Singularities of Lagrangian Mean Curvature Flow

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I, Albert Wood, confirm that the work presented in this thesis is my own. Where information has been derived from other sources, I confirm that this has been indicated in the work.

Abstract

Lagrangian mean curvature flow is a promising tool in the study of special Lagrangians. Though the flow has been of interest now for several decades, singularities of the flow are still not well understood, and extensions and generalisations of the flow are still being uncovered and explored. The work of this thesis investigates two distinct topics, both of which shed light on the structure and behaviour of Lagrangian mean curvature flow.

We first demonstrate the existence of a boundary condition for Lagrangian mean curvature flow in Calabi-Yau manifolds which preserves the Lagrangian condition. The boundary condition is a generalisation of the constant Lagrangian angle difference between intersecting special Lagrangian submanifolds. This work applies and extends the original work of Smoczyk [60] which proves that the class of closed Lagrangian submanifolds in Calabi-Yau manifolds is preserved.

We also investigate singularities of equivariant and almost-calibrated Lagrangian mean curvature flow – flows in $\mathbb{C}^n$ with an $O(n)$-symmetry and a pinching condition on the Lagrangian angle. Given a singularity, we prove that any Type I blowup is a unique pair of planes $P_1 \cup P_2$, any Type II blowup is the Lawlor neck Σ_{Law} with asymptotes $P_1 \cup P_2$, and any 'intermediate' blowup is $P_1 \cup P_2$. We also prove conditions for long-time existence and singularity formation of the flow.

Finally, we investigate the relationship between these topics. We prove that any almost-calibrated equivariant Lagrangian mean curvature flow with boundary on the Lawlor neck converges in infinite time to a special Lagrangian disc, and that the same is true for any rescaled Lagrangian mean curvature flow with boundary on the Clifford torus (assuming extra conditions on the Lagrangian angle).

Impact Statement

Geometric flows are an indispensable tool in solving problems, in science and mathematics alike. On the scientific end, the physically-inspired nature of geometric flows means they are the most natural way to describe many physical phenomena; for example the motion of interfaces by surface tension is described by a modified mean curvature flow. Geometric flows have also been used to solve conjectures of theoretical physics, for example the Riemannian Penrose conjecture, solved by H. Bray using a specially constructed flow of metrics. Further afield, the mean curvature flow has been employed in computer science as a novel tool for image processing.

In pure mathematics, the most famous application of geometric flows is the use of the Ricci flow in R. Hamilton and G. Perelman's work in proving the Poincaré conjecture. However, there are many more examples of geometric flows with applications in pure mathematics, including the Yamabe flow, the Yang-Mills flow, the Willmore flow and the mean curvature flow.

This thesis explores singularities and generalisations of Lagrangian mean curvature flow, a geometric flow of Lagrangian submanifolds. Though the work of this thesis focuses on one particular flow, the techniques and themes of the thesis are of wider interest. Extending the definition of geometric flows to include a suitable boundary condition is a physically natural problem, and in this thesis I demonstrate how this is achieved in our special case. Additionally, to apply geometric flows to physical and computational problems, an understanding of singularities is important, as numerical methods often break down at singular points. The study of singularities of Lagrangian mean curvature flow in the latter half of this thesis will make easier such applications. Finally, my results contribute to the general understanding of geometric flows, and as can be seen from the impressive list of problems solved using flow techniques, the development of their theory is of great value to mathematicians, physicists and computer scientists alike.

I consider the exposition of mathematics to be of similar importance to its creation. I have therefore taken a great effort to make the work accessible to those with a general geometric background, and an interest in the subject and its themes. As a result, I expect it will be of use to anyone who wishes to learn about the subject of Lagrangian mean curvature flow, or apply the results of the thesis in their own work.

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Chapter 1

Introduction

The search for a 'canonical representative' is a common refrain of geometry. Hodge theory is built on the observation that every de Rham cohomology class of a closed Riemannian manifold has a unique harmonic representative. The Calabi conjecture implies the existence of a unique Ricci-flat Kähler metric on any compact Kähler manifold with vanishing first Chern class. The Poincaré conjecture states that any simply-connected closed 3-fold is represented homeomorphically by the sphere.

Though these problems concern different objects in distinct fields of geometry, they are all connected by a common proof strategy: the use of flows. The Hodge theorem may be resolved by noting that closed forms remain closed under the heat flow, and converge in infinite time to the harmonic representative. The Ricci flow preserves the Kähler condition, and in certain cases can be shown to converge to Kähler metrics with constant Ricci curvature – Kähler-Einstein metrics. And most famously, the Poincaré conjecture was proven correct by Richard Hamilton and Grigori Perelman using the Ricci flow with surgeries – to date the only Millennium prize problem to have been resolved.

The use of flows to solve geometric problems is a powerful and modern strategy, and the driving motivation behind this work is to apply this philosophy and emulate the above successes in the case of a relatively new and surprising geometric flow – Lagrangian mean curvature flow.

1.1 Historical Context

Geometric Flows and Singularities

Geometric flows describe the gradient descent of a manifold with respect to a given functional. Since the important development of regularity theory for nonlinear parabolic PDE by N. Krylov and M. Safanov [38], they have become a popular and powerful alternative to the more traditional technique of direct minimisation of a functional when solving minimisation problems. However, the history of geometric flows predates these developments. The first use of a geometric flow was by J. Eells and J. Sampson in 1964 [19], where the harmonic map heat flow was used to find harmonic maps into Riemannian manifolds of non-positive curvature.

The use of flows offers several advantages. In analogy with the heat equation on functions, the smoothing effect of a gradient flow can provide control over the submanifold's curvature quantities, and lead to long-time existence and convergence of the flow. Whereas direct minimisation requires the development of a compactness theorem to provide a subsequential limit of the minimising subsequence (at which point regularity must often be proven separately), a geometric flow directly provides a homotopy from the initial condition to the final state, and the smoothing effect of the flow provides regularity. Even if one applies a surgery procedure to continue the flow past singularities, the topological changes can be tracked, and so the resulting manifold can be related to the initial condition. The use of flows also opens up the toolbox of techniques from parabolic PDE, such as the maximum principle, which is an invaluable tool for bounding and constraining solutions.

One of the first geometric flows to be studied in detail was the *mean curvature flow*, which moves a submanifold of a Riemannian manifold in the direction of the mean curvature vector (see Section 2.2). Originally proposed as a model for moving grain boundaries in annealing metals by W. Mullins [49], the mean curvature flow is the gradient descent for one of the most basic and well-studied functionals in mathematics: the volume functional of submanifolds in Riemannian manifolds.

Geometric flows are not without their problems. The nonlinearities of geometric flows often lead to singularity formation during the flow. In fact for compact manifolds, the best case scenario is often that the entire manifold collapses simultaneously to a point at the final time. This is true for the most basic example of Ricci flow and mean curvature flow: a shrinking sphere of constant curvature. An analysis of the possible singularities of a flow is a vital cornerstone of the theory, as it often makes possible a smooth continuation of the flow by topological 'surgeries', as in the work of R. Hamilton and G. Perelman on Ricci flow with surgeries [27, 53–55] employed in the proof of the Poincaré conjecture.

In the case of mean curvature flow, there is a large class of singularities that are more easily studied. By proving an elegant monotonicity formula, G. Huisken was able to demonstrate that so called *Type I* singularities are modelled on self-similarly shrinking solutions of the flow (see Section 2.2.2). Type I singularities are not rare, in fact it is expected that generic embedded mean curvature flows can only form singularities modelled on spheres and cylinders, both of which are Type I singularities. Aside from computational evidence, progress on this problem includes the work of T. Colding and W. Minicozzi [13] which proves that cylinders, spheres and planes are the only stable self-shrinkers, and more recent work of O. Chodosh, K. Choi, C. Mantoulidis and F. Schulze [12] which considers generic initial data. This extremely limited set of singularities makes possible the kind of analysis performed by G. Huisken and C. Sinestrari in [33], where the mean curvature flow with surgeries is applied to classify 2-convex immersed submanifolds of Euclidean space.

For mean curvature flow in general, other types of singularities may occur, known as *Type II* singularities, which are not modelled on self-shrinkers. These singularities are much more difficult to study, since it is not simply the local behaviour of the flow that is causing the singularity to occur. For example, an immersed figure eight curve in $\mathbb{R}^2$ forms a singularity modelled on the translating solution known as the 'grim reaper' (see [5] and Figure 2.1), and there even exist embedded spheres that form Type II 'degenerate neck-pinch' singularities modelled on the translating 'bowl soliton' [6]. Perhaps counter-intuitively, Type II singularities of the flow can even be modelled on minimal submanifolds, which are themselves static under mean curvature flow. This was exemplified in the hypersurface case by J. Velazquez [71], and it has recently been shown by M. Stolarski [66] that there exist flows with singularities modelled on his example whose mean curvature remains bounded as the singularity forms.

Lagrangian Mean Curvature Flow

An exciting new side of mean curvature flow was uncovered by K. Smoczyk [60] when he proved that the mean curvature flow preserves Lagrangian submanifolds of Calabi-Yau manifolds, a phenomenon now known as *Lagrangian mean curvature flow*. Lagrangian submanifolds of Kähler manifolds are those for which the symplectic form vanishes (see Section 2.3 and Section 2.4 for introductions to Kähler and Lagrangian geometry). The preservation of the Lagrangian condition is unexpected, since mean curvature flow is a concept of Riemannian submanifold geometry, rather than one of symplectic or Kähler geometry! Since mean curvature flow is the gradient descent of volume, an immediate and natural conjecture is that the Lagrangian mean curvature flow enjoys long-time existence and convergence to a minimal Lagrangian representative – a *special Lagrangian*.

Special Lagrangians are indeed special. As well as being minimal they are *calibrated submanifolds* of Calabi-Yau manifolds and so are volume-minimising in their homology class. Understanding special Lagrangians is therefore an important problem from both Riemannian and symplectic viewpoints, and the technique of mean curvature flow could simplify what is otherwise a difficult nonlinear PDE problem.

However, the above conjecture is quickly seen to be overambitious. Just as in mean curvature flow in general, the Lagrangian mean curvature flow forms singularities. In fact, a result of A. Neves [51] shows that any Lagrangian may be deformed within its Hamiltonian isotopy class to one that encounters such a singularity during the flow. Even more disappointingly, given a homology class of Lagrangian submanifolds, there may not be a minimal representative to find! An example found by J. Wolfson [77] demonstrates the existence of 2-dimensional Lagrangians which minimise volume amongst other Lagrangians in their homology class, but which are not branched immersions. Surfaces in 4-manifolds minimising volume in their homology class must be branched immersions, so it follows that there cannot be a special Lagrangian in this class.

The method of geometric flows has seen success before, in very similar contexts.

For example, Ricci flow preserves the class of Kähler metrics. The resulting *Kähler-Ricci flow* has for a while been used in the study of Kähler geometry, for example by H-D. Cao [10] to demonstrate the existence of Kähler-Einstein metrics on manifolds with negative first Chern class. S. Donaldson has also used the *Yang-Mills flow* to demonstrate the existence of Hermitian Yang-Mills connections on certain holomorphic vector bundles [14], a result now known as the Donaldson-Uhlenbeck-Yau theorem. In this latter case, the long-time existence and convergence of the flow is equivalent to an algebraic 'stability' condition on the holomorphic vector bundle.

The success of the flow approach in finding Hermitian Yang-Mills connections is particularly promising for Lagrangian mean curvature flow. The complex geometry and symplectic geometry of Calabi-Yau manifolds are related by a series of conjectural equivalences known as *Mirror Symmetry*. Under this framework, Lagrangians correspond to holomorphic vector bundles, and special Lagrangians to those bundles carrying a Hermitian Yang-Mills connection. These equivalences inspired R. Thomas and S-T. Yau [69] to conjecture that there is a 'stability condition' for Lagrangian submanifolds, that implies long-time existence of the flow and convergence to a special Lagrangian, just as there is on the other side of the mirror for Yang-Mills flow. This conjecture, now known as the Thomas-Yau conjecture, has recently been refined further by D. Joyce [36], who has rephrased the stability condition in terms of Fukaya categories of Lagrangian submanifolds. The Thomas-Yau conjecture has been proven for special cases in [36, 43, 69], and these works provide more detail on the precise nature of the stability condition for Lagrangian mean curvature flow.

Stability aside, there are more fundamental reasons for a Lagrangian submanifold not to flow to a special Lagrangian representative. Considering a Lagrangian submanifold of a Calabi-Yau manifold $L \subset \mathscr{Y}$, the normal bundle is related to the tangent bundle via the almost-complex structure J. Using this correspondence, the mean curvature may be represented as a 1-form, H, which is the derivative of a multi-valued function θ - the *Lagrangian angle*. The 1-form H is identically zero for special Lagrangians, since they are minimal - in particular θ is a single-valued function, a constant. It follows that if a Lagrangian flows to a special Lagrangian under mean curvature flow, then the Lagrangian angle must be single-valued at all times, and so the *Maslov class* $[H]$ of L must be trivial. In this case we say the Lagrangian is *zero Maslov*.

In fact, in order to properly define the stability condition suggested by Thomas and Yau, we need the stronger condition of *almost-calibrated*, which demands not just that H is exact, but that the Lagrangian angle θ has variation less than π. Therefore, the study of almost-calibrated Lagrangian submanifolds and their singularities is of vital importance in resolving the Thomas-Yau conjecture.

Singularities of almost-calibrated Lagrangian mean curvature flow are quite unlike those of mean curvature flow in general. Type I singularities are no longer the norm, in fact M-T. Wang shows in [72] that they cannot occur! However, there is still structure to be found. In [50], A. Neves gives an example of a Type II singularity modelled on an important special Lagrangian submanifold, the *Lawlor neck* (see Example 3.3.1), and it has been conjectured by D. Joyce [36] that singularities modelled on Lawlor necks are generic for Lagrangian mean curvature flow. Proving this result would go a long way towards making surgery feasible for Lagrangian mean curvature flow.

1.2 Summary of Results

This thesis builds on the groundbreaking work of mathematicians such as K. Smoczyk, A. Neves, R. Thomas, S-T. Yau and D. Joyce, by investigating the characteristics of Lagrangian mean curvature flow. We are motivated by the Thomas-Yau conjecture to understand singularities of the flow, and so this is our main focus. The other question that we explore is that of how one may generalise the flow, in order to provide a wider class of Lagrangian flows for use in symplectic geometry, and to build understanding of the mechanisms by which the flow preserves the Lagrangian condition. We now give a summary of the results of the thesis.

Lagrangian Mean Curvature Flow with Boundary

Given that the Lagrangian mean curvature flow is of intrinsic interest, a natural question is ‘How may Lagrangian mean curvature flow be generalised?’ A historically important generalisation, which has been considered for many other geometric flows and PDE more generally, is to find a suitable boundary condition for the flow. Chapter 4 provides an exposition of joint work with C. Evans and B. Lambert [20], in which we consider this question, and succeed in finding a boundary condition for mean curvature flow which preserves the Lagrangian condition.

Aside from flows with boundary being a geometrically natural problem to consider, it has been suggested by D. Joyce that to resolve the Thomas-Yau conjecture, one should work in an isomorphism class of a conjectural enlarged version of the derived Fukaya category $D^b\mathscr{F}(M)$ rather than the Hamiltonian isotopy class of L. In particular, the standard derived Fukaya category (as developed by Fukaya–Oh–Ohta–Ono and P. Seidel [59]) should be expanded to include immersed and singular Lagrangians. In order to work within this category, it is necessary to work with a larger class of Lagrangian mean curvature flows than have been previously considered. A full generalisation would include flows of Lagrangian networks (see [47] for the equiangular 1-dimensional version of this phenomenon), but Lagrangian mean curvature flow with boundary is a first step towards such a generalisation.

Boundary conditions for codimension 1 mean curvature flow have been considered in a variety of contexts, for example by K. Ecker [15], B. Priwitzer [57] and B. Thorpe [70] in the Dirichlet case, by J. Buckland [9], N. Edelen [17] [18], G. Huisken [31], B. Lambert [39] [40], J. Lira and G. Wanderley [42], A. Stahl [64] [65] and V. Wheeler [74] [75] in the Neumann case, and by G. Wheeler and V. Wheeler [73] in a mixed Dirichlet–Neumann case. Generalisations of Lagrangian mean curvature flow have also been considered. For example, generalisations which relax the condition of Calabi-Yau on the ambient manifold have been considered by K. Smoczyk [63], T. Behrndt [7] and J. Sun and L. Yang [68], and J. Lotay and T. Pacini [44] have considered flows which relax the Lagrangian condition. However, no prior work considers boundary conditions for mean curvature flow which preserve the Lagrangian condition.

Boundary conditions which preserve the class of Lagrangian submanifolds are exceptional; standard Dirichlet and Neumann conditions do not have this property. One might be tempted to consider instead boundary conditions on a potential function (see Example 2.4.1), but these are not natural on a geometric level. The first main result of this thesis provides the first example of a suitable geometric boundary condition that preserves the Lagrangian condition.

We now give some intuition as to how one may define the condition. Consider a Lagrangian submanifold of a Calabi-Yau manifold, $L \subset \mathscr{Y}$. Even if the Lagrangian is not zero Maslov, it can be shown that there exists a Lagrangian angle function $\theta : L \to \mathbb{R}/2\pi\mathbb{Z}$ with the property that the mean curvature 1-form is given by $H = d\theta$. The angle θ is therefore constant for special Lagrangians, and if two special Lagrangians intersect, their Lagrangian angles must differ by a constant along the intersection. Inspired by this, we impose this condition for Lagrangian submanifolds. Fixing a constant Lagrangian angle difference α for a flowing Lagrangian L_t (with Lagrangian angle θ) with boundary on a Lagrangian submanifold Σ_t (with Lagrangian angle $\tilde{\theta}$), we have a geometrically natural mixed Dirichlet–Neumann boundary problem:

$$\begin{cases} \left(\frac{d}{dt}F(x,t)\right)^{\perp} = \vec{H}(x,t) & \text{for all } (x,t) \in L^n \times [0,T), \\ F(x,0) = F_0(x) & \text{for all } x \in L^n, \\ \partial L_t \subset \Sigma_t & \text{for all } t \in [0,T), \\ e^{i(\tilde{\theta}-\theta)}(x,t) = ie^{i\alpha} & \text{for all } (x,t) \in \partial L^n \times [0,T). \end{cases} \tag{1.1}$$

In order for the Lagrangian condition to be preserved, it is in fact necessary for the boundary Lagrangian submanifold Σ_t to be moving by mean curvature flow. In this case are we able to show that enough of the symmetry of the second fundamental form of the boundary is inherited by the flow, which is required for the flow to remain Lagrangian. The fact that this works is quite remarkable!

It should be noted, however, that the Lagrangian angle is usually only defined for Lagrangian submanifolds. During the proof that L_t remains Lagrangian under this flow, the concept of Lagrangian angle must therefore be generalised, else (1.1) does not make sense.

The main results of Chapter 4 are Theorem 4.0.1 and Theorem 4.0.2, the former of which states that mean curvature flow with boundary problem (1.1) preserves the Lagrangian condition, and the latter that any initially smooth configuration of Lagrangian submanifolds is the initial condition for a unique Lagrangian mean curvature flow with boundary. For brevity, we do not include the full proof of the short-time existence result here, but it is included in the work [20].

Singularities of Equivariant Lagrangian Mean Curvature Flow

In Chapter 5, we investigate in detail the singularities of Lagrangian mean curvature flow. We focus our attention to almost-calibrated flows, as this condition is required in the statement of the Thomas-Yau conjecture. The results of Chapter 5 appear in the preprint [78].

The study of singularities of mean curvature flow in general is now very well established. The work of G. Huisken [32], S. Altschuler [1,2] and R. Hamilton [26] were among the first to consider blowups of mean curvature flows, and a recent detailed examination of a Type II singularity was carried out by J. Velazquez [71] and N. Sesum and S-H. Guo [25]. The Lagrangian case has previously been studied by J. Chen and J. Li [11], M-T. Wang [72], and more recently by A. Neves [50–52], and X. Han and J. Li [28]. Analysis of specific cases has also been undertaken: for the Clifford torus by C. Evans, J. Lotay and F. Schulze [21], and for the Whitney sphere by A. Savas-Halilaj and K. Smoczyk [58]. A recent paper by W-B. Su [67] studies the long-time behaviour of equivariant Lagrangian mean curvature flow.

Despite this prior work, the singularities of Lagrangian mean curvature flow are in general not well understood. However, a few powerful structure theorems have been proven. For instance, in [50], A. Neves proves that any singularity of almost-calibrated Lagrangian mean curvature flow is modelled on a union of special Lagrangian cones; this is achieved by utilising the elementary evolution equation for the Lagrangian angle, and the monotonicity formula of G. Huisken. Though this result is general and useful, it does not give information on the finer structure of the singularity, and nor does it give information on when singularities occur, or their stability.

Though these latter questions are difficult to answer in general, in this work we give an almost comprehensive answer in the specific case of equivariant Lagrangian mean curvature flow in $\mathbb{C}^n$, where a submanifold of $\mathbb{C}^n$ is *equivariant* if it has an $O(n)$ symmetry (see Section 5.1). Such submanifolds are automatically Lagrangian, and in the almost-calibrated case must be embedded and non-compact. We may study these flows by studying the *profile curve* obtained by a quotient of the group action, which makes the analysis much simpler than that of higher codimension flow. Therefore, they provide a good first case to more easily analyse and understand the phenomena of singularity formation in Lagrangian mean curvature flow.

Our main results are as follows. Theorem 5.0.1 and Theorem 5.0.4 state that all singularities in the equivariant, almost-calibrated case must be modelled on not just a union of special Lagrangian cones but a *single* such cone, on a Type I level, and on a Type II level they are modelled on a Lawlor neck. This confirms the prediction of D. Joyce that the Lawlor neck 'neck-pinch' singularity is a generic one, in the case of equivariant Lagrangians, and demonstrates a surprising rigidity in the singularity formation of Lagrangian mean curvature flow. The proofs revolve around singularity analysis and geometric arguments, using the almost-calibrated condition and embeddedness to argue that certain configurations of the profile curve are not possible. To show that the Type II blowup is a Lawlor neck, it is sufficient to demonstrate that the blowup retains the equivariant symmetry of the flow. This could be done by proving a curvature bound, but we instead argue by contradiction and use our knowledge of the possible singularity models to rule out the possibility that the centre of rotation becomes unbounded under the sequence of rescalings.

We additionally investigate long-time behaviour of the flow, and give open conditions for when singularities must form and for when they cannot. Theorem 5.0.2 states that if the initial Lagrangian lies in a sufficiently tight cone, then no singularity can form, and if it spans a sufficiently large cone, then it must. These theorems are inspired by A. Neves' examples of singularity formation [50], but are much more general, painting an almost complete picture of the long-time behaviour of equivariant Lagrangian mean curvature flow.

Finally, Theorem 5.0.5 investigates the link between the Type I and Type II models, and states that the asymptotes of any intermediate blowup match those of the Type I and Type II blowups. This shows there is no distinct intermediate behaviour, and gives hope to the idea that we may be able to derive information about the Type II blowup from just knowledge of the Type I blowup. Given A. Neves' structure theorem and the results by Y. Imagi, D. Joyce and J. dos Santos [35] on uniqueness of special Lagrangians with given asymptotes this would be a very powerful technique.

Equivariant Lagrangian Mean Curvature Flow with Boundary

In Chapter 6, we use similar techniques to investigate the long-time behaviour of Lagrangian mean curvature flow with boundary, in the equivariant case. This chapter appeared as part of joint work with B. Lambert and C. Evans [20].

The aim is to demonstrate the behaviour of the Lagrangian mean curvature flow with boundary through the use of two model boundary conditions, the Lawlor neck (see Example 3.3.1) and the Clifford torus (see Example 3.3.2). The former is a static special Lagrangian, and the latter is a self-similarly shrinking solution to mean curvature flow, so examining these two cases gives a good overview of how one might expect the flow to behave in general.

We first prove that the Lagrangian mean curvature flow with boundary has extremely good behaviour in the case of the Lawlor neck boundary condition. Specifically, in Theorem 6.0.1, we prove that any almost-calibrated Lagrangian mean curvature flow with boundary on the Lawlor neck converges in infinite time to a special Lagrangian disc. It should be noted that without the almost-calibrated condition one cannot expect long-time existence, by the examples of A. Neves [51].

The case of the Clifford torus boundary condition is more interesting. As a self-shrinker, it collapses to the origin in finite time. So, the question is no longer one of long-time existence, but of the nature of the forced singularity. We prove in Theorem 6.0.2 that if one demands a perpendicular angle for the boundary condition of the profile curves, and a suitable 'almost-calibrated-like' condition on the Lagrangian angle, then the Type I blowup of the flow is a special Lagrangian disc. However, the more general behaviour in this case is very different; numerical evidence shows that any other angle will lead to convergence of the blowups to an eternal, *rotating* solution to mean curvature flow. Since Type I blowups of mean curvature flow are usually self-shrinkers, this result demonstrates the difference in the theory of singularities between the standard flow and the flow with boundary.

Chapter 2

Preliminaries

In this chapter we cover well known definitions and theorems that will be required for the rest of the thesis, for reference or as an introduction to those who may not work in these fields.

Section 2.1 provides background theorems and notation relating to submanifold geometry in Riemannian manifolds. Section 2.2 reviews mean curvature flow, including the important monotonicity formula of G. Huisken (Theorem 2.2.7) and regularity theorem of B. White (Theorem 2.2.10) – both of which will be of key importance for our work on singularities in Chapters 5 and 6. The analysis of singularities of mean curvature flow is given particular attention. Section 2.3 reviews complex, symplectic, Kähler and Calabi-Yau geometry. Section 2.4 reviews the theory of Lagrangian submanifolds in Kähler and Calabi-Yau manifolds, in particular the important notions of the Lagrangian angle and mean curvature 1-form.

Sections 2.1 and 2.3 in particular are elementary in nature, and should only be referred to if and when necessary.

2.1 Riemannian Geometry

Let $F : N^n \hookrightarrow (M^m, \overline{g})$ be a smooth Riemannian immersion of a smooth manifold into a Riemannian manifold with metric $\overline{g}$. The image $F(N)$ is then an immersed submanifold. We will often abuse notation by denoting $F(N)$ by N, and the tangent and normal bundles $TN, TN^\perp$ will be considered sub-bundles of TM. Pulling back the Riemannian metric

The results of Chapter 2 do not comprise original work, and are individually attributed as appropriate.

$\overline{g}$ gives a metric on the manifold N, denoted by $g := F^*(\overline{g})$. Both $g(X,Y)$ and $\overline{g}(X,Y)$ will be denoted by $\langle X,Y\rangle$, and since they agree on vectors in TN there should be no confusion.

When working in a coordinate system with local basis $\{\partial_1,\ldots,\partial_m\}$ of TM, the metric is written as $\overline{g}_{\alpha\beta}dx^\alpha \otimes dx^\beta$, and the components of the inverse are denoted $\overline{g}^{\alpha\beta}$. The metric and its inverse will be used to 'raise and lower' indices according to the Einstein notation convention.

The metric induces an inner product on higher order tensors, for example if T,S are $(1,1)$ tensors, then

$$\langle T,S\rangle = g_{\alpha\gamma}g^{\beta\delta}T^{\alpha}{}_{\beta}\,S^{\gamma}{}_{\delta}.$$

The metric also induces a natural norm on all tensor bundles:

$$|T| = \sqrt{\langle T,T\rangle},$$

and a natural volume form, $vol_M := \sqrt{\det g}\cdot dx^1\wedge\ldots\wedge dx^m$.

The ambient Riemannian manifold M has a canonical Levi-Civita connection $\overline{\nabla}$. The components of the Levi-Civita connection in coordinates are known as the **Christoffel symbols** $\overline{\Gamma}^{\alpha}{}_{\beta\gamma}$, and are given by

$$(\overline{\nabla}_{\partial_\alpha}\partial_\beta) = \overline{\Gamma}^{\gamma}{}_{\alpha\beta}\partial_\gamma.$$

The connection allows us to define the **covariant derivative** of tensors. For example, the covariant derivative of a $(1,1)$-tensor T is denoted $\overline{\nabla}T$, and has components $\overline{\nabla}_\alpha T_\beta{}^\gamma$. This is not to be confused with the derivatives of the components of T, which will be denoted $\partial_\alpha T_\beta{}^\gamma$. The covariant derivative satisfies the tensor product rule, therefore

$$\begin{aligned}&\overline{\nabla}_X(T(Y)) = (\overline{\nabla}_X T)(Y) + T(\overline{\nabla}_X Y)\\ \implies\ &\overline{\nabla}_\alpha T_\beta{}^\gamma = \partial_\alpha(T_\beta{}^\gamma) + T_\beta{}^\delta\overline{\Gamma}^{\gamma}{}_{\alpha\delta} - T_\delta{}^\gamma\overline{\Gamma}^{\delta}{}_{\alpha\beta}.\end{aligned}$$

The Levi-Civita connection $\overline{\nabla}$ has the important property that $\overline{\nabla}\overline{g} = 0$, and in general, if a tensor T satisfies $\overline{\nabla}T = 0$ we say that it is **parallel**.

Higher covariant derivatives of tensors are defined inductively, for example

$$(\overline{\nabla}^2 T)(X,Y,Z) = \overline{\nabla}_X(\overline{\nabla}T)(Y,Z) = (\overline{\nabla}_X\overline{\nabla}_Y T)(Z) - (\overline{\nabla}_{\overline{\nabla}_X Y} T)(Z),$$

with components $\overline{\nabla}_\alpha\overline{\nabla}_\beta T_\gamma{}^\delta$.

The connection $\overline{\nabla}$ induces a connection on the tangent bundle of the submanifold, TN, which we denote ∇. This is in fact the Levi-Civita connection for (N,g), so that $\nabla g = 0$. It is given by

$$\nabla_X Y = \left(\overline{\nabla}_X Y\right)^\top.$$

On the normal bundle $TN^\perp$, there is also an induced normal connection $\widetilde{\nabla}$, which is defined on normal vectors $\nu \in TN^\perp$ by

$$\widetilde{\nabla}_X \nu = \left(\overline{\nabla}_X \nu\right)^\perp.$$

These two connections may be combined to form canonical connections on all mixed tensor bundles – for convenience all of these connections will be simply denoted by ∇.

We will usually use Roman indices i,j when working with a basis of the tangent space TN of the submanifold, and Greek indices α,β for a basis of the normal space $TN^\perp$ or of the ambient manifold TM, depending on the context. Usually, coordinates of the submanifold will use variables x^i, and coordinates of the ambient manifold will use variables y^α. We will use interchangeably the notations ∂_i, e_i, $\frac{\partial}{\partial x^i}$ and $\frac{\partial F}{\partial x^i}$ for the coordinate tangent vectors of N.

2.1.1 The Laplacian Operator

We will now define the various notions of Laplacian for the submanifold N. The most important for us is the **tensor Laplacian operator** Δ, which is defined as a trace of the second covariant derivative,

$$\Delta T := g^{pq}\nabla_p\nabla_q T.$$

There are however two alternative natural definitions of the Laplacian. For the first, we define the **gradient** of a function f and the **divergence** of a vector field X on N by

$$\begin{aligned}\nabla f &:= g^{ij}\partial_i f \partial_j,\\ \mathrm{div}^N X &:= \frac{1}{\sqrt{\det g}}\partial_i\left(\sqrt{\det g}X^i\right).\end{aligned}$$

To avoid confusion, the notation ∇f will always denote the gradient (as opposed to the covariant derivative of tensors) when f is a function. These are the natural extensions of the Euclidean concepts of gradient and divergence, and satisfy many of the same properties. We note some of them in the following two lemmas.

Lemma 2.1.1. *Let N be a Riemannian manifold, and f, X be a function and vector field on N respectively. Then:*

$$\begin{aligned}\mathrm{div}^N(fX) &= \langle\nabla f, X\rangle + f\mathrm{div}^N(X),\\ df(X) &= \langle\nabla f, X\rangle,\\ \mathrm{div}^N(X)vol_N &= \mathscr{L}_X(vol_N).\end{aligned}$$

Theorem 2.1.2. *Let N be a Riemannian manifold with boundary ∂N, and f, X be a function and vector field on N respectively. Then, if μ is an outward pointing normal vector,*

$$\int_N \mathrm{div}^N(f)vol_M = \int_{\partial N}\langle\nabla, \mu\rangle vol_{\partial N}.$$

We may then define the **Laplace-Beltrami operator** in the same way as the Laplacian is defined in Euclidean space:

$$\Delta^{LB} f := \mathrm{div}^N(\nabla f).$$

For the other important Laplacian, we define the **Hodge star operator** on basic k-forms to satisfy the following property:

$$\omega \wedge *\eta = \langle \omega, \eta \rangle vol_M,$$

and then extend linearly to define a map on k-forms,

$$* : \Lambda^k(T^*N) \to \Lambda^{n-k}(T^*N).$$

This operator is introduced so as to define the **codifferential** on forms $d^* : \Omega^j \to \Omega^{j-1}$,

$$d^*\omega := (-1)^j *^{-1} d * \omega,$$

which is the L^2-adjoint operator of d. With this, the **Hodge Laplacian** may be defined on forms by

$$\Delta^H = dd^* + d^*d.$$

Importantly, these three Laplacians coincide on functions:

Theorem 2.1.3. *Let N be a Riemannian manifold, and f a function on M. Then,*

$$\Delta f = \Delta^{LB} f = -\Delta^H f.$$

2.1.2 Curvature

We now introduce the various notions of curvature associated with a Riemannian manifold. The most important intrinsic curvature quantity for the submanifold N is the **Riemann curvature tensor**, which we define as

$$\langle R(X,Y)Z, W \rangle := \langle \nabla^2_{X,Y} Z - \nabla^2_{Y,X} Z, W \rangle$$

with components R_{ijkl}. One may also consider the **Riemann curvature operator**, which measures the obstruction to commutativity of the second covariant derivative of general tensors. For example, for a $(1,1)$-tensor T it is defined as

$$\begin{aligned} R(X,Y)T &:= \nabla^2_{X,Y}T - \nabla^2_{Y,X}T \\ \implies (\nabla^2_{X,Y}T)(W) &= (\nabla^2_{Y,X}T)(W) + R(X,Y)(T(W)) - T(R(X,Y)W) \\ \implies \nabla_i\nabla_j T_k{}^l &= \nabla_j\nabla_i T_k{}^l + R_{ijn}{}^l T_k{}^n - R_{ijk}{}^n T_n{}^l. \end{aligned} \tag{2.1}$$

Also important are the **Ricci curvature**, and **scalar curvature**, defined as traces of the Riemann curvature tensor:

$$\begin{aligned} R_{ij} &:= R_{kij}{}^k, \\ scal &:= R_k{}^k = R_{lk}{}^{kl}. \end{aligned}$$

The Riemannian curvature tensor and Ricci tensor have the following symmetries:

$$R_{ijkl} = -R_{jikl} = -R_{ijlk} = R_{klij}, \tag{2.2}$$

$$R_{ij} = R_{ji}, \tag{2.3}$$

$$R_{ijkl} + R_{jkil} + R_{kijl} = 0, \quad \text{(1st Bianchi Identity)} \tag{2.4}$$

$$\nabla_a R_{ijkl} + \nabla_i R_{jakl} + \nabla_j R_{aikl} = 0. \quad \text{(2nd Bianchi Identity)} \tag{2.5}$$

These curvature quantities are all defined also for the ambient manifold, and will be denoted with a bar, for example $\overline{R}_{ij}$ for the Ricci tensor.

Equally important for us are the following extrinsic curvature tensors, associated with the immersion F. The **vector-valued second fundamental form** of the immersion is the normal part of the ambient covariant derivative,

$$A(X,Y) := \overline{\nabla}^{\perp}_X Y, \quad A \in \Gamma(T^*N \otimes T^*N \otimes F^*TM).$$

It is symmetric in its two tangent vector arguments. In local coordinates, where $\{\nu_\alpha\}$ is an orthonormal basis of the normal space, and $e_i := \frac{\partial F}{\partial x^i}$, we have:

$$A = A^\alpha_{ij} dx^i \otimes dx^j \otimes \nu_\alpha, \qquad A^\alpha_{ij} = \langle \overline{\nabla}_{e_i} e_j, \nu_\alpha \rangle, \qquad A_{ij} = \left(\overline{\nabla}_{e_i} e_j \right)^{\perp}$$

The covariant derivative of the second fundamental form is defined using ∇ and $\widetilde{\nabla}$ by

$$(\nabla_X A)(Y,Z) = \widetilde{\nabla}_X A(Y,Z) - A(\nabla_X Y, Z) - A(Y, \nabla_X Z). \tag{2.6}$$

Finally, the **mean curvature vector** is defined as the trace of the second fundamental form:

$$\vec{H} = H^\alpha \nu_\alpha = g^{ij} A_{ij}^\alpha \nu_\alpha.$$

In the case of a codimension 1 submanifold, one may make a choice of unit normal vector ν and define the mean curvature scalar as $H := \langle \vec{H}, \nu \rangle$. Throughout the thesis we work with higher codimension submanifolds, so this will not be possible.

The link between the intrinsic curvature and the extrinsic curvature of a Riemannian immersion is encapsulated by the Gauss and Codazzi equations. Explicitly, they express the tangential and normal parts of the ambient Riemannian curvature tensor restricted to TN in terms of the second fundamental form. They will be useful again and again to convert curvature terms, and will take on an especially beautiful form in the context of Lagrangian geometry.

Theorem 2.1.4. *Let $F : (N, g) \to (M, \overline{g})$ be a Riemannian immersion. Then for $X, Y, Z, W \in TN$:*

- $\left\langle \overline{R}(X,Y)Z, W \right\rangle = \langle R(X,Y)Z, W \rangle + \langle A(X,Z), A(Y,W) \rangle - \langle A(X,W), A(Y,Z) \rangle,$
- $(\overline{R}(X,Y)Z)^\perp = (\nabla_X A)(Y,Z) - (\nabla_Y A)(X,Z).$

These two equations are known as the Gauss and Codazzi equations respectively.

2.2 Mean Curvature Flow

Consider a compactly supported normal variation ν of an immersed submanifold given by an immersion $F : N^n \to M^m$, and let Φ_t, $t \in (-\varepsilon, \varepsilon)$ be a family of compactly supported diffeomorphisms of M such that:

$$\begin{cases} \frac{d\Phi}{dt}\big|_N & = \nu \in TN^{\perp}, \\ \Phi_0 & = Id. \end{cases}$$

For small t, this gives a family $F_t : N^n \to M^m$ of immersed submanifolds defined by $F_t := \Phi_t \circ F$. By first calculating the change in the induced metric and volume form of N, we can calculate the first variation of volume. Using $\nabla g = 0$, note first the identities

$$0 = \overline{\nabla}_\alpha \overline{g}_{\beta\gamma} = \partial_\alpha \overline{g}_{\beta\gamma} - \overline{\Gamma}^{\delta}{}_{\alpha\beta}\overline{g}_{\delta\gamma} - \overline{\Gamma}^{\delta}{}_{\alpha\gamma} g_{\beta\delta},$$
$$\left(\nabla_{\frac{\partial F_t}{\partial x^i}} \nu\right)^\gamma = \left(\frac{\partial F_t}{\partial x^i \partial t}\right)^\gamma = \frac{\partial^2 F_t^\gamma}{\partial x^i \partial t} + \frac{\partial F_t^\alpha}{\partial x^i}\frac{\partial F_t^\beta}{\partial t}\overline{\Gamma}^{\gamma}{}_{\alpha\beta}.$$

The evolution equations then follow from a calculation. Denoting $e_i := \frac{\partial F_t}{\partial x^i}$, and remembering that $\overline{\nabla}\overline{g} = 0$:

$$\begin{aligned}
\frac{d}{dt} g_{ij} &= \frac{d}{dt}\left(\overline{g}_{\beta\gamma}\frac{\partial F_t^\beta}{\partial x^i}\frac{\partial F_t^\gamma}{\partial x^j}\right) \qquad (2.7)\\
&= \partial_\alpha \overline{g}_{\beta\gamma}\frac{\partial F_t^\alpha}{\partial t}\frac{\partial F_t^\beta}{\partial x^i}\frac{\partial F_t^\gamma}{\partial x^j} + \overline{g}_{\beta\gamma}\frac{\partial^2 F_t^\beta}{\partial t \partial x^i}\frac{\partial F_t^\gamma}{\partial x^j} + \overline{g}_{\beta\gamma}\frac{\partial F_t^\beta}{\partial x^i}\frac{\partial^2 F_t^\gamma}{\partial t \partial x^j}\\
&= \overline{\nabla}_\alpha \overline{g}_{\beta\gamma} + \left\langle \frac{\partial^2 F_t}{\partial t \partial x^i}, \frac{\partial F_t}{\partial x^j}\right\rangle + \left\langle \frac{\partial^2 F_t}{\partial t \partial x^j}, \frac{\partial F_t}{\partial x^i}\right\rangle\\
&= \left\langle \overline{\nabla}_{e_i}\nu, e_j\right\rangle + \left\langle \overline{\nabla}_{e_j}\nu, e_i\right\rangle\\
&= -2\langle \nu, A_{ij}\rangle,\\
\frac{d}{dt}\, vol_g &= \frac{d}{dt}\sqrt{\det g}\, dx^1 \wedge \ldots \wedge dx^n\\
&= \frac{1}{2\sqrt{\det g}}\frac{d}{dt}\det g\, dx^1 \wedge \ldots \wedge dx^n\\
&= -\frac{1}{\sqrt{\det g}}\det g\, g^{ij}\left\langle \nu, A_{ij}\right\rangle dx^1 \wedge \ldots \wedge dx^n\\
&= -\langle \nu, \vec{H}\rangle\, vol_g.\\
\frac{d}{dt} Vol(N) &= \frac{d}{dt}\int_N vol_g\\
&= -\int_N \langle \nu, \vec{H}\rangle vol_g.
\end{aligned}$$

Therefore, a submanifold is a critical point for the volume functional if and only if $\vec{H} = 0$. In this case it is known as a **minimal submanifold**. Note that this terminology is slightly misleading as a critical point for the volume functional need not be a minimiser; an example is the equator of the standard 2-sphere, S^2.

The above calculation also shows that the gradient descent for the volume functional is given by the normal variation $\nu = \vec{H}$. Motivated by this, we say that a family $F_t : N^n \hookrightarrow M^m$ of Riemannian immersions is a **mean curvature flow** (henceforth often abbreviated to MCF) if

$$\frac{dF}{dt} = \vec{H}, \tag{2.8}$$

which is a (degenerate) quasilinear parabolic partial differential equation. We have already seen the derivation of the two most important evolution equations for this flow:

Lemma 2.2.1. *Let $F_t : N^n \to M^m$ be a mean curvature flow. Then the following evolution equations hold for the induced metric $g = g(t)$ on N:*

- $\frac{d}{dt} g_{ij} = -2\langle \vec{H}, A_{ij} \rangle$,
- $\frac{d}{dt} vol_g = -|\vec{H}|^2 vol_g$.

The evolution of other quantities, such as the mean curvature and second fundamental form, are significantly more complicated in general. For a full treatment see [62].

2.2.1 Short and Long-Time Existence

As with any parabolic problem, the first natural question to ask is that of well-posedness: Given a smooth initial condition, does the flow exist for a short time, and is the flow unique? If the initial immersion is closed and smooth, then the curvature must be bounded. It is then possible to write the evolving surface locally as a graph over the initial immersion, and apply quasilinear PDE theory to obtain the following short-time existence result:

Theorem 2.2.2 (Short-time Existence for MCF)**.** *Let $F : N^n \to M^m$ be an immersion of a smooth closed manifold into a Riemannian manifold (M^m, g). Then there exists a unique smooth solution F_t to (2.8) with $F_0 = F$ on a time interval $[0, T)$ for some $T > 0$.*

It is possible for a singular initial condition to evolve smoothly in several different ways – see [4] for an example in $\mathbb{R}^3$. For non-compact submanifolds the question is also more complicated. Unless assumptions are made on the asymptotic behaviour, a smooth initial condition may result in several different evolutions under mean curvature flow.

The next natural question to ask is whether any mean curvature flow may be extended to exist for all time. Since mean curvature flow decreases the volume of submanifolds, it may be hoped that the mean curvature flow converges in infinite time to a volume-minimising representative in the homology class. In general this is not true, as can be seen by the most simple example of mean curvature flow.

Example 2.2.3 (The Sphere in $\mathbb{R}^{n+1}$)**.** *Given the constant curvature sphere $S^n_r \subset \mathbb{R}^{n+1}$ with radius r, the mean curvature vector points towards the origin and has size $\frac{n}{r}$. Since the symmetry of the sphere must be preserved under mean curvature flow, the flow starting with the sphere of radius R is characterised by the evolution of the radius:*

$$\frac{dr}{dt} = -\frac{n}{r}$$
$$\implies r = \sqrt{R - 2nt}.$$

The sphere therefore shrinks under mean curvature flow and becomes extinct at the origin at time $t = \frac{R}{2n}$.

