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Nordhaus–Gaddum-Type Results for the Steiner Gutman Index of Graphs

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Abstract: Building upon the notion of the Gutman index $\mathrm{SGut}(G)$, Mao and Das recently introduced the Steiner Gutman index by incorporating Steiner distance for a connected graph G. The Steiner Gutman k-index $\mathrm{SGut}_k(G)$ of G is defined by $\mathrm{SGut}_k(G) = \sum_{S\subseteq V(G),\ |S|=k} (\prod_{v\in S} deg_G(v))\, d_G(S)$, in which $d_G(S)$ is the Steiner distance of S and $deg_G(v)$ is the degree of v in G. In this paper, we derive new sharp upper and lower bounds on SGut_k, and then investigate the Nordhaus-Gaddum-type results for the parameter SGut_k. We obtain sharp upper and lower bounds of $\mathrm{SGut}_k(G) + \mathrm{SGut}_k(\overline{G})$ and $\mathrm{SGut}_k(G) \cdot \mathrm{SGut}_k(\overline{G})$ for a connected graph G of order n, m edges, maximum degree Δ and minimum degree δ.

Keywords: distance; Steiner distance; Gutman index; Steiner Gutman k-index

MSC: 05C05; 05C12; 05C35

1. Introduction

We consider simple, undirected graphs in this paper. For the standard theoretical graph terminology and notation not defined here, follow [1]. For a graph G, let $V(G)$ and $E(G)$ represent its sets of vertices and edges, respectively. Let $|E(G)| = m$ be the size of G. The complement of G is conventionally denoted by $\overline{G}$. For a vertex $v \in V(G)$, $deg_G(v)$ is the degree of v. The maximum and minimum degrees are, respectively, denoted by Δ and δ. Like degrees, distance is a fundamental concept of graph theory [2]. For two vertices $u, v \in V(G)$ with connected G, the distance $d(u,v) = d_G(u,v)$ between these two vertices is defined as the length of a shortest path connecting them. An excellent survey paper on this subject can be found in [3].

The above classical graph distance was extended by Chartrand et al. in 1989 to the Steiner distance, which since then has become an essential concept of graph theory. Given a graph $G(V,E)$ and a vertex set $S \subseteq V(G)$ containing no less than two vertices, an S-Steiner tree (or an S-tree, a Steiner tree connecting S) is defined as a subgraph $T(V', E')$ of G, which is a subtree satisfying $S \subseteq V'$. If G is connected with order no less than 2 and $S \subseteq V$ is nonempty, the Steiner distance $d(S)$ among the vertices of S (sometimes simply put as the distance of S) is the minimum size of connected subgraph whose vertex sets contain the set S. Clearly, for a connected subgraph $H \subseteq G$ with $S \subseteq V(H)$ and $|E(H)| = d(S)$, H is a tree. When T is subtree of G, we have $d(S) = \min\{|E(T)|, S \subseteq V(T)\}$. For $S = \{u, v\}$, $d(S) = d(u,v)$ reduces to the classical distance between the two vertices u and v. Another basic observation is that if $|S| = k$, $d(S) \geq k-1$. For more results regarding varied properties of the Steiner distance, we refer to the reader to [3–8].

In [9], Li et al. generalized the concept of Wiener index through incorporating the Steiner distance. The Steiner k-Wiener index $\mathrm{SW}_k(G)$ of G is defined by

$$\mathrm{SW}_k(G) = \sum_{\substack{S \subseteq V(G) \\ |S|=k}} d(S)\,.$$

For $k = 2$, it is easy to see the Steiner Wiener index coincides with the ordinary Wiener index. The interesting range of the Steiner k-Wiener index SW_k resides in $2 \le k \le n-1$, and the two trivial cases give $\mathrm{SW}_1(G) = 0$ and $\mathrm{SW}_n(G) = n-1$.

Gutman [10] studied the Steiner degree distance, which is a generalization of ordinary degree distance. Formally, the k-center Steiner degree distance $\mathrm{SDD}_k(G)$ of G is given as

$$\mathrm{SDD}_k(G) = \sum_{\substack{S \subseteq V(G) \\ |S|=k}} \left(\sum_{v \in S} deg_G(v) \right) d_G(S)\,.$$

The Gutman index of a connected graph G is defined as

$$\mathrm{Gut}(G) = \sum_{u,v \in V(G)} deg_G(u)\, deg_G(v)\, d_G(u,v).$$

The Gutman index of graphs attracted attention very recently. For its basic properties and applications, including various lower and upper bounds, see [11–13] and the references cited therein. Recently, Mao and Das [14] further extended the concept of the Gutman index by incorporating Steiner distance and considering the weights as multiplications of degrees. The Steiner k-Gutman index $\mathrm{SGut}_k(G)$ of G is defined by

$$\mathrm{SGut}_k(G) = \sum_{\substack{S \subseteq V(G) \\ |S|=k}} \left(\prod_{v \in S} deg_G(v) \right) d_G(S)\,.$$

Note that this index is a natural generalization of the classical Gutman index—in particular, for $k = 2$, $\mathrm{SGut}_k(G) = Gut(G)$. This is the reason the product of the degrees comes to the definition of Steiner k-Gutman index. The weighting of multiplication of degree or expected degree has also been extensively explored in, for example, the field of random graphs [15,16] and proves to be very prolific. For more results on Steiner Wiener index, Steiner degree distance and Steiner Gutman index, we refer to the reader to [9,10,14,17–19].

For a given a graph parameter $f(G)$ and a positive integer n, the well-known Nordhaus–Gaddum problem is to determine sharp bounds for: (1) $f(G) + f(\overline{G})$ and (2) $f(G) \cdot f(\overline{G})$ over the class of connected graph G, with order n, m edges, maximum degree Δ and minimum degree δ characterizing the extremal graphs. Many Nordhaus–Gaddum type relations have attracted considerable attention in graph theory. Comprehensive results regarding this topic can be found in e.g., [20–24].

In Section 2, we obtain sharp upper and lower bounds on SGut_k of graph G. In Section 3, we obtain sharp upper and lower bounds of $\mathrm{SGut}_k(G) + \mathrm{SGut}_k(\overline{G})$ and $\mathrm{SGut}_k(G) \cdot \mathrm{SGut}_k(\overline{G})$ for a connected graph G in terms of n, m, maximum degree Δ and minimum degree δ.

