A Matrix Integral Solution to $[P, Q] = P$ and Matrix Laplace Transforms

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Abstract

In this paper we solve the following problems: (i) find two differential operators P and Q satisfying $[P, Q] = P$, where P flows according to the KP hierarchy $\partial P/\partial t_n = [(P^{n/p})_+, P]$, with $p := \operatorname{ord} P \geq 2$; (ii) find a matrix integral representation for the associated τ-function. First we construct an infinite dimensional space $\mathcal{W} = \operatorname{span}_{\mathbb{C}}\{\psi_0(z), \psi_1(z), \dots\}$ of functions of $z \in \mathbb{C}$ invariant under the action of two operators, multiplication by z^p and $A_c := z\,\partial/\partial z - z + c$. This requirement is satisfied, for arbitrary p, if ψ_0 is a certain function generalizing the classical Hänkel function (for $p = 2$); our representation of the generalized Hänkel function as a double Laplace transform of a simple function, which was unknown even for the $p = 2$ case, enables us to represent the τ-function associated with the KP time evolution of the space $\mathcal{W}$ as a "double matrix Laplace transform" in two different ways. One representation involves an integration over the space of matrices whose spectrum belongs to a wedge-shaped contour $\gamma := \gamma^+ + \gamma^- \subset \mathbb{C}$ defined by $\gamma^\pm = \mathbb{R}_+ e^{\pm\pi i/p}$. The new integrals above relate to the matrix Laplace transforms, in contrast with the matrix Fourier transforms, which generalize the Kontsevich integrals and solve the operator equation $[P, Q] = 1$.

Introduction

It is a long-standing puzzle in the theory of $2d$-gravity to find an adequate description of gravitational coupling of (p, q) minimal models. One part of it is to find two differential operators P and Q of order p and q respectively, such that $[P, Q] = f(P)$ for some function f. In the simplest case of $q = 1$ and $f \equiv 1$, such description is provided by 1-matrix models, especially by the Kontsevich integral and their generalizations; see [1, 19, 25]. Going along the chain, $2d$-gravity $\to$ equilateral triangles $\to$ discrete matrix models $\to$ Kontsevich models, this approach has lead to the discovery of integrable structures for non-perturbative partition functions, which take the form of τ-functions of the KP hierarchy (see [7, 25, 31] for review and references). While similar results are believed to be true in the general (p, q)-case, the Kontsevich integral counterparts are still unknown. Note that a minor modification of the generalized Kontsevich integral can be interpreted as a duality transformation between (p, q) and (q, p)-models [18].

So far the most promising approach for finding integrable structures in the general (p, q)-case seems to be the one initiated by Kac–Schwarz in the case $q = 1$ and $f = 1$. So, the general problem comes in two stages: (1) Find a point in Sato's Grassmannian invariant under two symmetry operators,

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satisfying some commutation relation; the existence of such a plane leads to a system of differential equations specifying the wave function Ψ and thus to an algebra of constraints for the τ-function. (2) Find a matrix integral representation for this τ-function. Note a matrix representation, beyond the case $q = 1$ and $f = 1$, if it exists at all, was unknown.

The purpose of this paper is to find a τ-function and a matrix integral representation for the equation $[P, Q] = P$ for $q = 1$ and arbitrary p . Remarkably, the matrix integral representation can still be found, but it is far less straightforward and considerably more involved, than the ordinary Kontsevich integral.

The message is the following: whereas the case $[P, Q] = 1$ is described by general *matrix Fourier transforms*, a solution to $[P, Q] = P$ is related to *double Laplace transforms*. While it is not known whether this solution has immediate physical relevance, it may help to shed some light on the (p, q)-case and on the matrix representations of the corresponding τ-functions. In particular, what are the proper multimatrix generalizations of the Kontsevich integrals?

Note this problem has come up in the physical literature, in various different contexts: unitary matrix models have been written down, leading to equations $[P, Q] = P$ for differential operators P and Q in the double scaling limit; see the studies of Dalley, Johnson, Periwal, Minahan, Morris, Shevitz, and Wätterstam [4, 5, 27, 28, 22, 23]). In the mathematical context (inverse scattering and monodromy preserving transformations), see Ablowitz, Flaschka, Fokas and Newell [11, 9, 10]). The solution provided in our paper is new and does *not* require any scaling limit.

Consider the problem of finding a differential operator P of order p and another differential operator Q satisfying

$$[P, Q] = f(P)\,, \quad \text{with } 0 \neq f(z) \in \mathbb{C}[z]\,. \tag{1}$$

When P is (formally) deformed with respect to the KP flows, i.e., $\partial P/\partial t_n = [(P^{n/p})_+, P]$, one can introduce the corresponding deformation of Q which preserves Eq. (1). Hence (1) can be considered as a condition on a solution of the p-reduced KP hierarchy.

The basic ingredients of this construction are[1]

- $\psi_0 \in 1 + z^{-1}\mathbb{C}[[z^{-1}]]$,
- $A\colon \mathbb{C}((z^{-1})) \to \mathbb{C}((z^{-1}))$ which increases the order of an element of $\mathbb{C}((z^{-1}))$ in z exactly by one,

so that $\mathcal{W} := \mathrm{span}_{\mathbb{C}}\{\psi_0, A\psi_0, A^2\psi_0, \ldots\}$ belongs to the big stratum of the Sato Grassmannian and satisfies $A\mathcal{W} \subset \mathcal{W}$, such that

- ψ_0 satisfies the differential equation $v(z)\psi_0 = F(A)\psi_0$ for some $v(z) \in \mathbb{C}((z^{-1}))$ and $F(Z) \in \mathbb{C}[Z]$, so that $v(z)\mathcal{W} \subset \mathcal{W}$ also holds.

Let Ψ be the KP wave function corresponding to $\mathcal{W}$. The above conditions lead to the existence of differential operators Q and P in x such that $Q\Psi = A\Psi$ and $P\Psi = v(z)\Psi$. If A coincides with $\partial/\partial v = (1/v')\partial/\partial z$ up to the conjugation by a function, then we have $[P, Q] = 1$. And if ψ_0 is defined by a Fourier transform and the action of A on it can be expressed in a suitable way, then the corresponding Hermitian *matrix* Fourier transform, properly normalized, is the corresponding τ-function. See Sect. 3 for details.

The matrix integral approach to (1) has so far needed $\mathrm{ord}\, Q = 1$ at the initial point of the formal KP time flows, requiring $\deg_z f(z) \leq 1$. The degree 0 case can be reduced to $[P, Q] = 1$. In this paper, we provide a solution to the degree 1 case, or the next simplest instance of (1), which can clearly be reduced to

$$[P, Q] = P\,, \tag{2}$$

with differential operators P and Q. As in the case of $[P, Q] = 1$, we write the τ-function of its formal KP deformation explicitly in terms of a matrix integral.

[1] $\mathbb{C}[[x]] := \{\sum_{n=0}^{\infty} a_n x^n \mid a_n \in \mathbb{C}\}$ is the ring of formal power series in x, and $\mathbb{C}((x)) := \{\sum_{-\infty \ll n < \infty} a_n x^n \mid a_n \in \mathbb{C}\}$ is the ring of formal Laurent series in x.

Definition 1 *Let $-1 < c < 0$, $p \in \mathbb{Z}$, $p \geq 2$. Let $\mathcal{W}$ be the linear span*

$$\mathcal{W} = \operatorname{span}_{\mathbb{C}}\{\psi_0(z), \psi_1(z), \psi_2(z), \ldots\},$$

of generalized Hänkel functions,

$$\psi_k(z) = \frac{p^{c+1}}{\Gamma(-c)} \int_1^\infty \frac{z^{-c}(uz)^k e^{-(u-1)z}}{(u^p-1)^{c+1}}\, du\,, \quad k = 0, 1, 2, \ldots, \tag{3}$$

also representable as double Laplace transforms

$$\psi_k(z) = \frac{p^{c+1}}{2\pi i} z^{(p-1)(c+1)} e^z \int_0^\infty dx\, x^c e^{-xz^p} \int_0^\infty dy\, f_k(y) e^{-xy^p} \tag{4}$$

of the functions

$$f_k(y) = (\zeta^{k+1} e^{-\zeta y} - \zeta^{-k-1} e^{-\zeta^{-1} y}) y^k, \quad k = 0, 1, 2, \ldots, \quad \text{with} \quad \zeta := e^{\pi i/p}. \tag{5}$$

Using the asymptotic expansion $\psi_k(z) = z^k\big(1 + O(1/z)\big) \in \mathbb{C}((z^{-1}))$ as $\Re z \to \infty$, $\mathcal{W}$ defines a point of the Sato Grassmannian Gr. Let Ψ and τ be the wave (formal Baker–Akhiezer) function and τ-function, respectively, associated with the KP time evolution $\mathcal{W}^t = e^{-\sum t_i z^i}\mathcal{W}$; see Sects. 1 and 2. Then we have

Theorem 1

$$\Psi(x, 0, z) = e^{xz}\psi_0\big((1-x)z\big)\,, \tag{6}$$

and it satisfies

$$\left(L\left(x-1, \frac{\partial}{\partial x}\right) - z^p\right)\Psi(x,0,z) = 0 \quad \text{and} \quad \left(L\left(z, \frac{\partial}{\partial z} - 1\right) - (x-1)^p\right)\Psi(x,0,z) = 0\,, \tag{7}$$

where $L(z, \partial/\partial z)$ is the monic differential operator

$$L\left(z, \frac{\partial}{\partial z}\right) := \frac{1}{z^p}\left(\prod_{i=0}^{p-1}\left(z\frac{\partial}{\partial z} + c - i\right) - cp\prod_{i=0}^{p-2}\left(z\frac{\partial}{\partial z} + c - i\right)\right) = \left(\frac{\partial}{\partial z}\right)^p + \cdots.$$

Note that for $p = 2$, $L(z, \partial/\partial z) = (\partial/\partial z)^2 - (c^2 + c)/z^2$.

Theorem 2 *Let $\mathcal{H}_N$ be the space of $N \times N$ Hermitian matrices, and $\mathcal{H}_N^+$ the subspace of $\mathcal{H}_N$ of positive definite Hermitian matrices. The corresponding τ-function evaluated at*

$$t_n := -\frac{1}{n}\operatorname{tr} Z^{-n}\,, \quad \text{for } n = 1, 2, \ldots, \text{ and with an } N \times N \text{ diagonal } Z,$$

is given by the following (normalized) double matrix *Laplace transform:*

$$\tau(t) = S_1(t)\frac{\int_{\mathcal{H}_N^+} dX \det X^c e^{-\operatorname{tr} Z^p X} \int_{\mathcal{H}_N^+} dY\, S_0(y) e^{-\operatorname{tr} XY^p}}{\int_{\mathcal{H}_N} dX \exp\operatorname{tr}\left(-\frac{(X+Z)^{p+1}}{p+1}\right)_2}\,,$$

where $(\,)_2$ denotes the terms quadratic in X,

$$S_0(y) := \frac{\Delta(y^p)}{\Delta(y)^2}\det\big(f_{k-1}(y_i)\big)_{1\leq i,k\leq N} \quad \text{and} \quad S_1(t) := \det(Z^{(p-1)(c+1/2)})e^{\operatorname{tr} Z},$$

where $y = (y_1, \ldots, y_N)$ are the eigenvalues of Y, $y^p = (y_1^p, \ldots, y_N^p)$, and $\Delta(y) := \prod_{i>j}(y_i - y_j) = \det(y_i^{j-1})_{i,j}$, and f_{k-1} are as in (5).

