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Super Quantum Airy Structures

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Abstract: We introduce super quantum Airy structures, which provide a supersymmetric generalization of quantum Airy structures. We prove that to a given super quantum Airy structure one can assign a unique set of free energies, which satisfy a supersymmetric generalization of the topological recursion. We reveal and discuss various properties of these supersymmetric structures, in particular their gauge transformations, classical limit, peculiar role of fermionic variables, and graphical representation of recursion relations. Furthermore, we present various examples of super quantum Airy structures, both finite-dimensional—which include well known superalgebras and super Frobenius algebras, and whose classification scheme we also discuss—as well as infinite-dimensional, that arise in the realm of vertex operator super algebras.

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1. Introduction

In recent years we have learnt that solutions to a plethora of problems in physics and mathematics that involve some form of quantization arise from a universal system of recursive equations, referred to as the Chekhov–Eynard–Orantin topological recursion [24, 37, 38]. The topological recursion was originally discovered in the realm of matrix models as a way of solving loop equations, which enables computation of the free energy to all orders in the large N expansion, based on the information encoded in the

spectral curve of the matrix model [24]. Soon after, Eynard and Orantin realized that the topological recursion can be formulated independently of matrix models, as a tool that assigns symplectic invariants to a large class of algebraic curves (which play the role of spectral curves in the context of matrix models) [37, 38]. More recently, a more general and abstract reformulation of the topological recursion was provided in the form of quantum Airy structures [4, 47, 52, 62] (see also the lecture notes [10]).

To date, the study of the topological recursion proceeded along two (of course interrelated) main lines. First, various generalizations thereof have been discovered, such as those mentioned above, as well as its β -deformed version [25], the formulation for curves with higher ramifications and the global version [14, 17], the blobbed version [13], the geometric recursion [5], etc. Second, a lot of effort has been put in identifying various systems and problems whose solutions are captured by the topological recursion. This led to the simplification or better understanding of the structure of previous solutions, or to new solutions of those problems. The topological recursion turned out to play such a role in areas such as topological strings and Gromov–Witten theory [18, 22, 34, 40, 41], the theory of quantum curves [15, 46, 58], knot theory [12, 29], Hurwitz theory [2, 16, 19, 31, 36], just to name a few.

The main aim of this paper is to follow the first line of research mentioned above, and to introduce a supersymmetric generalization of quantum Airy structures, which we call super quantum Airy structures. After defining a super quantum Airy structure, as in the original (non-supersymmetric) case we assign to it free energies, and prove their existence and uniqueness. We write down recursion relations satisfied by these free energies: these recursion relations generalize the original (non-supersymmetric) topological recursion. Among various interesting features of super quantum Airy structures, we reveal a peculiar role played by fermionic variables. We provide several finite- and infinite-dimensional examples of super quantum Airy structures, and hint how their classification could be conducted. Furthermore, we expect that super quantum Airy structures should have interesting applications in various contexts, which we briefly mention in what follows.¹

In view of the matrix model origin of the (non-supersymmetric) topological recursion and its reformulation in terms of quantum Airy structures, one may expect similar connections in the supersymmetric context. However, such relations are obscure, even though corresponding supersymmetric structures in matrix models are known. Indeed, supersymmetric generalizations of matrix models, referred to supereigenvalue models, have been introduced and discussed some time ago [3, 8], and also more recently [20, 26–28, 61]. By construction, loop equations for such supereigenvalue models can be rewritten in the form of super-Virasoro constraints. This generalizes the reformulation of loop equations in terms of Virasoro constraints in the non-supersymmetric case, and thus one might hope that super-Virasoro constraints for supereigenvalue models lead immediately to supersymmetric topological recursion. However, in [20, 61] it was shown that such a generalization is not automatic: one can indeed write down a recursive system that determines the partition function of a supereigenvalue model, but it is augmented by an auxiliary equation, which does not have a simple interpretation. In a sense, this makes the super quantum Airy structures that we introduce here even more interesting, and revealing their meaning in the context of supereigenvalue models is an important task.

A similar situation arises in the context of quantum curves. On the one hand, for large classes of spectral curves it was shown that the topological recursion can be used

¹ For the subsequent progress in this context see [21].

to reconstruct the wave-function associated to a given classical curve, and at the same time the corresponding quantum curve [15,46,58]. On the other hand, various types of supersymmetric quantum curves (also called super quantum curves) have been constructed recently in the formalism of supereigenvalue models, and reinterpreted from the conformal field theory point of view [26–28]. Therefore one might expect that super quantum Airy structures provide a more general framework to construct super quantum curves, and to develop their theory further. We hope to address this issue in future work.

Super quantum Airy structures may also have interesting connections with various problems in enumerative geometry: we postpone this analysis for future work. They may be related to enumerative problems involving odd cohomology classes, such as Gromov–Witten theory of non-singular target curves. For instance, in [60] Okounkov and Pandharipande show that these invariants are encoded in a set of bosonic and fermionic operators forming a representation of a super-Virasoro algebra. These operators could potentially be related to super quantum Airy structures.

Super quantum Airy structures could also play a role in the theory surrounding Mirzakhani’s recursion relations for the Weil–Petersson volumes of the moduli spaces of Riemann surfaces [55]. In the non-supersymmetric case, it was shown that Mirzakhani’s recursion relations can be transformed into the form of the topological recursion [39,57], which plays a fundamental role in the connection between Jackiw–Teitelboim (JT) gravity and matrix models [63]. Very recently, Stanford and Witten generalized this fascinating story to the supersymmetric realm [64]. In the process, they found a generalization of Mirzakhani’s recursion relations to the volumes of the moduli spaces of super Riemann surfaces. Those new recursion relations could presumably be related to the supersymmetric topological recursion that we present here, and the corresponding super quantum Airy structures.

We discuss many more open problems and potential applications of super quantum Airy structures in Section 5, which we believe deserve further investigation.

1.1. Outline. The plan of this paper is as follows. In Section 2 we define super quantum Airy structures, and prove the existence and uniqueness of the corresponding free energies. We also show that these free energies satisfy a recursion relation, which is a supersymmetric version of the bosonic topological recursion, and we provide its graphical interpretation. We also introduce and discuss gauge transformations and the classical limit of super quantum Airy structures.

In Section 3 we present several finite-dimensional examples of super quantum Airy structures. We also discuss a classification scheme for such finite-dimensional structures. We illustrate this discussion by presenting the $\mathfrak{osp}(1|2)$ example, and conclude this section by constructing super quantum Airy structures from super Frobenius algebras.

In Section 4 we construct examples of infinite-dimensional super quantum Airy structures. The construction follows along the lines of [11,54]. We construct our examples as untwisted and \mathbb{Z}_2 -twisted representations for the free boson–fermion vertex operator super algebra (VOSA). In the process, we also generalize slightly the bosonic construction of [11], by considering larger families of subalgebras of the algebra of modes to construct our quantum Airy structures (from the point of view of vertex operator algebras, this should be related to the construction of Whittaker modules for the (super-)Virasoro algebra). As a byproduct, we obtain a realization of the “topological recursion without branched covers” of [4] in terms of untwisted representations for the free boson vertex operator algebra.

We conclude the paper with Section 5, where we list several open problems and potential applications of super quantum Airy structures. Finally, for completeness we provide a computational proof of existence of the free energies associated to a super quantum Airy structure in “Appendix A”.

2. Super Quantum Airy Structures

In this section we define super quantum Airy structures. We associate to them a unique free energy, which can be calculated recursively. We study their classical limits, and explain how super quantum Airy structures can be obtained as quantizations of super classical Airy structures. We also provide a graphical interpretation for the recursive computation of the free energy associated to a super quantum Airy structure.

2.1. Definition of Super Quantum Airy Structures. Quantum Airy structures were introduced in [4,52] (see also [10]) as an abstract framework underlying the Chekhov–Eynard–Orantin topological recursion [24,37,38]. Just as the Chekhov–Eynard–Orantin topological recursion can be generalized to spectral curves with arbitrary ramification [14,17], quantum Airy structures admit a natural generalization as higher quantum Airy structures: those were studied in [11].² In this section, we propose a further generalization, by introducing fermionic degrees of freedom.

2.1.1. Background and Notation The starting point of quantum Airy structures is a vector space V , with dimension either finite or countably infinite,³ over $\mathbb{K} = \mathbb{R}$ or \mathbb{C} . We introduce fermionic degrees of freedom by considering instead a super vector space V : that is, a \mathbb{Z}_2 -graded vector space V . We denote the even and odd subspaces of V by V_0 and V_1 , and elements of V_0 and V_1 will be called respectively even and odd. We write $|v| = \alpha$ for the parity of homogeneous elements $v \in V_\alpha, \alpha = 0, 1$.

We will only use bases composed of homogeneous elements. For a basis $\{x^i\}_{i \in I}$ in V , we denote by $\{y_i\}_{i \in I}$ the corresponding dual set in $V^* = \text{Hom}_{\mathbb{K}}(V, \mathbb{K})$. Here, $I = \{1, 2, \dots\}$ is a (possibly countably infinite) index set. We abbreviate $|i| = |x^i|$.

Remark 2.1. In order to distinguish the \mathbb{Z}_2 -grading on V from a \mathbb{Z} -grading of different structures that will appear in what follows, we denote the latter ones with a superscript. For instance, for any \mathbb{Z} -graded superalgebra A we have:

$$A = \bigoplus_{n \in \mathbb{Z}} A^n, \tag{2.1a}$$

$$A^n \cdot A^m \subseteq A^{n+m}. \tag{2.1b}$$

We also define $A^{\leq n} = \bigoplus_{m \leq n} A^m$ and $A^{\geq n} = \bigoplus_{m \geq n} A^m$. For any $a \in A$ we write $a = \sum_n a^n$ with $a^n \in A^n$. Moreover we set $a^{\leq n} = \sum_{m \leq n} a^m$ and $a^{\geq n} = \sum_{m \geq n} a^m$.

² In the nomenclature used in this paper, *higher quantum Airy structures* would be simply *quantum Airy structures*, while the original *quantum Airy structures* would be particular quantum Airy structures that are *quadratic*.

³ In fact this restriction is not essential, but it simplifies the notation a bit. It is satisfied in all interesting examples explored until now.

We define $\mathbb{K}[V, \hbar]$ as the space of all polynomials in x^i and \hbar . Similarly, we let $\mathcal{W}_\hbar(V)$ be the space of differential operators which can be written as a sum of finitely many terms of the form

$$\hbar^{m+k} x^{i_1} \dots x^{i_n} c_{i_1 \dots i_n}^{j_1 \dots j_m} \partial_{j_1} \dots \partial_{j_m}, \tag{2.2}$$

where for each fixed j_1, \dots, j_m there exist finitely many i_1, \dots, i_n such that $c_{i_1 \dots i_n}^{j_1 \dots j_m} \neq 0$.⁴ This condition does not depend on the choice of basis. It is equivalent to the statement that the set of coefficients c represents an element of $\text{Hom}(V^{\otimes m}, V^{\otimes n})$.

We introduce a \mathbb{Z} -grading on $\mathcal{W}_\hbar(V)$ by declaring an expression of the form (2.2) to be homogeneous of degree $n + m + 2k$. In particular we have

$$\text{deg}(x^i) = 1, \quad \text{deg}(\hbar \partial_i) = 1, \quad \text{deg}(\hbar) = 2. \tag{2.3}$$

The same rule defines a grading on the space $\mathbb{K}[V, \hbar]$.

So far we have defined the space $\mathcal{W}_\hbar(V)$ of differential operators acting on $\mathbb{K}[V, \hbar]$. An element of $\mathcal{W}_\hbar(V)$ may be written as a sum of finitely many terms of the form (2.2). We will also need to consider formal series in variables \hbar, x^i , and act on these with operators which are infinite sums of terms of the form (2.2). In order to define this precisely, we will introduce a topology on $\mathbb{K}[V, \hbar]$.

Let R be a ring and $I \subseteq R$ an ideal. We define on R the I -adic topology by declaring the collection $\{I^n\}_{n=0}^\infty$ to be the neighbourhood basis of zero. With this topology R is a topological ring.⁵ It satisfies the Hausdorff axiom if and only if $\bigcap_{n=0}^\infty I^n = \{0\}$.

Using this notion we equip $\mathbb{K}[V, \hbar]$ with a $\mathbb{K}[V, \hbar]^{\geq 1}$ -adic topology and denote its completion with respect to this topology by $\mathbb{K}[[V, \hbar]]$.

Example 2.2. To see explicitly what this means, let's take the simplest example: $V \cong \mathbb{K}$. Consider the algebra $\mathbb{K}[V]$ of polynomials in x together with the ideal $I = \mathbb{K}[V]^{\geq 1}$ of polynomials with vanishing constant term. The difference of two polynomials $p, q \in \mathbb{K}[V]$ belongs to I^{n+1} if $p(x)$ and $q(x)$ agree up to the term of the order x^n . The space $\mathbb{K}[[V]]$ obtained by completing $\mathbb{K}[V]$ with respect to the I -adic topology contains, besides polynomials, also formal power series in x .

$\mathcal{W}_\hbar(V)$ can now be regarded as a space of operators on $\mathbb{K}[[V, \hbar]]$ and endowed with the topology of pointwise convergence. This means that a generalized sequence T_α in $\mathcal{W}_\hbar(V)$ converges to an element T if and only if for every $f \in \mathbb{K}[[V, \hbar]]$ we have $T_\alpha f \rightarrow T f$ in $\mathbb{K}[[V, \hbar]]$. We claim that with this topology, multiplication on $\mathcal{W}_\hbar(V)$ is continuous. Indeed, let T_α and S_α be two generalized sequences in $\mathcal{W}_\hbar(V)$, with $T_\alpha \rightarrow T$ and $S_\alpha \rightarrow S$. Choose $f \in \mathbb{K}[[V, \hbar]]$. We have

$$(S_\alpha T_\alpha - ST)f = S_\alpha(T - T_\alpha)f + (S_\alpha - S)Tf. \tag{2.4}$$

By definition, $(S_\alpha - S)Tf \rightarrow 0$. Now given an $n \in \mathbb{N}$, there exists α_0 such that $(T - T_\alpha)f$ belongs to $\mathbb{K}[[V, \hbar]]^{\geq n}$ for $\alpha \geq \alpha_0$. Then also $S_\alpha(T - T_\alpha)f \in \mathbb{K}[[V, \hbar]]^{\geq n}$ for $\alpha \geq \alpha_0$, so $(S_\alpha T_\alpha - ST)f \rightarrow 0$. Since f was arbitrary, we obtain $S_\alpha T_\alpha \rightarrow ST$. Continuity of multiplication on $\mathcal{W}_\hbar(V)$ guarantees that it extends uniquely to a continuous multiplication on the completion, which will be denoted by $\widehat{\mathcal{W}_\hbar(V)}$. Explicitly, one has $TS = \lim_{n \rightarrow \infty} T^{\leq n} S^{\leq n} = \sum_{n=0}^\infty \sum_{k=0}^n T^k S^{n-k}$ for $T, S \in \widehat{\mathcal{W}_\hbar(V)}$.

⁴ Here and henceforth we use the Einstein summation convention, in which repeated indices are summed over.

⁵ This means that R is equipped with a topology such that addition and multiplication, regarded as maps $R \times R \rightarrow R$ (with $R \times R$ given the product topology), are continuous.

2.1.2. Super Quantum Airy Structures We are now ready to define super quantum Airy structures. To this end, let us introduce a little more notation. Let V be a super vector space. As above, we choose a basis $\{x^i\}_{i \in I}$ in V , with $I = \{1, 2, \dots\}$, and the corresponding dual set $\{y_i\}_{i \in I}$ in V^* . Let $\tilde{V} = V \oplus \mathbb{K}^{0|1}$. We let x^0 be a basis for $\mathbb{K}^{0|1}$. In other words, \tilde{V} has one more fermionic dimension than V .

To clearly distinguish between V and \tilde{V} , we denote indices which take value in the set $\{0, 1, 2, \dots\}$ by small letters from the beginning of the alphabet, i.e. $a, b, c, d = 0, 1, 2, \dots$, while the indices denoted by letters from the “middle” of the alphabet, i, j, k , take values in the set $I = \{1, 2, \dots\}$. Consequently, we denote by $\{x^a\}_{a \geq 0}$ a basis in $\tilde{V} = \mathbb{K}^{0|1} \oplus V$.

Definition 2.3. Let V be a super vector space, and $\tilde{V} = V \oplus \mathbb{K}^{0|1}$. A *super quantum Airy structure* is a pair (V, L) , with $L : V^* \rightarrow \widehat{\mathcal{W}}_{\hbar}(\tilde{V})$ an even (that is, grade-preserving) continuous linear operator such that:

- (1) The left ideal $\mathcal{L} \subseteq \widehat{\mathcal{W}}_{\hbar}(V)$ generated by $L(V^*)$ is involutive, i.e. $[\mathcal{L}, \mathcal{L}] \subseteq \hbar \cdot \mathcal{L}$,
- (2) $L(y_i)^{\leq 1} = \hbar \delta_i$.

Here $[\cdot, \cdot]$ is the super-commutator, which for two homogeneous elements $v, v' \in V^*$ is given by

$$[L(v), L(v')] = L(v)L(v') - (-1)^{|v||v'|}L(v)L(v'). \tag{2.5}$$

It is convenient to abbreviate $L_i = L(y_i)$. The condition (1) guarantees that

$$[L_i, L_j] = \hbar f_{ij}^k L_k \tag{2.6}$$

for some elements $f_{ij}^k \in \widehat{\mathcal{W}}_{\hbar}(V)$. Continuity of L implies that for fixed k there are only finitely many i, j such that $f_{ij}^k \neq 0$. It also follows from continuity of L that all $L(\phi)$ are determined by L_i

$$L(\phi) = \lim_{J \in \mathcal{P}_{\text{fin}}(I)} \sum_{j \in J} \phi(x^j) L_j, \tag{2.7}$$

where $\mathcal{P}_{\text{fin}}(I)$ is the set of all finite subsets of I , ordered by inclusion.

Remark 2.4. We note that in the definition of super quantum Airy structures, the domain is the super vector space V^* , while the image consists of differential operators acting on the space $\mathbb{K}[[\tilde{V}, \hbar]]$, where $\tilde{V} = V \oplus \mathbb{K}^{0|1}$. In other words, the linear operator L acts on a space with one more fermionic dimension than V^* . That is, the operators $\{L_i\}_{i \geq 1}$ can depend on the extra fermionic variable x^0 , but there is no operator L_0 . This possibility is a peculiar feature of super quantum Airy structures, which is not present for traditional quantum Airy structures. It turns out to be crucial in many interesting examples of super quantum Airy structures.

Remark 2.5. We remark that in the particular case where $L(V^*) \subset \widehat{\mathcal{W}}_{\hbar}(V) \subset \widehat{\mathcal{W}}_{\hbar}(\tilde{V})$ (i.e. the L_i do not depend on the extra fermionic variable x^0 in \tilde{V}), and $V = V_0$ (V is an even vector space with no fermionic dimension), Definition 2.3 becomes the standard definition of higher Airy structures of [11]. We will call such super quantum Airy structures *bosonic*.

The definition 2.3 is rather general, as it allows operators that are infinite sums of terms of the form (2.2), and also differential operators of infinite order. Nonetheless, in practice we will mostly consider finite order differential operators.

Definition 2.6. Let (V, L) be a super quantum Airy structure. If $L(\phi) \subseteq \widehat{\mathcal{W}}_{\hbar}(\tilde{V})^{\leq n}$ for some positive integer n , we call the smallest such n the *order* of L . If $n = 2$, we say that the super quantum Airy structure is *quadratic*. If there is no such n , we say that the super quantum Airy structure has *infinite order*.

We will also be interested in the particular case of super quantum Airy structures where the stronger requirement that the L_i themselves span a Lie superalgebra is satisfied, instead of the milder constraint (1) in Definition 2.3.

Definition 2.7. We say that a super quantum Airy structure (V, L) is *subalgebraic* if the stronger requirement that $[L(V^*), L(V^*)] \subseteq \hbar \cdot L(V^*)$ is satisfied. In other words, the L_i span a Lie superalgebra.

Condition $[L(V^*), L(V^*)] \subseteq \hbar \cdot L(V^*)$ is equivalent to existence of $f_{ij}^k \in \mathbb{K}$ such that

$$[L_i, L_j] = \hbar f_{ij}^k L_k. \tag{2.8}$$

Continuity of L implies that for fixed k there are finitely many i, j , such that $f_{ij}^k \neq 0$. This means that if we endow V with the discrete topology, and V^* with the topology of pointwise convergence, then the expression $[y_i, y_j] = f_{ij}^k y_k$ extends uniquely to a continuous Lie bracket on V^* .

In some cases we will also be interested in restricting to super quantum Airy structures such that the L_i can be written as finite sums of terms of the form (2.2). That is, we want to consider the particular case where the L_i are in $\mathcal{W}_{\hbar}(\tilde{V})$.

Lemma 2.8. *Let (V, L) be a super quantum Airy structure with $L(V^*) \subseteq \mathcal{W}_{\hbar}(\tilde{V})$. Then there exists $n \in \mathbb{N}$ such that $L(V^*) \subseteq \mathcal{W}_{\hbar}(\tilde{V})^{\leq n}$ for some n .*

Proof. Each L_i is an element of $\mathcal{W}_{\hbar}(\tilde{V})$, so it is actually in $\mathcal{W}_{\hbar}(\tilde{V})^{\leq n_i}$ for some n_i . Suppose that n_i are not uniformly bounded. Then for $\phi \in V^*$ such that $\phi(x^i) = 1$ for all i the right-hand side of (2.7) is manifestly divergent. Contradiction. Hence there is an $n \in \mathbb{N}$ such that $L_i \in \mathcal{W}_{\hbar}(\tilde{V})^{\leq n}$ for each i . Since $\mathcal{W}_{\hbar}(\tilde{V})^{\leq n}$ is closed in $\mathcal{W}_{\hbar}(\tilde{V})$, the result follows from the formula (2.7). \square

Remark 2.9. With all these definitions, we recover the original definition of quantum Airy structures in [4, 52] as the particular case of a super quantum Airy structure that is bosonic (there is no fermionic variable), quadratic ($L_i \in \mathcal{W}_{\hbar}(V)^{\leq 2}$), and subalgebraic ($[L(V^*), L(V^*)] \subseteq \hbar \cdot L(V^*)$). Note that we do not require that V is finite-dimensional.

2.1.3. Free Energy and Partition Function Perhaps the most important reason to study quantum Airy structures is that one can assign to them a unique free energy, which for special choice of L_i turns out to be a generating function for some interesting enumerative invariants [4, 10, 11, 52]. In this section we generalize this construction to super quantum Airy structures.

To every super quantum Airy structure we assign a partition function $Z(x) = e^{\frac{F(x)}{\hbar}}$, with free energy $F(x) \in (\tilde{V} \cdot \mathbb{K}[[\tilde{V}, \hbar]])_0^{\geq 3}$, which is defined, loosely speaking, as a solution to the equation

$$L_i \cdot Z(x) = 0. \tag{2.9}$$

The requirement that $F(x) \in (\tilde{V} \cdot \mathbb{K}[[\tilde{V}, \hbar]])_0^{\geq 3}$ means that:

- $F(x)$ is a formal power series in the variables x^a and \hbar with no term independent of the x^a ;
- $F(x)$ is even with respect to the \mathbb{Z}_2 -grading;
- $F(x)$ only has terms of degree ≥ 3 with respect to the \mathbb{Z} -grading.

Following [52] we will demonstrate that there is a unique such $Z(x)$. To this end it is useful to reformulate (2.9) slightly.

Every $F \in (\tilde{V} \cdot \mathbb{K}[[\tilde{V}, \hbar]])_0^{\geq 3}$ induces a continuous automorphism

$$\widehat{\mathcal{W}}_{\hbar}(\tilde{V}) \ni D \mapsto \psi_F(D) = \sum_{n=0}^{\infty} \frac{1}{n!} \left(-\frac{1}{\hbar} [F, \cdot] \right)^n (D) \in \widehat{\mathcal{W}}_{\hbar}(\tilde{V}). \tag{2.10}$$

In particular acting on topological generators⁶ of $\widehat{\mathcal{W}}_{\hbar}(\tilde{V})$ one gets:

$$\psi_F(\hbar) = \hbar, \tag{2.11a}$$

$$\psi_F(x^a) = x^a, \tag{2.11b}$$

$$\psi_F(\hbar \partial_a) = \hbar \partial_a + \partial_a F. \tag{2.11c}$$

Since $(-\frac{1}{\hbar} [F, \cdot])^n (D) \in \widehat{\mathcal{W}}_{\hbar}(\tilde{V})^{\geq n}$, the series in the definition of $\psi_F(D)$ converges. Linearity of ψ_F is obvious. Continuity follows from the fact that the subspaces $\widehat{\mathcal{W}}_{\hbar}(\tilde{V})^{\geq n}$ are ψ_F -invariant. $\psi_F(D_1 D_2) = \psi_F(D_1) \psi_F(D_2)$ is a consequence of $-\frac{1}{\hbar} [F, \cdot]$ being a derivation and ψ_F^{-1} is given explicitly as ψ_{-F} , what proves that ψ_F is indeed an automorphism. We thus say that F is the free energy associated to the super quantum Airy structure (V, L) if

$$\forall \phi \in V^*, \quad \psi_F(L(\phi)) \cdot 1 = 0. \tag{2.12}$$

We can now formulate the main result of this section:

Theorem 2.10. *Every super quantum Airy structure admits a unique free energy.*

Proof. Pick some $F \in (\tilde{V} \cdot \mathbb{K}[[\tilde{V}, \hbar]])_0^{\geq 3}$ and let $E(\phi) = \psi_F(L(\phi)) \cdot 1$. Then $E(\phi)^{\leq 1} = 0$. We make an inductive hypothesis that $F^{\leq n}$ may be chosen in a unique way so that $E(\phi)^{\leq n-1} = 0$. Acting with $\psi_F(L_i) = \hbar \partial_i + \partial_i F + \psi_F(L_i^{\geq 2})$ on 1 we obtain

$$(E(y_i))^n = \partial_i (F^{n+1}) + H_i^n, \tag{2.13}$$

where H_i^n is a function of $F^{\leq n}$. Continuity of L and ψ_F guarantees that for fixed n there are only finitely many i such that $H_i^n \neq 0$. Evaluating $\psi_F([L_i, L_j] - \hbar f_{ij}^k L_k) \cdot 1$ we get

$$\partial_i H_j^n - (-1)^{|i||j|} \partial_j H_i^n = 0, \tag{2.14}$$

so the equation $(E(y_i))^n = 0$ can be solved for F^{n+1} as

$$F^{n+1} = -(1 + x^i \partial_i)^{-1} x^j H_j^n, \tag{2.15}$$

up to the addition of an arbitrary integration constant in the extra fermionic variable x^0 in \tilde{V} . But F is required to be even, and hence this integration constant must vanish. Therefore, the solution is unique, and hence F^{n+1} is uniquely determined by the condition $E(\phi)^{\leq n} = 0$. \square

⁶ For a topological ring R , subset $S \subseteq R$ is said to be a set of topological generators if the smallest closed subring of R containing S coincides with R .

Remark 2.11. One could ask whether we cannot enlarge \tilde{V} further. For instance, one could consider $L : V^* \rightarrow \widehat{\mathcal{W}}_{\hbar}(V \oplus X)$ and $F \in \mathbb{K}[[V \oplus X, \hbar]]$, for more general X . In general, the proof of Theorem 2.10 would then guarantee existence of the free energy, but not uniqueness. Existence and uniqueness are obtained only when $\dim X = 0|1$, since in this case the requirement that F is even is sufficient to guarantee that $\partial_i F^{n+1} + H_i^n = 0$ has a unique solution.

In particular, as is clear from this argument, there is no such freedom of enlarging V for bosonic quantum Airy structures. The possibility of having an extra coordinate x^0 is a purely fermionic phenomenon.

2.1.4. A Little More Structure In the previous section we showed that we can associate a unique free energy $F \in (\tilde{V} \cdot \mathbb{K}[[\tilde{V}, \hbar]])_0^{\geq 3}$ to every super quantum Airy structure, by requiring that $\psi_F(L(\phi)) \cdot 1 = 0$ for all $\phi \in V^*$. However, we started the section by saying that the free energy was defined such that the partition function $Z(x) = e^{\frac{F(x)}{\hbar}}$ is a solution to the system of equations $L_i \cdot Z(x) = 0$. Let us now explore the connection between the two statements more precisely. To this end, we now define a convenient ring of series.

Definition 2.12. Let R be the super \mathbb{K} -vector space of formal series of the form

$$f = \sum_{a=-\infty}^{\infty} \sum_{b=0}^{\infty} \hbar^a f_{a,b}(x), \tag{2.16}$$

where $f_{a,b}(x)$ is a polynomial of degree b in the variables x^i and $f_{a,b} = 0$ if $3a + b < 0$. Note that we allow both positive and negative powers of \hbar here. Each term $\hbar^a f_{a,b}(x)$ is declared to be homogeneous of degree $2a + b$, as is consistent with our \mathbb{Z} -grading.⁷

It is easy to check that the condition $3a + b \geq 0$ combined with $b \geq 0$ entail that $2a + b \geq 0$, so the degree of each term is always non-negative. For fixed $d = 2a + b$ we have inequalities $2a \leq d$ and $a \geq -d$, so f may be rewritten as

$$f = \sum_{d=0}^{\infty} \sum_{a=-d}^{2a \leq d} \hbar^a f_{a,d-2a}(x) = \sum_{d=0}^{\infty} f^d, \tag{2.17}$$

where we introduced the homogeneous components f^d of f . Notice that each f^d is a polynomial in x, \hbar and \hbar^{-1} with f^0 being a constant. Therefore:

Lemma 2.13. R is a supercommutative \mathbb{K} -algebra, with a product given by

$$fg = \sum_{d=0}^{\infty} \sum_{k=0}^d f^k g^{d-k}. \tag{2.18}$$

Remark 2.14. Since we are considering series which include both positive and negative powers of \hbar , the condition $f_{a,b} = 0$ for $3a + b < 0$ is needed to make the multiplication in R well defined. Let us remark that a weaker condition $f_{a,b} = 0$ for $2a + b < 0$ would also work here, but it is insufficient to make sense of gauge transformations which will be discussed in Section 2.3.

⁷ This is not a typo: we use both combinations $2a + b$ and $3a + b$ in our considerations.

We can go further:

Lemma 2.15. *R is a local superring with maximal ideal $\mathfrak{m} = \{f \in R \mid f^0 = 0\}$.*

Proof. Since the degree is non-negative, elements of \mathfrak{m} cannot be invertible. On the other hand, for any $\epsilon \in \mathfrak{m}$ and $u \in \mathbb{K}^\times$, the inverse of the element $u + \epsilon$ is given explicitly by

$$\frac{1}{u + \epsilon} = u^{-1} \sum_{k=0}^{\infty} (-u^{-1}\epsilon)^k. \tag{2.19}$$

The infinite sum on the right hand side is well-defined because only the first d terms contribute to the homogeneous component of degree d . The identity $(u + \epsilon) \cdot u^{-1} \sum_{k=0}^{\infty} (-u^{-1}\epsilon)^k = 1$ is then quite obvious, and we conclude that all elements of $R \setminus \mathfrak{m}$ are invertible. \square

It is easy to see that R equipped with the \mathfrak{m} -adic topology is a complete Hausdorff space. In particular, for any $f \in \mathfrak{m}$ and $g \in \mathbb{K}[[t]]$ we have an element $g(f) \in R$. The most important for us (perhaps except for polynomials) examples of this construction are

$$\exp(f) = \sum_{k=0}^{\infty} \frac{f^k}{k!}, \tag{2.20a}$$

$$\log(1 + f) = - \sum_{k=1}^{\infty} \frac{(-f)^k}{k}. \tag{2.20b}$$

We note that $\log(\exp(f)) = f$ and $\exp(\log(1 + f)) = 1 + f$.