2. Sharp Bounds for the Steiner Gutman Index

In [14], the following results have been obtained:

Lemma 1 ([14])**.** *Let K_n, S_n and P_n be the complete graph, star graph and path graph of order n, respectively, and let k be an integer such that $2 \le k \le n$. Then*

(1) $SGut_k(K_n) = \binom{n}{k}(n-1)^n(k-1)$;

(2) $SGut_k(S_n) = (kn-2k+1)\binom{n-1}{k-1}$;

(3) $SGut_k(P_n) = 2^k(k-1)\binom{n}{k+1}$.

For connected graph G of order n with m edges, the authors in [14] derived the following upper and lower bounds on $\mathrm{SGut}_k(G)$.

Lemma 2 ([14])**.** *Let G be a connected graph of order n with m edges, and let k be an integer with $2 \le k \le n$. Then*

$$(n-1)\left(\frac{2m}{k}\right)^k \binom{n-1}{k-1}^k \ge \mathrm{SGut}_k(G) \ge \begin{cases} 2m(k-1)\binom{n-1}{k-1} & \text{if } \delta \ge 2 \\ (k-1)\binom{n}{k} & \text{if } \delta = 1. \end{cases}$$

We now give lower and upper bounds for $\mathrm{SGut}_k(G)$ in terms of n, m, maximum degree Δ and minimum degree δ:

Proposition 1. *Let G be a connected graph of order $n \ge 3$ with m edges and maximum degree Δ, minimum degree δ. Additionally, let k be an integer with $2 \le k \le n$. Then*

$$2m(n-1)\binom{n-1}{k-1}\frac{\Delta^{k-1}}{k} \ge \mathrm{SGut}_k(G) \ge \begin{cases} 2m(k-1)\binom{n-1}{k-1}\frac{\delta^{k-1}}{k} & \text{if } \delta \ge 2 \\ k\binom{p}{k} + 2^q(k-1)\left[\binom{n}{k} - \binom{p}{k}\right] & \text{if } \delta = 1, \end{cases}$$

where p is the number of pendant vertices in G, and $q = \max\{k-p, 1\}$. The equality of upper bound holds if and only if G is a regular graph with $k = n$. The equality of lower bound holds if and only if G is a regular $(n-k+1)$-connected graph of order n ($\delta \ge 2$), or $G \cong P_n$ and $k = n > 3$ ($\delta = 1$), or $G \cong P_3$ and $k = 2$ ($\delta = 1$).

Proof. Upper bound: For any $S \subseteq V(G)$ and $|S| = k$, we have $k-1 \le d_G(S) \le n-1$, and hence

$$(k-1)\sum_{\substack{S\subseteq V(G)\\|S|=k}}\left(\prod_{v\in S} deg_G(v)\right) \le \mathrm{SGut}_k(G) \le (n-1)\sum_{\substack{S\subseteq V(G)\\|S|=k}}\left(\prod_{v\in S} deg_G(v)\right). \tag{1}$$

Let

$$M = \sum_{\substack{S\subseteq V(G)\\|S|=k}}\left(\prod_{v\in S} deg_G(v)\right) = \sum_{\{v_1,v_2,\ldots,v_k\}\subseteq V(G)} deg_G(v_1)deg_G(v_2)\cdots deg_G(v_k).$$

and

$$N = \sum_{\{v_1,v_2,\ldots,v_k\}\subseteq V(G)} [deg_G(v_1) + deg_G(v_2) + \cdots + deg_G(v_k)].$$

We first prove the upper bound. Without loss of generality, we can assume that $deg_G(v_1) \le deg_G(v_2) \le \ldots \le deg_G(v_k)$. Since

$$deg_G(v_1)deg_G(v_2)\ldots deg_G(v_k) \le \Delta^{k-1}deg_G(v_1) \tag{2}$$

$$\le \frac{\Delta^{k-1}}{k}(deg_G(v_1) + deg_G(v_2) + \cdots + deg_G(v_k)), \tag{3}$$

it follows that

$$\begin{aligned} M &= \sum_{\{v_1,v_2,\ldots,v_k\}\subseteq V(G)} deg_G(v_1)deg_G(v_2)\ldots deg_G(v_k) \\ &\leq \frac{\Delta^{k-1}}{k} \sum_{\{v_1,v_2,\ldots,v_k\}\subseteq V(G)} [deg_G(v_1)+deg_G(v_2)+\cdots+deg_G(v_k)] \\ &\leq \frac{\Delta^{k-1}}{k} N. \end{aligned}$$

For each $v \in V(G)$, there are $\binom{n-1}{k-1}$ k-subsets in G such that each of them contains v. The contribution of vertex v is exactly $\binom{n-1}{k-1}deg_G(v)$. From the arbitrariness of v, we have

$$N = \binom{n-1}{k-1} \sum_{v\in V(G)} deg_G(v) = 2m\binom{n-1}{k-1},$$

and hence

$$\mathrm{SGut}_k(G) \leq (n-1)M \leq (n-1)\frac{\Delta^{k-1}}{k}N = 2m(n-1)\binom{n-1}{k-1}\frac{\Delta^{k-1}}{k}. \quad (4)$$

Suppose that the left equality holds. Then all the inequalities in the above must be equalities. From the equality in (3), one can easily see that G is a regular graph. From the equality in (4), we have $d(S) = n-1$ for any $S \subseteq V(G)$, $|S| = k$. Since G is connected, then there exists an $S \subseteq V(G)$ such that $|d_G(S)| = k-1$. If $k \leq n-1$, then one can easily see that the upper bound is strict as $|d_G(S)| = k-1 \leq n-2$ for some S. Otherwise, $k = n$. Since G is connected, we have $|d_G(S)| = n-1$ for any $S \subseteq V(G)$. Hence G is a regular graph with $k = n$.

Conversely, one can see easily that the left equality holds for regular graph with $k = n$.