In fact, long-time existence for mean curvature flow is atypical, in contrast to simpler parabolic equations of functions such as the heat equation in $\mathbb{R}^n$. For example, closed submanifolds of Euclidean space *always* become singular in a finite time. In the hypersurface case, this can be proven with the maximum principle, using the sphere example as a barrier surrounding the flow. In higher codimension, it is still possible to use maximum principle techniques, see for example the work of F. Schulze and J. Lotay [45, Theorem A.1]. On the other hand, non-compact submanifolds of Euclidean space can exist for all time; one of the most famous examples is the translating **grim reaper**:

Example 2.2.4 (The Grim Reaper in $\mathbb{R}^2$)**.** *If a mean curvature flow in $\mathbb{R}^2$ is graphical over the x-axis, then the flow may be reparametrised as $\gamma(x,t) = (x, u(x,t))$. It then* *follows from the mean curvature flow equation (2.8) that*

$$\frac{du}{dt} = \frac{u''}{1+(u')^2}.$$

If we assume that our solution translates upwards with a speed of 1*, then we find that the function* u *satisfies*

$$u(x) = -\ln(\cos(x+a)) + b + t,$$

for constants a *and* b*. This is an eternally translating solution to mean curvature flow (see bottom-left panel of Figure 2.2).*

The question of long-time existence is less clear when the ambient space is not flat, the most basic case of which is the classical **curve shortening flow** on surfaces. The following theorem is from a paper of Grayson [24, Theorem 0.1].

Theorem 2.2.5 (Grayson's Theorem)**.** *Let* M^2 *be a smooth Riemannian surface which is convex at infinity. Let*

$$F : S^1 \hookrightarrow M^2$$

be a smooth embedded curve. Then the unique solution F_t *to mean curvature flow (2.8) with* $F_0 = F$ *either shrinks to a round point in finite time, or smoothly converges to a geodesic as* $t \to \infty$*.*

The long-time existence case of Grayson's theorem is possible. For example, it can be shown by the Gauss-Bonnet theorem that any curve $\gamma \subset S^2 \subset \mathbb{R}^3$ that divides the area of the sphere equally continues to do so under the mean curvature flow, and converges to an equator in infinite time. However, the question of which initial conditions result in singularities and which result in long-time existence is very difficult in general.

2.2.2 Type I Blowups

Our primary interest is in studying finite-time singularities of MCF. The following theorem, proven in the hypersurface case by Huisken [30, Theorem 8.1], helps us to understand what happens at these singular times.

Theorem 2.2.6. *Let $F_t : N^n \hookrightarrow M^m$ be a mean curvature flow of a closed submanifold, with corresponding second fundamental form A_t. If T denotes the maximal time of existence, then*

$$\limsup_{t\to T} \max_{p\in N_t} |A_t(p)|^2 \to \infty.$$

In the case where $M = \mathbb{R}^m$, this rate has a lower bound. Explicitly, there exists $c > 0$ such that

$$\max_{p\in N_t} |A_t(p)|^2 \geq \frac{c}{T-t}.$$

The important step in the proof is to show that a bound on $|A|$ implies a bound on all higher derivatives, implying that the only way a singularity can form is if the curvature becomes unbounded.

We say a mean curvature flow $F_t : N^n \to M^m$ has a **Type I singularity** at time T if, for some $C \geq 1$,

$$\max_{p\in M} |A_t(p)|^2 \leq \frac{C}{T-t}. \tag{2.9}$$

By Theorem 2.2.6, this is the 'best possible' blowup rate in Euclidean space. On the other hand, if for any C there exists t such that

$$\max_{p\in M} |A_t(p)|^2 > \frac{C}{T-t},$$

we say the flow has a **Type II singularity**.

The quintessential Type I singularity is the shrinking sphere, $S^n \subset \mathbb{R}^{n+1}$. As we saw in Example 2.2.3, if we translate in time so that the singularity happens at time 0, the radius of the sphere at time t is given by

$$r = \sqrt{-2nt}.$$

The norm of the second fundamental form of the sphere takes the constant value $\frac{n}{r^2}$, therefore

$$\max_{p\in N_t} |A_t(p)|^2 = \frac{1}{-2t}.$$

This shows that the shrinking sphere has a Type I singularity at time 0.

To analyse Type I singularities in general, we use a subsequential blowup, which we illustrate in the Euclidean case of $M = \mathbb{R}^m$. Take a flow $F_t : N \to \mathbb{R}^m$ with a Type I singularity at the space-time point (x_0, T), and consider the parabolic rescaling around this point,

$$
\begin{aligned}
F_s^\lambda &:= \lambda \left(F_{T+\lambda^{-2}s} - x_0\right), \qquad (2.10)\\
N_s^\lambda &:= F_s^\lambda(N),
\end{aligned}
$$

which can be shown to be a mean curvature flow with time coordinate s and a Type I singularity at the space-time point $(O, 0)$. Taking a sequence $\lambda_i \to \infty$, we can use the bound (2.9) on $|A|$ (which implies a bound on higher derivatives of A) to show that the flows $F_s^{\lambda_i}$ converge subsequentially and locally smoothly to an ancient mean curvature flow F_s^∞, which we call a **Type I blowup** of F_t (see Figure 2.1). The Type I blowup is analogous to the construction of tangent cones in geometric measure theory, and so is often referred to as the **tangent flow**.

If instead we have a Type II singularity, we can still perform this sequence of parabolic rescalings, and we will still have convergence to a limiting flow (a Type I blowup) in a weak sense. The limiting object is no longer smooth but a flow of rectifiable varifolds, known as a Brakke flow [8]. We will study an example in Chapter 5 – see Figure 5.2.

There are two very important tools that will help us understand the nature of the Type I blowup, namely the monotonicity formula of Huisken [32, Theorem 3.1] and the regularity theorem of White [76, Theorem 3.1]. In what follows, we will need the following modified backwards heat kernel, defined on the ambient $\mathbb{R}^m$:

$$
\Phi_{(x_0,t_0)}(x,t) := (4\pi(t_0 - t))^{-\frac{n}{2}} e^{-\frac{|x-x_0|^2}{4(t_0-t)}}.
$$

Theorem 2.2.7 (Huisken's Monotonicity Formula)**.** *Let $F_t : N^n \hookrightarrow \mathbb{R}^m$ be a smooth solution of MCF, where $N_t := F_t(N)$ has bounded area ratios. Then:*

$$
\frac{d}{dt}\int_{N_t} \Phi_{(x_0,t_0)}\, d\mathscr{H}^n = -\int_{N_t} \left|\vec{H} + \frac{(x-x_0)^\perp}{2(t_0-t)}\right|^2 \Phi_{(x_0,t_0)}\, d\mathscr{H}^n.
$$

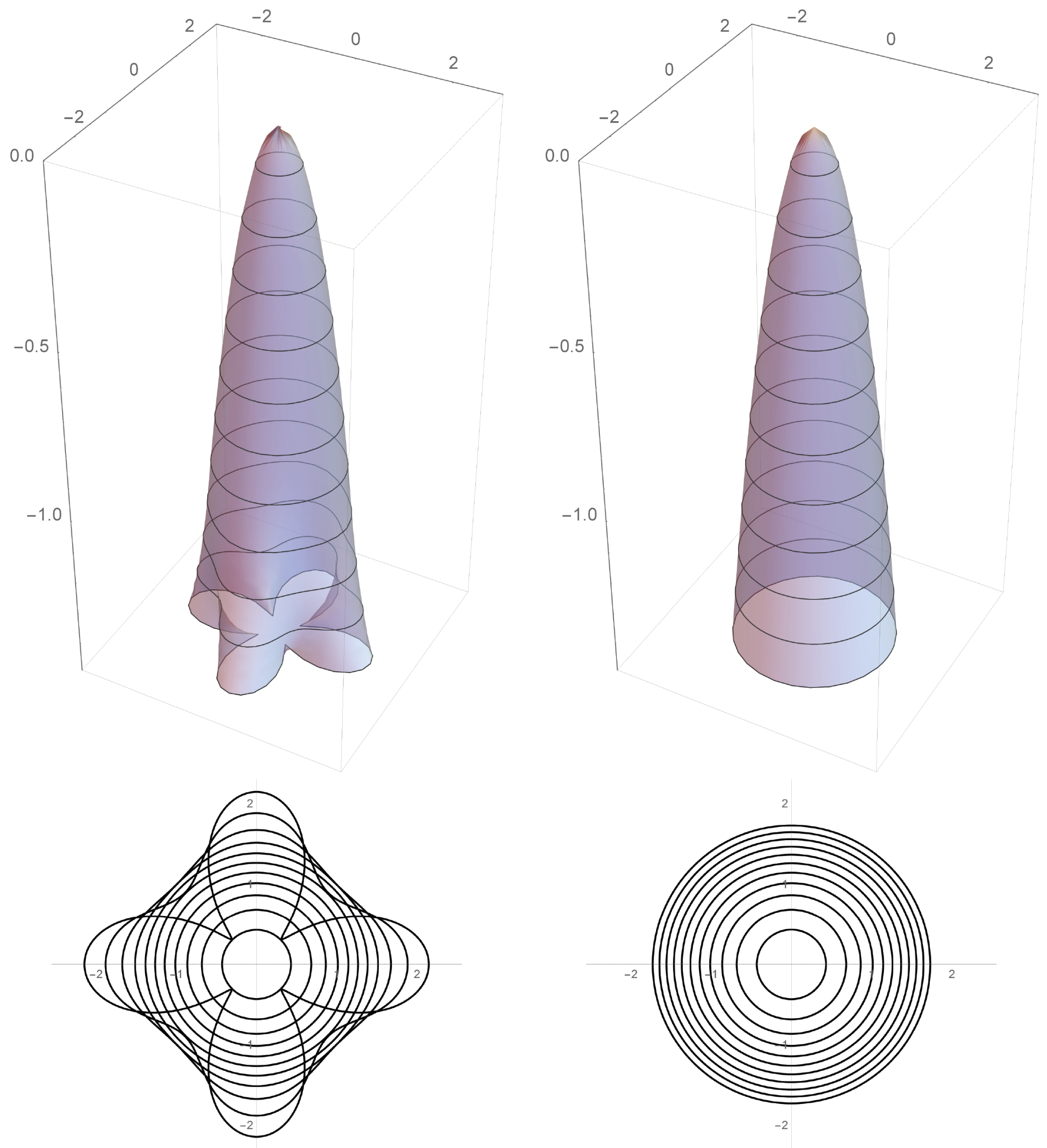

Figure 2.1: A depiction of a Type I blowup of a mean curvature flow of a curve in the plane, which forms a singularity at the space-time point $(O, 0)$.
On the left, we have the space-time track of the flow with certain time slices marked, and the same time slices drawn in the plane.
On the right, we have the Type I blowup of this singularity, i.e. a limit of parabolic rescalings centred at the space-time point $(O, 0)$. Note that close to the singularity, the flow resembles the singularity model more and more closely.

More generally, if $\phi_t : M \to \mathbb{R}$ is any smooth function with polynomial growth at infinity, then

$$\frac{d}{dt}\int_{N_t} \phi_t \, \Phi_{(x_0,t_0)} \, d\mathscr{H}^n = \int_{N_t} \left(\frac{d\phi_t}{dt} - \Delta\phi_t - \phi_t \left| \vec{H} + \frac{(x-x_0)^\perp}{2(t_0-t)} \right|^2 \right) \Phi_{(x_0,t_0)} \, d\mathscr{H}^n.$$

Huisken's monotonicity formula inspires the following quantities, known respectively as the **Gaussian density ratio** and **weighted Gaussian density ratio**, for a space-time point $X = (x_0,t_0)$ and arbitrary function $\phi : M \to \mathbb{R}$:

$$\Theta(F,X,r) := \int_{N_{t_0-r^2}} \Phi_X \, d\mathscr{H}^n, \qquad \Theta(F,X,r,\phi) := \int_{N_{t_0-r^2}} \phi \, \Phi_X \, d\mathscr{H}^n.$$

The monotonicity formula implies that this quantity is increasing in r. Importantly, the Huisken integral (and therefore the Gaussian density ratio) satisfies the following scaling invariance, which will make it useful in analysing blowups of singularities.

Lemma 2.2.8. *If $F_s^\lambda(N) = N_s^\lambda$ is a parabolic rescaling of the flow $F_t(N) = N_t$ around the point (x_0,t_0) with $s = \lambda^2(t-t_0)$, then for $X = (\bar{x},\bar{t})$, and any function f on N,*

$$\int_{N_t} f \, \Phi_X \, d\mathscr{H}^n = \int_{N_s^\lambda} f \, \Phi_{\left(\lambda(\bar{x}-x_0),\lambda^2(\bar{t}-t_0)\right)} \, d\mathscr{H}^n.$$

Additionally, the above implies the following symmetry of the Gaussian density ratio (i.e. in the case where $X = (\bar{x},\bar{t}) = (x_0,t_0)$):

$$\Theta(F^\lambda,O,r) = \Theta(F,X,\lambda^{-1}r),$$

where $O = (O,0)$ is the space-time origin.

Proof.

$$\begin{aligned}
\int_{N_t} f\,\Phi_X\,d\mathscr{H}^n &= \int_{N_t} f\,(4\pi(\bar{t}-t))^{-\frac{n}{2}}\cdot e^{-\frac{|\bar{x}-x|^2}{2(\bar{t}-t)}}\,d\mathscr{H}^n(x)\\
&= \int_{\lambda(N_t-x_0)} f\,\lambda^{-n}(4\pi(\bar{t}-t))^{-\frac{n}{2}}\cdot e^{-\frac{|\bar{x}-(\lambda^{-1}x+x_0)|^2}{2(\bar{t}-t)}}\,d\mathscr{H}^n(x)\\
&= \int_{\lambda(N_{\lambda^{-2}s+t_0}-x_0)} f\,(4\pi\left(\lambda^2(\bar{t}-t_0)-s\right))^{-\frac{n}{2}}\cdot e^{-\frac{|\lambda(\bar{x}-x_0)-x|^2}{2(\lambda^2(\bar{t}-t_0)-s)}}\,d\mathscr{H}^n(x)\\
&= \int_{N_s^\lambda} f\,\Phi_{\left(\lambda(\bar{x}-x_0),\lambda^2(\bar{t}-t_0)\right)}\,d\mathscr{H}^n.
\end{aligned}$$

□

The possibility of using a function ϕ in Theorem 2.2.7 allows for localisation of the monotonicity formula with a suitable cutoff function. In particular, following K. Ecker [16, Remark 4.8] we define

$$\phi_{(x_0,t_0),\rho}(x,t) = \left(1-\frac{|x-x_0|^2+2n(t-t_0)}{\rho^2}\right)_+^3,$$

which is a subsolution to the heat equation along a mean curvature flow, is compactly supported, and has value 1 at the point (x_0,t_0). Therefore, the weighted Gaussian density ratio

$$\Theta^\rho(F,X,r) := \Theta(F,X,r,\phi_{X,\rho}) = \int_{N_{t_0-r^2}} \Phi_X^\rho d\mathscr{H}^m = \int_{N_{t_0-r^2}} \Phi_X\phi_{X,\rho} d\mathscr{H}^m \qquad (2.11)$$

is monotonic by Theorem 2.2.7, with the same limit as the unweighted Gaussian density ratio as $r\to 0$.

The scaling invariance together with the monotonicity formula gives the most important fact about Type I blowups – they are modelled on self-similarly shrinking solutions to mean curvature flow. This is to be expected, since self-similarly shrinking solutions are invariant under parabolic rescaling, in much the same way that cones are invariant under homothetic scaling.

Proposition 2.2.9. *Consider a mean curvature flow $F_t : N^n \to \mathbb{R}^m$, which forms a singularity at the space-time point $X=(x_0,T)$. Consider the sequence of parabolic rescalings*

$F_s^{\lambda_i}$ with factor λ_i, as defined in (2.10).

Then the flows subsequentially converge smoothly to a self-similarly shrinking flow, i.e. a flow F_s satisfying

$$N_s = \sqrt{-s}N_{-1}. \tag{2.12}$$

Proof. Firstly, we show that (2.12) is equivalent to the following alternative characterisation of self-shrinkers:

$$\vec{H} - \frac{F_s^{\perp}}{2s} = 0. \tag{2.13}$$

In the case that (2.13) holds, the vectors H and $\frac{F}{2s}$ differ only by a tangential vector. We may therefore reparametrise mean curvature flow tangentially so that $\frac{dF}{ds} = \frac{F}{2s}$, and integrating this equation gives (2.12). For the other direction, defining $F_s(N) = N_s$ gives a tangential reparametrisation of the flow, and so

$$\vec{H} = \left(\frac{dF}{ds}\right)^{\perp} = \left(\frac{-1}{2\sqrt{-s}}N_{-1}\right)^{\perp} = \left(\frac{\sqrt{-s}N_{-1}}{2s}\right)^{\perp} = \frac{F_s^{\perp}}{2s}.$$

We now use Theorem 2.2.7 to prove the result. Passing to a subsequence if necessary, the flows $F_s^{\lambda_i}$ locally smoothly converge to a smooth Type I blowup F_s^{∞}. Since $\Theta(F,X,r)$ is increasing in r and bounded below, there must exist a limiting density as $r \to 0$, which we denote $\Theta(F,X)$. Then, by Lemma 2.2.8 and Theorem 2.2.7,

$$\begin{aligned}\Theta(F,X,\lambda_i^{-1}r) - \Theta(F,X) &= \Theta(F^{\lambda_i},0,r) - \Theta(F^{\lambda_i},0)\\ &= \int_{-r^2}^{0}\int_{M_s^{\lambda}}\left|\vec{H} + \frac{x^{\perp}}{2s}\right|\Phi_X d\mathscr{H}^n.\end{aligned}$$

Limiting $i \to \infty$, since $\lambda_i \to \infty$ the left hand side of this equation converges to 0, and so the result is proven. □

The other theorem of central importance to the study of singularities of mean curvature flow is White's regularity theorem, proven by B. White [76, Theorem 3.1]. In geometric measure theory, Allard's regularity theorem states that controlling the density ratios close to 1 for a stationary integral varifold implies a bound on the second fundamental form. White's theorem essentially states that the Gaussian density ratio performs the same role in mean curvature flow.

Theorem 2.2.10 (White's Regularity Theorem)**.** *Denote by $P_r(x,t)$ the parabolic cylinder $B_r(x) \times (t-r^2,t]$.*

Then there exist $\varepsilon > 0$, $C > 0$ depending on n such that if $F_t : M^m \to \mathbb{R}^n$ is a smooth mean curvature flow, and if

$$\forall X \in P_r(X_0), \quad \Theta(M,X,r) \leq 1+\varepsilon,$$

then

$$\sup_{X \in P(X_0, \frac{r}{2})} |A(X)| \leq \frac{C}{r}.$$

This theorem also holds if we replace the Gaussian density ratio Θ with the localised Gaussian density ratio Θ^ρ defined in (2.11), as long as $r < \rho$. It also follows from standard theory of elliptic PDE that under the same conditions, for similar universal constants C_k,

$$\sup_{X \in P(X_0, \frac{r}{4})} |\nabla^k A| \leq \frac{C_k}{r^{m+1}}.$$

This regularity theorem is useful for demonstrating that singularities do not occur, since the curvature must blow up at a singularity.

2.2.3 Type II Blowups

The Type I blowup procedure results only in a weak flow for Type II singularities. The trick to resolving these singularities smoothly is to take a sequence of space-time points (x_i,t_i) maximising the norm of the second fundamental form A_{t_i}, and then to perform a parabolic rescaling with factor $|A_{t_i}(p_i)|$ around that point to normalise its value to 1. There is a complication, however. In order to have a smooth convergence to the blowup in space-time, we need control on $|A|$ for a period of time around t_i. To achieve this, we choose a sequence of times $t_k \in [0, T-\frac{1}{k}]$ and points $p_k \in M$ such that:

$$|A_{t_k}(p_k)|^2 \left(T - \tfrac{1}{k} - t_k\right) = \max_{t \in [0, T-\frac{1}{k}],\, p \in M} \left(|A_t(p)|^2 \left(T - \tfrac{1}{k} - t\right)\right). \tag{2.14}$$

Note that the second fundamental form at time t_k is maximised at the point p_k. It then follows from the Type II condition (see for example [46, Chapter 4]) that one can choose a subsequence such that:

- $|A_{t_k}(p_k)| \to \infty$ monotonically,
- $|A_{t_k}(p_k)|^2 \left(T - \frac{1}{k} - t_k\right) \to \infty$,
- $p_k \to p$ for some $p \in M$,

where the last point is immediate if our manifold is compact, and otherwise must be proven.

Now we rescale the flow F_t, restricted to the time interval $[0, T - \frac{1}{k}]$, parabolically with factor $A_k := |A_{t_k}(p_k)|$ around $(x_k, t_k) := (F_{t_k}(p_k), t_k)$:

$$F_\tau^{(x_k,t_k)}(p) := A_k \left(F_{t_k + A_k^{-2}\tau}(p) - x_k\right).$$

This flow is defined for $\tau \in I_k := \left[-A_k^2 t_k, A_k^2 \left(T - \frac{1}{k} - t_k\right)\right]$. Since we choose the blowup factors to normalise $|A|$, these rescalings will converge locally smoothly to a limiting eternal flow: a **Type II blowup**. The value of $|A|$ for this blowup takes a maximum of 1 over time and space, and by the definition of the rescalings, this maximum value is achieved at the space-time point $(O, 0)$.

Type II blowups of Type II singularities do not result in self-similar shrinkers, since these necessarily satisfy the Type I curvature bound, and are not eternal solutions. Instead, there are several different possibilities, for example translating solutions and static (minimal) solutions. Note that a Type II blowup is not necessarily unique, and may depend on the sequence of points chosen.

An example of a flow which forms a Type II singularity is the figure-eight curve, shown in Figure 2.2. Here, the Type II blowup is an eternal solution known as the Grim Reaper. We will also see many examples in Chapter 5, see Figure 5.1.

2.3 Complex, Symplectic and Kähler Geometry

Our main setting for this work is Kähler manifolds; manifolds for which there are compatible Riemannian, symplectic and complex structures. In this section, we go over the definitions of these important objects.

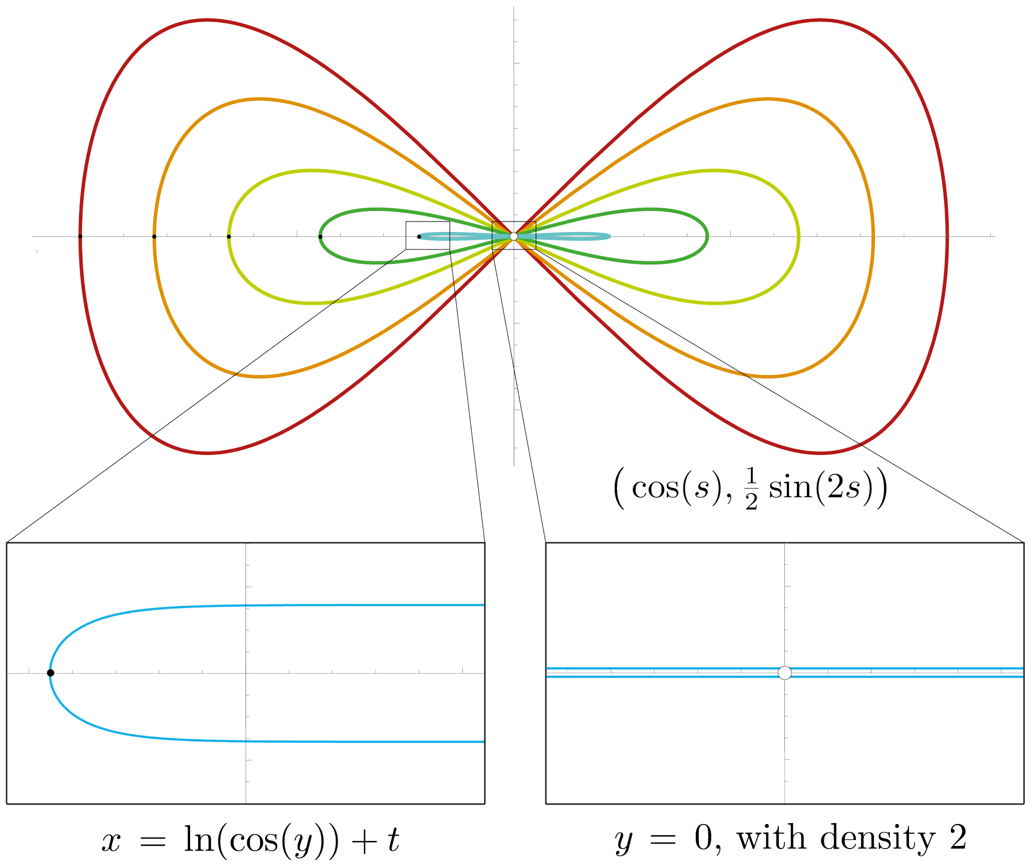

Figure 2.2: An illustration of a mean curvature flow with initial condition $(\cos(s), \frac{1}{2}\sin(2s))$, which forms a Type II singularity at the origin.
The Type I blowup at this singularity does not converge to a smooth flow but instead a Brakke flow of varifolds – in this case a double density plane, depicted on the lower right.
There are two possible Type II blowups, as there are two points of highest curvature at any given time (one such sequence of points is highlighted in black). Both Type II blowups are given by the eternal translating solution to mean curvature flow $x = \ln(\cos(y)) + t$ (known as the Grim Reaper) up to a reflection, one of which is shown on the lower left.

2.3.1 Complex and Hermitian Manifolds

An n-dimensional **complex manifold** is a smooth manifold with an atlas of charts to $\mathbb{C}^n$ (with complex coordinates $z^k = x^k + iy^k$) such that the transition maps are holomorphic. The tangent bundle then has a natural automorphism $J \in \Gamma(TM \otimes T^*M)$, given by

$$J\left(\frac{\partial}{\partial x^k}\right) = \frac{\partial}{\partial y^k}, \qquad J\left(\frac{\partial}{\partial y^k}\right) = -\frac{\partial}{\partial x^k}, \qquad J^2 = -Id.$$

This map is known as the **almost-complex structure**, since the existence of such a J on a real manifold is weaker than the existence of compatible holomorphic charts.

If we consider the complexified tangent bundle, $TM \otimes \mathbb{C}$, then the map J has $2n$ eigenvectors:

$$\begin{aligned} J\left(\frac{\partial}{\partial z^k}\right) &= i\frac{\partial}{\partial z^k}, \qquad \text{where} \quad \frac{\partial}{\partial z^k} = \frac{1}{2}\left(\frac{\partial}{\partial x^k} - i\frac{\partial}{\partial y^k}\right), \\ J\left(\frac{\partial}{\partial \bar{z}^k}\right) &= -i\frac{\partial}{\partial \bar{z}^k}, \qquad \text{where} \quad \frac{\partial}{\partial \bar{z}^k} = \frac{1}{2}\left(\frac{\partial}{\partial x^k} + i\frac{\partial}{\partial y^k}\right). \end{aligned} \tag{2.15}$$

The span of the n eigenvectors of the form $\frac{\partial}{\partial z^k}$ (the i-eigenspace of J) is known as the **holomorphic tangent bundle**, and denoted $T^{1,0}M$, and the span of the others (the $-i$-eigenspace of J) is the **antiholomorphic tangent bundle**, $T^{0,1}M$. Therefore, we have the splitting

$$TM \otimes \mathbb{C} = T^{1,0}M \oplus T^{0,1}M.$$

There is a similar splitting for forms:

$$T^*M \otimes \mathbb{C} = \Omega^{1,0} \oplus \Omega^{0,1},$$

where $\Omega^{1,0}$ is spanned by the forms $dz^i = dx^i + idy^i$, and $\Omega^{0,1}$ by the forms $d\bar{z}^i = dx^i - idy^i$ (respectively the i and $-i$ eigenspaces of J^*). Higher tensor bundles are defined using the wedge product, for example

$$\Omega^{p,q} = \Omega^{1,0} \wedge \ldots \wedge \Omega^{1,0} \wedge \Omega^{0,1} \wedge \ldots \wedge \Omega^{0,1},$$

where there are p and q factors of the respective 1-form bundles. Of particular importance is the **canonical bundle**, $\Omega^{n,0}M$, which is 1-dimensional.

The equivalent of a Riemannian metric on a complex manifold is a **Hermitian metric** – a positive-definite Hermitian form $\overline{h} \in \Gamma(T^{1,0}M \otimes T^{0,1}M)$ such that, for $X,Y \in T^{1,0}M$,

$$\begin{aligned} \overline{h}(X,Y^*) &= \overline{h}(Y,X^*)^* \\ \overline{h}(X,X^*) &\geq 0, \end{aligned}$$

where X^* denotes the complex conjugate, to avoid confusion. A complex manifold with a choice of Hermitian metric is known as a **Hermitian manifold**. The Hermitian metric splits into real and imaginary parts:

$$\overline{h} = \overline{g} - i\overline{\omega}, \qquad \overline{g} = \frac{1}{2}\left(\overline{h} + \overline{h}^*\right), \qquad \overline{\omega} = \frac{i}{2}\left(\overline{h} - \overline{h}^*\right), \tag{2.16}$$

and since both $\overline{g}$ and $\overline{\omega}$ are closed under complex conjugation, they descend to tensors on the real bundle TM. $\overline{g}$ is a Riemannian metric for the manifold M, considered as a real smooth manifold. ω on the other hand is a differential 2-form, known as the **Hermitian form**. The defining structures $\overline{g}$, $\overline{\omega}$, J of a Hermitian manifold are important and intimately connected, and we record here some important relationships between them.

Proposition 2.3.1. *Let $(M^{2n}, \overline{g}, \overline{\omega}, J)$ be a Hermitian manifold with Hermitian metric $\overline{h}$. Then:*

- *h, g, and ω are preserved by J, in the sense that for $X,Y \in TM \otimes \mathbb{C}$,*

$$\overline{h}(X,Y) = \overline{h}(JX,JY), \quad \overline{g}(X,Y) = \overline{g}(JX,JY), \;\; \textit{and} \;\; \overline{\omega}(X,Y) = \overline{\omega}(JX,JY).$$

- *$\overline{g}$ and $\overline{\omega}$ are related by $\overline{\omega}(X,Y) = \overline{g}(JX,Y)$, or $\overline{g}(X,Y) = \overline{\omega}(X,JY)$.*

It should be noted that a choice of any of $\overline{g}, \overline{h}$ and $\overline{\omega}$ for a complex manifold M defines the others, as long as it is preserved by the almost-complex structure J. In particular, a choice of J-preserving Riemannian metric $\overline{g}$ for a complex manifold M

fixes an associated Hermitian metric $\overline{h}$ and a Hermitian form $\overline{\omega}$.

2.3.2 Symplectic Manifolds

A **symplectic manifold** is a $2n$-dimensional smooth manifold M^{2n} with a choice of closed, non-degenerate 2-form $\overline{\omega} \in \Omega^2(M)$, known as the **symplectic form**. Symplectic manifolds are a generalisation of phase spaces in Hamiltonian mechanics in the following sense. Given a Hamiltonian function $H : M \to \mathbb{R}$, there is a unique corresponding vector field X_H, a 'symplectic gradient', such that

$$\overline{\omega}(X_H, \cdot) = dH(\cdot),$$

this vector field integrates to the 'flow' of the Hamiltonian system. Since the form $\overline{\omega}$ is alternating, the Hamiltonian is constant along the flow:

$$dh(X_H) = \overline{\omega}(X_H, X_H) = 0.$$

The condition that $\overline{\omega}$ is closed corresponds to the requirement that $\overline{\omega}$ does not change under the flow, since by Cartan's formula,

$$\mathscr{L}_{X_H}\overline{\omega} = \iota_{X_H} d\overline{\omega} + d(\iota_{X_H}\overline{\omega}) = d(dH) = 0.$$

2.3.3 Kähler and Calabi-Yau Manifolds

For most of this thesis, we will be working with **Kähler manifolds** – Hermitian manifolds $(M^{2n}, \overline{g}, \overline{\omega}, J)$ such that the Hermitian form $\overline{\omega}$ is closed. A Kähler manifold is therefore simultaneously a Riemannian manifold with metric $\overline{g}$, a symplectic manifold with symplectic form $\overline{\omega}$, and a complex manifold, with compatible structures. The condition on $\overline{\omega}$ has implications for the other structures:

Proposition 2.3.2. *Let $(M^{2n}, \overline{g}, \overline{\omega}, J)$ be a Hermitian manifold, and let $\overline{\nabla}$ be the Levi-Civita connection on M corresponding to $\overline{g}$. Then the following are equivalent, and in the case any of them hold we say that M is Kähler:*

- $\overline{\nabla}\overline{\omega} = 0$,

- $\overline{\nabla} J = 0$,
- $d\overline{\omega} = 0$.

It will often be useful to take Riemannian normal coordinates at a point p in a Kähler manifold, that are also holomorphic coordinates of the form $\{\frac{\partial}{\partial x^1}, \frac{\partial}{\partial y^1}, \ldots, \frac{\partial}{\partial x^n}, \frac{\partial}{\partial y^n}\}$. We call these **Kähler normal coordinates**. This is in fact possible precisely when the metric is Kähler!

Calabi-Yau manifolds are Kähler manifolds with additional structure. Though there are many definitions, we will say that a smooth manifold $\mathscr{Y}$ is a **Calabi-Yau manifold** if it is a Kähler manifold $(\mathscr{Y}, \overline{g}, \overline{\omega}, J)$ with vanishing Ricci tensor, and trivial canonical bundle $\Omega^{(n,0)}T\mathscr{Y}$. It follows that there exists an everywhere non-zero holomorphic n-form $\Omega \in \Omega^{(n,0)}T\mathscr{Y}$, and we may choose it so that

$$\frac{\overline{\omega}^n}{n!} = (-1)^{\frac{n(n-1)}{2}} \left(\frac{i}{2}\right)^n \Omega \wedge \Omega^*, \tag{2.17}$$

where Ω^* denotes the complex conjugate of Ω. The reason for the chosen factor will become clear in the following example.

Example 2.3.3 (The Kähler Manifold $\mathbb{C}^n$)**.** *Working in standard complex coordinates* $\{\frac{\partial}{\partial z^1}, \frac{\partial}{\partial \overline{z}^i}, \ldots, \frac{\partial}{\partial z^n}, \frac{\partial}{\partial \overline{z}^n}\}$, *the Hermitian metric on* $\mathbb{C}^n$ *is given by*

$$h = \sum_{i=1}^{n} dz^i \otimes d\overline{z}^i,$$

and the almost-complex structure is as always given by (2.15). It follows by (2.16, taking real and imaginary parts, that

$$\begin{aligned}
\overline{\omega} &= \frac{i}{2} \sum_{k=1}^{n} dz^k \wedge d\overline{z}^k, \\
\overline{g} &= \frac{1}{2} \sum_{k=1}^{n} \left(dz^k \otimes d\overline{z}^k + d\overline{z}^k \otimes dz^k \right).
\end{aligned}$$

Substituting real and imaginary coordinates gives the real expressions for these objects:

$$\overline{\omega} = \sum_{i=1}^{n} dx^i \wedge dy^i,$$
$$\overline{g} = \sum_{i=1}^{n} \left(dx^i \otimes dx^i + dy^i \otimes dy^i\right).$$

Since $\mathbb{C}^n$ is flat, and has trivial canonical bundle, we consider it to be a Calabi-Yau manifold (some authors exclude non-compact examples). The most simple choice of holomorphic volume form is

$$\Omega = dz^1 \wedge \ldots \wedge dz^n,$$

and this choice satisfies (2.17), justifying the choice of coefficient there.

2.4 Lagrangian Submanifolds

A particularly important class of submanifolds of a symplectic manifold $(M^{2n}, \overline{\omega})$ are the **Lagrangian** submanifolds. These are defined as the n-dimensional submanifolds L^n on which the symplectic form $\overline{\omega}$ vanishes:

$$\omega := \overline{\omega}|_L = 0.$$

Note that since the symplectic form is non-degenerate, it is not possible for the symplectic form to vanish on a submanifold of higher dimension than n.

Even when the ambient manifold has no additional structure, Lagrangian submanifolds have many interpretations and uses. In the phase space analogy, a Lagrangian submanifold is the level set of the integrals of motion, by the Arnold-Liouville theorem. Additionally, the graph of a symplectomorphism between two symplectic manifolds is a Lagrangian submanifold, with respect to a 'twisted' symplectic structure. An understanding of the Lagrangian submanifolds of a space also provides insight into the topology of the manifold, through Lagrangian Floer homology (originally developed by A. Floer [23]).

Example 2.4.1 (Lagrangians in $\mathbb{C}^n$)**.** *Consider $\mathbb{C}^n$, with its standard basis*

$\{\frac{\partial}{\partial x^1}, \frac{\partial}{\partial y^1}, \dots, \frac{\partial}{\partial x^n}, \frac{\partial}{\partial y^n}\}$, and symplectic form

$$\overline{\omega} = \sum_{k=1}^{n} dx^k \wedge dy^k.$$

The most basic Lagrangian submanifold is the span of the real axes, $\mathbb{R}^n \subset \mathbb{C}^n$. Which Lagrangians are near to this one? If we consider a function $F : \mathbb{R}^n \to \mathbb{R}^n$ with components F_i, then we may consider the graph of this function over $\mathbb{R}^n \subset \mathbb{C}^n$:

$$N(x^1, \dots, x^n) = \left(x^1, F_1(x^1, \dots, x^n), \dots, x^n, F_n(x^1, \dots, x^n)\right).$$

The tangent vectors to this surface are given by

$$\frac{\partial N}{\partial x^i} = \left(0, \frac{\partial F_1}{\partial x^i}, \dots, 1, \frac{\partial F_i}{\partial x^i}, \dots, 0, \frac{\partial F_n}{\partial x^i}\right),$$

and therefore the symplectic form restricted to L has components $\frac{\partial F_i}{\partial x^j} - \frac{\partial F_j}{\partial x^i}$. Viewing the submanifold as the graph of a 1-form $\alpha := F_i dx^i$ lends this condition a natural interpretation: N is Lagrangian precisely when α is a closed 1-form.

There are other Lagrangians in $\mathbb{C}^n$ with different global topology; for example we will later discuss in detail the Clifford torus (Example 3.3.2) and Lawlor neck (Example 3.3.1).

2.4.1 Lagrangians in Kähler Manifolds

Lagrangians in Kähler geometry have especially nice properties. In particular, if $L^n \subset M^{2n}$ is a Lagrangian submanifold of a Kähler manifold $(M^{2n}, \overline{g}, \overline{\omega}, J)$ and $X \in TL$ is a tangent vector, then JX is a normal vector, since for $Y \in TL$,

$$\overline{g}(JX, Y) = \overline{\omega}(X, Y) = 0.$$

Since $\overline{g}$ is preserved by J, it follows that in fact J is an isomorphism and an isometry between the normal and tangent bundles,

$$J : TL \to TL^{\perp}.$$

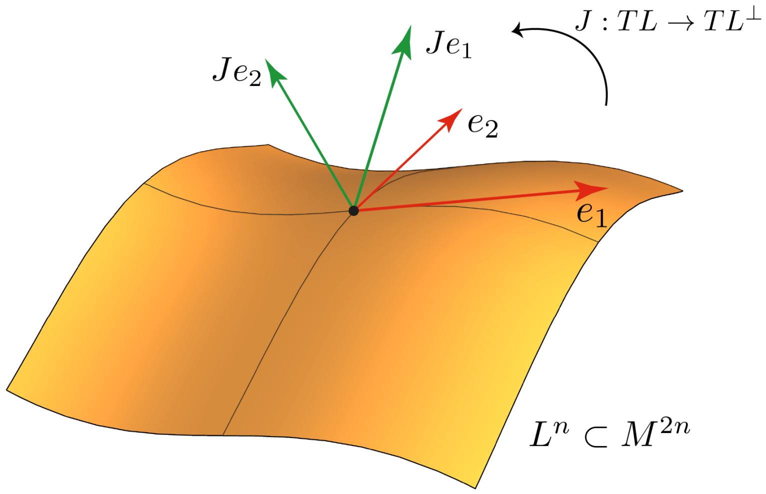

Figure 2.3: An illustration of a Lagrangian surface L^n, the isomorphism J between the tangent and normal spaces, and the basis $\{e_1, e_2, Je_1, Je_2\}$ of the ambient Kähler manifold M^{2n} along the surface.

So, if we take a basis $\{e_1, \ldots, e_n\}$ for T_pL, then $\{Je_1, \ldots Je_n\}$ is a basis for $T_pL^{\perp}$, and $\{e_1, Je_1, \ldots e_n, Je_n\}$ is a basis for T_pM (see Figure 2.3). When using this basis, or any basis of this form (for example induced by the complex coordinate charts of the Kähler manifold), we use an underline to denote components in the Je_i directions. For example,

$$\begin{aligned} \langle \overline{R}(e_i, e_j)e_k, Je_l \rangle &= \overline{R}_{ijk\underline{l}} \,, \\ \overline{g}^{ij} &= \overline{g}^{\underline{ij}} \,. \end{aligned} \tag{2.18}$$

Additionally, we will use Greek indices α, β, γ to denote components of an arbitrary basis of the ambient space, which will range from 1 to $2n$, to contrast with Roman indices i, j, k, which will always range from 1 to n. Underlined Greek indices $\underline{\alpha}$ will indicate a J-rotation of this basis. The automorphism J has the useful property of turning 'mixed tensors' on the normal and tangent bundles into tensors defined purely on the tangent bundle. Of particular importance to us are the second fundamental form, which may now be considered a $(0,3)$-tensor, and the **mean curvature 1-form**. For tangent vectors

$X,Y,Z \in TL$, these are defined respectively as

$$h(X,Y,Z) := \left\langle \overline{\nabla}_X Y, JZ \right\rangle = -\overline{\omega}(\overline{\nabla}_X Y, Z), \tag{2.19}$$
$$H(X) := \left\langle \vec{H}, JX \right\rangle = -\overline{\omega}(\vec{H}, X). \tag{2.20}$$

In coordinates, if $H = H_i dx^i$, then $\vec{H} = H^i J e_i$. By the usual definition of the second fundamental form, the components of h and H are related by $g^{ij} h_{ijk} = h^i{}_{ik} = H_k$.

The automorphism J may also be used to turn the Ricci curvature tensor of the ambient Kähler manifold into a 2-form, which we call the **Ricci form**:

$$\overline{\rho}(X,Y) := \overline{\mathrm{Ric}}(JX,Y).$$

Proposition 2.4.2. $\overline{\rho}(X,Y) = -\overline{\rho}(Y,X)$.

Proof. Firstly, as a consequence of $\overline{\nabla} J = 0$ (Proposition 2.3.2) we have the following symmetry of the Riemannian curvature tensor:

$$\left\langle \overline{R}(X,Y)Z, JW \right\rangle = -\left\langle \overline{R}(X,Y)JZ, W \right\rangle$$
$$\implies \overline{R}_{\alpha\beta\gamma\underline{\delta}} = -\overline{R}_{\alpha\beta\underline{\gamma}\delta}, \qquad \overline{R}_{\alpha\beta\underline{\gamma}\underline{\delta}} = \overline{R}_{\alpha\beta\gamma\delta}. \tag{2.21}$$

Using (2.18) and (2.21), along with the usual symmetries of the Riemannian curvature tensor, it follows that J preserves the Ricci tensor:

$$\overline{R}_{\underline{\alpha}\underline{\beta}} = \overline{g}^{\gamma\delta}\overline{R}_{\gamma\underline{\alpha}\underline{\beta}\delta} + \overline{g}^{\underline{\gamma}\underline{\delta}}\overline{R}_{\underline{\gamma}\underline{\alpha}\underline{\beta}\underline{\delta}} = \overline{g}^{\gamma\delta}\overline{R}_{\underline{\gamma}\alpha\beta\underline{\delta}} + \overline{g}^{\gamma\delta}\overline{R}_{\gamma\alpha\beta\delta} = \overline{R}_{\alpha\beta}.$$

This, along with the usual symmetry of the Ricci tensor, implies the result:

$$\overline{\rho}(X,Y) = \overline{\mathrm{Ric}}(JX,Y) = \overline{\mathrm{Ric}}(Y,JX) = -\overline{\mathrm{Ric}}(JY,X) = -\overline{\rho}(Y,X).$$

□

We may restrict $\overline{\rho}$ to give a 2-form on TL, which we denote ρ. The three curvature quantities h, H and ρ have symmetries and relationships that are stronger than in the general Riemannian case. Firstly, the second fundamental form in its new form h is symmetric in all three arguments, and the Gauss and Codazzi equations may be written as tensor identities on TL.

Proposition 2.4.3. *Let $L^n \subset M^{2n}$ be a Lagrangian submanifold of a Kähler manifold, and let h be the $(0,3)$-version of the second fundamental form. Then:*

- $h_{ijk} = h_{jik} = h_{ikj}$,
- $\overline{R}_{ijkl} = R_{ijkl} + A_{ikjl} + A_{iljk}$, *(Gauss Equation)*
- $\overline{R}_{ijk\underline{l}} = \nabla_i h_{jkl} - \nabla_j h_{ikl}$, *(Codazzi Equation)*

where we define the tensor $A_{ijkl} := g^{nm} h_{ijn} h_{mkl} = \left\langle A\left(e_i, e_j\right), A\left(e_k, e_l\right)\right\rangle$.

Proof. For the first equation, note that the symmetry in the first two indices is a standard fact about the second fundamental form. For the other symmetry, bearing in mind that $\overline{\nabla} J = 0$ and $\langle JX, JY\rangle = \langle X, Y\rangle$,

$$\left\langle A(e_i, e_j), Je_k\right\rangle = \left\langle \overline{\nabla}_{e_i} e_j, Je_k\right\rangle = e_i\left\langle e_j, Je_k\right\rangle - \left\langle e_j, J\overline{\nabla}_{e_i} e_k\right\rangle = \langle A(e_i, e_k), Je_k\rangle.$$

The second equation is simply the Gauss equation from Theorem 2.1.4, using the new notation. Finally, the last equation follows from the Codazzi equation (Theorem 2.1.4) and the following calculation:

$$\begin{aligned}
(\nabla_X h)(Y,Z,W) &= X(h(Y,Z,W)) - h(\nabla_X Y, Z, W) - h(Y, \nabla_X Z, W) - h(Y, Z, \nabla_X W)\\
&= X\langle A(Y,Z), JW\rangle - \langle A(\nabla_X Y, Z), JW\rangle - \langle A(Y, \nabla_X Z), JW\rangle\\
&\qquad - \langle A(Y,Z), \nabla_X JW\rangle\\
&= \langle (\nabla_X A)(Y,Z), JW\rangle,\\
\implies \nabla_i h_{jkl} &= \langle (\nabla_{e_i} A)(e_j, e_k), Je_l\rangle.
\end{aligned}$$

□

Next, we prove an important and surprising relationship between the mean curvature form and the Ricci curvature.