Since elements of each degree in R are polynomials, we can act on them by elements of the completed Weyl algebra $\widehat{\mathcal{W}}_{\hbar}(\tilde{V})$. After projecting to homogeneous terms, this reduces to the computation of a finite sums of terms in which a differential operator of finite degree acts on a polynomial. We conclude that:

Lemma 2.16. *R, equipped with the \mathfrak{m} -adic topology, is a topological $\widehat{\mathcal{W}}_{\hbar}(\tilde{V})$ -module.*

Now let us go back to the free energy and partition function associated to a super quantum Airy structure. Following [52], we defined the free energy associated to a super quantum Airy structure as an automorphism $\psi_F = \exp\left(\frac{1}{\hbar}[F, \cdot]\right)$ with $F \in (\tilde{V} \cdot \mathbb{K}[[\tilde{V}, \hbar]])_0^{\geq 3}$, and such that for all i the operator $\psi_F(L_i)$ annihilates 1. Observe now that F (and hence also $Z = e^{\frac{F}{\hbar}}$) is a well-defined element of the ring R . Moreover $Z^0 = 1$, so Z is invertible. Simple manipulation with the involved series shows that the identity⁸

$$Z^{-1} (L_i \cdot Z) = \psi_F(L_i) \cdot 1 \tag{2.21}$$

indeed holds, as expected. Therefore $Z \in R$ is annihilated by all L_i , as claimed originally.

An explicit formula for the partition function in terms of the free energy reads

$$Z = 1 + \sum_{k=1}^{\infty} \sum_{a=-k}^{2a \leq k-1} \hbar^a \sum_{\{(g_\alpha, n_\alpha)\}_{\alpha \in A}} \prod_{\alpha \in A} F_{g_\alpha, n_\alpha}, \tag{2.22}$$

⁸ After projecting to terms of given total degree we always get finite sums, so this is a purely combinatorial problem. It boils down to identity $[A, \cdot]^n(B) = \sum_{k=0}^n \binom{n}{k} A^k B(-A)^{n-k}$ for even A .

where the last sum is taken over all finite sets of pairs $(g, n) \in \mathbb{N}^2$ such that $n \geq 1$, $\sum_{\alpha \in A} (g_\alpha - 1) = a$, and $\sum_{\alpha \in A} [2(g_\alpha - 1) + n_\alpha] = k$. This sum is always finite. Two remarks are in order.

Remark 2.17. In the definition of the ring R , we allow both positive and negative powers of \hbar . One may then revisit the proof of existence and uniqueness of the free energy (Theorem 2.10) with slightly more general assumptions. Instead of requiring from the start that $F \in (\tilde{V} \cdot \mathbb{K}[[\tilde{V}, \hbar]])_0^{\geq 3}$, one could consider $F \in R^{\geq 3}$, with the requirement that $F = 0$ for $x = 0$. The difference here is that we allow terms with negative powers of \hbar , as long as they are accompanied with sufficiently many powers of x^a so that the degree of each term is ≥ 3 . Then, following the same steps as in the proof of Theorem 2.10, one sees that existence and uniqueness of the free energy associated to a super quantum Airy structure is still true, and hence it must belong to $(\tilde{V} \cdot \mathbb{K}[[\tilde{V}, \hbar]])_0^{\geq 3}$. In other words, the lack of negative powers of \hbar in F is a result, rather than an assumption.

Remark 2.18. One may ask why we did not prove existence and uniqueness of the partition function Z directly by solving the system of equations $L_i Z = 0$ subject to the condition that Z evaluated at $x = 0$ is equal to 1. This is certainly possible: the proof technique is exactly the same. Then one may define $F = \hbar \log Z$. In this approach however it is not clear to us how to prove directly that the free energy does not contain negative powers of \hbar . (Of course, this must still be true, since it is the same unique free energy as the one obtained above.)

2.2. Recursive System. For simplicity, in this section we focus on super quantum Airy structures that are quadratic and subalgebraic, and thus can be regarded as supersymmetric analogs of the original quantum Airy structures of [4, 52]. In the spirit of [4], we derive explicit conditions for the coefficients of the operators L_i such that the L_i form a super quantum Airy structure. We also compute a recursive system for the coefficients of the free energy uniquely associated to a super quantum Airy structure.

2.2.1. Constraints on the Coefficients As in the previous section, we choose a basis $\{x^i\}_{i \geq 1}$ for the super vector space V and denote by x^0 a basis vector for $\mathbb{K}^{0|1}$. We let $\tilde{V} = \mathbb{K}^{0|1} \oplus V$, and denote by $\{x^a\}_{a \geq 0}$ a basis in \tilde{V} . We denote indices which take value in the set $\{0, 1, 2, \dots\}$ by small letters from the beginning of the alphabet, i.e. $a, b, c, d = 0, 1, 2, \dots$, while indices denoted by the letters from the “middle” of the alphabet, i, j, k , take values in the set $\{1, 2, \dots\}$.

Super quantum Airy structures were introduced in Definition 2.3. To construct a quadratic super quantum Airy structure, we need to find an even continuous linear operator $L : V^* \rightarrow \widehat{\mathcal{W}}_{\hbar}(\tilde{V})^{\leq 2}$ such that:

- for each $i \geq 1$ we have $L_i = L(y_i) = \hbar \partial_i + L_i^2$ where $L_i^2 \in \widehat{\mathcal{W}}_{\hbar}(\tilde{V})^2$;
- $[L_i, L_j] = \hbar f_{ij}^k L_k$ for some structure constants $f_{ij}^k \in \mathbb{K}$.

Explicitly, we can write

$$L_i = \hbar \partial_i - \frac{1}{2} A_{iab} x^a x^b - \hbar B_{ia}^b x^a \partial_b - \frac{1}{2} \hbar^2 C_i^{ab} \partial_a \partial_b - \hbar D_i \tag{2.23}$$

$$=: L_i^{\leq 1} + L_i^A + L_i^B + L_i^C + L_i^D, \tag{2.24}$$

with the coefficients $A_{iab}, B_{ia}^b, C_i^{ab}, D_i \in \mathbb{K}$. As we have already remarked, continuity of L imposes that, if \tilde{V} is infinite-dimensional, for fixed i only finitely many A_{iab} are non-zero, and for fixed i and b only finitely many B_{ia}^b are non vanishing. Clearly, we may assume the symmetry conditions

$$A_{iab} = (-1)^{|a||b|} A_{iba}, \quad C_i^{ab} = (-1)^{|a||b|} C_i^{ba}. \tag{2.25}$$

Since $L : V^* \rightarrow \widehat{\mathcal{W}}_{\hbar}(\tilde{V})^{\leq 2}$ is assumed to be even, we can think of the coefficients $A_{iab}, B_{ia}^b, C_i^{ab}$, and D_i as components of even tensors

$$A \in V \otimes \tilde{V} \otimes \tilde{V}, \quad B \in \text{Hom}(\tilde{V}, V \otimes \tilde{V}), \quad C \in \text{Hom}(\tilde{V} \otimes \tilde{V}, V), \quad D \in V. \tag{2.26}$$

In the spirit of [4], we can reformulate the Lie superalgebra requirement $[L_i, L_j] = \hbar f_{ij}^k L_k$, as a set of constraints on the tensors A, B, C, D . This is to be compared with Lemma 2.2 of [4].

Lemma 2.19. *The differential operators L_i in (2.23) form a super quantum Airy structure if and only if the following conditions are satisfied*

$$A_{jia} = (-1)^{|i||j|} A_{ija}, \tag{2.27a}$$

$$f_{ij}^k = (-1)^{|i||j|} B_{ij}^k - B_{ji}^k, \tag{2.27b}$$

$$0 = (-1)^{|i||j|} B_{ij}^0 - B_{ji}^0, \tag{2.27c}$$

and

$$B_{ia}^c A_{jcb} + (-1)^{|a||b|} B_{ib}^c A_{jca} + (-1)^{|i||j|} B_{ij}^k A_{kab} = (-1)^{|i||j|} (i \leftrightarrow j), \tag{2.28a}$$

$$B_{ia}^c B_{jc}^b + (-1)^{|a||b|} C_i^{bc} A_{jca} + (-1)^{|i||j|} B_{ij}^k B_{ka}^b = (-1)^{|i||j|} (i \leftrightarrow j), \tag{2.28b}$$

$$C_i^{ac} B_{jc}^b + (-1)^{|a||b|} C_i^{bc} B_{jc}^a + (-1)^{|i||j|} B_{ij}^k C_k^{ab} = (-1)^{|i||j|} (i \leftrightarrow j), \tag{2.28c}$$

$$\frac{1}{2} C_i^{ba} A_{jab} + (-1)^{|i||j|} B_{ij}^k D_k = (-1)^{|i||j|} (i \leftrightarrow j). \tag{2.28d}$$

Proof. These conditions are very similar to those of Lemma 2.2 in [4], with appropriate signs, and range of indices to take into account the extra fermionic variable. The proof is also a straightforward computation. We simply expand the super-commutator $[L_i, L_j]$ and collect terms with respect to x^a and ∂_a . Then by comparing with $\hbar f_{ij}^k L_k$, we obtain the set of constraints. \square

2.2.2. Topological Recursion In Section 2.1.3 we associated a unique free energy $F \in (\tilde{V} \cdot \mathbb{K}[[\tilde{V}, \hbar]])_0^{\geq 3}$ to every super quantum Airy structure. In this section, we show how its coefficients can be calculated recursively. In the spirit of [4], and for the sake of completeness, we also show how existence and uniqueness of the free energy can be proven computationally from the recursive structure.

Theorem 2.20. *Let (V, L) be a quadratic super quantum Airy structure, and let $F \in (\tilde{V} \cdot \mathbb{K}[[\tilde{V}, \hbar]])_0^{\geq 3}$ be its associated free energy. We can expand F in the basis $\{x^a\}_{a \geq 0}$ for \tilde{V} as:*

$$F = \sum_{g \geq 0} \sum_{n \geq 1} \sum_{a_1, \dots, a_n \geq 0} \frac{\hbar^g}{n!} F_{g,n}[a_1, \dots, a_n] x^{a_1} \dots x^{a_n}, \tag{2.29}$$

where the coefficients $F_{g,n}[a_1, \dots, a_n] \in \mathbb{K}$ are \mathbb{Z}_2 -symmetric (with signs) under permutations of the indices $\{a_1, \dots, a_n\}$. Then the coefficients $F_{g,n}[a_1, \dots, a_n]$ satisfy the recursive system:

$$\begin{aligned}
 F_{g,n+1}[i, \Phi] &= A_{i a_1 a_2} \delta_{g,0} \delta_{n,2} + D_i \delta_{n,0} \delta_{g,1} \\
 &+ \sum_{k=1}^n \sigma_{a_k \subset \Phi} \sum_{b \geq 0} B_{i a_k}^b F_{g,n}[b, \Phi \setminus a_k] + \frac{1}{2} \sum_{b,c \geq 0} C_i^{bc} F_{g-1,n+2}[c, b, \Phi] \\
 &+ \frac{1}{2} \sum_{b,c \geq g_1+g_2=g} \sum_{\Phi_1 \cup \Phi_2 = \Phi} \sigma_{\Phi_1 \subset \Phi} C_i^{bc} F_{g_1,n_1+1}(b, \Phi_1) F_{g_2,n_2+1}[c, \Phi_2],
 \end{aligned}
 \tag{2.30}$$

with the auxiliary equation

$$F_{g,n+1}[0, a_1, a_2, \dots, a_n] = (-1)^{|a_1|} F_{g,n+1}[a_1, 0, a_2, \dots, a_n].
 \tag{2.31}$$

Here, $\Phi = \{a_1, \dots, a_n\}$ is an ordered set, and $\sigma_{\Phi_1 \subset \Phi}$ denotes the sign of the permutation from Φ to $\Phi_1 \cup (\Phi \setminus \Phi_1)$.

We remark that the recursive formula makes sense for infinite-dimensional V , since by induction one can show that for any g and n only finitely many $F_{g,n}[a_1, \dots, a_n]$ are non-vanishing, and hence the sums on the right-hand-side are all finite.

Proof. By Theorem 2.10, there exists a unique $F \in (\widetilde{V} \cdot \mathbb{K}[[\widetilde{V}, \hbar]])_0^{\geq 3}$ such that $\psi_F(L_i) \cdot 1 = 0$. We now show that this implies the recursive system (2.30) and the auxiliary equation (2.31).

To derive (2.30) from $\psi_F(L_i) \cdot 1 = 0$, we consider for $n \geq 0$

$$\partial_{a_n} \dots \partial_{a_1} \cdot \left(\psi_F(L_i) \cdot 1 \right) \Big|_{x=0} = 0.
 \tag{2.32}$$

Note that the order of the derivatives is important to have the correct sign. As $\psi_F(L_i)$ is linear, we list the computational results for each term in $L_i = L_i^{\leq 1} + L_i^A + L_i^B + L_i^C + L_i^D$ for completeness:

$$\begin{aligned}
 \partial_{a_n} \dots \partial_{a_1} \cdot \left(\psi_F \left(L_i^{\leq 1} \right) \cdot 1 \right) \Big|_{x=0} &= \sum_{g \geq 0} \hbar^g F_{g,n+1}[i, a_1, \dots, a_n] \\
 \partial_{a_n} \dots \partial_{a_1} \cdot \left(\psi_F \left(L_i^{(A)} \right) \cdot 1 \right) \Big|_{x=0} &= - \sum_{g \geq 0} \hbar^g (A_{i a_1 a_2} \delta_{n,2} \delta_{g,0}) \\
 \partial_{a_n} \dots \partial_{a_1} \cdot \left(\psi_F \left(L_i^{(B)} \right) \cdot 1 \right) \Big|_{x=0} &= - \sum_{g \geq 0} \hbar^g \sum_{k=1}^n \sum_{b \geq 0} \sigma_{a_k \subset \Phi} B_{i a_k}^b F_{g,n}[b, \Phi \setminus a_k] \\
 \partial_{a_n} \dots \partial_{a_1} \cdot \left(\psi_F \left(L_i^{(C)} \right) \cdot 1 \right) \Big|_{x=0} &= - \sum_{g \geq 0} \frac{\hbar^g}{2} \sum_{p,q \geq 0} C_i^{bc} \\
 &\quad \left(F_{g-1,n+2}[c, b, \Phi] + \sum_{g_1+g_2=g} \sum_{\Phi_1 \cup \Phi_2 = \Phi} \sigma_{\Phi_1 \subset \Phi} \right. \\
 &\quad \left. F_{g_1,n_1+1}[b, \Phi_1] F_{g_2,n_2+1}[c, \Phi_2] \right)
 \end{aligned}$$

$$\partial_{a_n} \dots \partial_{a_1} \cdot \left(\psi_F \left(L_i^{(D)} \right) \cdot 1 \right) \Big|_{x=0} = - \sum_{g \geq 0} \hbar^g (D_i \delta_{n,0} \delta_{g,1}). \tag{2.33}$$

Collecting terms order by order in \hbar , we obtain (2.30).

As for the auxiliary equation, it is necessary because $F_{g,n+1}[0, a_1, a_2, \dots, a_n]$ is not fixed by the recursive system, since there is no L_0 . However, $F_{g,n+1}[a_1, 0, a_2, \dots, a_n]$ is fixed, and hence $F_{g,n+1}[0, a_1, a_2, \dots, a_n]$ is uniquely fixed by symmetry as in (2.31).

□

Remark 2.21. We can in fact prove existence and uniqueness of the free energy from this computational point of view, in the spirit of [4]. This provides an alternative proof of Theorem 2.10. The proof proceeds in three steps:

- (1) We first show that $\psi_F(L_i) \cdot 1 = 0$ implies the recursive system (2.30) and (2.31), as in Theorem 2.20.
- (2) Assuming existence of a free energy $F \in (\tilde{V} \cdot \mathbb{K}[[\tilde{V}, \hbar]])_0^{\geq 3}$, with expansion given by (2.29), we show that (2.30) and (2.31) uniquely reconstructs it. This is clear, since (2.30) is a recursive system on $2g + n$ that reconstructs (in conjunction with (2.31)) all coefficients $F_{g,n}[a_1, \dots, a_n]$ from the initial conditions

$$F_{0,3}(i, a, b) = A_{iab}, \quad F_{0,3}(0, i, a) = (-1)^{|i|} A_{i0a}, \quad F_{1,1}(i) = D_i, \tag{2.34}$$

where we used (2.31) for the second condition.

- (3) What remains to be proved is that the free energy F actually exists, which is the difficult part. As in [4], the idea is to start from the recursive system (2.30) and (2.31), and show that, while it is not manifestly symmetric, the coefficients $F_{g,n}[a_1, \dots, a_n]$ that it constructs indeed are \mathbb{Z}_2 -symmetric. Therefore, the recursive system does reconstruct a free energy $F \in (\tilde{V} \cdot \mathbb{K}[[\tilde{V}, \hbar]])_0^{\geq 3}$ through its expansion (2.29). By (1), it is a solution to $\psi_F(L_i) \cdot 1 = 0$, and by (2), it is unique, and hence we have proven existence and uniqueness. The essence of the proof of existence thus consists in showing that the recursive system (2.30) and (2.31) reconstructs \mathbb{Z}_2 -symmetric coefficients. An interesting aspect of this computational proof is that it highlights the importance of the Lie superalgebra requirement. For completeness, we present this computational proof of existence in ‘‘Appendix A’’.

2.2.3. Graphical Interpretation The recursive system (2.30) has a nice interpretation in terms of sums of trivalent graphs, in parallel to the graphical interpretation presented for bosonic quantum Airy structures in [4, 10]. For completeness, we present this graphical interpretation in this section, focusing on quadratic super quantum Airy structures without the extra fermionic variable. We follow very closely the presentation in [10]. It remains to be seen whether the graphical interpretation can be extended to super quantum Airy structures with an extra fermionic variable.

The graphical interpretation is very similar to the one presented in Section 1.3 of [10]. In fact, the graphs are the same, but we need to change slightly the assignment of weights to the graphs to take into account the signs arising from \mathbb{Z}_2 -symmetry.

Let us first define the set $\mathbb{G}_{g,n+1}$, which is the same as Definition 1.4 of [10].

Definition 2.22. [Definition 1.4 of [10]] For $g \geq 0$ and $n \geq 0$ such that $\chi_{g,n} := 2g + n - 2 \geq 0$, we form the set $\mathbb{G}_{g,n+1}$ consisting of pairs $\Gamma = (G, T)$ where:

- G is a connected trivalent graph with $2g - 1 + n$ trivalent vertices, $n + 1$ ordered one-valent leaves, and first Betti number $b_1(G) = g$. We single out a leaf and call it the root of G .
- $T \subseteq G$ is a spanning tree that includes the root of G , but none of the leaves.
- The edges $e = \{v, v'\}$ of G which are not in the spanning tree T connect parent vertices, i.e. the common ancestor of v and v' in the rooted spanning tree T is either v or v' .

We denote the ordered leaves by $\ell, \ell_1, \ell_2, \dots, \ell_n$ in counterclockwise order,⁹ with ℓ being the root. We denote by $E'(\Gamma)$ the set, consisting of leaves (including the root) of G and edges of G that are not loops.

Definition 2.23. An *automorphism* of $\Gamma = (G, T) \in \mathbb{G}_{g,n+1}$ is a permutation of the edges in G that preserves the graph structure. We denote by $\text{Aut}(\Gamma)$ the set of automorphisms of $\Gamma = (G, T)$.

By convention, we set $\mathbb{G}_{0,1} = \mathbb{G}_{0,2} = \emptyset$. $\mathbb{G}_{0,3}$ and $\mathbb{G}_{1,1}$ both contain only one element, which are shown in Section 1.3 of [10].

Furthermore, as explained in [10], $\mathbb{G}_{g,n+1}$ has a recursive structure on $\chi_{g,n}$. If we remove from a given graph $\Gamma \in \mathbb{G}_{g,n+1}$ the vertex incident to the root ℓ with two more edges/leaves $\{e_1, e_2\}$, the resulting graph falls into one of the following three cases:

- I** a graph $\Gamma' \in \mathbb{G}_{g,n}$, if one of e_1 or e_2 (we call it e_2 without loss of generality) is a leaf of Γ . We let e_1 be the root of Γ' .
- I'** a graph $\Gamma' \in \mathbb{G}_{g-1,n+1}$. We let e_1 be the root of Γ' , and e_2 its first leaf. Note that we need to specify which of e_1 and e_2 is the root and the first leaf here, unlike [10], because of \mathbb{Z}_2 -symmetry.
- II** a non-ordered disjoint union of $\Gamma'_1 \cup \Gamma'_2$ where $\Gamma'_i \in \mathbb{G}_{g_i,|\Phi_i|+1}$ is a graph with root e_i and leaves L_i such that $g_1 + g_2 = g$ and $L_1 \cup L_2$ are leaves of Γ distinct from the root ℓ .

We would like to assign a weight to each graph $\Gamma = (G, T) \in \mathbb{G}_{g,n+1}$. To do this, we need to equip Γ with a colouring.

Definition 2.24. A *colouring* of $\Gamma = (G, T) \in \mathbb{G}_{g,n+1}$ is a map $\gamma : E'(\Gamma) \rightarrow I$, where I is the index set $I = \{1, 2, 3, \dots\}$. In other words, it assigns a positive integer to all leaves of G and edges of G that are not loops.

To each graph $\Gamma \in \mathbb{G}_{g,n+1}$ with colouring γ , with $\chi_{g,n} \geq 0$, we assign a weight $\omega(\Gamma, \gamma)$ as follows, using the recursive decomposition (**I**, **I'**, **II**). First of all, we define the cases with $\chi_{g,n} = 0$ as

$$\omega(\Gamma_{0,3}, \gamma) = A_{\gamma(l_1)\gamma(l_2)\gamma(l_3)}, \quad \omega(\Gamma_{1,1}, \gamma) = D_{\gamma(l_1)}. \tag{2.35}$$

For graphs with $\chi_{g,n} \geq 1$, we recursively determine the weight in terms of the decomposition (**I**, **I'**, **II**):

$$\mathbf{I} \quad \omega(\Gamma, \gamma) = \sigma_{\gamma(e_1) \subset \gamma(L)} B_{\gamma(\ell_1)\gamma(e_1)}^{\gamma(e_2)} \omega(\Gamma', \gamma'), \tag{2.36}$$

$$\mathbf{I}' \quad \omega(\Gamma, \gamma) = C_{\gamma(\ell_1)}^{\gamma(e_2)\gamma(e_1)} \omega(\Gamma', \gamma'), \tag{2.37}$$

$$\mathbf{II} \quad \omega(\Gamma, \gamma) = \sigma_{\gamma(L_1) \subset \gamma(L)} C_{\gamma(\ell_1)}^{\gamma(e_1)\gamma(e_2)} \omega(\Gamma_1, \gamma_1) \omega(\Gamma_2, \gamma_2), \tag{2.38}$$

⁹ In [10] the counterclockwise requirement is not specified, because the assigned weights are symmetric, whereas in our case we need to be a little more careful because the weights are only \mathbb{Z}_2 -symmetric.

where $L = (\ell_1, \dots, \ell_n)$ and $\gamma(L) = (\gamma(\ell_1), \dots, \gamma(\ell_n))$ is the colouring of leaves. The sign factor σ is defined in Theorem 2.20. γ' denotes the colouring for the corresponding decomposed graph, that is, the restriction of γ to $E'(\Gamma')$. L_i, γ_i are similarly defined. Let us emphasize that the order of edges/leaves is important, because of the sign factors.

Since this weight assignment precisely captures the recursive equation (2.30), we obtain the following lemma:

Lemma 2.25. *For any $g, n \geq 0$, and $i, i_1, \dots, i_n \in I$, we have*

$$F_{g,n+1}[i, i_1, \dots, i_n] = \sum_{\Gamma \in \mathbb{G}_{g,n+1}} \sum_{\substack{\gamma \in I^{E'(\Gamma)} \\ \gamma(\ell_j)=i_j}} \frac{\omega(\Gamma, \gamma)}{|\text{Aut}(\Gamma)|}. \tag{2.39}$$

We refer the reader to [4] and [10] for pretty pictures of the graphical representation of topological recursion.

2.3. Gauge Transformations. In this section we study gauge transformations of super quantum Airy structures.

Recall that $\tilde{V} = V \oplus \mathbb{K}^{0|1}$. We consider the symplectic super vector space $W = \tilde{V} \oplus \tilde{V}^*$ equipped with the product topology and the symplectic form given by $\omega(\phi, v) = \phi(v)$ for $\phi \in \tilde{V}^*$ and $v \in \tilde{V}$.

Topology on W was chosen in such a way that the following properties hold:

- if $\{a_n\}_{n=0}^\infty$ is a sequence of distinct indices, $\lim_{n \rightarrow \infty} y_{a_n} = 0$,
- if $\{v_n\}_{n=0}^\infty \subseteq V$ is a convergent sequence, then v_n are eventually constant.

2.3.1. Linear Gauge Transformations With this under our belt, we can define gauge transformations for super quantum Airy structures. We proceed as follows. Let T be a linear operator $W \rightarrow W$ represented by matrices

$$Tx^a = t_b^a x^b + b^{ab} y_b, \tag{2.40a}$$

$$Ty_a = c_{ab} x^b + d_a^b y_b. \tag{2.40b}$$

Then for each fixed a there are finitely many b such that $t_b^a \neq 0$ or $c_{ab} \neq 0$. Moreover T is continuous if and only if the following conditions are satisfied:

- there are finitely many pairs a, b such that¹⁰ $c_{ab} \neq 0$,
- for each fixed b there are finitely many a such that¹¹ $d_a^b \neq 0$.

If $T : W \rightarrow W$ is an even, linear and continuous symplectomorphism ($T \in \text{Aut}(W)$ for short), there exists a unique continuous automorphism \tilde{T} of $\mathcal{W}_{\hbar}(\tilde{V})$ which acts on generators as in formulas (2.40) with y_a replaced by $\hbar \partial_a$ and $\tilde{T}(\hbar) = \hbar$. It may be extended uniquely to a continuous automorphism of $\widetilde{\mathcal{W}}_{\hbar}(V)$. Indeed, it is obvious that proposed transformation preserves algebraic relations in $\mathcal{W}_{\hbar}(\tilde{V})$. By the preceding discussion, also finiteness conditions in the definition of $\mathcal{W}_{\hbar}(V)$ are preserved. Continuity of \tilde{T} is easy to see.

¹⁰ In other words $c_{ab} x^a \otimes x^b \in \tilde{V} \otimes \tilde{V}$.

¹¹ This means that d_a^b represents the transpose of some operator $V \rightarrow V$.

Now consider an even, linear and continuous symplectomorphism $T_s \in \text{Aut}(W)$ that acts trivially on \widetilde{V}^* . In terms of generators, it takes the form

$$T_s(x^a) = x^a + s^{ab} y_b, \tag{2.41a}$$

$$T_s(y_a) = y_a, \tag{2.41b}$$

where s is even and \mathbb{Z}_2 -symmetric (that is $s^{ab} = (-1)^{|a||b|} s^{ba}$). We think of s as $s \in \text{Hom}(\widetilde{V}, \widetilde{V}^*)_0^{S_2}$.

Let (V, L) be a super quantum Airy structure, and let \widetilde{T}_s be the continuous automorphism of $\widehat{\mathcal{W}}_{\hbar}(\widetilde{V})$ uniquely induced by T_s . Then $(V, \widetilde{T}_s \circ L)$ is also a super quantum Airy structure.

Definition 2.26. We say that the super quantum Airy structure $(V, \widetilde{T}_s \circ L)$ is *gauge equivalent* to (V, L) , and we call \widetilde{T}_s a *linear gauge transformation*.

Remark 2.27. We note here that linear gauge transformations do not change the order of a super quantum Airy structure, since it preserves the \mathbb{Z} -grading on $\widehat{\mathcal{W}}_{\hbar}(V)$.

Let us now explain the geometric meaning of gauge transformations. For any subspace $L \subseteq W$ we define its symplectic complement $L^\perp = \{w \in W \mid \omega(l, w) = 0 \text{ for all } l \in L\}$. We say that L is Lagrangian if $L = L^\perp$. If L is a Lagrangian subspace, every Lagrangian subspace C such that $W = L \oplus C$ is called a Lagrangian complement of L . There is a bijection between $\text{Hom}(\widetilde{V}, \widetilde{V}^*)^{S_2}$ and the set of Lagrangian complements of \widetilde{V}^* , given by $s \mapsto T_s(\widetilde{V})$.

2.3.2. Non-linear Gauge Transformations There are more general gauge transformations that one can consider. Since the L_i can have arbitrary order, we do not need to insist that gauge transformations preserve the order of a super quantum Airy structure. Thus we can also consider non-linear gauge transformations.

Consider a formal series in y of the form

$$s = \sum_{k=2}^{\infty} \frac{1}{k} s^{a_1 \dots a_k} y_{a_1} \dots y_{a_k}. \tag{2.42}$$

We require that each term in (2.42) is even and that each $s^{a_1 \dots a_k}$ is \mathbb{Z}_2 -symmetric. The largest k such that $s^{a_1 \dots a_k} \neq 0$ is called the order of s . We also allow s of infinite order, since the L_i can have infinite order. Then the polynomial transformation

$$T_s(x^a) = x^a + \frac{\partial s}{\partial y_a}, \tag{2.43a}$$

$$T_s(y_a) = y_a, \tag{2.43b}$$

induces (by replacing y_a with $\hbar \partial_a$) an automorphism \widetilde{T}_s of $\widehat{\mathcal{W}}_{\hbar}(\widetilde{V})$, which leaves the defining properties of super quantum Airy structures invariant.

Definition 2.28. We say that the super quantum Airy structure $(V, \widetilde{T}_s \circ L)$ is *gauge equivalent* to (V, L) , and if the order of s is greater than two, we call \widetilde{T}_s a *non-linear gauge transformation*.

2.3.3. *The Partition Functions of Gauge Equivalent Super Quantum Airy Structures* Given two gauge equivalent super quantum Airy structures, what is the relation between their partition functions? To answer this question we need to understand gauge transformations as conjugations of the differential operators L_i of a super quantum Airy structure. Let $D_s \in \widetilde{\mathcal{W}}_{\hbar}(\widetilde{V})^{\geq 2}$ be of the form

$$D_s = \sum_{k=2}^{\infty} \hbar^k s^{a_1 \dots a_k} \partial_{a_1} \dots \partial_{a_k}, \tag{2.44}$$

where the s are even \mathbb{Z}_2 -symmetric tensors. We claim that the L'_i of the super quantum Airy structure $(V, \widetilde{T}_s \circ L)$ gauge equivalent to (V, L) are given by the conjugated differential operators

$$L'_i = \exp\left(\frac{D_s}{\hbar}\right) L_i \exp\left(-\frac{D_s}{\hbar}\right). \tag{2.45}$$

But for this we need to make sense of this expression.

Clearly $\frac{D_s}{\hbar}$, regarded as an operator on the ring R introduced in the Section 2.1.4, satisfies:

$$\forall n \in \mathbb{N}, \exists n_0 \in \mathbb{N}, n' \geq n_0 \implies \left(\frac{D_s}{\hbar}\right)^{n'} (R) \subseteq \mathfrak{m}^n. \tag{2.46}$$

Therefore $\exp\left(\frac{D_s}{\hbar}\right)$ makes sense as an operator on R . More precisely, for every $f \in R$ let

$$\exp\left(\frac{D_s}{\hbar}\right) f = \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{D_s}{\hbar}\right)^n f. \tag{2.47}$$

After projecting the right-hand-side onto terms of fixed degree in \hbar and x , only finitely many terms are nonzero, so this is well defined. We have $\exp\left(\frac{D_s}{\hbar}\right) \exp\left(\frac{D_t}{\hbar}\right) = \exp\left(\frac{D_{s+t}}{\hbar}\right)$. Moreover, for any $f \in R$ we have the identity

$$\exp\left(\frac{[D_s, \cdot]}{\hbar}\right) (L_i) \cdot f = \exp\left(\frac{D_s}{\hbar}\right) L_i \exp\left(-\frac{D_s}{\hbar}\right) f. \tag{2.48}$$

Therefore, (2.45) makes sense, and indeed reconstructs the differential operators L'_i of the gauge equivalent Airy structure $(V, \widetilde{T}_s \circ L)$.