Lower bound: Without loss of generality, we can assume that $deg_G(v_1) \leq deg_G(v_2) \leq \ldots \leq deg_G(v_k)$. First we assume that $\delta \geq 2$. Then

$$\begin{aligned} deg_G(v_1)deg_G(v_2)\cdots deg_G(v_k) &\geq \delta^{k-1}deg_G(v_k) \\ &\geq \frac{\delta^{k-1}}{k}(deg_G(v_1)+deg_G(v_2)+\cdots+deg_G(v_k)), \end{aligned} \quad (5)$$

since $deg_G(v_1) \leq deg_G(v_2) \leq \cdots \leq deg_G(v_k)$. Furthermore, we have

$$\begin{aligned} \mathrm{SGut}_k(G) &\geq (k-1) \sum_{\{v_1,v_2,\ldots,v_k\}\subseteq V(G)} deg_G(v_1)deg_G(v_2)\ldots deg_G(v_k) \quad (6) \\ &\geq (k-1)\frac{\delta^{k-1}}{k} \sum_{\{v_1,v_2,\ldots,v_k\}\subseteq V(G)} [deg_G(v_1)+deg_G(v_2)+\cdots+deg_G(v_k)] \quad (7) \\ &= (k-1)\frac{\delta^{k-1}}{k}N \\ &= 2m(k-1)\binom{n-1}{k-1}\frac{\delta^{k-1}}{k}. \end{aligned}$$

Next we assume that $\delta = 1$. If $deg_G(v_1) = deg_G(v_2) = \cdots = deg_G(v_k) = 1$, then $d_G(S) \geq k$ and $deg_G(v_1)deg_G(v_2)\ldots deg_G(v_k) = 1$. If there exists some v_i such that $deg_G(v_i) \geq 2$, then $d_G(S) \geq k-1$ and $deg_G(v_1)deg_G(v_2)\ldots deg_G(v_k) \geq 2^{\max\{k-p,1\}} = 2^q$, where $1 \leq i \leq k$. Therefore, we have

$$
\begin{aligned}
\mathrm{SGut}_k(G) \;\geq\; & k \sum_{\substack{\{v_1,v_2,\ldots,v_k\}\subseteq V(G),\\ deg_G(v_1)=deg_G(v_2)=\cdots=deg_G(v_k)=1}} deg_G(v_1)deg_G(v_2)\ldots deg_G(v_k) \\
& +(k-1)\sum_{\substack{\{v_1,v_2,\ldots,v_k\}\subseteq V(G),\\ some\ deg_G(v_i)\geq 2}} deg_G(v_1)deg_G(v_2)\ldots deg_G(v_k) \qquad (8)\\
\geq\; & k\binom{p}{k}+2^q(k-1)\left[\binom{n}{k}-\binom{p}{k}\right]. \qquad (9)
\end{aligned}
$$

Suppose that the right equality holds. Then all the inequalities in the above must be equalities. Suppose that $\delta \geq 2$. From the equality in (6), $d_G(S) = k-1$ for any $S \subseteq V(G)$ and $|S| = k$, that is, $G[S]$ is connected for any $S \subseteq V(G)$ and $|S| = k$, and hence G is $(n-k+1)$-connected. From the equality in (7), we have $deg_G(v_1) = deg_G(v_2) = \cdots = deg_G(v_k)$ for any $S = \{v_1, v_2, \ldots, v_k\} \subseteq V(G)$, and hence G is a regular graph. Thus, G is a regular $(n-k+1)$-connected graph of order n.

Next suppose that $\delta = 1$. From the equality in (9), we obtain $deg_G(v_i) = 1$ or $deg_G(v_i) = 2$ for any vertex $v_i \in V(G)$. Since G is connected, $G \cong P_n$ and $p = 2$. If $k \geq 3$, then $q = k-p \geq 1$. In this case $d_G(S) = k-1$ for any $S \subseteq V(G)$ and $|S| = k$. One can easily see that $G \cong P_n$ and $k = n > 3$ (otherwise, $d_G(S) > k-1$ for some $S \subseteq V(G)$ as $q = k-p$). Otherwise, $k = p = 2$ and hence $q = 1$. In this case $G \cong P_3$ and $k = 2$.

Conversely, one can see easily that the equality holds on lower bound for a regular $(n-k+1)$-connected graph of order n $(\delta \geq 2)$, or $G \cong P_n$ and $k = n > 3$ $(\delta = 1)$, or $G \cong P_3$ and $k = 2$ $(\delta = 1)$. □

Example 1. *Let $G \cong K_n$ with $k = n$. Then*

$$\mathrm{SGut}_k(G) = (n-1)^{n+1} = 2m(n-1)\binom{n-1}{k-1}\frac{\Delta^{k-1}}{k}.$$

Let $G \cong K_n \backslash sK_2$ $(n = 2s)$ with $k = 3$. Then G is a $n-2$ regular graph of order n. Then

$$\mathrm{SGut}_k(G) = 2(n-2)^3\binom{n}{3} = 2m(k-1)\binom{n-1}{k-1}\frac{\delta^{k-1}}{k}.$$

Let $G \cong P_n$ with $k = n > 3$. Then

$$\mathrm{SGut}_k(G) = 2^{n-2}(n-1) = k\binom{p}{k}+2^q(k-1)\left[\binom{n}{k}-\binom{p}{k}\right] \text{ as } p = 2.$$

Let $G \cong P_n$ with $k = 2$. Then

$$\mathrm{SGut}_k(G) = 6 = k\binom{p}{k}+2^q(k-1)\left[\binom{n}{k}-\binom{p}{k}\right] \text{ as } p = 2.$$

3. Nordhaus–Gaddum-Type Results on $\mathrm{SGut}_k(G)$

We are now in a position to give the Nordhaus–Gaddum-type results on $\mathrm{SGut}_k(G)$.