The function $\tau(t)$ also has the following matrix integral representation

$$\tau(t)=\frac{\int_{\mathcal{H}_N^\gamma} m(dW)\int_{\mathcal{H}_N^+} dX\,\det X^c\big(\Delta(w^p)/\Delta(w)\big)e^{\operatorname{tr}(Z-W)}e^{\operatorname{tr}X(W^p-Z^p)}}{\int_{\mathcal{H}_N} dX\exp\operatorname{tr}\left(-\frac{(X+Z)^{p+1}}{p+1}\right)_2},$$

integrated over the space of matrices

$$\mathcal{H}_N^\gamma=\left\{W=UD_\gamma U^{-1}\;\middle|\;U\in\mathbf{U}(N),D_\gamma:=\operatorname{diag}(w_1,\ldots,w_N)\in(\gamma)^N\right\},$$

where γ denotes a wedge-shaped contour in $\mathbb{C}$, defined in Sect. 4 (see Fig. 1), in terms of a complex-valued measure

$$m(dW)=dU\,dw\prod_{1\le i<j\le N}(w_i-w_j)^2.$$

Theorem 3 *(i) The algebra of stabilizers of $\mathcal{W}$,*

$$S_{\mathcal{W}}:=\left\{\phi(z,\partial/\partial z)\in\mathbb{C}((z^{-1}))[\partial/\partial z]\ \text{such that}\ \phi\mathcal{W}\subset\mathcal{W}\right\},$$

is generated by $A_c:=z\frac{\partial}{\partial z}-z+c$, z^p and $\xi:=z^{-p}F(A_c)$, where $F(u)=\prod_0^{p-1}(u-i)-cp\prod_0^{p-2}(u-i)$:

$$S_{\mathcal{W}}=\mathbb{C}[A_c,z^p,\xi]\subset\mathbb{C}((z^{-1}))[\partial/\partial z].$$

Moreover, $\mathcal{W}=\mathbb{C}[A_c]\psi_0$, and ψ_0 satisfies the differential equation

$$F(A_c)\psi_0=(-z)^p\psi_0(z)\,. \tag{8}$$

(ii) A family of solutions to the operator equation $[P,Q]=P$ is given by the differential operators P and Q, defined equivalently by

$$P\Psi=z^p\Psi\,,\quad Q\Psi=\frac{1}{p}A_c\Psi\,, \tag{9}$$

or by

$$P=S\left(\frac{d}{dx}\right)^pS^{-1}\quad\text{and}\quad Q=\frac{1}{p}(MP^{1/p}-P^{1/p}+c)\,,$$

where $M=S\left(\sum_1^\infty k\bar{t}_k(d/dx)^{k-1}\right)S^{-1}$, $\bar{t}_k=t_k+\delta_{k,1}x$, with wave operator

$$S=\frac{\tau(\bar{t}-[(d/dx)^{-1}])}{\tau(\bar{t})}\,.$$

(iii) The function $\tau(t)$ satisfies, in terms of the W-generators in Eq. (20), the following constraints

$$\sum_{\substack{0\le i\le m\\0\le j\le i}}\alpha_{m,i}\binom{i}{j}\frac{(-1)^{i-j}}{j+1}W^{(j+1)}_{i+np-j}\tau(t)=a_{m,n,c}\tau(t)\,,\quad m,n=0,1,2,\ldots, \tag{10}$$

for some constants $a_{m,n,c}$, where the constants $\alpha_{n,i}$ are defined by the formula[2] $(x\cdot d/dx)^n=\sum_{i=0}^n\alpha_{n,i}x^i(d/dx)^i$. *In particular, setting $m=1$, $\tau(t)$ satisfies Virasoro constraints of the form (with $W^{(2)}_{np}=\sum_{i+j=np}{:}J^{(1)}_iJ^{(1)}_j{:}$)*

$$\left(\frac{1}{2}W^{(2)}_{np}-\frac{\partial}{\partial t_{np+1}}-a_{1,n,c}\right)\tau=0\,,\quad n=0,1,2,\ldots. \tag{11}$$

Remark 1 *The constants $a_{m,n,c}$ in (10) can all be calculated; in particular, the Virasoro constraint (11) for $n=0$ becomes:*

$$\left(\sum_1^\infty it_i\frac{\partial}{\partial t_i}-\frac{\partial}{\partial t_1}-\frac{c(1+c)(p-1)}{2}\right)\tau=0\,.$$

[2] More explicitly, $\alpha_{n,i}=\frac{1}{i!}\sum_{j=0}^i\binom{i}{j}(-1)^{i-j}j^n$. Note that it vanishes if $n>0$ and $i=0$.

Contents

1 The KP Hierarchy

Throughout, x is a formal scalar variable near 0, and z is a formal scalar variable near ∞. If $g(z) = cz^q\big(1 + O(z^{-1})\big)$, $c \neq 0$, then $\operatorname{ord}_z g(z) := q$ is the *order* of $g(z)$.

Throughout, we denote $\partial/\partial x$ by D. The algebra of ordinary pseudodifferential operators in x is denoted by $\mathcal{D}$ (the word "in x" may be dropped if there is no fear of confusion), with its splitting $\mathcal{D} = \mathcal{D}_+ + \mathcal{D}_-$ into the subalgebras of ordinary differential operators and of ordinary pseudodifferential operators of negative order:

$$\mathcal{D} = \left\{ \sum_{-\infty < i \leq n} a_i D^i \;\middle|\; n \in \mathbb{Z} \text{ arbitrary, } \; a_i = a_i(x) \right\},$$

$$A = \sum a_i D^i \in \mathcal{D} \quad \Rightarrow \quad A_+ = \sum_{i \geq 0} a_i D^i \in \mathcal{D}_+ \;\text{ and }\; A_- = A - A_+ \in \mathcal{D}_- \,.$$

The ring $\mathcal{D}$ acts on the space of functions of the form $\sum_{-\infty < i \ll \infty} a_i(x) z^i e^{xz}$ simply by extending the formulas $D^n e^{xz} = z^n e^{xz}$ and $A(Be^{xz}) = (A \circ B)e^{xz}$, A, $B \in \mathcal{D}$. When $A \in \mathcal{D}_+$, this definition of $A(Be^{xz})$ coincides with the usual action of A, as a differential operator, on Be^{xz} as a formal series in x with z-dependent coefficients.

A pseudodifferential operator in x may depend on the KP time variables $t = (t_1, t_2, \dots)$ introduced below, but not on z unless otherwise noted. We are not specific about the regularity of the coefficients of pseudodifferential operators. The operators S, L, M etc., associated to a point $\mathcal{W}$ of the big stratum Gr^0 of the Sato Grassmannian (see below) have regular (i.e., formal power series) coefficients; otherwise, the singularities of those operators can be controled by the Schubert stratum to which $\mathcal{W} \in \mathrm{Gr}$ belongs. In particular, there exist n, $m \geq 0$ such that $x^n S$ and $S^{-1} x^m$ at $t = 0$ have regular coefficients. See [29] for details.

As in [2], we set $\bar{t} = (x + t_1, t_2, t_3, \dots)$, and

$$\tilde{\partial} = \left(\frac{\partial}{\partial t_1}, \frac{1}{2}\frac{\partial}{\partial t_2}, \frac{1}{3}\frac{\partial}{\partial t_3}, \dots \right).$$

The elementary Schur functions p_n are defined by $\exp\left(\sum_1^\infty t_n z^n\right) = \sum_0^\infty p_n(t) z^n$.

1.1 KP hierarchy

The operator $L = L(t) = D + \sum_{j=-\infty}^{-1} a_j(x,t) D^j \in \mathcal{D}$, with $t = (t_1, t_2, \dots)$, subjected to the KP equations

$$\frac{\partial L}{\partial t_n} = [(L^n)_+, L] \,, \quad n = 1, 2, \dots,$$

is known to have the following representation in terms of an operator $S \in 1+\mathcal{D}_-$ called the wave operator, and the associated, formally infinite order pseudodifferential operator

$$W := Se^{\sum_{i=1}^{\infty} t_i D^i},$$

as follows:

$$L = SDS^{-1} = WDW^{-1}, \tag{12}$$

$$\frac{\partial S}{\partial t_n} = -(L^n)_- S\,, \quad \text{and} \quad \frac{\partial W}{\partial t_n} = (L^n)_+ W\,.$$

The wave function

$$\Psi(t,z) := \Psi(x,t,z) := We^{xz} = Se^{\sum_{i=1}^{\infty} \bar{t}_i z^i}\,, \tag{13}$$

where $\bar{t}_i = t_i + \delta_{i,1}x$, satisfies

$$L\Psi = z\Psi \quad \text{and} \quad \frac{\partial \Psi}{\partial t_n} = (L^n)_+\Psi\,, \tag{14}$$

and has the following representation in terms of a scalar-valued function associated to S called the tau function τ:

$$\begin{aligned} \Psi(t,z) &= \frac{\tau(\bar{t} - [z^{-1}])}{\tau(\bar{t})} e^{\sum_1^\infty \bar{t}_i z^i} \\ &= \sum_{n=0}^{\infty} \frac{p_n(-\tilde{\partial})\tau(\bar{t})}{\tau(\bar{t})} z^{-n} e^{\sum_1^\infty \bar{t}_i z^i} \\ &= \sum_{n=0}^{\infty} \frac{p_n(-\tilde{\partial})\tau(\bar{t})}{\tau(\bar{t})} D^{-n} e^{\sum_1^\infty \bar{t}_i z^i}, \end{aligned}$$

implying in view of (13)

$$S = \frac{\tau(\bar{t} - [D^{-1}])}{\tau(\bar{t})} := \sum_{n=0}^{\infty} \frac{p_n(-\tilde{\partial})\tau(\bar{t})}{\tau(\bar{t})} D^{-n}. \tag{15}$$

Moreover, using (13), we have

$$\frac{\partial}{\partial z}\Psi = \frac{\partial}{\partial z} We^{xz} = W\frac{\partial}{\partial z}e^{xz} = Wxe^{xz} = WxW^{-1}\Psi\,,$$

thus leading to the operator

$$\begin{aligned} M &:= WxW^{-1} = Se^{\sum t_k D^k} x e^{-\sum t_k D^k} S^{-1} = S\left(x + \sum_1^\infty k t_k D^{k-1}\right) S^{-1} \\ &= S\left(\sum_1^\infty k\bar{t}_k D^{k-1}\right) S^{-1} \end{aligned} \tag{16}$$

satisfying

$$M\Psi = (\partial/\partial z)\Psi \quad \text{and} \quad [L,M] = W[D,x]W^{-1} = 1\,,$$

and for any formal series $f = f(x,\xi)$,

$$f(M,L) = Wf(x,D)W^{-1}\,. \tag{17}$$

1.2 Symmetries

Consider the Lie algebra w_∞ of operators

$$w_\infty := \mathbb{C}[z,z^{-1}][d/dz] = \mathrm{span}_{\mathbb{C}}\left\{ z^\alpha \left(\frac{\partial}{\partial z}\right)^\beta \;\middle|\; \alpha,\beta\in\mathbb{Z}, \beta\geq 0\right\},$$

and its completion $\overline{w}_\infty := \mathbb{C}((z^{-1}))[\partial/\partial z]$ in the z^{-1}-adic topology, for the customary commutation relation $[\ ,\ ]$. Acting on Ψ, we have

$$z^\alpha(\partial/\partial z)^\beta \Psi = M^\beta L^\alpha \Psi\,, \tag{18}$$

motivating the definition of the following vector fields, called symmetries, on Ψ:

$$\mathbb{Y}_{z^\alpha(\partial/\partial z)^\beta}\Psi := (M^\beta L^\alpha)_-\Psi\,.$$

We require that these flows act trivially on parameters x, t, and hence on $S^{-1}MS = \sum k\bar{t}_k D^{k-1}$, for instance.

Lemma 1 *There is an injective homomorphism of Lie algebras*

$$\begin{array}{ccc} \overline{w}_\infty/\mathbb{C} & \longrightarrow & \left\{\begin{array}{l} \textit{Lie algebra of vector fields} \\ \textit{on the manifold of wave functions } \Psi \\ \textit{commuting with the KP flows } \partial/\partial t_n \end{array}\right\} \\ z^\alpha\left(\dfrac{\partial}{\partial z}\right)^\beta & \longmapsto & \mathbb{Y}_{z^\alpha(\partial/\partial z)^\beta}\Psi = (M^\beta L^\alpha)_-\Psi\,, \end{array}$$

i.e.,

$$\left[\mathbb{Y}_{z^\alpha(\partial/\partial z)^\beta}, \mathbb{Y}_{z^{\alpha'}(\partial/\partial z)^{\beta'}}\right] = \mathbb{Y}_{[z^\alpha(\partial/\partial z)^\beta, z^{\alpha'}(\partial/\partial z)^{\beta'}]}\,.$$

This definition differs from the one in [2] by the sign. Here this definition is chosen to make it consistent with the natural action of $\overline{w}_\infty$ on the Grassmannian discussed in the next section, rather than its negative. These vector fields induce vector fields on S and $L = SDS^{-1}$, as

$$\mathbb{Y}_{z^\alpha(\partial/\partial z)^\beta}(S) = (M^\beta L^\alpha)_- S$$

and

$$\mathbb{Y}_{z^\alpha(\partial/\partial z)^\beta}(L) = [(M^\beta L^\alpha)_-, L]\,.$$

Proposition 1 ([2]) *We have*

$$-\frac{(M^n L^{n+\ell})_-\Psi}{\Psi} = (e^{-\eta}-1)\left.\frac{\frac{1}{n+1}W_\ell^{(n+1)}(\tau)}{\tau}\right|_{t_1\to t_1+x}, \quad n,\ell\in\mathbb{Z},\ n\geq 0, \tag{19}$$

where the $W_\ell^{(n+1)}$, the generators of the W_∞-algebra, are the coefficients in the expansion of the vertex operator

$$\begin{aligned} X(t,\lambda,\mu) \quad &:= \quad \exp\left(\sum_{i=1}^\infty(\mu^i-\lambda^i)t_i\right)\exp\left(\sum_{i=1}^\infty \frac{\lambda^{-i}-\mu^{-i}}{i}\frac{\partial}{\partial t_i}\right) \\ &= \quad \sum_{k=0}^\infty \frac{(\mu-\lambda)^k}{k!}\sum_{\ell=-\infty}^\infty \lambda^{-\ell-k}W_\ell^{(k)}, \quad \textit{with}\ \ W_\ell^{(0)} = \delta_{\ell,0}. \end{aligned} \tag{20}$$

2 Grassmannian

Let $H := \mathbb{C}((z^{-1}))$, $H_+ := \mathbb{C}[z]$, and $H_- := z^{-1}\mathbb{C}[[z^{-1}]]$, so that $H = H_+ \oplus H_-$. We denote by Gr the Grassmannian manifold of linear subspaces $\mathcal{W}$ of H of relative dimension 0 with respect to H_+, i.e., the natural map

$$\pi_{\mathcal{W}} : \mathcal{W} \hookrightarrow H \xrightarrow{\pi} H/H_- \simeq H_+$$

being Fredholm of index 0. $\mathrm{Gr}^0 := \{\mathcal{W} \in \mathrm{Gr} \mid \pi_{\mathcal{W}} \text{ is isomorphism}\}$ is the big (open) Schubert stratum of Gr.