Now let Z be the partition function associated to the super quantum Airy structure L_i , that is, Z is annihilated by the L_i , and Z evaluated at $x = 0$ is equal to 1. Then $\exp\left(\frac{D_s}{\hbar}\right) Z$ is annihilated by the gauge transformed operators L'_i . Let \mathcal{N} be equal to $\exp\left(\frac{D_s}{\hbar}\right) Z$ evaluated at $x = 0$. It is easy to check that $\mathcal{N} = 1 + O(\hbar)$. Therefore $\mathcal{N}^{-1} \in R$ exists and

$$Z' = \mathcal{N}^{-1} \exp\left(\frac{D_s}{\hbar}\right) Z \in R \tag{2.49}$$

is a solution of equation $L'_i Z' = 0$ such that Z' evaluated at $x = 0$ is equal to 1. By the uniqueness of partition function, Z' coincides with the partition function associated to

the Airy structure L'_i . This gives us the relation between the partition function Z' of the gauge transformed super quantum Airy structure $(V, \tilde{T}_s \circ L)$ to the partition function Z of the original super quantum Airy structure (V, L) .

Remark 2.29. In [4] an alternative formula for Z' in terms of formal gaussian integrals was given. A similar formula also works also in the supersymmetric case. However, it is only valid for linear gauge transformations. We are not aware of its generalization for non-linear gauge transformations.

2.4. Classical Versus Quantum. Quantum Airy structures were originally defined in [52] in terms of quantizations of classical Airy structures. In this section we explore the parallel story for super Airy structures.

2.4.1. Classical Limit We now define the classical limit of super quantum Airy structures. We introduce the algebras $\mathbb{K}[W] = \frac{\mathcal{W}_\hbar(\tilde{V})}{\hbar \cdot \mathcal{W}_\hbar(\tilde{V})}$ and $\mathbb{K}[[W]] = \frac{\widehat{\mathcal{W}}_\hbar(\tilde{V})}{\hbar \cdot \widehat{\mathcal{W}}_\hbar(\tilde{V})}$.

Definition 2.30. We denote the quotient map by Cl , and call it the *classical limit*.

Taking the classical limit thus amounts to replacing all $\hbar\partial_a$ by y_a , and setting all terms of higher order in \hbar to zero. We interpret $\mathbb{K}[W]$ and $\mathbb{K}[[W]]$ as the superalgebras of polynomials and formal series respectively on some classical phase space with linear coordinate system x^a, y_a . The natural symplectic structure on W induces a Poisson bracket on $\mathbb{K}[W]$ and $\mathbb{K}[[W]]$, which may also be computed as

$$\{\text{Cl}(f), \text{Cl}(g)\} = \text{Cl}\left(\frac{1}{\hbar}[f, g]\right). \tag{2.50}$$

Now let (V, L) be a super quantum Airy structure. Define $L_i^{\text{cl}}(x, y) = \text{Cl}(L_i)$ as being the classical limit of the operators L_i . The equation $L_i^{\text{cl}} = 0$ may then be thought of as defining a “characteristic variety” Σ in phase space, even if this is slightly artificial in the presence of odd coordinates. Nevertheless, some geometric notions may still be defined, e.g. the Zariski cotangent space. It may be identified naturally with $V \oplus \mathbb{K}^{0|2}$, with $\mathbb{K}^{0|2}$ spanned by the additional Grassman variable together with its conjugate momentum. In particular, it is coisotropic (and in fact Lagrangian if there is no extra Grassman variable). The classical limit of the free energy may be interpreted as a parametrisation of Σ in terms of the variables x in a formal neighbourhood of zero. In fact, by repeating the steps performed in the proof of existence and uniqueness of the partition function associated to a super quantum Airy structure, one may show the following fact:

Theorem 2.31. *There exists a unique $F_{\text{cl}} \in \mathbb{K}[[\tilde{V}]]^{\geq 3}$ such that $L_i^{\text{cl}}(x, dF_{\text{cl}}) = 0$. It coincides with the classical limit of the free energy associated to L_i .*

2.4.2. Super Classical Airy Structures and Quantization In the previous subsection we defined the classical limit of super quantum Airy structures. In fact, we could have started by defining super classical Airy structures, and think of super quantum Airy structures as quantizations of classical structures, in the spirit of [52].

Definition 2.32. Let V be a super vector space, $\tilde{V} = V \oplus \mathbb{K}^{0|1}$, and $W = \tilde{V} \oplus \tilde{V}^*$. A *super classical Airy structure* is a pair (V, L^{cl}) , with $L^{\text{cl}} : V^* \rightarrow \mathbb{K}[[W]]$ an even continuous linear operator such that:

- (1) The left ideal $\mathcal{L} \subseteq \mathbb{K}[[W]]$ generated by $L^{\text{cl}}(V^*)$ is involutive under the Poisson bracket, i.e. $\{\mathcal{L}, \mathcal{L}\} \subseteq \mathcal{L}$;
- (2) $L^{\text{cl}}(y_i)^{\leq 1} = y_i$.

We have seen that every super quantum Airy structure determines uniquely its classical limit. But given a super classical Airy structure, does there exist a quantization that is a super quantum Airy structure? This turns out to be a difficult problem in general.

Nevertheless, it admits a simple solution in the special case of quadratic L^{cl} . Define L initially by replacing $y_a \mapsto \hbar \partial_a$, with all derivatives to the right of x^a . Then all $L(\phi)$ are well-defined elements of the Weyl algebra, but they need not satisfy correct commutation relations. Simple calculation shows that

$$\zeta(\phi, \psi) = \frac{1}{\hbar} L([\phi, \psi]) - \frac{1}{\hbar^2} [L(\phi), L(\psi)] \tag{2.51}$$

is a number for each $\phi, \psi \in V^*$. It follows from the Jacobi identity that ζ is a Lie superalgebra cocycle (see [43] for the relevant definitions). Clearly ζ is even and continuous. Upon replacement

$$L(\phi) \mapsto L(\phi) + \phi(v), \quad v \in V_0, \tag{2.52}$$

ζ changes by a coboundary. It follows that the cohomology class $[\zeta] \in H^2(V^*, \mathbb{K})_0$ (continuous, even Lie superalgebra cohomology) doesn't depend on the operator ordering prescription. Moreover quantization of L^{cl} exists if and only if $[\zeta] = 0$. In the finite-dimensional case it is guaranteed that ζ is a coboundary, because Weyl quantization is always possible. In this ordering scheme mixed terms $x^a y_b$ are replaced by $\frac{\hbar}{2} (x^a \partial_b + (-1)^{|a||b|} \partial_b x^a)$.

Now, given a quantization, one can still ask if other quantizations may be obtained by performing transformations of the form (2.52). This is possible only if v is such that $[\phi, \psi](v) = 0$ for any ϕ, ψ . In other words, v has to be a cocycle. Therefore if a quantization exists, its ambiguity is measured by the cohomology group $H^1(V^*, \mathbb{K})_0$.

2.4.3. Bosonic Classical Airy Structures

Definition 2.33. We call a classical Airy structure $L^{\text{cl}} : V^* \rightarrow \mathbb{K}[[W]]$ *bosonic*, if $V = V_0$ and L^{cl} does not depend on the extra fermionic variable coming from $\mathbb{K}^{0|1}$.

Given any super classical Airy structure $L^{\text{cl}} : V^* \rightarrow \mathbb{K}[[W]]$, one can always produce a bosonic one based on V_0 . Indeed, consider a restriction of this map to $L_0^{\text{cl}} : (V_0)^* \rightarrow \mathbb{K}[[W]]$. Since L^{cl} is even, so is its restriction. Therefore image of L_0^{cl} consists of even elements. Those elements are linear combination of monomials, which can be of two types: either have no fermionic variables or have an even number of them. Observe that both of these subsets are closed under the Poisson bracket. Let $W_0 = V_0 \oplus V_0^*$ and let $\pi_0 : \mathbb{K}[[W]] \rightarrow \mathbb{K}[[W_0]]$ be the projection onto the subspace spanned by those elements which have no fermionic variables. We define a bosonic Airy structure as a composition $L^{\text{cl},b} = \pi_0 \circ L_0^{\text{cl}} : (V_0)^* \rightarrow \mathbb{K}[[W_0]]$. The subspace spanned by those monomials in $\mathbb{K}[[W]]$, which have even and nonzero number of fermionic variables, is a left ideal with respect to the Poisson bracket. Therefore the condition $\{L_i^{\text{cl}}, L_j^{\text{cl}}\} = \hbar f_{ij}^k L_k^{\text{cl}}$, for some $f_{ij}^k \in \mathbb{K}[[W]]$, implies that $\{L_i^{\text{cl},b}, L_j^{\text{cl},b}\} = \hbar \pi_0(f_{ij}^k) L_k^{\text{cl},b}$. This proves that $L^{\text{cl},b}$ is a classical Airy structure.

Remark 2.34. The analog of the construction outlined above doesn't work for quantum super Airy structures in general. The reason is that the commutator of two terms with even, nonzero number of fermionic variables may contain terms with no fermionic variables. Instead one may consider the classical limit, remove fermionic variables and generators and then try to quantize again. This is always possible to carry out (but is possibly ambiguous) in the finite-dimensional case.

3. Finite-Dimensional Examples

In this section we study examples of finite-dimensional quadratic super quantum Airy structures. Moreover we propose a classification scheme for these objects.

3.1. Low-Dimensional Examples. Our first step is to consider super vector spaces V of low dimension. The "purely bosonic" case, with $\dim V_1 = 0$ and no extra fermionic variable, is not the subject of our current studies. The case with $\dim V_1 = 0$ but with an extra fermionic variable, i.e. $\tilde{V} = V_0 \oplus \mathbb{K}^{0|1}$ is straightforward. Thus we will assume that $\dim V_1 \geq 1$. The "purely fermionic" case, with $\dim V_0 = 0$, is essentially trivial, since the only allowed super quantum Airy structure has the form

$$L_i = \hbar \partial_{\theta_i}, \tag{3.1}$$

with θ_i odd variables. Thus we may also assume that $\dim V_0 \geq 1$.

3.1.1. The Superalgebras of Dimension (1|1) The simplest non-trivial case then consists in the superalgebras of dimension (1|1). In this section we classify all quadratic super quantum Airy structures that can be constructed from these superalgebras, assuming that we have no extra fermionic variables. We leave the case with an extra fermionic variables for future work.

Remark 3.1. We remark that from the point of view of the partition function, all examples based on superalgebras of dimension (1|1) with no extra fermionic variables are rather trivial, since the partition function is purely bosonic (i.e. does not depend on the single Grassmann variable entering in the construction). This follows directly from the requirement that the free energy is \mathbb{Z}_2 -even. Nevertheless, from an algebraic viewpoint it is interesting to classify which quadratic super quantum Airy structures can be constructed based on superalgebras of dimension (1|1).

Up to isomorphism, there exist three distinct complex Lie superalgebras of dimension (1|1). We denote the bosonic generator by L and the fermionic generator by G .

- (1) The abelian superalgebra, with commutation relations:

$$[L, L] = 0, \quad [L, G] = 0, \quad [G, G] = 0. \tag{3.2}$$

- (2) The algebra of affine automorphisms of $\mathbb{C}^{0|1}$, with commutation relations:

$$[L, L] = 0, \quad [L, G] = \hbar G, \quad [G, G] = 0. \tag{3.3}$$

- (3) The $\mathcal{N} = 1, d = 1$ supersymmetry (SUSY) algebra, with commutation relations:

$$[L, L] = 0, \quad [L, G] = 0, \quad [G, G] = \hbar L. \tag{3.4}$$

Below we provide all quadratic super quantum Airy structures that can be constructed as representations of these algebras, up to changes of bases and gauge transformations. This classification can be proved directly, by brute force calculations. For the sake of brevity, we omit the details.

In the following, $A, B, C, D \in \mathbb{C}$ always stand for arbitrary constants. We denote the bosonic variable by x and the fermionic variable by θ .

- (1) For the abelian superalgebra, there are three families of quadratic super quantum Airy structures. The first one takes the form:

$$G = (1 - x)\hbar\partial_\theta, \quad (3.5a)$$

$$L = \hbar\partial_x - \frac{1}{2}Ax^2 - \hbar(x\partial_x + \theta\partial_\theta) - \hbar D, \quad (3.5b)$$

while the second one is:

$$G = (1 - \hbar\partial_x)\hbar\partial_\theta, \quad (3.6a)$$

$$L = \hbar\partial_x - \frac{\hbar^2}{2}C\partial_x^2 - \hbar D. \quad (3.6b)$$

The third possibility is

$$G = \hbar\partial_\theta, \quad (3.7a)$$

$$L = L_0, \quad (3.7b)$$

where L_0 is an arbitrary θ -independent bosonic generator.

- (2) For the algebra of affine automorphisms of $\mathbb{C}^{0|1}$, there are also three families of quadratic super quantum Airy structures. The first one is:

$$G = (1 - x)\hbar\partial_\theta, \quad (3.8a)$$

$$L = \hbar\partial_x - \frac{1}{2}Ax^2 - \hbar x\partial_x - 2\hbar\theta\partial_\theta - \hbar D. \quad (3.8b)$$

The second family reads:

$$G = (1 - \hbar\partial_x)\hbar\partial_\theta, \quad (3.9a)$$

$$L = \hbar\partial_x - \hbar\theta\partial_\theta - \frac{\hbar^2}{2}C^{xx}\partial_x^2 - \hbar D, \quad (3.9b)$$

while the third one is:

$$G = \hbar\partial_\theta, \quad (3.10a)$$

$$L = -\hbar\theta\partial_\theta + L_0, \quad (3.10b)$$

where L_0 is an arbitrary θ -independent bosonic generator.

- (3) For the $\mathcal{N} = 1, d = 1$ SUSY algebra we have $L = \frac{2}{\hbar}G^2$, so it is sufficient to provide the form of G . There are again three possibilities:

$$G = \hbar\partial_\theta + \frac{\hbar}{2}(\theta\partial_x + x\partial_\theta), \quad (3.11a)$$

$$G = \hbar\partial_\theta + \frac{\hbar}{2}\theta\partial_x - \hbar^2\partial_x\partial_\theta, \quad (3.11b)$$

$$G = \hbar\partial_\theta + \frac{\hbar}{2}\theta\partial_x. \quad (3.11c)$$

3.1.2. *Superalgebras of Dimensions (2|1) and (1|2)* The brute force classification of quadratic super quantum Airy structures that can be obtained as representations of superalgebras of dimensions (2|1) and (1|2) is already a little tedious. We will thus only present a few interesting examples here.

Example 3.2. Our first example starts with the (2|1)-dimensional $\mathcal{N} = 1, d = 1$ SUSY algebra extended by a dilatation operator. We denote the two bosonic generators by L_1, L_2 and the fermionic generator by G . It has commutation relations:

$$[L_1, L_2] = \hbar L_2, \quad [L_1, G] = \frac{\hbar}{2}G, \quad [L_2, G] = 0, \quad [G, G] = \hbar L_2. \quad (3.12)$$

We construct a quadratic super quantum Airy structure with no extra fermion as a representation of this algebra. We denote the bosonic variables by x, y and the fermionic variable by θ .

The quadratic super quantum Airy structure reads:

$$L_1 = \hbar \partial_x - \frac{1}{2}Ax^2 - \hbar \left(\frac{1}{2} + B \right) \theta \partial_\theta - \hbar(1 + B)y \partial_y - \hbar Bx \partial_x - \hbar D, \quad (3.13a)$$

$$L_2 = \hbar \partial_y - \hbar x \partial_y, \quad (3.13b)$$

$$G = \hbar \partial_\theta + \frac{\hbar}{2} \theta \partial_y - \hbar Bx \partial_\theta, \quad (3.13c)$$

where $A, B, D \in \mathbb{C}$ are arbitrary constants.

Note that, as for the (1|1) examples, the partition function here does not depend on the fermionic variable θ , since it must be \mathbb{Z}_2 -even.

Let us now study examples where the partition function depends on fermionic variables.

Example 3.3. For our next example, we start with the same (2|1)-dimensional $\mathcal{N} = 1, d = 1$ SUSY algebra extended by a dilatation operator, with commutation relations (3.12), but we construct a quadratic super quantum Airy structures with an extra fermionic variable. We denote the bosonic variables by x^1, x^2 and the fermionic variables by θ^0, θ^1 .

The representation reads:

$$L_1 = \hbar \partial_{x^1} - 2\hbar x^1 \partial_{x^1} - \hbar x^2 \partial_{x^2} - \frac{\hbar}{2} \theta^1 \partial_{\theta^1} + \frac{3\hbar}{2} \theta^0 \partial_{\theta^0}, \quad (3.14a)$$

$$L_2 = \hbar \partial_{x^2} - \beta \hbar x^2 \partial_{x^1} - \theta^0 \theta^1, \quad (3.14b)$$

$$G = \hbar \partial_{\theta^1} + \theta^0 x^2 + \frac{\hbar}{2} \theta^1 \partial_{x^2} - \frac{\hbar^2}{2} \beta \partial_{\theta^0} \partial_{x^1}, \quad (3.14c)$$

with $\beta \in \mathbb{C}$ an arbitrary constant.

The constraints $L_1 Z = L_2 Z = GZ = 0$ uniquely fix the partition function Z . The result is

$$Z = \exp \left(\frac{1}{\hbar} x^2 \theta^0 \theta^1 \right). \quad (3.15)$$

Example 3.4. Our last example in this section is a quadratic super quantum Airy structure with no extra fermionic variable, but such that its partition function depends on fermionic variables. To this end, we start with a superalgebra of dimension $(1|2)$. We denote its bosonic generator by L and its fermionic generators by G_1, G_2 , and the corresponding bosonic and fermionic variables by x and θ_1, θ_2 respectively.

We choose the superalgebra with commutation relations:

$$[L, G_i] = \hbar G_i, \quad [G_i, G_j] = 0, \quad i, j = 1, 2. \quad (3.16)$$

Our representation is:

$$L = \hbar \partial_x - x^2 - \theta^1 \theta^2 + \hbar^2 \partial_x^2 + \hbar^2 \partial_{\theta_1} \partial_{\theta_2}, \quad (3.17a)$$

$$G_1 = \hbar \partial_{\theta_1} - x \theta^2 + \hbar x \partial_{\theta_1} - \hbar \theta^2 \partial_x + \hbar^2 \partial_x \partial_{\theta_1}, \quad (3.17b)$$

$$G_2 = \hbar \partial_{\theta_2} + x \theta^1 + \hbar x \partial_{\theta_2} + \hbar \theta^1 \partial_x + \hbar^2 \partial_x \partial_{\theta_2}. \quad (3.17c)$$

From the constraints $LZ = G_1 Z = G_2 Z = 0$, it is straightforward to compute the free energy perturbatively. Up to terms of order five in the variables x, θ^1, θ^2 , we get:

$$F = \frac{1}{3} x^3 - \frac{1}{5} x^5 + (x - x^3) \theta^1 \theta^2 - \hbar (x^2 + \theta^1 \theta^2) + \hbar^2 x + \dots \quad (3.18)$$

Thanks to the simplicity of the Airy structure (3.17b) the classical free energy can be in this case expressed in terms of elementary functions:

$$F_{\text{cl}} = \frac{1}{8} \log \left(2x + \sqrt{1 + 4x^2} \right) + \frac{x}{4} \sqrt{1 + 4x^2} - \frac{x}{2} + \frac{\sqrt{1 + 4x^2} + 2x - 1}{\sqrt{1 + 4x^2} + 2x + 1} \theta^1 \theta^2. \quad (3.19)$$

We will use this example to illustrate the action of the gauge symmetry on the partition function. Notice that under the gauge transformation

$$x \rightarrow x + \hbar \partial_x, \quad \theta^1 \rightarrow \theta^1 - \hbar \partial_{\theta_2}, \quad \theta^2 \rightarrow \theta^2 + \hbar \partial_{\theta_1}, \quad (3.20)$$

the operators go to

$$L \rightarrow L' = \hbar \partial_x - x^2 - 2\hbar x \partial_x - \hbar - \theta^1 \theta^2 - \hbar \theta^1 \partial_{\theta_1} - \hbar \theta^2 \partial_{\theta_2}, \quad (3.21a)$$

$$G_1 \rightarrow G'_1 = \hbar \partial_{\theta_1} - x \theta^2 - 2\hbar \theta^2 \partial_x, \quad (3.21b)$$

$$G_2 \rightarrow G'_2 = \hbar \partial_{\theta_2} + x \theta^1 + 2\hbar \theta^1 \partial_x. \quad (3.21c)$$

In particular, L', G'_1 and G'_2 are now all first order differential operators, and we can solve explicitly for the free energy at the quantum level. We get:

$$F' = -\frac{1}{4} x(x+1) - \frac{1}{8} (1+4\hbar) \log(1-2x) + \frac{x+2\hbar}{1-2x} \theta^1 \theta^2. \quad (3.22)$$

According to our discussion in Section 2.3—see (2.49)—we have an identity

$$e^{\frac{1}{\hbar} F} = \sum_{k=0}^{\infty} \frac{(-\hbar)^k}{k!} \left(\left[\frac{1}{2} \partial_x^2 + \partial_1 \partial_2, \cdot \right] \right)^k e^{\frac{1}{\hbar} F'}. \quad (3.23)$$

Using a simple Mathematica code, validity of (3.23) may be checked to any required order (at least for the explicitly known classical part of the l.h.s. of (3.23)).

3.2. Classification Scheme. So far we have studied a few examples of low-dimensional quadratic super quantum Airy structures. In this section we present a classification scheme for finite-dimensional super quantum Airy structures whose vector space V^* is a finite-dimensional Lie superalgebra \mathfrak{g} . We focus on finite-dimensional super quantum Airy structures, but we remark here that part of the discussion is also valid for the infinite-dimensional case.

Let \mathfrak{g} be a finite-dimensional Lie superalgebra. We denote its structure constants by f . The process of finding all quadratic super quantum Airy structures for \mathfrak{g} can be divided into three steps:

- (1) Classify at most quadratic classical hamiltonians L_i^{cl} up to affine automorphism.
- (2) Look for points of the zero locus $L_i^{cl} = 0$ at which gradients of L_i^{cl} are linearly independent in order to rewrite L_i^{cl} in the form required by the definition of super classical Airy structures.
- (3) Quantize the system.

Parts of this procedure are expressed as classical problems in representation theory (classify all representations of a given dimension, find all invariant symplectic forms, compute certain cohomology groups) or algebraic geometry (describe the zero locus of a given set of polynomials), whose solutions are known at least in certain special cases.

Remark 3.5. For clarity in this section we will write L_i for the classical hamiltonians, dropping the superscript cl.

3.2.1. Purely Quadratic Hamiltonians As a first step one has to classify all representations of \mathfrak{g} by purely quadratic hamiltonians:

$$\{L_i^2, L_j^2\} = f_{ij}^k L_k^2, \tag{3.24}$$

where $\{\cdot, \cdot\}$ is the Poisson bracket. L_i^2 have to be polynomials in $2 \dim \mathfrak{g}$ variables (or with one extra Grassman variable together with its conjugate momentum) z_a subject to the elementary Poisson bracket relations

$$\{z_a, z_b\} = \omega_{ab}, \tag{3.25}$$

where ω is a symplectic form on the super vector space W with basis $\{z_a\}$. Hence

$$L_i^2 = \frac{1}{2} z_a M_i^{ab} z_b, \tag{3.26}$$

with $M_i^{ab} = (-1)^{|a||b|} M_i^{ba}$. This symmetry condition is equivalent to demanding that the linear operator on W given by $z_a \mapsto z_b M_i^{bc} \omega_{ca}$ is an infinitesimal symplectomorphism. The Poisson bracket relations (3.24) are satisfied if and only if the operators M_i furnish a linear representation of \mathfrak{g} on W . Conversely, given any representation of \mathfrak{g} with an invariant symplectic form ω we may construct the space $\mathbb{K}[z]$, with Poisson bracket given by (3.25), and then we define the L_i^2 by (3.26).

To summarize, the classification of purely quadratic hamiltonians (depending on a given number of variables) representing the algebra \mathfrak{g} is equivalent to classifying all symplectic representations¹² of \mathfrak{g} of dimension $2 \dim \mathfrak{g}$ or $2 \dim \mathfrak{g} + 0|2$.

¹² Two symplectic representations W_1, W_2 are regarded as equivalent if there exists an even \mathfrak{g} -intertwiner $W_1 \rightarrow W_2$ which is also a symplectomorphism.

3.2.2. *Affine Extensions* Now fix a representation of \mathfrak{g} by quadratic hamiltonians L_i^2 . We ask if we can add linear and constant terms L_i^1, L_i^0 such that the commutation relations are preserved:

$$L_i = L_i^2 + L_i^1 + L_i^0, \tag{3.27a}$$

$$\{L_i, L_j\} = f_{ij}^k L_k. \tag{3.27b}$$

Assuming that Eq. (3.24) holds, (3.27b) reduces to the following system of equations:

$$\{L_i^2, L_j^1\} + \{L_i^1, L_j^2\} - f_{ij}^k L_k^1 = 0, \tag{3.28a}$$

$$\{L_i^1, L_j^1\} - f_{ij}^k L_k^0 = 0. \tag{3.28b}$$

We observe that given any solutions of these equations, a new solution may be obtained by shifting even variables:

$$\begin{aligned} L'_i &= \exp(\{\epsilon(z), \cdot\})(L_i) \\ &= L_i + \{\epsilon(z), L_i^1 + L_i^2\} + \frac{1}{2}\{\epsilon(z), \{\epsilon(z), L_i^2\}\}, \end{aligned} \tag{3.29}$$

where $\epsilon(z)$ is an arbitrary linear combination of even z_a . This is equivalent to the replacement:

$$L_i^1 \mapsto L_i^1 + \{\epsilon(z), L_i^2\}, \tag{3.30a}$$

$$L_i^0 \mapsto L_i^0 + \{\epsilon(z), L_i^1\} + \frac{1}{2}\{\epsilon(z), \{\epsilon(z), L_i^2\}\}. \tag{3.30b}$$

Shifting even variables is an isomorphism of the Poisson algebra. Solutions related by such transformations should be regarded as equivalent at this stage.

Definition 3.6. We say that two solutions of (3.27b) are *equivalent* if they are related by a shift of even variables as in (3.30).

Our plan to classify solutions of equations (3.28) up to equivalence is as follows: first solve the constraint (3.28a), then find solutions of (3.28b) for a given L_i^1 .

Lemma 3.7. *Given L_i^2 that satisfy (3.24), the space of all solutions to (3.28a) up to equivalence coincides with the even subspace of the cohomology group $H^1(\mathfrak{g}, W)$.*

Proof. Observe that equation (3.28a) is linear in L^1 . Moreover it has the form of a 1-cocycle condition for \mathfrak{g} valued in the module W . Solutions of the form $L_i^1 = \{\epsilon(z), L_i^2\}$ for some linear combination of even variables $\epsilon(z)$ may be identified with coboundaries. Therefore the space of all solutions up to equivalence coincides with the even subspace of the cohomology group $H^1(\mathfrak{g}, W)$. \square

Let us now turn to (3.28b). Equation (3.28b) is quadratic in L^1 . If L^1 is already chosen such that (3.28a) is satisfied, (3.28b) is a linear equation for L^0 . For some L^1 it may turn out that there are no solutions at all.

Definition 3.8. We call linear terms L_i^1 such that there exists some solution to (3.28b) *admissible*.

Admissibility depends only on the equivalence class of L^1 . Indeed, if L^1 is such that a consistent L^0 can be chosen, then for any L'^1 of the form $L'^1 = L^1 + \{\epsilon(z), L^2\}$, a consistent L^0 can be found in the form of (3.30b).

Lemma 3.9. *Given an admissible L^1_i that satisfies (3.28a), the space of all solutions to $\text{refeq:L0spsconstraint}$ is an affine space over $\left(\frac{\mathfrak{g}}{[\mathfrak{g}, \mathfrak{g}]}\right)^*_0$.*

Proof. Suppose that L^1 is admissible and pick one L^0 satisfying (3.28b). Then L^0 is another solution if and only if

$$f^k_{ij} \left(L^0_k - L^0_k \right) = 0. \tag{3.31}$$

This equation means that the even linear functional $L^0 - L^0 \in \mathfrak{g}^*_0$ vanishes on the commutator ideal $[\mathfrak{g}, \mathfrak{g}]$. Therefore it may be regarded as an element of $\left(\frac{\mathfrak{g}}{[\mathfrak{g}, \mathfrak{g}]}\right)^*_0$. \square

3.2.3. Choice of the Origin and a Lagrangian Complement Suppose that we have chosen a set of at most quadratic hamiltonians L_i satisfying (3.27b). Now we look for hamiltonians in the same equivalence class that are in the form required by the definition of super classical Airy structures. In other words, we want to perform a shift (3.29) to bring the hamiltonians in a form such that $L^0_i = 0$ and the L^1_i are linearly independent. After such shift, L^0_i changes according to (3.30b). Requiring that the new L^0_i vanishes, we get an equation for $\epsilon(z)$:

$$L^0_i + \{\epsilon(z), L^1_i\} + \frac{1}{2}\{\epsilon(z), \{\epsilon(z), L^2_i\}\} = 0. \tag{3.32}$$

Its set of all solutions may be identified with the zero locus $\Sigma = \{z \in W_0 \mid (L_i)_0(z) = 0\}$ (here the subscript 0 means that we ignore odd variables and odd generators of \mathfrak{g}). Not all solutions are admissible, because we must ensure that after the shift, the linear terms L^1_i are linearly independent. This means that we must keep only those points for which the matrix of partial derivatives

$$D^a_i = \left. \frac{\partial}{\partial z_a} L_i \right|_{z=0} \tag{3.33}$$

has rank $\dim(\mathfrak{g})$. The set of all elements of Σ satisfying this condition will be denoted by Σ_s . It is a Zariski open subset of Σ . In [52] elements of Σ_s were called the smooth points of Σ . We note that this is not completely consistent with the standard terminology. For example the zero locus $Z(f)$ of the polynomial $f(x, y) = x^2 \in \mathbb{K}[x, y]$ is nonsingular, even though we have $df|_{Z(f)} = 0$.

Once a point of Σ_s is chosen and the generators L_i are put in a form with $L^0_i = 0$ with linearly independent L^1_i , we define

$$y_i = L^1_i. \tag{3.34}$$

Equation (3.28b) combined with $L^0_i = 0$ gives

$$\{y_i, y_j\} = 0. \tag{3.35}$$

Therefore the variables y_i span an isotropic subspace in W . If there is no “extra” odd variable, that is $\dim W = 2 \dim \mathfrak{g}$, this subspace is Lagrangian. For simplicity, for the remainder of this section we restrict attention to this special case. The analysis in the situation with an additional fermion has to be slightly adjusted. The linear span of y_i will be denoted by V^* .

We may now find elements x^i such that

$$\{y_i, x^j\} = \delta_i^j, \tag{3.36a}$$

$$\{x^i, x^j\} = 0. \tag{3.36b}$$

The set of all solutions to these conditions is in one-to-one correspondence with the set of Lagrangian complements V of the subspace $V^* \subset W$ spanned by y_i . Once the L_i are expressed in terms of the y_i and x^i , we have a super classical Airy structure.

Let us end this section by specifying how the super classical Airy structures constructed by the procedure outlined above depend on a number of arbitrary choices made along the way.