Proposition 2.4.4. *Let $L^n \subset M^{2n}$ be a Lagrangian submanifold of a Kähler manifold, let H be the mean curvature* 1*-form, and $\rho = \overline{\rho}|_L$ be the restriction of the Ricci form to L. Then:*

$$dH = -\rho.$$

Proof. The result follows by expressing the Ricci form in terms of the second fundamental form, using Proposition 2.4.3 as well as (2.21):

$$\begin{aligned} \overline{R}_{k\underline{i}j}{}^{k} &= g^{km}\overline{R}_{k\underline{i}jm} \\ &= g^{km}(\nabla_j h_{mki} - \nabla_m h_{jki}) \\ &= g^{km}(\nabla_j h_{mki} - \nabla_i h_{mkj} + \overline{R}_{imk\underline{j}}) \\ &= \nabla_j H_i - \nabla_i H_j - \overline{R}_{\underline{ki}j}{}^{\underline{k}} \\ \implies \rho_{ij} = \overline{R}_{k\underline{i}j}{}^{k} + \overline{R}_{\underline{ki}j}{}^{\underline{k}} &= \nabla_j H_i - \nabla_i H_j = -dH_{ij}. \end{aligned}$$

□

Note that in the case that the ambient manifold is Ricci-flat, we have $\overline{\rho} = 0$, and so in this case the mean curvature form is closed. In fact, even if the manifold M is only Einstein (constant Ricci curvature), then for tangent vectors $X, Y \in TL$ we have:

$$\overline{\mathrm{Ric}} = \lambda \overline{g}$$

$$\implies \rho(X,Y) = \overline{\mathrm{Ric}}(JX,Y) = \lambda \overline{g}(JX,Y) = \omega(X,Y) = 0 \tag{2.22}$$

$$\implies dH = 0. \tag{2.23}$$

2.4.2 Lagrangians in Calabi-Yau Manifolds

We now assume that our ambient manifold is a Calabi-Yau manifold $\mathscr{Y}$, i.e. it is Ricci-flat with a choice of parallel holomorphic volume form $\Omega \in \Omega^{n,0}\mathscr{Y}$. Since the Ricci form satisfies $\overline{\rho} = 0$, the mean curvature form of any Lagrangian submanifold $L \subset \mathscr{Y}$ is closed – see (2.22) and (2.23). We may therefore locally find a function θ whose exterior derivative is the mean curvature form. In fact, the holomorphic volume form gives us a canonical choice of multivalued function θ on L, such that $d\theta = H$. In Proposition 2.4.5, we define this function, and in Proposition 2.4.6 we prove the above key relationship that it enjoys with the mean curvature.

Proposition 2.4.5. *Let $L^n \subset \mathscr{Y}^{2n}$ be a Lagrangian submanifold of a Calabi-Yau manifold. Then there exists a smooth function $\theta : L \to \frac{\mathbb{R}}{2\pi}$ such that*

$$\Omega|_L = e^{i\theta} vol_L.$$

*We call θ the **Lagrangian angle** at the point p.*

Proof. Choose normal coordinates for the Lagrangian $\{\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n}\}$. At a point $p \in L$, the vectors $\{\frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^i}\}$ form a basis for T_pM, where by an abuse of notation we denote $\frac{\partial}{\partial y^i} := J\frac{\partial}{\partial x^i}$, and the elements of the corresponding dual basis by dx^i and dy^i. It follows that at this point, the Kähler form is given by

$$\overline{\omega} = \sum_{i=1}^{n} dx^i \wedge dy^i.$$

Now define

$$\Omega' := \bigwedge_{k=1}^{n} (dx^k + idy^k) = \bigwedge_{k=1}^{n} dz^k \tag{2.24}$$

which is a holomorphic volume form since each of the 1-forms $dx^k + idy^k$ are linearly independent and belong to the i-eigenspace of J, $\Omega^{1,0}\mathscr{Y}$:

$$\begin{aligned} J^*(dx^k + idy^k)(\tfrac{\partial}{\partial x^k}) &= i \\ J^*(dx^k + idy^k)(J\tfrac{\partial}{\partial x^k}) &= -1 \\ \implies J^*(dx^k + idy^k) &= i(dx^k + idy^k). \end{aligned}$$

Since it is a holomorphic n-form, it must be a multiple of Ω, i.e. $\lambda\Omega' = \Omega$. In fact, $|\lambda| = 1$, since by (2.17),

$$\begin{aligned} \Omega' \wedge (\Omega')^* &= (-1)^{\frac{n(n-1)}{2}} (-2i)^n \frac{\overline{\omega}^n}{n!} = \Omega \wedge \Omega^* \\ \implies |\lambda|^2 &= 1. \end{aligned}$$

Therefore

$$\Omega = e^{i\theta}\Omega' \tag{2.25}$$

for some function θ. Finally, note that $\Omega'|_L = \bigwedge_{i=1}^n dx^i = vol_L$ is the volume form for L, and so it follows that

$$\Omega|_L = e^{i\theta} vol_L. \tag{2.26}$$

Since the tensors Ω and vol_L are smooth and defined globally on L, it follows that θ may be uniquely defined everywhere up to multiples of 2π, and is smooth as a multivalued function. $\square$

Proposition 2.4.6. *Let $L^n \subset \mathscr{Y}^{2n}$ be a Lagrangian submanifold of a Calabi-Yau manifold, let θ be its Lagrangian angle, and let H be its mean curvature 1-form. Then $d\theta = H$, or equivalently, $J\nabla\theta = \vec{H}$.*

Proof. We work using normal coordinates of the Lagrangian L at $p \in L$, $\{\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n}\}$, and by a slight abuse of notation we denote $\frac{\partial}{\partial y^i} := J\frac{\partial}{\partial x^i}$, the elements of the corresponding dual basis by dx^k and dy^k, and $dz^k = dx^k + idy^k$.

Since Ω is parallel, $\overline{\nabla}\Omega = 0$. Therefore, considering the volume form Ω' as defined in (2.24), it follows from (2.25) that for any $X \in TL$,

$$i\,d\theta(X)\Omega' + (\overline{\nabla}_X dz^1) \wedge \ldots \wedge dz^n + \ldots + dz^1 \wedge \ldots \wedge (\overline{\nabla}_X dz^n) = 0. \tag{2.27}$$

Working with the basis $\{dz^1, d\bar{z}^1, \ldots, dz^n, d\bar{z}^n\}$ of $T_p\mathscr{Y} \otimes \mathbb{C}$, the only components of $\overline{\nabla}_X dz^i$ in $dz^1 \wedge \ldots \wedge (\overline{\nabla}_X dz^i) \wedge \ldots \wedge dz^n$ that are not killed by the wedge product are the dz^i and $d\bar{z}^j$ components, for $1 \leq j \leq n$. By (2.27), the $d\bar{z}^j$ terms must also be zero, since none of the other terms in the sum are of this form. Therefore,

$$\begin{aligned} i\,d\theta(X) + \sum_{k=1}^{n} (\overline{\nabla}_X dz^k)(\tfrac{\partial}{\partial z^k}) &= 0 \\ \implies i\,d\theta(X) - \sum_{k=1}^{n} dz^k(\overline{\nabla}_X \tfrac{\partial}{\partial z^k}) &= 0 \\ \implies i\,d\theta(X) - \sum_{k=1}^{n} (dx^k + i dy^k)(\tfrac{1}{2}\overline{\nabla}_X \tfrac{\partial}{\partial x^k} - \tfrac{i}{2}\overline{\nabla}_X \tfrac{\partial}{\partial y^k}) &= 0. \end{aligned} \tag{2.28}$$

Since we are working with an orthonormal basis, $dx^k(Y) = \langle \frac{\partial}{\partial x^k}, Y\rangle$. It therefore follows that

$$dx^k(\overline{\nabla}_X \tfrac{\partial}{\partial x^k}) = \langle \tfrac{\partial}{\partial x^k}, \overline{\nabla}_X \tfrac{\partial}{\partial x^k}\rangle = \tfrac{1}{2}\overline{\nabla}_X \langle \tfrac{\partial}{\partial x^k}, \tfrac{\partial}{\partial x^k}\rangle = 0,$$

and similarly for the $dy^k(\overline{\nabla}\frac{\partial}{\partial y^k})$ term. Therefore working from (2.28),

$$\begin{aligned} i\,d\theta(X) &= \sum_{k=1}^{n} \tfrac{i}{2} dx^k(\overline{\nabla}_X \tfrac{\partial}{\partial y^k}) - \tfrac{i}{2} dy^k(\overline{\nabla}_X \tfrac{\partial}{\partial x^k}) \\ \implies i\,d\theta(X) &= \sum_{k=1}^{n} \tfrac{i}{2}\langle \tfrac{\partial}{\partial x^k}, \overline{\nabla}_X J \tfrac{\partial}{\partial x^k}\rangle - \tfrac{i}{2}\langle J\tfrac{\partial}{\partial x^k}, \overline{\nabla}_X \tfrac{\partial}{\partial x^k}\rangle \\ \implies d\theta(X) &= \sum_{k=1}^{n} \langle \tfrac{\partial}{\partial x^k}, \overline{\nabla}_X J \tfrac{\partial}{\partial x^k}\rangle = H(X). \end{aligned} \tag{2.29}$$

□

If we are working in $\mathbb{C}^n$, then there is a simple formula for the Lagrangian angle. If $\{X_1, X_2, \ldots, X_n\} \subset \mathbb{C}^n$ are linearly independent vectors tangent to L at a point $p \in L$, then the Lagrangian angle can be calculated (up to a multiple of π) as:

$$\theta(p) = \arg(\det_{\mathbb{C}}(X_i^j)). \tag{2.30}$$

If we ensure that $vol_L(X_1, X_2, \ldots, X_n) = 1$, i.e. an orientation is chosen, then the Lagrangian angle is determined modulo 2π by this method.

There are several conditions one can impose on the Lagrangian angle which will be important for Lagrangian mean curvature flow. If the Lagrangian angle is a single-valued function, then L is known as a **zero-Maslov** Lagrangian. If additionally $\cos\theta > 0$, i.e. there exists ε such that $\theta \in (-\frac{\pi}{2} + \varepsilon, \frac{\pi}{2} - \varepsilon)$, then we say that the Lagrangian is **almost-**

calibrated. Occasionally a more general definition of almost-calibrated is used, where we demand instead that there exist $\overline{\theta}$ and $\varepsilon > 0$ such that $\theta \in \{\overline{\theta} - \frac{\pi}{2} + \varepsilon, \overline{\theta} + \frac{\pi}{2} - \varepsilon\}$. Note that the almost-calibrated condition implies that the Lagrangian is zero-Maslov, no matter which definition is used.

2.4.3 Calibrated Geometry and Special Lagrangians

Minimal Lagrangians in Calabi-Yau manifolds are of great importance. One reason is that any minimal Lagrangian L^n in a Calabi-Yau manifold $\mathscr{Y}^{2n}$ must be volume-minimising – a fact which is not generally true for minimal submanifolds. Another is that they are defined by the simple equation $\theta = c$ for $c \in \mathbb{R}$, which (unlike the standard minimal submanifold equation) depends only on first order quantities and is a single equation rather than a system. Both of these facts are special cases of the theory of *calibrated submanifolds*, introduced by R. Harvey and H. Lawson [29], which we give a brief exposition of here.

If M^m is a Riemannian manifold, a closed n-form ϕ is a **calibration** if, for any point $x \in M$ and any oriented n-plane $V \subset T_xM$, we have $\phi|_V \leq vol_V$. Then, a submanifold $N^n \subset M^m$ is a **calibrated submanifold, calibrated by** ϕ if $\phi|_{TN} = vol_N$.

The importance of calibrated submanifolds is that they are volume-minimising in their homology class:

Proposition 2.4.7. *If N^n is a calibrated submanifold of a compact Riemannian manifold M^m, calibrated by the n-form ϕ, then N is volume-minimising in its homology class.*

Proof. Consider another submanifold N' in the same homology class, so that there exists a region $R \subset M$ with $\partial R = N' - N$. Then by Stokes' theorem and the definition of a calibration:

$$\begin{aligned} 0 &= \int_R d\phi = \int_{N'} \phi - \int_N \phi \\ \implies &\int_N vol_N \leq \int_{N'} vol_{N'} \\ \implies &vol(N) \leq vol(N'). \end{aligned}$$

□

For a Calabi-Yau manifold $\mathscr{Y}^{2n}$, there is a natural calibration.

Proposition 2.4.8. *Let $\mathscr{Y}^{2n}$ be a Calabi-Yau manifold, with holomorphic volume form Ω. Then $\mathscr{R}e(e^{i\theta}\Omega)$ is a calibration, and N is calibrated by $\mathscr{R}e(e^{i\theta}\Omega)$ if and only if N is a minimal Lagrangian with constant Lagrangian angle θ. In this case, N is known as a* ***special Lagrangian****.*

Proof. At any point in $\mathscr{Y}$, we may use Kähler normal coordinates $\{\frac{\partial}{\partial x^1}, \frac{\partial}{\partial y^1}, \ldots, \frac{\partial}{\partial x^n}, \frac{\partial}{\partial y^n}\}$ so that the holomorphic volume form is given by

$$\Omega = \bigwedge_{k=1}^{n} dz^k.$$

Then if $\{e_1, e_2, \ldots, e_n\}$ is an orthonormal basis for an n-plane at $x \in \mathscr{Y}$, and we denote by A the matrix mapping $\frac{\partial}{\partial x^i}$ to e^i, then

$$|\Omega(e_1, \ldots, e_n)|^2 = |det_{\mathbb{C}} A|^2 = |det_{\mathbb{R}} A| = |e_1 \wedge Je_1 \wedge \ldots \wedge e_n \wedge Je_n| \leq 1.$$

Therefore $\mathscr{R}e(\mathscr{Y})$ is a calibration. If N is a calibrated submanifold with the same orthonormal basis at $x \in N$, then

$$\begin{aligned} \mathscr{R}e(e^{i\theta}\Omega)(e_1, \ldots, e_n) &= 1 \\ \Longrightarrow |\Omega(e_1, \ldots, e_n)| &= 1 \\ \Longrightarrow |e_1 \wedge Je_1 \wedge \ldots \wedge e_n \wedge Je_n| &= 1, \end{aligned}$$

which can only be true if all these vectors are orthogonal, implying that N is Lagrangian. Finally, denoting by α its Lagrangian angle,

$$\begin{aligned} & 1 = \mathscr{R}e(e^{-i\theta}\Omega(e_1, \ldots, e_n)) \\ \iff & 1 = \mathscr{R}e(e^{-i\theta} e^{i\alpha} vol_L(e_1, \ldots, e_n) \\ \iff & 1 = \mathscr{R}e(e^{i(\alpha-\theta)}) \\ \iff & \alpha = \theta. \end{aligned}$$

Therefore, N is a minimal Lagrangian with constant Lagrangian angle θ. The above calculation runs in reverse, which proves the converse statement. □

One of the most important questions in Calabi-Yau geometry is: does there exist a special Lagrangian representative in a given homology class of Lagrangians? Since the mean curvature 1-form H is closed for a Lagrangian in a Kähler-Einstein manifold by Proposition 2.4.4, there is a natural associated cohomology class, $[H]$, known as the **Maslov class**. Since $H = 0$ for a special Lagrangian, a necessary condition for L to be homologous to a special Lagrangian is that this class vanishes. This is equivalent to the Lagrangian angle being a single valued function: the **zero-Maslov** condition.

Chapter 3

Lagrangian Mean Curvature Flow

We now introduce the central object of study: Lagrangian mean curvature flow (henceforth often abbreviated to LMCF). This is the name given to the phenomenon, first demonstrated by K. Smoczyk [60], that Lagrangian submanifolds remain Lagrangian under the mean curvature flow in Kähler-Einstein manifolds (Kähler manifolds such that $\overline{\rho} = \lambda \overline{\omega}$). The fact that Lagrangian mean curvature flow works is very surprising, since mean curvature flow is a concept of Riemannian submanifold geometry, rather than one of symplectic geometry!

Since Chapter 4 is concerned with introducing Lagrangian mean curvature flow with boundary, we first take the opportunity to provide a full exposition of Smoczyk's original result of preservation of the Lagrangian condition under mean curvature flow in a Kähler-Einstein manifold in Section 3.1. As well as being expository, the interior estimates will be used for our proofs in Chapter 4. The key to the preservation of the condition $\omega = 0$ is the closure of the mean curvature 1-form, as was shown in Proposition 2.4.4. As we will show in Lemma 3.1.1, only deformations corresponding to closed 1-forms under the identification $J : TL \to TL^{\perp}$ have a chance of preserving this condition. The remainder of the work is in calculating the evolution equations for $|\omega|^2$.

The rest of the chapter examines properties of the flow. In Section 3.2, we look at evolution equations of various quantities, most importantly that of the Lagrangian angle in Lemma 3.2.1. The Lagrangian angle is a tool of enormous importance, due to its simple evolution equation and the fact that it is a first-order quantity, in contrast

The results of Chapter 3 do not comprise original work. In particular, Section 3.1 is an exposition of the work of K. Smoczyk [60].

to the mean curvature form H. Section 3.3 introduces some examples of Lagrangian mean curvature flow that will be central to our work of Chapters 5 and 6, as singularity models and boundary conditions for the flow. Finally, Section 3.4 considers singularity analysis of Lagrangian mean curvature flow. This will provide the context for Chapters 5 and 6, which consider these questions in the special case of equivariant Lagrangian mean curvature flow. In particular, we review A. Neves' important work classifying singularities of zero-Maslov Lagrangian mean curvature flow.

3.1 Preservation of the Lagrangian Condition

The driving question behind Smoczyk's seminal work [60] is 'Does there exist a canonical way to deform a Lagrangian submanifold?' To answer this question, consider a Lagrangian submanifold L and a normal variation $\vec{N}$, which can be identified with a one-form $N = N_i dx^i$ by $N(X) := \overline{g}(\vec{N}, JX)$. What properties must such a variation have, in order to preserve the Lagrangian condition $\omega = 0$? In the following Lemma, we see that N must be a closed form.

Lemma 3.1.1. *If $\vec{N}$ is a normal variation of a Lagrangian immersion $F : L^2 \to M^{2n}$, then the derivative of the Kähler form $\omega := \overline{\omega}|_L$ is given by*

$$\frac{d\omega}{dt} = -dN,$$

where N is the associated 1-form to the normal vector $\vec{N}$.

Proof. Using $\overline{\nabla}\overline{\omega} = 0$, an almost identical calculation to (2.7) gives

$$\frac{d\omega_{ij}}{dt} = \frac{d}{dt}\left(\overline{\omega}\left(\frac{\partial F}{\partial x^i}, \frac{\partial F}{\partial x^j}\right)\right) = \overline{\omega}\left(\frac{\partial^2 F}{\partial t \partial x^i}, \frac{\partial F}{\partial x^j}\right) + \overline{\omega}\left(\frac{\partial F}{\partial x^i}, \frac{\partial^2 F}{\partial t \partial x^j}\right).$$

Remembering that $\frac{\partial F}{\partial t} = \vec{N} = N^i J\left(\frac{\partial}{\partial x^i}\right)$, it follows using the antisymmetry of $\overline{\omega}$ and the relationship $\omega(X,Y) = g(JX,Y)$ that

$$\begin{aligned}\frac{d\omega_{ij}}{dt} &= \frac{\partial F}{\partial x^i}\overline{\omega}\left(\frac{\partial F}{\partial t},\frac{\partial F}{\partial x^j}\right) - \overline{\omega}\left(\frac{\partial F}{\partial t},\frac{\partial^2 F}{\partial x^i \partial x^j}\right) - \frac{\partial F}{\partial x^j}\overline{\omega}\left(\frac{\partial F}{\partial t},\frac{\partial F}{\partial x^i}\right)\\ &\qquad + \overline{\omega}\left(\frac{\partial F}{\partial t},\frac{\partial^2 F}{\partial x^i \partial x^j}\right) \qquad (3.1)\\ &= -\frac{\partial F}{\partial x^i}\overline{g}\left(N^k\frac{\partial F}{\partial x^k},\frac{\partial F}{\partial x^j}\right) + \frac{\partial F}{\partial x^j}\overline{g}\left(N^k\frac{\partial F}{\partial x^k},\frac{\partial F}{\partial x^i}\right)\\ &= N_{i,j} - N_{j,i}\\ &= (dN)_{ji}.\end{aligned}$$

□

We have already seen that the mean curvature one-form is closed for a Lagrangian submanifold in a Kähler-Einstein manifold, in Proposition 2.4.4 and equation (2.23). Therefore, taking Lemma 3.1.1 into consideration, one may hope that mean curvature flow preserves the Lagrangian condition. The above calculation is not enough to show that this is true, however, as it is only valid when L is a Lagrangian submanifold (else J is not an isometry between the tangent bundle and the normal bundle), and if ω immediately became non-zero under the flow, the Lagrangian condition would be broken. To prove that the Lagrangian condition is preserved, we must consider submanifolds nearby to Lagrangians as well, and calculate the evolution of $\omega|_L$ for these.

3.1.1 Geometry of Totally Real Submanifolds

The above discussion motivates the following definitions. An n-dimensional submanifold N^n of a $2n$-dimensional Kähler manifold M^{2n} is **totally real** if $J(T_pN) \cap T_pN = \{0\}$ for all points $p \in N$, i.e. if TN and $J(TN)$ intersect transversally. Taking a local basis $\{e_1, \ldots, e_n\}$ of TN, it follows that the function

$$\tilde{J} : TN \to TN^{\perp}, \qquad \tilde{J}(X) := J(X) - g^{ij}\langle J(X), e_i\rangle e_j \qquad (3.2)$$

is a vector space isomorphism, since if $\tilde{J}(X) = \tilde{J}(Y)$, then (3.2) implies that $J(X-Y) \in TN$, contradicting the totally real assumption. Note the equalities

$$\tilde{J}e_k = Je_k - \omega_k{}^l e_l \qquad J\tilde{J}e_k = -e_k - \omega_k{}^l Je_l. \tag{3.3}$$

Unlike J for Lagrangians, $\tilde{J}$ is not an isometry, as $\langle X, Y\rangle$ and $\langle \tilde{J}X, \tilde{J}Y\rangle$ are not necessarily equal. However it is true that

$$\eta(X,Y) := \langle \tilde{J}X, \tilde{J}Y\rangle, \tag{3.4}$$

$$\eta_{ij} = g_{ij} + \omega_i{}^l \omega_{lj} \tag{3.5}$$

is a positive-definite, symmetric $(0,2)$-tensor on TN. We denote the components of the inverse η^{ij}. In the basis $\{e_1, \ldots, e_n, \tilde{J}e_1, \ldots, \tilde{J}e_n\}$, it follows that the matrices $\overline{g}$, $\overline{g}^{-1}$ take the form

$$\overline{g} = \begin{pmatrix} g_{ij} & 0 \\ 0 & \eta_{ij} \end{pmatrix}, \qquad \overline{g}^{-1} = \begin{pmatrix} g^{ij} & 0 \\ 0 & \eta^{ij} \end{pmatrix}.$$

When compared to Lagrangian submanifolds, what changes in the totally real case? Firstly, $\omega = \overline{\omega}|_N$ no longer vanishes, and nor do the derivatives $\nabla\omega$, $\nabla^2\omega$. We may still define the second fundamental form and mean curvature form as tensors on the tangent bundle:

$$\tilde{h}(X,Y,Z) := \langle \overline{\nabla}_X Y, \tilde{J}Z\rangle,$$
$$\tilde{H}(X) := \langle \vec{H}, \tilde{J}X\rangle,$$
$$\tilde{H}_i = g^{kl} h_{kli} = h^k{}_{ki},$$

and if the mean curvature vector is expressed in the new basis as $\vec{H} = H^i \tilde{J}e_i$, then the components of the new mean curvature 1-form are given by

$$\tilde{H}_i = \eta_{ij} H^j.$$

Unlike the Lagrangian case, $\tilde{h}$ is no longer fully symmetric, and the Gauss and Codazzi equations take slightly more complicated forms. In the following proposition, we record the identities describing the obstructions to symmetry for the curvature tensors in the totally real case.

Lemma 3.1.2. *Let $N^n \subset M^{2n}$ be a totally real submanifold of a Kähler manifold, and define the tensors $\tilde{h}$, $\tilde{J}$ as above. Then:*

a) $\tilde{h}_{ijk} = \tilde{h}_{jik} = \tilde{h}_{jki} + \nabla_j \omega_{ki}$

b) $\nabla_k \nabla_j \omega_{li} - \nabla_l \nabla_j \omega_{ki} - \nabla_j \nabla_i \omega_{lk} = R_{jli}{}^s \omega_{ks} + R_{kji}{}^s \omega_{ls} + R_{lkj}{}^s \omega_{is}$

c) $\overline{R}_{ijkl} = R_{ijkl} + \eta^{nm} \left(\tilde{h}_{ikm} \tilde{h}_{jln} - \tilde{h}_{ilm} \tilde{h}_{jkn} \right)$

d) $\nabla_i \tilde{h}_{jkl} - \nabla_j \tilde{h}_{ikl} = \overline{R}_{ijk\tilde{l}} + \omega_m{}^p \eta^{nm} \left(\tilde{h}_{jln} \tilde{h}_{ikp} - \tilde{h}_{iln} \tilde{h}_{jkp} \right)$
$+ \omega_l{}^p \eta^{nm} \left(\tilde{h}_{ipm} \tilde{h}_{jkn} - \tilde{h}_{jpm} \tilde{h}_{ikn} \right)$

e) $\nabla_i \tilde{h}_{jkl} - \nabla_l \tilde{h}_{jki} = \overline{R}_{ilk\tilde{j}} + \nabla_k \nabla_j \omega_{li} + \omega_i{}^s R_{kljs} + \omega_l{}^s R_{ikjs} + \omega_j{}^s \overline{R}_{ilsk}$
$+ \omega_m{}^p \eta^{nm} (\tilde{h}_{ljn} \tilde{h}_{ikp} - \tilde{h}_{ijn} \tilde{h}_{lkp}),$

where an index with a tilde $\tilde{l}$ here represents the component in the direction of $\tilde{J}e_l$.

Proof. For equation a), note that $\tilde{h}$ is still symmetric in the first two arguments by the usual symmetries of the second fundamental form. The rest follows from a calculation:

$$
\begin{aligned}
\tilde{h}_{ijk} = \tilde{h}_{jik} &= \overline{g}(\overline{\nabla}_{e_j} e_i, \tilde{J}(e_k)) \\
&= \overline{g}(\overline{\nabla}_{e_j} e_i, J e_k) - \overline{g}(\overline{\nabla}_{e_j} e_i, \omega_k{}^l e_l) \\
&= e_j(\overline{g}(e_i, J e_k)) + \overline{g}(\overline{\nabla}_{e_j} e_k, e_i) - g(\nabla_{e_j} e_i, \omega_k{}^l e_l) \\
&= \partial_j \omega_{ki} + \tilde{h}_{jki} + g(\nabla_{e_j} e_k, \omega_i{}^l) - g(\nabla_{e_j} e_i, \omega_k{}^l e_l) \\
&= \tilde{h}_{jki} + \partial_j \omega_{ki} - \Gamma^n{}_{jk} \omega_{ni} - \Gamma^n{}_{ji} \omega_{kn} \\
&= \tilde{h}_{jki} + \nabla_j \omega_{ki}.
\end{aligned}
$$

For equation b), note that since ω is closed, we have the equality

$$\nabla_l \omega_{ik} + \nabla_i \omega_{kl} + \nabla_k \omega_{li} = 0.$$

Therefore, using the rule for exchanging second covariant derivatives (2.1) and the Bianchi identity (2.4) for the last line,

$$\begin{aligned}
\nabla_k\nabla_j\omega_{li} &= \nabla_j\nabla_k\omega_{li} - R_{kjl}{}^n\omega_{ln} - R_{kji}{}^n\omega_{ni}\\
&= -\nabla_j\nabla_l\omega_{ik} - \nabla_j\nabla_i\omega_{kl}R_{kjl}{}^n\omega_{ni} - R_{kji}{}^n\omega_{ln}\\
&= \nabla_l\nabla_j\omega_{ki} - R_{jlk}{}^n\omega_{ni} - R_{jli}{}^n\omega_{kn} + \nabla_j\nabla_i\omega_{lk} - R_{kjl}{}^n\omega_{ni} - R_{kji}{}^n\omega_{ln}\\
\implies \nabla_k\nabla_j\omega_{li} &= \nabla_l\nabla_j\omega_{ki} + \nabla_j\nabla_i\omega_{lk} + R_{jli}{}^s\omega_{ks} + R_{kji}{}^s\omega_{ls} - R_{lkj}{}^s\omega_{si}.
\end{aligned}$$

Equation c) is simply the Gauss equation (Theorem 2.1.4), noting that

$$A_{ik} = \eta^{nm}\langle\overline{\nabla}_{e_i}e_k, \tilde{J}e_m\rangle\tilde{J}e_n = \eta^{nm}\tilde{h}_{ikm}\tilde{J}e_n. \tag{3.6}$$

For equation d), we use the Codazzi equation (Theorem 2.1.4) and (3.6) to expand and simplify the left hand side:

$$\begin{aligned}
\nabla_i\tilde{h}_{jkl} - \nabla_j\tilde{h}_{ikl} &= \partial_i\langle A(e_j,e_k),\tilde{J}e_l\rangle - \langle A(\nabla_{e_i}e_j,e_k),\tilde{J}e_l\rangle - \langle A(e_j,\nabla_{e_i}e_k),\tilde{J}e_l\rangle\\
&\quad - \partial_j\langle A(e_i,e_k),\tilde{J}e_l\rangle + \langle A(\nabla_{e_j}e_i,e_k),\tilde{J}e_l\rangle + \langle A(e_i,\nabla_{e_j}e_k),\tilde{J}e_l\rangle\\
&= \langle\nabla_iA_{jk} - \nabla_jA_{ik},\tilde{J}e_l\rangle + \langle A_{jk},\overline{\nabla}_{e_i}\tilde{J}e_l\rangle - \langle A_{ik},\overline{\nabla}_{e_j}\tilde{J}e_l\rangle\\
&\quad - \langle A_{jk},\tilde{J}\nabla_{e_i}e_l\rangle + \langle A_{ik},\tilde{J}\nabla_{e_j}e_l\rangle\\
&= \overline{R}_{ijk\tilde{l}} + \langle A_{jk},\overline{\nabla}_{e_i}\tilde{J}e_l - \tilde{J}\nabla_{e_i}e_l\rangle - \langle A_{ik},\overline{\nabla}_{e_j}\tilde{J}e_l - \tilde{J}\nabla_{e_j}e_l\rangle.
\end{aligned}$$

To simplify the right hand side further, note by (3.3) that for any normal vector ν,

$$\langle J(\tilde{J}e_p),\nu\rangle = \langle -\omega_p{}^q\tilde{J}e_q,\nu\rangle,$$

and therefore,

$$\begin{aligned}
\langle \overline{\nabla}_{e_i}\tilde{J}e_l - \tilde{J}\nabla_{e_i}e_l\,,\,\nu\rangle &= \langle \overline{\nabla}_{e_i}(Je_l - \omega_l{}^p e_p) - J\nabla_{e_i}e_l\,,\,\nu\rangle \\
&= \langle J(A(e_i,e_l)) - \omega_l{}^p A(e_i,e_p)\,,\,\nu\rangle \\
&= \langle -\eta^{nm}\tilde{h}_{iln}\omega_m{}^p\tilde{J}e_p \;-\; \eta^{nm}\tilde{h}_{ipn}\omega_l{}^p\tilde{J}e_m\,,\,\nu\rangle. \\
\Longrightarrow\ \nabla_i\tilde{h}_{jkl} - \nabla_j\tilde{h}_{ikl} &= \overline{R}_{ijk\tilde{l}} - \langle \eta^{rs}\tilde{h}_{jkr}\tilde{J}e_s\,,\,\eta^{nm}(\tilde{h}_{iln}\omega_m{}^p\tilde{J}e_p + \tilde{h}_{ipn}\omega_l{}^p\tilde{J}e_m)\rangle \\
&\qquad + \langle \eta^{rs}\tilde{h}_{ikr}\tilde{J}e_s\,,\,\eta^{nm}(\tilde{h}_{jln}\omega_m{}^p\tilde{J}e_p + \tilde{h}_{jpn}\omega_l{}^p\tilde{J}e_m)\rangle \\
&= \overline{R}_{ijk\tilde{l}} + \omega_m{}^p\eta^{nm}\left(\tilde{h}_{jln}\tilde{h}_{ikp} - \tilde{h}_{iln}\tilde{h}_{jkp}\right) \\
&\qquad + \omega_l{}^p\eta^{nm}\left(\tilde{h}_{ipm}\tilde{h}_{jkn} - \tilde{h}_{jpm}\tilde{h}_{ikn}\right).
\end{aligned}$$

Finally, the last equation can be obtained by combining equations a)-d) to simplify the left hand side. □

3.1.2 Mean Curvature Flow of Totally Real Submanifolds

We now flow these submanifolds by mean curvature flow. Note that the totally real condition is an open condition, so mean curvature flow of totally real submanifolds is well-defined for a short time.

Lemma 3.1.3. *Consider a mean curvature flow $F_t : N^n \to M^{2n}$ such that the image N is totally real for all $t \in [0,T]$. Then the following evolution equations hold:*

- $\frac{d}{dt}g_{ij} = -2\eta^{nm}\tilde{H}_n\tilde{h}_{ijm}$
- $\frac{d}{dt}g^{ij} = 2\eta^{nm}g^{il}g^{kj}\tilde{h}_{kln}\tilde{H}_m$
- $\frac{d}{dt}vol_g = -\eta^{nm}\tilde{H}_n\tilde{H}_m\, vol_g$
- $\frac{d}{dt}\omega = -d\tilde{H}.$

Proof. The first and third evolution equations follow from Lemma 2.2.1, and the second from differentiating $g^{ij}g_{jk}$ in time. The final equation essentially follows from Lemma 3.1.1, however since we are now working with totally real geometry, the latter proof must be modified slightly. Our deformation vector is now $H^i\tilde{J}e_i$. Everything before

(3.1) is the same, and subsequently the calculation proceeds as

$$
\begin{aligned}
&= e_i \overline{g}\left(H^k \tilde{J} e_k , e_j\right) - e_j \overline{\omega}\left(H^k \tilde{J} e_k , e_i\right) \\
&= e_j \overline{g}\left(H^k \tilde{J} e_k , \tilde{J} e_i\right) - e_i \overline{g}\left(H^k \tilde{J} e_k , \tilde{J} e_j\right) \\
&= e_j \tilde{H}_i - e_i \tilde{H}_j = -(d\tilde{H})_{ij}.
\end{aligned}
$$

□

We can use these to calculate the evolution of the key quantity $|\omega|^2$. The aim is to estimate in terms of $|\omega|^2$, so that an initially vanishing ω can be shown to stay zero. We can then prove the main theorem of the section; mean curvature flow in Kähler-Einstein manifolds preserves the class of Lagrangians.

Lemma 3.1.4. *Consider a smooth mean curvature flow $F_t : N^n \to M^{2n}, \quad t \in [0,T]$ of a compact manifold N in a Kähler-Einstein manifold M^{2n} (so that the Ricci form ρ vanishes). Assume that the image N is totally real for all $t \in [0,T]$. Then the following evolution equation holds:*

$$
\frac{d}{dt}|\omega|^2 \leq \Delta|\omega|^2 + c|\omega|^2,
$$

where $c > 0$ is a positive constant.

Proof. Throughout, we work in normal coordinates at a point. Using Lemma 3.1.2, we first calculate the components of the exterior derivative of the mean curvature form:

$$
\begin{aligned}
d\tilde{H}_{il} &= \partial_i \tilde{H}_l - \partial_l \tilde{H}_i = \nabla_i \tilde{H}_l - \nabla_l \tilde{H}_i \\
&= g^{jk}\left(\nabla_i \tilde{h}_{jkl} - \nabla_l \tilde{h}_{jki}\right) \\
&= g^{jk}\nabla_k \nabla_j \omega_{li} + \omega_l{}^s R_{is} - \omega_i{}^s R_{ls} + g^{jk}\omega_j{}^s \overline{R}_{ilsk} + g^{jk}\overline{R}_{ilk\tilde{j}} \\
&\qquad + \omega_m{}^p \eta^{nm} g^{jk}\left(\tilde{h}_{ljn}\tilde{h}_{ikp} - \tilde{h}_{ijn}\tilde{h}_{lkp}\right).
\end{aligned}
$$

We may estimate the coefficient ω_{ij} by $|\omega|$ in normal coordinates:

$$
\omega_{ij}^2 \leq \sum_{i,j=1}^{n} \omega_{ij}^2 = \omega_{ij}\omega^{ij} = |\omega|^2.
$$

Therefore, since the quantities $\tilde{H}$, $\tilde{h}$, g, R, $\overline{R}$ are bounded on our smooth flow, we may estimate any term that depends quadratically on ω by $c|\omega|^2$ for a positive constant c depending on the flow. We use this, along with Lemma 3.1.3, to estimate the evolution of $|\omega|^2$.

$$\begin{aligned}\frac{d}{dt}|\omega|^2 &= \frac{d}{dt}\left(g^{ij}g^{kl}\omega_{il}\omega_{jk}\right)\\ &= 2\frac{d}{dt}(g^{ij})g^{kl}\omega_{il}\omega_{jk} + 2g^{ij}g^{kl}\frac{d}{dt}(\omega_{il})\omega_{jk}\\ &= 4\eta^{nm}g^{ir}g^{js}\tilde{h}_{rsn}\tilde{H}_m g^{kl}\omega_{ik}\omega_{jl} - 2d\tilde{H}_{il}g^{ij}g^{kl}\omega_{jk}\\ &\leq c|\omega|^2 - 2g^{ij}g^{kl}\omega_{jk}g^{pq}\nabla_p\nabla_q\omega_{li} - 2g^{ij}g^{kl}\omega_{jk}g^{pq}\overline{R}_{ilq\tilde{p}}\\ &\leq c|\omega|^2 + \Delta|\omega|^2 + \omega^{il}\overline{R}_{il\tilde{p}}{}^{p}.\end{aligned}$$

It is left to estimate this final term, which will be possible due to the Kähler-Einstein assumption. We work using an orthonormal basis of eigenvectors for η, so that

$$\overline{g} = \begin{pmatrix} g & 0 \\ 0 & \eta \end{pmatrix} = \left(\begin{array}{c|ccc} Id & & 0 & \\ \hline & 1-a_1 & \cdots & 0 \\ 0 & \vdots & \ddots & \vdots \\ & 0 & \cdots & 1-a_n \end{array}\right),$$

where by (3.5), $a_i = \sum_{l=1}^{n}\omega_{il}^2$. Since M is Kähler-Einstein,

$$\begin{aligned}0 = \rho_{kj} &= R_{i\underline{k}j}{}^{i} + R_{\tilde{i}\underline{k}j}{}^{\tilde{i}}\\ &= R_{i\underline{k}ji} + \frac{1}{1-a_i}R_{\tilde{i}\underline{k}j\tilde{i}}\\ &= R_{i\underline{k}ji} + R_{\tilde{i}\underline{k}j\tilde{i}} + \frac{a_i}{1-a_i}R_{\tilde{i}\underline{k}j\tilde{i}}.\end{aligned} \tag{3.7}$$

Now, using (2.21) to swap the underlined components, and subsequently (3.3) to exchange some J terms for $\tilde{J}$ terms,

$$0 = -R_{\tilde{i}kji} - \omega_{is}R_{skji} + R_{ikj\tilde{i}} + \omega_{is}R_{\underline{s}kj\tilde{i}} + \frac{a_i}{1-a_i}R_{\tilde{i}\underline{k}j\tilde{i}}.$$

Finally, using the Bianchi identity (2.4) and the fact that $a_i = \sum_{l=1}^n \omega_{il}^2$,

$$\begin{aligned} 0 &= R_{kji\tilde{i}} + R_{ikj\tilde{i}} + R_{jik\tilde{i}} \\ \implies R_{kji\tilde{i}} &= -\omega_{is}R_{skji} + \omega_{is}R_{\underline{s}kj\tilde{i}} + \frac{a_i}{1-a_i}R_{\tilde{i}\underline{k}j\tilde{i}} \\ \implies \omega^{il}\overline{R}_{il\tilde{p}}{}^{p} &\leq c|\omega|^2, \end{aligned}$$

and the estimation is complete. □

Theorem 3.1.5. *Consider a smooth mean curvature flow $F_t : N^n \to M^{2n}$ of a compact manifold N in a Kähler-Einstein manifold M^{2n}, so that the Ricci form ρ of M vanishes. Assume that the initial immersion N_0 is a Lagrangian submanifold of M. Then for all $t \in [0,T]$, N_t is a Lagrangian submanifold.*

Proof. The condition of a submanifold being totally real is an open condition. Therefore, if N_t is Lagrangian, then for a short time interval $[t, t+\varepsilon)$ it remains totally real under the mean curvature flow. We prove that $|\omega| = 0$ on this interval, and then since the condition $|\omega| = 0$ is also a closed condition, it follows by a continuity argument that N is Lagrangian for all time.

Consider the function $f_\varepsilon := |\omega|^2 - c\varepsilon e^{-2ct}$, which is initially negative and has evolution equation

$$\frac{df_\varepsilon}{dt} \leq \Delta|\omega|^2 + c|\omega|^2 - 2c^2e^{-2ct} < \Delta f_\varepsilon + cf_\varepsilon.$$

An application of the parabolic maximum principle now implies that f_ε remains negative for $t \in [0,T]$. Letting $\varepsilon \to 0$ proves that $|\omega|^2 \leq 0$ for $t \in [0,T]$, and since it is a positive quantity the result is proven. □

Note that this theorem as stated works only for compact Lagrangians, as we need to bound the mean curvature, second fundamental form and metric. However, as long as these quantities remain finite (i.e. a singularity does not occur), we are able to apply the same reasoning, and so in fact any smooth mean curvature flow with Lagrangian initial condition remains Lagrangian.

Remark 3.1.6. *Though it is necessary for the ambient manifold to be Kähler-Einstein for the Lagrangian condition to be preserved under mean curvature flow, K. Smoczyk shows in [61] that if the ambient manifold is Kähler and simultaneously flows by the Kähler-Ricci flow, then mean curvature flow preserves the Lagrangian condition. The extra Ricci curvature term in (3.7) which disappears by the Einstein condition is instead cancelled by the extra Ricci flow terms.*

3.2 Evolution of the Lagrangian Angle

For our purposes, by far the most important evolution equation under Lagrangian mean curvature flow in a Calabi-Yau manifold is that of the Lagrangian angle. It satisfies the most simple parabolic equation of all when pulled back to the abstract manifold L - the heat equation. This makes it extremely useful as a test function, for example to demonstrate convergence to a special Lagrangian.

Proposition 3.2.1. *Let $F_t : L^n \to \mathscr{Y}^{2n}$ be a Lagrangian mean curvature flow in a Calabi-Yau manifold $\mathscr{Y}$ with holomorphic volume form Ω. Let θ_t be the Lagrangian angle of $F_t(L)$, so that*

$$\Omega|_{L_t} = e^{i\theta_t} vol_{L_t}. \tag{3.8}$$

Then:

$$\begin{aligned} \frac{d\theta}{dt} &= \Delta\theta, \\ \frac{d\theta^2}{dt} &= \Delta\theta^2 - |H|^2. \end{aligned}$$

Proof. We calculate the evolution of both sides of (3.8). Note that the flow of L in $\mathscr{Y}$ may be considered as induced by a vector field on the ambient space, which agrees with $\vec{H}$ on L_t at time t. Denoting the associated diffeomorphisms by ϕ_t, it follows from Proposition 2.4.6 that:

$$\begin{aligned} &\frac{d}{dt}\Omega|_{L_t} = \frac{d}{dt}\left(F_0^*\phi_t^*\Omega\right) = F_0^*\mathscr{L}_{\vec{H}}\Omega = F_0^*\left(d(\iota_{\vec{H}}\Omega)\right) = d(F_0^*(i\cdot\iota_{\nabla\theta}\Omega)) \\ \Longrightarrow\ &\frac{d}{dt}\Omega|_{L_t} = d\left(ie^{i\theta}\iota_{\nabla\theta} vol_{L_t}\right). \end{aligned}$$

Since $\iota_{\nabla\theta} vol_L = *d\theta$ (by the definition of the Hodge star), it follows that

$$\frac{d}{dt}\Omega|_L = d(e^{i\theta} i * d\theta) = -e^{i\theta} d\theta \wedge *d\theta + ie^{i\theta} d(*d\theta).$$

Then, working instead with the right-hand side of (3.8),

$$\frac{d}{dt}\Omega|_L = ie^{i\theta}\frac{d\theta}{dt} vol_L + e^{i\theta}\frac{d}{dt} vol_L.$$

Equating imaginary parts, and using the equivalence of the Hodge and Laplace-Beltrami operators (Theorem 2.1.3),

$$\begin{aligned} \frac{d\theta}{dt} vol_L &= d(*d\theta) \\ \implies \frac{d\theta}{dt} &= *^{-1} d * d\theta = -d^* d\theta = \Delta\theta. \end{aligned}$$

For the second equation, using the above, Proposition 2.3.1 and Proposition 2.4.6:

$$\frac{d}{dt}\theta^2 - \Delta(\theta^2) = 2\theta\frac{d\theta}{dt} - \mathrm{div}(2\theta\nabla\theta) = -2\langle\nabla\theta, \nabla\theta\rangle = -2|H|^2.$$

□

One immediate application of the above theorem is that it makes sense to talk of **almost-calibrated Lagrangian mean curvature flow**, since the almost-calibrated condition introduced in 2.4.3 is preserved (by the maximum principle applied to the heat equation of Proposition 3.2.1).