Given a wave function $\Psi = \Psi(x, t, z)$, let $\mathcal{W}$ be the point of Gr defined by[3]

$$\begin{aligned} \mathcal{W} &= \mathrm{span}_{\mathbb{C}}\left\{ \frac{\partial^j}{\partial x^j}\Psi(0,0,z) \,\middle|\, j = 0,1,2,\dots \right\} \\ &= \mathrm{span}_{\mathbb{C}}\left\{ \frac{\partial^{j_1+\cdots+j_N}}{\partial t_1^{j_1}\dots\partial t_N^{j_N}}\Psi(0,0,z) \,\middle|\, N \geq 0,\ j_1,\dots,j_N \geq 0 \right\}. \end{aligned}$$

The first line guarantees $\mathcal{W} \in \mathrm{Gr}$, and the second line follows from the first by using the second equation in (14), i.e., the KP time evolutions of Ψ. Hence up to the t-adic completion we have

$$\mathcal{W} = \mathrm{span}_{\mathbb{C}}\left\{ \left(\frac{\partial}{\partial x}\right)^j \Psi(0,t,z) \,\middle|\, j = 0,1,2,\dots \right\},$$

so that, letting $\psi = e^{-\sum t_i z^i}\Psi$ and

$$\mathcal{W}^t := e^{-\sum t_i z^i}\mathcal{W} = \mathrm{span}_{\mathbb{C}}\left\{ (\partial/\partial x)^j \psi(0,t,z) \mid j = 0,1,2,\dots \right\},$$

we have $\psi = (\pi_{\mathcal{W}^t})^{-1}(1)$, i.e., ψ is the preimage of 1 by the map $\pi_{\mathcal{W}^t} : \mathcal{W}^t \to H_+$.

The corresponding τ-function $\tau(t)$ is the determinant of the composite map

$$\mathcal{W} \xrightarrow{g} \mathcal{W}^t \xrightarrow{\pi_{\mathcal{W}^t}} H/H_- \simeq H_+, \tag{21}$$

where g denotes the multiplication by $e^{-\sum t_i z^i}$. Given $\mathcal{W}$, the determinant is well-defined up to a constant which is determined by the choice of a basis $\{\psi_k\}_{k=0}^{\infty}$, $\psi_k = z^k\bigl(1 + O(z^{-1})\bigr)$ for $k \gg 0$, of $\mathcal{W}$. We take $\{z^k\}_{k=0}^{\infty}$ as the basis of H_+. More specifically, $\tau(t)$ is defined as the limit as $n \to \infty$ of the determinant of

$$\mathcal{W}_n \hookrightarrow \mathcal{W} \to H_+ \to H_+/z^n H_+, \tag{22}$$

where the middle arrow is the composite map in (21), $\mathcal{W}_n = \mathrm{span}_{\mathbb{C}}\{\psi_k\}_{k=0}^{n-1}$, and the determinant is computed with respect to the bases $\{\psi_k\}_{k=0}^{n-1}$ of $\mathcal{W}_n$ and $\{z^k\}_{k=0}^{n-1}$ of $H_+/z^n H_+$. The limit exists in the t-adic topology of $\mathbb{C}[[t]]$, i.e., for any multi-index α, there exists a positive integer n_α such that, if $n \geq n_\alpha$, then the coefficient of t^α in the determinant of (22) is independent of n, and gives the coefficient of t^α in $\tau(t)$. This finiteness property is an immediate consequence of the fact that, expanding $\tau(t)$ in terms of Schur functions, the coefficients give the Plücker coordinates of $\mathcal{W}$. See [29] for details.

The $\overline{w}_\infty$-action on Ψ becomes the natural action of $\overline{w}_\infty$ on Gr: As an ordinary differential operator in z, each $A \in \overline{w}_\infty$ acts on H, which defines a vector field on Gr.

2.1 Stabilizers

Given $\mathcal{W} \in \mathrm{Gr}$, we shall call

$$S_{\mathcal{W}} := \left\{ Q := Q(z, \partial/\partial z) \in \overline{w}_\infty \mid Q\mathcal{W} \subset \mathcal{W} \right\}$$

the *stabilizer* of $\mathcal{W}$. In this subsection we shall observe basic properties of the stabilizer which can be obtained without referring to matrix integrals.

[3] If Ψ is singular at $(x,t) = 0$, we need to replace $(\partial/\partial x)^j\Psi(0,0,z)$ in the first line by $(\partial/\partial x)^j\left(x^n\Psi(x,0,z)\right)\Big|_{x=0}$ for some $n > 0$, and make a similar replacement in the second line (see [29] for details). We chose to write the formulas for $\mathcal{W} \in \mathrm{Gr}^0$ for simplicity.

Lemma 2 *Let* $\mathcal{W} \in \mathrm{Gr}$ *and* $A := \sum_{-\infty < i \ll \infty, 0 \le j \ll \infty} c_{ij} z^i (\partial/\partial z)^j \in \overline{w}_\infty$. *If*

$$A\mathcal{W} \subset \mathcal{W}\,, \tag{23}$$

then

$$Q_A := \sum_{\substack{-\infty < i \ll \infty \\ 0 \le j \ll \infty}} c_{ij} M^j L^i \in \mathcal{D}_+\,.$$

Conversely, if $Q \in \mathcal{D}_+$ *is of this form, i.e.,* $Q = Q_A$ *for some* $A \in \overline{w}_\infty$*, then this* A *satisfies (23).*

Proof: We have

$$A\Psi(t,z) = Q_A \Psi(t,z) \tag{24}$$

by definition. Since $A\mathcal{W} \subset \mathcal{W}$, and since the Taylor coefficients (or Laurent coefficients if $\mathcal{W} \notin \mathrm{Gr}^0$) in x of Ψ generates $\mathcal{W}$, $A\Psi$ is a $\mathbb{C}[[x,t]]$-linear combination of Ψ, $D\Psi$, $D^2\Psi$, $\dots$, i.e., $A\Psi = Q\Psi$ for some $Q \in \mathcal{D}_+$. Hence, since (24) determines Q_A uniquely, Q_A itself must be in $\mathcal{D}_+$. Conversely, suppose $Q_A \in \mathcal{D}_+$, and let $\Psi(x,0,z) = \sum f_n(z) x^n$ be the Taylor (or Laurent) expansion of $\Psi(x,0,z)$ at $x = 0$. Then each Taylor coefficient in x of $Q_A\Psi$ is a linear combination of $\{f_n(z)\}$, and hence it belongs to $\mathcal{W}$, so that by (24) $Af_n \in \mathcal{W}$ for every n (the action of A on f_n is well-defined since A is a differential operator in z). Since $\{f_n\}$ is a basis of $\mathcal{W}$, we have $A\mathcal{W} \subset \mathcal{W}$. ∎

Corollary 1 *Let* $p \neq 0$ *be an integer, and let* $Q \in \mathcal{D}_+$ *such that* $\mathrm{ad}(L^p)^N Q = 0$ *for* $N \gg 0$. *Then* $Q = Q_A$ *for some* $A \in \overline{w}_\infty$ *such that* $A\mathcal{W} \subset \mathcal{W}$ *holds. In particular, a solution to the string equation (1) always comes from a pair of* $A \in \overline{w}_\infty$ *and* $\mathcal{W} \in \mathrm{Gr}$*, such that* $A\mathcal{W} \subset \mathcal{W}$ *(and* $z^p\mathcal{W} \subset \mathcal{W}$ *due to the extra assumption* $P = L^p \in \mathcal{D}_+$*).*

Proof: Writing $Q = \sum_{ij} c_{ij} M^j L^i$, let $A = \sum_{ij} c_{ij} z^i (\partial/\partial z)^j$. Since $\mathrm{ad}(L^p)^N Q = 0$ we have $\mathrm{ad}(z^p)^N A = 0$, which implies that A is a differential operator in z. Hence the "converse" part of Lemma 2 applies. ∎

Lemma 3 *Let* A, $B \in \overline{w}_\infty$, $\psi_0 = 1 + O(z^{-1}) \in 1 + H_-$ *and* $\mathcal{W} \in \mathrm{Gr}$. *Suppose* A *acts on the monomials* z^k, $k \in \mathbb{Z}$, *as*

$$Az^k = z^{k+1}\big(c_k + O(z^{-1})\big)\,,$$

and $c_k \neq 0$ *if* $k \ge 0$. *Then the following conditions are equivalent:*

(i) $\psi_0 \in \mathcal{W}$, $A\mathcal{W} \subset \mathcal{W}$ *and* $B\mathcal{W} \subset \mathcal{W}$*;*

(ii) $\mathcal{W} = \mathrm{span}_{\mathbb{C}}\{\psi_0, A\psi_0, A^2\psi_0, \dots\}$*, and* ψ_0 *satisfies the differential equations*

$$BA^n\psi_0 = F_n(A)\psi_0\,, \quad n = 0, 1, \dots \tag{25}$$

for some $F_n(s) \in \mathbb{C}[s]$.

In particular, under these conditions $\mathcal{W}$ *belongs to the big stratum* Gr^0 *of* Gr. *If, moreover,* A *and* B *satisfy a commutation relation of the form*

$$[A,B] = a(A)B + b(A) \tag{26}$$

for some $a(s)$, $b(s) \in \mathbb{C}[s]$*, then in (25) it suffices to assume only the* $n = 0$ *case, i.e.,*

$$B\psi_0 = F(A)\psi_0 \tag{27}$$

for some $F(s) \in \mathbb{C}[s]$.

Proof: Since $\psi_0 \in \mathcal{W}$, $A\mathcal{W} \subset \mathcal{W}$ implies $\mathcal{W}' := \mathrm{span}_{\mathbb{C}}\{\psi_0, A\psi_0, A^2\psi_0, \ldots\} \subset \mathcal{W}$. Since $\psi_0 = 1 + O(z^{-1})$ and A raises the order of a function in z by 1, the map $\mathcal{W}' \to H_+$ is bijective, and $\mathcal{W}' \in \mathrm{Gr}^0$. In particular, both $\mathcal{W}$ and $\mathcal{W}'$ are of relative dimension 0, so that $\mathcal{W} = \mathcal{W}'$. Conversely, $\mathcal{W} = \mathcal{W}'$ clearly implies $\psi_0 \in \mathcal{W}$ and $A\mathcal{W} \subset \mathcal{W}$. Assume these equivalent conditions. Then $B\mathcal{W} \subset \mathcal{W}$ if and only if $B\mathcal{W}' \subset \mathcal{W}'$ if and only if the differential equations of the form (25) are satisfied. Finally, when A and B satisfy a commutation relation of the form (26), the n^{th} equation in (25) implies the $(n+1)^{\mathrm{st}}$ one, so that (27) suffices. ∎

The following propositions take a closer look at the $[P,Q] = 1$ case and $[P,Q] = P$ case, to show that essentially those elements in $\overline{w}_\infty$ which give rise to P and Q in the sense of Lemma 2, and their polynomials, are the only elements of the stabilizer.

Proposition 2 *Let $p \in \mathbb{Z}$, $p > 0$. Let $A \in \overline{w}_\infty$ be such that $[A, z^p] = 1$. If $\mathcal{W} \in \mathrm{Gr}$ satisfies $z^p\mathcal{W} \subset \mathcal{W}$ and $A\mathcal{W} \subset \mathcal{W}$, then the stabilizer of $\mathcal{W}$ is generated by z^p and A, i.e., $S_{\mathcal{W}} = \mathbb{C}[A, z^p]$.*

Proof: Since $[A, z^p] = 1$, A is a first order differential operator in z, so that any $C \in S_{\mathcal{W}}$ can be written as $C = \sum_{-\infty < i \ll \infty, 0 \le j \le N} a_{ij} z^i A^j$ for some $N \ge 0$. It suffices to prove that $a_{ij} = 0$ if $i < 0$ or if $i \not\equiv 0 \bmod p$. Suppose A raises the order of a function in z by k: $\mathrm{ord}_z\, Az^\ell = \ell + k$. Let I be the set of pairs (i,j) such that $i < 0$ or $i \not\equiv 0 \bmod p$, $a_{ij} \ne 0$, and $i + kj$ is maximum among all such a_{ij}'s. We have $|I| < \infty$, and we only need to prove $|I| = 0$. Suppose this is not true. Let $C_0 := \sum_{(i,j)\in I} a_{ij} z^i A^j$. Noting

$$[A, z^i A^j] = [A, (z^p)^{i/p}]A^j = (i/p) z^{i-p} A^j,$$

so that $\mathrm{ad}(A)^n (z^i A^j) = 0$ for $n \gg 0$ if and only if $i \ge 0$ and $i \equiv 0 \bmod p$, we see that for $n \gg 0$ the leading terms of $\mathrm{ad}(A)^n C$ are $\mathrm{ad}(A)^n C_0$, which lowers the order of a function in z, and does not annihilate the function for a general n. This cannot happen since $\mathrm{ad}(A)^n C\mathcal{W} \subset \mathcal{W}$, and since in $\mathcal{W}$ the order of functions in z are bounded from below. ∎