First of all, there is an ambiguity in the choice of x^i , or equivalently, in the choice of the Lagrangian complement V of the subspace $V^* \subseteq W$. As described earlier, this ambiguity is precisely the so-called gauge freedom. We regard Airy structures related by a gauge transformation as equivalent. Indeed, not only gauge-transformed generators are related by an explicit automorphism of the Poisson algebra, but also (after quantization) partition functions are related by a formal gaussian smearing transformation.

Secondly, we have chosen a point of Σ_s . We can get more such points by exponentiating the action of \mathfrak{g}_0 on $\mathbb{K}[z]_0$. This is possible, because action of a Lie algebra on a finite-dimensional vector space exponentiates to an action of the corresponding Lie group, and each $\{L_i, \cdot\}$ preserves a filtration of $\mathbb{K}[z]$ by finitely-dimensional subspaces $\{\mathbb{K}[z]^{\leq n}\}_{n \in \mathbb{N}}$ of all polynomials of degree at most n . Clearly hamiltonians related by G -transformations give rise to isomorphic Airy structures. It may happen that the set Σ_s is disconnected. Then it is possible that several non-isomorphic Airy structures may be obtained from the same set of hamiltonians.

Since the dimension of Σ_s coincides with the dimension of \mathfrak{g}_0 and the stabilizer of each point of Σ_s in G is discrete, G acts locally transitively on Σ_s . Therefore G -orbits are open in Σ_s (with respect to the analytic topology). Since set Σ_s is semialgebraic, it has finitely many connected components. Therefore its connected components are clopen. Combining these two facts we conclude that the orbits of the G -action on Σ_s are precisely the connected components of Σ_s . Notice that this is not necessarily true for the G -action on the whole Σ , since the G -orbits in Σ are in general not open in Σ . For example if L_i are homogeneous of degree 2, then $z = 0$ is always a solution which is a fixed point for G , hence an orbit. Except for some trivial cases it is not a discrete point of Σ , hence we have orbits of G in Σ which are proper subsets of their connected components. For any orbit $O \subseteq \Sigma_s$ the stabiliser Γ of any $p \in O$ is discrete. This means that O may be identified with the homogeneous space $\frac{G}{\Gamma}$ and that G is the universal covering space of O . In particular, we conclude that the homotopy groups of O are given by $\pi_1(O) \cong \Gamma$ and $\pi_k(O) \cong \pi_k(G)$ for $k \geq 2$. Finally, we observe that the hamiltonian vector fields generated by L_i provide a global framing for the tangent bundle of O , so $TO \rightarrow O$ is trivial.

3.2.4. Quantization Suppose that we have constructed a quadratic super classical Airy structure through the procedure outlined above. We then ask whether this classical Airy

structure may be lifted to the quantum level. We will see that, in the finite-dimensional case, the answer is always affirmative. (In the infinite-dimensional case, there is a possible cohomological obstruction). Moreover it turns out that the set of all consistent quantizations is an affine space over $\left(\frac{\mathfrak{g}}{[\mathfrak{g}, \mathfrak{g}]}\right)_0^*$.

Lemma 3.10. *Let the L_i form a quadratic super classical Airy structure, and define*

$$\zeta_{ij} = \frac{1}{\hbar^2} \left([L_i, L_j] - \hbar f_{ij}^k L_k \right), \tag{3.37}$$

which is a constant, independent of x_i, y_i and \hbar . Then the L_i can be lifted to a quadratic super quantum Airy structure if and only if the cohomology class $[\zeta] \in H^2(\mathfrak{g}, \mathbb{K})_0$ vanishes. If this is the case, the space of all consistent quantizations is an affine space over $\left(\frac{\mathfrak{g}}{[\mathfrak{g}, \mathfrak{g}]}\right)_0^$.*

Proof. To quantize L_i , we need to replace all y_i by $\hbar \frac{\partial}{\partial x^i}$. The meaning of this operation is ambiguous for mixed terms, i.e. for products $x^i y_j$, which supercommute on the classical level but not after quantization. Due to this ambiguity, in general we need to introduce constants of order \hbar in our quantum hamiltonians. These terms are invisible at the classical level.

Note that it is always possible to replace $x^i y_j$ by the “normally-ordered” expressions $\hbar x^i \frac{\partial}{\partial x^i}$, but unfortunately this ordering prescription does not lead to differential operators satisfying the commutation relations of the algebra \mathfrak{g} in general. However, a simple calculation shows that this is almost true, in the sense that the quantity

$$\zeta_{ij} = \frac{1}{\hbar^2} \left([L_i, L_j] - \hbar f_{ij}^k L_k \right) \tag{3.38}$$

does not depend on x, y or \hbar . By the Jacobi identity, it satisfies the 2-cocycle condition

$$\zeta_{k[i} f_{j]l}^k = 0, \tag{3.39}$$

where square bracket denotes \mathbb{Z}_2 -graded skew-symmetrization. One may try to get rid of the problematic ζ by shifting the generators by constants proportional to \hbar :

$$L_i \mapsto L_i + \hbar D_i, \tag{3.40}$$

with $D_i = 0$ for $|i| = 1$. This transformation has the following effect on ζ :

$$\zeta_{ij} \mapsto \zeta'_{ij} = \zeta_{ij} - f_{ij}^k D_k. \tag{3.41}$$

This means that ζ changes by a coboundary. In other words, the cohomology class of ζ depends only on the classical generators and not on the choice of the ordering prescription. In order to get the commutation relations of \mathfrak{g} at the quantum level, we have to impose the condition $\zeta'_{ij} = 0$. By the preceding discussion, this is possible if and only if the cohomology class $[\zeta] \in H^2(\mathfrak{g}, \mathbb{K})_0$ vanishes. Now suppose that we have found a particular solution D_i such that $\zeta'_{ij} = 0$. We ask if other choices D'_i are possible. It follows from the formula (3.41) that D'_i is a consistent constant term for L_i if and only if

$$f_{ij}^k (D'_i - D_i) = 0. \tag{3.42}$$

This condition means that the functional $D \in \mathfrak{g}_0^*$ vanishes on the commutator ideal $[\mathfrak{g}, \mathfrak{g}]$. Therefore the space of all consistent constant terms is affine over $\left(\frac{\mathfrak{g}}{[\mathfrak{g}, \mathfrak{g}]}\right)_0^*$. \square

The situation is simpler when \mathfrak{g} is finite-dimensional, in which case there is no cohomological obstruction to quantization.

Lemma 3.11. *If \mathfrak{g} is finite-dimensional, then $[\zeta] = 0$, and all quadratic super classical Airy structures can be quantized.*

Proof. If \mathfrak{g} is finite-dimensional, there exists a simpler quantization procedure, namely Weyl quantization. In this scheme every term of the form $x^i y_j$ in the classical hamiltonians is replaced by $\frac{\hbar}{2} \left(x^i \frac{\partial}{\partial x^j} + \frac{\partial}{\partial x^j} x^i \right)$ at the quantum level. This has the advantage that the commutation relations between quantum L_i are automatically satisfied. Thus a quantization always exists: it not necessary to shift further with D_i -terms (although it is still possible to construct other quantizations as in the previous Lemma). Since the space of consistent quantizations now has a distinguished origin, it is a vector (rather than affine) space. This has the corollary that the cohomology class $[\zeta]$ described in the previous paragraph vanishes identically, since Weyl quantization guarantees existence of a quantization for finite-dimensional quadratic super classical Airy structures. \square

Remark 3.12. In the infinite-dimensional case, Weyl quantization fails in general. Indeed, in order to have well-defined operators L_i , we need to be able to use commutation relations to put all $\frac{\partial}{\partial x^j}$ to the right of all x^i in such a way that the coefficient in front of each derivative is finite. If we insist on Weyl ordering, this may turn out to be impossible. Indeed, we would like to replace expressions of the form $b_i^j x^i y_j$ with $\frac{\hbar}{2} b_i^j \left(x^i \frac{\partial}{\partial x^j} + \frac{\partial}{\partial x^j} x^i \right) = \frac{\hbar}{2} b_i^j x^i \frac{\partial}{\partial x^j} + \frac{\hbar}{2} b_i^j$. Unfortunately, the contraction b_i^j is meaningless in general, as it contains an infinite sum. Therefore $[\zeta] \neq 0$ is possible.

We have claimed earlier that Weyl quantization, whenever possible, is distinguished among all quantization schemes. The main reason for this is that it is canonical, with no room for arbitrary choices. Another pleasant property is that it is covariant with respect to symplectic transformations, in the sense that classical hamiltonians expressed in different coordinates are quantized to the same (up to isomorphism) quantum operators. This is not true for all quantization schemes. We illustrate this feature with the simplest possible example.

Example 3.13. Consider the purely quadratic hamiltonian

$$L = \frac{1}{2}x^2 - \frac{1}{2}y^2. \tag{3.43}$$

In this situation Weyl quantization and normal ordering quantizations agree. Both give

$$L_q = \frac{1}{2}x^2 - \frac{\hbar^2}{2}\partial_x^2. \tag{3.44}$$

Define now $a = \frac{x+y}{\sqrt{2}}$ and $b = \frac{x-y}{\sqrt{2}}$. The classical hamiltonian takes the form

$$L' = ab. \tag{3.45}$$

The Weyl quantization of this hamiltonian differs from the normal ordering quantization (and infinitely many different quantization prescriptions). It gives

$$L'_q = \hbar b \frac{\partial}{\partial b} + \frac{\hbar}{2}. \tag{3.46}$$

We introduce new generators $x = \frac{b+\partial_b}{\sqrt{2}}$, $\partial_x = \frac{-b+\partial_b}{\sqrt{2}}$. They satisfy the same algebraic relations as b, ∂_b . Reexpressing¹³ L'_q in terms of x, ∂_x we recover the hamiltonian L_q . This doesn't happen if L' is quantized with any other ordering prescription.

3.2.5. An Example We now illustrate the steps outlined in the classification scheme in a specific example. Our starting point is the Lie superalgebra with a single bosonic generator H and two fermionic generators Q_1, Q_2 , with the only nonzero commutator $[Q_1, Q_2] = H$. We note that the simply connected Lie group generated by the bosonic part of this algebra is isomorphic to \mathbb{C} .

- (1) The first step in the classification is the construction of all representations of the superalgebra by purely quadratic hamiltonians. For the sake of brevity, we will not perform this step here. Rather, we will focus on one choice of a purely quadratic representation of the superalgebra.

Let us denote the bosonic variables by x, y satisfying $\{y, x\} = 1$, and the fermionic variables by $\theta_1, \theta_2, \xi_1, \xi_2$ satisfying $\{\theta_i, \theta_j\} = \{\xi_i, \xi_j\} = 0$ and $\{\xi_i, \theta_j\} = \delta_{ij}$. We will choose the following purely quadratic representation of the superalgebra:

$$Q_1^2 := x\theta_1, \tag{3.47a}$$

$$Q_2^2 := y\xi_1, \tag{3.47b}$$

$$H^2 := \{Q_1^2, Q_2^2\} = xy - \theta_1\xi_1. \tag{3.47c}$$

- (2) The second step consists in adding linear and constant terms in a way that preserves the Poisson brackets. We first construct the most general linear terms by solving (3.28a), up to equivalences generated by shifts of even variables (see Definition 3.6). We then solve for constant terms using (3.28b).

In our context, (3.28a) becomes the equations:

$$\{Q_1^2, Q_2^1\} + \{Q_1^1, Q_2^2\} = H^1, \tag{3.48a}$$

$$\{Q_1^2, Q_1^1\} = 0, \tag{3.48b}$$

$$\{Q_2^2, Q_2^1\} = 0, \tag{3.48c}$$

with Q_1^1, Q_2^1 odd, and H^1 even. The most general solution to these equations is

$$Q_1^1 = \alpha_1\theta_1 + \beta_1\theta_2 + \gamma_1\xi_2, \tag{3.49a}$$

$$Q_2^1 = \alpha_2\xi_1 + \beta_2\xi_2 + \gamma_2\theta_2, \tag{3.49b}$$

$$H^1 = \alpha_2x + \alpha_1y, \tag{3.49c}$$

for constants $\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2 \in \mathbb{C}$. However, the terms $\alpha_1\theta_1$ and $\alpha_2\xi_1$ in Q_1^1 and Q_2^1 respectively (and the corresponding terms in H^1) can be obtained by shifting the even variables $x \mapsto x + \alpha_1$ and $y \mapsto y + \alpha_2$. According to the classification scheme, we consider these solutions as equivalent (see Definition 3.6), and so we can set $\alpha_1 = \alpha_2 = 0$ without loss of generality, and we get the linear terms

$$Q_1^1 = \beta_1\theta_2 + \gamma_1\xi_2, \tag{3.50a}$$

¹³ This transformation may be implemented by an automorphism of the Weyl algebra.

$$Q_2^1 = \beta_2 \xi_2 + \gamma_2 \theta_2, \quad (3.50b)$$

$$H^1 = 0. \quad (3.50c)$$

We then need to add constant terms using (3.28b). First, since Q_1 and Q_2 are fermionic, they cannot have constant terms. From (3.28b), the equations that we have to solve are

$$\{Q_1^1, Q_1^1\} = \{Q_2^1, Q_2^1\} = 0, \quad \{Q_1^1, Q_2^1\} = H^0. \quad (3.51)$$

The first two equalities impose that $\beta_1 \gamma_1 = \beta_2 \gamma_2 = 0$. In particular, we must have that either $\beta_1 = 0$ or $\gamma_1 = 0$. In the case with β_1 non-zero, we can always use the transformation $\theta_2 \mapsto \xi_2$, $\xi_2 \mapsto \theta_2$ (which does not modify the original quadratic hamiltonians and preserves Poisson brackets) to make β_1 vanish. Thus we can assume without loss of generality that $\beta_1 = 0$. The linear terms become

$$Q_1^1 = \gamma_1 \xi_2, \quad (3.52a)$$

$$Q_2^1 = \beta_2 \xi_2 + \gamma_2 \theta_2, \quad (3.52b)$$

$$H^1 = 0. \quad (3.52c)$$

Then, from the last equality in (3.51), we get:

$$H^0 = \gamma_1 \gamma_2. \quad (3.53)$$

As a result, up to equivalences we have found the general representation:

$$Q_1 = \gamma_1 \xi_2 + x \theta_1, \quad (3.54a)$$

$$Q_2 = \beta_2 \xi_2 + \gamma_2 \theta_2 + y \xi_1, \quad (3.54b)$$

$$H = \gamma_1 \gamma_2 + xy - \theta_1 \xi_1, \quad (3.54c)$$

with $\beta_2 \gamma_2 = 0$. By changing values of the coefficients $\gamma_1, \beta_2, \gamma_2$ one may obtain several genuinely different Airy structures.

For clarity we now pick one particular example. We remark that the “trivial” case with $\gamma_1 = \beta_2 = \gamma_2 = 0$ is rather boring; indeed, in this case it turns out that Σ_s is empty. Thus we consider the slightly more complicated case with $\gamma_1 = \gamma_2 = 1$ and $\beta_2 = 0$. The generators take the form:

$$Q_1 = \xi_2 + x \theta_1, \quad (3.55a)$$

$$Q_2 = \theta_2 + y \xi_1, \quad (3.55b)$$

$$H = 1 + xy - \theta_1 \xi_1. \quad (3.55c)$$

- (3) The third step is to shift even variables to find an equivalent representation with the generators in the form of a super quantum Airy structure (if possible). As explained in Section 3.2.3, this corresponds to finding these points of the zero locus

$$(H)_0 = 1 + xy = 0 \quad (3.56)$$

where the gradient of $(H)_0$ is nonzero. They take the form

$$x = a, \quad y = -\frac{1}{a}, \quad (3.57)$$

with $a \in \mathbb{C} \setminus \{0\}$. We see that Σ_s is connected and isomorphic to $\mathbb{C}^\times \cong \frac{\mathbb{C}}{\mathbb{Z}}$, so its fundamental group is \mathbb{Z} . Hence the stabiliser of any point with respect to the \mathbb{C} -action is isomorphic to \mathbb{Z} . We can double check that this is correct by solving for the Hamiltonian flow generated by $(H)_0$. To this end we evaluate the Poisson brackets

$$\{H, x\} = x, \tag{3.58a}$$

$$\{H, y\} = -y. \tag{3.58b}$$

Therefore the solution of the Hamilton equations $\frac{df}{dt} = \{H, f\}$ takes the form

$$x(t) = e^t x(0), \tag{3.59a}$$

$$y(t) = e^{-t} y(0). \tag{3.59b}$$

Clearly we can obtain any element of Σ_s by flowing from any given initial point, say $x(0) = 1, y(0) = -1$. The relation $xy + 1 = 0$ is explicitly preserved by the flow. Moreover we have $(x(t), y(t)) = (x(0), y(0))$ if and only if $t \in 2\pi i\mathbb{Z}$, confirming that the stabiliser of $(x(0), y(0))$ is infinite cyclic.

Since Σ_s is connected, we are free to choose $a = 1$. Performing the corresponding shifts of the even variables $x \mapsto x + 1$ and $y \mapsto y - 1$ brings the generators in the form:

$$Q_1 = \theta_1 + \xi_2 + x\theta_1, \tag{3.60a}$$

$$Q_2 = \theta_2 - \xi_1 + y\xi_1, \tag{3.60b}$$

$$H = y - x + xy - \theta_1\xi_1. \tag{3.60c}$$

- (4) Finally, the last step in the classification is to choose a Lagrangian complement. First, we define new canonical momenta $\pi_1 = Q_1^1, \pi_2 = Q_2^1$, and $p = H^1$ to bring the linear terms in the form of a super quantum Airy structure. More explicitly,

$$\pi_1 = \theta_1 + \xi_2, \tag{3.61a}$$

$$\pi_2 = \theta_2 - \xi_1, \tag{3.61b}$$

$$p = y - x. \tag{3.61c}$$

Then, we have to choose a Lagrangian complement, i.e. odd linear generators κ^1, κ^2 and an even q such that $\{\pi_i, \kappa^j\} = \delta_i^j, \{p, q\} = 1, \{\kappa^i, \kappa^j\} = 0$. The following choice is convenient:

$$\kappa^1 = \frac{1}{2}(\theta_2 + \xi_1), \tag{3.62a}$$

$$\kappa^2 = \frac{1}{2}(\xi_2 - \theta_1), \tag{3.62b}$$

$$q = \frac{1}{2}(x + y). \tag{3.62c}$$

In terms of these variables, the generators become:

$$Q_1 = \pi_1 + \frac{1}{4}(2q - p)(\pi_1 - 2\kappa^2), \tag{3.63a}$$

$$Q_2 = \pi_2 + \frac{1}{4}(2q + p)(2\kappa^1 - \pi_2), \tag{3.63b}$$

$$H = p + q^2 - \frac{1}{4}p^2 - \frac{1}{4}(\pi_1 - 2\kappa^2)(2\kappa^1 - \pi_2). \tag{3.63c}$$

This completes the construction of the super classical Airy structure. It can be quantized as usual to get a super quantum Airy structure. For instance, using Weyl quantization, the generators become the following differential operators in the variables κ^1, κ^2 and q :

$$Q_1 = \hbar\partial_{\kappa^1} - q\kappa^2 + \frac{\hbar}{2}q\partial_{\kappa^1} + \frac{\hbar}{2}\kappa^2\partial_q - \frac{\hbar^2}{4}\partial_q\partial_{\kappa^1}, \tag{3.64a}$$

$$Q_2 = \hbar\partial_{\kappa^2} + q\kappa^1 - \frac{\hbar}{2}q\partial_{\kappa^2} + \frac{\hbar}{2}\kappa^1\partial_q - \frac{\hbar^2}{4}\partial_q\partial_{\kappa^2}, \tag{3.64b}$$

$$H = \hbar\partial_q + q^2 - \kappa^1\kappa^2 + \frac{\hbar}{2}\kappa^1\partial_{\kappa^1} - \frac{\hbar}{2}\kappa^2\partial_{\kappa^2} - \frac{\hbar^2}{4}\partial_q^2 + \frac{\hbar^2}{4}\partial_{\kappa^1}\partial_{\kappa^2}. \tag{3.64c}$$

By construction, those differential operators have the form of a super quantum Airy structure, and they form a representation of the original superalgebra, with the only non-zero commutator $[Q_1, Q_2] = H$.

3.3. The $\mathfrak{osp}(1|2)$ Example. In this section we explain how we can construct quadratic super quantum Airy structures using representation theory of Lie superalgebras, following [4]. We then apply the procedure to the particular case of the $\mathfrak{osp}(1|2)$ Lie superalgebra.

Let $(V^*, [\cdot, \cdot])$ be a finite dimensional Lie superalgebra. A quadratic super classical Airy structure L^{cl} can be understood as a Lie superalgebra homomorphism: for any $\phi, \psi \in V^*$, and with $\{\cdot, \cdot\}$ denoting the canonical Poisson bracket on $W = V^* \oplus V$, we have

$$\{L^{\text{cl}}(\phi), L^{\text{cl}}(\psi)\} = L^{\text{cl}}([\phi, \psi]). \tag{3.65}$$

For a quadratic Airy structure $L^{\text{cl}}(\phi) = \phi + \mathcal{L}(\phi)$, $\mathcal{L}(\phi) \in \mathbb{K}[W]^2$, (3.65) is equivalent to

$$\{\mathcal{L}(\phi), \mathcal{L}(\psi)\} = \mathcal{L}([\phi, \psi]), \tag{3.66a}$$

$$\{\phi, \mathcal{L}(\psi)\} + \{\mathcal{L}(\phi), \psi\} = [\phi, \psi]. \tag{3.66b}$$

In particular \mathcal{L} is itself a Lie superalgebra homomorphism, and the formula

$$\rho_W(\phi)(w) = \{\mathcal{L}(\phi), w\}, \quad \text{for } \phi \in V^*, \quad w \in W, \tag{3.67}$$

defines a representation of $(V^*, [\cdot, \cdot])$ on W .

Following [4], we now show how one can use representation theory to construct a super classical Airy structure. Suppose that ρ_M is a representation of the Lie superalgebra V^* on a $2 \dim V$ dimensional space M equipped with a V^* invariant symplectic form ω_M . Suppose further that there exists $\Omega \in M$ such that $\rho_M(V^*) \cdot \Omega$ is a Lagrangian subspace of M . Thus, in particular, $\dim \rho_M(V^*) \cdot \Omega = \dim V$ and the map

$$I : V^* \ni \phi \mapsto I(\phi) = \rho_M(\phi) \cdot \Omega \in \rho_M(V^*) \cdot \Omega \tag{3.68}$$

is an even isomorphism. Choose a Lagrangian complement Σ of $\rho_M(\phi) \cdot \Omega$, and define a map $K : \Sigma \rightarrow V^{**} \simeq V$ by the formula

$$\{\phi, K(s)\} = \omega_M(\rho_M(\phi) \cdot \Omega, s), \quad \text{for } \phi \in V^*, \quad s \in \Sigma. \quad (3.69)$$

Clearly K is also an even isomorphism and consequently

$$J = I^{-1} \oplus K : M = \rho_M(V^*) \cdot \Omega \oplus \Sigma \mapsto V^* \oplus V = W, \quad (3.70)$$

is an even isomorphism as well. J is a symplectomorphism, which follows from the fact that the four (sub)spaces $V, V^*, \rho_M(V^*) \cdot \Omega$ and Σ are Lagrangian, and for $s \in \Sigma, \phi \in V^*$ we have

$$\{J(\rho_M(\phi) \cdot \Omega), J(s)\} = \left\{ I^{-1}(I(\phi)), K(s) \right\} = \omega_M(\rho_M(\phi) \cdot \Omega, s) \quad (3.71)$$

by (3.69). We can thus define a symplectic representation of V^* on W by the formula

$$\rho_W = J \circ \rho_M \circ J^{-1}, \quad (3.72)$$

and consequently, via (3.67), quadratic hamiltonians $\mathcal{L}(\phi)$ satisfying (3.66a).

Equation (3.66b) is also satisfied, since with our definitions (viewing $\phi, \psi \in V^*$ via the natural embedding $V^* \simeq V^* \oplus 0 \hookrightarrow V^* \oplus V = W$ as elements of W), we have for homogeneous ϕ and ψ ,

$$\begin{aligned} \{\mathcal{L}(\phi), \psi\} + \{\phi, \mathcal{L}(\psi)\} &= \rho_W(\phi)(\psi) - (-1)^{|\mathcal{L}(\psi)||\phi|} \rho_W(\psi)(\phi) \\ &= J(\rho_M(\phi)I(\psi)) - (-1)^{|\mathcal{L}(\psi)||\phi|} J(\rho_M(\psi)I(\phi)) \\ &= J(\rho_M(\phi)\rho_M(\psi) \cdot \Omega) - (-1)^{|\mathcal{L}(\psi)||\phi|} J(\rho_M(\psi)\rho_M(\phi) \cdot \Omega) \\ &= I^{-1}(\rho_M([\phi, \psi]) \cdot \Omega) = [\phi, \psi]. \end{aligned} \quad (3.73)$$

So to summarize, the construction goes as follows:

- (1) We start with a Lie superalgebra V^* and a representation $\rho_M(V^*)$ on some $2 \dim V$ -dimensional space M ;
- (2) We construct a symplectomorphism $J : M \rightarrow W = V^* \oplus V$, and a representation $\rho_W = J \circ \rho_M \circ J^{-1}$ on W ;
- (3) By (3.67), this gives us a quadratic super classical Airy structure. It may be then quantized using Weyl prescription.

Let us now apply this construction to the $\mathfrak{osp}(1|2)$ Lie superalgebra. Recall that $\mathfrak{osp}(1|2)$ is generated by three even and two odd vectors, l_0, l_{\pm} and q_{\pm} respectively, satisfying the relations

$$\begin{aligned} [l_0, l_{\pm}] &= \pm l_{\pm} & [l_+, l_-] &= 2l_0. \\ [l_0, q_{\pm}] &= \pm \frac{1}{2} q_{\pm}, & [l_{\pm}, q_{\mp}] &= q_{\pm}, \\ [q_{\pm}, q_{\pm}] &= \pm l_{\pm}, & [q_+, q_-] &= -l_0. \end{aligned} \quad (3.74)$$

Proposition 3.14. *Let V be a $(3|2)$ -dimensional super vector space, and let $\tilde{V} = V \oplus \mathbb{K}^{0|1}$. We choose a basis $\{x^1, x^2, x^3\}$ and $\{\theta^1, \theta^2\}$ for the even and odd subspaces of V respectively, with dual basis $\{y_1, y_2, y_3\}$ and $\{\xi_1, \xi_2\}$. We let θ^0 be a basis for $\mathbb{K}^{0|1}$. Then the linear operator $L : V^* \rightarrow \widehat{\mathcal{W}}(\tilde{V})$ defined by*

$$L(y_1) = \hbar \partial_{x^1} - \sqrt{3} \theta^0 \theta^1 - 12 (x^1)^2 - \frac{3}{2} \hbar x^2 \partial_{x^1} - \frac{10}{3} \hbar x^3 \partial_{x^2} + 2\sqrt{3} \hbar \theta^1 \partial_{\theta^0} + \hbar \theta^2 \partial_{\theta^1}, \tag{3.75a}$$

$$L(y_2) = \hbar \partial_{x^2} - \frac{1}{2} \hbar x^1 \partial_{x^1} - \frac{3}{2} \hbar x^2 \partial_{x^2} - \frac{5}{2} \hbar x^3 \partial_{x^3} - \hbar \theta^1 \partial_{\theta^1} - 2\hbar \theta^2 \partial_{\theta^2} - \frac{3}{4} \hbar, \tag{3.75b}$$

$$L(y_3) = \hbar \partial_{x^3} - \frac{16}{3} \hbar x^1 \partial_{x^2} - \frac{3}{2} \hbar x^2 \partial_{x^3} + \frac{\sqrt{3}}{2} \hbar \theta^0 \partial_{\theta^1} + 4\hbar \theta^1 \partial_{\theta^2} + \frac{3}{16} \hbar^2 \partial_{x^1}^2 - \sqrt{3} \hbar^2 \partial_{\theta^1} \partial_{\theta^0}, \tag{3.75c}$$

and

$$L(\xi_1) = \hbar \partial_{\theta^1} + \sqrt{3} x_1 \theta^0 - \frac{1}{2} \hbar \theta^1 \partial_{x^1} + \frac{1}{3} \hbar \theta^2 \partial_{x^2} + 2\sqrt{3} \hbar x^1 \partial_{\theta^0} - \frac{3}{2} \hbar x_2 \partial_{\theta^1} + 5\hbar x_3 \partial_{\theta^2}, \tag{3.76a}$$

$$L(\xi_2) = \hbar \partial_{\theta^2} + \frac{\sqrt{3}}{8} \hbar \theta^0 \partial_{x^1} - \frac{4}{3} \hbar \theta^1 \partial_{x^2} + \frac{1}{2} \hbar \theta^2 \partial_{x^3} + 2\hbar x^1 \partial_{\theta^1} - \frac{3}{2} \hbar x^2 \partial_{\theta^2} + \frac{\sqrt{3}}{4} \hbar^2 \partial_{\theta^0} \partial_{x^1}, \tag{3.76b}$$

is a quadratic super quantum Airy structure (with an extra fermionic variable), realized as a differential representation of the $\mathfrak{osp}(1|2)$ Lie superalgebra under the identification

$$\{l_-, l_0, l_+, q_-, q_+\} \mapsto \{L(y_1), L(y_2), L(y_3), L(\xi_1), L(\xi_2)\}. \tag{3.77}$$

Remark 3.15. Clearly, the operator L satisfies the second property in the definition of super quantum Airy structures (see Definition 2.3), and it is quadratic. All that remains to be checked is that it forms a representation of the $\mathfrak{osp}(1|2)$ Lie superalgebra. This could be checked by brute force calculation. Let us instead construct this super quantum Airy structure using the general approach presented above.

Proof. We apply the construction described above for the particular case of the $\mathfrak{osp}(1|2)$ Lie superalgebra, with generators satisfying the relations (3.74). We let V be a $(3|2)$ -dimensional super vector space. We choose a basis $\{x^1, x^2, x^3\}$ and $\{\theta^1, \theta^2\}$ for the even and odd subspaces of V respectively, with dual basis $\{y_1, y_2, y_3\}$ and $\{\xi_1, \xi_2\}$. We identify V^* with the algebra $\mathfrak{osp}(1|2)$ by:

$$\{y_1, y_2, y_3, \xi_1, \xi_2\} \leftrightarrow \{l_-, l_0, l_+, q_-, q_+\}. \tag{3.78}$$

The irreducible representations of $\mathfrak{osp}(1|2)$ are odd-dimensional. We will modify the construction above slightly, to allow our representation space M to be odd-dimensional. We enlarge W by adding an extra fermionic variable. We define $\tilde{V} = V \oplus \mathbb{K}^{0|1}$, and

let $\tilde{W} = \tilde{V} \oplus \tilde{V}^*$. We will construct a symplectomorphism J as a map between the representation space M and a suitably chosen, odd-dimensional subspace of the space \tilde{W} . The result of the construction will be a quadratic super classical Airy structure with an extra fermionic variable.

Since V is five-dimensional, we take M to be 11-dimensional and denote its basis vectors by f_m and e_k , where $m = -2, -1, \dots, 2$ and $k = -\frac{5}{2}, -\frac{3}{2}, \dots, \frac{5}{2}$. The representation of $\mathfrak{osp}(1|2)$ in question is given by

$$\rho_M(\xi_2) f_m = \sqrt{\frac{3+m}{2}} e_{m+\frac{1}{2}}, \quad \rho_M(\xi_1) f_m = \sqrt{\frac{3-m}{2}} e_{m-\frac{1}{2}}, \quad (3.79)$$

and

$$\rho_M(\xi_2) e_k = \sqrt{\frac{\frac{5}{2}-k}{2}} f_{k+\frac{1}{2}}, \quad \rho_M(\xi_1) e_k = -\sqrt{\frac{\frac{5}{2}+k}{2}} f_{k+\frac{1}{2}}, \quad (3.80)$$

so that $\rho_M(\xi_2) e_{\frac{5}{2}} = \rho_M(\xi_1) e_{-\frac{5}{2}} = 0$. The action of other generators can be computed from (3.74).