Theorem 1. *Let G be a connected graph of order n with m edges, maximum degree Δ, minimum degree δ and a connected $\overline{G}$. Additionally, let k be an integer with $2 \leq k \leq n$. Then*
(1)

$$\mathrm{SGut}_k(G) + \mathrm{SGut}_k(\overline{G}) \leq (n-1)^2\binom{n}{k}s_1^{k-1}$$

and

$$\mathrm{SGut}_k(G)\cdot\mathrm{SGut}_k(\overline{G})\le 2m(n^2-n-2m)(n-1)^2\binom{n-1}{k-1}^2\frac{\Delta^{k-1}\,(n-\delta-1)^{k-1}}{k^2},$$

where $s_1=\max\{\Delta,\,n-\delta-1\}$. *Moreover, the upper bounds are sharp.*
(2)

$$\begin{aligned}&\mathrm{SGut}_k(G)+\mathrm{SGut}_k(\overline{G})\\ \ge\;&\begin{cases}(n-1)(k-1)\binom{n}{k}\,t_1^{k-1} & \text{if } \delta\ge 2,\ \Delta\le n-3\\ 2m(k-1)\binom{n-1}{k-1}\frac{\delta^{k-1}}{k}+k\binom{n}{k} & \text{if } \delta\ge 2,\ \Delta=n-2\\ k\binom{n}{k}+[n(n-1)-2m](k-1)\binom{n-1}{k-1}\frac{(n-\Delta-1)^{k-1}}{k} & \text{if } \delta=1,\ \Delta\le n-3\\ 2k\binom{n}{k} & \text{if } \delta=1,\ \Delta=n-2,\end{cases}\end{aligned}$$

where $t_1=\min\{\delta,\,n-\Delta-1\}$.
(3)

$$\begin{aligned}&\mathrm{SGut}_k(G)\cdot\mathrm{SGut}_k(\overline{G})\\ \ge\;&\begin{cases}2m(n^2-n-2m)(k-1)^2\binom{n-1}{k-1}^2\frac{\delta^{k-1}\,(n-\Delta-1)^{k-1}}{k^2} & \text{if } \delta\ge 2,\ \Delta\le n-3\\ 2m(k-1)\binom{n}{k}\binom{n-1}{k-1}\delta^{k-1} & \text{if } \delta\ge 2,\ \Delta=n-2\\ [n(n-1)-2m](k-1)\binom{n}{k}\binom{n-1}{k-1}(n-\Delta-1)^{k-1} & \text{if } \delta=1,\ \Delta\le n-3\\ k^2\binom{n}{k}^2 & \text{if } \delta=1,\ \Delta=n-2.\end{cases}\end{aligned}$$

Proof. (1) From Proposition 1, we have

$$\mathrm{SGut}_k(G)\le 2m(n-1)\binom{n-1}{k-1}\frac{\Delta^{k-1}}{k}$$

and

$$\mathrm{SGut}_k(\overline{G})\le [n(n-1)-2m](n-1)\binom{n-1}{k-1}\frac{(n-\delta-1)^{k-1}}{k},$$

and hence

$$\mathrm{SGut}_k(G)+\mathrm{SGut}_k(\overline{G})\le (n-1)^2\binom{n}{k}\,s_1^{k-1}$$

and

$$\mathrm{SGut}_k(G)\cdot\mathrm{SGut}_k(\overline{G})\le 2m(n^2-n-2m)(n-1)^2\binom{n-1}{k-1}^2\frac{\Delta^{k-1}(n-\delta-1)^{k-1}}{k^2}.$$

(2) From Proposition 1, if $\delta\ge 2$ and $\Delta\le n-3$, then

$$\begin{aligned}&\mathrm{SGut}_k(G)+\mathrm{SGut}_k(\overline{G})\\ \ge\;&2m(k-1)\binom{n-1}{k-1}\frac{\delta^{k-1}}{k}+[n(n-1)-2m](k-1)\binom{n-1}{k-1}\frac{(n-\Delta-1)^{k-1}}{k}\\ \ge\;&(n-1)(k-1)\binom{n}{k}\,t_1^{k-1}.\end{aligned}$$

If $\delta(G) \geq 2$ and $\Delta = n-2$, then

$$\begin{aligned}
& \mathrm{SGut}_k(G) + \mathrm{SGut}_k(\overline{G}) \\
\geq\ & 2m(k-1)\binom{n-1}{k-1}\frac{\delta^{k-1}}{k} + k\binom{p'}{k} + 2^{q'}(k-1)\left[\binom{n}{k} - \binom{p'}{k}\right] \\
\geq\ & 2m(k-1)\binom{n-1}{k-1}\frac{\delta^{k-1}}{k} + k\binom{p'}{k} + 2(k-1)\left[\binom{n}{k} - \binom{p'}{k}\right] \\
\geq\ & 2m(k-1)\binom{n-1}{k-1}\frac{\delta^{k-1}}{k} + k\binom{p'}{k} + k\left[\binom{n}{k} - \binom{p'}{k}\right] \\
=\ & 2m(k-1)\binom{n-1}{k-1}\frac{\delta^{k-1}}{k} + k\binom{n}{k},
\end{aligned}$$

where p' is the number of pendant vertices in G, and $q' = \max\{k-p', 1\}$.

If $\delta = 1$ and $\Delta \leq n-3$, then

$$\begin{aligned}
& \mathrm{SGut}_k(G) + \mathrm{SGut}_k(\overline{G}) \\
\geq\ & k\binom{p}{k} + 2^{q}(k-1)\left[\binom{n}{k} - \binom{p}{k}\right] + [n(n-1)-2m](k-1)\binom{n-1}{k-1}\frac{(n-\Delta-1)^{k-1}}{k} \\
\geq\ & k\binom{n}{k} + [n(n-1)-2m](k-1)\binom{n-1}{k-1}\frac{(n-\Delta-1)^{k-1}}{k},
\end{aligned}$$

where p is the number of pendant vertices in $\overline{G}$, and $q = \max\{k-p, 1\}$.

If $\delta = 1$ and $\Delta = n-2$, then

$$\begin{aligned}
& \mathrm{SGut}_k(G) + \mathrm{SGut}_k(\overline{G}) \\
\geq\ & k\binom{p}{k} + 2^{q}(k-1)\left[\binom{n}{k} - \binom{p}{k}\right] + k\binom{p'}{k} + 2^{q'}(k-1)\left[\binom{n}{k} - \binom{p'}{k}\right] \\
\geq\ & k\binom{n}{k} + k\binom{n}{k} \geq 2k\binom{n}{k},
\end{aligned}$$

where p, p' are the number of pendant vertices in $G, \overline{G}$, respectively, and $q = \max\{k-p, 1\}$, $q' = \max\{k-p', 1\}$.

