a time derivative of given order of each of the original degrees of freedom. In this manner, the local structure of phase space and its local symplectic geometry, is made perhaps much more transparent than in Ostrogradsky's approach.

More importantly however, the auxiliary formulation presents the additional advantage that the auxiliary Lagrangian depends on time derivatives of first order only. Therefore, *any* higher order Lagrangian system—*be it regular or not*—can always be brought into the realm of those Lagrangian systems for which a wealth of methods—classical and quantum—have been developed over the years. Due to the presence of auxiliary degrees of freedom, the auxiliary formulation always leads to constraints, requiring the techniques of constrained dynamics[2].

Finally, in contradistinction to Ostrogradsky's construction which applies to *regular* higher order systems *only*, the auxiliary formulation, being already a constrained one, does not require to distinguish between regular and singular higher order Lagrangian systems. Hence, the quantisation of such systems, including the BRST quantisation of singular ones, *does not necessitate a separate and generalised formalism not yet developed*. All the readily available methods of ordinary constrained quantisation—*and nothing more*—suffice for the Hamiltonian formulation and the quantisation of *any* higher order Lagrangian system. As this note has established, Ostrogradsky's construction is thereby recovered exactly in the case of regular systems. The case of singular systems however, is beyond the scope of the latter approach, and the auxiliary formulation then becomes unavoidable. In effect, precisely this method has been applied already to rigid particles for example, with important conclusions as to their quantum consistency[6].

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References

[1] See for example,
C. Batlle, J. Gomis, J.M. Pons and N. Román-Roy, *J. Phys.* **A21** (1988) 2693;
V.V. Nesterenko, *J. Phys.* **A22** (1989) 1673;
X. Gracia, J.M. Pons and N. Román-Roy, *J. Math. Phys.* **32** (1991) 2744.

[2] For a recent review, see Ref.[3].

[3] J. Govaerts, *Hamiltonian Quantisation and Constrained Dynamics*, (Leuven University Press, Leuven, 1991).

[4] A.M. Polyakov, *Nucl. Phys.* **B268** (1986) 406.

[5] H. Kleinert, *Phys. Lett.* **B174** (1986) 335.

[6] For an extended list of references, see
J. Govaerts, *Phys. Lett.* **B293** (1992) 327; *Z. Phys.* **C57** (1993) 59.

[7] A. Ostrogradsky, *Mémoires de l'Académie de Saint Petersbourg* **VI** (1850) 385.

[8] J.M. Pons, *Lett. Math. Phys.* **17** (1989) 181.

[9] See Ref.[3], page 2.

[10] P.A.M. Dirac, *Lectures on Quantum Mechanics* (Belfer Graduate School of Science, Yeshiva University, New York, 1964).