Proposition 3 *Let $p \in \mathbb{Z}$, $p > 0$. Let $A = z\partial/\partial z - a(z)$, where $a(z) \in z + \mathbb{C}[[z^{-1}]]$, and $\psi_0 = 1 + O(z^{-1}) \in 1 + H_-$. Let $\mathcal{W} \in \mathrm{Gr}$ be the point of the Grassmannian determined by the conditions $\psi_0 \in \mathcal{W}$ and $A\mathcal{W} \subset \mathcal{W}$. Suppose $\mathcal{W}$ also satisfies $z^p\mathcal{W} \subset \mathcal{W}$. Let $F(s) = c\prod_{i=1}^p (s - c_i) \in \mathbb{C}[s]$, where $c_i \in \mathbb{C}$, $c \in \mathbb{C}^*$, be the polynomial of degree p as in (27) with $B = z^p$, i.e., ψ_0 satisfies the equation*

$$F(A)\psi_0 = z^p\psi_0\,. \tag{28}$$

Then if F satisfies the following genericity condition:

(G) For any $n \not\equiv 0 \bmod p$, we have $(F) + n := \sum(c_i + n) \not\equiv (F) \bmod p$, i.e., $\pi_p\big((F) + n\big) \ne \pi_p(F)$, where $(F) = \sum_{i=1}^p (c_i)$ is the divisor of F, and $\pi_p\colon \mathbb{C} \to \mathbb{C}/p\mathbb{Z}$ is the natural projection,

then the stabilizer of $\mathcal{W}$ is generated by A, z^p and $\xi := z^{-p}F(A)$, i.e.,

$$S_{\mathcal{W}} = \mathbb{C}[A, z^p, \xi]\,. \tag{29}$$

Remark 2 *Condition (G) is equivalent to*

(G′) There does not *exist $n \mid p$, $0 < n < p$, and $H(s) \in \mathbb{C}[s]$ of degree n such that $F(s) = \prod_{i=0}^{p/n} H(s - in)$;*

and if it is not satisfied, i.e., if $F(s) = \prod_{i=0}^{p/n} H(s - in)$ for some $n \mid p$ and H, then taking such (n, H) of the smallest n, we observe from our proof below that $\mathbb{C}[A, z^p, \xi] \subset S_{\mathcal{W}} \subset \mathbb{C}[A, z^n, \xi']$, where $\xi' = z^{-n}H(A)$.

Remark 3 *The right-hand side of (29) equals* $\sum_{i,j,k\geq 0}\mathbb{C}a^ib^jc^k$*, where* (a,b,c) *is any permutation of* (A,z^p,ξ)*; the order does not matter because*

$$[A,z^p]=pz^p\,,\quad [A,\xi]=-pF(A)\quad \textit{and}\quad [z^p,\xi]=F(A)-F(A-p)\,. \tag{30}$$

Remark 4 *Condition (G) is satisfied by the* F *in Theorem 3: Since*

$$F(s)=\left(\prod_{i=0}^{p-2}(s-i)\right)\big(s-(p-1+cp)\big)$$

and $-1<c<0$*, there is no period less than* p *in the divisor of* F *modulo* p*.*

Proof of Prop. 3. Using the commutation relations (30), the definition of $\mathcal{W}$, and Eq. (28), we observe easily that $S_{\mathcal{W}}\supset\mathbb{C}[A,z^p,\xi]$. We prove the converse inclusion in two steps. Only Step 2 needs Condition (G).

Step 1. We observe that $S_{\mathcal{W}}$ is spanned by the z-homogeneous elements in $S_{\mathcal{W}}$, i.e., the elements of $S_{\mathcal{W}}$ of the form $z^nf(A)$, where $n\in\mathbb{Z}$ and $f(s)\in\mathbb{C}[s]$.

Indeed, let $S'\subset S_{\mathcal{W}}$ be the subspace of $S_{\mathcal{W}}$ spanned by the z-homogeneous elements, and suppose that $S'':=S_{\mathcal{W}}\setminus S'\neq\emptyset$. Let N be a nonnegative integer such that

$$S''^{(N)}:=\left\{C\in S''\;\middle|\;\operatorname{ord}_{\partial/\partial z}C\leq N\right\}$$

is nonempty. Let $C\in S''^{(N)}$ be such that, writing

$$C=\sum z^nf_n(A)\,, \tag{31}$$

$n_0(C):=\max\{n\mid f_n\not\equiv 0\}$ is the smallest in $S''^{(N)}$. Such a C exists because

Claim 1 $\left\{n_0(C)\;\middle|\;C\in S''^{(N)}\right\}$ *is bounded below.*

Proof: Indeed it is bounded from below by $-2N+1$: since $C\in S''^{(N)}$ is an ordinary differential operator of order $\leq N$, and since ψ_0, $A\psi_0,\ldots,A^{N-1}\psi_0$ are linearly independent, we have $CA^i\psi_0\not\equiv 0$ for some i, $0\leq i<N$. Since $A^i\psi_0=(-1)^iz^i\big(1+O(z^{-1})\big)$, since $C\mathcal{W}\subset\mathcal{W}$, and since $\mathcal{W}$ is a span of $A^j\psi_0$ for $j\geq 0$, we observe that C does not decrease the order of $A^i\psi_0$ in z by more than $N-1$. This implies, using the notation of (31), that $n+\deg f_n\geq-(N-1)$ for some n. Hence $n_0(C)\geq n\geq-\deg f_n-(N-1)\geq-(2N-1)$. ∎

Now let

$$C':=[A,C]-n_0(C)C=\sum\big(n-n_0(C)\big)z^nf_n(A)\,. \tag{32}$$

Clearly $C'\in S_{\mathcal{W}}$. We have $\operatorname{ord}_{\partial/\partial z}C'\leq\operatorname{ord}_{\partial/\partial z}C\leq N$, and $n_0(C')\leq n_0(C)-1$. Hence by the minimality of $n_0(C)$, we must have $C'\notin S''^{(N)}$, so that $C'\in S'$. Thus each term $\big(n-n_0(C)\big)z^nf_n(A)$ in (32) belongs to S', and only finitely many f_n are non-zero. As a finite linear combination of such, we have $C'':=C-z^{n_0(C)}f_{n_0(C)}(A)\in S'$, so that $z^{n_0(C)}f_{n_0(C)}(A)=C-C''$ must also belong to $S_{\mathcal{W}}$, and hence to S', since it is z-homogeneous. This implies $C=C''+(C-C'')\in S'$, which is a contradiction.

Step 2. Let $f(s)\not\equiv 0$ be any constant coefficient polynomial, and let n be an integer. We prove that

$$z^nf(A)\in S_{\mathcal{W}}\quad\text{implies}\quad p\mid n\,,$$

and that, when $n<0$, $z^nf(A)\in S_{\mathcal{W}}$ must have the form $\xi^kh(A)$ for $k:=-n/p>0$ and some $h(s)\in\mathbb{C}[s]$.

Suppose $z^nf(A)\in S_{\mathcal{W}}$. We assume $n\neq 0$ without loss of generality. Since $z^nf(A)\psi_0\in\mathcal{W}$, by Lemma 3 there exists another polynomial $g(s)\in\mathbb{C}[s]$, such that

$$z^nf(A)\psi_0=g(A)\psi_0\,. \tag{33}$$

First assume $n > 0$. Let $\ell > 0$ be the least common multiple of p and n. Noting

$$z^{2p}\psi_0 = z^p F(A)\psi_0 = F(A-p)z^p\psi_0 = F(A-p)F(A)\psi_0$$

etc., we have

$$\left(\prod_{i=0}^{\ell/p-1} F(A-ip)\right)\psi_0 = z^\ell\psi_0 \tag{34}$$

from (28), and

$$\left(\prod_{j=0}^{\ell/n-1} G(A-jn)\right)\psi_0 = z^\ell\psi_0 \tag{35}$$

from (33), where $G(s) = g(s)/f(s-n)$ is a rational function in s, and $G(A-jn)$ in (35) is understood as an element of the field of fractions of $\mathbb{C}[A]$; this makes sense because, since $\{A^n\psi_0\}_{n=0,1,\ldots}$ is linearly independent, the representation

$$\mathbb{C}[s] \ni f(s) \mapsto f(A)\psi_0 \in \mathcal{W}$$

is faithful.

Comparing the left-hand sides of (34) and (35), we thus have the equality

$$\prod_{i=0}^{\ell/p-1} F(s-ip) = \prod_{j=0}^{\ell/n-1} G(s-jn) \tag{36}$$

of rational functions in s. Since the left-hand side of it is a polynomial of s, so is the right-hand side. Let D be the divisor of this polynomial, and let π_ℓ be the natural map $\mathbb{C} \to \mathbb{C}/\ell\mathbb{Z}$. From the left-(resp. right-)hand side of (36) the image $\pi_\ell(D)$ of divisor D on the cylinder $\mathbb{C}/\ell\mathbb{Z}$ is invariant under the translation by p (resp. n). But the genericity condition (G) implies that if $\pi_\ell(D)$ is invariant under the translation by $k \in \mathbb{Z}$, then $p \mid k$. Hence $p \mid n$.

Note here that, since ℓ is the least common multiple of p and n, this implies $\ell = n$, so that the right-hand side of (36) is $G(s)$ itself. Hence

$$g(s)/f(s-n) = G(s) = \prod_{i=0}^{n/p-1} F(s-ip)\,.$$

In particular, $g(s)/f(s-n)$ is a polynomial.

In the case where $n < 0$, after rewriting (33) as

$$z^{-n}g(A)\psi_0 = f(A)\psi_0\,,$$

we switch the roles of f and g, and n and $-n$, to proceed exactly the same way to prove $p \mid n$ and

$$f(s)/g(s+n) = \prod_{i=0}^{-n/p-1} F(s-ip)\,.$$

Thus we have

$$\begin{aligned} z^n f(A) &= z^n\left(\prod_{i=0}^{-n/p-1} F(A-ip)\right)g(A+n) \\ &= \left(z^{-p}F(A)\right)^{-n/p} g(A+n) \\ &= \xi^k g(A+n) =: \xi^k h(A)\,, \end{aligned}$$

proving the last assertion of Step 2, and hence completing the proof of Prop. 3.

2.2 Symmetric functions and matrix integrals

In this subsection, we prove a number of lemmas regarding symmetric functions.

Lemma 4 *Let s and N be positive integers. Let $F\big(x^{(1)},\dots,x^{(s)}\big)$ be a function which is symmetric in each $x^{(r)} := \big(x_1^{(r)},\dots,x_N^{(r)}\big)\in\mathbb{C}^N$, $r=1,\dots,s$; let $f_1,\dots,f_s$ be functions of two variables, and let $B\big(x^{(s)}\big)$ be a skew-symmetric function of $x^{(s)}$. If $C_1,\dots,C_s$ denote s fixed contours in $\mathbb{C}$, then the integral*

$$\begin{aligned}\Phi\big(x^{(0)}\big) \;&:=\; \int\cdots\int_{(C_1)^N\times\cdots\times(C_s)^N}\prod_{r=1}^{s}\prod_{i=1}^{N}dx_i^{(r)}\cdot\\ &\quad\cdot F\big(x^{(1)},\dots,x^{(s)}\big)B\big(x^{(s)}\big)\prod_{r=1}^{s}\det\Big(f_r\big(x_i^{(r-1)},x_j^{(r)}\big)\Big)_{1\le i,j\le N}\,,\end{aligned}$$

where $x^{(0)}\in\mathbb{C}^N$ comes in as the first argument of f_1, is skew-symmetric in $x^{(0)}$, and

$$\begin{aligned}\Phi\big(x^{(0)}\big) \;&=\; (N!)^s\int\cdots\int_{(C_1)^N\times\cdots\times(C_s)^N}\prod_{r,i}dx_i^{(r)}\cdot\\ &\quad\cdot F\big(x^{(1)},\dots,x^{(s)}\big)B\big(x^{(s)}\big)\prod_{r=1}^{s}\prod_{i=1}^{N}f_r\big(x_i^{(r-1)},x_i^{(r)}\big)\,.\end{aligned}$$

Proof: For any (good) functions $A=A\big(x^{(1)},\dots,x^{(s)}\big)$ and $h=h\big(x^{(1)},\dots,x^{(s)}\big)$, let

$$\langle Ah\rangle := \int\cdots\int_{(C_1)^N\times\cdots\times(C_s)^N}\prod_{r,i}dx_i^{(r)}\cdot A\big(x^{(1)},\dots,x^{(s)}\big)h\big(x^{(1)},\dots,x^{(s)}\big)\,.$$

For any $\sigma_r\in\mathfrak{S}_N$, let $x_{\sigma_r}^{(r)} := \big(x_{\sigma_r 1}^{(r)},\dots,x_{\sigma_r N}^{(r)}\big)$, and $h^{(\sigma_1,\dots,\sigma_s)}\big(x^{(1)},\dots,x^{(s)}\big) := h\big(x_{\sigma_1}^{(1)},\dots,x_{\sigma_s}^{(s)}\big)$. Clearly $\langle Ah\rangle = \big\langle A^{(\sigma_1,\dots,\sigma_s)}h^{(\sigma_1,\dots,\sigma_s)}\big\rangle$. If, moreover, A is symmetric in each of $x^{(1)},\dots,x^{(s-1)}$, and skew-symmetric in $x^{(s)}$, i.e., $A^{(\sigma_1,\dots,\sigma_s)} = (-1)^{\varepsilon(\sigma_s)}A$, then we have

$$\langle Ah\rangle = \big\langle A^{(\sigma_1,\dots,\sigma_s)}h^{(\sigma_1,\dots,\sigma_s)}\big\rangle = (-1)^{\varepsilon(\sigma_s)}\big\langle Ah^{(\sigma_1,\dots,\sigma_s)}\big\rangle \quad \forall\sigma_r\in\mathfrak{S}_N\,.$$