From the above argument, we have

$$\mathrm{SGut}_k(G) + \mathrm{SGut}_k(\overline{G}) \geq \begin{cases} (n-1)(k-1)\binom{n}{k}\, t_1^{k-1} & \text{if } \delta \geq 2,\ \Delta \leq n-3 \\ 2m(k-1)\binom{n-1}{k-1}\frac{\delta^{k-1}}{k} + k\binom{n}{k} & \text{if } \delta \geq 2,\ \Delta = n-2 \\ k\binom{n}{k} + [n(n-1)-2m](k-1)\binom{n-1}{k-1}\frac{(n-\Delta-1)^{k-1}}{k} & \text{if } \delta = 1,\ \Delta \leq n-3 \\ 2k\binom{n}{k} & \text{if } \delta = 1,\ \Delta = n-2. \end{cases}$$

For (3), from Proposition 1, if $\delta \geq 2$ and $\Delta \leq n-3$, then

$$\mathrm{SGut}_k(G)\cdot\mathrm{SGut}_k(\overline{G})\geq 2m(n^2-n-2m)(k-1)^2\binom{n-1}{k-1}^2\frac{\delta^{k-1}\,(n-\Delta-1)^{k-1}}{k^2}.$$

If $\delta\geq 2$ and $\Delta=n-2$, then

$$\begin{aligned}&\mathrm{SGut}_k(G)\cdot\mathrm{SGut}_k(\overline{G})\\ \geq\;&\left[2m(k-1)\binom{n-1}{k-1}\frac{\delta^{k-1}}{k}\right]\left[k\binom{p'}{k}+2^{q'}(k-1)\left[\binom{n}{k}-\binom{p'}{k}\right]\right]\\ \geq\;&2m(k-1)\binom{n}{k}\binom{n-1}{k-1}\delta^{k-1},\end{aligned}$$

where p' is the number of pendant vertices in $\overline{G}$, and $q'=\max\{k-p',1\}$.

If $\delta=1$ and $\Delta\leq n-3$, then

$$\begin{aligned}&\mathrm{SGut}_k(G)\cdot\mathrm{SGut}_k(\overline{G})\\ \geq\;&\left[[n(n-1)-2m](k-1)\binom{n-1}{k-1}\frac{(n-\Delta-1)^{k-1}}{k}\right]\left[k\binom{p}{k}+2^{q}(k-1)\left[\binom{n}{k}-\binom{p}{k}\right]\right]\\ \geq\;&[n(n-1)-2m](k-1)\binom{n}{k}\binom{n-1}{k-1}(n-\Delta-1)^{k-1},\end{aligned}$$

where p is the number of pendant vertices in G, and $q=\max\{k-p,1\}$.

If $\delta(G)=1$ and $\Delta=n-2$, then

$$\begin{aligned}&\mathrm{SGut}_k(G)\cdot\mathrm{SGut}_k(\overline{G})\\ \geq\;&\left[k\binom{p}{k}+2^{q}(k-1)\left[\binom{n}{k}-\binom{p}{k}\right]\right]\left[k\binom{p'}{k}+2^{q'}(k-1)\left[\binom{n}{k}-\binom{p'}{k}\right]\right]\\ \geq\;&k^2\binom{n}{k}^2,\end{aligned}$$

where p, p' are the number of pendant vertices in G and $\overline{G}$, respectively, and $q=\max\{k-p,1\}$, $q'=\max\{k-p',1\}$.

From the above argument, we have

$$\mathrm{SGut}_k(G)\cdot\mathrm{SGut}_k(\overline{G})$$
$$\geq\begin{cases}2m(n^2-n-2m)(k-1)^2\binom{n-1}{k-1}^2\frac{\delta^{k-1}\,(n-\Delta-1)^{k-1}}{k^2} & \text{if } \delta(G)\geq 2,\ \Delta\leq n-3\\ 2m(k-1)\binom{n}{k}\binom{n-1}{k-1}\delta^{k-1} & \text{if } \delta(G)\geq 2,\ \Delta=n-2\\ [n(n-1)-2m](k-1)\binom{n}{k}\binom{n-1}{k-1}(n-\Delta-1)^{k-1} & \text{if } \delta(G)=1,\ \Delta\leq n-3\\ k^2\binom{n}{k}^2 & \text{if } \delta(G)=1,\ \Delta=n-2.\end{cases}$$

To show the sharpness of the upper bound and the lower bound for $\delta(G)\geq 2$, $\Delta\leq n-3$, we let G and $\overline{G}$ be two $\frac{n-1}{2}$-regular graphs of order n, where n is odd. If $k=n$, then $\mathrm{SGut}_k(G)=(n-1)(\frac{n-1}{2})^n$, $\mathrm{SGut}_k(\overline{G})=(n-1)(\frac{n-1}{2})^n$, $s_1=\max\{\Delta,\,n-\delta-1\}=\frac{n-1}{2}$, $\Delta\,(n-\delta-1)=(\frac{n-1}{2})^2$, $t_1=\min\{\delta,\,n-\Delta-1\}=\frac{n-1}{2}$ and $\delta\,(n-\Delta-1)=(\frac{n-1}{2})^2$. Furthermore, we have $\mathrm{SGut}_k(G)+\mathrm{SGut}_k(\overline{G})=2(n-1)(\frac{n-1}{2})^n=(n-1)^2\binom{n}{k}s_1^{k-1}$, $\mathrm{SGut}_k(G)\cdot\mathrm{SGut}_k(\overline{G})=(n-1)^2(\frac{n-1}{2})^{2n}=$

$2m(n^2 - n - 2m)(n-1)^2\binom{n-1}{k-1}^2 \frac{\Delta^{k-1}\,(n-\delta-1)^{k-1}}{k^2}$, $\mathrm{SGut}_k(G) + \mathrm{SGut}_k(\overline{G}) = 2(n-1)(\frac{n-1}{2})^n = (n-1)(k-1)\binom{n}{k}t_1^{k-1}$ and $\mathrm{SGut}_k(G) \cdot \mathrm{SGut}_k(\overline{G}) = (n-1)^2(\frac{n-1}{2})^{2n} = 2m(n^2-n-2m)(k-1)^2\binom{n-1}{k-1}^2 \frac{\delta^{k-1}\,(n-\Delta-1)^{k-1}}{k^2}$. □

The following corollary is immediate from the above theorem.