3.3 Examples of LMCF in $\mathbb{C}^n$

In this section, we list some important examples of Lagrangian mean curvature flows in $\mathbb{C}^n$.

In $\mathbb{C}^1$, the 1-dimensional case of mean curvature flow (curve shortening flow) is an example of Lagrangian mean curvature flow, since any 1-dimensional submanifold is Lagrangian. Curve shortening flow is well studied, for example Grayson's theorem (Theorem 2.2.5) gives a complete classification of long-time behaviour of embedded curve shortening flow.

For simple higher-dimensional examples, we consider equivariant submanifolds. These are Lagrangian submanifolds which may be described as the orbit of a profile curve $\gamma \subset \mathbb{C} \times \{0\}^{n-1}$ under the following action of $O(n)$:

$$A \in O(n), \quad (x,y) \in \mathbb{R}^n \times \mathbb{R}^n \cong \mathbb{C}^n \qquad \Longrightarrow \qquad A \cdot (x,y) = (Ax, Ay).$$

Under mean curvature flow, we may study the evolution of the profile curve γ instead of the entire Lagrangian, which means that equivariant Lagrangian mean curvature flow can be studied in a similar way to curve shortening flow, and is simpler than the general case. For more details on equivariant Lagrangians, see Section 5.1.

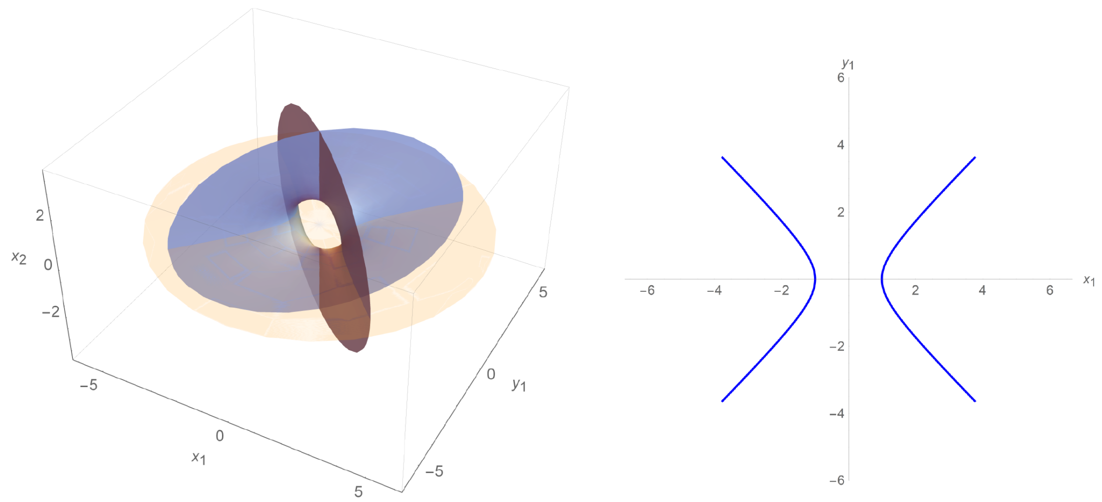

Figure 3.1: The Lawlor neck, $\Sigma_{\text{Law}} \subset \mathbb{C}^2$. The left hand image is the projection of Σ_{Law} onto $\mathbb{R}^3 \times \{0\} \subset \mathbb{C}^2$, and the self-intersections are an artefact of this projection. The right hand image is the profile curve $\sigma_{\text{Law}} \subset \mathbb{C}$ of the Lawlor neck, which may be viewed either as the intersection with the span of the real axes (depicted here as a translucent plane), or the quotient under the $SO(n)$ action. It is static under mean curvature flow as it is a special Lagrangian, and therefore minimal.

Example 3.3.1. *A **Lawlor neck**, $\Sigma_{\text{Law}} \subset \mathbb{C}^n$, is an equivariant special Lagrangian with*

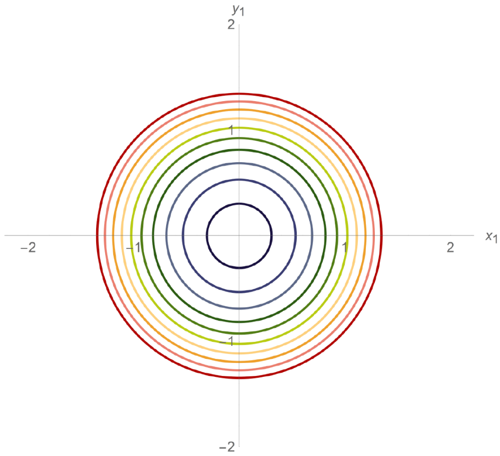

Figure 3.2: The profile curve $\sigma_{\text{Cliff}} \subset \mathbb{C}$ of the shrinking Clifford torus, $\Sigma_{\text{Cliff}} \subset \mathbb{C}^2$.

Lagrangian angle $\overline{\theta}$. Its profile curve may be expressed in polar coordinates as

$$r(\alpha) = \frac{B}{\sqrt[n]{\sin(\overline{\theta} - n\alpha)}}$$

for $B \geq 0$. Given any pair of equivariant planes that span an angle of $\frac{\pi}{n}$, there exists a Lawlor neck (unique up to scaling) asymptotic to these planes. For $n = 2$ and $\overline{\theta} = \frac{\pi}{2}$, the profile curve is the hyperbola (see Figure 3.1):

$$\sigma_{\text{Law}} : \mathbb{R} \to \mathbb{C}, \qquad \sigma_{\text{Law}}(s) := (\cosh(s), \sinh(s)).$$

Since it is a special Lagrangian, it remains static under Lagrangian mean curvature flow. There are also non-equivariant Lawlor necks spanning any special Lagrangian pair of planes in $\mathbb{C}^n$, but they will not be important in this work.

Example 3.3.2. *The **Clifford torus**, $\Sigma_{\text{Cliff}} \subset \mathbb{C}^n$, is an equivariant Lagrangian whose profile curve is a circle of radius $\sqrt{2n}$,*

$$\sigma_{\text{Cliff}} : S^1 \to \mathbb{C}, \qquad \sigma_{\text{Cliff}}(s) := (\sqrt{2n}\cos(s), \sqrt{2n}\sin(s)).$$

By the symmetries it is clear that the Clifford torus is a self-shrinking soliton for mean curvature flow, and the radius is chosen here so that it solves the self-shrinker equation (2.13), $\vec{H} = -\frac{X^\perp}{2}$. Figure 3.2 depicts the Clifford torus in $\mathbb{C}^2$.

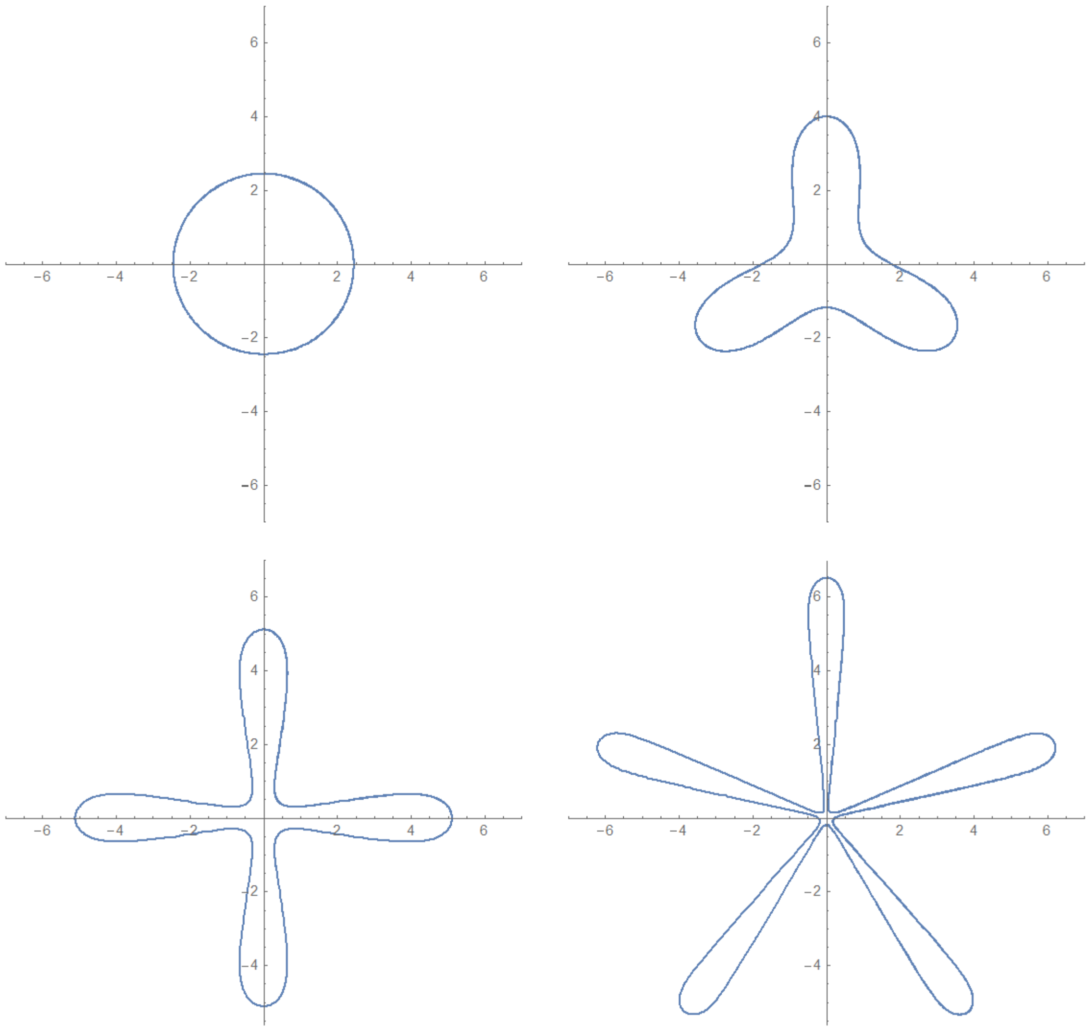

Figure 3.3: The profile curves of the four embedded self-shrinking Lagrangian tori in $\mathbb{C}^3$. The top-left curve is the Clifford torus, $\Sigma_{\text{Cliff}} \subset \mathbb{C}^3$.

It is the simplest example of a monotone flow, and of a Lagrangian self-shrinker, since for example there are no self-shrinking Lagrangian spheres in $\mathbb{C}^2$. There are however other self-shrinking tori which were first discovered by Anciaux [3]; Figure 3.3 depicts the profile curves of the four embedded toric self-shrinkers in $\mathbb{C}^3$.

Example 3.3.3. *The* ***Anciaux expander*** *is an equivariant self-expander, proven to exist by H. Anciaux in [3]. They are the equivariant case of the self-expanders found by D. Joyce, Y-I. Lee and M-P. Tsui in [37]. Though it is not easy to express these expanders explicitly, it is enough to know that for any pair of lines spanning an angle $\alpha < \frac{\pi}{n}$, there exists an Anciaux expander asymptotic to these lines, unique up to scaling.*

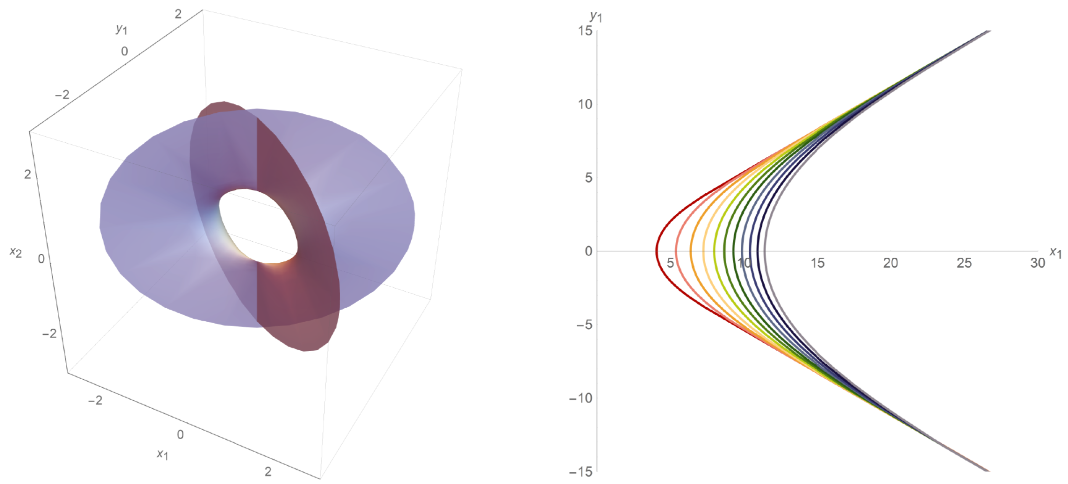

Figure 3.4: The Anciaux expander in $\mathbb{C}^2$. As with Figure 3.1, the left hand image is the projection onto $\mathbb{R}^3 \times \{0\} \subset \mathbb{C}^2$, and the right hand image is the flow of the profile curve under mean curvature flow.

3.4 Finite-time Singularities of LMCF

Our primary interest is the study of singularities of zero-Maslov Lagrangian mean curvature flow. This is a distinctly different subject to the study of singularities of generic mean curvature flow, as is highlighted by the following theorem of M-T. Wang [72, Remark 5.1], which states that an open class of Lagrangian mean curvature flows cannot form Type I singularities, i.e. the parabolic rescalings around singularities do not converge smoothly to a smooth self-shrinking soliton. This result contrasts starkly with the work of T. Colding and W. Minicozzi [13] and of O. Chodosh, K. Choi, C. Mantoulidis and F. Schulze [12], which suggest that generic singularities of hypersurface mean curvature flow are Type I.

Theorem 3.4.1. *An almost-calibrated Lagrangian mean curvature flow L_t in a Calabi-Yau manifold $\mathscr{Y}$ cannot form a Type I singularity.*

Proof. Assume for a contradiction that the flow L_t forms a Type I singularity, which without loss of generality we may assume is at the space-time point $X_0 = (0,0)$. Using Proposition 3.2.1, the evolution equation for $*Re(\overline{\Omega}|_L) = \cos\theta$ is given by

$$\frac{d}{dt}\cos\theta = \Delta\cos\theta + |H|^2\cos\theta.$$

Combining this with the weighted monotonicity formula, Theorem 2.2.7, and considering a sequence F^i of Type I rescalings with factors $\lambda_i \to \infty$, it follows that:

$$\begin{aligned}
\frac{d}{dt}\Theta(F^i, X_0, \sqrt{-t}, \cos\theta) &= \frac{d}{dt}\int_{L_t^i} (\cos\theta)\,\Phi_{X_0}\, d\mathscr{H}^n \\
&= \int_{L_t^i} \left(|\vec{H}|^2 - \left|\vec{H} - \frac{x^\perp}{2t}\right|^2 \right) \cos\theta\, \Phi_{X_0}\, d\mathscr{H}^n \\
\implies \Theta(F^i, X_0, \sqrt{-b}, \cos\theta) &- \Theta(F^i, X_0, \sqrt{-a}, \cos\theta) \\
&= \int_a^b \int_{F_t^i} \left(|\vec{H}|^2 - \left|\vec{H} - \frac{x^\perp}{2t}\right|^2 \right) \cos\theta\, \Phi_{X_0}\, d\mathscr{H}^n \\
\implies \Theta(F, X_0, \frac{\sqrt{-b}}{\lambda_i}, \cos\theta) &- \Theta(F, X_0, \frac{\sqrt{-a}}{\lambda_i}, \cos\theta) \\
&= \int_a^b \int_{F_t^i} \left(|\vec{H}|^2 - \left|\vec{H} - \frac{x^\perp}{2t}\right|^2 \right) \cos\theta\, \Phi_{X_0}\, d\mathscr{H}^n.
\end{aligned}$$

Since we have assumed that the singularity is Type I, we may choose the scaling factors such that L_t^i smoothly converges to a self-similarly shrinking mean curvature flow L_t^∞, with $|\vec{H} - \frac{x^\perp}{2t}| = 0$. Therefore, if we can show that $\Theta(F, X_0, r, \cos\theta)$ has a limit as $r \to 0$, then it follows by limiting the above equation that $H = 0$ for this limiting shrinker. Then,

$$\left|\vec{H} - \frac{x^\perp}{2t}\right| = 0 \implies x^\perp = 0,$$

implying that L_t^∞ is a plane. It then follows from White's regularity theorem, Theorem 2.2.10, that no singularity occurs, giving us a contradiction.

To show that $\Theta(F, X_0, r, \cos\theta)$ has a limit, note that by the almost-calibrated condition,

$$0 \le 1 - \cos\theta < 1,$$

$$\left(\frac{d}{dt} - \Delta\right)(1 - \cos\theta) = -|\vec{H}|^2 \cos\theta.$$

Huisken's monotonicity formula implies

$$\begin{aligned}
&\frac{d}{dt}\int_{L_{-r^2}} (1 - \cos\theta)\Phi_{X_0}\, d\mathscr{H}^n \\
&\quad = \int_{L_{-r^2}} \left(-|H|^2 \cos\theta - \left|\vec{H} - \frac{x^\perp}{2t}\right|^2 (1 - \cos\theta) \right) \Phi_{X_0} d\mathscr{H}^n,
\end{aligned}$$

and since the right hand side is negative, it follows that $\int_{L_{-r^2}}(1-\cos\theta)\Phi_{X_0}\,d\mathscr{H}^n$ has a limit as $r \to 0$. Since also the density $\int_{L_{-r^2}}\Phi_{X_0}d\mathscr{H}^n$ has a limiting value by Huisken's monotonicity formula, the result follows. □

It is possible in general for singularities of Lagrangian mean curvature flow to be of Type I. For example, any compact embedded curve shortening flow in $\mathbb{C}^1$ forms a Type I singularity modelled on a shrinking circle, by Grayson's theorem (Theorem 2.2.5). Additionally, the Clifford torus of Example 3.3.2 is a self-shrinking monotone solution to mean curvature flow, and so therefore provides an example of a Type I singularity in any number of dimensions. However, it has been shown that it is unstable as a Lagrangian self-shrinker by C. Evans, J. Lotay and F. Schulze [21, Theorem 1.1], in the sense that generic perturbations do not flow to Clifford torus singularities. This suggests that the Clifford torus singularity is not a generic behaviour of the flow.

Further work on singularities of zero-Maslov and almost-calibrated LMCF has been carried out by A. Neves in [50], which contains two important theorems on analysis of singularities. These theorems utilise the excellent evolution of the Lagrangian angle, Lemma 3.2.1, to prove that singularities of certain Lagrangian flows are modelled on special Lagrangian cones.

Theorem A tells us that any Type I blowup of a zero-Maslov LMCF looks like a union of special Lagrangian cones.

Theorem 3.4.2 (Neves' Theorem A). *If L_0 is a zero-Maslov class Lagrangian with bounded Lagrangian angle, then for any sequence of Type I rescaled flows $(L^i_s)_{s<0}$ at a singularity, with Lagrangian angle θ^i_s, there exist a finite set $\{\overline{\theta}_1,\dots,\overline{\theta}_N\}$ and integral special Lagrangian cones $\{L_1,\dots,L_N\}$ such that on passing to a subsequence, for every $f \in C^2(\mathbb{R})$, $\phi \in C^\infty_c(\mathbb{C}^n)$ and $s<0$,*

$$\lim_{i\to\infty}\int_{L^i_s} f(\theta^i_s)\phi\, d\mathscr{H}^n = \sum_{j=1}^N m_j f(\overline{\theta}_j)\mu_j(\phi),$$

where μ_j, m_j denote the Radon measure of the support and multiplicity of L_j respectively. Furthermore, the set $\{\overline{\theta}_1,\dots,\overline{\theta}_N\}$ doesn't depend on the sequence chosen.

Theorem B tells us that cones corresponding to the same local connected component in fact have the same Lagrangian angle, if we assume that the flow is almost-calibrated and rational. **Rational** here means that for some $a \in \mathbb{R}$,

$$\lambda(H_1(L_0,\mathbb{Z})) \in \{a2k\pi | k \in \mathbb{Z}\},$$

for $\lambda := \sum_{i=0}^{n} x^i dy^i - y^i dx^i$ the Liouville form. The rational condition is a generalisation of exactness; the form $\lambda|_L$ being exact is precisely L being rational with $a = 0$, and is preserved under the flow. See [50, Section 6] for a proof of preservation of rationality.

Theorem 3.4.3 (Neves' Theorem B)**.** *If L_0 is almost-calibrated and rational, then after passing to a subsequence of the rescaled flows L_s^i, with Lagrangian angle θ_s^i, the following holds for all $R > 0$ and almost all $s < 0$.*

For any convergent subsequence (in the Radon measure sense) Σ^i of connected components of $B_{4R}(0) \cap L_s^i$ intersecting $B_R(0)$, there exists a special Lagrangian cone L in $B_{2R}(0)$ with Lagrangian angle $\overline{\theta}$ such that for every $f \in C(\mathbb{R})$ and every $\phi \in C_c^\infty(B_{2R}(0))$,

$$\lim_{i\to\infty} \int_{\Sigma^i} f(\theta_s^i)\phi d\mathscr{H}^n = mf(\overline{\theta})\mu(\phi),$$

where μ and m denote the Radon measure of the support of L and the multiplicity respectively.

An important aspect of Theorem 3.4.3 to note is that it concerns a sequence of connected components of $L_s^i \cap B_{4R}$ in the Type I rescaling, which corresponds to a sequence of connected components in a shrinking ball for the original flow.

Chapter 4

LMCF with Boundary in Calabi-Yau Manifolds

In Section 3.1, we gave an exposition of Smoczyk's result [60] that in Kähler-Einstein manifolds, mean curvature flow preserves Lagrangian submanifolds. A natural follow-up question to this result is: 'Does there exist a well-defined boundary condition for mean curvature flow that preserves the Lagrangian condition?' In this chapter we answer this question in the affirmative, in the case that the ambient manifold is a Calabi-Yau manifold, i.e. a Ricci flat Kähler manifold.

Consider a family of immersed compact-with-boundary Lagrangian submanifolds $F_t : L^n \to \mathscr{Y}$, and an immersed Lagrangian mean curvature flow Σ_t in $\mathscr{Y}$ for $t \in [0, T_\Sigma)$, which will be our boundary submanifold. Denote $L_t := F_t(L^n)$, and suppose that $\partial L_t \subset \Sigma_t$; this may be thought of as $(n-1)$ Dirichlet boundary conditions for the mean curvature flow problem on L_t. For a well-posed PDE problem, one more boundary condition is required. For this, we fix the difference between the Lagrangian angles $\tilde{\theta}$ and θ of Σ_t and L_t respectively on ∂L_t – this is a natural generalisation of the fact that there is a constant Lagrangian angle difference between intersecting special Lagrangians. We

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The results of Chapter 4 comprise original joint work with B. Lambert and C. Evans, and appear in the preprint [20].

now have a well-posed boundary value problem:

$$\begin{cases} \left(\frac{d}{dt}F(x,t)\right)^{NL} = \vec{H}(x,t) & \text{for all } (x,t) \in L^n \times [0,T) \\ F(x,0) = F_0(x) & \text{for all } x \in L^n \\ \partial L_t \subset \Sigma_t & \text{for all } t \in [0,T) \\ e^{i(\tilde{\theta}-\theta)}(x,t) = ie^{i\alpha} & \text{for all } (x,t) \in \partial L^n \times [0,T), \end{cases} \tag{4.1}$$

where NL is the normal bundle of L, θ and $\tilde{\theta}$ are the Lagrangian angles of L and Σ respectively, and $\alpha \in (-\pi/2, \pi/2)$ is a constant angle. In the case where Σ_t and L_t are zero-Maslov, the final condition may be written as $\tilde{\theta} - \theta = \alpha + \frac{\pi}{2}$.

However, a priori the Lagrangian angle is not well-defined for L_t for $t > 0$, since the mean curvature flow does not necessarily preserve the Lagrangian condition. We must therefore generalise the Neumann boundary condition in equation (4.1) to a statement that holds for any 'totally real' n-dimensional manifold M^n intersecting along an $(n-1)$-dimensional manifold, see equation (4.4) in Section 4.1. In the case $M_t = L_t$ is Lagrangian, (4.4) and (4.1) are equivalent.

We remark that the Lagrangian mean curvature flow with boundary together with its boundary condition does not constitute a Brakke flow, and in particular the blow-up of a singularity does not need to be a self-similar shrinker. For example, Figure 4.2 depicts an example where the Type I blowup is a rotating soliton solution to (4.4).

Our first main theorem concerns preservation of the Lagrangian condition, so that (4.4) and (4.1) are equivalent when the initial immersion L_0 is Lagrangian, and we call a solution of this problem a **Lagrangian mean curvature flow with boundary**. In the following theorem, $\text{inj}_{\partial M}$ denotes the boundary injectivity radius, defined in Section 4.3.2.

Theorem 4.0.1. *Let Σ_t be a smooth Lagrangian mean curvature flow in a Calabi-Yau manifold $\mathscr{Y}$. Suppose M_t is a solution of (4.4) with M_0 Lagrangian and*

$$inj_{\partial M} > \delta > 0,$$

for $t \in [0,T)$. Then M_t is Lagrangian for all $t \in [0,T)$.

The proof of this result follows much the same strategy as the original proof of preservation of the Lagrangian condition by K. Smoczyk in [60], which we gave an exposition of in Section 3.1. Denoting as before $\omega := \overline{\omega}|_L$ for the restriction of the ambient Kähler form to M_t, by a careful analysis of the boundary condition we are able to apply a maximum principle to estimate the rate of increase of $|\omega|^2$ in terms of its initial value. Since the initial condition is Lagrangian, this implies that $|\omega|^2$ is identically zero.

The second major theorem is the short-time existence result:

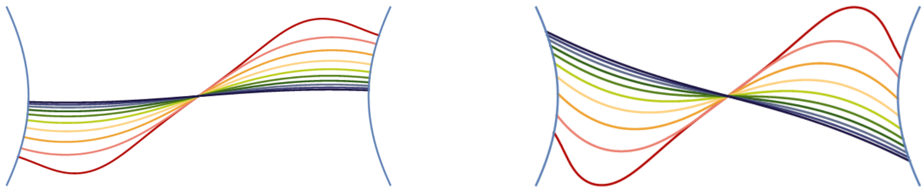
Theorem 4.0.2. *Let Σ_t be a smooth oriented Lagrangian mean curvature flow in a Calabi-Yau manifold $\mathscr{Y}$, and let M_0 be an oriented smooth compact Lagrangian submanifold of $\mathscr{Y}$ with boundary satisfying the boundary conditions in (4.4). Then there exists a $T \in (0,\infty)$ such that a unique solution of (4.4) exists for $t \in [0,T)$, and this solution is smooth for $t > 0$. Furthermore, if we assume this T is maximal, then at T at least one of the following hold:*

1. ***Boundary flow curvature singularity:*** $\sup_{\Sigma_t} |\tilde{A}|^2 \to \infty$ *as* $t \to T$.
2. ***Flowing curvature singularity:*** $\sup_{M_t} |A|^2 \to \infty$ *as* $t \to T$.
3. ***Boundary injectivity singularity:*** *The boundary injectivity radius $inj_{\partial M}$ of ∂M_t in M_t converges to zero as $t \to T$.*

Remark 4.0.3. *Whilst singularity options 1 and 2 in Theorem 4.0.2 are standard singularities, the boundary injectivity singularity is new and a result of the flowing boundary condition.*

For brevity, we exclude the proof of this result here, though it can be found in [20]. The boundary condition is a geometric mixed Dirichlet-Neumann boundary condition, which are not well-covered in the literature.

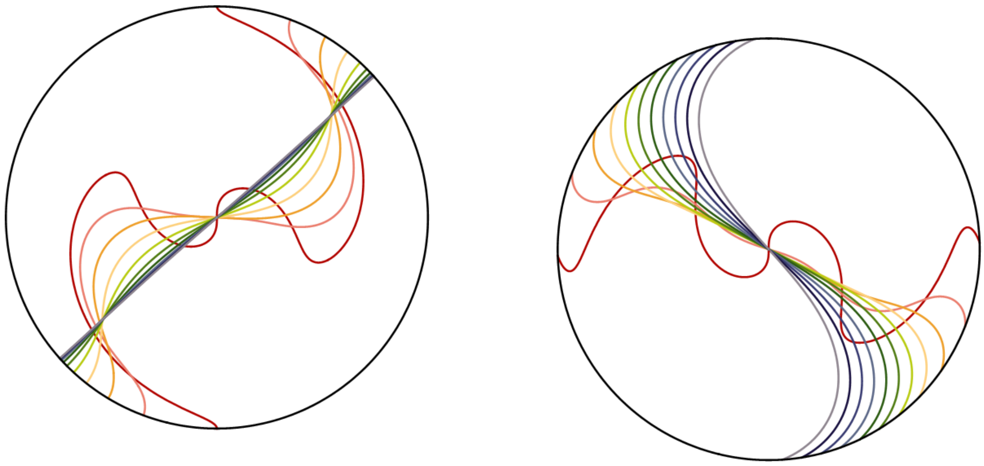
Finally, we prove a convergence result that will be useful later in Chapter 6, when investigating the behaviour of specific examples of LMCF with boundary. Explicitly, we prove that in the case where our flow is almost-calibrated with bounded second fundamental form, and the boundary is a special Lagrangian, the flow converges in infinite time to a special Lagrangian.



(a) An example of LMCF with boundary on the Lawlor neck Σ_{Law}, $\alpha = 0$.

(b) An example of LMCF with boundary on the Lawlor neck Σ_{Law}, $\alpha = 0.8$.

Figure 4.1: Two examples of LMCF with boundary on the Lawlor neck Σ_{Law}. Shown here are the 'profile curves' of the equivariant flow, i.e. the intersection of the flowing Lagrangians with $\mathbb{C} \times \{0\}$. They will both be studied in detail in Chapter 6.



(a) An example of rescaled LMCF with boundary on the Clifford torus Σ_{Cliff}, $\alpha = 0$.

(b) Another example on the Clifford torus Σ_{Cliff}, $\alpha = -\frac{2\pi}{5}$.

Figure 4.2: Two examples of LMCF with boundary on the Clifford torus Σ_{Cliff}. Shown here are the 'profile curves' of the equivariant flow, i.e. the intersection of the flowing Lagrangians with $\mathbb{C} \times \{0\}$. The first example will be studied in detail in Chapter 6.

Proposition 4.0.4. *Suppose that:*

- $\Sigma_t = \Sigma$ *is a special Lagrangian with Lagrangian angle* $\frac{\pi}{2}$,
- L_0 *is almost-calibrated, that is* $\theta_0 \in (-\frac{\pi}{2} + \varepsilon, \frac{\pi}{2} - \varepsilon)$,
- *and the solution to* (4.1), L_t, *exists for* $t \in [0, \infty)$ *with uniform estimates* $|\nabla^k A|^2 < C_k$.

Then L_t *converges smoothly to a special Lagrangian with Lagrangian angle* α.

4.1 The Boundary Conditions

In this section, we generalise (4.1) to a boundary problem that holds for any totally real M_t^n (as defined in Section 3.1.1) with $\partial M_t \subset \Sigma_t$. Note that we use the notation M here for the submanifold, and the ambient Calabi-Yau manifold is denoted $\mathscr{Y}$. Let Σ_t, $t \in [0,T)$ be a Lagrangian mean curvature flow in $\mathscr{Y}^{2n}$, and let M be a submanifold that satisfies the Dirichlet boundary condition $\partial M \in \Sigma_t$.

Throughout, we will employ the following notational conventions. We distinguish between quantities on the mean curvature flow M_t, the Lagrangian mean curvature flow Σ_t and the ambient Calabi-Yau manifold $\mathscr{Y}$ by diacritical marks: for instance, the ambient connection on $\mathscr{Y}$ is $\overline{\nabla}$, the induced connection on M is ∇, and the induced connection on Σ is $\widetilde{\nabla}$. We extend this convention in the natural way to other quantities such as the second fundamental form and the mean curvature. For any submanifold $Z \subset \mathscr{Y}$, $p \in Z$ and a general vector $V \in T_p\mathscr{Y}$ we will denote orthogonal projection of V onto the tangent space and normal space of Z by V^{TZ} and V^{NZ} respectively. Finally, throughout we will use the Einstein summation convention of summing over repeated indices, where we assume that lower case Roman letters sum $1 \leq i, j, k, \ldots \leq n$ and upper case Roman letters sum $1 \leq I, K, L, \ldots \leq n-1$.

We now set up the orthonormal bases that we will be working with to simplify our calculations; see Figure 4.3 for a diagram. At a point $p \in \partial M$, there exists tangent vectors $e_1, \ldots, e_{n-1}$ of $T_p\partial M$, $\mu \in T_pM$ an 'outward pointing' tangent vector and $\nu \in T_p\Sigma$ so that $\{e_1, \ldots, e_{n-1}, \mu\}$ is an orthonormal basis of T_pM and $\{e_1, \ldots, e_{n-1}, \nu\}$ is an orthonormal basis of $T_p\Sigma$. By the Lagrangian condition, $\{e_1, \ldots, e_{n-1}, \nu, Je_1, \ldots, Je_{n-1}, J\nu\}$ is an orthonormal basis for $T_p\mathscr{Y}$.

Define also $\tau = \tau^I Je_I$ as the projection of μ onto $\operatorname{span}\{Je_1, \ldots, Je_{n-1}\}$. Then, since μ has no components in any of the e_I directions, μ is of the form

$$\mu = \tau + \langle \nu, \mu \rangle \nu + \langle J\nu, \mu \rangle J\nu \tag{4.2}$$

$$\implies |\tau|^2 = 1 - \langle \nu, \mu \rangle^2 - \langle J\nu, \mu \rangle^2. \tag{4.3}$$

This yields that the Calabi–Yau form Ω' 'relative to $T_p\Sigma$' (as defined in (2.24)) restricted

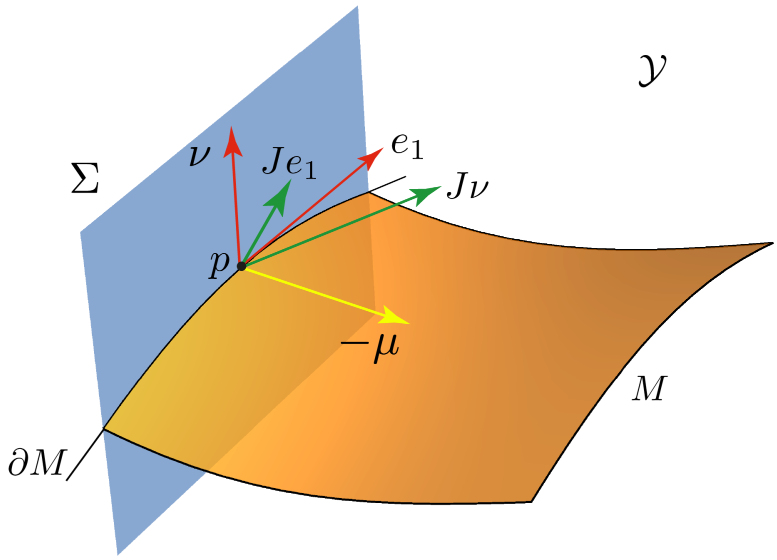


Figure 4.3: An illustration of the bases we will be working with, in the simple case of a 4-dimensional ambient Calabi-Yau manifold $\mathscr{Y}^4$ and 2-dimensional submanifolds M^2 and Σ^2. In this case, $\{e_1, \mu\}$ is an orthonormal basis of T_pM, $\{e_1, \nu\}$ is an orthonormal basis of $T_p\Sigma$, and $\{e_1, \nu, Je_1, J\nu\}$ is a basis of $T_p\mathscr{Y}$. Note that μ is the *outward pointing* tangent vector to M normal to the boundary at p.

to T_pM is given by

$$\Omega'(e_1, \ldots, e_{n-1}, \mu) = \det \left(\begin{array}{c|c} Id & i\tau^I \\ \hline 0 & \langle \nu, \mu \rangle + i \langle J\nu, \mu \rangle \end{array} \right) = \langle \nu, \mu \rangle + i \langle J\nu, \mu \rangle \, .$$

This complex number has modulus 1 if and only if the tangent space of M is Lagrangian at p, as M can only be Lagrangian if $\mu \in \operatorname{span}\{\nu, J\nu\}$, and the boundary condition of (4.1) is equivalent to this unit complex number being constant. We may therefore extend the boundary condition in (4.1) by simply assuming that the *argument* of this complex number is constant, that is we impose that there exists a constant $\alpha \in (-\frac{\pi}{2}, \frac{\pi}{2})$ so that

$$\langle \nu, \mu \rangle = \tan \alpha \, \langle J\nu, \mu \rangle \, .$$

If both Σ_t and M_t are Lagrangian submanifolds, this corresponds to a phase difference $ie^{i\alpha} = e^{i(\tilde{\theta} - \theta)}$. We note that a value $\alpha = 0$ corresponds to the submanifolds being 'perpendicular' at the intersection, and the excluded value $\alpha = \frac{\pi}{2}$ would correspond to the submanifolds being tangent at the intersection.

Remark 4.1.1. *We believe an analogous boundary condition could be defined in the non-Ricci-flat setting since we have only used the existence of a* relative *Calabi–Yau form. Hence the results of this chapter should be applicable with some modification to Lagrangian mean curvature flows in general Kähler–Einstein manifolds.*

Now let $F : M^n \times [0,T) \to \mathscr{Y}$ be a one parameter family of immersions, and write $M_t = F(M,t)$. We define a reparametrised mean curvature flow as follows:

$$\begin{cases} \left(\frac{d}{dt}F(x,t)\right)^{NM} = \vec{H}(x,t) & \text{for all } (x,t) \in M \times [0,T), \\ F(x,0) = F_0(x) & \text{for all } x \in M, \\ \partial M_t \subset \Sigma_t & \text{for all } t \in [0,T), \\ \cos\alpha\,\langle \nu,\mu\rangle - \sin\alpha\,\langle J\nu,\mu\rangle = 0 & \text{for all } (x,t) \in \partial M \times [0,T). \end{cases} \tag{4.4}$$

Note that (4.4) is exactly (4.1) when $F_t(M)$ is Lagrangian.

We now assume that M_t is a mean curvature flow that satisfies the boundary conditions in (4.4). It follows from (4.2) that at a boundary point we have

$$\begin{aligned} T_p\Sigma &= \text{span}\{e_1,\ldots,e_{n-1},\nu\}\ , \\ T_p\Sigma^\perp &= \text{span}\{Je_1,\ldots,Je_{n-1},J\nu\}\ . \\ T_pM &= \text{span}\{e_1,\ldots,e_{n-1},\langle \nu,\mu\rangle\,\nu + \langle J\nu,\mu\rangle\,J\nu + \tau\}, \end{aligned}$$

where $\tau \in JT_p\partial M$. As in Section 3.1, we will throughout work with totally real submanifolds, i.e. submanifolds M such that $J(T_pM)\cap T_pM = 0$. As shown in Section 3.1.1 (Equation 3.2), on such submanifolds there is an isomorphism

$$\tilde{J} : TM \to TM^\perp, \qquad \tilde{J}(X) := J(X) - g^{ij}\langle J(X), e_i\rangle e_j.$$

We may then use the following basis for the normal space:

$$T_pM^\perp = \text{span}\{f_1,\ldots,f_n\}\ ,$$

where for $1 \leq I \leq n-1$,

$$f_I := \tilde{J}e_I = Je_I - \langle Je_I, \mu \rangle \, \mu, \qquad f_n := \tilde{J}\mu = -\langle J\nu, \mu \rangle \, \nu + \langle \nu, \mu \rangle \, J\nu \,. \tag{4.5}$$

As before, this yields a positive-definite matrix

$$\eta_{ij} = \left\langle f_i, f_j \right\rangle = \begin{pmatrix} \delta_{IJ} - \tau^I \tau^J & 0 \\ 0 & 1 - |\tau|^2 \end{pmatrix}$$

where we write $\tau^I = \langle \tau, Je_I \rangle = \langle \mu, Je_I \rangle$, and used (4.3) for the bottom-right component. This has inverse

$$\eta^{ij} = \begin{pmatrix} \delta_{IJ} + \frac{\tau^I \tau^J}{1-|\tau|^2} & 0 \\ 0 & \frac{1}{1-|\tau|^2} \end{pmatrix} \,.$$

We now rewrite some of our vector quantities in terms of normal vectors, which will later be useful to convert covariant derivatives into second fundamental form terms. We may write

$$\mu = \mu^{N\Sigma} + \langle \nu, \mu \rangle \, \nu, \qquad \nu = \nu^{NM} + \langle \nu, \mu \rangle \, \mu \,.$$

Substituting back into the last terms and rearranging yields

$$\mu = \frac{1}{1 - \langle \nu, \mu \rangle^2} \left[\mu^{N\Sigma} + \langle \nu, \mu \rangle \, \nu^{NM} \right] \tag{4.6}$$

$$\nu = \frac{1}{1 - \langle \nu, \mu \rangle^2} \left[\nu^{NM} + \langle \nu, \mu \rangle \, \mu^{N\Sigma} \right] \,. \tag{4.7}$$

We have that $\tau = \tau^I Je_I \in T_p\Sigma^\perp$ satisfies

$$\langle \tau, f_I \rangle = \langle \tau, Je_I - \tau^I \mu \rangle = \tau^I (1 - |\tau|^2),$$

and so it follows by (4.5) that

$$\begin{aligned}\tau^{NM} &= \eta^{ij}\langle\tau, f_i\rangle f_j = \eta^{IJ}\langle\tau, f_I\rangle f_J = \eta^{IJ}(1-|\tau|^2)\tau^I f_J = \tau^J f_J \\ &= \tau - |\tau|^2\mu, \qquad (4.8)\\ \nu^{NM} &= \eta^{ij}\langle\nu, f_i\rangle f_j = -\eta^{IJ}\langle\nu,\mu\rangle\tau^I f_J - \frac{\langle J\nu,\mu\rangle}{1-|\tau|^2} f_n \\ &= -\frac{\langle\nu,\mu\rangle}{1-|\tau|^2}\tau^{NM} - \frac{\langle J\nu,\mu\rangle}{1-|\tau|^2} f_n. \qquad (4.9)\end{aligned}$$

In the following sections, we will assume that the vectors $e_1,\dots,e_{n-1}$, μ and ν are extended locally to a neighbourhood in $U \subset \partial M_t$ of p so that at every $q \in U$, $\{e_1,\dots,e_{n-1},\mu\}$ is an orthonormal basis of T_qM and $\{e_1,\dots,e_{n-1},\nu\}$ is an orthonormal basis of $T_q\Sigma$.

4.2 Derivatives of the Boundary Conditions

In this section, we derive identities that arise by differentiating the boundary conditions. Throughout, we use the notation $A_{XY} := A(X,Y) = (\overline{\nabla}_X Y)^\perp$ as shorthand for the components of the second fundamental form, and we bear in mind that the Lagrangian condition implies full symmetry of the second fundamental form $\tilde{A}$, i.e. $X,Y,Z \in T_p\Sigma$,

$$\langle\tilde{A}_{XY}, JZ\rangle = \langle\tilde{A}_{YX}, JZ\rangle = \langle\tilde{A}_{XZ}, JY\rangle. \qquad (4.10)$$

4.2.1 Space Derivatives of the Dirichlet Boundary Condition

We first use the Dirichlet condition to compare first order boundary derivatives.