Applying this to $h\big(x^{(1)},\dots,x^{(s)}\big) := \prod_r\prod_i f_r\big(x_i^{(r-1)},x_i^{(r)}\big)$, and summing it up over $(\sigma_1,\dots,\sigma_s)\in(\mathfrak{S}_N)^s$, we obtain

$$\begin{aligned}&(N!)^s\Big\langle A\prod_r\prod_i f_r\big(x_i^{(r-1)},x_i^{(r)}\big)\Big\rangle\\ &= \Big\langle A\sum_{\sigma_1,\dots,\sigma_s}(-1)^{\varepsilon(\sigma_s)}\prod_r\prod_i f_r\big(x_{\sigma_{r-1}i}^{(r-1)},x_{\sigma_r i}^{(r)}\big)\Big\rangle, \quad \text{with } \sigma_0=\mathrm{id},\\ &= \Big\langle A\sum_{\sigma_1,\dots,\sigma_s}\prod_r(-1)^{\varepsilon(\sigma_r)-\varepsilon(\sigma_{r-1})}\prod_i f_r\big(x_{\sigma_{r-1}i}^{(r-1)},x_{\sigma_r i}^{(r)}\big)\Big\rangle\\ &= \Big\langle A\prod_r\sum_{\sigma\in\mathfrak{S}_N}(-1)^{\varepsilon(\sigma)}\prod_i f_r\big(x_i^{(r-1)},x_{\sigma i}^{(r)}\big)\Big\rangle\\ &= \Big\langle A\prod_r\det\Big(f_r\big(x_i^{(r-1)},x_j^{(r)}\big)\Big)_{i,j}\Big\rangle\,.\end{aligned}$$

Setting here $A=F\big(x^{(1)},\dots,x^{(s)}\big)B\big(x^{(s)}\big)$ proves the identity in Lemma 4. Finally, $\Phi\big(x^{(0)}\big)$ is skew-symmetric in $x^{(0)}$ since $\det\Big(f_1\big(x_i^{(0)},x_j^{(1)}\big)\Big)$ is. ■

Lemma 5 *(See [19, Lemma 4.2], [17, Eq. (2.21)], [26, Theorem 8.18].) Let*

$$\mathcal{W} = \operatorname{span}_{\mathbb{C}}\{\psi_0(z), \psi_1(z), \psi_2(z), \ldots\} \in \mathrm{Gr}$$

with functions

$$\psi_k(z) = \sum_{-\infty<j\le k} a_{j,k} z^j, \qquad k = 0, 1, 2, \ldots,$$

such that $a_{kk} = 1$ for $k \gg 0$, i.e., $\operatorname{ord}_z \psi_k(z) \le k$, and $\psi_k(z) = z^k\big(1 + O(z^{-1})\big)$ for $k \gg 0$. Let $N > 0$ be any integer such that this condition holds for $k \ge N$. Let $z_1, \ldots, z_N$ be formal scalar variables near ∞. Then the τ-function $\tau(t)$ at

$$t_n := -\frac{1}{n}\sum_{i=1}^{N} z_i^{-n}, \quad n = 1, 2, \ldots, \tag{37}$$

is given by

$$\tau(t) = \frac{\det\big(\psi_{j-1}(z_i)\big)_{1\le i,j\le N}}{\det(z_i^{j-1})_{1\le i,j\le N}}\,. \tag{38}$$

Proof: Our proof is based on Kontsevich's idea in [19]; see [17, Sect. 2.3] for a proof using free fermions. To keep the notation simple, let us denote by $(1-z)^{-1}$ and $(-z+1)^{-1}$ the geometric series $\sum_0^\infty z^n$ and $-\sum_{-\infty}^{-1} z^n$, respectively. Let $\delta(z) := (1-z)^{-1} - (-z+1)^{-1} = \sum_{-\infty}^{\infty} z^n$, which plays the role of delta function, in the sense that

$$\delta(z/y)f(z) = \delta(z/y)f(y)\,, \tag{39}$$

as is obvious by taking $f(z) = z^m$ (see [6]). Let $\sigma := \prod_{i=1}^N(-z_i) = (-1)^N z_1 \ldots z_N$. Let $\sigma_i := 1/\prod_{j(\ne i)}(1 - z_i/z_j)$, $i = 1, \ldots, N$, understood as rational functions of z_j's, so that we have the following identity of formal power series in z:

$$\prod_{i=1}^{N}\left(1-\frac{z}{z_i}\right)^{-1} = \sum_{i=1}^{N}\sigma_i\left(1-\frac{z}{z_i}\right)^{-1}.$$

From (37) we have

$$\begin{aligned}
g := \exp\left(-\sum_{n=1}^{\infty} t_n z^n\right) &= \prod_{i=1}^{N}\left(1-\frac{z}{z_i}\right)^{-1} = \sum_{i=1}^{N}\sigma_i\left(1-\frac{z}{z_i}\right)^{-1} \\
&= \sum_{i=1}^{N}\sigma_i\delta(z/z_i) + \sum_{i=1}^{N}\sigma_i\left(-\frac{z}{z_i}+1\right)^{-1} \\
&= \sum_{i=1}^{N}\sigma_i\delta(z/z_i) + \prod_{i=1}^{N}\left(-\frac{z}{z_i}+1\right)^{-1},
\end{aligned}$$

so that by using (39), we have

$$\begin{aligned}
g\psi_j(z) &= \sum_{i=1}^{N}\sigma_i\delta(z/z_i)\psi_j(z) + \left(\prod_{i=1}^{N}\left(-\frac{z}{z_i}+1\right)^{-1}\right)\psi_j(z) \\
&= \sum_{i=1}^{N}\sigma_i\delta(z/z_i)\psi_j(z_i) + z^{-N}\big(\sigma + O(z^{-1})\big)\psi_j(z)\,.
\end{aligned}$$

Denoting by B the matrix of the composite map in (21) with respect to the bases $\{\psi_j\}_{j=0}^\infty$ and $\{z^k\}_{k=0}^\infty$, we have thus $B = B^0 + B^1$, where

$$B^0 = \begin{pmatrix} 1 & \cdots & 1 \\ z_1^{-1} & \cdots & z_N^{-1} \\ z_1^{-2} & \cdots & z_N^{-2} \\ \vdots & \cdots & \vdots \end{pmatrix} S_N \begin{pmatrix} \psi_0(z_1) & \psi_1(z_1) & \cdots \\ \vdots & \vdots & \vdots \\ \psi_0(z_N) & \psi_1(z_N) & \cdots \end{pmatrix},$$

$$B^1 = \left(\begin{array}{ccc|ccccc} \cdots & 0 & 0 & \overbrace{0 \cdots 0}^{N} & \sigma & & * \\ \cdots & 0 & 0 & 0 \cdots 0 & 0 & \sigma & \\ \cdots & \vdots & \vdots & \vdots \cdots \vdots & \vdots & \ddots & \ddots \\ \cdots & \vdots & \vdots & \vdots \cdots \vdots & \vdots & \cdots & \ddots \end{array}\right) \left(\begin{array}{ccc} \vdots & \vdots & \cdots \\ a_{-2,0} & a_{-2,1} & \cdots \\ a_{-1,0} & a_{-1,1} & \cdots \\ \hline a_{00} & a_{01} & \cdots \\ 0 & a_{11} & \cdots \\ 0 & 0 & \ddots \\ \vdots & \vdots & \ddots \end{array}\right),$$

S_N is the diagonal matrix $\mathrm{diag}(\sigma_1, \ldots, \sigma_N)$, and a_{kj}, $-\infty < k < \infty$, $0 \le j < \infty$, are the Laurent coefficients of $\psi_j = \sum_k a_{kj} z^k$.

Let us apply some column operations on B. Adding an appropriate linear combination of first N columns to the $(N+i)^{\text{th}}$ column ($i > 0$), we can eliminate the column ${}^t\big(\psi_{N+i}(z_1), \ldots, \psi_{N+i}(z_N)\big)$, $i > 0$, from B^0. Since N is large enough so that $a_{jj} = 1$ for $j \ge N$, B^1 has the form

$$\left(\begin{array}{c|ccc} & \sigma & & * \\ O_{\infty \times N} & & \sigma & \\ & & & \ddots \\ & 0 & & \end{array}\right),$$

so that the "$*$" part can be eliminated by further column operations on columns $N+1$, $N+2$, $\ldots$, which do not alter the B^0-part. Here $O_{m \times n}$ is the $m \times n$ zero matrix. The matrix B can thus be reduced to $B' = B'^0 + B'^1$, where

$$B'^0 = \begin{pmatrix} 1 & \cdots & 1 \\ z_1^{-1} & \cdots & z_N^{-1} \\ z_1^{-2} & \cdots & z_N^{-2} \\ \vdots & \cdots & \vdots \end{pmatrix} S_N \left(\begin{array}{ccc|c} \psi_0(z_1) & \cdots & \psi_N(z_1) & \\ \vdots & \vdots & \vdots & O_{N \times \infty} \\ \psi_0(z_N) & \cdots & \psi_N(z_N) & \end{array}\right),$$

$$B'^1 = \left(O_{\infty \times N} \;\middle|\; \sigma I_\infty\right).$$

Let n, $n \ge N$, be an integer. Note that the column operations needed to bring B into B' only adds linear combinations of lower numbered columns to higher ones. Hence, denoting by B_n, B'_n, B'^0_n and B'^1_n the matrices of the first n rows and columns in B, B', B'^0 and B'^1, respectively, we have $\det B_n = \det B'_n = \det(B'^0_n + B'^1_n)$, with

$$B'^0_n = \begin{pmatrix} 1 & \cdots & 1 \\ z_1^{-1} & \cdots & z_N^{-1} \\ \vdots & \cdots & \vdots \\ z_1^{-n+1} & \cdots & z_N^{-n+1} \end{pmatrix} S_N \left(\begin{array}{ccc|c} \psi_0(z_1) & \cdots & \psi_N(z_1) & \\ \vdots & \vdots & \vdots & O_{N \times (n-N)} \\ \psi_0(z_N) & \cdots & \psi_N(z_N) & \end{array}\right),$$

and

$$B'^1_n = \left(\begin{array}{c|c} O_{(n-N) \times N} & \sigma I_{n-N} \\ \hline O_{N \times N} & O_{N \times (n-N)} \end{array}\right).$$

Since the last $n - N$ columns of B'^0_n are 0, we have

$$B'_n = \left(\begin{array}{c|c} * & \sigma I_{n-N} \\ \hline Z & O_{N \times (n-N)} \end{array}\right),$$

where Z consists of the last N rows and the first N columns of B'^0_n:

$$Z = \begin{pmatrix} z_1^{-n+N} & \cdots & z_N^{-n+N} \\ \vdots & \cdots & \vdots \\ z_1^{-n+1} & \cdots & z_N^{-n+1} \end{pmatrix} S_N \begin{pmatrix} \psi_0(z_1) & \cdots & \psi_N(z_1) \\ \vdots & \vdots & \vdots \\ \psi_0(z_N) & \cdots & \psi_N(z_N) \end{pmatrix}.$$

Hence we have, using $\sigma = (-1)^N z_1 \dots z_N$,

$$\begin{aligned}\det B_n = \det B'_n &= (-1)^{N(n-N)} \det Z \det(\sigma I_{n-N}) \\ &= (z_1 \dots z_N)^{n-N} \det Z \\ &= (z_1 \dots z_N)^{1-N} \det Z',\end{aligned}$$

where

$$Z' = \begin{pmatrix} z_1^{N-1} & \cdots & z_N^{N-1} \\ \vdots & \cdots & \vdots \\ z_1^1 & \cdots & z_N^1 \\ 1 & \cdots & 1 \end{pmatrix} S_N \begin{pmatrix} \psi_0(z_1) & \cdots & \psi_N(z_1) \\ \vdots & \vdots & \vdots \\ \psi_0(z_N) & \cdots & \psi_N(z_N) \end{pmatrix}.$$

Noticing

$$\det(z_j^{N-i})_{1\le i,j\le N} = (-1)^{N(N-1)/2} \det(z_j^{i-1})_{1\le i,j\le N},$$

and

$$\det S_N = \prod_1^N \sigma_i = \frac{\left(\prod_{j=1}^N z_j\right)^{N-1}}{\prod_{i,j\ne i}(z_j - z_i)} = \frac{(z_1 \dots z_N)^{N-1}}{(-1)^{N(N-1)/2} \det(z_j^{i-1})^2_{1\le i,j\le N}},$$

we observe that $\det B_n$ coincides with the right-hand side of (38). Since $n \ge N$ is arbitrary, this completes the proof of Lemma 5. ∎

Lemma 6 *Let $Z := \operatorname{diag}(z_1, \dots, z_N)$. Let $\lambda := \big((p-1)(N-1), (p-1)(N-2), \dots, p-1\big)$. For a polynomial $f(y,z)$, let us denote by $\big(f(y,z)\big)_2$ the terms in $f(y,z)$ which are quadratic in y. Then we have*[4]

$$\begin{aligned}\frac{\Delta(z^p)}{\Delta(z)} &= F_\lambda\left(-\operatorname{tr} Z, -\frac{1}{2}\operatorname{tr} Z^2, -\frac{1}{3}\operatorname{tr} Z^3, \dots\right) \\ &= c \prod z_i^{-\frac{p-1}{2}} \left(\int_{\mathcal{H}_N} dY \exp \operatorname{tr}\left(-\frac{(Y+Z)^{p+1}}{p+1}\right)_2\right)^{-1},\end{aligned}$$

where c is a non-zero constant which depends only on N and p.