Corollary 1. *Let G be a connected graph of order $n \geq 4$ with maximum degree Δ and minimum degree δ. Then*
(1)

$$(n-1)^2\binom{n}{k}s_1^{k-1} \geq \mathrm{SGut}_k(G) + \mathrm{SGut}_k(\overline{G})$$

$$\geq \begin{cases} (n-1)(k-1)\binom{n}{k}\, t_1^{k-1} & \text{if } \delta \geq 2,\ \Delta \leq n-3 \\ n(k-1)\binom{n-1}{k-1}\frac{\delta^k}{k} + k\binom{n}{k} & \text{if } \delta \geq 2,\ \Delta = n-2 \\ k\binom{n}{k} + n(k-1)\binom{n-1}{k-1}\frac{(n-\Delta-1)^k}{k} & \text{if } \delta = 1,\ \Delta \leq n-3 \\ 2k\binom{n}{k} & \text{if } \delta = 1,\ \Delta = n-2, \end{cases}$$

where $s_1 = \min\{\Delta, n-\delta-1\}$, $t_1 = \min\{\delta, n-\Delta-1\}$;
(2)

$$n^2\binom{n-1}{k-1}^2 \frac{\Delta^{k-1}\,(n-\delta-1)^{k-1}\,(n-1)^4}{4k^2} \geq \mathrm{SGut}_k(G) \cdot \mathrm{SGut}_k(\overline{G})$$

$$\geq \begin{cases} n^2(k-1)^2\binom{n-1}{k-1}^2 \frac{\delta^k\,(n-\Delta-1)^k}{k^2} & \text{if } \delta \geq 2,\ \Delta \leq n-3 \\ n(k-1)\binom{n}{k}\binom{n-1}{k-1}\delta^k & \text{if } \delta \geq 2,\ \Delta = n-2 \\ n(k-1)\binom{n}{k}\binom{n-1}{k-1}\,(n-\Delta-1)^k & \text{if } \delta = 1,\ \Delta \leq n-3 \\ k^2\binom{n}{k}^2 & \text{if } \delta = 1,\ \Delta = n-2. \end{cases}$$

The following is the famous inequality by Pólya and Szegö:

Lemma 3. (Pólya–Szegö inequality) [25] *Let $(a_1, a_2, \ldots, a_r)$ and $(b_1, b_2, \ldots, b_r)$ be two positive r-tuples such that there exist positive numbers M_1, m_1, M_2, m_2 satisfying:*

$$0 < m_1 \leq a_i \leq M_1,\ 0 < m_2 \leq b_i \leq M_2,\ 1 \leq i \leq r.$$

Then

$$\frac{\sum_{i=1}^{r} a_i^2 \sum_{i=1}^{r} b_i^2}{\left(\sum_{i=1}^{r} a_i\, b_i\right)^2} \leq \frac{1}{4}\left(\sqrt{\frac{M_1\,M_2}{m_1\,m_2}} + \sqrt{\frac{m_1\,m_2}{M_1\,M_2}}\right)^2. \tag{10}$$

We now give more lower and upper bounds for $\mathrm{SGut}_k(G) \cdot \mathrm{SGut}_k(\overline{G})$ in terms of n, Δ and δ.

Theorem 2. *Let G be a connected graph of order n with maximum degree Δ, minimum degree δ and a connected $\overline{G}$. Additionally, let k be an integer with $2 \leq k \leq n$. Then*

$$\mathrm{SGut}_k(G)\cdot \mathrm{SGut}_k(\overline{G}) \geq \begin{cases} (k-1)^2\,\delta^k\,(n-\delta-1)^k\,\binom{n}{k}^2 & \text{if } \Delta+\delta\leq n-1, \\ (k-1)^2\,\Delta^k\,(n-\Delta-1)^k\,\binom{n}{k}^2 & \text{if } \Delta+\delta\geq n-1 \end{cases} \tag{11}$$

with equality holding if and only if G is a regular graph with $d_G(S) = d_{\overline{G}}(S) = k-1$ for any $S \subseteq V(G)$, $|S| = k$, and

$$\mathrm{SGut}_k(G)\cdot \mathrm{SGut}_k(\overline{G}) \leq \frac{(n-1)^{2k+2}}{2^{2k+2}}\binom{n}{k}^2\left[\left(\frac{\Delta\,(n-\delta-1)}{\delta\,(n-\Delta-1)}\right)^k + \left(\frac{\delta\,(n-\Delta-1)}{\Delta\,(n-\delta-1)}\right)^k + 2\right],$$

Moreover, the equality holds if and only if G is a $\left(\frac{n-1}{2}\right)$-regular graph with $k = n$, n is odd.

Proof. Lower bound: By Cauchy–Schwarz inequality with (1), we have

$$\begin{aligned}
\mathrm{SGut}_k(G)\cdot \mathrm{SGut}_k(\overline{G}) &\geq (k-1)^2 \sum_{\substack{S\subseteq V(G)\\ |S|=k}}\left(\prod_{v\in S} deg_G(v)\right)\sum_{\substack{S\subseteq V(\overline{G})\\ |S|=k}}\left(\prod_{v\in S} deg_{\overline{G}}(v)\right) && (12)\\
&\geq (k-1)^2\left(\sum_{\substack{S\subseteq V(G)\\ |S|=k}}\left(\prod_{v\in S} deg_G(v)\prod_{v\in S} deg_{\overline{G}}(v)\right)^{1/2}\right)^2 && (13)\\
&\geq (k-1)^2\left(\sum_{\substack{S\subseteq V(G)\\ |S|=k}}\left(\prod_{v\in S} deg_G(v)\,(n-1-deg_G(v))\right)^{1/2}\right)^2.
\end{aligned}$$