Lemma 4.2.1. *Suppose that Σ is Lagrangian, and M is a n-dimensional totally real submanifold with boundary $\partial M \subset \Sigma$. At a point $p \in \partial M$, we have that for any $X,Y \in T_p\partial M$,*

$$\frac{\langle J\nu,\mu\rangle^2}{1-\langle\nu,\mu\rangle^2}\langle\tilde{A}_{XY},\tau\rangle = \langle A_{XY},\tau\rangle + \frac{|\tau|^2}{1-\langle\nu,\mu\rangle^2}\left[\langle J\nu,\mu\rangle\langle\tilde{A}_{XY},J\nu\rangle + \langle\nu,\mu\rangle\langle A_{XY},\nu\rangle\right].$$

Proof. Since $X, Y \in T_p\partial M$, we may write $\overline{\nabla}_X Y$ in two ways, namely

$$\begin{aligned}\overline{\nabla}_X Y &= \langle \tilde{A}_{XY}, Je_I\rangle Je_I + \langle \tilde{A}_{XY}, J\nu\rangle J\nu + \left\langle \overline{\nabla}_X Y, e_I\right\rangle e_I + \left\langle \overline{\nabla}_X Y, \nu\right\rangle \nu,\\ \overline{\nabla}_X Y &= \eta^{ik}\langle A_{XY}, f_i\rangle f_k + \left\langle \overline{\nabla}_X Y, e_I\right\rangle e_I + \left\langle \overline{\nabla}_X Y, \mu\right\rangle \mu,\end{aligned}$$

where the f_i are the basis of N_pM as above. Taking an inner product with Je_I, and noting that $\langle f_i, Je_j\rangle = \langle f_i, f_j\rangle = \eta_{ij}$ since f_j, Je_j are the same up to a component tangential to T_pM, this equality yields

$$\begin{aligned}\langle \tilde{A}_{XY}, Je_I\rangle &= \eta^{ik}\eta_{kI}\langle A_{XY}, f_i\rangle + \left\langle \overline{\nabla}_X Y, \mu\right\rangle \langle \mu, Je_I\rangle\\ &= \langle A_{XY}, f_I\rangle + \left\langle \overline{\nabla}_X Y, \mu\right\rangle \tau^I . \end{aligned} \tag{4.11}$$

Due to equation (4.6),

$$\begin{aligned}\left\langle \overline{\nabla}_X Y, \mu\right\rangle &= \frac{1}{1-\langle \nu,\mu\rangle^2}\left[\left\langle \overline{\nabla}_X Y, \mu^{N\Sigma} + \langle \nu,\mu\rangle \nu^{NM}\right\rangle\right]\\ &= \frac{1}{1-\langle \nu,\mu\rangle^2}\left[\langle \tilde{A}_{XY}, \mu\rangle + \langle \nu,\mu\rangle\langle A_{XY},\nu\rangle\right] .\end{aligned} \tag{4.12}$$

Equation (4.11) now yields

$$\langle \tilde{A}_{XY}, Je_I\rangle - \frac{\tau^I}{1-\langle \nu,\mu\rangle^2}\langle \tilde{A}_{XY}, \mu\rangle = \langle A_{XY}, f_I\rangle + \frac{\langle \nu,\mu\rangle \tau^I}{1-\langle \nu,\mu\rangle^2}\langle A_{XY},\nu\rangle .$$

Multiplying by τ^I and summing, we have that (using (4.8)),

$$\left\langle \tilde{A}_{XY}, \tau - \frac{|\tau|^2}{1-\langle \nu,\mu\rangle^2}\mu^{N\Sigma}\right\rangle = \langle A_{XY}, \tau\rangle + \frac{|\tau|^2\langle \nu,\mu\rangle}{1-\langle \nu,\mu\rangle^2}\langle A_{XY},\nu\rangle .$$

By (4.2) and (4.3), we have that

$$\begin{aligned}\tau - \mu + \langle \nu,\mu\rangle\nu &= \langle J\nu,\mu\rangle J\nu\\ \implies \tau - \frac{|\tau|^2}{1-\langle \nu,\mu\rangle^2}(\mu - \langle \nu,\mu\rangle\nu) &= \frac{\langle J\nu,\mu\rangle^2}{1-\langle \nu,\mu\rangle^2}\tau - \frac{\langle J\nu,\mu\rangle|\tau|^2}{1-\langle \nu,\mu\rangle^2}J\nu.\end{aligned}$$

Thus we conclude

$$\frac{\langle J\nu,\mu\rangle^2}{1-\langle\nu,\mu\rangle^2}\left\langle\tilde{A}_{XY},\tau\right\rangle = \langle A_{XY},\tau\rangle + \frac{|\tau|^2}{1-\langle\nu,\mu\rangle^2}\left[\langle J\nu,\mu\rangle\left\langle\tilde{A}_{XY},J\nu\right\rangle + \langle\nu,\mu\rangle\langle A_{XY},\nu\rangle\right] .$$

□

4.2.2 Time Derivatives of the Dirichlet Boundary Condition

We now consider time derivatives. We denote the mean curvature vector of M by $\vec{H}$ as before, and the mean curvature vector of Σ by $\widetilde{H}$.

Lemma 4.2.2. *Let $\Sigma_t \subset \mathscr{Y}$ be a smooth solution of LMCF and assume that $M_t \subset \mathscr{Y}$ is a totally real solution to (4.4). Suppose that $\partial M_t \subset \Sigma_t$ for all $t \geq 0$, then for all $t > 0$,*

$$\left\langle \vec{H}-\widetilde{H},\tau\right\rangle\langle J\nu,\mu\rangle = \left\langle \vec{H}-\widetilde{H},J\nu\right\rangle|\tau|^2 .$$

Proof. We consider a smooth function $p(t) = F(p^1(t),\dots,p^n(t),t)$ such that p stays in Σ_t (such a point exists by assumption). Then we must have that

$$\widetilde{H} = \left(\frac{dp}{dt}\right)^{N_p\Sigma} = \left(P+\vec{H}\right)^{N_p\Sigma}$$

where $P = \frac{\partial p^i}{\partial t}X_i$ is a tangent vector to M. Fixing t and writing $P = P^I e_I + P^\mu\mu$ we see that

$$\left\langle\widetilde{H},Je^I\right\rangle = \tau^I P^\mu + \left\langle Je^I,\vec{H}\right\rangle, \qquad \left\langle\widetilde{H},J\nu\right\rangle = \langle J\nu,\mu\rangle P^\mu + \left\langle J\nu,\vec{H}\right\rangle.$$

This is equivalent to the statement that

$$\vec{H}^{N\Sigma} - \widetilde{H} = -P^\mu\left[\tau + \langle J\nu,\mu\rangle J\nu\right] .$$

We also see that

$$\left\langle\vec{H}-\widetilde{H},\tau\right\rangle = -P^\mu|\tau|^2, \qquad \left\langle\vec{H}-\widetilde{H},J\nu\right\rangle = -P^\mu\langle J\nu,\mu\rangle$$

which yields the claim. □

4.2.3 Space Derivatives of the Neumann Boundary Condition

We will see that at a point $p \in \partial M$ such that the Neumann boundary condition holds and $\frac{1}{2} > |\omega|^2(p) = \max_{q \in \partial M} |\omega|^2(q)$ we have that

$$\nabla_I \langle \nu, \mu \rangle = 0 = \nabla_I \langle J\nu, \mu \rangle .$$

We will now investigate the implications of these equalities.

Lemma 4.2.3. *Let $\Sigma_t \subset \mathscr{Y}$ be a smooth solution of LMCF and assume that $M_t \subset \mathscr{Y}$ is a totally real solution to (4.4). Suppose that at some $p \in \partial M$*

$$\nabla_I \langle \nu, \mu \rangle = 0. \tag{4.13}$$

Then

$$\left\langle \nu, A_{I\mu} \right\rangle + \left\langle \mu, \tilde{A}_{I\nu} \right\rangle = 0.$$

Proof. Using (4.13), we have

$$0 = \left\langle \overline{\nabla}_I \nu, \mu \right\rangle + \left\langle \overline{\nabla}_I \mu, \nu \right\rangle,$$

and so using equations (4.6) and (4.7),

$$\begin{aligned}
0 &= \left\langle \overline{\nabla}_I \nu^{NM} + \langle \nu, \mu \rangle \overline{\nabla}_I \mu^{N\Sigma}, \mu \right\rangle + \left\langle \overline{\nabla}_I \mu^{N\Sigma} + \langle \nu, \mu \rangle \overline{\nabla}_I \nu^{NM}, \nu \right\rangle \\
&= -\left\langle \nu, A_{I\mu} \right\rangle - \left\langle \mu, \tilde{A}_{I\nu} \right\rangle + \langle \nu, \mu \rangle \left[\left\langle \overline{\nabla}_I \mu^{N\Sigma}, \mu \right\rangle + \left\langle \overline{\nabla}_I \nu^{NM}, \nu \right\rangle \right] \\
&= -\left\langle \nu, A_{I\mu} \right\rangle - \left\langle \mu, \tilde{A}_{I\nu} \right\rangle \\
&\quad + \frac{\langle \nu, \mu \rangle}{1 - \langle \nu, \mu \rangle^2} \left[\left\langle \overline{\nabla}_I \mu^{N\Sigma}, \mu^{N\Sigma} + \langle \nu, \mu \rangle \nu^{NM} \right\rangle + \left\langle \overline{\nabla}_I \nu^{NM}, \nu^{NM} + \langle \nu, \mu \rangle \mu^{N\Sigma} \right\rangle \right] \\
&= -\left\langle \nu, A_{I\mu} \right\rangle - \left\langle \mu, \tilde{A}_{I\nu} \right\rangle \\
&\quad + \frac{\langle \nu, \mu \rangle}{1 - \langle \nu, \mu \rangle^2} \left[\frac{1}{2} (\nabla_I \left| \mu^{N\Sigma} \right|^2 + \nabla_I \left| \nu^{NM} \right|^2) + \langle \nu, \mu \rangle \left(\nabla_I \left\langle \mu^{N\Sigma}, \nu^{NM} \right\rangle \right) \right]
\end{aligned}$$

However, we see that

$$\left|\mu^{N\Sigma}\right|^2 = 1 - \langle \nu, \mu \rangle^2 = \left|\nu^{NM}\right|^2$$

and

$$\left\langle \mu^{N\Sigma}, \nu^{NM} \right\rangle = \langle \mu - \langle \nu, \mu \rangle \nu, \nu - \langle \nu, \mu \rangle \mu \rangle = \langle \nu, \mu \rangle^2 - \langle \nu, \mu \rangle$$

and so by (4.13) the square bracket vanishes. □

Lemma 4.2.4. *Let $\Sigma_t \subset \mathscr{Y}$ be a smooth solution of LMCF and assume that $M_t \subset \mathscr{Y}$ is a totally real solution to (4.4). Suppose that at $p \in \partial M$ we have that*

$$0 = \nabla_I \langle \nu, \mu \rangle = \nabla_I \langle J\nu, \mu \rangle .$$

Then

$$0 = \left\langle A_{I\mu}, J\nu \right\rangle - \left\langle \tilde{A}_{I\nu}, J\mu \right\rangle + \frac{1}{1 - \langle \nu, \mu \rangle^2} \left[\langle A_{I\sigma}, \nu \rangle + \langle \nu, \mu \rangle \left\langle \tilde{A}_{I\sigma}, \mu \right\rangle \right] ,$$

where we define $\sigma := J\tau = -\tau^I e_I$ to simplify notation.

Proof. We expand the statement $\nabla_I \langle J\nu, \mu \rangle = 0$. We first note that

$$\begin{aligned} \left\langle \overline{\nabla}_I \mu, J\nu \right\rangle &= \left\langle \overline{\nabla}_I \mu, (J\nu)^{NM} + (J\nu)^{TM} \right\rangle \\ &= \left\langle A_{I\mu}, J\nu \right\rangle + \langle J\nu, \mu \rangle \left\langle \overline{\nabla}_I \mu, \mu \right\rangle \\ &= \left\langle A_{I\mu}, J\nu \right\rangle , \end{aligned}$$

as $|\mu|^2 = 1$. We also calculate using (4.2) that

$$\begin{aligned} \left\langle \overline{\nabla}_I \nu, J\mu \right\rangle &= \left\langle \overline{\nabla}_I \nu, (J\mu)^{N\Sigma} + (J\mu)^{T\Sigma} \right\rangle \\ &= \left\langle \tilde{A}_{I\nu}, J\mu \right\rangle + \left\langle \overline{\nabla}_I \nu, \sigma - \langle J\nu, \mu \rangle \nu \right\rangle \\ &= \left\langle \tilde{A}_{I\nu}, J\mu \right\rangle + \left\langle \overline{\nabla}_I \nu, \sigma \right\rangle \\ &= \left\langle \tilde{A}_{I\nu}, J\mu \right\rangle + \frac{1}{1 - \langle \nu, \mu \rangle^2} \left\langle \overline{\nabla}_I \left(\nu^{NM} + \langle \nu, \mu \rangle \mu^{N\Sigma} \right), \sigma \right\rangle \\ &= \left\langle \tilde{A}_{I\nu}, J\mu \right\rangle - \frac{1}{1 - \langle \nu, \mu \rangle^2} \left[\langle A_{I\sigma}, \nu \rangle + \langle \nu, \mu \rangle \left\langle \tilde{A}_{I\sigma}, \mu \right\rangle \right] . \end{aligned}$$

Putting these together we have that

$$\begin{aligned}\nabla_I \langle J\nu,\mu\rangle &= \left\langle J\nu,\overline{\nabla}_I\mu\right\rangle - \left\langle \overline{\nabla}_I\nu, J\mu\right\rangle \\ &= \left\langle A_{I\mu}, J\nu\right\rangle - \left\langle \tilde{A}_{I\nu}, J\mu\right\rangle + \frac{1}{1-\langle\nu,\mu\rangle^2}\left[\left\langle A_{I\sigma},\nu\right\rangle + \langle\nu,\mu\rangle\left\langle\tilde{A}_{I\sigma},\mu\right\rangle\right] .\end{aligned}$$

□

4.3 Preservation of the Lagrangian Condition

In this section, we prove Theorem 4.0.1, i.e. that the Lagrangian condition is preserved (assuming short-time existence of the flow).

4.3.1 Boundary Estimates

In preparation for this proof, we calculate some important quantities using the coordinate system introduced in Section 4.1. Using the Neumann boundary condition of (4.4),

$$\cos\alpha\,\langle\nu,\mu\rangle - \sin\alpha\,\langle J\nu,\mu\rangle = 0\ , \tag{4.14}$$

it follows from (4.2) that we may write μ as

$$\mu = \frac{\langle J\nu,\mu\rangle}{\cos\alpha}\left(\sin\alpha\nu + \cos\alpha J\nu\right) + \tau, \tag{4.15}$$

and from (4.3) that we may write $|\tau|^2$ as

$$|\tau|^2 = 1 - \frac{\langle J\nu,\mu\rangle^2}{\cos^2\alpha} = 1 - \frac{\langle\nu,\mu\rangle^2}{\sin^2\alpha}. \tag{4.16}$$

Let ω be the restriction of $\overline{\omega}$ to M. We wish to consider $|\omega|^2 = \omega_{ij}\omega^{ij}$ where $\omega_{ij} = \left\langle JX_i, X_j\right\rangle$. Calculating on the boundary in the basis $\{e_1,\ldots,e_{n-1},\mu\}$ of Section 3.1.1 we have that

$$\omega = \begin{pmatrix} & & \tau^1 \\ & 0 & \vdots \\ & & \tau^{n-1} \\ -\tau^1 \ldots - \tau^{n-1} & & 0 \end{pmatrix}$$

and so at the boundary

$$|\omega|^2 = 2|\tau|^2 \; = 2 - \frac{2\langle J\nu,\mu\rangle^2}{\cos^2(\alpha)}. \tag{4.17}$$

As a result, if $|\omega|^2 < \frac{1}{2}$ at a boundary point then

$$\frac{\langle J\nu,\mu\rangle^2}{\cos^2\alpha} > \frac{3}{4}\,, \tag{4.18}$$

and so at such a point, since $\nu^{NM} = \nu - \langle\nu,\mu\rangle\,\mu$,

$$\left|\nu^{NM}\right|^2 = \left|\mu^{N\Sigma}\right|^2 = 1 - \langle\nu,\mu\rangle^2 = |\tau|^2 + \langle J\nu,\mu\rangle^2 > \frac{3}{4}\cos^2\alpha > 0\,.$$

Finally, by Lemma 3.1.2 (remembering that $\tilde{h}(X,Y,Z) := \langle\overline{\nabla}_X Y, \tilde{J}Z\rangle$),

$$\begin{aligned}\nabla_j\omega_{ki} &= \tilde{h}_{ijk} - \tilde{h}_{jki}\\ &= \left\langle A_{ij}, JX_k\right\rangle - \left\langle A_{jk}, JX_i\right\rangle,\end{aligned}$$

and so, denoting $\sigma := J\tau$,

$$\begin{aligned}\nabla_\mu|\omega|^2 &= 2\left[\left\langle A_{j\mu}, JX_i\right\rangle - \left\langle A_{i\mu}, JX_j\right\rangle\right]\omega^{ij}\\ &= -4\left\langle A_{I\mu}, J\mu\right\rangle\tau^I + 4\left\langle A_{\mu\mu}, Je_I\right\rangle\tau^I\\ &= 4\left\langle A_{\sigma\mu}, J\mu\right\rangle + 4\left\langle A_{\mu\mu}, \tau\right\rangle\,.\end{aligned}$$

We now prove the key estimate for the proof of Theorem 4.0.1.

Lemma 4.3.1. *Let $\Sigma_t \subset \mathscr{Y}$ be a smooth solution of LMCF and assume that $M_t \subset \mathscr{Y}$ is a totally real solution to (4.4). Let p be a boundary maximum of $|\omega|^2$ where $|\omega| < \frac{1}{2}$. Then we have that*

$$\begin{aligned}\nabla_\mu|\omega|^2 = 2|\omega|^2\Bigg[&-\tan^2\alpha\langle A_{\mu\mu},\tau\rangle + \frac{1-\langle\nu,\mu\rangle^2}{\cos^2\alpha\,\langle J\nu,\mu\rangle}\left\langle H-\widetilde{H}, J\nu\right\rangle + \frac{1}{\cos^2\alpha}\left\langle\widetilde{H},\tau\right\rangle\\ &+\frac{1}{\cos^2\alpha}\left[\langle J\nu,\mu\rangle\left\langle\tilde{A}^I{}_I, J\nu\right\rangle + \langle\nu,\mu\rangle\left\langle A^I{}_I,\nu\right\rangle\right] - \frac{\langle J\nu,\mu\rangle}{|\sigma|^2}\left\langle\tilde{A}_{\sigma\sigma}, J\nu\right\rangle\\ &-\frac{\tan\alpha\,\langle J\nu,\mu\rangle}{|\sigma|^2\left(1-\langle\nu,\mu\rangle^2\right)}\left[\langle A_{\sigma\sigma},\nu\rangle + \langle\nu,\mu\rangle\left\langle\tilde{A}_{\sigma\sigma},\mu\right\rangle\right]\Bigg].\end{aligned}$$

and in particular, if $|A_p| < C_M$, $|\tilde{A}_p| < C_\Sigma$ *then there exists a constant* $C = C(n,\alpha)$ *so that*

$$\nabla_\mu |\omega|^2 = C(C_M + C_\Sigma)|\omega|^2 \ .$$

Proof. We first prove that

$$0 = \nabla_I \langle \nu, \mu \rangle = \nabla_I \langle J\nu, \mu \rangle ; \tag{4.19}$$

this will allow us to apply Lemmas 4.2.3 and 4.2.4. By (4.17), p is a boundary maximum of $|\tau|^2$, and so using (4.14),

$$\begin{aligned} 0 = \frac{1}{2}\nabla_I |\tau|^2 &= -\langle \nu, \mu \rangle \nabla_I \langle \nu, \mu \rangle - \langle J\nu, \mu \rangle \nabla_I \langle J\nu, \mu \rangle \\ &= -\frac{\langle J\nu, \mu \rangle}{\cos\alpha} \left[\sin\alpha \nabla_I \langle \nu, \mu \rangle + \cos\alpha \nabla_I \langle J\nu, \mu \rangle \right]. \end{aligned}$$

By (4.18) we have

$$\sin\alpha \nabla_I \langle \nu, \mu \rangle + \cos\alpha \nabla_I \langle J\nu, \mu \rangle = 0,$$

and differentiating (4.14) yields

$$\cos\alpha \nabla_I \langle \nu, \mu \rangle - \sin\alpha \nabla_I \langle J\nu, \mu \rangle = 0.$$

These together imply equation (4.19). We now wish to estimate $\frac{1}{4}\nabla_\mu |\omega|^2 = \left\langle A_{\sigma\mu}, J\mu \right\rangle + \left\langle A_{\mu\mu}, \tau \right\rangle$ at the boundary in terms of $|\omega|^2$ or equivalently $|\tau|^2 = |\sigma|^2$. Using (4.14) and Lemmas 4.2.3 and 4.2.4:

$$\begin{aligned} \left\langle A_{\sigma\mu}, J\mu \right\rangle &= \left\langle A_{\sigma\mu}, -\langle J\nu, \mu \rangle \nu + \langle \nu, \mu \rangle J\nu \right\rangle \\ &= \frac{\langle J\nu, \mu \rangle}{\cos\alpha} \left\langle A_{\sigma\mu}, -\cos\alpha \, \nu + \sin\alpha \, J\nu \right\rangle \\ &= \frac{\langle J\nu, \mu \rangle}{\cos\alpha} \left\langle \tilde{A}_{\sigma\nu}, \cos\alpha \, \mu + \sin\alpha \, J\mu \right\rangle \\ &\qquad - \frac{\tan\alpha \langle J\nu, \mu \rangle}{1 - \langle \nu, \mu \rangle^2} \left[\left\langle A_{\sigma\sigma}, \nu \right\rangle + \langle \nu, \mu \rangle \left\langle \tilde{A}_{\sigma\sigma}, \mu \right\rangle \right]. \end{aligned}$$

We may extract a $|\tau|^2$ from the second of these terms, so working with the first term:

$$\begin{aligned}
&\frac{\langle J\nu,\mu\rangle}{\cos\alpha}\left\langle \tilde{A}_{\sigma\nu}, \cos\alpha\,\mu + \sin\alpha\, J\mu\right\rangle \\
&\quad= \frac{\langle J\nu,\mu\rangle}{\cos\alpha}\Big\langle \tilde{A}_{\sigma\nu}\,,\, \cos\alpha\,\langle J\nu,\mu\rangle\, J\nu + \cos\alpha\,\langle \nu,\mu\rangle\,\nu + \cos\alpha\,\tau \\
&\qquad\qquad - \sin\alpha\,\langle J\nu,\mu\rangle\,\nu + \sin\alpha\,\langle \nu,\mu\rangle\, J\nu + \sin\alpha J\tau\Big\rangle \\
&\quad= \frac{\langle J\nu,\mu\rangle^2}{\cos^2\alpha}\left\langle \tilde{A}_{\sigma\nu}, \cos^2\alpha\, J\nu + \sin^2\alpha\, J\nu\right\rangle + \langle J\nu,\mu\rangle\left\langle \tilde{A}_{\sigma\nu},\tau\right\rangle \\
&\quad= \frac{\langle J\nu,\mu\rangle^2}{\cos^2\alpha}\left\langle \tilde{A}_{\sigma\nu}, J\nu\right\rangle - \langle J\nu,\mu\rangle\left\langle \tilde{A}_{\sigma\sigma},\nu\right\rangle.
\end{aligned}$$

The second term contains a $|\tau|^2$, so we work with the first term. Using (4.10) and Lemma 4.2.1 for the third line:

$$\begin{aligned}
\frac{\langle J\nu,\mu\rangle^2}{\cos^2\alpha}\left\langle \tilde{A}_{\sigma\nu}, J\nu\right\rangle &= -\frac{\langle J\nu,\mu\rangle^2}{\cos^2\alpha}\left\langle \tilde{A}_{\nu\nu},\tau\right\rangle \\
&= \frac{\langle J\nu,\mu\rangle^2}{\cos^2\alpha}\left\langle \tilde{A}^I{}_I,\tau\right\rangle - \frac{\langle J\nu,\mu\rangle^2}{\cos^2\alpha}\left\langle \widetilde{H},\tau\right\rangle \\
&= \frac{1-\langle \nu,\mu\rangle^2}{\cos^2\alpha}\left\langle A^I{}_I,\tau\right\rangle - \frac{\langle J\nu,\mu\rangle^2}{\cos^2\alpha}\left\langle \widetilde{H},\tau\right\rangle \\
&\qquad + \frac{|\tau|^2}{\cos^2\alpha}\left[\langle J\nu,\mu\rangle\left\langle \tilde{A}^I{}_I, J\nu\right\rangle + \langle \nu,\mu\rangle\left\langle A^I{}_I,\nu\right\rangle\right].
\end{aligned}$$

The final term contains a $|\tau|^2$, so we work with only the first two terms. Using Lemma 4.2.2:

$$\begin{aligned}
&\frac{1-\langle \nu,\mu\rangle^2}{\cos^2\alpha}\left\langle A^I{}_I,\tau\right\rangle - \frac{\langle J\nu,\mu\rangle^2}{\cos^2\alpha}\left\langle \widetilde{H},\tau\right\rangle \\
&\quad= \frac{1-\langle \nu,\mu\rangle^2}{\cos^2\alpha}\left[\left\langle \vec{H},\tau\right\rangle - \left\langle A_{\mu\mu},\tau\right\rangle\right] - \frac{\langle J\nu,\mu\rangle^2}{\cos^2\alpha}\left\langle \widetilde{H},\tau\right\rangle \\
&\quad= \frac{(1-\langle \nu,\mu\rangle^2)|\tau|^2}{\cos^2\alpha\,\langle J\nu,\mu\rangle}\left\langle \vec{H}-\widetilde{H}, J\nu\right\rangle - \frac{1-\langle \nu,\mu\rangle^2}{\cos^2\alpha}\left\langle A_{\mu\mu},\tau\right\rangle + \frac{|\tau|^2}{\cos^2\alpha}\left\langle \widetilde{H},\tau\right\rangle.
\end{aligned}$$

Finally we note after rewriting $\left\langle A_{\sigma\mu}, J\mu\right\rangle$ following all the steps as above, the coefficient of $\left\langle A_{\mu\mu},\tau\right\rangle$ in the overall equation for $\frac{1}{4}\nabla_\mu|\omega|^2$ is now

$$1-\left(\frac{1-\langle \nu,\mu\rangle^2}{\cos^2\alpha}\right) = \frac{-\sin^2\alpha + \langle \nu,\mu\rangle^2}{\cos^2\alpha} = -\tan^2\alpha\left(1-\frac{\langle \nu,\mu\rangle^2}{\sin^2\alpha}\right) = -\tan^2\alpha\,|\tau|^2$$

if $\alpha \neq 0$, and vanishes in the case that $\alpha = 0$. Putting all of this together, we obtain the result. □

4.3.2 Distance to the Boundary

We need a function ρ with a bounded evolution such that $\nabla_\mu \rho = -1$ for all boundary points, to use for our comparison principle argument. A natural choice would be the ambient distance to Σ, but unfortunately this is not smooth at Σ and we cannot in general avoid intersections of the interior of M with Σ due to the lack of geometric comparison principles for mean curvature flow in higher codimension. We instead consider a function based on the intrinsic distance to Σ.

We also need the notion of **boundary injectivity radius**, which we define here. Let μ be the outward pointing unit vector to ∂M. For $p \in \partial M$ let $\gamma_p(s)$ be the unit speed geodesic starting at $p \in \partial M$ with tangent vector $-\mu(p)$. We define the *boundary injectivity radius* to be

$$\mathrm{inj}_{\partial M} = \frac{1}{2}\min\left\{\lambda > 0 \,\middle|\, \exists p \in \partial M \text{ such that } \gamma_p((0,\lambda)) \subset M, \text{ but } \gamma_p(\lambda) \subset \partial M\right\} .$$

If M is compact then $\mathrm{inj}_{\partial M} > 0$ and in this case $\mathrm{inj}_{\partial M}$ coincides with the maximal collar region such that the distance to the boundary function is smooth.

Lemma 4.3.2. *Suppose Σ_t satisfies LMCF and M_t satisfies (4.4) such that there exist constants C_Σ and C_M so that*

$$\sup_{M\times[0,T)} |A| < C_M, \qquad \sup_{\Sigma\times[0,T)} |\tilde{A}| < C_\Sigma .$$

Let $\mathrm{inj}_{\partial M} > \delta > 0$ on $[0,T)$. Then there exists a function $\rho : M_t \to \mathbb{R}$ which is smooth and has the properties that

$$\begin{cases} \left(\frac{d}{dt} - \Delta\right)\rho \le C_\rho & \text{on } M_t \\ \nabla_\mu \rho = -1 & \text{on } \partial M_t \end{cases}$$

where C_ρ depends only on $\tilde{A}$, A, and δ.

Proof. Let $r(p,t) = \mathrm{dist}^{M_t}(p, \partial M_t),\ r : M \times [0,T) \to \mathbb{R}$ be the intrinsic distance to the boundary. Note that r satisfies $\nabla_\mu r = -1$ at the boundary. Define the collar region $U_R \subset M$ by

$$U_R = \{p \in M \,:\, r(p,t) \leq R,\ \forall t \in [0,T)\},$$

and denote by g_t the pullback metric on U_R at time t. Since A and $\tilde{A}$ are uniformly bounded, we can guarantee that r is smooth on U_R by choosing $R < \delta$ sufficiently small (dependent on $\tilde{A}, A$) so that $F_t(U_R)$ contains no focal or conjugate points for all times $t \in [0,T)$. We write the metric on U_R as a product metric $g_t = dr^2 + g_r$, and note that since r is a non-singular distance function, we have the fundamental equation

$$\partial_r g_r = 2\,\mathrm{Hess}(r), \tag{4.20}$$

(see for instance [56, Section 3.2.4]). Since (4.20) is linear, the Hessian cannot blow-up on U_R unless the metric degenerates. However, since U_R contains no focal points, g_r cannot degenerate and hence

$$|\,\mathrm{Hess}(r)| \leq C(\tilde{A}, A).$$

We now consider the time derivative of r for $r < \frac{1}{2}R$. For any p,t we have that there exists a unique geodesic $\gamma_{(p,t)} : [0,1] \to M$ such that $\ell(\gamma_{(p,t)}) = r$, $\gamma_{(p,t)}(0) = p$ and $\gamma_{(p,t)}(1) \in \partial M$. $\gamma_{(p,t)}$ must vary smoothly with time as otherwise it would contain conjugate points which are disallowed by the restriction of r. Remembering that the energy of the curve $\gamma_{(p,t)}$ is given by

$$E(\gamma) = \frac{1}{2}\int_0^1 \langle \gamma', \gamma' \rangle_{g(t)}$$

since $\gamma_{(p,t)}(s)$ is a minimiser of energy for the metric g_t we have

$$0 = \int_0^1 \left\langle \frac{d\gamma'}{dt}, \gamma' \right\rangle_{g_t} ds,$$

where for convenience we write $\gamma_{(p,t)} = \gamma$. Therefore, using the fact that $2E(\gamma) = \ell(\gamma)^2$

for geodesics and Lemma 2.2.1, we calculate that

$$\frac{dr}{dt}\bigg|_{(p,t)} = \frac{1}{\ell(\gamma)}\int_0^1 \left(\left\langle \frac{d\gamma'}{dt},\gamma'\right\rangle_{g_t} + \frac{1}{2}(\gamma')^i(\gamma')^j\frac{dg_{ij}}{dt}\right)ds = -\frac{1}{\ell(\gamma)}\int_0^1 \left\langle \vec{H},A(\gamma',\gamma')\right\rangle ds.$$

We therefore have that for $r < \frac{1}{2}R$,

$$\left(\frac{d}{dt}-\Delta\right) r \leq C(\tilde{A},A)$$

and at the boundary

$$\nabla_\mu r = -1\ .$$

The lemma is achieved by setting $\rho = \eta(r)$ where η is a smooth cutoff function so that

$$\begin{cases} \eta(x) = x & \text{for } x \in [0,\frac{R}{8}] \\ \eta(x) = \frac{R}{4} & \text{for } x \in [\frac{R}{2},\infty) \\ \frac{\partial\eta}{\partial x}(x) < 8 & \text{for } x \in \mathbb{R}. \end{cases}$$

□

4.3.3 Proof of Theorem 4.0.1

We now combine the evolution for $|\omega|^2$ derived in Section 3.1 and the estimates at the boundary from Section 4.3.2 to prove by comparison principle that $|\omega|^2$ is bounded by its initial value. This will complete the proof of preservation of the Lagrangian condition.

Lemma 4.3.3. *Suppose that Σ_t satisfies LMCF and M_t is a totally real solution to (4.4) on the time interval $[0,T)$. Suppose that there exist constants C_M, C_Σ and δ as in Lemma 4.3.2. Suppose that $\sup_{M_0}|\omega|^2 < \frac{1}{2}$ and $\widetilde{T}$ is chosen so that for all $t \in [0,\widetilde{T})$, $\sup_{M_t}|\omega|^2 < \frac{1}{2}$. Then, there exists constants $C_1 = C_1(C_M,C_\Sigma,n)$, $C_2 = C_2(C_M,C_\Sigma,n)$ such that for all $t \in [0,\widetilde{T})$,*

$$|\omega|^2 \leq C_1\, e^{C_2 t}\, \sup_{M_0}|\omega|^2.$$

Proof. For ρ as in Lemma 4.3.2, we now consider

$$f = |\omega|^2 e^{A\rho - Bt}$$

where $0 < A, B \in \mathbb{R}$. At the boundary we note that using Lemmas 4.3.1 and 4.3.2

$$\nabla_\mu f \leq |\omega|^2 e^{A\rho - Bt}(C(C_\Sigma + C_M) - A)$$

which is negative if we set $A = C(C_\Sigma + C_M) + 1$. Therefore f has no boundary maxima.

Using Lemma 3.1.4, originally derived by K. Smoczyk [61, Lemma 3.2.8], there exists a $C_2 = C_2(C_M)$ so that

$$\left(\frac{d}{dt} - \Delta\right)|\omega|^2 \leq C_2|\omega|^2 .$$

As a result, at an increasing maximum of f we may estimate

$$\begin{aligned}
0 &\leq \left(\frac{d}{dt} - \Delta\right) f \\
&= |\omega|^2 e^{A\rho - Bt}\Bigg[\frac{1}{|\omega|^2}\left(\frac{d}{dt} - \Delta\right)|\omega|^2 + A\left(\frac{d}{dt} - \Delta\right)\rho - A^2|\nabla\rho|^2 \\
&\qquad - 2\left\langle \frac{\nabla|\omega|^2}{|\omega|^2}, A\nabla\rho \right\rangle - B\Bigg] \\
&= |\omega|^2 e^{A\rho - Bt}\left[\frac{1}{|\omega|^2}\left(\frac{d}{dt} - \Delta\right)|\omega|^2 + A\left(\frac{d}{dt} - \Delta\right)\rho + A^2|\nabla\rho|^2 - B\right] \\
&\leq |\omega|^2 e^{A\rho - Bt}\left[C_2 + AC_\rho + A^2 - B\right]
\end{aligned}$$

where we used that as at a maximum $\nabla f = 0$, we have that $\frac{\nabla|\omega|^2}{|\omega|^2} = -A\nabla\rho$. Clearly, making B sufficiently large now yields a contradiction, implying that

$$f \leq \sup_{M_0} f ,$$

completing the proof. □

Proof of Theorem 4.0.1. Suppose M_t is a solution of (4.4) with M_0 Lagrangian and $\mathrm{inj}_{\partial M} > \delta > 0$, for $t \in [0, T)$. Then for any $\widehat{T} \in (0, T)$, there exists a constant C_M so that

$$\sup_{L^n\times[0,\widehat{T})} |A| < C_M, \qquad \sup_{L^n\times[0,\widehat{T})} |\tilde{A}| < C_\Sigma\ .$$

There also exists a maximal time $\widetilde{T} \leq \widehat{T}$ such that for all $t \in [0, \widetilde{T})$, $\sup_{M_t} |\omega|^2 < \frac{1}{2}$ and M_t is totally real. We may therefore apply Lemma 4.3.3 to see that for all $t \in (0, \widetilde{T})$, $|\omega|^2 = 0$ and so $\widetilde{T} = \widehat{T}$. As $\widehat{T}$ was arbitrary we see that for all $t \in [0, T)$, $|\omega|^2 \equiv 0$. □

4.4 Long-Time Convergence to a Special Lagrangian

In aid of the material in Chapter 6, we finish this chapter with a proposition on long-time convergence of LMCF with boundary to a special Lagrangian.

Proposition 4.4.1. *Suppose that:*

- $\Sigma_t = \Sigma$ *is a special Lagrangian with Lagrangian angle* $\frac{\pi}{2}$*,*
- L_0 *is almost-calibrated, that is* $\theta_0 \in (-\frac{\pi}{2} + \varepsilon, \frac{\pi}{2} - \varepsilon)$*,*
- *and the solution to* (4.1)*,* L_t*, exists for* $t \in [0, \infty)$ *with uniform estimates* $|\nabla^k A|^2 < C_k$*.*

Then L_t *converges smoothly to a special Lagrangian with Lagrangian angle* α*.*

To begin, we calculate the following evolution equation:

Lemma 4.4.2. *Suppose* L_0 *is zero-Maslov and* L_t *is a solution to* (4.1)*. Then for any be a smooth function* f *on* L_t*,*

$$\frac{d}{dt}\int_{L_t} f d\mathscr{H}^n = \int_{L_t} \frac{df}{dt} - |H|^2 f d\mathscr{H}^n + \int_{\partial L_t} f \left[\left\langle \widetilde{H}, J\nu \right\rangle \langle J\nu, \mu\rangle^{-1} - \tan\alpha \nabla_\mu \theta \right] d\mathscr{H}^{n-1}.$$

Proof. Here we have to distinguish between the standard mean curvature flow F

$$\frac{dF}{dt} = \vec{H}$$

which may "flow through the boundary" and a reparametrised mean curvature flow $X : L^n \to \mathscr{Y}$ such that $X(\partial L, t) \subset \Sigma_t$ and $\left(\frac{dX}{dt}\right)^\perp = H$, say

$$\frac{dX}{dt} = \vec{H} + V,$$

where V is a time dependent tangential vector field on L_t. We write $\frac{\partial f}{\partial t}$ for time differentiation with respect to X (as opposed to F, for which we write $\frac{df}{dt}$), and note the following evolution equations:

$$\frac{\partial}{\partial t} g_t = -2H^k h_{ijk} + \left\langle \overline{\nabla}_{X_j} V, X_i \right\rangle + \left\langle \overline{\nabla}_{X_i} V, X_j \right\rangle,$$
$$\frac{df}{dt} = \frac{\partial f}{\partial t} - \langle \nabla f, V \rangle .$$

We therefore see that for a general smooth function f (remembering that μ is outward pointing),

$$\begin{aligned}\frac{d}{dt}\int_{L_t} f d\mathscr{H}^n &= \int_{L_t} \left(\frac{\partial f}{\partial t} + f \operatorname{div}^{L_t}(V) - |H|^2 f \right) d\mathscr{H}^n \\ &= \int_{L_t} \left(\frac{\partial f}{\partial t} - \langle V, \nabla f \rangle - |H|^2 f \right) d\mathscr{H}^n + \int_{\partial L_t} f \langle V, \mu \rangle \, d\mathscr{H}^{n-1} \\ &= \int_{L_t} \left(\frac{df}{dt} - |H|^2 f \right) d\mathscr{H}^n + \int_{\partial L_t} f \langle V, \mu \rangle \, d\mathscr{H}^{n-1}.\end{aligned}$$

At the boundary, $\vec{H} - \widetilde{H} + V \in T\Sigma_t$, and as in the proof of Lemma 4.2.2, noting that $\tau = 0$ since L is Lagrangian:

$$\vec{H}^{N\Sigma} - \widetilde{H} = CJ\nu.$$

Writing V in the basis from Section 4.1,

$$\langle V, \mu \rangle \langle \mu, J\nu \rangle = \langle V, J\nu \rangle = \left\langle \widetilde{H} - \vec{H}, J\nu \right\rangle .$$

We observe that due to our boundary condition, $\left\langle \vec{H}, J\nu \right\rangle = \left\langle \vec{H}, J\mu \right\rangle \langle \mu, \nu \rangle = \langle \mu, \nu \rangle \nabla_\mu \theta$, and recall that $\frac{\langle \nu, \mu \rangle}{\langle J\nu, \mu \rangle} = \tan\alpha$, completing the Lemma. □

Corollary 4.4.3. *If Σ is special Lagrangian with Lagrangian angle $\frac{\pi}{2}$, then*

$$\begin{aligned}\frac{d}{dt}\int_{L_t} f d\mathscr{H}^n = \int_{L_t} \left(\left(\frac{d}{dt} - \Delta \right) f - |H|^2 f \right) d\mathscr{H}^n \\ + \int_{\partial L_t} \left(\nabla_\mu f - f \tan\alpha \nabla_\mu \theta \right) d\mathscr{H}^{n-1} ,\end{aligned}$$

and if $f = f(\theta)$ then

$$\frac{d}{dt}\int_{L_t} f d\mathscr{H}^n = \int_{L_t} -|H|^2(f'' + f)d\mathscr{H}^n + \int_{\partial L_t}(f' - f\tan\alpha)\nabla_\mu\theta d\mathscr{H}^{n-1} .$$

We now make the following observation

Lemma 4.4.4. *If Σ is special Lagrangian with Lagrangian angle $\frac{\pi}{2}$, and $\theta_0 \in (-\frac{\pi}{2} + \varepsilon, \frac{\pi}{2} - \varepsilon)$ then while the flow exists*

$$\frac{d}{dt}\int_{L_t} \cos(\theta)d\mathscr{H}^n = 0 .$$

In particular, $|L_t|$ is bounded from above and below.

Proof. Due to the boundary condition on ∂L, $\theta = -\alpha$, and so the maximum principle implies that the bounds on θ are preserved. Set $f(x) = \cos(x)$, then $f'' = -f$ and $f'(-\alpha) - \tan(\alpha)f(-\alpha) = 0$. $|L_t|$ is bounded as $\cos(\theta)$ is bounded from above and below away from 0 (depending on ε). □

Lemma 4.4.5. *If Σ is special Lagrangian with Lagrangian angle $\frac{\pi}{2}$, L_0 is zero Maslov and there exists a constant V such that $|L_t| < V$. Then there exists a constant $c = c(n,V)$ such that*

$$\int_{L_t}(\theta + \alpha)^2 d\mathscr{H}^n \le Ce^{-ct}, \qquad \int_0^\infty \int_{L_t} |H|^2 e^{\frac{c}{2}t} d\mathscr{H}^n dt \le C.$$

Proof. We apply Corollary 4.4.3 with $f(\theta) = (\theta + \alpha)^{2p}$ for some $p \ge 1$. In particular, at the boundary $f = f' = 0$ and so

$$\frac{d}{dt}\int_{L_t}(\theta + \alpha)^{2p} d\mathscr{H}^n = -\int_{L_t} |H|^2(\theta + \alpha)^{2p} + \frac{2p(2p-1)}{p^2}|\nabla(\theta + \alpha)^p|^2 d\mathscr{H}^n. \quad (4.21)$$

We recall that the Michael–Simon Sobolev inequality [48] implies that

$$\left(\int_{L_t} \phi^{\frac{2n}{n-1}}\right)^{\frac{n-1}{2n}} \le C(n,|L_t|)\sqrt{\int_{L_t} |\nabla\phi|^2 + |H|^2|\phi|^2 d\mathscr{H}^n},$$

and we note that as $\theta + \alpha$ is zero on ∂L_t, it is a function of compact support on the interior of L_t and this theorem applies to $\phi = (\theta+\alpha)^p$ for all $p \geq 1$.

We see that by choosing $\phi = (\theta+\alpha)^p$ then

$$\begin{aligned}\frac{d}{dt}\int_{L_t}(\theta+\alpha)^{2p}d\mathscr{H}^n &\leq -\tilde{c}(n,|L_t|)\left(\int_{L_t}\left[(\theta+\alpha)^{2p}\right]^{\frac{n}{n-1}}d\mathscr{H}^n\right)^{\frac{n-1}{n}}\\ &\leq -c(n,|L_t|)\int_{L_t}(\theta+\alpha)^{2p}d\mathscr{H}^n, \qquad (4.22)\end{aligned}$$

and so

$$\frac{d}{dt}\int_{L_t}(\theta+\alpha)^{2p}e^{ct}d\mathscr{H}^n \leq 0.$$

Finally, setting $p = 1$ and using (4.21) and (4.22):

$$\begin{aligned}\frac{d}{dt}\int_{L_t}(\theta+\alpha)^2 e^{\frac{c}{2}t}d\mathscr{H}^n &= \frac{c}{2}\int_{L_t}(\theta+\alpha)^2 e^{\frac{c}{2}t}d\mathscr{H}^n + e^{\frac{c}{2}t}\frac{d}{dt}\int_{L_t}(\theta+\alpha)^2 d\mathscr{H}^n\\ &\leq \frac{1}{2}e^{\frac{c}{2}t}\frac{d}{dt}\int_{L_t}(\theta+\alpha)^2 d\mathscr{H}^n\\ &\leq -\frac{1}{2}e^{\frac{c}{2}t}\int_{L_t}|H|^2(\theta+\alpha)^2+2|\nabla\theta|^2 d\mathscr{H}^n\\ &\leq -e^{\frac{c}{2}t}\int_{L_t}|H|^2 d\mathscr{H}^n\,.\end{aligned}$$

Integrating implies the final claim. □

Proof of Proposition 4.4.1. Due to Lemma 4.4.5 and the above regularity assumptions, there exists a $T > 0$ such that for all $t > T$, $|H| < e^{-\frac{c}{4}t}$. This bounds the normal velocity of the parametrisation F, and as a result we see that for $s,t > T$, $\operatorname{dist}(L_s, L_t) < \frac{4}{c}e^{-\frac{c}{4}\min\{s,t\}}$. Clearly, as $t \to \infty$, $\vec{H} \to 0$, and so we see that L_t converges to a special Lagrangian, first subsequentially by Arzela–Ascoli, then uniformly by the above, then smoothly by interpolation. □

Chapter 5

Equivariant Lagrangian Mean Curvature Flow in $\mathbb{C}^n$

We now turn our attention to the formation and structure of singularities and criteria for long-time existence in Lagrangian mean curvature flow. To make progress on this daunting program, we choose in this chapter to focus in detail on a particular case: the flow of $O(n)$-equivariant submanifolds of $\mathbb{C}^n$. These are submanifolds with an $O(n)$-symmetry, and which therefore may be represented as a 'profile curve' in $\mathbb{C}$ symmetric across the origin, upon a quotient by the group action. As we shall see, these submanifolds are necessarily Lagrangian (Lemma 5.1.1), and so they provide a model case for Lagrangian mean curvature flow. The equivariance essentially reduces the codimension of the flow to 1, making the analysis significantly more tractable, and the diagrams easier to draw!