The symplectic form on M , invariant with respect to the $\mathfrak{osp}(1|2)$ algebra, i.e. satisfying

$$\begin{aligned} \omega_M(\rho_M(y_i) e_k, e_l) + \omega_M(e_k, \rho_M(y_i) e_l) \\ = \omega_M(\rho_M(y_i) f_m, f_n) + \omega_M(f_m, \rho_M(y_i) f_n) = 0, \end{aligned} \quad (3.81a)$$

$$\omega_M(\rho_M(\xi_j) e_k, f_n) + \omega_M(e_k, \rho_M(\xi_j) f_n) = 0, \quad (3.81b)$$

is determined uniquely up to an overall normalization. Its non-zero elements read

$$\omega_M(e_{\frac{1}{2}}, e_{-\frac{1}{2}}) = -\omega_M(e_{\frac{3}{2}}, e_{-\frac{3}{2}}) = \omega_M(e_{\frac{5}{2}}, e_{-\frac{5}{2}}) = 1, \quad (3.82a)$$

$$\omega_M(f_0, f_0) = -\omega_M(f_1, f_{-1}) = \omega_M(f_2, f_{-2}) = 1, \quad (3.82b)$$

$$\omega_M(f_i, f_j) - \omega_M(f_j, f_i) = 0, \quad (3.82c)$$

$$\omega_M(e_i, e_j) + \omega_M(e_j, e_i) = 0. \quad (3.82d)$$

The maximal isotropic subspaces of M are of dimension 5 and we can choose one of such subspaces to be generated by the action of $\rho_M(V^*)$ on $e_{\frac{3}{2}}$ (which thus plays the role of the vector Ω from the initial paragraphs of this subsection):

$$M_+ = \rho_M(V^*) e_{\frac{3}{2}} = \text{span} \left\{ e_{\frac{1}{2}}, f_1, e_{\frac{3}{2}}, f_2, e_{\frac{5}{2}} \right\}. \quad (3.83)$$

Defining the map $I : V^* \mapsto M_+$ as in (3.68) we thus get

$$\begin{aligned} I(y_1) &= 2\sqrt{2} e_{\frac{1}{2}}, & I(\xi_1) &= -\sqrt{2} f_1, & I(y_2) &= \frac{3}{2} e_{\frac{3}{2}}, \\ I(\xi_2) &= \frac{1}{\sqrt{2}} f_2, & I(y_3) &= \sqrt{5} e_{\frac{5}{2}}. \end{aligned} \quad (3.84)$$

As a complement of M_+ , we take the space $M_0 \oplus M_-$, where $M_0 = \mathbb{K} f_0$ and

$$M_- = \text{span} \left\{ e_{-\frac{1}{2}}, f_{-1}, e_{-\frac{3}{2}}, f_{-2}, e_{-\frac{5}{2}} \right\}. \quad (3.85)$$

Note that M_- is also a maximal isotropic subspace of M . We can now define the map $K : M_- \mapsto V$ as in (3.69) with the result

$$\begin{aligned} K(e_{-\frac{1}{2}}) &= 2\sqrt{2}x^1, & K(f_{-1}) &= \sqrt{2}\theta^1, & K(e_{-\frac{3}{2}}) &= -\frac{3}{2}x^2, \\ K(f_{-2}) &= \frac{1}{\sqrt{2}}\theta^2, & K(e_{-\frac{5}{2}}) &= \sqrt{5}x^3. \end{aligned} \tag{3.86}$$

Let us now denote the basis of the $\mathbb{K}^{0|1}$ subspace appearing in the decomposition $\tilde{V} = V \oplus \mathbb{K}^{0|1}$ by θ^0 , and the corresponding element of the dual basis in \tilde{V}^* by ξ_0 . Taking $\eta = \xi_0 + \frac{1}{2}\theta^0$, so that

$$\{\eta, \eta\} = \{\xi_0, \theta^0\} = \xi_0(\eta^0) = 1 = \omega_M(f_0, f_0), \tag{3.87}$$

and defining $J_0(f_0) = \eta$, we construct the symplectomorphism

$$J = I^{-1} \oplus J_0 \oplus K : M_+ \oplus \mathbb{K}f_0 \oplus M_- \mapsto V^* \oplus \mathbb{K}\eta \oplus V \tag{3.88}$$

satisfying the properties required for $\rho_{\tilde{W}} = J^{-1} \circ \rho_M \circ J$ to define via (3.67) a quadratic super classical Airy structure on $\mathbb{K}[\tilde{W}]$ (depending on an extra fermionic variable). Calculating the matrix elements of $\rho_{\tilde{W}}$, and using (3.67), we get

$$L^{cl}(y_1) = y_1 - \frac{3}{2}x^2y_1 - \frac{10}{3}x^3y_2 - 12(x^1)^2 + 2\sqrt{3}\theta^1\eta + \theta^2\xi_1, \tag{3.89a}$$

$$L^{cl}(y_2) = y_2 - \frac{1}{2}x^1y_1 - \frac{3}{2}x^2y_2 - \frac{5}{2}x^3y_3 - \theta^1\xi_1 - 2\theta^2\xi_2 - \frac{3}{2}\eta^2, \tag{3.89b}$$

$$L^{cl}(y_3) = y_3 + \frac{3}{16}(y_1)^2 - \frac{16}{3}x^1y_2 - \frac{3}{2}x^2y_3 + \sqrt{3}\eta\xi_1 + 4\theta^1\xi_2, \tag{3.89c}$$

as well as

$$L^{cl}(\xi_1) = \xi_1 - \frac{1}{2}\theta^1y_1 + \frac{1}{3}\theta^2y_2 + 2\sqrt{3}x^1\eta - \frac{3}{2}x^2\xi_1 + 5x^3\xi_2, \tag{3.90a}$$

$$L^{cl}(\xi_2) = \xi_2 - \frac{4}{3}\theta^1y_2 + \frac{1}{2}\theta^2y_3 + \frac{\sqrt{3}}{4}\eta y_1 + 2x^1\xi_1 - \frac{3}{2}x^2\xi_2. \tag{3.90b}$$

Finally, applying Weyl quantization, we get

$$\begin{aligned} L(y_1) &= \hbar\partial_{x^1} - \sqrt{3}\theta^0\theta^1 - 12(x^1)^2 - \frac{3}{2}\hbar x^2\partial_{x^1} - \frac{10}{3}\hbar x^3\partial_{x^2} + 2\sqrt{3}\hbar\theta^1\partial_{\theta^0} \\ &\quad + \hbar\theta^2\partial_{\theta^1}, \end{aligned} \tag{3.91a}$$

$$\begin{aligned} L(y_2) &= \hbar\partial_{x^2} - \frac{1}{2}\hbar x^1\partial_{x^1} - \frac{3}{2}\hbar x^2\partial_{x^2} - \frac{5}{2}\hbar x^3\partial_{x^3} - \hbar\theta^1\partial_{\theta^1} \\ &\quad - 2\hbar\theta^2\partial_{\theta^2} - \frac{3}{4}\hbar, \end{aligned} \tag{3.91b}$$

$$\begin{aligned} L(y_3) &= \hbar\partial_{x^3} - \frac{16}{3}\hbar x^1\partial_{x^2} - \frac{3}{2}\hbar x^2\partial_{x^3} + \frac{\sqrt{3}}{2}\hbar\theta^0\partial_{\theta^1} + 4\hbar\theta^1\partial_{\theta^2} + \frac{3}{16}\hbar^2\partial_{x^1}^2 \\ &\quad - \sqrt{3}\hbar^2\partial_{\theta^1}\partial_{\theta^0}, \end{aligned} \tag{3.91c}$$

and

$$L(\xi_1) = \hbar\partial_{\theta^1} + \sqrt{3}x_1\theta^0 - \frac{1}{2}\hbar\theta^1\partial_{x^1} + \frac{1}{3}\hbar\theta^2\partial_{x^2} + 2\sqrt{3}\hbar x^1\partial_{\theta^0} - \frac{3}{2}\hbar x_2\partial_{\theta^1} + 5\hbar x_3\partial_{\theta^2}, \tag{3.92a}$$

$$L(\xi_2) = \hbar\partial_{\theta^2} + \frac{\sqrt{3}}{8}\hbar\theta^0\partial_{x^1} - \frac{4}{3}\hbar\theta^1\partial_{x^2} + \frac{1}{2}\hbar\theta^2\partial_{x^3} + 2\hbar x^1\partial_{\theta^1} - \frac{3}{2}\hbar x^2\partial_{\theta^2} + \frac{\sqrt{3}}{4}\hbar^2\partial_{\theta^0}\partial_{x^1}. \tag{3.92b}$$

□

We would like to make several remarks about the constructed super quantum Airy structure:

- It is not possible to construct an additional odd generator, say Q , which together with $L(y_i)$ and $L(\xi_i)$ forms a super quantum Airy structure for some superalgebra of dimension $3|3$. Indeed, suppose that such an extension exists. Writing down the Jacobi identity for the extended algebra and using the fact that the first cohomology group of $\mathfrak{osp}(1|2)$ valued in the adjoint module vanishes, one can show that with no loss of generality Q may be assumed to be central. Similar arguments show that Q has to annihilate the whole $6|6$ -dimensional module $M \oplus \mathbb{K}^{0|1}$. This means that we have to take the corresponding Hamiltonian to be purely linear. The only linear variable which supercommutes with all L is $\xi_0 - \frac{1}{2}\theta^0$, which does not work because it is not nilpotent.
- It is not possible to construct a super quantum Airy structure for $\mathfrak{osp}(1|2)$ which does not involve an additional fermionic variable. The reason is that there is no $6|4$ -dimensional representation whose bosonic part is the irreducible 6-dimensional representation of $\mathfrak{sl}(2, \mathbb{C})$, which is known [4] to be necessary to construct an Airy structure for the even subalgebra. For similar reasons, the irreducible representation that we chose is the only one which allows the construction of a super quantum Airy structure.
- In the language of the classification scheme for super quantum Airy structures that we have outlined in Section 3.2, the fact that the Lagrangian embedding of $\mathfrak{osp}(1|2)$ into $M \oplus \mathbb{K}^{0|1}$ may be obtained by acting with the generators on some reference vector Ω is equivalent to the statement that the linear terms of our generators are trivial, i.e. can be obtained by an affine shift of coordinates. There is no need to consider more general linear terms, because the relevant cohomology groups vanish [48].
- We have $[\mathfrak{osp}(1|2), \mathfrak{osp}(1|2)] = \mathfrak{osp}(1|2)$, so the quantization procedure is unique.

3.4. Super Frobenius Algebras. For our last finite-dimensional examples, we construct super quantum Airy structures from super Frobenius algebras, in the spirit of [4] (see also [10]).

We define super Frobenius algebras following [50], where a definition of G -twisted Frobenius algebras is given. We concentrate on the $G = \mathbb{Z}_2$ case, which corresponds to super Frobenius algebras.

Definition 3.16. A super Frobenius algebra $\mathbb{A}_s = \mathbb{A}_0 \oplus \mathbb{A}_1$ over \mathbb{K} is a finite-dimensional \mathbb{Z}_2 -graded vector space equipped with a super-commutative, associative product $\mathbb{A}_g \otimes \mathbb{A}_h \rightarrow \mathbb{A}_{gh}$ respecting grading, and a non-degenerate bilinear form $\phi : \mathbb{A}_g \otimes \mathbb{A}_h \rightarrow \mathbb{K}$ where $\phi = 0$ unless $|g| + |h| = 0$ and $g, h \in \{0, 1\}$.

Given a super Frobenius algebra, for any choice of even elements $\theta_A, \theta_B, \theta_C \in \mathbb{A}_0$, we can construct a super quantum Airy structure as follows:

Lemma 3.17. *Let $\{e_i\}$ be a basis for a super Frobenius algebra \mathbb{A}_s , and $\{e^j\}$ be the dual basis, i.e.*

$$\phi(e_i, e^j) = (-1)^{|i||j|} \phi(e^j, e_i) = \delta_i^j. \tag{3.93}$$

Then, for any even $\theta_A, \theta_B, \theta_C \in \mathbb{A}_0$, the coefficients

$$A_{ijk} = \phi(\theta_A e_i e_j e_k), \quad B_{ij}^k = \phi(\theta_B e_i e_j e^k), \quad C_i^{jk} = \phi(\theta_C e_i e^j e^k), \tag{3.94}$$

together with an arbitrary coefficient D_i , define a super quantum Airy structure on $V = \mathbb{A}_s$ with vanishing structure constants $f_{ij}^k = 0$.

Remark 3.18. We note that this super quantum Airy structure does not depend on an extra fermionic variable, i.e. the number of differential operators match with the dimension of the super vector space \mathbb{A}_s .

Proof. To prove that this is a super quantum Airy structure, we will show that the coefficients satisfy the conditions of Lemma 2.19.

Note that (3.93) implies that every $a \in \mathbb{A}_s$ can be written as $a = \phi(a, e^i) e_i$. This gives

$$\begin{aligned} B_{ik}^p A_{jpl} &= \phi(\theta_B e_i e_k e^p) \phi(\theta_A e_j e_p e_l) = \phi(\theta_A e_j \phi(\theta_B e_i e_k e^p) e_p e_l) \\ &= \phi(\theta_A e_j \theta_B e_i e_k e_l) = \phi(\theta_A \theta_B e_j e_i e_k e_l). \end{aligned} \tag{3.95}$$

Thus, we have

$$\begin{aligned} B_{ik}^p A_{jpl} + (-1)^{|k||l|} B_{il}^p A_{jpk} + (-1)^{|i||j|} B_{ij}^p A_{pkl} \\ = \phi(\theta_A \theta_B e_j e_i e_k e_l) + (-1)^{|k||l|} \phi(\theta_A \theta_B e_j e_i e_l e_k) + (-1)^{|i||j|} \phi(\theta_A \theta_B e_i e_j e_k e_l) \\ = 3\phi(\theta_A \theta_B e_j e_i e_k e_l) = (-1)^{|i||j|} (i \leftrightarrow j), \end{aligned} \tag{3.96}$$

and hence (2.28a) is satisfied.

Similarly, we find

$$B_{ik}^p B_{jp}^l = \phi(\theta_B^2 e_j e_i e_k e^l), \tag{3.97}$$

$$C_i^{lp} A_{jpk} = \phi(\theta_A \theta_C e_j e_i e^l e_k), \tag{3.98}$$

$$C_i^{kp} B_{jp}^l = \phi(\theta_B \theta_C e_j e_i e^k e^l), \tag{3.99}$$

$$B_{ij}^p C_p^{kl} = \phi(\theta_B \theta_C e_i e_j e^k e^l). \tag{3.100}$$

These ensure that all terms in the remaining conditions (2.28b), (2.28c) and (2.28d) are \mathbb{Z}_2 -symmetrical under $i \leftrightarrow j$, and hence the conditions of Lemma 2.19 are satisfied. \square

A natural question then is to determine what this super quantum Airy structure associated to a super Frobenius algebra calculates. What is the meaning of the $F_{g,n}$?

In the standard, bosonic case, it is well known that a two-dimensional topological quantum field theory (2D TQFT) naturally defines the structure of a Frobenius algebra, and that, conversely, to any Frobenius algebra can be associated a unique 2D TQFT.

In this case, it is shown in [4, 10] that the quantum Airy structure naturally associated to a Frobenius algebra, with the choice $\theta_A = \theta_B = \theta_C = 1$ and $D_i = \phi(e_i \cdot H)$ for $H = \sum_j e_j e^j$, solves the corresponding TQFT, in the sense that its $F_{g,n}$ compute the amplitudes of the 2D TQFT (up to a simple combinatorial factor).

In the general setting, it is known that a G -equivariant 2D TQFT defines the structure of a G -twisted Frobenius algebra, from which it can be recovered [44]. Super Frobenius algebras correspond to the case with $G = \mathbb{Z}_2$. We expect that a story analogous to the bosonic case holds here as well, namely, that the $F_{g,n}$ associated to the super quantum Airy structure naturally constructed from a super Frobenius algebra with $\theta_A = \theta_B = \theta_C = 1$ and choice of D_i as above, compute the amplitudes of the corresponding 2D super TQFT. This correspondence should be made precise and investigated further.

4. Infinite-Dimensional Examples

In this section we construct examples of infinite-dimensional, quadratic, subalgebraic, super quantum Airy structures, as representations of subalgebras of super Virasoro algebras. The construction follows along the lines of [11, 54]. In the bosonic case, many infinite-dimensional quantum Airy structures compute interesting enumerative invariants, such as intersection numbers over the moduli space of curves, Hurwitz numbers, Gromov–Witten invariants, etc. We expect the super quantum Airy structures that we construct in this section to also have interesting enumerative interpretations, which we leave for future work. In particular, they may be related to the recent supersymmetric generalization of JT gravity presented in [64].

In this section we focus on constructing quadratic super quantum Airy structures. But the construction can naturally be generalized to higher order super quantum Airy structures along the lines of [11], which we also leave for future work.

We do not review the well known definitions for vertex operator algebras (VOAs), vertex operator super algebras (VOSAs) and their representations. See [11] for relevant definitions in the context of the construction presented here, and for instance [6, 7, 9, 42, 49, 53] for more details on VOAs and VOSAs.

4.1. Quantum Airy Structures from The Free Boson VOA. Before we study super quantum Airy structures constructed as representations of subalgebras of the super Virasoro algebra, let us review the standard bosonic construction for the Virasoro algebra. In the process we will generalize the construction of [11], discovering a VOA realization of the “topological recursion without branched covers” presented in [4].

Our goal in this section is to construct quantum Airy structures as representations of subalgebras of the Virasoro algebra with central charge $c = 1$:

$$[L_m, L_n] = (m - n)L_{m+n} + \delta_{m,-n} \frac{1}{12} m(m^2 - 1). \tag{4.1}$$

This algebra arises as the algebra of modes for the energy–momentum tensor of the free boson VOA, which is central in our construction. So let us start by reviewing the main features of the free boson VOA.

4.1.1. The Free Boson VOA We study the free boson VOA (also called the Heisenberg VOA). It is generated by a single vector $b_{-1}|0\rangle \in V$, where V is the space of states. Here $|0\rangle \in V$ is the vacuum vector. The state-operator correspondence reads

$$b(z) := Y(b_{-1}|0\rangle, z) = \sum_{m \in \mathbb{Z}} b_m z^{-m-1}, \quad (4.2)$$

where the modes of the bosonic field generate the Heisenberg algebra

$$[b_m, b_n] = m\delta_{m,-n}. \quad (4.3)$$

The vacuum vector $|0\rangle$ is annihilated by all b_k with $k \geq 0$, and the space of states V is the Fock space consisting of all excited modes

$$b_{-k_1} \dots b_{-k_n} |0\rangle, \quad k_1, \dots, k_n \in \mathbb{Z}_{>0}. \quad (4.4)$$

Their corresponding operators are

$$Y(b_{-k_1} \dots b_{-k_n} |0\rangle, z) = \circ \prod_{i=1}^n \frac{1}{(k_i - 1)!} \left(\frac{d}{dz} \right)^{k_i-1} Y(b_{-1}|0\rangle, z) \circ, \quad (4.5)$$

where $\circ \dots \circ$ denotes normal ordering, i.e., all modes b_k with negative k are on the left and those with positive k are on the right.

The conformal vector $|\omega\rangle$ for the free boson VOA is:

$$|\omega\rangle = \frac{1}{2} b_{-1} b_{-1} |0\rangle. \quad (4.6)$$

Its operator takes the form

$$T(z) := Y(|\omega\rangle, z) = \sum_{m \in \mathbb{Z}} L_m z^{-m-2}, \quad (4.7)$$

with its modes generating the Virasoro algebra with central charge $c = 1$:

$$[L_m, L_n] = (m - n)L_{m+n} + \delta_{m,-n} \frac{1}{12} m(m^2 - 1). \quad (4.8)$$

The modes of the conformal field can be related to the Heisenberg modes as follows. From (4.5), we have:

$$T(z) = \frac{1}{2} Y(b_{-1} b_{-1} |0\rangle) = \frac{1}{2} \circ Y(b_{-1}|0\rangle, z) Y(b_{-1}|0\rangle, z) \circ = \frac{1}{2} \circ b(z) b(z) \circ, \quad (4.9)$$

and hence

$$L_m = \frac{1}{2} \sum_{k \in \mathbb{Z}} \circ b_k b_{m-k} \circ. \quad (4.10)$$

4.1.2. Untwisted and Twisted Representations of the Free Boson VOA To construct quantum Airy structures as representations of subalgebras of the Virasoro algebra, we will start with two different representations of the free boson VOA: an untwisted representation, and a \mathbb{Z}_2 -twisted representation (see for instance [11] for more details on twisted representations for VOAs in the context of Airy structures).

In both cases, we will represent the bosonic modes b_m as endomorphisms of the space $\mathbb{K}[[V, \hbar]]$, where V is an infinite-dimensional vector space. In other words, we represent the modes of the Heisenberg algebra as differential operators in $\widehat{\mathcal{W}}_{\hbar}(V)$, which turns the Virasoro modes into differential operators in $\widehat{\mathcal{W}}_{\hbar}(V)$ as well.

The untwisted representation is basically the VOA itself. The state-field correspondence for the representation is

$$Y^M(b_{-1} | 0 \rangle, z) = \sum_{m \in \mathbb{Z}} b_m^M z^{-m-1}, \tag{4.11}$$

with the b_m^M endomorphisms of $\mathbb{K}[[V, \hbar]]$. The Virasoro modes of the untwisted representation then take the form:

$$L_m^M = \frac{1}{2} \sum_{k \in \mathbb{Z}} \circ b_k^M b_{m-k}^M \circ. \tag{4.12}$$

For the \mathbb{Z}_2 -twisted representation, we consider the order two automorphism that acts on the Fock space as:

$$\begin{aligned} \sigma : V &\rightarrow V \\ b_{-k_1} \dots b_{-k_n} | 0 \rangle &\mapsto (-1)^{\sum_{i=1}^n k_i} b_{-k_1} \dots b_{-k_n} | 0 \rangle. \end{aligned} \tag{4.13}$$

This automorphism preserves the vacuum vector $| 0 \rangle$ and the conformal vector $|\omega\rangle = \frac{1}{2} b_{-1} b_{-1} | 0 \rangle$. However, it does not preserve the fundamental vector $b_{-1} | 0 \rangle$, which picks a sign.

Thus the state-field correspondence for this \mathbb{Z}_2 -twisted representation takes the form

$$Y^\sigma(b_{-1} | 0 \rangle, z) = \sum_{r \in \mathbb{Z} + \frac{1}{2}} b_r^\sigma z^{-r-1}, \tag{4.14}$$

with fractional exponents, and twisted modes that generate the Heisenberg algebra:

$$[b_r^\sigma, b_s^\sigma] = r \delta_{r, -s}. \tag{4.15}$$

The Virasoro modes L_m^σ of the \mathbb{Z}_2 -twisted representation are related to the twisted bosonic modes b_r^σ as follows:

Lemma 4.1. *Consider the σ -twisted representation of the free boson VOA, with state-field correspondence (4.14). The conformal field*

$$Y^\sigma(|\omega\rangle, z) = \frac{1}{2} Y^\sigma(b_{-1} b_{-1} | 0 \rangle, z) = \sum_{m \in \mathbb{Z}} L_m^\sigma z^{-m-2} \tag{4.16}$$

has Virasoro modes given by

$$L_m^\sigma = \frac{1}{2} \sum_{r \in \mathbb{Z} + \frac{1}{2}} \circ b_r^\sigma b_{m-r}^\sigma \circ + \frac{1}{16} \delta_{m, 0}. \tag{4.17}$$

Proof. The proof goes as Lemma 4.2 of [11]. The conformal field is

$$Y^\sigma(|\omega\rangle, z) = \frac{1}{2} Y^\sigma(b_{-1}b_{-1}|0\rangle, z). \tag{4.18}$$

We want to rewrite it as a normal ordered product of twisted bosonic fields. For clarity, let us denote the twisted bosonic field by $b^\sigma(z) := Y^\sigma(b_{-1}|0\rangle, z)$.

For this we can use the product formula for twisted representations (see for instance Definition 3.18 in [11]):

$$Y^\sigma(b_{-1}b_{-1}|0\rangle, z) = \frac{1}{2} \left[\frac{\partial^2}{\partial z_1^2} \left((z_1 - z_2)^2 b^\sigma(z_1) b^\sigma(z_2) \right) \right]_{z_1=z_2=z}. \tag{4.19}$$

We have:

$$\begin{aligned} b^\sigma(z_1)b^\sigma(z_2) &= \circ b^\sigma(z_1)b^\sigma(z_2) \circ + \sum_{\substack{k_1, k_2 \in \mathbb{Z} + \frac{1}{2} \\ k_1 > 0, k_2 < 0}} [b_{k_1}^\sigma, b_{k_2}^\sigma] z_1^{-k_1-1} z_2^{-k_2-1} \\ &= \circ b^\sigma(z_1)b^\sigma(z_2) \circ + \sum_{\substack{k_1 \in \mathbb{Z} + \frac{1}{2} \\ k_1 > 0}} k_1 z_1^{-k_1-1} z_2^{k_1-1} \\ &= \circ b^\sigma(z_1)b^\sigma(z_2) \circ + \sum_{k \in \mathbb{Z}_{\geq 0}} \frac{\partial}{\partial z_2} \left(\frac{z_2^k}{z_1^{k+1}} \sqrt{\frac{z_2}{z_1}} \right) \\ &= \circ b^\sigma(z_1)b^\sigma(z_2) \circ + \frac{\partial}{\partial z_2} \left(\frac{1}{z_1 - z_2} \sqrt{\frac{z_2}{z_1}} \right). \end{aligned} \tag{4.20}$$

Substituting back in (4.19), we get:

$$\begin{aligned} &Y^\sigma(b_{-1}b_{-1}|0\rangle, z) \\ &= \frac{1}{2} \left[\frac{\partial^2}{\partial z_1^2} \left((z_1 - z_2)^2 \circ b^\sigma(z_1)b^\sigma(z_2) \circ + (z_1 - z_2)^2 \frac{\partial}{\partial z_2} \left(\frac{1}{z_1 - z_2} \sqrt{\frac{z_2}{z_1}} \right) \right) \right]_{z_1=z_2=z} \\ &= \circ b^\sigma(z)b^\sigma(z) \circ + \frac{1}{8z^2}, \end{aligned} \tag{4.21}$$

where the second line follows from a straightforward calculation. Therefore,

$$Y^\sigma(|\omega\rangle, z) = \frac{1}{2} \circ b^\sigma(z)b^\sigma(z) \circ + \frac{1}{16z^2} = \sum_{m \in \mathbb{Z}} L_m^\sigma z^{-m-2}. \tag{4.22}$$

Combining with (4.14), we extract the relation between the modes for the \mathbb{Z}_2 -twisted representation:

$$L_m^\sigma = \frac{1}{2} \sum_{r \in \mathbb{Z} + \frac{1}{2}} \circ b_r^\sigma b_{m-r}^\sigma \circ + \frac{1}{16} \delta_{m,0}. \tag{4.23}$$

□

4.1.3. *Quantum Airy Structures from Untwisted Representations of the Free Boson VOA*
 Let us now construct quantum Airy structures using these representations of the free boson VOA. We proceed in three steps:

- (1) We choose a representation (untwisted or twisted) of the free boson VOA to obtain a differential representation for the Virasoro modes: $L_n \in \widehat{\mathcal{W}}_{\hbar}(V)$. As we will see, the operators $\hbar L_n$ have degree 2 (with the notion of degree defined in (2.3)). We thus have constructed a representation of the Virasoro algebra by quadratic differential operators $\hbar L_i$.
- (2) We pick a subalgebra of the Virasoro algebra. Our quantum Airy structure will be obtained as a representation of this particular Virasoro subalgebra.
- (3) We shift some of the bosonic modes to create linear terms to the operators $\hbar L_i$, without changing the algebra, so that the second condition of Definition 2.3 is satisfied. This constructs a representation of the chosen Virasoro subalgebra in the form of a quantum Airy structure.

For convenience, let us define the following notation. For any integer R , we define

$$\delta_{i \leq R} = \begin{cases} 1 & \text{if } i \leq R \\ 0 & \text{if } i > R. \end{cases} \tag{4.24}$$

From untwisted representations of the free boson VOA, we construct the following three classes of quantum Airy structures:

Proposition 4.2. *We represent the Heisenberg algebra as*

$$\forall k \in \mathbb{Z}_{\geq 1}, \quad b_k^M = \sqrt{\hbar} \partial_k, \quad b_{-k}^M = \frac{1}{\sqrt{\hbar}} k x^k, \quad b_0^M = \sqrt{\hbar} \partial_0. \tag{4.25}$$

Let $\{x^0, x^1, x^2, \dots\}$ be a basis for V , with dual set $\{y_0, y_1, y_2, \dots\}$. Define the differential operators $H_i \in \widehat{\mathcal{W}}_{\hbar}(V)$, $i \in \mathbb{Z}_{\geq 0}$:

$$H_i = \hbar \partial_i + \frac{\hbar}{2} \sum_{k \in \mathbb{Z}} \circ b_k^M b_{i+N-1-k}^M \circ, \tag{4.26}$$

which form a representation of the Virasoro subalgebra

$$[H_i, H_j] = \hbar(i - j)H_{i+j+N-1}. \tag{4.27}$$

(1) Let N be any integer $N \geq 0$. The linear operator $H : V^* \rightarrow \widehat{\mathcal{W}}_{\hbar}(V)$ defined by:

$$\forall i \in \mathbb{Z}_{\geq 0}, \quad H(y_i) = H_i + \hbar D_i \delta_{i \leq N-1}, \tag{4.28}$$

for arbitrary constant D_i , $i = 0, \dots, N - 1$, forms a quantum Airy structure as a representation of the Virasoro subalgebra (4.27).

(2) Let N be any integer $N \geq -1$. The linear operator $H : V^* \rightarrow \widehat{\mathcal{W}}_{\hbar}(V)$ defined by

$$\forall i \in \mathbb{Z}_{\geq 1}, \quad H(y_i) = H_i + \hbar D_i \delta_{i \leq N+1}, \tag{4.29a}$$

$$H(y_0) = \hbar \partial_0 + \frac{\hbar^2}{2} C_0 \partial_0^2 + \hbar D_0, \tag{4.29b}$$

for arbitrary constants D_i , $i = 0, \dots, N+1$ and C_0 , forms a quantum Airy structure as a representation of the Virasoro subalgebra (4.27) extended by:

$$[H(y_0), H(y_i)] = 0. \tag{4.30}$$

(3) Let N be any integer $N \geq -1$. Let us now formally set the bosonic zero mode $b_0^M = 0$. Consider the subspace $V_{red} \subset V$ spanned by $\{x^1, x^2, \dots\}$. The linear operator $H : V_{red}^* \rightarrow \widehat{\mathcal{W}}_{\hbar}(V_{red})$ defined by

$$\forall i \in \mathbb{Z}_{\geq 1}, \quad H(y_i) = H_i \Big|_{b_0^M=0} + \hbar D_i \delta_{i \leq N+1}, \tag{4.31}$$

for arbitrary constants $D_i, i = 1, \dots, N + 1$, forms a quantum Airy structure as a representation of the Virasoro subalgebra (4.27).