Since $\delta \leq deg_G(v) \leq \Delta$, one can easily see that

$$deg_G(v)\,(n-1-deg_G(v)) \geq \begin{cases} \delta\,(n-\delta-1) & \text{if } \Delta+\delta\leq n-1, \\ \Delta\,(n-\Delta-1) & \text{if } \Delta+\delta\geq n-1. \end{cases} \tag{14}$$

From the above results, we have

$$\mathrm{SGut}_k(G)\cdot \mathrm{SGut}_k(\overline{G}) \geq \begin{cases} (k-1)^2\,\delta^k\,(n-\delta-1)^k\,\binom{n}{k}^2 & \text{if } \Delta+\delta\leq n-1, \\ (k-1)^2\,\Delta^k\,(n-\Delta-1)^k\,\binom{n}{k}^2 & \text{if } \Delta+\delta\geq n-1. \end{cases}$$

The equality holds in (12) if and only if $d_G(S) = d_{\overline{G}}(S) = k-1$ for any $S \subseteq V(G)$ with $|S| = k$. By the Cauchy–Schwarz inequality, the equality holds in (13) if and only if

$$\frac{\prod_{v\in S_1} deg_G(v)}{\prod_{v\in S_1} deg_{\overline{G}}(v)} = \frac{\prod_{v\in S_2} deg_G(v)}{\prod_{v\in S_2} deg_{\overline{G}}(v)} \quad \text{for any } S_1,\, S_2 \in V(G) \text{ with } |S_1| = |S_2| = k,$$

that is, if and only if $deg_G(u) = deg_G(v)$ for any $u, v \in V(G)$, that is, if and only if G is a regular graph. Hence the equality holds in (11) if and only if G is a regular graph with $d_G(S) = d_{\overline{G}}(S) = k-1$ for any $S \subseteq V(G)$, $|S| = k$.

Upper bound: Let $\overline{\Delta}$ and $\overline{\delta}$ be the maximum degree and the minimum degree of graph $\overline{G}$, respectively. Then $\overline{\Delta} = n-\delta-1$ and $\overline{\delta} = n-\Delta-1$. By (1) and (10), we have

$$\begin{aligned}
&\mathrm{SGut}_k(G)\cdot \mathrm{SGut}_k(\overline{G})\\
\leq\ & (n-1)^2 \sum_{\substack{S\subseteq V(G)\\|S|=k}} \left(\prod_{v\in S} deg_G(v)\right) \sum_{\substack{S\subseteq V(\overline{G})\\|S|=k}} \left(\prod_{v\in S} deg_{\overline{G}}(v)\right)\\
\leq\ & (n-1)^2 \left(\sum_{\substack{S\subseteq V(G)\\|S|=k}} \left(\prod_{v\in S} deg_G(v) \prod_{v\in S} deg_{\overline{G}}(v)\right)^{1/2}\right)^2 \frac{1}{4}\left(\left(\frac{\Delta\,\overline{\Delta}}{\delta\,\overline{\delta}}\right)^{k/2}+\left(\frac{\delta\,\overline{\delta}}{\Delta\,\overline{\Delta}}\right)^{k/2}\right)^2\\
\leq\ & \frac{(n-1)^2}{4} \left(\sum_{\substack{S\subseteq V(G)\\|S|=k}} \left(\prod_{v\in S} deg_G(v)\,(n-1-deg_G(v))\right)^{1/2}\right)^2 \left(\left(\frac{\Delta\,\overline{\Delta}}{\delta\,\overline{\delta}}\right)^{k/2}+\left(\frac{\delta\,\overline{\delta}}{\Delta\,\overline{\Delta}}\right)^{k/2}\right)^2.
\end{aligned}$$

One can easily see that

$$deg_G(v)\,(n-1-deg_G(v)) \leq \frac{(n-1)^2}{4} \text{ for any } v\in V(G).$$

Using this result in the above with $\overline{\Delta} = n-\delta-1$ and $\overline{\delta} = n-\Delta-1$, we get

$$\mathrm{SGut}_k(G)\cdot \mathrm{SGut}_k(\overline{G}) \leq \frac{(n-1)^{2k+2}}{2^{2k+2}}\binom{n}{k}^2 \left[\left(\frac{\Delta\,(n-\delta-1)}{\delta\,(n-\Delta-1)}\right)^k + \left(\frac{\delta\,(n-\Delta-1)}{\Delta\,(n-\delta-1)}\right)^k + 2\right].$$

Moreover, the above equality holds if and only if G is a $\left(\frac{n-1}{2}\right)$-regular graph with $k=n$, n is odd (very similar proof of the Proposition 1). □

Example 2. *Let $G \cong C_n$ with $k=n$. Then $\delta = 2$ and hence*

$$\mathrm{SGut}_k(G)\cdot \mathrm{SGut}_k(\overline{G}) = (n-1)^2(n-3)^n\,2^n = (k-1)^2\,\delta^k\,(n-\delta-1)^k\binom{n}{k}^2.$$

Let G be a $\left(\frac{n-1}{2}\right)$-regular graph of order n with $k=n$ and odd n. Then $\Delta = \delta = \frac{n-1}{2}$ and hence

$$\begin{aligned}
\mathrm{SGut}_k(G)\cdot \mathrm{SGut}_k(\overline{G}) &= \frac{(n-1)^{2n+2}}{2^{2n}}\\
&= \frac{(n-1)^{2k+2}}{2^{2k+2}}\binom{n}{k}^2 \left[\left(\frac{\Delta\,(n-\delta-1)}{\delta\,(n-\Delta-1)}\right)^k + \left(\frac{\delta\,(n-\Delta-1)}{\Delta\,(n-\delta-1)}\right)^k + 2\right].
\end{aligned}$$

We now give more lower and upper bounds of $\mathrm{SGut}_k(G) + \mathrm{SGut}_k(\overline{G})$ in terms of n, Δ and δ.