We also restrict to **almost-calibrated** Lagrangian mean curvature flow (as defined in Section 2.4.2), which is the condition that the Lagrangian angle θ_t satisfies $\theta_t \in (\overline{\theta} - \frac{\pi}{2} + \varepsilon, \overline{\theta} + \frac{\pi}{2} - \varepsilon)$ for constants $\varepsilon > 0$ and $\overline{\theta}$. This is a necessary condition for the Thomas-Yau conjecture as reformulated by D. Joyce in [36], which approximately states that long-time existence and convergence of Lagrangian mean curvature flow is equivalent to a 'stability condition', as is true for Hermitian-Yang-Mills flow.

When combined with the equivariance, it can be seen that the almost-calibrated condition implies that the flow is non-compact and embedded (Lemmas 5.1.2 and 5.1.3).

The results of Chapter 5 comprise original work, and appear in the preprint [78].

It is therefore most natural to work with asymptotically conical flows L_t, and in fact in this case by the equivariance the asymptotic cone must be a union of planes. To this end, we say that a Lagrangian mean curvature flow has **planar asymptotics** if outside a compact ball B_R, the flow L_t may be written as a graph over finitely many fixed planes through the origin which decay smoothly to those planes at infinity, and we will work with flows with planar asymptotics throughout. The questions of short-time existence and uniqueness for LMCF with planar asymptotics have been answered in the affirmative by W-B. Su [67], and will not be considered here.

We can already deduce much about what singularities of such a flow must look like. As we have seen in Theorem 3.4.1 (first proven by M-T. Wang in [72]), a singularity of almost-calibrated Lagrangian mean curvature flow must be Type II, meaning that the Type I blowup is not a smooth flow, and in fact Theorem 3.4.2 of A. Neves [50, Theorem A] implies that the Type I blowup must be a union of special Lagrangian cones. In the equivariant case, the only special Lagrangian cones are unions of planes through the origin. The profile curve of such a cone is a union of lines through the origin, which have pairwise argument differences of multiples of $\frac{\pi}{n}$. Therefore, any Type I blowup of an almost-calibrated equivariant Lagrangian mean curvature flow must consist of a union of special Lagrangian planes. However, these results still leave much unknown about the nature of singularities: we do not know where and when a singularity may occur, there may be any number of planes with any integer density comprising the Type I blowup, and no information is given on the finer structure of the singularity, for example the Type II blowups as defined in Section 2.2.3. In this chapter we thoroughly investigate these topics.

We now go through our results in detail. The first main result, Theorem 5.2.8, provides a complete classification of singularities for our considered flow. Explicitly, we prove that any singularity of an almost-calibrated equivariant LMCF with planar asymptotics must occur at the origin, and that its Type I blowup is a special Lagrangian transverse pair of planes $P_1 \cup P_2$ which does not depend on the rescaling sequence (see Figure 5.10).

Theorem 5.0.1. *Let L_t be an almost-calibrated, connected $O(n)$-equivariant mean cur-*

vature flow in $\mathbb{C}^n$ with planar asymptotics.

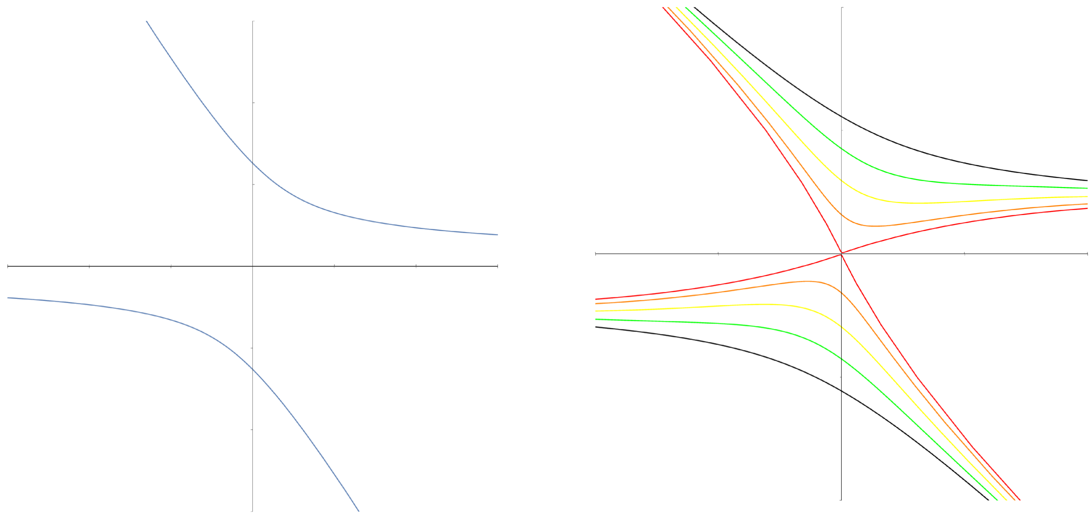
Then any finite-time singularity must occur at the origin. Additionally, a Type I blowup of such a singularity must be a special Lagrangian cone consisting of a transverse pair of planes $P_1 \cup P_2$ with identical Lagrangian angle, and the blowup does not depend on the rescaling sequence. The profile curves of these planes intersect at an angle of $\frac{\pi}{n}$.

We also analyse how the initial condition of the flow affects the long-time behaviour of the flow, and in doing so paint an almost complete picture of almost-calibrated equivariant mean curvature flow in $\mathbb{C}^n$. The profile curve l_0 of a connected, equivariant Lagrangian L_0 may have one of two different topologies - it can consist of a single curve that passes through the origin, asymptotic to a single line, or two mirrored curves that do not pass through the origin, asymptotic to two different lines spanning an angle α (see Figure 5.3). We use this profile curve to classify our flows into three groups, namely the flows that pass through the origin, the flows with $\alpha > \frac{\pi}{n}$, and the flows with $\alpha \leq \frac{\pi}{n}$. We prove in the first and third cases a long-time existence result (Theorems 5.2.9 and 5.2.12), and in the second case that a singularity must occur (Theorem 5.2.11). Here, we collate these results into one theorem.

Theorem 5.0.2. *Let L_t be an almost-calibrated, connected, $O(n)$-equivariant mean curvature flow in $\mathbb{C}^n$ with planar asymptotics. Then, if the profile curve of the initial embedding l_0 passes through the origin, then L_t exists for all time. Otherwise, denoting by α the angle between the asymptotes of a connected component of the profile curve l_0:*

- *If $\frac{\pi}{n} < \alpha < \frac{2\pi}{n}$, then a finite-time singularity must occur, with form given by Theorem 5.0.1.*
- *If $\alpha \leq \frac{\pi}{n}$, and l_0 is contained in the cone between the asymptotic lines, then L_t exists for all time.*

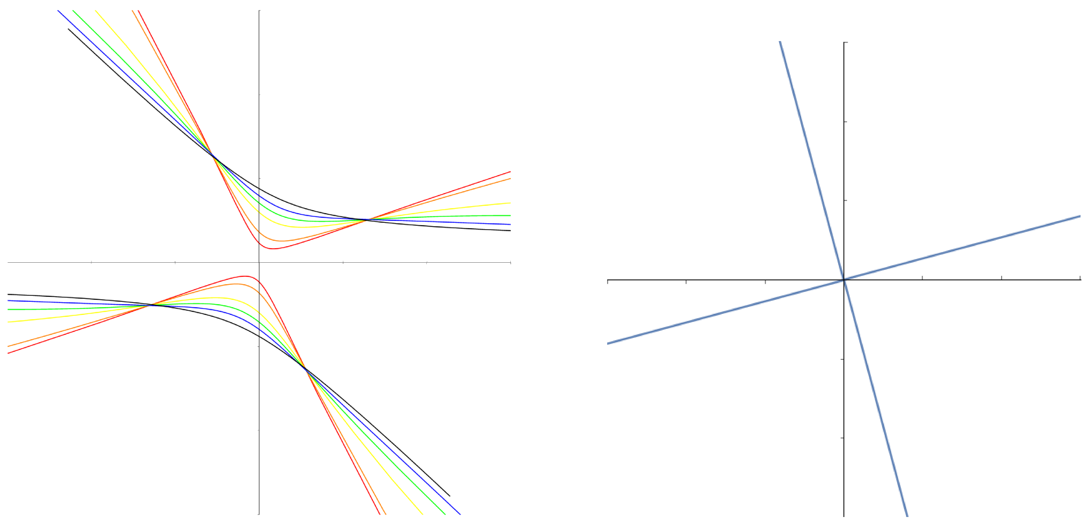
Remark 5.0.3. *In the case where the profile curve has two connected components, a calculation shows that the Lagrangian angle along these asymptotes differs by $n\alpha - \pi$. Therefore the angle α between the asymptotes must be less than $\frac{2\pi}{n}$ by the almost-calibrated condition (see Lemma 5.2.7 for a rigorous proof). This calculation also shows*



(a) Initial Condition

(b) MCF

Figure 5.1: The profile curve of Neves' equivariant mean curvature flow in $\mathbb{C}^2$ spanning an angle $\beta = \frac{2\pi}{3}$, which forms a singularity at the origin.



(a) Type I rescalings, at a fixed time (the initial condition is in black).

(b) The Type I blowup, i.e. the limiting Brakke flow of the Type I rescalings.

Figure 5.2: Convergence of the profile curves of the Type I rescalings of Neves' equivariant mean curvature flow in $\mathbb{C}^2$ spanning an angle of $\beta = \frac{2\pi}{3}$.

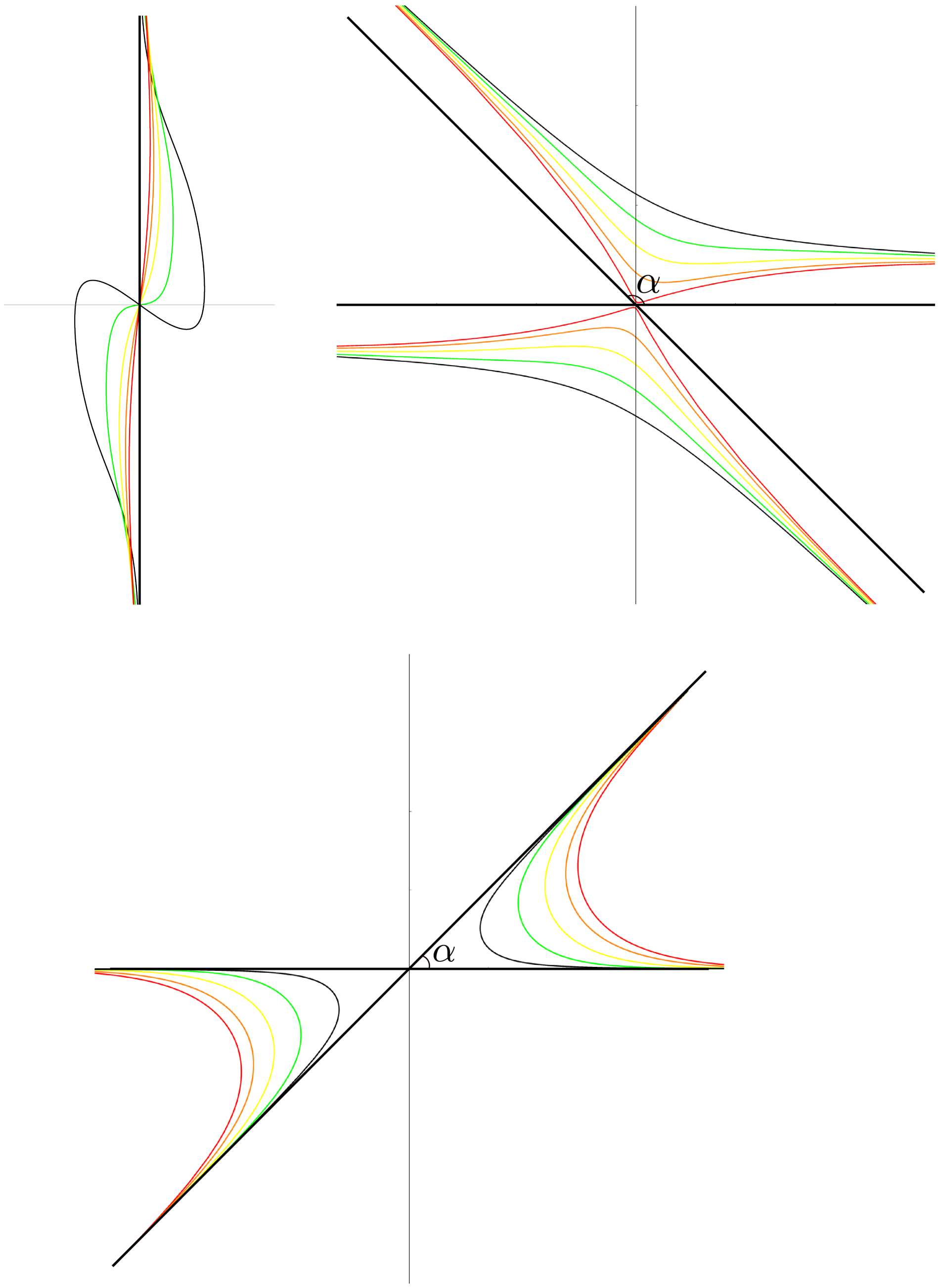


Figure 5.3: The flow of the profile curve of three different equivariant almost-calibrated Lagrangian mean curvature flows in $\mathbb{C}^2$, representing each of the three different cases of Theorem 5.0.2. In each diagram, the black curve is the initial condition, and the thick black lines are the asymptotes.
The top-left diagram depicts the flow of a Lagrangian which passes through the origin, and exists for all time. The top-right diagram depicts the $\alpha > \frac{\pi}{2}$ case, where a singularity forms under the flow. The lower diagram depicts the $\alpha < \frac{\pi}{2}$ case, where the flow exists for all time (in this case, it converges to an Anciaux expander).

that two lines with an angle of $\frac{\pi}{n}$ is the profile curve of a special Lagrangian pair of planes.

In the $\alpha \leq \frac{\pi}{n}$ case, long-time existence was proven by W-B. Su [67], and additionally that (with extra assumptions) the flow converges in infinite time to a Lawlor neck Σ_{Law} if $\alpha = \frac{\pi}{n}$ and the Anciaux expander if $\alpha < \frac{\pi}{n}$ (see Examples 3.3.1 and 3.3.3 and Figure 5.3). Note that for this case we require the stronger condition that l_0 lies entirely within a cone of angle $\frac{\pi}{n}$ - without this condition, both singularities and long-time existence are possible.

Finally, we examine the Type II blowups of the flow. It is conjectured that any Type II blowup should have the same asymptotes as a Type I blowup, i.e. the 'blowdown' of a Type II blowup should be a Type I blowup of the flow. Evidence for this is provided both by A. Savas-Halilaj and K. Smoczyk in [58], where it is shown that equivariant Lagrangian spheres develop Type II singularities with a double-density plane as the Type I blowup and the grim reaper as the Type II blowup, and by J. Velázquez in [71], in which he provides a MCF whose Type I blowup is the Simons' cone and whose Type II blowup is the unique minimal hypersurface tangent to it at infinity. Further analysis of the Velázquez example was undertaken by N. Sesum and S-H. Guo [25] and M. Stolarski [66], including explicit estimates for the mean curvature and second fundamental form, and an examination of the intermediate scales.

Recently, B. Lambert, J. Lotay and F. Schulze [41] proved that if the blowdown of a smooth Type II blowup is a pair of transverse planes $P_1 \cup P_2$, the blowup must be a Lawlor neck Σ_{Law}, which is the minimal hypersurface with asymptotes $P_1 \cup P_2$ (unique up to scaling). Therefore if the above conjecture were true we would expect by Theorem 5.0.1 that every Type II blowup of an almost-calibrated $O(n)$-equivariant flow to be a Lawlor neck. We verify this explicitly.

Theorem 5.0.4. *Let L_t be an almost-calibrated, connected, $O(n)$-equivariant mean curvature flow in $\mathbb{C}^n$ with planar asymptotics.*

Then up to a translation, a Type II blowup of any finite-time singularity is the unique Lawlor neck with the same Lagrangian angle as the unique Type I blowup $P_1 \cup P_2$ and $\max |A|^2 = 1$. *In particular, the asymptotes of this Type II blowup are the planes P_1 and*

P_2.

We also check the 'intermediate scales', to confirm that there is no different behaviour in between the Type I and Type II scales – this is the content of Section 5.2.6. We prove that, using the same sequence of times as a Type II rescaling, if we use blowup factors smaller than the second fundamental form then we still obtain the blowup $P_1 \cup P_2$.

Theorem 5.0.5. *Let L_t be an almost-calibrated, connected $O(n)$-equivariant mean curvature flow in $\mathbb{C}^n$ with planar asymptotics. Assume that L_t forms a singularity at the origin at time $t = 0$, and let*

$$L_\tau^{t_k,\lambda_k} := \lambda_k L_{t_k + T + \lambda_k^{-2}\tau}$$

be a sequence of rescalings satisfying

$$\delta_k := \frac{\lambda_k}{A_k} \to 0, \quad -\lambda_k^2 t_k \to \infty,$$

where $A_k := \max_{L_{t_k}}(|A|)$, and $0 > t_k \to 0$ satisfies (2.14) for $p_k \equiv 0$. Then for any R, ε and finite time interval I, there exists a subsequence such that $L_\tau^{t_k,\lambda_k} \cap (B_R \backslash B_\varepsilon)$ may be expressed as a graph over $P_1 \cup P_2$ for $\tau \in I$, and this graph converges in $C^{1;0}([\varepsilon, R] \times I)$ (C^1 in space, C^0 in time) to 0.

The proofs of these theorems are contained in Section 5.2. Section 5.1 contains material on $O(n)$-equivariant submanifolds, including descriptions of the Lawlor neck and convergence theorems for sequences of equivariant submanifolds.

5.1 $O(n)$-Equivariant Submanifolds in $\mathbb{C}^n$

Throughout this chapter, we will restrict our attention to connected $O(n)$-equivariant submanifolds in $\mathbb{C}^n$, where $\mathbb{C}^n$ is equipped with its usual Kähler structure (see Example 2.3.3). An $O(n)$-**equivariant submanifold** L is a submanifold $L \subset \mathbb{C}^n$ that may be expressed as the image of a function

$$L : M^1 \times S^{n-1} \to \mathbb{C}^n = \mathbb{R}^n \times \mathbb{R}^n, \quad L(s,\alpha) = (a(s)\alpha, b(s)\alpha), \tag{5.1}$$

for some smooth functions $a,b : M^1 \to \mathbb{R}$, where M is a 1-dimensional manifold. L is invariant under the $O(n)$ action

$$O(n) \circlearrowright \mathbb{C}^n, \qquad A\big((x,y)\big) = (Ax, Ay)$$

for $x,y \in \mathbb{R}^n$, $A \in O(n)$. Of particular importance is that $L(s,\alpha) = -L(s,-\alpha)$, implying that L has reflective symmetry through the origin. The **profile curve**

$$l : M^1 \to \mathbb{C}, \quad l(s) = a(s) + ib(s) \tag{5.2}$$

can therefore be chosen to have reflective symmetry across the origin. Throughout the remainder of the chapter, we will make this choice (we can think of the profile curve as the intersection $L_t \cap (\mathbb{C}\times\{0\}^{n-1})$, if we identify $\mathbb{C}$ with $\mathbb{C}\times\{0\}^{n-1}$). Since we demand that the manifold L is connected, if l passes through the origin then we must have a single connected component, and if it does not, then l has two connected components γ and $-\gamma$. In the case that l passes through the origin, note that by the reflective symmetry, $\vec{H} = 0$ there.

We will now prove some results regarding equivariant submanifolds. Firstly, we prove that $O(n)$-equivariant submanifolds are examples of Lagrangian submanifolds, and that the Lagrangian angle is preserved under the $O(n)$ rotations.

Lemma 5.1.1. *An immersed $O(n)$-equivariant surface $L \subset \mathbb{C}^n$ is a Lagrangian submanifold.*

Proof. We must show that $\omega|_L \equiv 0$. If we pick a local coordinate system $(\sigma^1, \ldots, \sigma^{n-1})$ for S^{n-1}, the derivatives of L are given by

$$\frac{\partial L}{\partial s} = \big(a'(s)\alpha, b'(s)\alpha\big), \quad \frac{\partial L}{\partial \sigma^i} = \left(a(s)\frac{\partial \alpha}{\partial \sigma^i}, b(s)\frac{\partial \alpha}{\partial \sigma^i}\right),$$

where we identify $\mathbb{C}^n$ with $\mathbb{R}^n \times \mathbb{R}^n$. Remembering that the almost complex structure J

in $\mathbb{C}^n$ is given by $J(x,y) = (-y,x)$, we can then calculate $\omega|_L$:

$$\begin{aligned}
\omega\left(\frac{\partial L}{\partial \sigma^i}, \frac{\partial L}{\partial \sigma^j}\right) &= \left\langle J\left(\frac{\partial L}{\partial \sigma^i}\right), \frac{\partial L}{\partial \sigma^j}\right\rangle \\
&= \left(-b(s)\frac{\partial \alpha}{\partial \sigma^i}, a(s)\frac{\partial \alpha}{\partial \sigma^i}\right) \cdot \left(a(s)\frac{\partial \alpha}{\partial \sigma^j}, b(s)\frac{\partial \alpha}{\partial \sigma^j}\right) = 0, \\
\omega\left(\frac{\partial L}{\partial s}, \frac{\partial L}{\partial \sigma^j}\right) &= \left(-b'(s)\alpha, a'(s)\alpha\right) \cdot \left(a(s)\frac{\partial \alpha}{\partial \sigma^j}, b(s)\frac{\partial \alpha}{\partial \sigma^j}\right) = 0, \\
\omega\left(\frac{\partial L}{\partial s}, \frac{\partial L}{\partial s}\right) &= \left(-b'(s)\alpha, a'(s)\alpha\right) \cdot \left(a'(s)\alpha, b'(s)\alpha\right) = 0,
\end{aligned}$$

where for the second line we use $\alpha \cdot \frac{\partial \alpha}{\partial \sigma^j} = 0$, which is a property of the sphere S^{n-1}. □

Since equivariant submanifolds are Lagrangian, we may consider the Lagrangian angle θ, as defined in (2.26). Locally and up to a multiple of 2π, the Lagrangian angle is given by (2.30) to be

$$\theta(s,\alpha) = (n-1)\arg(l(s)) + \arg(l'(s)). \tag{5.3}$$

Note that this implies the Lagrangian angle is well-defined for the profile curve l.

Throughout this chapter, we will be using the more general definition of 'special Lagrangian' and 'almost-calibrated'. Namely, a Lagrangian is said to be a **special Lagrangian** if its Lagrangian angle is constant, $\theta = \overline{\theta}$, and **almost-calibrated** if there exist $\overline{\theta}$ and $\varepsilon > 0$ such that

$$\theta \in \left(\overline{\theta} - \frac{\pi}{2} + \varepsilon, \overline{\theta} + \frac{\pi}{2} - \varepsilon\right).$$

Note that the latter condition implies that the Lagrangian is **zero-Maslov**, i.e. the Lagrangian angle is a function $\theta : L \to \mathbb{R}$.

We now show that embedded zero-Maslov $O(n)$-equivariant Lagrangians cannot be compact, and the embeddedness assumption follows if our Lagrangian is also almost-calibrated.

Lemma 5.1.2. *A connected, $O(n)$-equivariant, embedded, zero-Maslov Lagrangian submanifold L of $\mathbb{C}^n$ is non-compact and rational. Moreover, if the profile curve contains* *the origin, then $L \cong \mathbb{R}^n$, and if not, $L \cong \mathbb{R} \times S^{n-1}$.*

Proof. Assume that the profile curve l is compact, for a contradiction. Consider for simplicity a connected component γ of the profile curve l, which is embedded by assumption and homeomorphic to a circle. Firstly we claim that $O \notin \gamma$. Otherwise, we could parametrise γ by unit-speed such that

$$\gamma(0) = O, \quad \gamma(s) = -\gamma(-s)$$

by the $O(n)$-equivariance. But then since γ is compact, there must exist $S > 0$ such that $\gamma(S) = \gamma(-S)$, which implies that $\gamma(S) = O$. This contradicts embeddedness of γ.

Now consider the following integral:

$$\begin{aligned} \int_\gamma d\theta &= \int \frac{\partial}{\partial s} \arg(\gamma'(s))\, ds + (n-1) \int \frac{\partial}{\partial s} \arg(\gamma(s))\, ds \\ &= 2\pi T(\gamma) + 2(n-1)\pi W_O(\gamma), \end{aligned}$$

where $T(\gamma)$ is the turning number, and $W_O(\gamma)$ is the winding number around the origin. Since γ is embedded, it follows from standard theory that $T(\gamma) \in \{-1, 1\}$, and $W(\gamma) \in \{T(\gamma), 0\}$ (depending on whether the origin is contained in γ or not). It follows that $[d\theta][\gamma] \neq 0$, contradicting the zero-Maslov assumption.

Finally, since the submanifold has been shown to be non-compact, the domain for the profile curve M^1 must be homeomorphic to $\mathbb{R}$. So, by the equivariance, $L \cong \mathbb{R}^n$ or $\mathbb{R} \times S^{n-1}$ (depending on whether the profile curve contains the origin or not respectively). Since both of these have first homology generated by at most one element, rationality of L follows from the definition. □

Lemma 5.1.3. *A connected, $O(n)$-equivariant, almost-calibrated Lagrangian submanifold L of $\mathbb{C}^n$ must be embedded, non-compact and rational.*

Proof. Consider a connected component of the profile curve, $l : M^1 \to \mathbb{C}$. Parametrising M^1 by the real numbers (or a quotient of $\mathbb{R}$ if M^1 is compact), choose $a \in M^1$ sufficiently small such that there exists $c > a$ with $l([a, c])$ not embedded. Taking c as the infimum

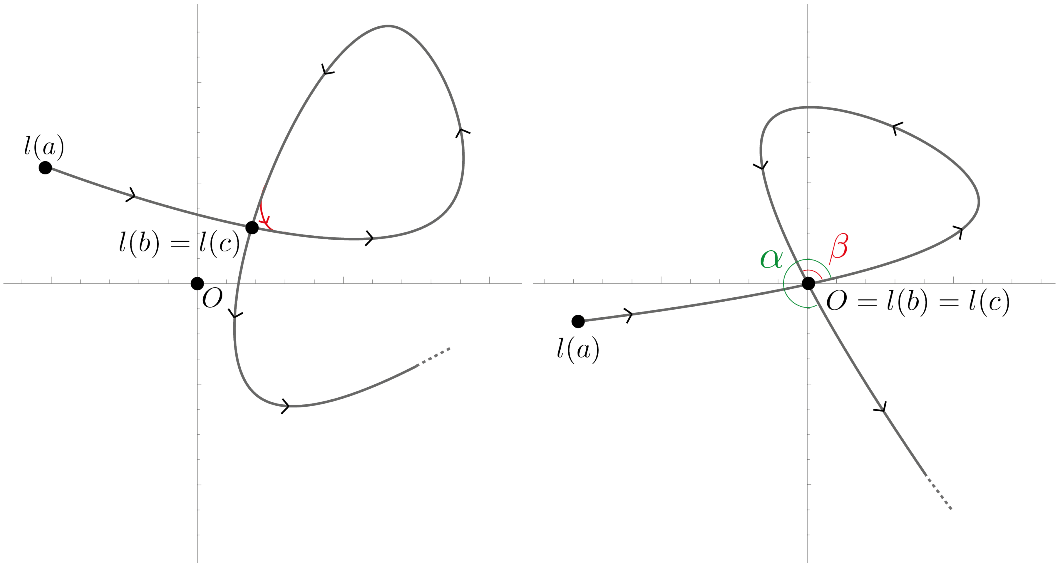


Figure 5.4: The proof of Lemma 5.1.3. The case where the intersection is away from the origin is depicted on the left, with the smoothing highlighted in red, and the case where the intersection is at the origin is depicted on the right.

of its possible values, it follows that there exists $a \leq b \leq c$ such that $l(b) = l(c)$, and $l([b,c])$ is a closed loop.

We consider two possibilities: either $l(b) = l(c) = O$, or $l(b), l(c) \neq O$ (see Figure 5.4). In the latter case, at the point $l(b)$ we may smooth the curve off to create a closed, embedded loop γ. By (5.3), we can do this so that the Lagrangian angle of the smoothed section lies in the interval $(\theta(b) - \varepsilon, \theta(c) + \varepsilon)$ for any given ε. This produces a compact, embedded, almost-calibrated loop, which contradicts Lemma 5.1.2. In the other case, assume without loss of generality that the loop is oriented anticlockwise, and define

$$\alpha := \int_{l([b,c])} \frac{\partial}{\partial s} \arg(\gamma'(s))\, ds, \qquad \beta := \int_{l([b,c])} \frac{\partial}{\partial s} \arg(\gamma(s))\, ds.$$

Note that due to the orientation and embeddedness, α, β must be positive, and $\alpha - \beta =$

π. It therefore follows that

$$\begin{aligned}\int_{[b,c]} d\theta &= \int_{l([b,c])} \frac{\partial}{\partial s}\arg(\gamma'(s))\,ds + (n-1)\int_{l([b,c])} \frac{\partial}{\partial s}\arg(\gamma(s))\,ds \\ &= \alpha+(n-1)\beta \\ &= \pi+n\beta \\ &> \pi,\end{aligned}$$

which is a contradiction to the almost-calibrated condition. □

We remark that there do exist non-embedded zero-Maslov equivariant curves, for example any equivariant Lagrangian with a 'figure 8' profile curve through the origin, such as the Whitney sphere studied in [58].

5.1.1 $O(n)$-Equivariant Special Lagrangians

We would now like to characterise the $O(n)$-equivariant special Lagrangian cones in $\mathbb{C}^n$ (potential Type I blowup models) and the $O(n)$-equivariant smooth special Lagrangians (potential Type II blowup models).

Lemma 5.1.4. *The only $O(n)$-equivariant special Lagrangian cones in $\mathbb{C}^n$ are unions of special Lagrangian planes, with profile curve consisting of unions of lines through the origin.*

Proof. The only special Lagrangian cones in $\mathbb{C}$ are lines through the origin, so this follows from the equivariance and (5.3). □

Lemma 5.1.5. *The only $O(n)$-equivariant surfaces in $\mathbb{C}^n$ with constant Lagrangian angle of $\overline{\theta}$ are those with profile curves given by either lines through the origin, or the parametrisation*

$$r(\alpha) = \frac{B}{\sqrt[n]{\sin(\overline{\theta}-n\alpha)}}$$

for $B \geq 0$. In the latter case, these are the ***Lawlor necks*** *of Example 3.3.1.*

Proof. If a connected component γ of the profile curve is not a line through the origin, then there is an open interval on which it may be parametrised by angle, i.e. as $\gamma(\alpha) = r(\alpha)e^{i\alpha}$. Then on this interval,

$$\dot{\gamma} = (\dot{r} + ir)e^{i\alpha} \quad \Longrightarrow \quad \overline{\theta} = n\alpha + \cot^{-1}(\tfrac{\dot{r}}{r}).$$

by equation (5.3). Integrating this gives the expression in the statement, and since this expression is valid until the value of r diverges to ∞, the entire connected component may be parametrised in this way. □

5.1.2 Limits of $O(n)$-Equivariant Submanifolds

When considering Type I and Type II blowups, we will be trying to understand the limit of sequences of submanifolds, L^i. Since they are translations and dilations of equivariant submanifolds, they will have rotational symmetry, but the centres of rotation x_i may not be the origin (though without loss of generality, we will be able to assume that $x_i \in \mathbb{C}\times\{0\}^{n-1}$). There are two possible behaviours: either $|x_i|$ stays bounded, or diverges to infinity. These two cases will correspond to equivariance and translation invariance respectively for the limiting object.

We formalise and prove these statements. Consider for the rest of this section a sequence L^i of submanifolds of $\mathbb{C}^n$, which converge in the sense of Radon measures to $\mu_\infty := m\mathscr{H}^n \llcorner L^\infty$, where m is a multiplicity function and L^∞ is the supporting set. Explicitly, for all $\phi \in C_c^\infty(\mathbb{C}^n)$, denoting the underlying Radon measures of the L^i by μ_i,

$$\mu_i(\phi) = \int_{L^i} \phi \, d\mathscr{H}^n \to \mu_\infty(\phi) = \int_{L^\infty} m\phi \, d\mathscr{H}^n.$$

Assume that L^i is a translation of an $O(n)$-invariant submanifold by $x_i \in \mathbb{C}\times\{0\}^{n-1}$, therefore invariant under the rotation mappings

$$R_{x_i} : S^{n-2} \times \mathbb{R} \times \mathbb{C}^n \to \mathbb{C}^n,$$

$$R_{x_i}(\alpha,\lambda,y) := \left(\begin{array}{cc|c} \cos\left(\frac{\lambda}{|y-x_i|}\right) & -\sin\left(\frac{\lambda}{|y-x_i|}\right) & 0 \\ \sin\left(\frac{\lambda}{|y-x_i|}\right) & \cos\left(\frac{\lambda}{|y-x_i|}\right) & \\ \hline \multicolumn{2}{c|}{0} & Id \end{array}\right)(y-x_i)+x_i, \tag{5.4}$$

where $\alpha \in S^{n-1} \cap \left(\{0\}\times\mathbb{R}^{n-1}\right) \cong S^{n-2}$ is an equatorial element of S^{n-1} (the direction of rotation), $\lambda \in \mathbb{R}$ is a distance factor, and for the matrix we have used an orthogonal basis of $\mathbb{R}^n$ starting with e_1 and α. Note that keeping α constant and varying λ creates a 1-parameter family of rotations that corresponds to the rotations of the $S^1 \subset S^{n-1}$ containing α and e_1. Define also the translation map

$$T_{x_i}(\alpha,\lambda,y) := y - \lambda \left(\begin{array}{cc|c} 0 & -1 & 0 \\ 1 & 0 & \\ \hline \multicolumn{2}{c|}{0} & 0 \end{array}\right)\frac{x_i}{|x_i|}, \tag{5.5}$$

using the same basis.

Lemma 5.1.6. *If $|x_i| \to \infty$, then we may pass to a subsequence such that as $i \to \infty$,*

$$\tfrac{x_i}{|x_i|} \to v, \qquad R_{x_i} \to T_v$$

in C^∞_{loc}. If $|x_i|$ remains bounded, then we may pass to a subsequence such that as $i \to \infty$,

$$x_i \to x, \qquad R_{x_i} \to R_x$$

in C^∞.

Proof. This is clear in the $|x_i|$ bounded case. If $|x_i| \to \infty$, then first pass to a subsequence such that $\frac{x_i}{|x_i|} \to v$. Fixing a compact region of the domain $U \subset S^{n-2}\times\mathbb{R}\times\mathbb{C}^n$ and taking

$(\alpha,\lambda,y)\in U$,

$$\frac{\lambda}{|y-x_i|}\to 0$$

$$\Longrightarrow R_{x_i}(\alpha,\lambda,y)-T_{x_i}(\alpha,\lambda,y)=\left(\begin{array}{cc|c}\cos\left(\frac{\lambda}{|y-x_i|}\right)-1 & -\sin\left(\frac{\lambda}{|y-x_i|}\right) & 0\\ \sin\left(\frac{\lambda}{|y-x_i|}\right) & \cos\left(\frac{\lambda}{|y-x_i|}\right)-1 & \\ \hline \multicolumn{2}{c|}{0} & 0\end{array}\right)y+$$

$$\left(\begin{array}{cc|c}1-\cos\left(\frac{\lambda}{|y-x_i|}\right) & \sin\left(\frac{\lambda}{|y-x_i|}\right)-\frac{\lambda}{|x_i|} & 0\\ -\sin\left(\frac{\lambda}{|y-x_i|}\right)+\frac{\lambda}{|x_i|} & 1-\cos\left(\frac{\lambda}{|y-x_i|}\right) & \\ \hline \multicolumn{2}{c|}{0} & 0\end{array}\right)x_i\longrightarrow 0,$$

where all convergences are uniform in U. Similarly, the derivatives converge uniformly. Since also

$$|T_{x_i}(\alpha,\lambda,y)-T_v(\alpha,\lambda,y)|\to 0$$

in C^∞, the result follows. □

What kind of invariance can we deduce for μ_∞ from this lemma? It is immediate that in both cases we can extract a measure-theoretic invariance. For example in the $|x_i|$ unbounded case, taking $\phi\in C_c^\infty$:

$$\mu_\infty(\phi\circ T_v(\alpha,\lambda,\cdot)) = \lim_{i\to\infty}\mu_i(\phi\circ R_{x_i}(\alpha,\lambda,\cdot)) = \lim_{i\to\infty}\mu_i(\phi) = \mu_\infty(\phi).$$

It follows that if L^∞ is a cone smooth away from the origin, or indeed a smooth manifold, then we have invariance of the supporting set, as well as of the multiplicity function.

One of the most useful aspects of $O(n)$-equivariant smooth manifolds is that they are characterised by the intersection with $\mathbb{C}\times\{0\}^{n-1}$. In particular, it is convenient to replace the $\mathscr{H}^n$ (Hausdorff) measure of our submanifolds with the $\mathscr{H}^1$ measure of their intersection with $\mathbb{C}\times\{0\}^{n-1}$. We wish to do this also with our limit μ_∞, in the case where L^∞ is a cone. In place of the Hausdorff measures $\mathscr{H}^n$ and $\mathscr{H}^1$, we work with the limiting measure μ_∞, and the limiting measure $\widetilde{\mu}_\infty$ of the profile curve $l^\infty = L^\infty\cap(\mathbb{C}\times\{0\}^{n-1})$

respectively:

$$\mu_\infty(A) = \int_{L^\infty \cap A} m\,\mathscr{H}^n, \qquad \widetilde{\mu}_\infty(A) = \int_{l^\infty \cap A} m\,\mathscr{H}^1.$$

From now on, we assume that $\frac{x_i}{|x_i|} \to e_1 = (1,0,\ldots,0)$, since this may be achieved by passing to a subsequence and applying a rotation.

Lemma 5.1.7. *Assume that the L^i are as above, converging to μ_∞ as Radon measures. If $|x_i| \to \infty$ and $\frac{x_i}{|x_i|} \to e_1$, then μ_∞ is supported on $\mathbb{C}\times\mathbb{R}^{n-1} \subset \mathbb{C}\times\mathbb{C}^{n-1}$. If instead x_i limits to x, then μ_∞ is supported on $\{R_x(\alpha,\lambda,z) : \alpha \in S^{n-2}, \lambda \in \mathbb{R}, z \in \mathbb{C}\times\{0\}^{n-1}\}$.*

Proof. Note that L^i is supported on $\{R_{x_i}(\alpha,\lambda,z) : \alpha \in S^{n-2}, \lambda \in \mathbb{R}, z \in \mathbb{C}\times\{0\}^{n-1}\}$, since the profile curve determines the entire submanifold. Therefore in the $|x_i|$ unbounded case, for any open set U disjoint from $\mathbb{C}\times\mathbb{R}^{n-1} = \{T_1(\alpha,\lambda,z) : \alpha \in S^{n-2}, \lambda \in \mathbb{R}, z \in \mathbb{C}\times\{0\}^{n-1}\}$, the submanifolds L^i are eventually disjoint from U, by the convergence of R_{x_i} to T_1. It follows that L^∞ is supported on U^c, and so the result follows since U was arbitrary. An identical argument works if $|x_i|$ is bounded. □

Lemma 5.1.8. *Assume that L^i are as above, converging to μ_∞ as Radon measures, $|x_i| \to \infty$ and $\frac{x_i}{|x_i|} \to e_1$. Assume L^∞ is a cone, smooth away from the origin, with profile curve l^∞ in $\mathbb{C}$. Then, denoting the ball of radius δ in $\mathbb{C}$ by $B^{\mathbb{C}}_\delta$, the surface area and enclosed volume of the unit sphere $S^{n-1} \subset \mathbb{R}^n$ by ω_{n-1} and V_n respectively,*

$$\frac{\widetilde{\mu}_\infty(B^{\mathbb{C}}_\delta)}{2\delta} = \frac{\mu_\infty(B_\delta)}{\delta^n V_n}.$$

Proof. By Lemmas 5.1.6 and 5.1.7, L^∞ and m are invariant under T_1, and supported on

$\mathbb{C}\times\mathbb{R}^{n-1}$. Remembering that L^∞ is a cone, and applying the coarea formula:

$$\begin{aligned}
\mu_\infty(B_\delta^{\mathbb{C}^n})\delta^n V_n &= \frac{1}{\delta^n V_n}\int_{B_\delta^{n-1}\subset\{0\}\times\mathbb{R}^{n-1}} \widetilde{\mu}_\infty\left(B^{\mathbb{C}}_{\sqrt{\delta^2-|\alpha|^2}}\right)d\alpha \\
&= \frac{\widetilde{\mu}_\infty(B_\delta^{\mathbb{C}})}{\delta^{n+1}V_n}\int_{B_\delta^{n-1}}\sqrt{\delta^2-|\alpha|^2}\,d\alpha \\
&= \frac{\widetilde{\mu}_\infty(B_\delta^{\mathbb{C}})}{\delta^{n+1}V_n}\int_0^\delta r^{n-2}\omega_{n-2}\sqrt{\delta^2-r^2}\,dr \\
&= \frac{\widetilde{\mu}_\infty(B_\delta^{\mathbb{C}})}{2\delta}.
\end{aligned}$$

□

Finally, we show that the $\mathscr{H}^1$ cross-sectional measures of the profile curves l^i converge to the $\widetilde{\mu}_\infty$ measure of the limiting profile curve l^∞. In the next section, this will allow us to consider the densities of the profile curve in $\mathbb{C}$, instead of the densities of the n-dimensional submanifolds in $\mathbb{C}^n$.

Lemma 5.1.9. *Assume L^i are as above, converging to μ_∞ as Radon measures, $|x_i|\to\infty$ and $\frac{x_i}{|x_i|}\to e_1$. Assume L^∞ is a cone, smooth away from the origin, with profile curve l^∞ in $\mathbb{C}$. Then denoting the profile curves by l^i,*

$$\mathscr{H}^1(l^i\cap B_\delta^{\mathbb{C}})\to\widetilde{\mu}_\infty(B_\delta).$$

Proof. Define the fattened disk sets:

$$\begin{aligned}
C^\infty_{\delta,\Lambda} &:= \{T_1(\alpha,\lambda,z):\lambda\in[-\Lambda,\Lambda],\,\alpha\in S^{n-2},\,z\in B_\delta^{\mathbb{C}}\},\\
C^i_{\delta,\Lambda} &:= \{R_{x_i}(\alpha,\lambda,z):\lambda\in[-\Lambda,\Lambda],\,\alpha\in S^{n-2},\,z\in B_\delta^{\mathbb{C}}\}.
\end{aligned}$$

By Radon measure convergence and Lemma 5.1.6, it follows that

$$\mathscr{H}^n(L^i\cap C^i_{\delta,\Lambda})\to\mu_\infty(C^\infty_{\delta,\Lambda}).$$

But also, by rotation invariance and the co-area formula, denoting by $A_{n-1}(r,\lambda)$ the

$(n-1)$-dimensional Hausdorff measure of the cap of S_r^{n-1} with polar angle of λ:

$$\begin{aligned}\lim_{i\to\infty}\mathscr{H}^n(L^i\cap C^i_{\delta,\Lambda}) &= \lim_{i\to\infty}\int_{l_i\cap B_\delta^{\mathbb{C}}} A_{n-1}\left(|x-x_i|,\frac{\Lambda}{|x-x_i|}\right)d\mathscr{H}^1\\ &\leq \lim_{i\to\infty}\left(A_{n-1}\left(|x_i|+\delta,\frac{\Lambda}{|x_i|-\delta}\right)\int_{l_i\cap B_\delta^{\mathbb{C}}} 1\, d\mathscr{H}^1\right)\\ &= \Lambda^{n-1}V_{n-1}\lim_{i\to\infty}\left(\mathscr{H}^1(l^i\cap B_\delta^{\mathbb{C}})\right),\end{aligned}$$

and an identical inequality holds in the other direction by changing the sign of δ in the $|x_i|+\delta$, $|x_i|-\delta$ terms. Since L^∞ is invariant under T_1, it follows that

$$\lim_{i\to\infty}\mathscr{H}^1(l^i\cap B_\delta^{\mathbb{C}}) = \frac{\mu_\infty(C^\infty_{\delta,\Lambda})}{\Lambda^{n-1}V_{n-1}} = \widetilde{\mu}_\infty(B_\delta^{\mathbb{C}}).$$

□

5.2 $O(n)$-Equivariant Mean Curvature Flow in $\mathbb{C}^n$

We now consider an almost-calibrated, connected $O(n)$-equivariant mean curvature flow in $\mathbb{C}^n$, which we denote L_t, with profile curve l_t in $\mathbb{C}$. By Lemmas 5.1.2 and 5.1.3, l_t is embedded and non-compact. We will denote the abstract manifold by L, and the Lagrangian angle at time t by θ_t. We use $s\in\mathbb{R}$ to denote the parameter along l_t (note that by Lemma 5.1.2, l_t is non-compact) and α for the spherical parameter, so that the Lagrangian submanifold is parametrised by

$$L_t(s,\alpha) = (a_t(s)\,\alpha,\, b_t(s)\,\alpha).$$

We will assume throughout that our flow has **planar asymptotics**. By this, we mean that our profile curve l_t is graphical over finitely many lines outside of some ball B_R, and the graph function converges smoothly to 0 at infinity. In fact due to the equivariance, it must be graphical over exactly one or two lines, depending on whether l_t passes through the origin or not (see Figure 5.3). This assumption provides uniformly bounded area ratios, which are necessary to use Neves' Theorem 3.4.2 and 3.4.3. Note that if the profile curve is asymptotic to two different lines, then for the curve to be almost-calibrated the angle between these lines must be less than or equal to $\frac{2\pi}{n}$. This follows from (5.3) by considering the value of θ as the profile curve decays to the asymptotes, and will be proven more rigorously in Lemma 5.2.7.