Proof: The Schur function associated with the partition λ is given by (see [21])

$$F_\lambda\left(-\sum_1^N y_i, -\frac{1}{2}\sum_1^N y_i^2, -\frac{1}{3}\sum_1^N y_i^3, \dots\right) := \frac{\Delta_{\lambda+\delta}(y)}{\Delta_\delta(y)},$$

where $\delta = (N-1 > N-2 > \dots > 1 > 0)$ and $\Delta_\mu(y) = \det(y_i^{\mu_j})_{1\le i,j\le N}$. Therefore we have, with $\lambda + \delta = \big(p(N-1) > p(N-2) > \dots > p > 0\big)$,

$$\frac{\Delta(z^p)}{\Delta(z)} = \frac{\Delta_{\lambda+\delta}(z)}{\Delta_\delta(z)} = F_\lambda\left(-\sum_1^N z_i, -\frac{1}{2}\sum_1^N z_i^2, -\frac{1}{3}\sum_1^N z_i^3, \dots\right),$$

establishing the first equality of Lemma 6. In order to establish the second one, note

$$\begin{aligned}\operatorname{tr}\left(\frac{(Y+Z)^{p+1}}{p+1}\right)_2 &= \frac{1}{2}\operatorname{tr}(Y^2 Z^{p-1} + YZYZ^{p-2} + \dots + YZ^{p-1}Y) \\ &= \frac{1}{2}\sum_{i,j} Y_{ij}Y_{ji}(z_i^{p-1} + z_i^{p-2}z_j + \dots + z_j^{p-1}) \\ &= \frac{1}{2}\sum_{i,j} Y_{ij}Y_{ji}\left(\frac{z_i^p - z_j^p}{z_i - z_j}\right).\end{aligned}$$

[4] F_λ is the Schur function for the partition λ.

Hence, performing a Gaussian integration, we find

$$\begin{aligned}
\int dY \exp\operatorname{tr}\left(-\frac{(Y+Z)^{p+1}}{p+1}\right)_2 &= \int dY \exp\left(-\frac{1}{2}\sum_{i,j} Y_{ij}Y_{ji}\frac{z_i^p - z_j^p}{z_i - z_j}\right) \\
&= (2\pi)^{N^2/2}\left(\prod_{1\le i,j\le N}\frac{z_i - z_j}{z_i^p - z_j^p}\right)^{1/2} \\
&= \frac{(2\pi)^{N^2/2}}{p^{N/2}}\prod_{1\le i<j\le N}\frac{z_i - z_j}{z_i^p - z_j^p}\prod_1^p z_i^{-\frac{p-1}{2}} \\
&= \frac{(2\pi)^{N^2/2}}{p^{N/2}}\frac{\Delta(z)}{\Delta(z^p)}\prod_1^N z_i^{-\frac{p-1}{2}},
\end{aligned}$$

establishing Lemma 6. ∎

Remark 5 *In general we have*

$$\int_{\mathcal{H}} dY\, e^{-\operatorname{tr}(V(Y+Z))_2} = (2\pi)^{N^2/2}\frac{\Delta(z)}{\Delta\big(V'(z)\big)}\frac{1}{\sqrt{\prod_1^N V''(z_i)}}\,.$$

The following lemma is due to Harish Chandra, Bessis–Itzykson–Zuber and Duistermaat–Heckman among others:

Lemma 7 *Given $N\times N$-diagonal matrices X and Y, we have*

$$\int_{\mathbf{U}(N)} e^{\operatorname{tr} XUYU^\dagger}dU = (2\pi)^{\frac{N(N-1)}{2}}\frac{\det(e^{x_i y_j})_{1\le i,j\le N}}{\Delta(X)\Delta(Y)}\,.$$

A proof can be found in [13].

3 Matrix Fourier Transforms

In this section we explain how generalized Kontsevich integrals (see [19, 1, 24]) are closely related to the theory of Fourier transforms. Indeed, if $V(x)$ grows sufficiently at infinity, any *Fourier transform*

$$a(y) = \int_{-\infty}^{\infty} e^{-V(x)+xy}dx \tag{40}$$

leads to a linear space of functions $\mathcal{W}$ *invariant under two operators* A *and* $V'(z)$ satisfying $[A, V'(z)] = 1$.

(i) The point is that $a(y)$ satisfies the differential equation

$$V'\left(\frac{\partial}{\partial y}\right)a(y) = ya(y)\,, \tag{41}$$

as seen from

$$\begin{aligned}
0 &= \int_{-\infty}^{\infty}\frac{\partial}{\partial x}e^{-V(x)+xy}dx = \int_{-\infty}^{\infty}\big(-V'(x)+y\big)e^{-V(x)+xy}dx \\
&= \left(-V'\left(\frac{\partial}{\partial y}\right)+y\right)a(y).
\end{aligned}$$

Thus setting $y = V'(z)$ in (41) and $A_0 := V''(z)^{-1}\partial/\partial z = \partial/\partial y\big|_{y=V'(z)}$, the function $a\big(V'(z)\big)$ satisfies the differential equation

$$V'(A_0)a\big(V'(z)\big) = V'(z)a\big(V'(z)\big)\,. \tag{42}$$

(ii) The method of stationary phase applied to integrals (40) and their derivatives leads to the following estimate, upon Taylor expanding $V(x)$ around $x=z$,

$$\begin{aligned}
&\left(\frac{\partial}{\partial y}\right)^n a(y)\Big|_{y=V'(z)} \\
&= \int_{-\infty}^{\infty} x^n e^{-V(x)+xV'(z)}dx \\
&= \int_{-\infty}^{\infty} x^n e^{-\left(V(z)+(x-z)V'(z)+(1/2)(x-z)^2V''(z)+O(x-z)^3\right)+xV'(z)}dx \\
&= e^{-V(z)+zV'(z)}\int_{-\infty}^{\infty} x^n e^{-(1/2)(x-z)^2V''(z)\left(1+(V'''/V'')O(x-z)\right)}dx \\
&= e^{-V(z)+zV'(z)}\frac{1}{\sqrt{V''}}\left(\int_{-\infty}^{\infty}\left(\frac{y}{\sqrt{V''}}+z\right)^n e^{-y^2/2}dy+O(1/z)\right) \\
&= \rho(z)^{-1}z^n\left(1+O(1/z)\right), \qquad (43)
\end{aligned}$$

with

$$\rho(z)=\frac{1}{\sqrt{2\pi}}e^{V(z)-zV'(z)}\sqrt{V''(z)}\,.$$

Therefore defining

$$A:=\rho(z)\frac{\partial}{\partial y}\Big|_{y=V'(z)}\circ\rho(z)^{-1}$$

and

$$\psi_n(z):=A^n\psi_0(z):=\rho(z)\frac{\partial^n}{\partial y^n}a(y)\Big|_{y=V'(z)}, \quad n=0,1,\dots,$$

the differential equation (42) implies

$$V'(A)\psi_0(z)=V'(z)\psi_0(z)\,.$$

This, combined with (43), proves that the linear span

$$\mathcal{W}:=\mathrm{span}_{\mathbb{C}}\left\{\psi_k(z)=z^k\left(1+O(1/z)\right)\;\middle|\;k=0,1,2,\dots\right\}$$

is invariant under the operators A and $V'(z)$, i.e.,

$$A\mathcal{W}\subset\mathcal{W}\quad\text{and}\quad V'(z)\mathcal{W}\subset\mathcal{W}\,,\quad\text{with}\quad [A,V'(z)]=1\,.$$

(iii) By Lemma 5, the τ-function corresponding to $\mathcal{W}$, at time t as in (37), is given by

$$\begin{aligned}
\tau(t) &= \frac{\det\left(A^{j-1}\psi_0(z_i)\right)_{1\le i,j\le N}}{\det(z_i^{j-1})_{1\le i,j\le N}} \\
&= \frac{1}{\Delta(z)}\det\left(\rho(z_i)\left(\frac{\partial}{\partial y}\right)^{j-1}\int_{-\infty}^{\infty}e^{-V(x)+xy}dx\Big|_{y=V'(z_i)}\right)_{1\le i,j\le N} \\
&= \frac{\prod_1^N\rho(z_i)}{\Delta(z)}\int_{\mathbb{R}^N}dx\,e^{-\sum_1^N V(x_i)}\Delta(x)\prod_1^N e^{x_\alpha V'(z_\alpha)} \\
&= \frac{\prod_1^N\rho(z_i)}{N!\Delta(z)}\int_{\mathbb{R}^N}dx\,e^{-\sum_1^N V(x_i)}\Delta(x)\det\left(e^{x_\alpha V'(z_\beta)}\right)_{1\le\alpha,\beta\le N}, \\
&\qquad\text{using Lemma 4 with } s=1 \text{ and the skew-symmetry of } \Delta(x), \\
&= \frac{\prod_1^N\rho(z_i)}{N!\Delta(z)/\Delta\left(V'(z)\right)}\int_{\mathbb{R}^N}dx\,e^{-\sum_1^N V(x_i)}\Delta^2(x)\frac{\det\left(e^{x_\alpha V'(z_\beta)}\right)_{1\le\alpha,\beta\le N}}{\Delta(x)\Delta\left(V'(z)\right)}
\end{aligned}$$

$$
\begin{aligned}
&= c\frac{\prod_1^N \rho(z_i)}{\Delta(z)/\Delta\big(V'(z)\big)}\int_{\mathbb{R}^N} dx\, e^{-\sum_1^N V(x_i)}\Delta^2(x)\int_{\mathbf{U}(N)} dU\, e^{\operatorname{tr} UXU^{-1}V'(Z)}\,,\\
&\qquad \text{using Lemma 7, with } X = \operatorname{diag}(x),\\
&= c' e^{\operatorname{tr}(V(Z)-ZV'(Z))}\frac{\int_{\mathcal{H}_N} dX\, e^{-\operatorname{tr} V(X)}e^{\operatorname{tr} XV'(Z)}}{\int_{\mathcal{H}_N} dX\, e^{-\operatorname{tr}(V(X+Z))_2}}\,,\quad \text{using Lemma 6,}\\
&= c''\frac{\int_{\mathcal{H}_N} dY\, e^{-\operatorname{tr}(V(Y+Z))_{\geq 2}}}{\int_{\mathcal{H}_N} dY\, e^{-\operatorname{tr}(V(Y+Z))_2}}\,,\quad \text{upon setting } X = Y+Z,
\end{aligned}
$$

for some constants c, c' and c'' depending on N.

4 Generalized Hänkel Functions, Differential Equations and Laplace Transforms

This section deals with the properties of Hänkel functions and their generalizations.

Lemma 8 *The family of integrals*

$$
\psi_k(z) = \frac{p^{c+1}}{\Gamma(-c)}\int_1^\infty \frac{z^{-c}(uz)^k e^{-(u-1)z}}{(u^p-1)^{c+1}}\,du\,, \qquad \begin{array}{l} -1 < c < 0, \\ k = 0,1,\ldots,\ p = 2,3,\ldots \end{array} \tag{44}
$$

admits, for large $z > 0$, an asymptotic expansion in $\mathbb{C}((z^{-1}))$ of the form

$$
\psi_k(z) = z^k\big(1 + O(1/z)\big)\,, \tag{45}
$$

with $\psi_0(z)$ satisfying the differential equation

$$
e^z z^{-c}\left(\prod_{i=0}^{p-1}\left(z\frac{\partial}{\partial z} - i\right) - cp\prod_{i=0}^{p-2}\left(z\frac{\partial}{\partial z} - i\right)\right) z^c e^{-z}\psi_0(z) = (-z)^p\psi_0(z)\,, \tag{8}
$$

or equivalently

$$
e^z z^{-c}\left(z^p\left(\frac{\partial}{\partial z}\right)^p - cp\, z^{p-1}\left(\frac{\partial}{\partial z}\right)^{p-1}\right) z^c e^{-z}\psi_0(z) = (-z)^p\psi_0(z)\,. \tag{8'}
$$

Moreover $\psi_k(z)$ admits the following representation in terms of a double integral[5]

$$
\begin{aligned}
\psi_k(z) &= \frac{p^{c+1}}{2\pi i} z^{(p-1)(c+1)}\int_\gamma dw \int_0^\infty dx\, e^{z-w} w^k x^c e^{x(w^p - z^p)}\\
&= \frac{p^{c+1}}{2\pi i} z^{(p-1)(c+1)} e^z \int_0^\infty dx\, x^c e^{-xz^p}\int_0^\infty dy\, f_k(y) e^{-xy^p}\,,
\end{aligned} \tag{46}
$$

where, in the first integral, $\gamma := \gamma^+ + \gamma^- \subset \mathbb{C}$ denotes the contour consisting of two half-lines $\gamma^\pm = \mathbb{R}_+\zeta^{\pm 1}$, $\zeta := e^{\pi i/p}$, through the origin making an angle $\pm\pi/p$ with the positive real axis, with the orientation given as to go from $\zeta^{-1}\cdot\infty$ to 0 to $\zeta\cdot\infty$ (see Fig. 1 (a)), and where in the second integral,

$$
f_k(y) = (\zeta^{k+1}e^{-\zeta y} - \zeta^{-k-1}e^{-\zeta^{-1}y})y^k = \sum_{j=0}^\infty \frac{(-1)^j}{j!} a_{j+k+1} y^{j+k},
$$

where $a_n = \zeta^n - \zeta^{-n} = 2i\sin(n\pi/p)$.