Theorem 3. *Let G be a connected graph of order n with maximum degree Δ, minimum degree δ and a connected $\overline{G}$. Additionally, let k be an integer with $2 \leq k \leq n$. Then*

$$\mathrm{SGut}_k(G) + \mathrm{SGut}_k(\overline{G}) \geq \begin{cases} 2\,(k-1)\,\delta^{k/2}\,(n-\delta-1)^{k/2}\binom{n}{k} & \text{if } \Delta+\delta \leq n-1,\\[2ex] 2\,(k-1)\,\Delta^{k/2}\,(n-\Delta-1)^{k/2}\binom{n}{k} & \text{if } \Delta+\delta \geq n-1 \end{cases} \tag{15}$$

with equality holding if and only if G is a $\left(\frac{n-1}{2}\right)$-regular graph with odd n and $d_G(S) = d_{\overline{G}}(S) = k-1$ for any $S \subseteq V(G)$, $|S| = k$, and

$$\mathrm{SGut}_k(G) + \mathrm{SGut}_k(\overline{G}) \le (n-1)\left[\Delta^k + (n-\delta-1)^k\right]\binom{n}{k} \tag{16}$$

with equality holding if and only if G is a regular graph with $k = n$.

Proof. For any two real numbers a, b, we have $(a-b)^2 \ge 0$, that is, $a^2 + b^2 \ge 2\,a\,b$ with equality holding if and only if $a = b$. Therefore we have

$$\begin{aligned}
\prod_{v\in S} deg_G(v) + \prod_{v\in S} deg_{\overline{G}}(v) &\ge 2\left(\prod_{v\in S} deg_G(v) \prod_{v\in S} deg_{\overline{G}}(v)\right)^{1/2} \\
&= 2\left(\prod_{v\in S} deg_G(v)\, deg_{\overline{G}}(v)\right)^{1/2} \\
&= 2\left(\prod_{v\in S} deg_G(v)\,(n - deg_G(v) - 1)\right)^{1/2}.
\end{aligned}$$

From the above result with (14), we get

$$\prod_{v\in S} deg_G(v) + \prod_{v\in S} deg_{\overline{G}}(v) \ge \begin{cases} 2\,\delta^{k/2}\,(n-\delta-1)^{k/2} & \text{if } \Delta+\delta \le n-1, \\ 2\,\Delta^{k/2}\,(n-\Delta-1)^{k/2} & \text{if } \Delta+\delta \ge n-1. \end{cases}$$

Now,

$$\begin{aligned}
\mathrm{SGut}_k(G) + \mathrm{SGut}_k(\overline{G}) &= \sum_{\substack{S\subseteq V(G)\\ |S|=k}} \left[\left(\prod_{v\in S} deg_G(v)\right) d_G(S) + \left(\prod_{v\in S} deg_{\overline{G}}(v)\right) d_{\overline{G}}(S)\right] \\
&\ge (k-1) \sum_{\substack{S\subseteq V(G)\\ |S|=k}} \left[\prod_{v\in S} deg_G(v) + \prod_{v\in S} deg_{\overline{G}}(v)\right] \\
&\ge \begin{cases} 2\,(k-1)\,\delta^{k/2}\,(n-\delta-1)^{k/2}\binom{n}{k} & \text{if } \Delta+\delta \le n-1, \\ 2\,(k-1)\,\Delta^{k/2}\,(n-\Delta-1)^{k/2}\binom{n}{k} & \text{if } \Delta+\delta \ge n-1. \end{cases}
\end{aligned}$$

From the above, one can easily see that the equality holds in (15) if and only if G is a $\left(\frac{n-1}{2}\right)$-regular graph with odd n and $d_G(S) = d_{\overline{G}}(S) = k-1$ for any $S \subseteq V(G)$, $|S| = k$.

Upper bound: By arithmetic-geometric mean inequality, we have

$$
\begin{aligned}
\mathrm{SGut}_k(G)+\mathrm{SGut}_k(\overline{G}) &= \sum_{\substack{S\subseteq V(G)\\|S|=k}}\left[\left(\prod_{v\in S} deg_G(v)\right)d_G(S)+\left(\prod_{v\in S} deg_{\overline{G}}(v)\right)d_{\overline{G}}(S)\right]\\
&\le (n-1)\sum_{\substack{S\subseteq V(G)\\|S|=k}}\left[\prod_{v\in S} deg_G(v)+\prod_{v\in S} deg_{\overline{G}}(v)\right]\\
&\le (n-1)\sum_{\substack{S\subseteq V(G)\\|S|=k}}\left[\left(\frac{\sum_{v\in S} deg_G(v)}{k}\right)^k+\left(\frac{\sum_{v\in S} deg_{\overline{G}}(v)}{k}\right)^k\right]\\
&= \frac{(n-1)}{k^k}\sum_{\substack{S\subseteq V(G)\\|S|=k}}\left[\left(\sum_{v\in S} deg_G(v)\right)^k+\left(\sum_{v\in S}(n-deg_G(v)-1)\right)^k\right]\\
&= \frac{(n-1)}{k^k}\sum_{\substack{S\subseteq V(G)\\|S|=k}}\left[\left(\sum_{v\in S} deg_G(v)\right)^k+\left(k\,(n-1)-\sum_{v\in S} deg_G(v)\right)^k\right]\\
&\le \frac{(n-1)}{k^k}\sum_{\substack{S\subseteq V(G)\\|S|=k}}\left[(k\,\Delta)^k+(k\,(n-1)-k\,\delta)^k\right]\\
&= (n-1)\left[\Delta^k+(n-\delta-1)^k\right]\binom{n}{k}.
\end{aligned}
$$

From the above, one can easily see that the equality holds in (16) if and only if G is a regular graph with $k = n$ (very similar proof of the Proposition 1). □

Example 3. *Let G be a $\left(\frac{n-1}{2}\right)$-regular graph with odd n and $k = n$. Then $\delta = \frac{n-1}{2}$ and hence*

$$
\mathrm{SGut}_k(G)+\mathrm{SGut}_k(\overline{G}) = \frac{(n-1)^{n+1}}{2^{n-1}} = 2\,(k-1)\,\delta^{k/2}\,(n-\delta-1)^{k/2}\binom{n}{k}
$$

Let $G \cong C_n$ with $k = n$. Then $\Delta = \delta = 2$, $\overline{\Delta} = \overline{\delta} = 2$ and hence

$$
\mathrm{SGut}_k(G)+\mathrm{SGut}_k(\overline{G}) = (n-1)\left[2^n+(n-3)^n\right] = (n-1)\left[\Delta^k+(n-\delta-1)^k\right]\binom{n}{k}.
$$

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