The curvature tensors are simpler in the equivariant case. Working with $L_t \cap (\mathbb{C} \times \{0\}^{n-1})$ for simplicity, the mean curvature vector of the Lagrangian can be divided into two key components:

$$\vec{H} = \vec{k} - (n-1)\vec{p},$$

where $\vec{k} = \frac{\gamma''^{\perp}}{|\gamma'|^2}$ is the curvature of the profile curve, and $\vec{p} = \frac{\gamma^{\perp}}{|\gamma|^2}$ is the curvature induced by the equivariance. Denoting by $\nu = \frac{J\dot{\gamma}}{|\dot{\gamma}|}$ the unit normal vector within $\mathbb{C} \times \{0\}^{n-1}$, we can also define the scalar quantities $\vec{k} = k\nu$ and $\vec{p} = p\nu$. The other curvature quantities may be expressed in terms of k and p. In the following tensor expressions, we will use normal coordinates e_i on the sphere centred at the point we are considering, where $i \in \{2, 3, \ldots, n\}$.

Lemma 5.2.1. *The extrinsic curvature quantities for equivariant Lagrangians as as follows, where all unmentioned components are equal to 0 ($i, j \neq 1$):*

$$\begin{aligned}
g_{ss} &= |l'|, & g_{ii} &= |l|^2, \\
h_{sss} &= |l'|^3 k, & h_{iis} &= -|l|^2|l'|p, \\
&H = |l'|\,(k-(n-1)p)\,ds, \\
&|H|^2 = (k-(n-1)p)^2, \\
&|A|^2 = k^2 + 3(n-1)p^2,
\end{aligned}$$

5.2.1 Evolution Equations for Equivariant Lagrangians

The equivariant condition significantly simplifies the study of mean curvature flow, since we may study the flow of the profile curve, given by

$$\frac{\partial l}{\partial t} = \vec{k} - (n-1)\vec{p}. \tag{5.6}$$

We refer to this as the **equivariant flow**. This is simpler than Lagrangian mean curvature flow in general as it is a codimension 1 flow, and therefore essentially a single PDE as opposed to a system.

There are several parametrisations of the profile curve that will come in useful. Firstly, if $u : \mathbb{R} \to \mathbb{R}$ is a graph function such that our flow may be expressed as $l_t(x) = (x, u_t(x))$, then the evolution of u_t under mean curvature flow is given by

$$\frac{\partial u}{\partial t} = \frac{u''}{1+(u')^2} + (n-1)\frac{xu'-u}{x^2+u^2} \tag{5.7}$$

$$= a(u')u'' + b(x,u,u'). \tag{5.8}$$

It is also often useful to parametrise in polar coordinates, $l_t(s) = r_t(s)e^{is}$, in which case the evolution of r_t under mean curvature flow is given by

$$\frac{\partial r}{\partial t} = -\frac{\theta'}{r}, \tag{5.9}$$

where θ' is the derivative of the Lagrangian angle with respect to the angle s.

5.2.2 Embeddedness and Avoidance Principle

We have already seen in Lemma 5.1.3 that almost-calibrated equivariant Lagrangians in $\mathbb{C}^n$ must be embedded. In this section, we derive some more general results about embeddedness and avoidance for equivariant Lagrangian mean curvature flow, without requiring the almost-calibrated hypothesis. This will be useful later so that we can use barriers to control the flow.

Though embeddedness does not typically hold for higher codimension MCF, it does for the equivariant case, since we may work with the profile curve and the equivariant flow (5.6). However, the equation (5.6) becomes singular at the origin, so we must treat the possibility of embeddedness breaking there separately. We cover this complication first by showing that embeddedness cannot break at the origin. Throughout, we denote the first time of non-embeddedness by T_{emb} and the singular time (first time of non-immersion) by T_{sing}.

Lemma 5.2.2. *Let L_t be a connected $O(n)$-equivariant mean curvature flow in $\mathbb{C}^n$, and* *assume that* $T_{emb} < \infty$. *Then if* $T_{emb} \neq T_{sing}$, *the embeddedness cannot break at the origin, i.e. there do not exist* $a, b \in L$ *such that* $L_{T_{emb}}(a) = L_{T_{emb}}(b) = O$.

Proof. We work with the profile curve for simplicity. Assume that there exist a, b such that

$$l_{T_{emb}}(a) = l_{T_{emb}}(b) = O,$$

therefore $l_t(a), l_t(b) \to O$ as $t \to T_{emb}$. Choose a sequence $t_n \to T_{emb}$, $t_n < T_{emb}$.

Claim. If $T_{emb} \neq T_{sing}$, then there exists $N \in \mathbb{N}$ such that for all $n \geq N$, there exists ε_n such that $l_{t_n}(a), l_{t_n}(b)$ lie in different connected components of $B_{\varepsilon_n}(0) \cap l_{t_n}$.

Proof of claim. We prove the contrapositive, so assume that there exists a subsequence n_k such that $l_{t_{n_k}}(a), l_{t_{n_k}}(b)$ lie in the same connected component of $B_\varepsilon(0) \cap l_{t_{n_k}}$ for all suitable ε. Therefore,

$$\forall x \in [a, b], \quad l_{t_{n_k}}(x) \to O$$

and so $l_{T_{emb}}$ is not immersed. Therefore $T_{emb} = T_{sing}$, and the claim is proven.

We may therefore find sequences of numbers ε_n and of connected components α_n, β_n in $B_{\varepsilon_n} \cap l_{t_n}$ such that $l_{t_n}(a) \in \alpha_n$, $l_{t_n}(b) \in \beta_n$. If there exists a time t and a point c with $l_t(c) = O$, then by the equivariant symmetry and uniqueness of the mean curvature flow, for all times $s > t$, $l_s(c) = O$. It follows that if there exist times $t < s < T_{emb}$ and points $c \neq d$ with $l_t(c) = l_s(d) = O$, then

$$l_s(c) = l_s(d) = O,$$

contradicting T_{emb} being the first time of non-embeddedness. Therefore (up to relabelling) at least one of the sequences of connected components α_n, β_n never includes the origin. Without loss of generality let it be α_n.

Now let $a_n \in \mathbb{R}$ be the point such that $p_n := l_{t_n}(a_n)$ is the closest point in α_n to the origin; note that $p_n \to 0$. Then

$$\left\langle \frac{\partial l_{t_n}}{\partial s}(a_n), p_n \right\rangle = 0 \quad \Longrightarrow \quad \langle p_n, \nu(a_n) \rangle = |p_n|,$$

where ν is the outward normal, so at the space-time point (p_n, t_n),

$$\begin{aligned} |A|^2 &= k^2 + 3(n-1)p^2 \\ &= k^2 + \frac{3(n-1)}{|p_n|^2}. \end{aligned}$$

This diverges to infinity as $n \to \infty$, and therefore, $T_{emb} = T_{sing}$. □

Theorem 5.2.3 (Preservation of Embeddedness/Avoidance Principle)**.** *Let L_t be a connected $O(n)$-equivariant mean curvature flow in $\mathbb{C}^n$ with planar asymptotics, such that each end is asymptotic to a different n-plane. Assume $T_{emb} < \infty$. Then $T_{emb} = T_{sing}$.*

Additionally, if L_t and $\overline{L}_t$ are two such flows, initially disjoint and embedded and with different asymptotes, then they remain disjoint until a singularity occurs.

Proof. We prove only preservation of embeddedness. The avoidance principle follows precisely the same argument, noting that the first point of contact for the flows cannot be at the origin else the curvature would blow up at that time. Assume that $T_{emb} < T_{sing}$, for a contradiction. Then we may take a sequence of points (x_n, t_n) and points $a_n, b_n \in \mathbb{R}$ such that $l_{t_n}(a_n) = l_{t_n}(b_n) = p_n$, where t_n is a decreasing sequence converging to T_{emb}. Since the ends of the profile curve have different asymptotes, there exists R such that $p_n \in B_R$ for all n, so passing to a subsequence there exist limits $a_n \to a$, $b_n \to b$, $p_n \to p$ such that $l_{T_{emb}}(a) = l_{T_{emb}}(b) = p$. By Lemma 5.2.2, p is not the origin.

Since p is the first point of contact, we must have (at T_{emb}) $l'(a) = l'(b)$, and so there is a unique line Λ through the origin parallel to the shared tangent space to l at p. Additionally we may take ε sufficiently small such that $B_\varepsilon(p) \cap l$ has two connected components for $t < T_{emb}$, which may be written as graphs u_1, u_2 over Λ, with $u_1 \geq u_2$ and $u_1 = u_2$ at a point $x \in \Lambda$. These graphs both satisfy the equivariant mean curvature flow equation (5.8).

We show that the difference $v := u_1 - u_2$ also satisfies a parabolic differential equation. Defining $u_s := su_1 + (1-s)u_2$, we interpolate between the equations:

$$\begin{aligned}\frac{\partial v}{\partial t} = \frac{\partial u_1}{\partial t} - \frac{\partial u_2}{\partial t} &= a(u_1')u_1'' + b(x,u_1,u_1') - a(u_2')u_2'' - b(x,u_2,u_2') \\ &= \int_0^1 \frac{\partial}{\partial s}\left(a(u_s')u_s'' + b(x,u_s,u_s')\right) ds \\ &= \left(\int_0^1 a(u_s')ds\right) v'' + \widetilde{b}(x)v' + \widetilde{c}(x)v.\end{aligned}$$

We may therefore apply the parabolic Harnack inequality or maximum principle ([22], Chapter 7) to this equation to conclude that $v = 0$ at some earlier time, contradicting the definition of T_{emb}. □

5.2.3 The Type I Blowup

We now return to almost-calibrated flows, and examine the Type I blowup. By Neves' Theorem A (Theorem 3.4.2), any Type I blowup of our LMCF must be a union of equivariant special Lagrangian planes, and due to Theorem 3.4.1, almost-calibrated LMCF cannot develop Type I singularities. Therefore we expect the Type I blowup to consist of a union of multiple equivariant planes through the origin. We will show in this section that in fact it must be a pair of planes, with the same Lagrangian angle.

Throughout we will use the notation L_s^i for a sequence of Type I rescalings, with factors λ_i, and profile curves l_s^i. As before, we assume that L_0 is asymptotically planar, and this implies the area bound

$$\mathscr{H}^n(L_0 \cap B_R(0)) \leq C_0 R^n.$$

This implies uniformly bounded area ratios for all time by Huisken's monotonicity formula, see for example [50].

The following main lemma proves that blowup sequences centered away from the centre of rotation converge to a single plane. This will be used to rule out singularities away from the origin, as well as double density planes for singularities at the origin. Throughout, we use the notation $B_a(x)$ for a ball of radius a centered at x, and B_a as shorthand for $B_a(O)$.

Lemma 5.2.4. *Let L^i be a sequence of uniformly almost-calibrated and connected Lagrangian submanifolds in $\mathbb{C}^n$, with the property that $L^i - x_i e_1$ is an $O(n)$-equivariant submanifold of $\mathbb{C}^n$ for a sequence $x_i \in \mathbb{C}$ and $e_1 = (1,0,\dots,0) \in \mathbb{C}^n$. Assume that x_i eventually lies outside of B_d for some d.*

Assume further that the conclusions to Theorem 3.4.2 and 3.4.3 hold locally in B_1 for the sequence L^i. Explicitly,

- *There exists a finite set $\{\overline{\theta}_1,\dots,\overline{\theta}_M\}$ and integral special Lagrangian cones $\{L_1,\dots,L_M\}$ such that for all $f \in C^2(\mathbb{R})$, $\phi \in C^\infty_c(B_1)$,*

$$\lim_{i\to\infty}\int_{L_i} f(\theta^i)\phi\, d\mathscr{H}^n = \sum_{j=1}^{M} m_j f(\overline{\theta_j})\mu_j(\phi), \tag{5.10}$$

- *For any convergent sequence Σ^i of connected components of $B_{2\delta}\cap L^i$ intersecting $B_{\frac{\delta}{2}}$, there exists a special Lagrangian cone L with Lagrangian angle $\overline{\theta}$ such that for all $f \in C^2(\mathbb{R})$, $\phi \in C^\infty_c(B_1)$,*

$$\lim_{i\to\infty}\int_{\Sigma_i} f(\theta^i)\phi\, d\mathscr{H}^n = m f(\overline{\theta})\mu(\phi). \tag{5.11}$$

Then there exists a single special Lagrangian plane P with angle $\overline{\theta}$ and underlying Radon measure μ_P such that for all $\phi \in C^\infty_c(B_1)$, $f \in C^2(\mathbb{R})$:

$$\lim_{i\to\infty}\int_{L^i} f(\theta^i)\phi\, d\mathscr{H}^n = f(\overline{\theta})\mu_P(\phi).$$

Proof. The proof is by a density argument (extending a similar argument of A. Neves in [50]), a sketch of which is as follows. By the work of Section 5.1 that allows us to work with the profile curve, we are done if we can prove that

$$\lim_{i\to\infty}\frac{\mathscr{H}^1(\gamma^i\cap B_\delta)}{2\delta} = 1,$$

where $B_\delta := B_\delta(O)$. Since we know already that the limit is a union of planes it follows from this that it must be a single plane. We therefore wish to estimate this density ratio.

Taking a sequence of connected components of l^i, which we label γ^i, (5.11) gives integral convergence of the Lagrangian angle in B_δ, and since the centre of $O(n)$ symmetry x_i is away from the origin, this implies a tight bound on the angle of $\dot{\gamma}^i$. We then use this to show the above density bound, for sufficiently small δ. However we are not done, as there may be another, different sequence of connected components that can increase the total density further – we must rule this out.

Considering two different connected components ξ^i and η^i, they can either converge to the same Lagrangian angle, or a different one. If the limiting Lagrangian angle is different, then we can show that ξ^i and η^i must collide, perhaps in a larger ball, since the angles of their derivatives are tightly bounded around different values. On the other hand if the Lagrangian angle is the same, then we can show by embeddedness that there must be another connected component in between with different Lagrangian angle, causing a collision as before. This shows that there is in fact only one connected component to consider, and we are done.

We now fill in the details. Let $B_{2\delta}$ be small enough so that for i large, $x_i \notin B_{2\delta}$, and consider a sequence Σ^i of connected components of $L^i \cap B_{2\delta}$ intersecting $B_{\frac{\delta}{2}}$, with profile curve γ^i. By (5.11), there exists a special Lagrangian cone L^∞ with underlying Radon measure μ_∞ and an integer multiplicity m such that for $\phi \in C_c^\infty(B_\delta), f \in C^2\mathbb{R}$:

$$\lim_{i\to\infty}\int_{\Sigma^i} f(\theta^i)\phi d\mathscr{H}^n = mf(\overline{\theta})\mu_\infty(\phi). \tag{5.12}$$

We first use this convergence to get a bound on $\arg(\dot{\gamma})$. For $\varepsilon > 0$, define the following "ε-good" and "ε-bad" subsets of $\gamma^i \cap B_\delta$:

$$S_\delta(\gamma^i) := \left\{ x \in \gamma^i \cap B_\delta \,\middle|\, |\theta^i(x) - \overline{\theta}(x)| \leq \varepsilon \right\},$$

$$T_\delta(\gamma^i) := \left\{ x \in \gamma^i \cap B_\delta \,\middle|\, |\theta^i(x) - \overline{\theta}(x)| > \varepsilon \right\},$$

note we suppress the dependence on ε for notational clarity. Then (5.12) implies that

$$\forall \varepsilon \ \exists N \text{ s.t } \forall i > N, \quad \mathscr{H}^1(T_\delta(\gamma^i)) < \varepsilon. \tag{5.13}$$

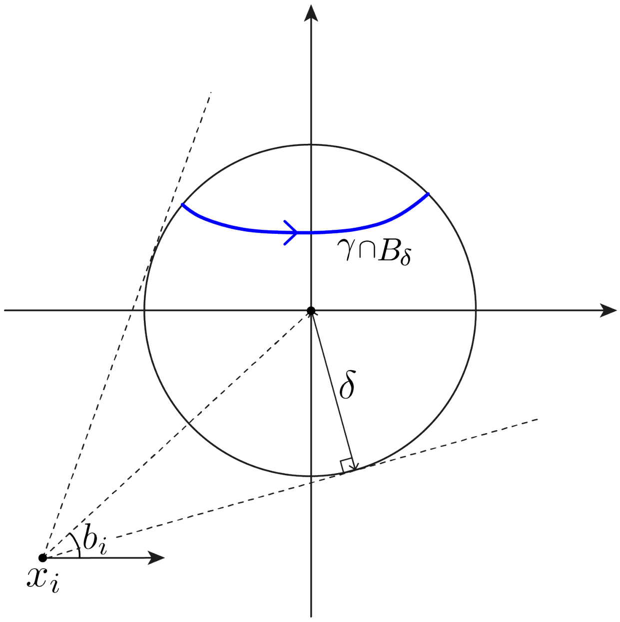


Figure 5.5: The setup of Lemma 5.2.4.

Our aim is therefore to estimate $\mathscr{H}^1(S_\delta(\gamma^i))$. Taking arguments with respect to the point x_i and the e_1 direction, by (5.3) the Lagrangian angle is given by

$$\theta = (n-1)\arg(\gamma) + \arg(\dot{\gamma}). \tag{5.14}$$

Denoting $b_i := \arg(O - x_i)$, b_i converges to some b (after passing to a subsequence if necessary). Then on B_δ we have the bound

$$\arg(\gamma^i) \in \left(b_i - \sin^{-1}\left(\frac{\delta}{|x_i|}\right), b_i + \sin^{-1}\left(\frac{\delta}{|x_i|}\right)\right),$$

(see Figure 5.5) and therefore on $S_\delta(\gamma^i)$, taking i sufficiently large so that $|b - b_i| < \varepsilon$, we obtain a bound on the argument of $\dot{\gamma}^i$:

$$|\arg(\dot{\gamma}^i) - \overline{\theta} + (n-1)b| \leq (n-1)\sin^{-1}\left(\frac{\delta}{|x_i|}\right) + 2\varepsilon =: \rho(\delta, \varepsilon). \tag{5.15}$$

Parametrise by unit speed, so that $\dot{\gamma}^i(s) = e^{i(\lambda(s) + \overline{\theta} - (n-1)b)}$ for an angle function $\lambda(s)$.

Then equation (5.15) implies $|\lambda(s)| \leq \rho$, and therefore

$$\left|\int_{S_\delta(\gamma^i)} \dot{\gamma}^i(s)\,ds\right| \geq \left|\int_{S_\delta(\gamma^i)} e^{i(\overline{\theta}-(n-1)b)}\cos(\lambda(s))\,ds\right| \geq \mathscr{H}^1(S_\delta(\gamma^i))\cos\rho. \qquad (5.16)$$

We'd like to use (5.16) to bound $\mathscr{H}^1(S_\delta(\gamma^i))$, so we need to bound $\left|\int_{S_\delta(\gamma^i)} \dot{\gamma}^i(s)\,ds\right|$. If $\gamma^i \cap B_\delta$ was a single connected component, this would be simple, as the integral of $\dot{\gamma}^i$ over B_δ would then be less than 2δ by the fundamental theorem of calculus. However there may be more connected components to worry about. The following lemma demonstrates that if we widen our ball slightly, we will only have to worry about one connected component. After its proof, we will resume the proof of Lemma 5.2.4.

Lemma 5.2.5. *Assume that we have the setup of Lemma 5.2.4. Then for sufficiently small δ, ε, there exists N such that for all $i > N$, there is only one connected component of $l^i \cap B_{\delta+3\varepsilon}$ intersecting B_δ.*

Proof. We demonstrate that for sufficiently small δ, ε, there exists N such that for all $i > N$:

1. Two distinct sequences of connected components of $l^i \cap B_{\delta+3\varepsilon}$ intersecting B_δ can't have different Lagrangian angles in the limit.

2. If two distinct sequences of connected components of $l^i \cap B_{\delta+3\varepsilon}$ intersecting B_δ have the *same* limiting Lagrangian angle, we can find a third connected component ζ^i with a different limiting Lagrangian angle.

Together, these two claims complete the proof.

Proof of 1. By (5.10), there are a finite number of possible limiting Lagrangian angles for these curves, $\{\overline{\theta}_1, \ldots, \overline{\theta}_M\}$. These correspond bijectively to a finite number of possible limiting values for the argument of the tangent vector, $\arg(\dot{\gamma}^i)$ (see (5.15)):

$$A = \{\alpha_1, \ldots, \alpha_M\} := \{\overline{\theta}_1 - (n-1)b, \ldots, \overline{\theta}_M - (n-1)b\}.$$

By the almost-calibrated condition, these angles are all different modulo π, and so any two straight lines representing different angles in A that intersect B_δ must intersect in a sufficiently large ball. We may therefore choose R large enough such that any two curves η and ξ in $B_{R\delta}$ intersecting B_δ such that $\arg(\dot\eta)$ and $\arg(\dot\xi)$ are ε-close to distinct values in A (outside a set of $\mathscr{H}^1$-measure ε) must collide inside $B_{R\delta}$.

Now for a contradiction, assume that, after passing to a subsequence, for all i there exist two distinct connected components η^i and ξ^i of $l^i \cap B_{\delta+3\varepsilon}$ intersecting B_δ whose Lagrangian angles converge to distinct values $\overline{\theta}_\eta$ and $\overline{\theta}_\xi$. Now extend η^i and ξ^i to the connected components in $B_{R\delta}$ that contain them (which may be the same): call these $\overline{\eta}^i$ and $\overline{\xi}^i$. For sufficiently small δ we can apply the same argument as in the proof (so far) of Lemma 5.2.4 and show that, for sufficiently large i, (5.15) holds for $\overline{\eta}^i$ and $\overline{\xi}^i$ in $B_{R\delta}$ outside a set of $\mathscr{H}^1$-measure ε, with the Lagrangian angles $\overline{\theta}_\eta$ and $\overline{\theta}_\xi$ respectively. This implies that the connected components must be distinct, but by the choice of R, $\overline{\eta}^i$ and $\overline{\xi}^i$ must then collide for i sufficiently large, contradicting embeddedness.

Proof of 2. Assume that (after passing to a subsequence) for all i there exist two distinct connected components η^i and ξ^i of $l^i \cap B_{\delta+3\varepsilon}$ intersecting B_δ, and that the Lagrangian angles of ξ, η converge to the same value $\overline{\theta}$; without loss of generality we assume that $\overline{\theta} - (n-1)b = 0$.

We first show that ξ^i must enter the ball $B_{\delta+3\varepsilon}$ on the left-hand side and leave on the right-hand side. Work with a unit-speed parametrisation, $\dot\xi^i(s) = e^{i\lambda(s)}$. Since ξ^i intersects B_δ, there is some s_0 such that $\xi^i(s_0) = p \in B_\delta$. By connectedness,

$$\mathscr{H}^1(\{\xi(s) : s \geq s_0\}) \geq 3\varepsilon.$$

Therefore by splitting the set into $S_{\delta+3\varepsilon}$ and $T_{\delta+3\varepsilon}$ we can calculate the horizontal and vertical distance travelled:

$$\begin{aligned}
\int_{\{s\geq s_0\}} \cos(\lambda(s))\,ds &= \int_{\{s\geq s_0\}\cap T_{\delta+3\varepsilon}} \cos(\lambda(s))\,ds + \int_{\{s\geq s_0\}\cap S_{\delta+3\varepsilon}} \cos(\lambda(s))\,ds \\
&\geq -\varepsilon + 3\varepsilon\cos(\rho) \geq \tfrac{\varepsilon}{2}, \\
\int_{\{s\geq s_0\}} |\sin(\lambda(s))|\,ds &= \int_{\{s\geq s_0\}\cap T_{\delta+3\varepsilon}} |\sin(\lambda(s))|\,ds + \int_{\{s\geq s_0\}\cap S_{\delta+3\varepsilon}} |\sin(\lambda(s))|\,ds \\
&\leq \varepsilon + 3\varepsilon\sin(\rho) \leq 2\varepsilon,
\end{aligned}$$

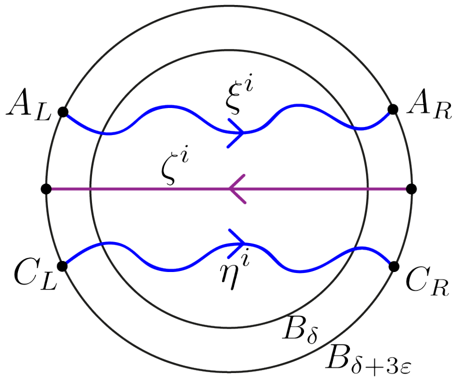


Figure 5.6: Two connected components with the same angle must have another between them.

by (5.15) and (5.13), since by taking δ, ε sufficiently small we may make $\rho(\delta, \varepsilon)$ as close to 0 as we like. This shows that ξ^i leaves the ball on the right-hand side, since p_0 must be to the left of the exit point, and less than 2ε vertically separated from it. An identical argument shows that ξ^i enters on the left-hand side. The same is true for η^i.

Now if these were the only connected components, we have the situation depicted in Figure 5.6. Since L^i is connected, either A_R joins to C_L or C_R joins to A_L. In both situations, one end of the curve must be trapped in a compact region of the plane by embeddedness (the C_R or A_L end in the former case, the C_L or A_R end in the latter), which is a contradiction. Therefore there must be another connected component ζ^i in $B_{\delta+3\varepsilon}$; to solve the above problem it must be a curve from right to left, in the middle of ξ^i and η^i (see Figure 5.6). By the above argument, since ζ^i does not enter on the left and leave on the right it must have a different limiting Lagrangian angle, and so the argument is complete. □

We now resume the proof of Lemma 5.2.4. Taking δ, ε sufficiently small, we know by Lemma 5.2.5 that for sufficiently large i, there is only one connected component of $\gamma^i \cap B_{\delta+3\varepsilon}$ that intersects B_δ: call it $\tilde{\gamma}^i$. Also for sufficiently large i, $\mathscr{H}^1(T_{\delta+3\varepsilon}(\tilde{\gamma}^i)) < \varepsilon$. Using this, (5.15) and (5.16), we estimate for sufficiently large i using a unit-speed parametrisation (suppressing the superscript i and defining $\tilde{\rho} := \rho(\delta + 3\varepsilon, \varepsilon)$, $\rho :=$

$\rho(\delta,\varepsilon)$ for readability):

$$\begin{aligned}\mathscr{H}^1(S_{\delta+3\varepsilon}(\tilde{\gamma}))\cos\tilde{\rho} &\leq \left|\int_{S_{\delta+3\varepsilon}(\tilde{\gamma})}\dot{\tilde{\gamma}}ds\right| \leq \left|\int_{B_{\delta+3\varepsilon}\cap\tilde{\gamma}}\dot{\tilde{\gamma}}ds\right| + \varepsilon \leq 2\delta+7\varepsilon \\ \implies \mathscr{H}^1(S_\delta(\gamma)) &\leq \mathscr{H}^1(S_{\delta+3\varepsilon}(\tilde{\gamma})) \leq \frac{2\delta+7\varepsilon}{\cos\tilde{\rho}}.\end{aligned}$$

Finally, using this and (5.13) we can estimate our density ratio:

$$\frac{\mathscr{H}^1(\gamma\cap B_\delta)}{2\delta} = \frac{\mathscr{H}^1(S_\delta(\gamma))}{2\delta} + \frac{\mathscr{H}^1(T_\delta(\gamma))}{2\delta} \leq \frac{1}{\cos\tilde{\rho}} + \frac{\varepsilon}{\delta}\left(\frac{7}{2\cos\tilde{\rho}} + \frac{1}{2}\right).$$

By (5.15), $\cos\tilde{\rho} = \cos(\rho(\delta+2\varepsilon,\varepsilon)) = 1+O(\delta,\varepsilon)$. Therefore, taking δ and $\frac{\varepsilon}{\delta}$ sufficiently small ensures that the density ratio is bounded away from 2. By Lemmas 5.1.4 and 5.1.9, we must have that this density ratio converges to an integer, which due to the bound must be 1. Therefore, the limit of the sequence Σ^i of connected components is a single plane.

Finally, Lemma 5.2.5 implies that there are no other connected components of $l^i \cap B_{2\delta}$ intersecting $B_{\frac{\delta}{2}}$, so we have in fact proven that L^i converges to a single Lagrangian plane. $\square$

Now we apply Lemma 5.2.4 to get our main results.

Theorem 5.2.6. *Let L_t be an almost-calibrated, connected $O(n)$-equivariant mean curvature flow in $\mathbb{C}^n$ with planar asymptotics. Then if L_t has a finite-time singularity, it must occur at the origin.*

Proof. Assume for a contradiction that such a flow has a singularity away from the origin. Without loss of generality, it is at a point $(ae^{ib},0,\ldots,0) \in \mathbb{C}\times\{0\}^{n-1}$, since otherwise we may just perform a rotation that leaves the flow unaffected. Note that the planar asymptotics imply uniformly bounded area ratios, and by Lemma 5.1.1 the flow is rational. Taking a sequence of rescalings L_s^j around ae^{ib} with factor λ^j, the conclusions to Theorems 3.4.2 and 3.4.3 therefore hold for almost all s. The centre of rotation for L_s^j is $x_j := -\lambda_j ae^{ib}$, whose size diverges to ∞.

We may therefore apply Lemma 5.2.4 to conclude that L_s^i converges to a density 1 Lagrangian plane for almost all s. This convergence is smooth by White regularity

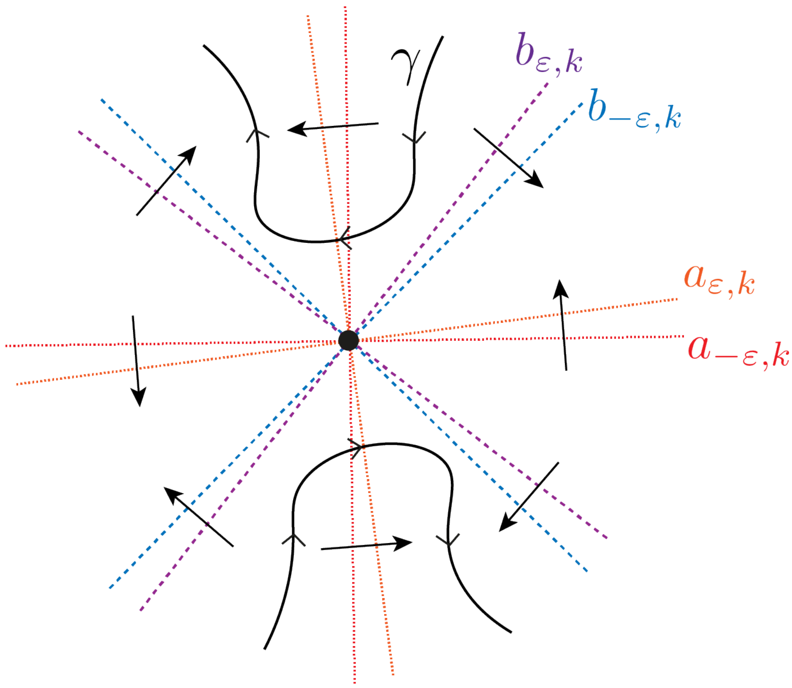


Figure 5.7: The half-lines $a_{\varepsilon,k}$, $a_{-\varepsilon,k}$, $b_{\varepsilon,k}$ and $b_{-\varepsilon,k}$ in the proof of Lemma 5.2.7, in the case where $n = 4$.

(Theorem 2.2.10), by the following argument. Choosing a space-time point $\overline{X} = (\overline{x}, \overline{s})$, for any $X \in P_r(\overline{X})$ we have by Huisken monotonicity (Theorem 2.2.7):

$$\begin{aligned}\lim_{i\to\infty} \Theta(L^i, X, r) &= \lim_{i\to\infty} \int_{L^i_{s-r^2}} \Phi_X(x, s-r^2)\, d\mathscr{H}^n \\ &\leq \lim_{i\to\infty} \int_{L^i_{\overline{s}-2r^2}} \Phi_X(x, \overline{s}-2r^2)\, d\mathscr{H}^n \leq 1,\end{aligned}$$

for any r such that $L^i_{\overline{s}-2r^2}$ converges to a density 1 Lagrangian plane. The last inequality holds since Φ integrates to 1 over a plane including X, and less than 1 on any other plane. It follows that by White regularity that the curvatures are all bounded, giving smooth convergence upon passing to a subsequence by Arzelà-Ascoli.

But since the singularity is Type II by Theorem 3.4.1 we should have that the curvature of these rescalings is diverging. So, we have a contradiction. □

Next, we prove the uniqueness of the Type I blowup. We will need the following lemma, which gives bounds on the argument of l_t.

Lemma 5.2.7. *Let L be a connected, $O(n)$-equivariant Lagrangian submanifold, with* *planar asymptotics. Assume that the profile curve l does not contain the origin, and that L is almost-calibrated; explicitly that there exist $\overline{\theta}$ and ε such that*

$$\theta \in \left(\overline{\theta} - \frac{\pi}{2} + \varepsilon, \overline{\theta} + \frac{\pi}{2} - \varepsilon\right).$$

Then for a connected component γ of l, there exists a cone of angular width strictly less than $\frac{2\pi}{n}$ containing γ.

Proof. Consider the following half-lines, for $k \in \mathbb{N}$ (see Figure 5.7 for a diagram):

$$\begin{aligned}
\arg(a_{\varepsilon,k}) &= \tfrac{\overline{\theta}}{n} + \tfrac{2\pi k}{n} + \tfrac{\pi}{2n} + \tfrac{\varepsilon}{n}, \\
\arg(a_{-\varepsilon,k}) &= \tfrac{\overline{\theta}}{n} + \tfrac{2\pi k}{n} + \tfrac{\pi}{2n} - \tfrac{\varepsilon}{n}, \\
\arg(b_{\varepsilon,k}) &= \tfrac{\overline{\theta}}{n} + \tfrac{2\pi k}{n} - \tfrac{\pi}{2n} + \tfrac{\varepsilon}{n}, \\
\arg(b_{-\varepsilon,k}) &= \tfrac{\overline{\theta}}{n} + \tfrac{2\pi k}{n} - \tfrac{\pi}{2n} - \tfrac{\varepsilon}{n}.
\end{aligned}$$

By the almost-calibrated condition, it can be shown that the curve may only pass through the lines $a_{\varepsilon,k}$ and $a_{-\varepsilon,k}$ in a clockwise direction, and the lines $b_{\varepsilon,k}$ and $b_{-\varepsilon,k}$ in an anti-clockwise direction. For example, for a contradiction assume that γ passes through $a_{\varepsilon,k}$ anticlockwise. Then at that point, by (2.30),

$$\begin{aligned}
\theta &\in \left(n\arg(a_{\varepsilon,k}),\, n\arg(a_{\varepsilon,k}) + \pi\right) \mod 2\pi \\
&\equiv \left(\overline{\theta} + \tfrac{\pi}{2} + \varepsilon,\, \overline{\theta} + \tfrac{3\pi}{2} + \varepsilon\right) \mod 2\pi,
\end{aligned}$$

which contradicts the almost-calibrated condition.

But now it is clear that the curve must remain in a cone of angle less than $\frac{2\pi}{n}$, as it is constrained by the above lines. □

Theorem 5.2.8. *Let L_t be an almost-calibrated, connected $O(n)$-equivariant mean curvature flow in $\mathbb{C}^n$ with planar asymptotics. Then the Type I blowup of any finite-time singularity is a special Lagrangian cone consisting of a transverse pair of planes $P_1 \cup P_2$ whose profile curves span an angle of $\frac{\pi}{n}$, and does not depend on the sequence of rescalings.*

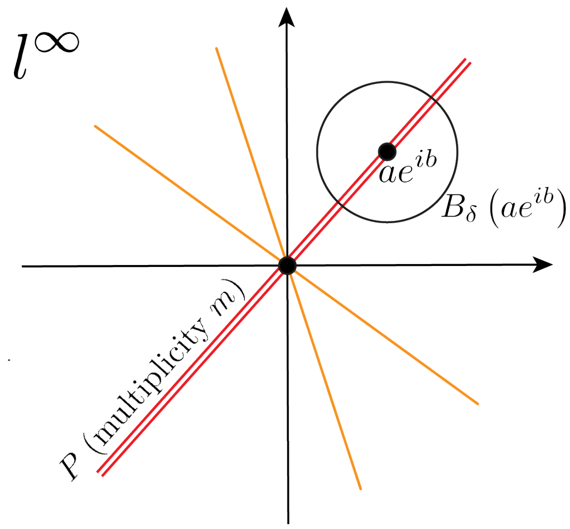


Figure 5.8: The profile curve l^∞ of a Type I blowup in the proof of Theorem 5.2.8.

Proof. We will first rule out planes with density greater than 1 in the limit, and then demonstrate that a single transverse pair of planes is the only option. We know from Theorem 5.2.6 that the singularity must occur at the origin, therefore the centre of rotation for L_s^i is O. We also know by Theorem 3.4.2 and Lemma 5.1.4 that any blowup sequence L_s^i converges subsequentially for almost all s to a finite number of special Lagrangian cones. Fix such an s; we suppress the subscript for clarity.

Assume that one of the limiting planes, P, has a multiplicity $m > 1$. Then there is a point ae^{ib} with $a < \frac{1}{4}$ and δ small enough such that all other planes in the blowup do not intersect $B_\delta(ae^{ib})$, and so

$$\lim_{i\to\infty} \frac{\mathscr{H}^1(l^i \cap B_\delta(ae^{ib}))}{2\delta} \to m$$

(see Figure 5.8). Now for $2\varepsilon < \delta$, any sequence of connected components of $L^i \cap B_{2\varepsilon}(ae^{ib})$ intersecting $B_{\frac{\varepsilon}{2}}(ae^{ib})$ may be extended to a sequence of connected components of B_1 intersecting $B_{\frac{1}{4}}$. These converge to a special Lagrangian in B_1 by Theorem 3.4.3, which must be P with some multiplicity.

It therefore follows that the conclusions to Theorems 3.4.2 and 3.4.3 apply to the flows obtained by translating ae^{ib} to the origin and scaling by $\frac{1}{\delta}$, locally inside the ball B_1. We may therefore apply Lemma 5.2.4 to the resulting sequence and conclude that $m = 1$.

Now we show that a special Lagrangian pair of planes is the only option for the Type I blowup, working with the profile curve throughout. ξ^i, η^i will denote the profile curves of sequences of different connected components of $L^i \cap B_\delta$ intersecting $B_{\frac{\delta}{4}}$. We will rule out a single line in the limit, 3 or more lines in the limit, and two separate lines coming from different connected components. This will leave the only option as a pair of lines coming from a single sequence of connected components.

One unit-density line. Assume ξ^i converges to a unit-density line; by White regularity (Theorem 2.2.10) this convergence is smooth. But then there is no curvature blowup in the Type I rescalings, and by Theorem 3.4.1 the singularity must be Type II, so this is a contradiction.

Two unit-density lines from different connected components. Assume ξ^i and η^i converge to distinct lines. By White regularity (Theorem 2.2.10) they must converge smoothly to the lines in any annulus, but this means that they must intersect each other at the origin for sufficiently large i by the reflective symmetry, which is impossible since l^i is embedded.

More than three unit-density lines. By White regularity (Theorem 2.2.10), we have smooth convergence to the Type I blowup in the annulus $B_\delta \setminus B_{\frac{\delta}{4}}$. Take N sufficiently large, so that for $i > N$ and inside this annulus, the profile curve l^i can be expressed as a graph over the limiting lines.

Giving l^i an orientation, label the first, second and third connected components of $l^i \cap (B_\delta \backslash B_{\frac{\delta}{4}})$ by γ_1, γ_2 and γ_3 respectively (If l^i has two disconnected components, we make this definition using one half of it, γ^i). By passing to a subsequence, we may assume that these curves always lie over the same limiting half-lines; we denote the limiting half-line over which γ_k is a graph by c_k, and the argument of c_k by α_k. Assume that γ_2 is clockwise from γ_1 along l^i (the other case follows by an identical argument). Note that the curve l^i does not pass through the origin between γ_1 and γ_2, by considering the reflective symmetry, and so the orientations of γ_1, γ_2 and γ_3 are towards, away from, and towards the origin respectively. (see Figure 5.9).

Since γ_1 and γ_2 are part of the same connected component, the limiting Lagrangian angle must be the same, and since we also have the argument bound from Lemma 5.2.7

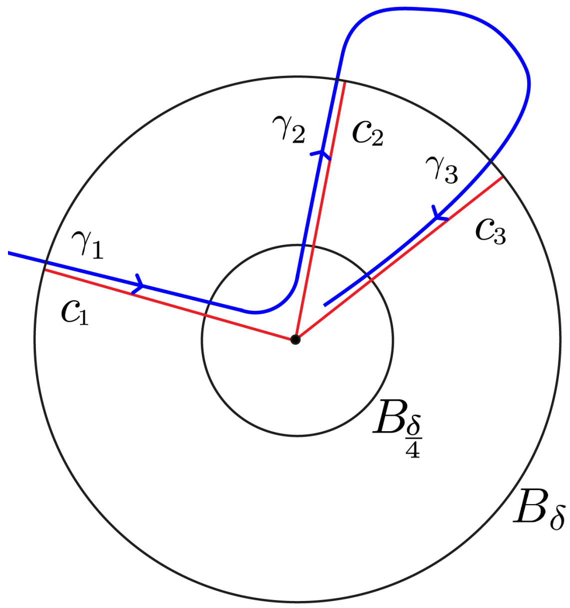


Figure 5.9: The curves γ_1, γ_2 and γ_3.

it follows that

$$\alpha_1 - \frac{\pi}{n} = \alpha_2. \tag{5.17}$$

Additionally, the curve γ_3 cannot be between γ_1 and γ_2. If it was, the curve l^i would have to leave B_δ again after γ_3 between these curves by embeddedness, and since it would be part of the same connected component as γ_3, it would make an angle of $\frac{\pi}{n}$ with it in the limit. But since the angle between c_1 and c_2 is $\frac{\pi}{n}$, this would imply that $c_3 = c_1$ or $c_3 = c_2$, and we have ruled out the possibility of double-density lines. It follows that

$$\alpha_3 \leq \alpha_2 \leq \alpha_1.$$

By the smooth convergence, for all ε we may take N large such that if $i > N$, then (keeping the orientation of the curves in mind):

$$|\arg(\dot{\gamma}_1) - \alpha_1 + \pi| \leq \varepsilon, \quad |\arg(\dot{\gamma}_2) - \alpha_2| \leq \varepsilon, \quad |\arg(\dot{\gamma}_3) - \alpha_3 + \pi| \leq \varepsilon,$$

$$|\arg(\gamma_k) - \alpha_k| < \varepsilon.$$

Therefore, denoting the Lagrangian angle of γ_i by θ_i,

$$\begin{aligned}
\theta_1 &= \arg(\dot{\gamma}_1) + (n-1)\arg(\gamma_1) \\
&\geq \alpha_1 - \pi - \varepsilon + (n-1)(\alpha_1 - \varepsilon) \\
&= n\alpha_1 - \pi - n\varepsilon, \\
\Longrightarrow \theta_3 &= \arg(\dot{\gamma}_3) + (n-1)\arg(\gamma_3) \\
&\leq n\alpha_3 - \pi + n\varepsilon \\
&\leq n\alpha_2 - \pi + n\varepsilon \\
&= \theta_1 - \pi + 2n\varepsilon.
\end{aligned}$$

Taking ε sufficiently small gives a contradiction to the almost-calibrated condition.

We therefore must have a single pair of lines in the limit, with the same Lagrangian angle $\overline{\theta}$. These lines must span an angle of $\frac{\pi}{n}$, by the same argument that gave (5.17). Uniqueness of the Type I blowup follows from the fact that there is only one such pair of lines with Lagrangian angle $\overline{\theta}$ in the cone given by Lemma 5.2.7, since this cone spans an angle of strictly less than $\frac{2\pi}{n}$. □

5.2.4 Singularity Formation and Long-Time Existence

In this section, we prove Theorem 5.0.2 on singularity formation and long-time existence (see Figure 5.3 for a diagram of the different cases). We will use our previous results on the nature of singularities of equivariant mean curvature flow to rule out singularities in certain cases, which implies long-time existence.

We first look at the two distinct topologies that our flow may take. One option is that the profile curve of the initial condition passes through the origin; in this case the singularity analysis implies long-time existence.

Theorem 5.2.9. *Let L_t be an almost-calibrated, connected, $O(n)$-equivariant mean curvature flow in $\mathbb{C}^n$ with planar asymptotics. Assume that the profile curve of the initial condition, l_0, passes through the origin. Then L_t exists for all time.*

Proof. Assume for a contradiction that there is a finite-time singularity. By Theorem 5.2.8, the profile curve of any Type I blowup must be a pair of lines. By White regularity, Theorem 2.2.10, we must have smooth convergence of the rescalings in an annulus to this pair of lines. This creates 4 'ends' for the profile curve of each rescaling on the outer boundary of the annulus.