Proof. We start with the untwisted representation of the free boson VOA. For our first two classes of quantum Airy structures, we represent the bosonic modes as endomorphisms on the space $M = \mathbb{K}[[V, \hbar]]$, where V is an infinite-dimensional vector space with basis $\{x^0, x^1, x^2, \dots\}$, as:

$$\forall k \in \mathbb{Z}_{\geq 1}, \quad b_k^M = \sqrt{\hbar} \partial_k, \quad b_{-k}^M = \frac{1}{\sqrt{\hbar}} k x^k, \quad b_0^M = \sqrt{\hbar} \partial_0, \tag{4.32}$$

which form a representation of the Heisenberg algebra. From (4.12), the Virasoro modes read:

$$\hbar L_m^M = \frac{\hbar}{2} \sum_{k \in \mathbb{Z}} \circ b_k^M b_{m-k}^M \circ. \tag{4.33}$$

We multiplied the Virasoro modes by \hbar here so that they have degree 2 according to the grading (2.3) on $\widehat{\mathcal{W}}_{\hbar}(V)$. This rescales the Virasoro algebra by \hbar .

We now choose the following infinite sequence of subalgebras of the Virasoro algebra:

$$[L_m^M, L_n^M] = (m - n)L_{m+n}^M, \quad m, n \geq N, \tag{4.34}$$

for arbitrary fixed integer $N \geq -1$. We will construct quantum Airy structures for each choice of such subalgebra, using the representation of the Virasoro algebra (4.33). To do so, we need to bring the operators $\hbar L_m^M$ in the form of quantum Airy structures by creating appropriate linear terms.

For our first class of quantum Airy structures, we shift indices so that for any choice of subalgebra (choice of N) our operators are indexed by integers $i = 0, 1, 2, \dots$. Thus, given an integer $N \geq -1$, we define the quadratic hamiltonians:

$$\forall i \in \mathbb{Z}_{\geq 0}, \quad H_i^2 := \hbar L_{i+N}^M = \frac{\hbar}{2} \sum_{k \in \mathbb{Z}} \circ b_k^M b_{i+N-k}^M \circ, \tag{4.35}$$

which have commutation relations:

$$[H_i^2, H_j^2] = \hbar(i - j)H_{i+j+N}^2. \tag{4.36}$$

To create appropriate linear terms, we do the shift $b_N \mapsto b_N + \frac{1}{\sqrt{\hbar}}$, which creates a linear term $H_i^1 = \sqrt{\hbar} b_i = \hbar \partial_i$ in the hamiltonians without changing the commutation relations. It however also creates an unwanted constant term $H_N^0 = \frac{1}{2}$, but since H_N

never appears in the right-hand-side of the commutation relations (4.36), we can get rid of this constant term without changing the algebra. We thus obtain hamiltonians:

$$\forall i \in \mathbb{Z}_{\geq 0}, \quad H_i = \hbar \partial_i + \frac{\hbar}{2} \sum_{k \in \mathbb{Z}} \circ b_k^M b_{i+N-k}^M \circ, \tag{4.37}$$

which have the form of a quantum Airy structure. However, this quantum Airy structure has no “A” or “D” terms (see (2.23)), and hence its associated partition function Z is trivial: that is, $Z = 1$ (see (2.30)). For $N = -1$, there is not much that we can do. However, for $N \geq 0$ we can make the partition function non-trivial. Looking at the commutation relations (4.36), we see that the operators $H_i, i = 0, \dots, N$ never appear on the right-hand-side of the commutation relations. Thus we can add quadratic “D” terms of the form $\hbar D_i$ to those operators without changing the commutation relations (since these terms commute with all H_i). We thus obtain our first class of quantum Airy structures. For convenience, in this case we redefine $N \mapsto N - 1$, so that the operators take the same form as for the other cases. We then have $N \geq 0$. We obtain the quantum Airy structure (4.28), as a representation of the algebra (4.27).

For our second class of quantum Airy structures, we shift indices so that for any N our operators are indexed by integers $i = 1, 2, \dots$:

$$\forall i \in \mathbb{Z}_{\geq 1}, \quad H_i^2 := \hbar L_{i+N-1}^M = \frac{\hbar}{2} \sum_{k \in \mathbb{Z}} \circ b_k^M b_{i+N-1-k}^M \circ, \tag{4.38}$$

with commutation relations:

$$[H_i^2, H_j^2] = \hbar(i - j)H_{i+j+N-1}^2. \tag{4.39}$$

To create linear terms, we do the shift $b_{N-1} \mapsto b_{N-1} + \frac{1}{\sqrt{\hbar}}$, which creates a linear term $H_i^1 = \sqrt{\hbar}b_i = \hbar \partial_i$ in the hamiltonians without changing the commutation relations. We get rid of the unwanted constant term $H_{N-1}^0 = \frac{1}{2}$ as above, and introduce non-trivial D -terms $\hbar D_i$ for $i = 1, \dots, N + 1$, since the corresponding $H_i, i = 1, \dots, N + 1$ do not appear on the right-hand-side of the commutation relations. This is not quite an Airy structure though, since the hamiltonians depend on the variable x^0 , which does not appear in the linear terms. We notice however that the H_i only depend on the derivative ∂_0 through the bosonic zero mode b_0^M , they do not depend on the variable x^0 itself. Thus, we can introduce an auxiliary operator

$$\mathcal{H}_0 = \hbar \partial_0 + \frac{\hbar^2}{2} C_0 \partial_0^2 + \hbar D_0, \tag{4.40}$$

which commutes with all $H_i, i \geq 0$, to get a quantum Airy structure. The result is the quantum Airy structure (4.29), as a representation of the algebra (4.30).

For our third class of quantum Airy structures, we modify the representation of the Heisenberg algebra slightly, and set the bosonic zero mode $b_0^M = 0$. That is, we consider a “momentum zero” representation. To get a quantum Airy structure, we do the same manipulations as for the second class above. We end up with the quantum Airy structure (4.31), as a representation of the algebra (4.27). \square

Before we proceed, a few remarks are in order.

Remark 4.3. Note that from the point of view of partition functions, the third class can be understood as a special case of the second class. Indeed, consider the second class of quantum Airy structures with $C_0 = D_0 = 0$. The constraint $\mathcal{H}_0 Z = 0$ implies that Z does not depend on the variable x^0 . Then any term in the constraints $H_i Z = 0$ that involves b_0^M identically vanishes, and hence the constraints $H_i Z = 0$ become identical to the constraints of the third class of quantum Airy structures. Therefore, the partition function of the third class is equal to the partition function of the second class with $C_0 = D_0 = 0$.

Remark 4.4. In the construction above we introduced the shifts $\hbar D_i$ as defining a new representation for the Virasoro subalgebra. We could however take a different viewpoint. We could stick with the representation without the added D -terms, and then, instead of solving the constraints $H_i Z = 0$ to define the partition function, we could solve the constraints

$$H_i Z = -\hbar D_i Z, \tag{4.41}$$

with possibly non-vanishing D_i in the allowed range (for instance $i = 0, \dots, N - 1$ in the first class). This is entirely equivalent to what we did above, but from the point of view of vertex operator algebras, by solving these constraints we would be constructing so-called “Whittaker modules” for the Virasoro algebra. We are currently investigating this connection further.

It turns out that the quantum Airy structures in the third class of Proposition 4.2 have appeared in the literature before, in a different form. Indeed, in Section 10 of [4], a variant of the Eynard–Orantin topological recursion was constructed, the so-called “topological recursion without branched covers”. We now show that the quantum Airy structures of Proposition 4.2 are examples of this topological recursion without branched covers. We thus obtain a realization of the topological recursion without branched covers in terms of representations of the free boson VOA, which sheds light on its origin. It remains to be seen however whether the associated partition functions compute interesting enumerative invariants, and whether the other quantum Airy structures of Proposition 4.2 have similar realizations.

Proposition 4.5. *The data of a spectral curve for the topological recursion without branched covers of Section 10 of [4] is given by:*

- A Riemann surface Σ ;
- A meromorphic one-form $\omega_{0,1}$ on Σ ;
- A bilinear differential $\omega_{0,2}$ on Σ^2 ;
- A finite subset $\tau \subset \Sigma$ such that $\omega_{0,1}$ has at most simple zeros on τ ;
- A meromorphic one-form $\omega_{1,1}$ on Σ such that for any point $p \in \tau$, $z^2 \frac{\omega_{1,1}(z)}{\omega_{0,1}(z)}$ is holomorphic at p , where z is a local coordinate near p .

Consider the spectral curve given by the data:

$$\begin{aligned} \Sigma = \mathbf{P}^1, \quad \omega_{0,1}(z) &= -\frac{dz}{z^N}, \quad \omega_{0,2}(z_1, z_2) = \frac{dz_1 dz_2}{(z_1 - z_2)^2}, \quad \tau = \{0\}, \\ \omega_{1,1}(z) &= -\sum_{k=1}^{N+1} D_k \frac{dz}{z^{k+1}}, \end{aligned} \tag{4.42}$$

where z is a coordinate on \mathbb{P}^1 and $N \in \mathbb{Z}_{\geq 0}$. Then, for any $N \geq 0$, the quantum Airy structure associated to the topological recursion without branched cover on this spectral curve, which was constructed in Section 10 of [4], precisely corresponds to the quantum Airy structure in the third class of Proposition 4.2.

Proof. Following [4], we define, for $k \in \mathbb{Z}_{\geq 1}$,¹⁴

$$\xi_k(z) = \frac{dz}{z^{k+1}}, \quad \xi_k^*(z) = z^k, \quad \theta(z) := \frac{1}{\omega_{0,1}(z)} = -\frac{z^N}{dz}. \tag{4.43}$$

According to the recipe of Section 10 in [4] (see also Proposition 8.13), the coefficients of the quantum Airy structure associated to topological recursion without branched covers can be calculated as follows. The D -coefficients can be obtained by expanding the one-form $\omega_{1,1}(z)$ in the basis of differentials $\xi_k(z)$:

$$\omega_{1,1}(z) = -\sum_{k=1}^{N+1} D_k \xi_k(z). \tag{4.44}$$

As for A, B, C , they can be calculated as:

$$A_{ijk} = \operatorname{Res}_{z \rightarrow 0}(\xi_i^*(z) d\xi_j^*(z) d\xi_k^*(z) \theta(z)), \tag{4.45}$$

$$B_{ij}^k = \operatorname{Res}_{z \rightarrow 0}(\xi_i^*(z) d\xi_j^*(z) \xi_k(z) \theta(z)), \tag{4.46}$$

$$C_i^{jk} = \operatorname{Res}_{z \rightarrow 0}(\xi_i^*(z) \xi_j(z) \xi_k(z) \theta(z)). \tag{4.47}$$

Substituting (4.43) in these equations, we obtain:

$$A_{ijk} = 0, \quad B_{ij}^k = -j \delta_{i+j+N-1,k}, \quad C_i^{jk} = -\delta_{i+N-j-1,k}. \tag{4.48}$$

Thus, the resulting differential operators are, for $i \in \mathbb{Z}_{\geq 1}$,

$$H_i = \hbar \frac{\partial}{\partial x^i} + \hbar \sum_{k-l=-i-N+1} kx^k \partial_l + \frac{\hbar^2}{2} \sum_{k+l=i+N-1} \partial_k \partial_l + \hbar D_i \delta_{i \leq N+1}, \tag{4.49}$$

with $k, l \in \mathbb{Z}_{\geq 1}$, which can be rewritten as

$$H_i = \hbar \partial_i + \frac{\hbar}{2} \sum_{k \in \mathbb{Z}} \circ b_k^M b_{i+N-1-k}^M \circ + \hbar D_i \delta_{i \leq N+1}, \tag{4.50}$$

with

$$b_k^M = \sqrt{\hbar} \partial_k, \quad b_{-k}^M = \frac{1}{\sqrt{\hbar}} kx^k, \quad b_0^M = 0, \tag{4.51}$$

for all $k \in \mathbb{Z}_{\geq 1}$. Those are precisely the differential operators H_i of Proposition 4.2. \square

¹⁴ Note that our definition for ξ_k and ξ_{k+1}^* is rescaled by $(k+1)^{-1}$ and k respectively with respect to [4].

4.1.4. *Quantum Airy Structures from \mathbb{Z}_2 -Twisted Representations of the Free Boson VOA*
 We now construct another class of quantum Airy structures obtained from the σ -twisted representation of the free boson VOA (see Lemma 4.1). This is an example of the general construction of [11] for W -algebras, although only the cases with $N = -1$ and $N = 0$ (and their generalizations to W -algebras) were considered there.

Proposition 4.6. *Let N be any fixed integer $N \geq -1$. We represent the Heisenberg algebra for the twisted bosonic modes as:*

$$\forall r \in \mathbb{Z}_{\geq 0} + \frac{1}{2}, \quad b_r^\sigma = \sqrt{\hbar} \partial_{r+\frac{1}{2}}, \quad b_{-r}^\sigma = \frac{r}{\sqrt{\hbar}} x^{r+\frac{1}{2}}. \tag{4.52}$$

Let $\{x^1, x^2, x^3, \dots\}$ be a basis for V , with dual set $\{y_1, y_2, y_3, \dots\}$. The linear operator $H : V^* \rightarrow \widehat{\mathcal{W}}_{\hbar}(V)$ defined by:

$$\forall i \in \mathbb{Z}_{\geq 1},$$

$$H_i := H(y_i) = \hbar \partial_i + \frac{\hbar}{2} \sum_{r \in \mathbb{Z} + \frac{1}{2}} \circ b_r^\sigma b_{i+N-1-r}^\sigma \circ + \frac{\hbar}{16} \delta_{i,1-N} + \hbar D_i \delta_{i \leq N+1}, \tag{4.53}$$

for arbitrary constants D_i , $i = 1, \dots, N + 1$, forms a quantum Airy structure as a representation of the Virasoro subalgebra

$$[H_i, H_j] = \hbar(i - j)H_{i+j+N-1}. \tag{4.54}$$

Proof. We start with the σ -twisted representation of the free boson VOA. The twisted bosonic modes, see (4.14), form the Heisenberg algebra $[b_r^\sigma, b_s^\sigma] = r \delta_{r,-s}$, where $r, s \in \mathbb{Z} + \frac{1}{2}$. We represent the twisted bosonic modes as endomorphisms on the space $\mathbb{K}[[V, \hbar]]$ as:

$$\forall r \in \mathbb{Z}_{\geq 0} + \frac{1}{2}, \quad b_r^\sigma = \sqrt{\hbar} \partial_{r+\frac{1}{2}}, \quad b_{-r}^\sigma = \frac{r}{\sqrt{\hbar}} x^{r+\frac{1}{2}}. \tag{4.55}$$

Note that there is no choice of bosonic zero mode here, because the bosonic field is twisted by the \mathbb{Z}_2 automorphism. From (4.23), the Virasoro modes take the form:

$$L_m^\sigma = \frac{1}{2} \sum_{r \in \mathbb{Z} + \frac{1}{2}} \circ b_r^\sigma b_{m-r}^\sigma \circ + \frac{1}{16} \delta_{m,0} \tag{4.56}$$

Now as in Proposition 4.2, we choose the subalgebra

$$[L_m^\sigma, L_n^\sigma] = (m - n)L_m^\sigma, \quad m, n \geq N, \tag{4.57}$$

for an arbitrary fixed integer $N \geq -1$. We now shift indices so that our operators are indexed by integers $i = 1, 2, 3, \dots$ for any N . For any $N \geq -1$, we define the quadratic hamiltonians:

$$H_i^2 := \hbar L_{i+N-1}^\sigma = \frac{\hbar}{2} \sum_{r \in \mathbb{Z} + \frac{1}{2}} \circ b_r^\sigma b_{i+N-1-r}^\sigma \circ + \frac{\hbar}{16} \delta_{i,1-N}. \tag{4.58}$$

Those have commutation relations:

$$[H_i^2, H_j^2] = \hbar(i - j)H_{i+j+N-1}^2. \tag{4.59}$$

To add linear terms, we consider the shift $b_{N-\frac{1}{2}} \mapsto b_{N-\frac{1}{2}} + \frac{1}{\sqrt{\hbar}}$, which creates linear terms $H_i^1 = \sqrt{\hbar} b_{i-\frac{1}{2}}^\sigma = \hbar \partial_i$. It also creates a constant term $H_N^0 = \frac{1}{2}$, but as usual we get rid of it without changing the algebra since H_N does not appear on the right-hand-side of the commutation relations. We also add D -terms $\hbar D_i$ for $i = 1, \dots, N + 1$ to the operators H_i , $i = 1, \dots, N + 1$, since they do not appear on the right-hand-side of the commutation relations. The resulting quantum Airy structures are (4.53), as representations of the algebra (4.54). \square

These quantum Airy structures are known to produce interesting enumerative invariants:

- For $N = -1$, the quantum Airy structure takes the explicit form

$$\begin{aligned} \forall i \in \mathbb{Z}_{\geq 1}, \\ H_i = \hbar \partial_i + \frac{\hbar^2}{2} \sum_{j=1}^{i-2} \partial_j \partial_{i-1-j} + \frac{\hbar}{2} \sum_{j=i-1}^{\infty} (2j - 2i + 3) x^{j-i+2} \partial_j \\ + \frac{1}{8} (x^1)^2 \delta_{i,1} + \frac{\hbar}{16} \delta_{i,2}, \end{aligned} \tag{4.60}$$

where the second sum is understood to vanish for terms with $j < 1$. This the quantum Airy structure associated to the Eynard–Orantin topological recursion on the Airy spectral curve (up to trivial rescaling of the variables x^i) [37,38]. The constraints $H_i Z = 0$ reproduce the well-known Virasoro constraints for the Kontsevich–Witten tau-function of the KdV hierarchy, and Z is a generating function for intersection numbers on the moduli space of curves [30,51,65].

- For $N = 0$, the quantum Airy structure takes the form:

$$\begin{aligned} \forall i \in \mathbb{Z}_{\geq 1}, \quad H_i = \hbar \partial_i + \frac{\hbar^2}{2} \sum_{j=1}^{i-1} \partial_j \partial_{i-j} + \frac{\hbar}{2} \sum_{j=i}^{\infty} (2j - 2i + 1) x^{j-i+1} \partial_j \\ + \frac{\hbar}{16} \delta_{i,1} + \hbar D_1 \delta_{i,1}. \end{aligned} \tag{4.61}$$

For $D_1 = 0$, this is the quantum Airy structure associated to the Eynard–Orantin topological recursion on the Bessel spectral curve (up to rescaling of variables x^i) [32]. In this case, the constraints $H_i Z = 0$ reproduce the Virasoro constraints for the Brézin–Gross–Witten tau-function of the KdV hierarchy [1,23,45,56]. Z is now a generating function for intersection numbers on the moduli space of curves involving Norbury’s cohomology class [33,59].

- For arbitrary $N \geq 1$, the quantum Airy structure reads ($\forall i \in \mathbb{Z}_{\geq 1}$):

$$\begin{aligned} H_i = \hbar \partial_i + \frac{\hbar^2}{2} \sum_{j=1}^{i+N-1} \partial_j \partial_{i+N-j} \\ + \frac{\hbar}{2} \sum_{j=i+N}^{\infty} (2j - 2i - 2N + 1) x^{j-i-N+1} \partial_j + \hbar D_i \delta_{i \leq N+1}. \end{aligned} \tag{4.62}$$

It is at the moment unknown whether the partition function that it computes has an interesting enumerative interpretation. Remark that for $N \geq 1$, the partition function

is non-trivial only if the D_i do not all vanish. In this case, is it a generating function for some intersection numbers on the moduli space curves? Is Z a tau-function for the KdV hierarchy? These questions certainly deserve further investigation.

4.2. Super Quantum Airy Structures from the Free Boson–Fermion VOSA. Along the same lines as the bosonic construction of the previous section, we now construct classes of examples of infinite-dimensional, quadratic, super quantum Airy structures as representations of subalgebras of the super Virasoro algebra with central charge $c = \frac{3}{2}$:

$$\begin{aligned} [L_m, L_n] &= (m - n)L_{n+m} + \delta_{m,-n} \frac{1}{8} m(m^2 - 1), \\ [L_n, G_r] &= \left(\frac{n}{2} - r\right) G_{n+r}, \\ \{G_r, G_s\} &= 2L_{r+s} + \delta_{r,-s} \frac{1}{2} \left(r^2 - \frac{1}{4}\right). \end{aligned} \tag{4.63}$$

We will construct super quantum Airy structures as representations of subalgebras of the super Virasoro algebra in both the Neveu-Schwarz (NS) sector (where r, s are half-integers) and the Ramond sector (where r, s are integers).

Our main tool is the free boson–fermion vertex operator super algebra (VOSA), which has $N = 1$ supersymmetry. We will construct our super quantum Airy structures from untwisted and twisted representations of the free boson–fermion VOSA. So let us first review the main features of this theory. We refer the reader to [6, 42, 49] for more details.

Remark 4.7. In this section we construct classes of super quantum Airy structures as representations of subalgebra of the super Virasoro algebra in the NS and Ramond sectors. A natural question then is whether the associated partition functions Z compute interesting enumerative invariants. This is unclear at the moment, and certainly deserves further investigation. For instance, it would be very interesting to see whether these super quantum Airy structures are related to the supersymmetric generalization of JT gravity and Mirzakhani’s recursion presented in [64].

We also remark that we only consider the free boson–fermion VOSA here, which has $N = 1$ supersymmetry. But it would interesting to investigate whether super quantum Airy structures can be constructed as representations of VOSAs with $N = 2$ supersymmetry as well.

4.2.1. The Free Boson–Fermion VOSA The free boson VOA was introduced in Section 4.1.1. Let us now introduce the free fermion VOSA. It is again generated by a single vector $\psi_{-\frac{1}{2}} |0\rangle \in V_f$, where V_f is the space of states. Here $|0\rangle \in V_f$ is the vacuum vector. The state-operator correspondence reads:

$$Y(\psi_{-\frac{1}{2}} |0\rangle, z) = \sum_{m \in \mathbb{Z}} \psi_{m+\frac{1}{2}} z^{-m-1} = \sum_{r \in \mathbb{Z}+\frac{1}{2}} \psi_r z^{-r-\frac{1}{2}}, \tag{4.64}$$

where the modes ψ_r generate the Clifford algebra $\{\psi_r, \psi_s\} = \delta_{r,-s}$.

The vacuum vector $|0\rangle$ is annihilated by all ψ_k with $k > 0$, and the space of states V_f is the Fock space of all excited states

$$\psi_{-k_1} \dots \psi_{-k_n} |0\rangle, \quad k_1, \dots, k_n \in \mathbb{Z}_{\geq 0} + \frac{1}{2}. \tag{4.65}$$

Define a normal ordering on the fermionic modes as:

$$\circ \psi_r \psi_s \circ = \begin{cases} \psi_r \psi_s & \text{for } r \leq s, \\ -\psi_s \psi_r & \text{for } r > s. \end{cases} \quad (4.66)$$

Then the operators corresponding to the states in the Fock space are:

$$\begin{aligned} & Y(\psi_{-k_1} \cdots \psi_{-k_n} | 0 \rangle, z) \\ &= \frac{1}{(k_1 - \frac{1}{2})!} \left(\frac{d}{dz} \right)^{k_1 - \frac{1}{2}} Y(\psi_{-\frac{1}{2}} | 0 \rangle, z) \cdots \\ & \frac{1}{(k_n - \frac{1}{2})!} \left(\frac{d}{dz} \right)^{k_n - \frac{1}{2}} Y(\psi_{-\frac{1}{2}} | 0 \rangle, z) \circ. \end{aligned} \quad (4.67)$$

The conformal vector $|\omega\rangle$ for the free fermion VOSA reads:

$$|\omega\rangle = \frac{1}{2} \psi_{-\frac{3}{2}} \psi_{-\frac{1}{2}} | 0 \rangle. \quad (4.68)$$

Its operator takes the form

$$Y(|\omega\rangle, z) = \sum_{m \in \mathbb{Z}} L_m z^{-m-2}, \quad (4.69)$$

with its modes generating the Virasoro algebra with central charge $c = \frac{1}{2}$:

$$[L_m, L_n] = (m - n)L_{m+n} + \frac{1}{24}m(m^2 - 1)\delta_{m, -n}. \quad (4.70)$$

The Virasoro modes are related to the modes of the fermionic field as follows. From (4.67), we have:

$$\begin{aligned} Y(|\omega\rangle, z) &= \frac{1}{2} \circ \left(\frac{d}{dz} Y(\psi_{-\frac{1}{2}} | 0 \rangle, z) \right) Y(\psi_{-\frac{1}{2}} | 0 \rangle, z) \circ \\ &= -\frac{1}{2} \sum_{r_1, r_2 \in \mathbb{Z} + \frac{1}{2}} \binom{r_1 + \frac{1}{2}}{r_2} \circ \psi_{r_1} \psi_{r_2} \circ z^{-r_1 - r_2 - 2}. \end{aligned} \quad (4.71)$$

Thus

$$\begin{aligned} L_m &= -\frac{1}{2} \sum_{r \in \mathbb{Z} + \frac{1}{2}} \binom{r + \frac{1}{2}}{r} \circ \psi_r \psi_{m-r} \circ \\ &= \frac{1}{2} \sum_{r \in \mathbb{Z} + \frac{1}{2}} \binom{r + \frac{m}{2}}{r} \circ \psi_{-r} \psi_{r+m} \circ, \end{aligned} \quad (4.72)$$

where the second equality follows from a straightforward calculation.

Now let V_b be a free boson VOA, and V_f be a free fermion VOSA. Let us consider their tensor product $V = V_b \otimes V_f$. More precisely, the vector space is a Fock space of all states excited by bosonic modes b_{-n} and fermionic modes ψ_{-r} , where we assume

$[b_n, \psi_r] = 0$ for any $n \in \mathbb{Z}$ and $r \in \mathbb{Z} + \frac{1}{2}$. The state-operator correspondence Y_V satisfies $Y_V(|u\rangle \otimes |v\rangle, z) = Y_{V_b}(|u\rangle, z) \otimes Y_{V_f}(|v\rangle, z)$ for $|u\rangle \in V_b$ and $|v\rangle \in V_f$.

We define the vacuum vector of the combined theory as $|0\rangle = |0\rangle_{V_b} \otimes |0\rangle_{V_f}$. From now on we will omit the tensor product symbols for clarity. The conformal vector $|\omega\rangle$ for V reads:

$$|\omega\rangle = \frac{1}{2} \left(b_{-1} b_{-1} + \psi_{-\frac{3}{2}} \psi_{-\frac{1}{2}} \right) |0\rangle. \tag{4.73}$$

Its modes are given by the sum of (4.10) and (4.72):

$$L_n = \frac{1}{2} \sum_{k \in \mathbb{Z}} \circ b_k b_{n-k} \circ + \frac{1}{2} \sum_{r \in \mathbb{Z} + \frac{1}{2}} \left(r + \frac{n}{2} \right) \circ \psi_{-r} \psi_{n+r} \circ. \tag{4.74}$$

Those generate the Virasoro algebra with central charge $c = 3/2$:

$$[L_m, L_n] = (m - n) L_{m+n} + \frac{1}{8} m(m^2 - 1) \delta_{m, -n}. \tag{4.75}$$

It turns out that this theory has $N = 1$ supersymmetry. This means that it has a superconformal vector $|\tau\rangle$, whose modes, together with the Virasoro modes, generate a super Virasoro algebra in the NS sector. For the free boson–fermion VOSA, the superconformal vector is [6]:

$$|\tau\rangle = b_{-1} \psi_{-\frac{1}{2}} |0\rangle. \tag{4.76}$$

Its operator reads:

$$Y(|\tau\rangle, z) = \sum_{m \in \mathbb{Z}} G_{m-\frac{1}{2}} z^{-m-1} = \sum_{r \in \mathbb{Z} + \frac{1}{2}} G_r z^{-r-\frac{3}{2}}, \tag{4.77}$$

with the modes G_r given by

$$G_r = \sum_{m \in \mathbb{Z}} \psi_{r-m} b_m. \tag{4.78}$$

These G_r , together with the L_m of (4.74), generate an $N = 1$ super Virasoro algebra in the NS sector with central charge $3/2$:

$$\begin{aligned} [L_m, L_n] &= (m - n) L_{n+m} + \delta_{m, -n} \frac{1}{8} m(m^2 - 1), \\ [L_n, G_r] &= \left(\frac{n}{2} - r \right) G_{n+r}, \\ \{G_r, G_s\} &= 2L_{r+s} + \delta_{r, -s} \frac{1}{2} \left(r^2 - \frac{1}{4} \right). \end{aligned} \tag{4.79}$$

We call this theory the *free boson–fermion VOSA*.

4.2.2. *Untwisted and Twisted Representations for the Free Boson–Fermion VOSA* Our goal is now to construct super quantum Airy structures as representations of subalgebras of the super Virasoro algebra (in both NS and Ramond sectors). To this end, we will construct untwisted and twisted representations for the free boson–fermion VOSA.

We will construct four different representations. The first one is the untwisted one, which is obtained directly from the natural representation of the super Heisenberg algebra. The super Virasoro modes take the form (4.74) and (4.78) in terms of the representation of the bosonic and fermionic modes.

For the three twisted ones, we will use three distinct \mathbb{Z}_2 automorphisms of the VOSA:

- (1) The σ -twisted representation is obtained using the \mathbb{Z}_2 automorphism $\sigma : V \rightarrow V$ that we already studied in Lemma 4.1, extended trivially to the fermionic sector. It acts on the Fock space of the free boson–fermion VOSA as follows:

$$\begin{aligned} \sigma : & b_{-k_1} \cdots b_{-k_m} \psi_{-r_1} \cdots \psi_{-r_n} |0\rangle \\ \mapsto & (-1)^{\sum_{i=1}^m k_i} b_{-k_1} \cdots b_{-k_m} \psi_{-r_1} \cdots \psi_{-r_n} |0\rangle. \end{aligned} \tag{4.80}$$

In other words, it acts on the bosons as before, but leaves the fermions invariant. It preserves the vacuum vector $|0\rangle$ and the conformal vector $|\omega\rangle$. However, it twists the boson $b_{-1} |0\rangle$, and also the superconformal vector $|\tau\rangle = b_{-1} \psi_{-\frac{1}{2}} |0\rangle$. Therefore, the σ -twisted representation will product a representation of the super Virasoro algebra in the Ramond sector.

- (2) The μ -twisted representation is obtained using the parity \mathbb{Z}_2 automorphism $\mu : V \rightarrow V$, which sends odd vectors to minus themselves and keeps even vectors invariant. On the Fock space, it acts as:

$$\begin{aligned} \mu : & b_{-k_1} \cdots b_{-k_m} \psi_{-r_1} \cdots \psi_{-r_n} |0\rangle \\ \mapsto & (-1)^n b_{-k_1} \cdots b_{-k_m} \psi_{-r_1} \cdots \psi_{-r_n} |0\rangle. \end{aligned} \tag{4.81}$$

It preserves the vacuum vector $|0\rangle$ and the conformal vector $|\omega\rangle$, but it twists the fermion $\psi_{-\frac{1}{2}} |0\rangle$ and the superconformal vector $|\tau\rangle = b_{-1} \psi_{-\frac{1}{2}} |0\rangle$. We will then again obtain a representation of the super Virasoro algebra in the Ramond sector.

- (3) Our last twisted representation is the ρ -twisted representation, with $\rho = \sigma \circ \mu : V \rightarrow V$, where we combine both automorphisms. The combined \mathbb{Z}_2 automorphism acts on the Fock space as:

$$\begin{aligned} \sigma \circ \mu : & b_{-k_1} \cdots b_{-k_m} \psi_{-r_1} \cdots \psi_{-r_n} |0\rangle \\ \mapsto & (-1)^{n+\sum_{i=1}^m k_i} b_{-k_1} \cdots b_{-k_m} \psi_{-r_1} \cdots \psi_{-r_n} |0\rangle. \end{aligned} \tag{4.82}$$

As usual, it keeps the vacuum and the conformal vectors invariant. While it twists both the boson vector $b_{-1} |0\rangle$ and the fermion vector $\psi_{-\frac{1}{2}} |0\rangle$, it keeps the superconformal vector $|\tau\rangle = b_{-1} \psi_{-\frac{1}{2}} |0\rangle$ invariant. Thus we will get a representation of the super Virasoro algebra in the NS sector.

Let us now calculate the super Virasoro modes for all three \mathbb{Z}_2 -twisted representations.