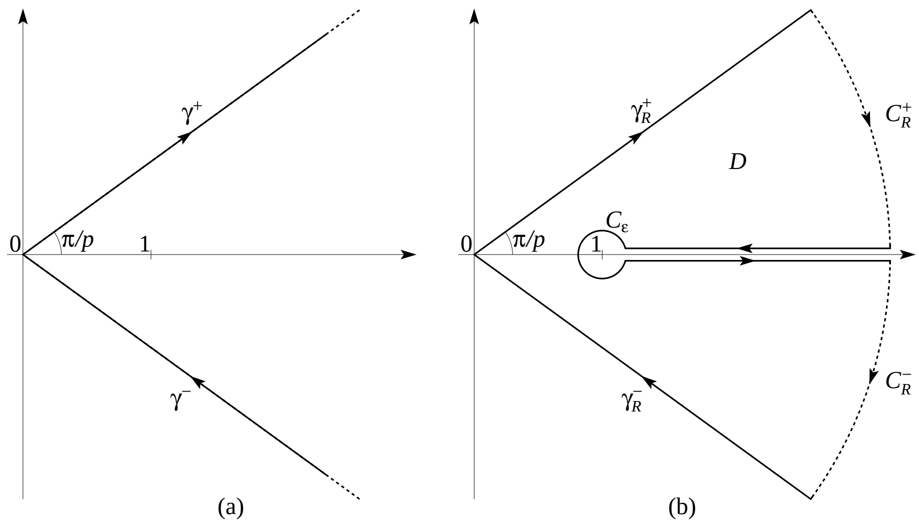

Figure 1: Contours of integration: (a) contour γ for (46); (b) closed contour for (48)

Proof: Setting $v = (u-1)z$, and using

$$\Gamma(-c) = \int_0^\infty \frac{e^{-v}}{v^{c+1}}\, dv \quad \text{for } c < 0,$$

we first observe that for each $n \geq 0$,

$$\begin{aligned} \psi_k(z) &= \frac{p^{c+1} z^k}{\Gamma(-c)} \int_0^\infty \frac{(1+v/z)^k e^{-v}}{v^{c+1} p^{c+1} \left(1 + \frac{1}{p}\left(\sum_{i=2}^p \binom{p}{i} (v/z)^{i-1}\right)\right)^{c+1}}\, dv \\ &= z^k \left(1 + \tilde{b}_{k,1} z^{-1} + \cdots + \tilde{b}_{k,n} z^{-n} + O(1/z^{n+1})\right) \end{aligned}$$

as $z \to \infty$, where the $\tilde{b}_{k,i} := \big(\Gamma(-c+i)/\Gamma(-c)\big) b_{k,i} = \left(\prod_{j=0}^{i-1}(-c+j)\right) b_{k,i}$ are obtained from the coefficients $b_{k,i}$ of the expansion[6]

$$\frac{(1+s)^k}{\left(1 + \frac{1}{p}\left(\sum_{i=2}^p \binom{p}{i} s^{i-1}\right)\right)^{c+1}} = 1 + \sum_{i=1}^\infty b_{k,i} s^i\,, \tag{47}$$

confirming the asymptotic expansion (45).

Moreover, setting

$$\varphi_0(z) = \int_1^\infty \frac{z^{-c} e^{-uz}}{(u^p-1)^{c+1}}\, du\,,$$

we have for $c < 0$ and Re $z > 0$,

$$\begin{aligned} 0 &= -z^{p-1-c} \frac{e^{-uz}}{(u^p-1)^c}\Bigg|_{u=1}^{u=\infty} \\ &= -z^{p-1} \int_1^\infty \frac{\partial}{\partial u}\left((u^p-1)\frac{z^{-c} e^{-uz}}{(u^p-1)^{c+1}}\right) du \end{aligned}$$

[5] If $p = 2$, so that γ becomes the imaginary axis, these integrals should be interpreted by replacing ζ by $\zeta_\varepsilon = e^{(\pi i/2)-\varepsilon}$, and γ by $\mathbb{R}_+\zeta_\varepsilon + \mathbb{R}_+\zeta_\varepsilon^{-1}$, and then taking the limit as $\varepsilon \downarrow 0$.

[6] Noting that the radius of convergence of this power series is $|\zeta - 1|$, one can get a precise growth estimate of the coefficients of $\psi_k(z)$ which implies that, in particular, as always with the string equation, $\mathcal{W}$ does not belong to the L^2-Grassmannian of Segal–Wilson [30].

$$
\begin{aligned}
&= (-1)^p \int_1^\infty \left((-zu)^p - cp(-zu)^{p-1} - (-z)^p\right) \frac{z^{-c}e^{-uz}}{(u^p-1)^{c+1}}\, du \\
&= (-1)^p z^{-c} \left(z^p \left(\frac{\partial}{\partial z}\right)^p - cp\, z^{p-1} \left(\frac{\partial}{\partial z}\right)^{p-1} - (-z)^p \right) z^c \varphi_0(z) \\
&= (-1)^p z^{-c} \left(\prod_{i=0}^{p-1} \left(z\frac{\partial}{\partial z} - i\right) - cp \prod_{i=0}^{p-2} \left(z\frac{\partial}{\partial z} - i\right) - (-z)^p \right) z^c \varphi_0(z) \,,
\end{aligned}
$$

using in the last line the operator identity

$$
\prod_{i=0}^{p-1} \left(z\frac{\partial}{\partial z} - i\right) = z^p \left(\frac{\partial}{\partial z}\right)^p ,
$$

thus showing that $\psi_0(z)$ satisfies the differential equation (8) or (8′).

Consider a bounded domain $D \subset \mathbb{C}$, whose boundary consists of the lines $\gamma_R^\pm$, making an angle $\pm\pi/p$ with the positive real axis, two circle segments $C_R^\pm$, about the origin, of large enough radius R and a small circle about 1 of radius ε connected to $C_R^\pm$, as in Fig. 1 (b). The function $e^{-uz}/(u^p-1)^{c+1}$ is univalued in D and all its singularities lie outside D. By Cauchy's theorem we have

$$
\left(\int_{\gamma_R^-} + \int_{\gamma_R^+} + \int_{C_R^+} + \int_R^{1+\varepsilon} + \int_{C_\varepsilon} + \int_{1+\varepsilon}^R + \int_{C_R^-} \right) \frac{(uz)^k e^{-(u-1)z}}{(u^p-1)^c}\, du = 0 \,. \tag{48}
$$

Observe that, for $z > 0$ and $p > 2$, we have $z\cos\theta \geq z\cos(\pi/p) > 0$ for $0 \leq \theta \leq \pi/p$, implying

$$
\int_{C_R^\pm} \frac{(uz)^k e^{-(u-1)z}}{(u^p-1)^{c+1}}\, du = O\big(R^{k-(c+1)p+1} e^{-Rz\cos(\pi/p)}\big) \to 0
$$

as $R \uparrow \infty$. Since $c < 0$, we also have

$$
\int_{C_\varepsilon} \frac{(uz)^k e^{-(u-1)z}}{(u^p-1)^{c+1}}\, du = O(\varepsilon^{-c}) \to 0
$$

as $\varepsilon \downarrow 0$. So, taking limits as $\varepsilon \downarrow 0$ and $R \uparrow \infty$ leads to

$$
\begin{aligned}
\int_\gamma \frac{(uz)^k e^{-(u-1)z}}{(u^p-1)^{c+1}}\, du &= -\left(\int_\infty^1 + \int_{1-i0}^{\infty - i0} \right) \frac{(uz)^k e^{-(u-1)z}}{(u^p-1)^{c+1}}\, du \\
&= (1 - e^{-2\pi i(c+1)}) \int_1^\infty \frac{(uz)^k e^{-(u-1)z}}{(u^p-1)^{c+1}}\, du \\
&= 2ie^{-\pi i c} \sin \pi c \int_1^\infty \frac{(uz)^k e^{-(u-1)z}}{(u^p-1)^{c+1}}\, du \,.
\end{aligned}
$$

Note that, since $u^p - 1 < 0$ along γ, we have the following Γ-function representation

$$
\frac{1}{(u^p-1)^{c+1}} = -\frac{e^{-\pi i c}}{\Gamma(c+1)} \int_0^\infty dx\, x^c e^{x(u^p-1)} ,
$$

and thus

$$
\begin{aligned}
\psi_k(z) &= \frac{p^{c+1}}{\Gamma(-c)} \int_1^\infty \frac{z^{-c}(uz)^k e^{-(u-1)z}}{(u^p-1)^{c+1}} \\
&= \frac{p^{c+1} e^{\pi i c}}{2i \sin \pi c\, \Gamma(-c)} z^{-c} \int_\gamma \frac{(uz)^k e^{-(u-1)z}}{(u^p-1)^{c+1}}\, du \\
&= -\frac{p^{c+1} z^{-c}}{2i \sin \pi c \Gamma(-c)\Gamma(c+1)} \int_\gamma du\, (uz)^k e^{-(u-1)z} \int_0^\infty dx\, x^c e^{x(u^p-1)} \\
&= \frac{p^{c+1}}{2\pi i} z^{-c} \int_\gamma du\, (uz)^k e^{-(u-1)z} \int_0^\infty dx\, x^c e^{x(u^p-1)} , \\
&= \frac{p^{c+1}}{2\pi i} z^{(p-1)(c+1)} \int_\gamma dw \int_0^\infty dx\, e^{z-w} w^k x^c e^{xw^p} e^{-xz^p} ,
\end{aligned}
$$

upon setting $w = uz$. Here we used the Γ-function duplication, $\Gamma(-c)\Gamma(c+1) = -\pi/\sin\pi c$, $-1 < c < 0$. Working out the integral over γ, interchanging the integrations and using $\zeta^{\pm p} = -1$, we find

$$\begin{aligned}\psi_k(z) &= \frac{p^{c+1}}{2\pi i} z^{(p-1)(c+1)} e^z \int_0^\infty dx\, x^c e^{-xz^p} \cdot \\ &\quad \cdot \left(\zeta^{-k-1} \int_\infty^0 dy\, e^{-\zeta^{-1}y} y^k e^{-xy^p} + \zeta^{k+1} \int_0^\infty dy\, e^{-\zeta y} y^k e^{-xy^p} \right) \\ &= \frac{p^{c+1}}{2\pi i} z^{(p-1)(c+1)} e^z \int_0^\infty dx\, x^c e^{-xz^p} \int_0^\infty dy\, f_k(y) e^{-xy^p}\end{aligned}$$

with

$$\begin{aligned}f_k(y) &= (\zeta^{k+1} e^{-\zeta y} - \zeta^{-k-1} e^{-\zeta^{-1} y}) y^k \\ &= \sum_{j=0}^\infty \frac{(-1)^j}{j!} (\zeta^{j+k+1} - \zeta^{-j-k-1}) y^{j+k},\end{aligned}$$

as announced in (46), thus ending the proof of Lemma 8. ∎

Lemma 9 *The linear space spanned by the generalized Hänkel functions,*

$$\mathcal{W} = \operatorname{span}_{\mathbb{C}} \left\{ \psi_k(z) = \frac{p^{c+1}}{\Gamma(-c)} \int_1^\infty \frac{z^{-c} (uz)^k e^{-(u-1)z}}{(u^p - 1)^{c+1}} du \;\middle|\; k = 0, 1, 2, \ldots \right\}$$

is invariant under

$$z^p \quad \text{and} \quad A_c := z^{-c} e^z z \frac{\partial}{\partial z} \circ e^{-z} z^c = z \frac{\partial}{\partial z} - z + c$$

(so that $[(1/p)A, z^p] = z^p$), with ψ_0 satisfying the differential equation (8).

Proof: The space $\mathcal{W}$ is invariant under A_c, because

$$\begin{aligned}A_c \psi_k(z) &= \frac{p^{c+1}}{\Gamma(-c)} z^{-c} e^z z \frac{\partial}{\partial z} z^c e^{-z} \int_1^\infty \frac{z^{-c} (uz)^k e^{-(u-1)z}}{(u^p - 1)^{c+1}} du \\ &= \frac{p^{c+1}}{\Gamma(-c)} z^{-c} e^z z \frac{\partial}{\partial z} \int_1^\infty \frac{(uz)^k e^{-uz}}{(u^p - 1)^{c+1}} du \\ &= k\psi_k(z) - \psi_{k+1}.\end{aligned}$$

Moreover, the operator

$$\prod_{i=0}^{p-1} (A_c - i) - cp \prod_{i=0}^{p-2} (A_c - i)$$

has the form $\sum_0^p \alpha_j A_c^j$, with $\alpha_p = 1$. From Lemma 8, the solution to the differential equation

$$\left(\prod_{i=0}^{p-1} (A_c - i) - cp \prod_{i=0}^{p-2} (A_c - i) \right) \psi_0(z) = (-z)^p \psi_0(z)$$

is given by the function in (44) or (46) for $k = 0$. An asymptotic expansion of the form

$$\psi_0(z) = 1 + O(z^{-1})$$

follows from (45). ∎

5 Proof of the Main Statements

5.1 Proof of Theorems 3 and 1 and Remark 1

In Lemma 9, we have constructed a space $\mathcal{W}$ and an operator $A = A_c$ such that

$$A\mathcal{W} \subset \mathcal{W} \quad \text{and} \quad z^p \mathcal{W} \subset \mathcal{W},$$

with the lowest order element $\psi_0 \in \mathcal{W}$ satisfying Eq. (8). Proposition 3 and Remark 4 imply that the stabilizer of $\mathcal{W}$ is $\mathbb{C}[A, z^p, z^{-p}F(A)]$, proving Theorem 3, Part (i).