Lemma 4.8. • *For the σ -twisted representation, the super Virasoro modes (in the Ramond sector) take the form*

$$L_m^\sigma = \frac{1}{2} \sum_{r \in \mathbb{Z} + \frac{1}{2}} \circ b_r^\sigma b_{m-r}^\sigma \circ + \frac{1}{2} \sum_{r \in \mathbb{Z} + \frac{1}{2}} \left(r + \frac{m}{2}\right) \circ \psi_{-r}^\sigma \psi_{m+r}^\sigma \circ + \frac{1}{16} \delta_{m,0}, \tag{4.83a}$$

$$G_m^\sigma = \sum_{r \in \mathbb{Z} + \frac{1}{2}} b_{m-r}^\sigma \psi_r^\sigma, \tag{4.83b}$$

with $m \in \mathbb{Z}$, in terms of the twisted bosonic modes and untwisted fermionic modes:

$$Y^\sigma(b_{-1}|0\rangle, z) = \sum_{r \in \mathbb{Z} + \frac{1}{2}} b_r^\sigma z^{-r-1}, \quad Y^\sigma(\psi_{-\frac{1}{2}}|0\rangle, z) = \sum_{r \in \mathbb{Z} + \frac{1}{2}} \psi_r^\sigma z^{-r-\frac{1}{2}}. \tag{4.84}$$

- For the μ -twisted representation, the super Virasoro modes (in the Ramond sector) take the form

$$L_m^\mu = \frac{1}{2} \sum_{i \in \mathbb{Z}} \circ b_i^\mu b_{m-i}^\mu \circ + \frac{1}{2} \sum_{i \in \mathbb{Z}} \left(i + \frac{m}{2}\right) \circ \psi_{-i}^\mu \psi_{m+i}^\mu \circ + \frac{1}{16} \delta_{m,0}, \tag{4.85a}$$

$$G_m^\mu = \sum_{i \in \mathbb{Z}} b_{m-i}^\mu \psi_i^\mu, \tag{4.85b}$$

with $m \in \mathbb{Z}$, in terms of the untwisted bosonic modes and twisted fermionic modes:

$$Y^\mu(b_{-1}|0\rangle, z) = \sum_{m \in \mathbb{Z}} b_m^\mu z^{-m-1}, \quad Y^\mu(\psi_{-\frac{1}{2}}|0\rangle, z) = \sum_{n \in \mathbb{Z}} \psi_n^\mu z^{-n-\frac{1}{2}}. \tag{4.86}$$

- For the $\rho = \sigma \circ \mu$ -twisted representation, the super Virasoro modes (in the NS sector) take the form:

$$L_m^\rho = \frac{1}{2} \sum_{r \in \mathbb{Z} + \frac{1}{2}} \circ b_r^\rho b_{m-r}^\rho \circ + \frac{1}{2} \sum_{i \in \mathbb{Z}} \left(i + \frac{m}{2}\right) \circ \psi_{-i}^\rho \psi_{m+i}^\rho \circ + \frac{1}{8} \delta_{m,0}, \tag{4.87a}$$

$$G_r^\rho = \sum_{s \in \mathbb{Z} + \frac{1}{2}} b_s^\rho \psi_{r-s}^\rho, \tag{4.87b}$$

with $m \in \mathbb{Z}$ and $r \in \mathbb{Z} + \frac{1}{2}$, in terms of the twisted bosonic modes and twisted fermionic modes:

$$Y^\rho(b_{-1}|0\rangle, z) = \sum_{r \in \mathbb{Z} + \frac{1}{2}} b_r^\rho z^{-r-1}, \quad Y^\rho(\psi_{-\frac{1}{2}}|0\rangle, z) = \sum_{n \in \mathbb{Z}} \psi_n^\rho z^{-n-\frac{1}{2}}. \tag{4.88}$$

Proof. For the σ -twisted representation, the boson is twisted, while the fermion is not:

$$Y^\sigma(b_{-1}|0\rangle, z) = \sum_{r \in \mathbb{Z} + \frac{1}{2}} b_r^\sigma z^{-r-1}, \quad Y^\sigma(\psi_{-\frac{1}{2}}|0\rangle, z) = \sum_{r \in \mathbb{Z} + \frac{1}{2}} \psi_r^\sigma z^{-r-\frac{1}{2}}. \tag{4.89}$$

The calculation of the conformal field $Y^\sigma(|\omega\rangle, z)$ is the same as in Lemma 4.1, since only the boson is twisted. The superconformal field $Y^\sigma(|\tau\rangle, z)$ is twisted, and we get:

$$Y^\sigma(|\tau\rangle, z) = \sum_{m \in \mathbb{Z}} G_m^\sigma z^{-m-\frac{3}{2}}$$

$$\begin{aligned}
 &= Y^\sigma(b_{-1} | 0 \rangle, z) Y^\sigma(\psi_{-\frac{1}{2}, z}) \\
 &= \sum_{r, s \in \mathbb{Z} + \frac{1}{2}} b_r^\sigma \psi_s^\sigma z^{-r-s-\frac{3}{2}}.
 \end{aligned} \tag{4.90}$$

For the μ -twisted representation, the boson is untwisted, while the fermion is:

$$Y^\mu(b_{-1} | 0 \rangle, z) = \sum_{m \in \mathbb{Z}} b_m^\mu z^{-m-1}, \quad Y^\mu(\psi_{-\frac{1}{2}} | 0 \rangle, z) = \sum_{n \in \mathbb{Z}} \psi_n^\mu z^{-n-\frac{1}{2}}. \tag{4.91}$$

The bosonic part of the conformal field is untwisted. For the fermionic part, the calculation follows along the same lines as in Lemma 4.1. We will omit it for brevity. The superconformal field $Y^\mu(|\tau\rangle, z)$ is twisted, and we get:

$$\begin{aligned}
 Y^\mu(|\tau\rangle, z) &= \sum_{m \in \mathbb{Z}} G_m^\mu z^{-m-\frac{3}{2}} \\
 &= Y^\mu(b_{-1} | 0 \rangle, z) Y^\mu(\psi_{-\frac{1}{2}, z}) \\
 &= \sum_{i, j \in \mathbb{Z}} b_i^\mu \psi_j^\mu z^{-i-j-\frac{3}{2}}.
 \end{aligned} \tag{4.92}$$

For the $\rho = \sigma \circ \mu$ -twisted representation, both boson and fermion are twisted:

$$Y^\rho(b_{-1} | 0 \rangle, z) = \sum_{r \in \mathbb{Z} + \frac{1}{2}} b_r^\rho z^{-r-1}, \quad Y^\rho(\psi_{-\frac{1}{2}} | 0 \rangle, z) = \sum_{n \in \mathbb{Z}} \psi_n^\rho z^{-n-\frac{1}{2}}. \tag{4.93}$$

The conformal field is calculated by combining the calculations for the σ -twisted and μ -twisted representation. As for the superconformal field, it is untwisted, and we get:

$$\begin{aligned}
 Y^\rho(|\tau\rangle, z) &= \sum_{r \in \mathbb{Z} + \frac{1}{2}} G_r^\rho z^{-r-\frac{3}{2}} \\
 &= Y^\rho(b_{-1} | 0 \rangle, z) Y^\rho(\psi_{-\frac{1}{2}, z}) \\
 &= \sum_{s \in \mathbb{Z} + \frac{1}{2}} \sum_{m \in \mathbb{Z}} b_s^\rho \psi_m^\rho z^{-m-s-\frac{3}{2}}.
 \end{aligned} \tag{4.94}$$

□

Let us now construct classes of super quantum Airy structures using these four representations.

4.2.3. Super Quantum Airy Structures from Untwisted Representations of the Free Boson–Fermion VOSA We first construct super quantum Airy structures from the untwisted representation of the free boson–fermion VOSA.

Proposition 4.9. *We represent the super Heisenberg algebra of untwisted bosonic and fermionic modes as:*

$$\forall m \in \mathbb{Z}_{\geq 1}, \quad b_m^M = \sqrt{\hbar} \frac{\partial}{\partial x^m}, \quad b_{-m}^M = \frac{m}{\sqrt{\hbar}} x^m, \quad b_0^M = \sqrt{\hbar} \frac{\partial}{\partial x^0},$$

$$\forall r \in \mathbb{Z}_{\geq 0} + \frac{1}{2}, \quad \psi_r^M = \sqrt{\hbar} \frac{\partial}{\partial \theta^{r+\frac{1}{2}}}, \quad \psi_{-r}^M = \frac{1}{\sqrt{\hbar}} \theta^{r+\frac{1}{2}}. \tag{4.95}$$

Let $\{x^0, x^1, \dots\}$ (even) and $\{\theta^1, \theta^2, \dots\}$ (odd) be a basis for the super vector space V , with dual sets $\{y_0, y_1, \dots\}$ and $\{\eta_1, \eta_2, \dots\}$. Define the operators $H_i, F_r \in \widehat{\mathcal{W}}_{\hbar}(V)$ for $i \in \mathbb{Z}_{i \geq 0}$ and $r \in \mathbb{Z}_{\geq 0} + \frac{1}{2}$:

$$\begin{aligned} H_i &= \hbar \frac{\partial}{\partial x^i} + \frac{\hbar}{2} \sum_{k \in \mathbb{Z}} \circ b_k^M b_{i+N-1-k}^M \circ \\ &+ \frac{\hbar}{2} \sum_{r \in \mathbb{Z} + \frac{1}{2}} \left(r + \frac{i + N - 1}{2} \right) \circ \psi_{-r}^M \psi_{i+N-1+r}^M \circ, \end{aligned} \tag{4.96a}$$

$$F_r = \hbar \frac{\partial}{\partial \theta^{r+\frac{1}{2}}} + \hbar \sum_{m \in \mathbb{Z}} \psi_{r+N-1-m}^M b_m^M, \tag{4.96b}$$

which generate the following subalgebra of the super Virasoro algebra in the NS sector:

$$\begin{aligned} [H_m, H_n] &= (m - n)H_{m+n+N-1}, \\ [H_m, F_r] &= \left(\frac{m - N + 1}{2} - r \right) F_{m+r+N-1}, \\ \{F_r, F_s\} &= 2H_{r+s+N-1}. \end{aligned} \tag{4.97}$$

(1) Let N be any integer $N \geq 0$. The linear operator $H : V^* \rightarrow \widehat{\mathcal{W}}_{\hbar}(V)$ defined by, for all $i \in \mathbb{Z}_{\geq 0}$ and $r \in \mathbb{Z}_{\geq 0} + \frac{1}{2}$,

$$H(y_i) = H_i + \hbar D_i \delta_{i \leq N-1}, \quad H(\eta_{r+\frac{1}{2}}) = F_r, \tag{4.98}$$

for arbitrary constants $D_i, i = 0, \dots, N - 1$, forms a super quantum Airy structure as a representation of the algebra (4.97).

(2) Let N be any integer $N \geq 1$. The linear operator $H : V^* \rightarrow \widehat{\mathcal{W}}_{\hbar}(V)$ defined by, for all $i \in \mathbb{Z}_{\geq 1}$ and $r \in \mathbb{Z}_{\geq 0} + \frac{1}{2}$,

$$\begin{aligned} H(y_i) &= H_i + \hbar D_i \delta_{i \leq N-1}, \quad H(\eta_{r+\frac{1}{2}}) = F_r, \\ H(y_0) &= \hbar \frac{\partial}{\partial x^0} + \frac{\hbar^2}{2} C_0 \frac{\partial^2}{\partial (x^0)^2} + \hbar D_0, \end{aligned} \tag{4.99}$$

for arbitrary constants $D_i, i = 0, \dots, N - 1$ and C_0 , forms a super quantum Airy structure as a representation of the algebra (4.97) extended by

$$[H(y_0), H(y_i)] = [H(y_0), H(\eta_{r+\frac{1}{2}})] = 0. \tag{4.100}$$

(3) Let N be any integer $N \geq -1$. Let $V_{red} \subset V$ be the subspace spanned by $\{x^0, x^1, \dots\}$ and $\{\theta^2, \theta^3, \dots\}$. The linear operator $H : V_{red}^* \rightarrow \widehat{\mathcal{W}}_{\hbar}(V)$ defined by, for all $i \in \mathbb{Z}_{\geq 1}$ and $r \in \mathbb{Z}_{\geq 1} + \frac{1}{2}$,

$$H(y_i) = H_i + \hbar D_i \delta_{i \leq N+1}, \quad H(\eta_{r+\frac{1}{2}}) = F_r,$$

$$H(y_0) = \hbar \frac{\partial}{\partial x^0} + \frac{\hbar^2}{2} C_0 \frac{\partial^2}{\partial (x^0)^2} + \hbar D_0, \tag{4.101}$$

for arbitrary constants $D_i, i = 0, \dots, N + 1$, and C_0 , form a super quantum Airy structure as a representation of the algebra (4.97) extended by

$$[H(y_0), H(y_i)] = [H(y_0), H(\eta_{r+\frac{1}{2}})] = 0. \tag{4.102}$$

This is a super quantum Airy structure with an extra fermionic variable, θ^1 .

We remark that we could have considered as separate cases setting the bosonic zero mode $b_0^M = 0$. But since the result will be equivalent to the cases with an auxiliary operator \mathcal{H}_0 with $C_0 = D_0 = 0$, we did not consider it separately.

Proof. We start with the untwisted representation, see Lemma 4.8. We represent the Heisenberg algebra for untwisted bosonic and fermionic modes as:

$$\begin{aligned} \forall m \in \mathbb{Z}_{\geq 1}, \quad b_m^M &= \sqrt{\hbar} \frac{\partial}{\partial x^m}, & b_{-m}^M &= \frac{m}{\sqrt{\hbar}} x^m, & b_0^M &= \sqrt{\hbar} \frac{\partial}{\partial x^0}, \\ \forall r \in \mathbb{Z}_{\geq 0} + \frac{1}{2}, \quad \psi_r^M &= \sqrt{\hbar} \frac{\partial}{\partial \theta^{r+\frac{1}{2}}}, & \psi_{-r}^M &= \frac{1}{\sqrt{\hbar}} \theta^{r+\frac{1}{2}}. \end{aligned} \tag{4.103}$$

The super Virasoro generators (in the NS sector) take the form:

$$L_n^M = \frac{1}{2} \sum_{k \in \mathbb{Z}} \circ b_k^M b_{n-k}^M \circ + \frac{1}{2} \sum_{r \in \mathbb{Z} + \frac{1}{2}} \left(r + \frac{n}{2} \right) \circ \psi_{-r}^M \psi_{n+r}^M \circ, \tag{4.104a}$$

$$G_r^M = \sum_{m \in \mathbb{Z}} \psi_{r-m}^M b_m^M. \tag{4.104b}$$

We consider the closed subalgebras $\{L_m^M, G_r^M\}$ with $m \geq N - 1$ and $r \geq N - \frac{1}{2}$ for any $N \geq 0$, which have no central term. We shift indices to index the bosonic generators with $0, 1, 2, \dots$ and the fermionic ones with $\frac{1}{2}, \frac{3}{2}, \dots$. We define the quadratic hamiltonians:

$$\begin{aligned} H_i^2 &:= \hbar L_{i+N-1}^M = \frac{\hbar}{2} \sum_{k \in \mathbb{Z}} \circ b_k^M b_{i+N-1-k}^M \circ \\ &+ \frac{\hbar}{2} \sum_{r \in \mathbb{Z} + \frac{1}{2}} \left(r + \frac{i+N-1}{2} \right) \circ \psi_{-r}^M \psi_{i+N-1+r}^M \circ, \end{aligned} \tag{4.105}$$

$$F_r^2 := \hbar G_{r+N-1}^M = \hbar \sum_{m \in \mathbb{Z}} \psi_{r+N-1-m}^M b_m^M, \tag{4.106}$$

which have the commutation relations

$$\begin{aligned} [H_m^2, H_n^2] &= (m - n) H_{m+n+N-1}^2, \\ [H_m^2, F_r^2] &= \left(\frac{m - N + 1}{2} - r \right) F_{m+r+N-1}^2, \end{aligned}$$

$$\{F_r^2, F_s^2\} = 2H_{r+s+N-1}^2. \tag{4.107}$$

We shift the bosonic mode $b_{N-1}^M \mapsto b_{N-1}^M + \frac{1}{\sqrt{\hbar}}$ to get linear terms $H_i^1 = \sqrt{\hbar}b_i^M = \hbar \frac{\partial}{\partial x^i}$ and $F_r^1 = \sqrt{\hbar}\psi_r^M = \hbar \frac{\partial}{\partial \theta^{r+\frac{1}{2}}}$. We also get a constant term $H_{N-1}^0 = \frac{1}{2}$, which we get rid of without changing the algebra. We add D -terms $\hbar D_i$ to the operators H_i with $i = 0, \dots, N - 1$. This gives our first class of super quantum Airy structures.

We may want to consider θ^1 as being an extra fermionic variable. For this we would like to consider the smaller algebra $\{L_m^M, G_r^M\}$ with $m \geq N$ and $r \geq N + \frac{3}{2}$. This is closed only for $N \geq 0$. Then we proceed as in the previous case, and the D -terms that we can add are the same. Thus, the operators are precisely the same as in the previous case, minus the operator $F_{\frac{1}{2}}$. By uniqueness of the partition function, it follows that the solution to the constraint is the same as in the previous case, and hence, in particular, it also satisfies the constraint $F_{\frac{1}{2}}Z = 0$. Therefore it is the same super quantum Airy structure.

The next thing that we can try is add an auxiliary operator \mathcal{H}_0 as in Proposition 4.2. For this, we consider the subalgebra $\{L_m^M, G_r^M\}$ with $m \geq N$ and $r \geq N - \frac{1}{2}$, which is closed for $N \geq 1$. We then shift indices to index the bosonic operators with $1, 2, \dots$ (without the 0) and the fermionic operators with $\frac{1}{2}, \frac{3}{2}, \dots$. To do this, we use the same quadratic hamiltonians as (4.105), with commutation relations (4.107). We shift the bosonic mode $b_{N-1}^M \mapsto b_{N-1}^M + \frac{1}{\sqrt{\hbar}}$ to create linear terms $H_i^1 = \sqrt{\hbar}b_i^M = \hbar \frac{\partial}{\partial x^i}$ and $F_r^1 = \sqrt{\hbar}\psi_r^M = \hbar \frac{\partial}{\partial \theta^{r+\frac{1}{2}}}$. We also get a constant term $H_{N-1}^0 = \frac{1}{2}$ which we get rid of as usual. We add D -terms $\hbar D_i$ for $i = 1, \dots, N - 1$. To get a quantum Airy structure, we need to supplement with the auxiliary operator

$$\mathcal{H}_0 = \hbar \frac{\partial}{\partial x^0} + \frac{\hbar^2}{2} C_0 \frac{\partial^2}{\partial (x^0)^2} + \hbar D_0, \tag{4.108}$$

which commutes with all other operators. This is our second class of super quantum Airy structures.

For the third class, we want to keep using \mathcal{H}_0 , but we would like to think of θ^1 as an extra fermionic variable. For this, we consider the subalgebra $\{L_m^M, G_r^M\}$ with $m \geq N$ and $r \geq N + \frac{1}{2}$ for any $N \geq -1$. We want to shift indices so that bosonic generators are indexed with $1, 2, \dots$ and fermionic ones with $\frac{3}{2}, \frac{5}{2}, \dots$. The same hamiltonians (4.105) will do the job, with commutation relations (4.107). To create appropriate linear terms, we shift the bosonic mode $b_{N-1}^M \mapsto b_{N-1}^M + \frac{1}{\sqrt{\hbar}}$ as usual. We add D -terms $\hbar D_i$ for $i = 1, \dots, N + 1$, and the auxiliary operator (4.108). This gives our third class of super quantum Airy structures.

We note that we could also consider separately the cases where we set the bosonic zero mode $b_0^M = 0$, but in the end it is equivalent to the special case of the auxiliary operator \mathcal{H}_0 with $C_0 = D_0 = 0$, and thus we do not consider it separately. \square

4.2.4. Super Quantum Airy Structures from σ -Twisted Representations of the Free Boson–Fermion VOSA We now consider the σ -twisted representation of the free boson–fermion VOSA. We construct the following two classes of super quantum Airy structures:

Proposition 4.10. *Let N be any integer $N \geq 0$. We represent the super Heisenberg algebra for the twisted bosonic modes and untwisted fermionic modes as:*

$$\begin{aligned} \forall r \in \mathbb{Z}_{\geq 0} + \frac{1}{2}, \quad b_r^\sigma &= \sqrt{\hbar} \frac{\partial}{\partial x^{r+\frac{1}{2}}}, & b_{-r}^\sigma &= \frac{r}{\sqrt{\hbar}} x^{r+\frac{1}{2}}, \\ \forall r \in \mathbb{Z}_{\geq 0} + \frac{1}{2}, \quad \psi_r^\sigma &= \sqrt{\hbar} \frac{\partial}{\partial \theta^{r+\frac{1}{2}}}, & \psi_{-r}^\sigma &= \frac{1}{\sqrt{\hbar}} \theta^{r+\frac{1}{2}}. \end{aligned} \tag{4.109}$$

Let $\{x^1, x^2, \dots\}$ (even) and $\{\theta^1, \theta^2, \dots\}$ (odd) be a basis for the super vector space V , with dual sets $\{y_1, y_2, \dots\}$ and $\{\eta_1, \eta_2, \dots\}$. Define the differential operators $H_i, F_i \in \widehat{\mathcal{W}}_\hbar(V)$, for $i \in \mathbb{Z}_{\geq 1}$,

$$\begin{aligned} H_i &= \hbar \frac{\partial}{\partial x^i} + \frac{\hbar}{2} \sum_{r \in \mathbb{Z} + \frac{1}{2}} \circ b_r^\sigma b_{i+N-1-r}^\sigma \circ \\ &+ \frac{\hbar}{2} \sum_{r \in \mathbb{Z} + \frac{1}{2}} \left(r + \frac{i+N-1}{2} \right) \circ \psi_{-r}^\sigma \psi_{i+N-1+r}^\sigma \circ, \end{aligned} \tag{4.110a}$$

$$F_i = \hbar \frac{\partial}{\partial \theta^i} + \hbar \sum_{r \in \mathbb{Z} + \frac{1}{2}} b_{i+N-1-r}^\sigma \psi_r^\sigma, \tag{4.110b}$$

which form a representation of the following subalgebra of the super Virasoro algebra in the Ramond sector:

$$\begin{aligned} [H_m, H_n] &= (m-n)H_{m+n+N-1}, \\ [H_m, F_n] &= \left(\frac{m-N+1}{2} - n \right) F_{m+n+N-1}, \\ \{F_m, F_n\} &= 2H_{m+n+N-1}. \end{aligned} \tag{4.111}$$

(1) The linear operator $H : V^* \rightarrow \widehat{\mathcal{W}}_\hbar(V)$ defined by, for all $i \in \mathbb{Z}_{\geq 1}$,

$$H(y_i) = H_i + \hbar D_i \delta_{i \leq N}, \quad H(\eta_i) = F_i, \tag{4.112}$$

for arbitrary constants $D_i, i = 1, \dots, N$, forms a super quantum Airy structure as a representation of the algebra (4.111)

(2) Let $V_{red} \subset V$ be the subspace spanned by $\{x^1, x^2, \dots\}$ and $\{\theta^2, \theta^3, \dots\}$. The linear operator $H : V_{red}^* \rightarrow \widehat{\mathcal{W}}_\hbar(V)$ defined by, for all $i \in \mathbb{Z}_{\geq 1}$ and $j \in \mathbb{Z}_{\geq 2}$,

$$H(y_i) = H_i + \hbar D_i \delta_{i \leq N+1}, \quad H(\eta_j) = F_j, \tag{4.113}$$

for arbitrary constants $D_i, i = 1, \dots, N+1$, forms a super quantum Airy structure with an extra fermionic coordinate θ^1 , as a representation of the algebra (4.111)

Proof. We start with the σ -twisted representation, see Lemma 4.8. We represent the twisted bosonic modes and untwisted fermionic modes as:

$$\forall r \in \mathbb{Z}_{\geq 0} + \frac{1}{2}, \quad b_r^\sigma = \sqrt{\hbar} \frac{\partial}{\partial x^{r+\frac{1}{2}}}, \quad b_{-r}^\sigma = \frac{r}{\sqrt{\hbar}} x^{r+\frac{1}{2}},$$

$$\forall r \in \mathbb{Z}_{\geq 0} + \frac{1}{2}, \quad \psi_r^\sigma = \sqrt{\hbar} \frac{\partial}{\partial \theta^{r+\frac{1}{2}}}, \quad \psi_{-r}^\sigma = \frac{1}{\sqrt{\hbar}} \theta^{r+\frac{1}{2}}. \quad (4.114)$$

Here there is no bosonic or fermionic zero mode. From Lemma 4.8, the super Virasoro generators take the form (4.83):

$$L_m^\sigma = \frac{1}{2} \sum_{r \in \mathbb{Z} + \frac{1}{2}} \circ b_r^\sigma b_{m-r}^\sigma \circ + \frac{1}{2} \sum_{r \in \mathbb{Z} + \frac{1}{2}} \left(r + \frac{m}{2} \right) \circ \psi_{-r}^\sigma \psi_{m+r}^\sigma \circ + \frac{1}{16} \delta_{m,0}, \quad (4.115a)$$

$$G_m^\sigma = \sum_{r \in \mathbb{Z} + \frac{1}{2}} b_{m-r}^\sigma \psi_r^\sigma. \quad (4.115b)$$

The super Virasoro algebra is in the Ramond sector.

For our first class of super quantum Airy structures, we consider the closed subalgebra $\{L_m^\sigma, G_n^\sigma\}$ with $m, n \geq N$, for any $N \geq 0$. It takes the form:

$$\begin{aligned} [L_m^\sigma, L_n^\sigma] &= (m-n)L_{m+n}^\sigma, \\ [L_m^\sigma, G_n^\sigma] &= \left(\frac{m}{2} - n \right) G_{m+n}^\sigma, \\ \{G_m^\sigma, G_n^\sigma\} &= 2L_{m+n}^\sigma - \frac{1}{8} \delta_{m,0} \delta_{n,0}. \end{aligned} \quad (4.116)$$

To get rid of the central term, we redefine $L_0^\sigma \mapsto L_0^\sigma - \frac{1}{16}$, which does not change the rest of the algebra. We shift indices as usual to define the quadratic hamiltonians, for $i \in \mathbb{Z}_{\geq 1}$:

$$\begin{aligned} H_i^2 &:= \hbar L_{i+N-1}^\sigma = \frac{\hbar}{2} \sum_{r \in \mathbb{Z} + \frac{1}{2}} \circ b_r^\sigma b_{i+N-1-r}^\sigma \circ \\ &\quad + \frac{\hbar}{2} \sum_{r \in \mathbb{Z} + \frac{1}{2}} \left(r + \frac{i+N-1}{2} \right) \circ \psi_{-r}^\sigma \psi_{i+N-1+r}^\sigma \circ, \\ F_i^2 &:= \hbar G_{i+N-1}^\sigma = \hbar \sum_{r \in \mathbb{Z} + \frac{1}{2}} b_{i+N-1-r}^\sigma \psi_r^\sigma, \end{aligned} \quad (4.117)$$

which satisfy the commutation relations:

$$\begin{aligned} [H_m^2, H_n^2] &= (m-n)H_{m+n+N-1}^2, \\ [H_m^2, F_n^2] &= \left(\frac{m-N+1}{2} - n \right) F_{m+n+N-1}^2, \\ \{F_m^2, F_n^2\} &= 2H_{m+n+N-1}^2. \end{aligned} \quad (4.118)$$

To create appropriate linear terms, we shift the bosonic modes $b_{N-\frac{1}{2}}^\sigma \mapsto b_{N-\frac{1}{2}}^\sigma + \frac{1}{\sqrt{\hbar}}$, which creates terms $H_i^1 = \sqrt{\hbar} b_{i-\frac{1}{2}}^\sigma = \hbar \frac{\partial}{\partial x^i}$ and $F_i^1 = \sqrt{\hbar} \psi_{i-\frac{1}{2}}^\sigma = \hbar \frac{\partial}{\partial \theta^i}$. This also creates a constant term $H_N^0 = \frac{1}{2}$ which we get rid of without changing the algebra. We can also add D -terms $\hbar D_i$ to H_i for $i = 1, \dots, N$ without changing the algebra, which gives our first class of super quantum Airy structures.

For the second class, we consider a smaller closed subalgebra $\{L_m^\sigma, G_n^\sigma\}$ with $m \geq N, n \geq N + 1$, for any $N \geq 0$. The algebra is still (4.116) but with no central term now. We shift indices as above: now F_i starts with $i = 2, 3, \dots$. We then shift the same bosonic modes to create appropriate linear terms, and introduce D -terms $\hbar D_i$ to H_i for $i = 1, \dots, N + 1$. Note that we can add one more D -term, since H_{N+1} does not appear anymore on the right-hand-side of the commutation relations. This gives a super quantum Airy structure, where θ^1 is considered as an extra fermionic variable (since it does not appear in the linear terms). Note that in (4.113) we absorbed the term $\frac{\hbar}{16}\delta_{i,1-N}$, which only appears for $N = 0$, into the arbitrary constant $\hbar D_1$. \square

Remark 4.11. Note that the case with $N = 0$ for the second class of super quantum Airy structures in Proposition 4.10 is interesting. Setting $D_1 = \frac{1}{16}$ to its natural value, we get a non-trivial partition function. The pure bosonic part of the Virasoro generators H_i is in this case equivalent to the Virasoro operators that annihilate the Brézin–Gross–Witten tau-function of the KdV hierachy (see (4.61) and the discussion around there). This suggests that the partition function associated to this super quantum Airy structure may be a supersymmetric analog of the BGW tau-function, which would be worth investigating further.

Remark 4.12. We also remark that for the second class of super quantum Airy structures in Proposition 4.10, if the last arbitrary constant D_{N+1} is set to zero, then we recover the same differential operators as the super quantum Airy structure in the first class, minus F_1 . Then, by uniqueness of the partition function, we conclude that they share the same partition function, and that F_1 must also annihilate the partition function of the second class. Thus the super quantum Airy structures in the second class differ from for the first class only when $D_{N+1} \neq 0$.

4.2.5. Super Quantum Airy Structures from μ -Twisted Representations of the Free Boson–Fermion VOSA We now consider the μ -twisted representation of the free boson–fermion VOSA.