Let Ψ and τ be the wave function and the τ-function, respectively, associated with the KP time evolution $\mathcal{W}^t = e^{-\sum t_i z^i}\mathcal{W}$ of $\mathcal{W}$. We now define the operators P and Q in the x-variable, via the operators A and z^p in the z-variable, by means of

$$z^p \Psi(t,z) = P\Psi(t,z) \quad \text{and} \quad (1/p)A\Psi(t,z) = Q\Psi(t,z).$$

According to Lemma 2, P and Q are differential operators. They satisfy $[P,Q] = P$ since $[(1/p)A, z^p] = z^p$. Note that P and Q can also be written:

$$P = L^p = SD^pS^{-1}$$

and

$$Q = \frac{1}{p}(ML - L + c) = \frac{1}{p}S\left(\sum_1^\infty k\bar{t}_k D^k - D + c\right)S^{-1},$$

where

$$S = \frac{\tau(t - [D^{-1}])}{\tau(t)}$$

in terms of the τ-function above, and L and M are as in (12) and (16), proving Theorem 3, Part (ii).

Since $(M-1)L = pQ - c$ is a differential operator, we also have, using the notation α_{ij} as in the statement of Theorem 3,

$$\begin{aligned}
\big((M-1)L\big)^m L^{np} &= \sum_{i=1}^m \alpha_{m,i}(M-1)^i L^{i+np} \\
&= \sum_{\substack{0\le i\le m \\ 0\le j\le i}} \alpha_{m,i}\binom{i}{j}(-1)^{i-j}M^j L^{i+np}
\end{aligned}$$

is a differential operator. Thus

$$\sum_{\substack{0\le i\le m \\ 0\le j\le i}} \alpha_{m,i}\binom{i}{j}(-1)^{i-j}(M^j L^{i+np})_- \Psi = 0,$$

implying (10), upon using (19), completing the proof of Theorem 3.

To prove Remark 1, we evaluate

$$\left(\sum i t_i \frac{\partial}{\partial t_i} - \frac{\partial}{\partial t_1} - a\right)\tau = 0$$

at $t = 0$ to find $-\big((\partial\tau/\partial t_1)/\tau\big)\big|_{t=0} = a$. Remember, on the one hand,

$$\Psi(0,0,z) = \psi_0(z) = (1 + \tilde{b}_{01}z^{-1} + \cdots),$$

and on the other hand

$$\begin{aligned}
\Psi(0,0,z) &= \left.\frac{\tau(t_1 + x - z^{-1}, \ldots)}{\tau(t_1 + x, \ldots)}\right|_{x=0,t=0} \\
&= \left.\left(1 - \tau^{-1}\frac{\partial\tau}{\partial x}z^{-1} + \cdots\right)\right|_{t=0}.
\end{aligned}$$

Therefore $a = \tilde{b}_{01} = (-c)b_{0,1} = c(1+c)(p-1)/2$ as stated in Remark 1, as implied by (47).

To prove Theorem 1, note that at $t = 0$,

$$\begin{aligned} Q|_{t=0} &= (1/p)S((x-1)(\partial/\partial x)+c)S^{-1} \\ &= (1/p)(x-1)(\partial/\partial x)+c+(\text{negative order terms})\,. \end{aligned}$$

Since Q must be a differential operator, the negative order terms vanish, and $Q|_{t=0} = (1/p)(x-1)(\partial/\partial x)+c$. Thus, from the second equation in (9), we have

$$0 = \big(Q|_{t=0} - A_c\big)\Psi(x,0,z) = \big((x-1)(\partial/\partial x) - z(\partial/\partial z - 1)\big)\Psi(x,0,z)\,. \tag{49}$$

Since this is a first order equation and the line $x = 0$ is noncharacteristic, $\Psi(x,0,z)$ is determined by (49) together with the initial condition $\Psi(0,0,z) = \psi_0(z)$. It is easy to check that the right-hand side of (6) satisfies these conditions. Finally, (7) follows from (8): Writing (8) as

$$F\left(z\frac{\partial}{\partial z}+c\right)\big(e^{-z}\psi_0(z)\big) = (-z)^p e^{-z}\psi_0(z)\,,$$

substituting $(1-x)z$ for z, using the scaling invariance of $z\,\partial/\partial z$, and dividing both sides by z^p, we get

$$\frac{1}{z^p}F\left(z\frac{\partial}{\partial z}+c\right)\big(e^{(x-1)z}\psi_0((1-x)z)\big) = (x-1)^p e^{(x-1)z}\psi_0((1-x)z)\,. \tag{50}$$

Multiplying both sides of this formula by e^z, and using the identity $e^z(z\,\partial/\partial z + c) = (z\,\partial/\partial z - z + c)\circ e^z$, we get the second formula in (7). Next, switching the roles of z and $1-x$ in (50), we get

$$\frac{1}{(x-1)^p}F\left((x-1)\frac{\partial}{\partial x}+c\right)\big(e^{(x-1)z}\psi_0((1-x)z)\big) = z^p e^{(x-1)z}\psi_0((1-x)z)\,.$$

Multiplying both sides of this formula by e^z, and using the fact that e^z commutes with $(x-1)\partial/\partial x + c$, we get the first formula in (7), completing the proof of Theorem 1.

5.2 Proof of Theorem 2

Setting $t_n = -\frac{1}{n}\sum_{i=1}^n z_i^{-n}$, $n = 1, 2, \ldots$, and using Lemma 5, and Lemma 4 with $s = 2$, we have

$$\begin{aligned} \tau(t) &= \frac{\det\big(\psi_{k-1}(z_i)\big)_{1\le k,i\le N}}{\Delta(z)} \\ &= \frac{a^N}{\Delta(z)}\det\left(z_i^{(p-1)(c+1)}e^{z_i}\int_0^\infty dx\int_0^\infty dy\, x^c e^{-xz_i^p} f_{k-1}(y)e^{-xy^p}\right)_{k,i} \\ &= \frac{a^N S_2(t)}{\Delta(z)}\int_{\mathbb{R}_+^N} dx\int_{\mathbb{R}_+^N} dy\left(\prod_1^N x_i^c\right)\cdot \\ &\quad \cdot\det\big(f_{k-1}(y_i)\big)_{k,i}\; e^{-\sum_1^N x_i z_i^p}e^{-\sum_1^N x_i y_i^p} \\ &= \frac{a^N S_2(t)}{(N!)^2\Delta(z)}\int_{\mathbb{R}_+^N} dx\int_{\mathbb{R}_+^N} dy\left(\prod_1^N x_i^c\right)\cdot \\ &\quad \cdot\det\left(f_{k-1}(y_i)\right)_{k,i}\det\left(e^{-x_i z_j^p}\right)_{i,j}\det\left(e^{-x_i y_j^p}\right)_{i,j} \\ &= \frac{a^N S_2(t)\Delta(z^p)}{(N!)^2\Delta(z)}\int_{\mathbb{R}_+^N} dx\int_{\mathbb{R}_+^N} dy\left(\prod_1^N x_i^c\right)\Delta(x)^2\Delta(y)^2\cdot \\ &\quad \cdot S_0(y)\frac{\det\left(e^{-x_i z_j^p}\right)_{i,j}}{\Delta(x)\Delta(z^p)}\frac{\det\left(e^{-x_i y_j^p}\right)_{i,j}}{\Delta(x)\Delta(y^p)}\,, \end{aligned}$$

where $a = p^{c+1}/2\pi i$,

$$S_2(t) = \prod_1^N \left(z_i^{(p-1)(c+1)} e^{z_i} \right) ,$$

and

$$S_0(y_1, y_2, \dots, y_N) = \frac{\Delta(y^p)}{\Delta(y)} \frac{\det\big(f_{k-1}(y_i)\big)_{1\le i,k\le N}}{\Delta(y)} .$$

So we have, for some constants C, C' and C'' depending on N, p and c,

$$\begin{aligned}
\tau(t) &= C\frac{S_2(t)\Delta(z^p)}{\Delta(z)} \int_{\mathbb{R}_+^N} dx\, \Delta(x)^2 \int_{\mathbb{R}_+^N} dy\, \Delta(y)^2 S_0(y) \cdot \\
&\quad \cdot \int_{\mathbf{U}(N)} dU_X\, e^{-\operatorname{tr} Z^p U_X^{-1} x U_X} \int_{\mathbf{U}(N)} dV_Y\, e^{-\operatorname{tr} x V_Y^{-1} y^p V_Y} \\
&\qquad \text{using Lemma 7} \\
&= C\frac{S_2(t)\Delta(z^p)}{\Delta(z)} \int_{\mathbb{R}_+^N} dx\, \Delta(x)^2 \left(\prod_1^N x_i^c \right) \int_{\mathbb{R}_+^N} dy\, \Delta(y)^2 S_0(y) \cdot \\
&\quad \cdot \int_{\mathbf{U}(N)} dU_X\, e^{-\operatorname{tr} Z^p U_X^{-1} x U_X} \int_{\mathbf{U}(N)} dU_Y\, e^{-\operatorname{tr} U_X^{-1} x U_X U_Y^{-1} y^p U_Y} \\
&\qquad \text{setting } U_Y = V_Y U_X \text{ for fixed } U_X \text{ in the last} \\
&\qquad \text{integral and noting that } dU_X\, dU_Y = dU_X\, dV_Y \\
&= C\frac{S_2(t)\Delta(z^p)}{\Delta(z)} \int_{\mathbb{R}_+^N} dx\, \Delta(x)^2 \left(\prod_1^N x_i^c \right) \int_{\mathbf{U}(N)} dU_X\, e^{-\operatorname{tr} Z^p U_X^{-1} x U_X} \cdot \\
&\quad \cdot \int_{\mathbb{R}_+^N} dy\, \Delta^2(y) S_0(y) \int_{\mathbf{U}(N)} dU_Y\, e^{-\operatorname{tr} U_X^{-1} x U_X U_Y^{-1} y^p U_Y} \\
&= C' \frac{S_2(t)\Delta(z^p)}{\Delta(z)} \int_{\mathcal{H}_N^+} dX\, \det(X^c) e^{-\operatorname{tr} Z^p X} \int_{\mathcal{H}_N^+} dY\, S_0(y) e^{-\operatorname{tr} X Y^p} \\
&= C'' S_1(t) \frac{\int_{\mathcal{H}_N^+} dX \det(X^c) e^{-\operatorname{tr} Z^p X} \int_{\mathcal{H}_N^+} dY S_0(y) e^{-\operatorname{tr} XY^p}}{\int_{\mathcal{H}_N} dX \exp\operatorname{tr}\left(-\frac{((X+Z)^{p+1})_2}{p+1} \right)} ,
\end{aligned}$$

where we used Lemma 6 in the last equality, and the definition of $S_1(t)$ in Theorem 2. A similar calculation, outlined below, implies the second formula for τ, upon using the first representation of $\psi_k(z)$ in (46):

$$\begin{aligned}
\tau(t) &= \frac{\det\left(A^{k-1}\Psi(0, z_i) \right)_{1\le k,i\le N}}{\Delta(z)} , \quad \text{with } t_n = -\tfrac{1}{n}\textstyle\sum_{i=1}^\infty z_i^{-n}, \\
&= \frac{1}{\Delta(z)} \det\left(a\, e^{z_i} z_i^{(p-1)(c+1)} \int_\gamma dw \int_0^\infty dx\, e^{-w} w^{k-1} x^c e^{xw^p} e^{-xz_i^p} \right)_{k,i} \\
&= \frac{a^N}{\Delta(z)} e^{\sum z_i} \prod z_i^{(p-1)(c+1)} \cdot \\
&\quad \cdot \int_\gamma \cdots \int_\gamma dw \int_0^\infty \cdots \int_0^\infty dx\, e^{-\sum w_i} \prod x_i^c \Delta(w) \prod_{i=1}^N e^{-z_i^p x_i} \prod_{i=1}^N e^{x_i w_i^p} \\
&= \frac{a^N}{(N!)^2} e^{\sum z_i} \prod z_i^{(p-1)(c+1)} \frac{1}{\Delta(z)} \int_{\gamma^N} dw \int_{\mathbb{R}_+^N} dx\, e^{-\sum w_i} \prod x_i^c \Delta(w) \cdot \\
&\quad \cdot \det\big(e^{-z_i^p x_j}\big)_{1\le i,j\le N} \det\big(e^{x_i w_j^p}\big)_{1\le i,j\le N} \\
&= \cdots
\end{aligned}$$

$$= \quad \frac{\int_{\mathcal{H}_N^\gamma} m(dW) \int_{\mathcal{H}_N^+} dX \det X^c \big(\Delta(w^p)/\Delta(w)\big) e^{\operatorname{tr}(Z-W)} e^{\operatorname{tr} X(W^p - Z^p)}}{\int_{\mathcal{H}_N} dX \exp\operatorname{tr}\left(-\frac{(X+Z)^{p+1}}{p+1}\right)_2},$$

ending the proof of Theorem 2.

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