Proposition 4.13. *We represent the super Heisenberg algebra for the untwisted bosonic and twisted fermionic modes as:*

$$\begin{aligned} \forall m \in \mathbb{Z}_{\geq 1}, \quad b_m^\mu &= \sqrt{\hbar} \frac{\partial}{\partial x^m}, & b_{-m}^\mu &= \frac{m}{\sqrt{\hbar}} x^m, & b_0^\mu &= \sqrt{\hbar} \frac{\partial}{\partial x^0}, \\ \forall m \in \mathbb{Z}_{\geq 1}, \quad \psi_m^\mu &= \sqrt{\hbar} \frac{\partial}{\partial \theta^m}, & \psi_{-m}^\mu &= \frac{1}{\sqrt{\hbar}} \theta^m, & \psi_0^\mu &= \frac{1}{\sqrt{2\hbar}} \left(\theta^0 + \hbar \frac{\partial}{\partial \theta^0} \right). \end{aligned} \tag{4.119}$$

Let $\{x^0, x^1, \dots\}$ (even) and $\{\theta^1, \theta^2, \dots\}$ (odd) be a basis for the super vector space V , with dual sets $\{y_0, y_1, \dots\}$ and $\{\eta_1, \eta_2, \dots\}$. Let $\tilde{V} = V \oplus \mathbb{K}^{0|1}$, with $\{\theta^0\}$ a basis for $\mathbb{K}^{0|1}$. Let us define the differential operators $H_i, F_j \in \mathcal{W}_\hbar(\tilde{V})$, for $i \in \mathbb{Z}_{\geq 0}$ and $j \in \mathbb{Z}_{\geq 1}$:

$$\begin{aligned} H_i &= \hbar \frac{\partial}{\partial x^i} + \frac{\hbar}{2} \sum_{j \in \mathbb{Z}} \circ b_j^\mu b_{i+N-1-j}^\mu \circ \\ &\quad + \frac{\hbar}{2} \sum_{j \in \mathbb{Z}} \left(j + \frac{i+N-1}{2} \right) \circ \psi_{-j}^\mu \psi_{i+N-1+j}^\mu \circ, \end{aligned} \tag{4.120a}$$

$$F_j = \hbar \frac{\partial}{\partial \theta^j} + \sum_{k \in \mathbb{Z}} b_{j+N-1-k}^\mu \psi_k^\mu, \tag{4.120b}$$

which form a representation of the subalgebra of the super Virasoro algebra in the Ramond sector:

$$\begin{aligned} [H_m, H_n] &= (m - n)H_{m+n+N-1}, \\ [H_m, F_n] &= \left(\frac{m - N + 1}{2} - n \right) F_{m+n+N-1}, \\ \{F_m, F_n\} &= 2H_{m+n+N-1}. \end{aligned} \tag{4.121}$$

(1) Let N be any integer $N \geq 1$. The linear operator $H : V^* \rightarrow \widehat{\mathcal{W}}_\hbar(\widetilde{V})$ defined by, for all $i \in \mathbb{Z}_{\geq 0}$ and $j \in \mathbb{Z}_{\geq 1}$:

$$H(y_i) = H_i + \hbar D_i \delta_{i \leq N-1}, \quad H(\eta_j) = F_j, \tag{4.122}$$

for arbitrary constants $D_i, i = 0, \dots, N - 1$, forms a super quantum Airy structure as a representation of the algebra (4.121)

(2) Let N be any integer $N \geq 0$. The linear operator $H : V^* \rightarrow \widehat{\mathcal{W}}_\hbar(\widetilde{V})$ defined by, for all $i \in \mathbb{Z}_{\geq 1}$:

$$\begin{aligned} H(y_i) &= H_i + \hbar D_i \delta_{i \leq N}, \quad H(\eta_i) = F_i, \\ H(y_0) &= \hbar \frac{\partial}{\partial x^0} + \frac{\hbar^2}{2} C_0 \frac{\partial^2}{\partial (x^0)^2} + \hbar D_0, \end{aligned} \tag{4.123}$$

for arbitrary constants $D_i, i = 0, \dots, N$ and C_0 , forms a super quantum Airy structure as a representation of the algebra (4.121) extended by:

$$[H(y_0), H(y_i)] = [H(y_0), H(\eta_i)] = 0. \tag{4.124}$$

We note that both of these super quantum Airy structures have an extra fermionic coordinate θ^0 . We remark that we could also consider a third case, where we set the bosonic zero mode $b_0^\mu = 0$, as in Proposition 4.2. But as it will be equivalent to the special case of case (2) with $C_0 = D_0 = 0$ we do not consider it separately.

Proof. We start with the μ -twisted representation, see Lemma 4.8. We represent the untwisted bosonic modes and twisted fermionic modes as:

$$\begin{aligned} \forall m \in \mathbb{Z}_{\geq 1}, \quad b_m^\mu &= \sqrt{\hbar} \frac{\partial}{\partial x^m}, & b_{-m}^\mu &= \frac{m}{\sqrt{\hbar}} x^m, & b_0^\mu &= \sqrt{\hbar} \frac{\partial}{\partial x^0}, \\ \forall m \in \mathbb{Z}_{\geq 1}, \quad \psi_m^\mu &= \sqrt{\hbar} \frac{\partial}{\partial \theta^m}, & \psi_{-m}^\mu &= \frac{1}{\sqrt{\hbar}} \theta^m, & \psi_0^\mu &= \frac{1}{\sqrt{2\hbar}} \left(\theta^0 + \hbar \frac{\partial}{\partial \theta^0} \right). \end{aligned} \tag{4.125}$$

We have both bosonic and fermionic zero modes. In particular, we expect to get only super quantum Airy structures with an extra fermionic coordinate, namely θ^0 .

From Lemma 4.8, the super Virasoro generators (in the Ramond sector) take the form:

$$L_m^\mu = \frac{1}{2} \sum_{i \in \mathbb{Z}} \circ b_i^\mu b_{m-i}^\mu \circ + \frac{1}{2} \sum_{i \in \mathbb{Z}} \left(i + \frac{m}{2} \right) \circ \psi_{-i}^\mu \psi_{m+i}^\mu \circ + \frac{1}{16} \delta_{m,0}, \tag{4.126a}$$

$$G_m^\mu = \sum_{i \in \mathbb{Z}} b_{m-i}^\mu \psi_i^\mu. \tag{4.126b}$$

For our first class of super quantum Airy structures, we consider the subalgebra $\{L_m, G_n\}$ with $m \geq N - 1, n \geq N$, for any $N \geq 1$. We shift indices so that bosonic operators are indexed by $\{0, 1, 2, \dots\}$ and fermionic operators by $\{1, 2, 3, \dots\}$. Thus we define quadratic hamiltonians:

$$\begin{aligned} H_i^2 &:= \hbar L_{i+N-1}^\mu = \frac{\hbar}{2} \sum_{j \in \mathbb{Z}} \circ b_j^\mu b_{i+N-1-j}^\mu \circ \\ &+ \frac{\hbar}{2} \sum_{j \in \mathbb{Z}} \left(j + \frac{i+N-1}{2} \right) \circ \psi_{-j}^\mu \psi_{i+N-1+j}^\mu \circ + \frac{\hbar}{16} \delta_{i,-N+1}, \tag{4.127} \\ F_i^2 &:= \hbar G_{i+N-1}^\mu = \sum_{j \in \mathbb{Z}} b_{i+N-1-j}^\mu \psi_j^\mu, \end{aligned}$$

with commutation relations:

$$\begin{aligned} [H_m^2, H_n^2] &= (m - n) H_{m+n+N-1}^2, \\ [H_m^2, F_n^2] &= \left(\frac{m - N + 1}{2} - n \right) F_{m+n+N-1}^2, \\ \{F_m^2, F_n^2\} &= 2 H_{m+n+N-1}^2. \end{aligned} \tag{4.128}$$

To create appropriate linear terms, we shift the bosonic modes $b_{N-1}^\mu \mapsto b_{N-1}^\mu + \frac{1}{\sqrt{\hbar}}$, which creates linear terms $H_i^1 = \sqrt{\hbar} b_i^\mu = \hbar \frac{\partial}{\partial x^i}$ and $F_i^1 = \sqrt{\hbar} \psi_i^\mu = \hbar \frac{\partial}{\partial \theta^i}$. It also creates a constant term $H_{N-1}^0 = \frac{1}{2}$ which we get rid of without changing the algebra. We also add D -terms $\hbar D_i, i = 0, \dots, N - 1$ without changing the algebra. This creates a super quantum airy structure with the extra fermionic variable θ^0 . Note that in (4.122), we absorbed the term $\frac{\hbar}{16} \delta_{i,-N+1}$, which only appears for $N = 1$, into the arbitrary constant D_0 .

For our second class, we consider the larger subalgebra $\{L_m, G_n\}$ with $m, n \geq N$, for any $N \geq 0$. As in Proposition 4.10, to get rid of the central term in the algebra we redefine $L_0^\mu \mapsto L_0^\mu - \frac{1}{16}$. We shift indices as in (4.127), but now we consider H_i and F_i with $i = 1, 2, 3, \dots$. We shift the bosonic modes $b_{N-1}^\mu \mapsto b_{N-1}^\mu + \frac{1}{\sqrt{\hbar}}$ as before, which creates the right linear terms. We can add D -terms $\hbar D_i, i = 1, \dots, N$ without changing the algebra.

This is not however a super quantum Airy structure, since x^0 does not appear in the linear terms (H_0 is not include in the algebra). But, as in Proposition 4.2, we notice that the H_m and F_m only depend on x^0 through the bosonic zero mode, i.e. through $\frac{\partial}{\partial x^0}$. Thus we can introduce an auxiliary bosonic operator

$$\mathcal{H}_0 = \hbar \frac{\partial}{\partial x^0} + \frac{\hbar^2}{2} C_0 \frac{\partial^2}{\partial (x^0)^2} + \hbar D_0, \tag{4.129}$$

which commutes with all $H_m, F_m, m \geq 1$. The result is a super quantum Airy structure with an extra fermionic variable θ^0 .

We could consider a third class, where we set the bosonic zero mode $b_0^\mu = 0$. But this will end up being equivalent to the second class with $C_0 = D_0 = 0$, and hence we do not consider it separately. \square

4.2.6. *Super Quantum Airy Structures from ρ -Twisted Representations of the Free Boson–Fermion VOSA* We finally consider the ρ -twisted representation of the free boson–fermion VOA. We get:

Proposition 4.14. *Let N be any integer $N \geq -1$. We represent the super Heisenberg algebra for the twisted bosonic and fermionic modes as:*

$$\begin{aligned} \forall r \in \mathbb{Z}_{\geq 0} + \frac{1}{2}, \quad b_r^\rho &= \sqrt{\hbar} \frac{\partial}{\partial x^{r+\frac{1}{2}}}, & b_{-r}^\rho &= \frac{r}{\sqrt{\hbar}} x^{r+\frac{1}{2}}, \\ \forall m \in \mathbb{Z}_{\geq 1}, \quad \psi_m^\rho &= \sqrt{\hbar} \frac{\partial}{\partial \theta^m}, & \psi_{-m}^\rho &= \frac{1}{\sqrt{\hbar}} \theta^m, & \psi_0^\rho &= \frac{1}{\sqrt{2\hbar}} \left(\theta^0 + \hbar \frac{\partial}{\partial \theta^0} \right). \end{aligned} \tag{4.130}$$

Let $\{x^1, x^2, \dots\}$ (even) and $\{\theta^1, \theta^2, \dots\}$ (odd) be a basis for the super vector space V , with dual sets $\{y_1, y_2, \dots\}$ and $\{\eta_1, \eta_2, \dots\}$. Let $\tilde{V} = V \oplus \mathbb{K}^{0|1}$, with $\{\theta^0\}$ a basis for $\mathbb{K}^{0|1}$. The linear operator $H : V^* \rightarrow \mathcal{W}_{\hbar}(\tilde{V})$ defined by, for all $i \in \mathbb{Z}_{\geq 1}$ and $r \in \mathbb{Z}_{\geq 0} + \frac{1}{2}$:

$$\begin{aligned} H_i := H(y_i) &= \hbar \frac{\partial}{\partial x^i} + \frac{\hbar}{2} \sum_{r \in \mathbb{Z} + \frac{1}{2}} \circ b_r^\rho b_{i+N-1-r}^\rho \circ \\ &+ \frac{\hbar}{2} \sum_{j \in \mathbb{Z}} \left(j + \frac{i+N-1}{2} \right) \circ \psi_{-j}^\rho \psi_{i+N-1+j}^\rho \circ \\ &+ \frac{\hbar}{8} \delta_{i,1-N} + \hbar D_i \delta_{i \leq N+1}, \end{aligned} \tag{4.131a}$$

$$F_r := H(\eta_{r+\frac{1}{2}}) = \hbar \frac{\partial}{\partial \theta^{r+\frac{1}{2}}} + \sum_{s \in \mathbb{Z} + \frac{1}{2}} b_s^\rho \psi_{r+N-s}^\rho, \tag{4.131b}$$

with arbitrary constants $D_i, i = 1, \dots, N + 1$, forms a super quantum Airy structure as a representation of the subalgebra of the super Virasoro algebra in the NS sector:

$$\begin{aligned} [H_m, H_n] &= (m - n) H_{m+n+N-1}, \\ [H_m, F_r] &= \left(\frac{m - N - 1}{2} - r \right) F_{m+r+N-1}, \\ \{F_r, F_s\} &= 2H_{r+s+N+1}. \end{aligned} \tag{4.132}$$

Note that this is a super quantum Airy structure with an extra fermionic variable θ^0 .

Proof. We start with the ρ -twisted representation, see Lemma 4.8. We represent the twisted bosonic modes and twisted fermionic modes as:

$$\begin{aligned} \forall r \in \mathbb{Z}_{\geq 0} + \frac{1}{2}, \quad b_r^\rho &= \sqrt{\hbar} \frac{\partial}{\partial x^{r+\frac{1}{2}}}, & b_{-r}^\rho &= \frac{r}{\sqrt{\hbar}} x^{r+\frac{1}{2}}, \\ \forall m \in \mathbb{Z}_{\geq 1}, \quad \psi_m^\rho &= \sqrt{\hbar} \frac{\partial}{\partial \theta^m}, & \psi_{-m}^\rho &= \frac{1}{\sqrt{\hbar}} \theta^m, & \psi_0^\rho &= \frac{1}{\sqrt{2\hbar}} \left(\theta^0 + \hbar \frac{\partial}{\partial \theta^0} \right). \end{aligned} \tag{4.133}$$

There is no bosonic zero mode here, but there is a fermionic one. Hence we expect to get only super quantum Airy structures with an extra fermionic coordinate, namely θ^0 .

From Lemma 4.8, the super Virasoro generators (in the NS sector) take the form:

$$L_m^\rho = \frac{1}{2} \sum_{r \in \mathbb{Z} + \frac{1}{2}} \circ b_r^\rho b_{m-r}^\rho \circ + \frac{1}{2} \sum_{i \in \mathbb{Z}} \left(i + \frac{m}{2} \right) \circ \psi_{-i}^\rho \psi_{m+i}^\rho \circ + \frac{1}{8} \delta_{m,0}, \quad (4.134a)$$

$$G_r^\rho = \sum_{s \in \mathbb{Z} + \frac{1}{2}} b_s^\rho \psi_{r-s}^\rho. \quad (4.134b)$$

We consider the closed subalgebras $\{L_m^\rho, G_r^\rho\}$ with $m \geq N$ and $r \geq N + \frac{1}{2}$, for any $N \geq -1$, which have no central term. We shift indices to index the bosonic generators with $1, 2, \dots$ and the fermionic ones with $\frac{1}{2}, \frac{3}{2}, \dots$. We define the quadratic hamiltonians, for $i \in \mathbb{Z}_{\geq 1}$ and $r \in \mathbb{Z}_{\geq 0} + \frac{1}{2}$:

$$H_i^2 := \hbar L_{i+N-1}^\rho = \frac{\hbar}{2} \sum_{r \in \mathbb{Z} + \frac{1}{2}} \circ b_r^\rho b_{i+N-1-r}^\rho \circ + \frac{\hbar}{2} \sum_{j \in \mathbb{Z}} \left(j + \frac{i+N-1}{2} \right) \circ \psi_{-j}^\rho \psi_{i+N-1+j}^\rho \circ + \frac{\hbar}{8} \delta_{i,1-N}, \quad (4.135)$$

$$F_r^2 := \hbar G_{r+N}^\rho = \sum_{s \in \mathbb{Z} + \frac{1}{2}} b_s^\rho \psi_{r+N-s}^\rho,$$

which satisfy the commutation relations:

$$\begin{aligned} [H_m^2, H_n^2] &= (m-n) H_{m+n+N-1}^2, \\ [H_m^2, F_r^2] &= \left(\frac{m-N-1}{2} - r \right) F_{m+r+N-1}^2, \\ \{F_r^2, F_s^2\} &= 2H_{r+s+N+1}^2. \end{aligned} \quad (4.136)$$

To create the appropriate linear terms, we shift the bosonic modes $b_{N-\frac{1}{2}}^\rho \mapsto b_{N-\frac{1}{2}}^\rho + \frac{1}{\sqrt{\hbar}}$, which creates terms $H_i^1 = \sqrt{\hbar} b_{i-\frac{1}{2}}^\rho = \hbar \frac{\partial}{\partial x^i}$ and $F_r^1 = \sqrt{\hbar} \psi_{r+\frac{1}{2}}^\rho = \hbar \frac{\partial}{\partial \theta^{r+\frac{1}{2}}}$. It also creates a constant term $H_N^0 = \frac{1}{2}$ which we get rid of as usual without changing the algebra. We can add D -terms $\hbar D_i$ to H_i for $i = 1, \dots, N+1$, and we get our class of super quantum Airy structures. Note that for all of those, θ^0 appears as an extra fermionic variable. \square

Remark 4.15. The cases with $N = -1$ and $N = 0$ of Proposition 4.14 are interesting. In the case $N = -1$, the bosonic part of the Virasoro generators L_i is almost the same as the $N = -1$ case of Proposition 4.6, which reproduces the Virasoro constraints satisfied by the Kontsevich–Witten tau function of KdV. The only difference is in the term $\frac{\hbar}{8}$, which is twice that of the bosonic case.

For $N = 0$ and $D_1 = 0$, the bosonic part of L_i also recovers almost exactly the $N = 0$ case of Proposition 4.6, which gives the Virasoro constraints satisfied by the BGW tau function of KdV. Again, the only difference is in the term $\frac{\hbar}{8}$ which is twice its

bosonic counterpart, but given that there is an arbitrary constant D_1 it could be adjusted accordingly.

Thus both cases may be thought of as supersymmetric generalizations of Kontsevich–Witten and BGW, which deserves further investigation.

5. Conclusion and Open Questions

In this paper we have defined super quantum Airy structures, as a natural supersymmetric generalization of quantum Airy structures. We showed existence and uniqueness of the associated free energy, which is computed by a topological recursion that can be understood as a supersymmetric generalization of the Chekhov–Eynard–Orantin (CEO) topological recursion. We constructed a number of examples of finite-dimensional and infinite-dimensional quadratic super quantum Airy structures.

There are many open questions that deserve further investigation. Here we propose a few, in random order:

- For quantum Airy structures that come from the CEO topological recursion, it is well known that the free energies are related to intersection numbers over the moduli space of curves [34, 35]. Is there an analogous statement for some classes of super quantum Airy structures? Do they provide invariants of moduli spaces of supersymmetric algebraic curves or super-Riemann surfaces?
- Super-Virasoro constraints have appeared in enumerative geometry in the context of invariants that involve odd cohomology classes, such as Gromov–Witten invariants for higher genus target curves [60]. Perhaps the appearance of odd cohomology classes is key to uncovering the enumerative meaning of super quantum Airy structures?
- Stanford and Witten very recently proposed a supersymmetric generalization of Mirzakhani’s recursion relation, in the context of JT gravity [64]. Since Mirzakhani’s recursion relation can be formulated as an example of a quantum Airy structure [39], are super quantum Airy structures related to the work of Stanford and Witten?
- The CEO topological recursion was originally discovered as a solution of the loop equations for matrix models [24, 37, 38]. Is the supersymmetric generalization that we propose in this paper related to supereigenvalue models?
- In the case of the Airy and Bessel spectral curves, the partition functions calculated by the associated quantum Airy structures construct the Kontsevich–Witten and Brézin–Gross–Witten tau functions for the KdV hierarchy [33]. Can an analogous statement be made for some of the examples of Section 4, perhaps with respect to the super KdV hierarchy?
- Super quantum Airy structures are \mathbb{Z}_2 -graded quantum Airy structures. Can G -graded quantum Airy structures be defined for more general finite groups G ?
- While the CEO topological recursion can be understood as an example of a quantum Airy structure, it was originally formulated in terms of complex analysis of a spectral curve. Is there a similar class of super quantum Airy structures that can be understood as coming from “super spectral curves”? If so, what is the formulation of the corresponding topological recursion in terms of the geometry of super spectral curves?
- In this spirit, for a large class of spectral curves, the CEO topological recursion can be used to reconstruct the quantum curve and its associated wave-function [15, 46]. Can our supersymmetric generalization of topological recursion be used to study super quantum curves [26–28] ?

- Can examples of infinite-dimensional super quantum Airy structures be constructed as representations of $N = 2$ vertex operators super algebras? Or as representations of VOSAs more generally?
- In this paper we focused on constructing examples of quadratic super quantum Airy structures. Are there interesting examples of higher order, finite-dimensional, super quantum Airy structures? In the infinite-dimensional case, it would certainly be interesting to generalize the higher order construction of [11] in terms of representations of W -algebras to the supersymmetric realm.
- In Section 3.2 we proposed a classification scheme for quadratic, finite-dimensional super quantum Airy structures. Can this classification be carried out?

This list is certainly not exhaustive. But what is clear is that super quantum Airy structures are interesting conceptually, and that many questions remain. The time is ripe to understand their properties and applications.¹⁵

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Appendix A: Computational Proof of Existence

To prove existence of the free energy associated to a super quantum Airy structure, we need to show that the recursive formulae (2.30) and (2.31) produce $F_{g,n}[a_1, \dots, a_n]$ that are \mathbb{Z}_2 -symmetric under permutations of indices.

Since \mathbb{Z}_2 -symmetry for permutations among indices in Φ for $F_{g,n+1}[i, \Phi]$ is obvious from (2.30), the only nontrivial symmetry is for the permutation of i and any other index in Φ . We thus have to show that

$$F_{g,n+2}(i, j, \Phi) = (-1)^{|i||j|} F_{g,n+2}(j, i, \Phi). \tag{A.1}$$

¹⁵ During the review process of this manuscript, (parts of) the third, fourth, fifth, and seventh open questions in the list above were subsequently addressed in [21]. In particular, it was realized that (suitably polarized) super quantum Airy structures could be used to compute (parts of the) correlation functions for a variety of examples related to 2d supergravity. [21] also showed a relation between certain families of super quantum Airy structures and non-super quantum Airy structures, which provides an interpretation of the work of Stanford and Witten [64] in terms of super quantum Airy structures. We are hoping to solve other open questions in the near future.

Let us prove (A.1) by induction on $2g+n \geq 1$. For $2g+n = 1$, we have $F_{0,3}(i, j, a) = A_{ija}$, hence (A.1) holds thanks to (2.27a). For $2g+n = 2$ there are two cases: $F_{0,4}(i, j, a_1, a_2)$ and $F_{1,2}(i, j)$. It follows from (2.30) that $F_{0,4}(i, j, a_1, a_2)$ becomes

$$\begin{aligned}
 &F_{0,4}(i, j, a_1, a_2) \\
 &= B_{ij}^0 F_{0,3}(0, a_1, a_2) + (-1)^{|i||j|} \\
 &\quad \left((-1)^{|i||j|} B_{ij}^k A_{ka_1a_2} + B_{ia_1}^b A_{jba_2} + (-1)^{|a_1||a_2|} B_{ia_2}^b A_{jba_1} \right) \\
 &= (-1)^{|i||j|} B_{ji}^0 F_{0,3}(0, a_1, a_2) \\
 &\quad + \left((-1)^{|i||j|} B_{ji}^k A_{ka_1a_2} + B_{ja_1}^b A_{iba_2} + (-1)^{|a_1||a_2|} B_{ja_2}^b A_{iba_1} \right) \\
 &= (-1)^{|i||j|} F_{0,4}(j, i, a_1, a_2), \tag{A.2}
 \end{aligned}$$

where we used (2.27c) and (2.28a) for the third equality. Similarly, for $F_{1,2}(i, j)$ we have:

$$\begin{aligned}
 &F_{1,2}(i, j) = (-1)^{|i||j|} \left(\frac{1}{2} C_i^{bc} A_{jcb} + (-1)^{|i||j|} B_{ij}^c D_c \right) \\
 &= (-1)^{|i||j|} (-1)^{|i||j|} \left(\frac{1}{2} C_j^{bc} A_{icb} + (-1)^{|i||j|} B_{ji}^c D_c \right) \\
 &= (-1)^{|i||j|} F_{1,2}(j, i), \tag{A.3}
 \end{aligned}$$

where we used $F_{1,1}(0) = 0$ and (2.27c) for the second equality and (2.28d) for the fourth equality. Therefore, the $F_{g,n+2}(i, j, \Phi)$ are \mathbb{Z}_2 -symmetric for $2g+n = 2$ as well.

Now let us assume \mathbb{Z}_2 -symmetry for $F_{h,m+2}(i, j, \Phi)$ up to $1 \leq 2h+m < 2g+n$. (2.30) can be rewritten as:

$$\begin{aligned}
 &F_{g,n+2}(i, j, \Phi) = \sum_{c \geq 0} B_{ij}^c F_{g,n+1}(c, \Phi) + \sum_{k=1}^n \sigma_{a_k \subset \{j, \Phi\}} \sum_{c \geq 0} B_{ia_k}^c F_{g,n+1}(c, j, \Phi \setminus a_k) \\
 &\quad + \frac{1}{2} \sum_{b, c \geq 0} C_i^{bc} F_{g-1, n+3}(c, b, j, \Phi) \\
 &\quad + \sum_{b, c \geq 0} C_i^{bc} \sum_{g_1+g_2=g} \sum_{\Phi_1 \cup \Phi_2 = \Phi} \sigma_{\Phi_1 \subset \Phi} F_{g_1, n_1+1}(b, j, \Phi_1) F_{g_2, n_2+1}(c, \Phi_2) \\
 &= B_{ij}^0 F_{g,n+1}(0, \Phi) + \sum_{q=1}^d B_{ij}^q F_{g,n+1}(q, \Phi) + \sum_{k=1}^n (-1)^{|i||j|} \sigma_{a_k \subset \Phi} \\
 &\quad \sum_{c \geq 0} B_{ia_k}^c F_{g,n+1}(j, c, \Phi \setminus a_k) \\
 &\quad + \frac{1}{2} (-1)^{|i||j|} \sum_{b, c \geq 0} C_i^{bc} F_{g-1, n+3}(j, c, b, \Phi) \\
 &\quad + \sum_{b, c \geq 0} C_i^{bc} (-1)^{|j||b|} \\
 &\quad \sum_{g_1+g_2=g} \sum_{\Phi_1 \cup \Phi_2 = \Phi} \sigma_{\Phi_1 \subset \Phi} F_{g_1, n_1+1}(j, b, \Phi_1) F_{g_2, n_2+1}(c, \Phi_2). \tag{A.4}
 \end{aligned}$$

The first term in (A.4) is \mathbb{Z}_2 -symmetric in (i, j) thanks to (2.27c). For the second term, we apply (2.30) to $F_{g,n+1}(q, \Phi)$. For the other terms, we substitute (2.30) into $F_{h,m'+1}(j, \Phi')$ for any h, Φ' whenever j is the first index. The computation becomes rather tedious; the final result after simplification is summarized below. The terms highlighted in red are \mathbb{Z}_2 -symmetric in (i, j) thanks to Lemma 2.19, while the other terms are manifestly \mathbb{Z}_2 -symmetric. Therefore, \mathbb{Z}_2 -symmetry of the $F_{g,n}[a_1, \dots, a_n]$ produced by (2.30) and (2.31) is proved by induction, and hence the unique free energy associated to a super quantum Airy structure exists.

$$\begin{aligned}
 &F_{g,n+2}(i, j, \Phi) \\
 &= B_{ij}^0 F_{g,n+1}(0, \Phi) + (-1)^{|i||j|} \\
 &\quad \left(B_{ia_1}^b A_{jba_2} + (-1)^{|a_1||a_2|} B_{ia_2}^b A_{jba_1} + (-1)^{|i||j|} B_{ij}^l A_{la_1a_2} \right) \delta_{n,2} \delta_{g,0} \\
 &\quad + (-1)^{|i||j|} \left(\frac{1}{2} C_i^{bc} A_{jcb} + (-1)^{|i||j|} B_{ij}^l D_l \right) \delta_{n,0} \delta_{g,1} \\
 &\quad + (-1)^{|i||j|} \sum_{k=1}^n \sum_{c \geq 0} \sigma_{a_k \subset \Phi} F_{g,n}(c, \Phi \setminus a_k) \\
 &\quad \left(B_{ia_k}^b B_{jb}^c + (-1)^{|c||a_k|} C_i^{cb} A_{jba_k} + (-1)^{|i||j|} B_{ij}^p B_{pa_k}^c \right) \\
 &\quad + \frac{1}{2} (-1)^{|i||j|} \sum_{d,c \geq 0} F_{g-1,n+2}(d, c, \Phi) \\
 &\quad \left(C_i^{cb} B_{jb}^d + (-1)^{|c||d|} C_i^{db} B_{jb}^c + (-1)^{|i||j|} B_{ij}^p C_p^{cd} \right) \\
 &\quad + \frac{1}{2} (-1)^{|i||j|} \sum_{g_1+g_2=g} \sum_{\Phi_1 \cup \Phi_2 = \Phi} \sum_{c,d \geq 0} \sigma_{\Phi_1 \subset \Phi} F_{g_1,n+1}(c, \Phi_1) F_{g_2,n+1}(d, \Phi_2) \\
 &\quad \times \left(C_i^{cb} B_{jb}^d + (-1)^{|c||d|} C_i^{db} B_{jb}^c + (-1)^{|i||j|} B_{ij}^p C_p^{cd} \right) \\
 &\quad + \frac{1}{2} \sum_{k,l=1}^n \sum_{b,c \geq 0} \sigma_{\{a_k, a_l\} \subset \Phi} (-1)^{|b||a_l|} F_{g,n}(c, b, \Phi \setminus \{a_k, a_l\}) \\
 &\quad \left((-1)^{|i||j|} B_{ia_k}^b B_{ja_l}^c + B_{ja_k}^b B_{ia_l}^c \right) \\
 &\quad + \frac{1}{2} \sum_{k=1}^n \sum_{g_1+g_2=g} \sum_{\Phi_1 \cup \Phi_2 = \Phi \setminus a_k} \sum_{b,c,d \geq 0} \sigma_{\{a_k, \Phi_1\} \subset \Phi} F_{g_1,n+2}(c, b, \Phi_1) F_{g_2,n+1}(d, \Phi_2) \\
 &\quad \times \left(B_{ja_k}^b C_i^{cd} + (-1)^{|j||i|} B_{ia_k}^b C_j^{cd} \right) \\
 &\quad + \frac{1}{2} \sum_{k=1}^n \sum_{b,c,d \geq 0} \sigma_{a_k \subset \Phi} F_{g-1,n+2}(b, d, c, \Phi \setminus a_k) \\
 &\quad \left(C_i^{cd} B_{ja_k}^b (-1)^{|i||b|} + (-1)^{|j||i|} B_{ia_k}^b C_j^{cd} (-1)^{|j||b|} \right) \\
 &\quad + \frac{1}{2} \sum_{g_1+g_2+g_3=g} \sum_{\Phi_1 \cup \Phi_2 \cup \Phi_3 = \Phi} \sum_{b,c,d,e \geq 0} \sigma_{\{\Phi_1, \Phi_2\} \subset \Phi} \\
 &F_{g_1,n+2}(b, d, \Phi_1) F_{g_2,n+1}(c, \Phi_2) F_{g_3,n+1}(e, \Phi_3)
 \end{aligned}$$

$$\begin{aligned}
& \times \left(C_i^{de} C_j^{bc} (-1)^{|j||e|} + (-1)^{|i||j|} C_j^{de} C_i^{bc} (-1)^{|i||e|} \right) \\
& + \sum_{b,c,d,e \geq 0} \left((-1)^{|i||j|} C_i^{de} C_j^{bc} + C_j^{de} C_i^{bc} \right) \\
& \times \left(\frac{1}{8} F_{g-2,n+4}(c, b, e, d, \Phi) \right. \\
& + \frac{1}{2} \sum_{g_1+g_2=g-1} \sum_{\Phi_1 \cup \Phi_2 = \Phi} \sigma_{\Phi_1 \subset \Phi} F_{g_1, n_1+3}(b, e, d, \Phi_1) F_{g_2, n_2+1}(c, \Phi_2) \\
& \left. + \frac{1}{4} \sum_{g_1+g_2=g-1} \sum_{\Phi_1 \cup \Phi_2 = \Phi} (-1)^{|e||b|} \sigma_{\Phi_1 \subset \Phi} F_{g_1, n_1+2}(b, d, \Phi_1) F_{g_2, n_2+2}(c, e, \Phi_2) \right). \tag{A.5}
\end{aligned}$$

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