Now in any rescaling, one connected component must go through the origin. Therefore, by the reflectional symmetry of the profile curve, this connected component must account for *opposite* ends. The other two ends can only be joined if the curve is not embedded, and so we have a contradiction. □

The other option is that the profile curve doesn't pass through the origin, then l_t has two different asymptotes. In the paper [50], Neves exhibits examples of almost-calibrated S^1-equivariant flows in $\mathbb{C}^2$ that are of this form. In particular, he studies the flow of the initial profile curve

$$\eta_0(s) := \left(\sin(\tfrac{\pi s}{\beta})\right)^{-\frac{\beta}{\pi}} e^{is},$$

which is given in polar form. If $\pi > \beta > \frac{\pi}{2}$, this flow forms a finite-time singularity at the origin, and if $\frac{\pi}{2} > \beta > 0$, then the flow is eternal, and flows outwards to infinity (see Figure 5.1).

We generalise Neves' constructions to $\mathbb{C}^n$, and prove that if $\frac{2\pi}{n} > \beta > \frac{\pi}{n}$ for the same initial curve then the flow forms a finite-time singularity. Note that Lemma 5.2.7 implies that this initial condition is only almost-calibrated if $\beta < \frac{2\pi}{n}$.

Lemma 5.2.10. *If $\frac{2\pi}{n} > \beta > \frac{\pi}{n}$, then the Lagrangian mean curvature flow L_t^n in $\mathbb{C}^n$ with profile curve η_t starting at the initial condition with profile curve η_0 forms a singularity at the origin in finite time.*

Proof. The curve η_t may be expressed in polar form, $r_t(s)e^{is}$, until a singularity forms. This can be proven using a Sturmian theorem – see [50] for details.

We may then look at the evolution of the area under the curve between angles ε and

$\beta - \varepsilon$, using the evolution equation (5.9):

$$\begin{aligned} A_{\varepsilon,t} &:= \frac{1}{2}\int_{\varepsilon}^{\beta-\varepsilon} r_t^2 ds. \\ \frac{dA_{\varepsilon,t}}{dt} &= \int_{\varepsilon}^{\beta-\varepsilon} \theta_t' ds = \theta(\varepsilon) - \theta(\beta-\varepsilon). \end{aligned}$$

Using the fact that

$$\theta_t(s) = \arg\left((r'(s) + ir(s))e^{is}\right) + (n-1)\arg\left(r(s)e^{is}\right) = ns + \arg(r' + ir),$$

it follows that $\theta_t(s) \in (ns, ns+\pi)$. Therefore, if $\pi - n\beta < 0$, we may choose ε sufficiently small such that

$$\frac{dA_{\varepsilon,t}}{dt} < 2n\varepsilon + \pi - n\beta < -C$$

for a positive constant C. It follows that a singularity must form in finite time, and by Theorem 5.2.6 it must occur at the origin. □

Theorem 5.2.11. *Let L_t be an almost-calibrated, connected, $O(n)$-equivariant mean curvature flow in $\mathbb{C}^n$ with planar asymptotics. Assume that the profile curve of the initial condition, l_0 does not pass through the origin, and that the angle α between the asymptotes of the profile curve are strictly between $\frac{\pi}{n}$ and $\frac{2\pi}{n}$. Then L_t forms a finite-time singularity.*

Proof. Working with a connected component γ of the profile curve, Lemma 5.2.7 gives us a cone that γ_t remains in until a singularity forms. We may then find a scaled and rotated copy of Neves' curve η_0 with angle $\frac{\pi}{n} < \beta < \alpha$ that also lies in this cone, further away from γ_0 than the origin and with different asymptotes, that does not intersect it. By the avoidance principle, Theorem 5.2.3, under equivariant MCF these curves do not intersect until one forms a singularity. Since η_t descends to the origin within the cone, γ_t is also forced to the origin; here the curvature blows up and so a singularity must occur. □

The final situation is one in which the asymptotes of the flow span an angle $\alpha \leq \frac{\pi}{n}$.

In this case, long-time existence is possible, as shown by the examples of the Lawlor neck Σ_{Law} 3.3.1 and the Anciaux expander 3.3.3. In fact, we may use these examples as barriers to prove long-time existence in the case where $\alpha < \frac{\pi}{n}$. The case where $\alpha = \frac{\pi}{n}$ is more difficult, as the avoidance principle as we have stated it does not hold if the asymptotes of the two flows are matching. To cover this case, we construct subsolutions to the flow with different asymptotes, and use those as barriers.

Theorem 5.2.12. *Let L_t be an almost-calibrated, connected, $O(n)$-equivariant mean curvature flow in $\mathbb{C}^n$ with planar asymptotics. Assume that the profile curve of the initial condition l_0 does not pass through the origin, and lies completely within a cone of angle $\alpha \leq \frac{\pi}{n}$. Then the flow exists for all time $t \in [0, \infty)$.*

Proof. We construct subsolutions to the radial mean curvature flow equation (5.9) that initially lie underneath l_0, with different asymptotics to our initial condition l_0. For simplicity, assume that the cone in question is $(-\frac{\pi}{2n}, \frac{\pi}{2n})$. Our aim is to find a family of functions $r_t(\alpha) : \left(-\frac{\pi}{2(n-\delta)}, \frac{\pi}{2(n-\delta)}\right) \to \mathbb{R}_+$ such that:

- The curve $r_0(\alpha)e^{i\alpha}$ lies beneath l_0,
- r_t diverges to infinity at the ends of the interval,
- $\frac{dr}{dt} \leq \frac{-\theta'}{r}$.

For our Ansatz we modify the Lawlor neck (Example 3.3.1) by stretching it so that the curvature k will be reduced relative to p, and the mean curvature vector will point towards the origin,

$$r_t(\alpha) := \frac{B(t)}{\sqrt[n]{\sin(\frac{\pi}{2} - (n-\delta)\alpha)}}. \tag{5.18}$$

We now aim to find a suitable function $B(t)$. By assumption, there exists a value B_0 such that (5.18) lies beneath l_0. By a calculation, we find that

$$\theta = n\alpha + \cot^{-1}\left(\frac{n-\delta}{n}\cot(\tfrac{\pi}{2} - (n-\delta)\alpha)\right),$$

$$\theta' = n \cdot \frac{1 - \left(\frac{n-\delta}{n}\right)^2}{1 + \left(\frac{n-\delta}{n}\right)^2 \cot(\frac{\pi}{2} - (n-\delta)\alpha)^2},$$

and so for sufficiently small δ,

$$\begin{aligned}
&\frac{dr}{dt} \leq -\frac{\theta'}{r} \\
\iff B'(t)B(t) &\leq -n \cdot \frac{1-\left(\frac{n-\delta}{n}\right)^2}{1+\left(\frac{n-\delta}{n}\right)\cot(\frac{\pi}{2}-(n-\delta)\alpha)} \sin(\tfrac{\pi}{2}-(n-\delta)\alpha)^{\frac{2}{n}} \\
\Longleftarrow (B(t)^2)' &\leq -(4\delta - \tfrac{\delta^2}{n}) \sin(\tfrac{\pi}{2}-(n-\delta)\alpha)^{2+\frac{2}{n}} \\
\Longleftarrow (B(t)^2)' &\leq -4\delta \\
\Longleftarrow B(t) &= \sqrt{B_0^2 - 4\delta t}.
\end{aligned}$$

With this choice of $B(t)$, we have a subsolution that exists until time $t = \frac{B_0^2}{4\delta}$, and this final time diverges to infinity as $\delta \to 0$. Therefore by using these as barriers below the flow of the profile curve l, l is prevented from reaching the origin by the avoidance principle (Lemma 5.1.3), and by Theorem 5.2.6 a singularity cannot occur.

□

5.2.5 The Type II Blowup

In this section we analyse the Type II blowup of a singularity of our equivariant LMCF. Since by Theorem 5.2.9 an initial profile curve through the origin cannot form a finite-time singularity under MCF, we assume throughout this section that the initial profile curve l avoids the origin, and therefore consists of two connected components, γ and $-\gamma$.

We first show that any Type II blowup of an LMCF must have the same Lagrangian angle as the Type I blowup, in particular that it must be a special Lagrangian (we actually prove a slightly more general theorem, so that it can also be used for intermediate rescalings later). Lemma 5.1.6 will rule out the possibility of the centre of rotation becoming unbounded under the rescalings, and then with a combination of Lemma 5.1.6 and Lemma 5.1.5 we will conclude that the only possibility for a Type II blowup is the Lawlor neck of Lemma 5.1.5.

Theorem 5.2.13. *Let L_t be an almost-calibrated LMCF in $\mathbb{C}^n$ with Lagrangian angle θ_t that forms a singularity at the space-time point $(O,0)$. Assume that any sequence of*

Type I rescalings $L_s^{\sigma_i}$ converge as flows to the same special Lagrangian cone C, with angle $\overline{\theta}$.

Let $X_i = (x_i, t_i)$ be a sequence of space-time points such that $(x_i, t_i) \to (O, 0)$, let $\lambda_i \in \mathbb{R}$ satisfy $-\lambda_i^2 t_i \to \infty$, and define the rescalings

$$L_\tau^{X_i,\lambda_i} := \lambda_i \left(L_{t_i + \lambda_i^2 \tau} - x_i \right)$$

with Lagrangian angle θ_τ^i.

Then for any bounded parabolic region $\Omega \times I \subset \mathbb{C}^n \times \mathbb{R}$,

$$\theta_\tau^i(\chi_i) \to \overline{\theta} \quad \textit{uniformly in } \Omega \times I, \tag{5.19}$$

where $\tau \in I$ and $\chi_i \in L_\tau^{X_i,\lambda_i} \cap \Omega$ is any sequence of points.

Proof. We will be considering the following three flows:

- L_t, the original LMCF
- $L_s^{\sigma_i}$, the Type I rescaled LMCF with factor σ_i (σ_i to be explicitly decided later)
- $L_\tau^{X_i,\lambda_i}$, the LMCF rescaled around $X_i = (x_i, t_i)$ with factor λ_i.

The time variables t, s, τ are related by

$$s = \sigma_i^2 t, \;\; \tau = \lambda_i^2 (t - t_i) = -t_i \lambda_i^2 \left(1 - \frac{s}{\sigma_i^2 t_i} \right).$$

For a suitable choice of σ_i, the result (5.19) can be shown using the following sequence of steps. For all ε, there exists N independent of $\tau, (\chi_i)_{i=1}^\infty$ such that for all

$i \geq N$:

$$
\begin{aligned}
|\theta^i_\tau(\chi_i) - \overline{\theta}|^2 &= \int_{L^{X_i,\lambda_i}_\tau} |\theta - \overline{\theta}|^2 \Phi_{(\chi_i,\tau)} d\mathscr{H}^2 && (5.20)\\
&\leq \int_{L^{X_i,\lambda_i}_{(-t_i\lambda_i^2)(1+t_i^{-1}\sigma_i^{-2})}} |\theta - \overline{\theta}|^2 \Phi_{(\chi_i,\tau)} d\mathscr{H}^2 && (5.21)\\
&= \int_{L_{-\sigma_i^{-2}}} |\theta - \overline{\theta}|^2 \Phi_{\left(\lambda_i^{-1}\chi_i + x_i, \lambda_i^{-2}\tau + t_i\right)} d\mathscr{H}^2 \\
&= \int_{L^{\sigma_i}_{-1}} |\theta - \overline{\theta}|^2 \Phi_{\left(\sigma_i\left(\lambda_i^{-1}\chi_i + x_i\right), \sigma_i^2\left(\lambda_i^{-2}\tau + t_i\right)\right)} d\mathscr{H}^2 && (5.22)\\
&\leq \int_{L^{\sigma_i}_{-1}} |\theta - \overline{\theta}|^2 \Phi_{(0,0)} d\mathscr{H}^2 + \tfrac{\varepsilon}{2} && (5.23)\\
&\leq \varepsilon. && (5.24)
\end{aligned}
$$

The idea is that we have convergence of the Type I rescalings $L^{\sigma_i}_{-1}$, as well as convergence of their Lagrangian angles. To change to an integral over $L^{\sigma_i}_{-1}$, we first change to an integral over $L^{X_i,\lambda_i}_{\overline{\tau}}$, for a suitable choice of $\overline{\tau}$, using Huisken monotonicity.

We now proceed to justify each of these steps. To prove (5.21), notice that $|\theta - \overline{\theta}|^2$ is a subsolution to the heat equation. Also, since by assumption $t_i\lambda_i^2 \to -\infty$ as $i \to \infty$, if we pick our σ_i such that $\sigma_i < \frac{1}{2\sqrt{-t_i}}$ then

$$(-t_i\lambda_i^2)\left(1 + \tfrac{1}{t_i\sigma_i^2}\right) \to -\infty$$

for sufficiently large i. In particular, this quantity is eventually less than $\inf(I)$, so we may pick a uniform N such that for any $i \geq N$,

$$(-t_i\lambda_i^2)\left(1 + \tfrac{1}{t_i\sigma_i^2}\right) \leq \tau$$

for all $\tau \in I$. Then we can directly apply Huisken's monotonicity formula, Theorem 2.2.7.

To prove (5.22), we relate the integral over the Type II rescaling to the integral over the Type I rescaling. To do this, we apply Lemma 2.2.8 twice to relate the integral to the original flow at time $-\sigma_i^{-2}$, and then to the Type I rescaled flow at time -1.

To prove (5.23), we show that we can replace our spacetime-shifted heat kernel with the stationary one at $(0,0)$. As long as

$$\left(\sigma_i\left(\lambda_i^{-1}\chi_i + x_i\right), \sigma_i^2\left(\lambda_i^{-2}\tau + t_i\right)\right) \to (0,0),$$

we get smooth convergence of $\Phi_{\left(\sigma_i\left(\lambda_i^{-1}\chi_i+x_i\right),\sigma_i^2\left(\lambda_i^{-2}\tau+t_i\right)\right)}$ to $\Phi_{(0,0)}$. Then by Theorem 3.4.3 and the Type I blowup assumption, it follows that:

$$\begin{aligned} &\left| \int_{L_{-1}^{\sigma_i}} |\theta - \overline{\theta}|^2 \left(\Phi_{\left(\sigma_i\left(\lambda_i^{-1}\chi_i+x_i\right),\sigma_i^2\left(\lambda_i^{-2}\tau+t_i\right)\right)} - \Phi_{(0,0)} \right) d\mathscr{H}^2 \right| \\ \leq &\left| \Phi_{\left(\sigma_i\left(\lambda_i^{-1}\chi_i+x_i\right),\sigma_i^2\left(\lambda_i^{-2}\tau+t_i\right)\right)} - \Phi_{(0,0)} \right|_\infty \cdot \int_{L_{-1}^{\sigma_i}} |\theta - \overline{\theta}|^2 d\mathscr{H}^2 \longrightarrow 0. \end{aligned}$$

Convergence of the space-time points $\left(\sigma_i\left(\lambda_i^{-1}\chi_i + x_i\right), \sigma_i^2\left(\lambda_i^{-2}\tau + t_i\right)\right)$ will follow if we pick our σ_i correctly. For example,

$$\sigma_i := \frac{1}{2}\min\left\{\frac{1}{\sqrt[4]{-t_i}}, \frac{1}{\sqrt{x_i}}, \sqrt{\lambda_i}\right\}$$

will work for this step and for step 1. All stated convergences are uniform in χ_i and τ, since Ω and I are bounded regions. Finally, (5.24) follows from Theorem 3.4.3, just as in step (5.23). □

In particular, $-t_iA_i^2 \to \infty$ for a Type II singularity (see Section 2.2). Since the Type II rescalings converge smoothly to the limiting flow, and the Lagrangian angles of the Type I rescalings converge to $\overline{\theta}$ by Theorem 5.2.8, Lemma 5.2.13 implies that any Type II blowup of our flow must be a special Lagrangian.

Corollary 5.2.14. *Let L_t be an almost-calibrated, connected, equivariant Lagrangian MCF with Lagrangian angle θ_t that forms a singularity at the space-time point $(O,0)$. Assume that any sequence of Type I rescalings $L_s^{\sigma_i}$ converge subsequentially as flows to the same special Lagrangian cone C, with angle $\overline{\theta}$.*

Then any sequence of Type II rescalings, $L_\tau^{(x_i,t_i)}$ converges subsequentially in C_{loc}^∞

to a special Lagrangian, with Lagrangian angle $\overline{\theta}$.

We are now ready to prove that any Type II blowup of our equivariant flow must be the unique Lawlor neck with asymptotic planes $P_1 \cup P_2$.

Theorem 5.2.15. *Let L_t be an almost-calibrated, connected, $O(n)$-equivariant mean curvature flow in $\mathbb{C}^n$ with planar asymptotics. Then up to a translation, a Type II blowup of any finite-time singularity is a Lawlor neck Σ_{Law} with the same Lagrangian angle as the (unique) Type I blowup $P_1 \cup P_2$, and is asymptotically planar with asymptotes P_1 and P_2. Additionally, the Type II blowup does not depend on the rescaling sequence.*

Proof. Consider a sequence of Type II rescalings, $L_\tau^{(x_i,t_i)}$, that converge to a Type II blowup L_τ^∞, and denote by A_i the rescaling factors. We first show that we may assume $x_i \in \mathbb{C}\times\{0\}^{n-1}$, so that we may apply the theory from Section 5.1. Apply a sequence of rotations $R^i(\cdot)$ centred on the origin so that $R_i(x_i) \in \mathbb{C}\times\{0\}^{n-1}$, and pass to a subsequence so that this sequence of rotations converges in C^∞ to a rotation R_∞. Then, since we are working with equivariant Lagrangians,

$$\begin{aligned}
L_\tau^{(R_i(x_i),t_i)} &= A_i\left(L_{t_i+A_i^{-2}\tau} - R_i(x_i)\right)\\
&= A_i\left(R_i(L_{t_i+A_i^{-2}\tau}) - R_i(x_i)\right)\\
&= R_i(L_\tau^{(x_i,t_i)})\\
&\to R_\infty(L_\tau^\infty),
\end{aligned}$$

so up to a rotation we obtain the same limit if we use the sequence $R_i(x_i)$ instead of x_i.

Now, we know from Corollary 5.2.14 that the Type II blowup L_τ^∞ is a special Lagrangian, i.e. a static flow with $\theta = \overline{\theta}$. So we only need look at L_0^∞ to understand the entire flow. There are now two cases to consider: either the image of the origin $-A_i x_i$ remains bounded under the Type II rescalings, or $|A_i x_i|$ diverges to ∞.

If $|A_i x_i| \to \infty$, then (on passing to a subsequence) L_0^∞ is invariant under translations T_v (as defined in Section 5.1) by Lemma 5.1.6, where $v = ze_1 := \lim_{i\to\infty} \frac{-A_i x_i}{|A_i x_i|} \in \mathbb{C}\times\{0\}^{n-1}$, for a constant $z \in \mathbb{C}$. Therefore, referring to (5.5) for the definition of T_v) at the point $l_0^\infty(s)e_1 \in L_0^\infty \cap (\mathbb{C}\times\{0\}^{n-1})$

$$\frac{\partial}{\partial\lambda}\bigg|_{\lambda=0} T_v(\alpha,\lambda,\gamma(s)) = -z\alpha$$

is a tangent direction, for any $\alpha \in S^{n-2}$. In particular, $ze_i = (0,\ldots,z,\ldots,0)$ is a tangent direction at every point in $L_0^\infty \cap (\mathbb{C}\times\{0\}^{n-1})$ for all $i \neq 1$. So, if the profile curve of the Type II blowup is $l_0^\infty(s) = a(s) + ib(s)$,

$$\arg\left(\begin{vmatrix} a'+ib' & 0 & \cdots & 0 \\ 0 & z & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & z \end{vmatrix}\right) = \overline{\theta} \implies \arg(a'+ib') = \overline{\theta} - (n-1)\arg(z),$$

which implies that l_0^∞ is a straight line through the origin, and that L_0^∞ is an n-plane. But since Type II blowups must satisfy $\max|A|^2 = 1$, this is a contradiction.

It follows that $|A_i x_i|$ remains bounded. In this case it follows from Lemma 5.1.6 and C^∞_{loc}-convergence of the Type II rescalings that L_0^∞ is an $O(n)$-equivariant submanifold of $\mathbb{C}^n$, after a translation by an element of $\mathbb{C}\times\{0\}^{n-1}$. Therefore by Lemma 5.1.5, it must be a Lawlor neck Σ_{Law}. The uniqueness follows from Lemma 5.2.7, as there is only one Lawlor neck with $\sup|A| = 1$ and Lagrangian angle $\overline{\theta}$ that fits in the cone containing γ. □

5.2.6 Intermediate Blowups

Finally, we examine the behaviour between the Type I and Type II scales of a finite time singularity of our LMCF. Assume our flow forms a singularity at the space-time point $(0,0)$, consider a sequence of times $t_i \to 0$ from a Type II rescaling sequence (i.e. satisfying (2.14)), and let A_i be the maximum value of the second fundamental form over L_{t_i}, as before. Let $\lambda_i \in \mathbb{R}$ be a sequence diverging to $+\infty$ such that

$$\delta_i := \frac{\lambda_i}{A_i} \to 0, \quad -\lambda_i^2 t_i \to \infty.$$

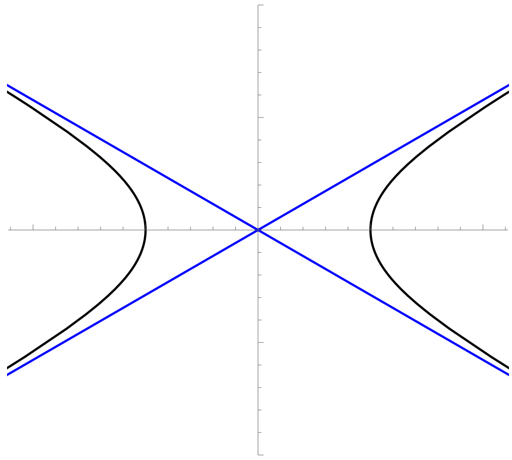


Figure 5.10: The profile curves of the Type I and Type II blowups for an equivariant LMCF in $\mathbb{C}^3$.

Then we define the **intermediate rescalings** corresponding to the sequence (t_i, λ_i) as

$$L_\tau^{t_i,\lambda_i} := \lambda_i L_{t_i+\lambda_i^{-2}\tau}.$$

Note that we need not translate the rescaling to centre on the point of highest curvature - we proved in Theorem 5.2.15 that the origin remains bounded along the sequence, so any convergence will be unaffected by such translations. The assumptions that $\delta_i := \frac{\lambda_i}{A_i} \to 0$ and $-\lambda_i^2 t_i \to \infty$ are made since otherwise the resulting blowup will just be a scaling of a Type II blowup or a time-translation of a Type I blowup respectively. We prove the following:

Theorem 5.2.16. *Let L_t be an almost-calibrated, connected, $O(n)$-equivariant mean curvature flow in $\mathbb{C}^n$ with planar asymptotics. Assume that L_t forms a singularity at the origin at time $t = T$, with Type I blowup $P_1 \cup P_2$. Then for any R, ε and finite time interval I, there exists a subsequence such that $L_\tau^{t_i,\lambda_i} \cap (B_R \backslash B_\varepsilon)$ may be expressed as a graph over $P_1 \cup P_2$ for $\tau \in I$, and this graph converges in $C^{1;0}$ to $P_1 \cup P_2$.*

Proof. Extend I so that it contains 0 in its interior, and pass to a subsequence so that the Type II rescalings centred at the space-time points (O, t_i) converge smoothly to a Type II blowup. By Theorem 5.2.13, on the cylinder $B_R \times I$ the Lagrangian angle θ^i of the intermediate rescalings is converging uniformly to a constant $\overline{\theta}$, the same value as the Lagrangian angles of the Type I and Type II blowups. For convenience we assume $\overline{\theta} =$

$\frac{\pi}{2}$, that the profile curve of the Type I blowup is the pair of lines at $\alpha = \frac{\pi}{2n}$ and $\alpha = -\frac{\pi}{2n}$ (and their reflections in O), and that the Type II blowup is the unique Lawlor neck with $\sup|A| = 1$ asymptotic to these planes, as in Figure 5.10 (this can all be achieved by a single rotation of the plane $\mathbb{C}\times\{0\}^{n-1}$).

If ε is small enough, and we take i large enough so that $|\theta^i - \frac{\pi}{2}| < \varepsilon$, then on $B_R \times I$ there is at most one intersection of each component of l^i with the real axis. Denote by b^i_τ the sequence of intersections on the positive real axis at time τ, where it exists, and by γ^i_τ the component of l^i_τ containing b^i_τ. We first prove that we have the expected convergence on individual time slices.

Lemma 5.2.17. *Fix a sequence $\tau_i \in I$.*

- *If $b^i_{\tau_i} \to 0$, then for all ε, the profile curves $\gamma^i_{\tau_i}$ parametrised by arc-length converge in C^1 on $B_R \backslash B_\varepsilon$ to the half-lines at $\alpha = \frac{\pi}{2n}$ and $\alpha = -\frac{\pi}{2n}$.*
- *If $b^i_{\tau_i} \to B > 0$, then the profile curves $\gamma^i_{\tau_i}$ parametrised by arc-length or by argument converge in C^1 on B_R to the profile curve of the Lawlor neck σ_{law}, with asymptotes given by these same lines.*

Proof. Throughout, we suppress the subscript τ_i, as nothing depends on it.

We tackle case 2 first, so $b^j \to B > 0$. Take N large enough such that on $B_R \times I$ for $j > N$,

$$|\theta^j - \tfrac{\pi}{2}| < \varepsilon, \quad |b^j - B| < \varepsilon.$$

Note that, close to $\alpha = 0$, by the above condition on θ^j, we may parametrise γ^j by angle:

$$\begin{aligned} \gamma^j(\alpha) &= r^j(\alpha)e^{i\alpha} \\ \implies \dot{\gamma}^j &= (\dot{r}^j + ir^j)e^{i\alpha} \\ \implies \dot{r}^j &= r^j \cot(\theta^j - n\alpha), \end{aligned}$$

where $\cot : (0,\pi) \to \mathbb{R}$. In fact, by this gradient equation, we see that it is parametrisable in this fashion for

$$\alpha \in \left(-\frac{\pi}{2n} + \frac{\varepsilon}{n}, \frac{\pi}{2n} - \frac{\varepsilon}{n}\right).$$

Integrating the inequality obtained by using the bound on θ^j, it follows that for $\alpha > 0$:

$$\frac{(B-\varepsilon)\sqrt[n]{\sin\left(\frac{\pi}{2}+\varepsilon\right)}}{\sqrt[n]{\sin\left(\frac{\pi}{2}+\varepsilon-n\alpha\right)}} \leq r^j(\alpha) \leq \frac{B+\varepsilon}{\sqrt[n]{\sin\left(\frac{\pi}{2}-\varepsilon-n\alpha\right)}}.$$

This implies that, as $\varepsilon \to 0$ and $j \to \infty$,

$$r^j\left(\frac{\pi}{2n}-\frac{\varepsilon}{n}\right) \to \infty.$$

An identical argument shows that the same is true for the $\alpha < 0$ half of the curve. Therefore, on B_R the curve may be fully parametrised by angle for sufficiently large j, and so this parametrisation converges in C^1 to

$$r^\infty = \frac{B}{\sqrt[n]{\sin\left(\frac{\pi}{2}-n\alpha\right)}},$$

which is the Lawlor neck described in the statement of the lemma.

Now assume that $b^j \to 0$. By the same method as above, we see that the curve is parametrisable by angle for the same range of α, and for $\alpha > 0$ in this range,

$$\frac{b^j\sqrt[n]{\sin\left(\frac{\pi}{2}+\varepsilon\right)}}{\sqrt[n]{\sin\left(\frac{\pi}{2}+\varepsilon-n\alpha\right)}} \leq r^j(\alpha) \leq \frac{b^j}{\sqrt[n]{\sin\left(\frac{\pi}{2}-\varepsilon-n\alpha\right)}}.$$

Since $b^j \to 0$, for each ε we may choose N large so that for $j > N$, $r^j < \varepsilon$ on the angle range $\left(-\frac{\pi}{2n}+\varepsilon, \frac{\pi}{2n}-\varepsilon\right)$ and $|\theta^j - \frac{\pi}{2}| < \varepsilon$ on $B_R \times I$. Therefore for $j > N$, the curve enters the cone $\Gamma := \{\alpha \in \left(\frac{\pi}{2n}-\varepsilon, \frac{\pi}{2n}+\varepsilon\right)\}$ within the ball B_ε. Now we show that it remains there while in B_R (an identical argument holds for the cone on the other side of the real axis, $\Gamma' := \{\alpha \in \left(-\frac{\pi}{2n}-\varepsilon, -\frac{\pi}{2n}+\varepsilon\right)\}$). Once the curve has entered the cone Γ, if it intersected the line $\alpha = \frac{\pi}{2n}-\varepsilon$ again, then at this point we would have

$$\arg(\dot{\gamma}^j) \leq \frac{\pi}{2n}-\varepsilon \quad \implies \quad \theta^j \leq \frac{\pi}{2}-n\varepsilon,$$

which is a contradiction. A similar contradiction is reached if we assume that the curve intersects the line $\alpha = \frac{\pi}{2n} + \varepsilon$, therefore the curve must remain in the cone Γ once it enters. Now, parametrising the curve by arc-length so that

$$\gamma^j(s) = r^j(s)e^{i\alpha^j(s)} \quad \Longrightarrow \quad \dot{\gamma}^j = e^{i(\theta^j-(n-1)\alpha^j)};$$

limiting $\varepsilon \to 0$ shows that our curves γ^j converge in C^1 away from the origin to the specified half-lines. □

To finish the proof, we need to show that $b^i_\tau \to 0$ uniformly in I. The above lemma will then show that our intermediate rescalings converge uniformly to the pair of planes we expect. We know from the Type II convergence that $b^i_0 \to 0$, and so it suffices to show that the value b^i is a C^0-Cauchy sequence as a function of time. Intuitively, the argument is that if the Lagrangian angle is converging uniformly to a constant, then the 'average' value of H also is. This puts a limit on how far the profile curve can travel between times, which prevents b^i converging to two different values.

Lemma 5.2.18. *b^i_τ is a C^0-Cauchy sequence of functions in τ, converging to* 0.

Proof. Assume for a contradiction that it isn't a Cauchy sequence. We know that $b^i_0 \to 0$, so this means that there exists $B \in \mathbb{R}^+$, $B < \frac{R}{2}$ such that, on passing to a subsequence,

$$\sup_{\tau \in I} |b^i_\tau| > B.$$

Take a sequence $\tau_i \in I$ such that $b^i_{\tau_i} = B$; we assume for notational convenience that τ_i is negative. Denote by σ_B the profile curve of the Lawlor neck intersecting the real axis at B, and by v the two half-lines, both as described in Lemma 5.2.17. Then by this lemma we may take N sufficiently large such that for all $i \geq N$:

- $|\theta - \frac{\pi}{2}| < \varepsilon$ in $B_R \times I$,
- $d_{Haus}(\gamma^i_{\tau_i} \cap B_R, \sigma_B \cap B_R) < \varepsilon$,
- $d_{Haus}(\gamma^i_0 \cap B_R, v \cap B_R) < \varepsilon$.

If we denote by $B_\varepsilon(A)$ the ε-fattening of the set A, then this means that for $i \geq N$, $\gamma^i_0 \subset B_\varepsilon(v)$ and $\gamma^i_{\tau_i} \subset B_\varepsilon(\sigma_B)$. Let $d(\varepsilon) := d_{Haus}(B_\varepsilon(\sigma_B) \cap B_R, B_\varepsilon(v) \cap B_R)$, and notice that it is a decreasing function of ε.

Taking ε sufficiently small, we may find $p_1 \leq p_2 \in \mathbb{R}$ such that $\gamma^i_\tau(p_1), \gamma^i_\tau(p_2) \in B_R$ for all $\tau \in I$. Such points must exist, else by an identical argument to the one given below, the integral of $\vec{H}$ over an escaping region of the curve would be large, contradicting the uniform bound on θ. Now take ν to be the outward pointing normal and s the arc-length parameter. Then since the flow must travel from $B_\varepsilon(\sigma_B)$ to $B_\varepsilon(v)$ between times 0 and τ_i, it follows by the definition of mean curvature flow that

$$\begin{aligned}
&\forall p \in [p_1, p_2],\ \int_{\tau_i}^0 H(p) \cdot \nu(p)\, d\tau \geq d(\varepsilon) \\
&\quad \Longrightarrow \int_{\tau_i}^0 \int_{\gamma^i_\tau([p_1,p_2])} H \cdot \nu\, ds\, d\tau \geq d(\varepsilon) \cdot \min_{\tau \in [\tau_i, 0]} \mathscr{H}^1(\gamma^i_\tau[p_1, p_2]).
\end{aligned}$$

But on the other hand, since $\vec{H} = J\nabla\theta$,

$$\begin{aligned}
\left| \int_{\tau_i}^0 \int_{\gamma^i_\tau([p_1,p_2])} H \cdot \nu\, ds\, d\tau \right| &= \left| \int_{\tau_i}^0 \int_{\gamma^i_\tau([p_1,p_2])} \frac{\partial \theta^i_\tau}{\partial s}\, ds\, d\tau \right| \\
&= \left| \int_{\tau_i}^0 \theta^i_\tau(p_2) - \theta^i_\tau(p_1)\, d\tau \right| \leq -2\tau_i \varepsilon,
\end{aligned}$$

which is a contradictory inequality if ε is taken small. □

This completes the proof of Theorem 5.2.16. □

Chapter 6

Equivariant Examples of LMCF with Boundary in $\mathbb{C}^2$

We now combine the interests of the previous chapters, and illustrate the behaviour of the Lagrangian mean curvature flow with boundary (Chapter 4) in the particular case of S^1-equivariant Lagrangian submanifolds of $\mathbb{C}^2$ (Chapter 5). To do this, we focus on two particular boundary conditions – the Lawlor neck Σ_{Law} (Example 3.3.1) and the Clifford torus Σ_{Cliff} (Example 3.3.2). Our boundary condition for Lagrangian mean curvature flow can expressed in terms of the Lagrangian angle θ, and θ takes a simple form on these two flows; in particular it is not time-dependent. These are therefore natural and relatively simple examples to study. In both cases, we prove a long-time existence and smooth convergence result – of the original flow in the case of the Lawlor neck, and of a rescaled flow in the case of the Clifford torus.

We now go over the results of this chapter. The Lawlor neck Σ_{Law} (see Example 3.3.1 and Figure 4.1) is the only non-flat equivariant special Lagrangian in $\mathbb{C}^2$, and is therefore static under the mean curvature flow; this makes it a good choice of boundary manifold for our flow. We prove that any solution to (4.1) satisfying the almost-calibrated condition (defined in Section 2.4) with boundary on the static Lawlor neck exists for all time and converges smoothly to a special Lagrangian. This convergence is depicted for two specific initial conditions in Figure 4.1.

The results of Chapter 6 comprise original joint work with B. Lambert and C. Evans, and appear in the preprint [20].

Theorem 6.0.1. *Let F_0 be an almost-calibrated S^1-equivariant Lagrangian embedding of the disc D^2 into $\mathbb{C}^2$ with boundary on the static Lawlor neck, Σ_{Law}, such that the Lagragian angle of L_0, θ_0, satisfies $\theta_0|_{\partial L_0} = -\alpha$. Then there exists a unique, immortal solution to the LMCF problem (4.1), and it converges smoothly in infinite time to a special Lagrangian disc.*

Our other choice of boundary manifold is the Clifford torus (see Example 3.3.2 and Figure 4.2). The symmetry of the Clifford torus is preserved under mean curvature flow, so it is a self-shrinking solution, and is therefore static under the rescaled flow (defined in Section 6.2). Here, the condition $\theta - 2\arg(\gamma) \in (-\frac{\pi}{2}+\varepsilon, \frac{\pi}{2}-\varepsilon)$ is a natural preserved condition to consider in place of the almost-calibrated condition, as $\theta - 2\arg(\gamma)$ always vanishes on the boundary. Given this condition, we show a long-time existence and convergence result for the rescaled flow in the $\alpha = 0$ case, as depicted in Figure 4.2a.

Theorem 6.0.2. *Let $\overline{F}_0 : D \to \mathbb{C}$ be an S^1-equivariant Lagrangian embedding of a disc D, with boundary on the* Clifford torus, *Σ_{Cliff}. Assume that its Lagragian angle θ_0 satisfies*

$$\theta_0(s) - 2\arg(\overline{\gamma}_0(s)) \in (-\tfrac{\pi}{2}+\varepsilon, \tfrac{\pi}{2}-\varepsilon)$$

for some $\varepsilon > 0$, and that $\theta_0 - 2\arg(\gamma_0) = 0$ on ∂L_0. Then there exists a unique, eternal solution to the rescaled LMCF problem (6.2.1) *(corresponding to (4.1) with $\alpha = 0$), which converges smoothly in infinite time to a special Lagrangian disc.*

In the case of the Clifford torus, numerical evidence suggests that a rescaled solution of (4.1) with $\alpha \neq 0$ exists for all time and converges to a unique rotating soliton - see Figure 4.2b. This is in contrast to a standard Lagrangian mean curvature flow, where the blow-up of a singularity would result in a self-shrinking flow.

6.1 The Lawlor Neck

Our first example is an LMCF with boundary on the Lawlor neck, which has constant Lagrangian angle $\tilde{\theta} = \frac{\pi}{2}$. It follows that the boundary condition of (4.1) is equivalent to

$$\theta\big|_{\partial L} = -\alpha.$$

We prove the following long-time existence result, depicted in Figure 4.1.

Theorem 6.1.1. *Let L_0 be an S^1-equivariant Lagrangian embedding of the disc D^2 into $\mathbb{C}^2$ with Lagrangian angle θ_0 satisfying*

$$\theta_0(s) \in (-\tfrac{\pi}{2}+\varepsilon, \tfrac{\pi}{2}-\varepsilon)$$

for some $\varepsilon > 0$, with boundary on the Lawlor neck with profile curve $\sigma_{\text{Law}} = \{(\pm\cosh(\phi), \sinh(\phi)) : \phi \in \mathbb{R}\}$, and with $\theta_0|_{\partial L_0} = -\alpha$ (as in Figures 4.1a and 4.1b). Then there exists a unique, immortal solution to the LMCF problem:

$$\begin{cases} \left(\frac{d}{dt}F(x,t)\right)^{NM} = H(x,t) & \textit{for all } (x,t) \in D \times [t_0,\infty) \\ F(x,t_0) = L_0(x) & \textit{for all } x \in D \\ \partial L_t \subset \Sigma_{\text{Law}} & \textit{for all } t \in [t_0,\infty) \\ \theta_t|_{\partial L_t} = -\alpha & \textit{for all } (x,t) \in \partial D \times [t_0,\infty), \end{cases} \tag{6.1}$$

and it converges smoothly in infinite time to the disc with profile curve $\gamma^\infty(s) = (s, s\tan(\frac{-\alpha}{2}))$.

Remark 6.1.2. *The 'almost-calibrated' condition $\theta_0 \in (-\frac{\pi}{2}+\varepsilon, \frac{\pi}{2}-\varepsilon)$ is necessary, as there exist Lagrangian discs which are not almost-calibrated but which form a finite-time singularity under the flow – see [51] for an example.*

If the profile curve γ_t does not pass through the origin, i.e. if the topology of the flow is not a disc, then a finite-time singularity will form. For example one can prove using the barriers of this section that any curve that does not initially pass through the origin must approach the origin as $t \to \infty$, and therefore by the equivariance the curvature $|A|^2$ must blow up.

6.1.1 Parametrisation

For simplicity, we work throughout with the profile curves of our flow and the boundary manifold, and we will work with the following parametrisation for the profile curve. Consider the foliation

$$Y(s,\phi) := (s\cosh(\phi), s\sinh(\phi))$$

and graphs of the form

$$\begin{aligned}\gamma_t(s) &= Y(s, v_t(s)) = (s\cosh(v_t(s)), s\sinh(v_t(s))) \\ \gamma_t'(s) &= (\cosh(v_t(s)) + s v_t'(s)\sinh(v_t(s)), \sinh(v_t(s)) + s v_t'(s)\cosh(v_t(s))).\end{aligned} \tag{6.2}$$

In this parametrisation, using the equation for equivariant mean curvature flow (5.6), the problem (6.1) is reduced to the following boundary value problem:

$$\begin{cases} \frac{\partial v}{\partial t} = \frac{v'' + 2s^{-1}v' - s(v')^3}{|\gamma'|^2} + \frac{v'}{s\cosh(2v)} & \text{for } s \in [-1,1],\ t \geq t_0, \\ v(s,t_0) = v_0 & \text{for } s \in [-1,1], \\ sv'(s,t) = \frac{\tan(-\alpha)}{\cosh(2v(s,t))} - \tanh(2v(s,t)) & \text{for } s \in \{-1,1\},\ t \geq t_0. \end{cases} \tag{6.3}$$

Note that this PDE problem is uniformly parabolic away from the origin, if we can bound $|\gamma'|$ and $|\gamma| = s\cosh(2v)$. We must also show that this parametrisation is valid for our problem.

6.1.2 The Lagrangian Angle and C^1 Bounds

The Lagrangian angle for an equivariant LMCF is given by

$$\theta(s) = \arg(\gamma) + \arg(\gamma').$$

It is an important quantity, because on the interior of the abstract manifold it has very simple evolution equations:

$$\frac{\partial \theta}{\partial t} = \Delta\theta, \qquad \frac{\partial(\theta)^2}{\partial t} = -2|H|^2 + \Delta(\theta)^2. \tag{6.4}$$

Lemma 6.1.3. *A solution of (6.1) on $[t_0, T)$ which satisfies $\theta \in (-\frac{\pi}{2} + \varepsilon, \frac{\pi}{2} - \varepsilon)$ at the initial time, satisfies this condition for all $t \in [t_0, T)$.*

Proof. The boundary conditions on our flow are $\theta\big|_{\partial L} = -\alpha$. Therefore by (6.4), θ solves the Dirichlet problem for the heat equation on the abstract manifold, and by the parabolic maximum principle must be bounded by its initial values. □

We will now show that our flow may be parametrised using the parametrisation (6.2) for as long as the flow exists, and derive C^1 bounds on the graph function v away from the origin. Certainly it may be parametrised in this way on a small ball B around the origin, since at the origin we have the identity

$$\theta = 2\arg(\gamma'),$$

and so it follows from the almost-calibrated condition for θ that, on B, the curve intersects the Lawlor neck foliation $Y(s,\phi)$ transversely. On this ball B, using (6.2) and the almost-calibrated condition:

$$\begin{aligned}
\theta(s) &= \arg(\gamma) + \arg(\gamma') = \arg(\gamma\gamma') \\
&= \arg\left(s + i\left(s\sinh(2v) + s^2 v' \cosh(2v)\right)\right) \\
\implies \tan(\theta) &= \sinh(2v) + sv'\cosh(2v) \\
\implies sv' &= \frac{\tan(\theta)}{\cosh(2v)} - \tanh(2v) \qquad (6.5) \\
\implies |\gamma'(s)| &\le \left(1 + s|v'|\right)\left(\cosh(v) + |\sinh(v)|\right) \\
&\le \left(1 + \frac{|\tan(\theta)|}{\cosh(2v)} + |\tanh(2v)|\right)(\cosh(v) + |\sinh(v)|). \qquad (6.6)
\end{aligned}$$

This will give us a uniform C^1 bound for v on any annulus centred at the origin, if we can parametrise globally in this way, and bound the function v.

Lemma 6.1.4. *Let γ be the profile curve of an equivariant Lagrangian submanifold $L \subset \mathbb{C}^2$ with boundary on the Lawlor neck, satisfying $\theta \in (-\frac{\pi}{2} + \varepsilon, \frac{\pi}{2} - \varepsilon)$. Then one connected component of the curve $\gamma \setminus \{O\}$ is parametrisable using the parametrisation* (6.2)*, and satisfies*

$$\arg(\gamma) \in (-\tfrac{\pi}{4} + \tfrac{\varepsilon}{2}, \tfrac{\pi}{4} - \tfrac{\varepsilon}{2}),$$
$$v \in (-V, V),$$

for $V = \tanh^{-1}(\tan(\frac{\pi}{4} - \frac{\varepsilon}{2})) < \infty$. The other connected component satisfies analogous bounds.

Proof. At the origin, we must have $\arg(\gamma'(0)) \in (-\frac{\pi}{4} + \frac{\varepsilon}{2}, \frac{\pi}{4} - \frac{\varepsilon}{2})$ (for one choice of orientation) by the bound on θ, therefore for small s the curve is parametrisable by (6.2), and the first bound holds. If there was some smallest s_0 such that

$$\arg(\gamma(s_0)) = \frac{\pi}{4} - \frac{\varepsilon}{2},$$

that at this point,

$$\arg(\gamma'(s_0)) \geq \frac{\pi}{4} - \frac{\varepsilon}{2} \implies \theta(s_0) \geq \frac{\pi}{2} - \varepsilon$$

which is a contradiction. An identical argument works for the lower bound, and so the first statement is proven.

For the second, note that in the foliation $Y(s,\phi) = (s\cosh(\phi), s\sinh(\phi))$, the line of constant argument α satisfies

$$\tan(\alpha) = \frac{\sinh(\phi)}{\cosh(\phi)} = \tanh(\phi) \implies \phi = \tanh^{-1}(\tan(\alpha)),$$

therefore lines of constant angle are equivalent to lines of constant ϕ, with the above correspondence. The first bound then implies the second, for as long as the parametrisation is valid. Finally, this bound on v, along with (6.5), proves that v' is bounded on any annulus – therefore the parametrisation is valid for all $s > 0$. The other half of the curve γ is a reflection of the first in the origin, by the equivariance, and so analogous results hold. □

Using this lemma, (6.6) implies that $|\gamma'(s)| < C_1$, for some uniform constant C_1. We can use this to derive the following density bound on small balls, which will be useful later:

$$\int_{B_\delta \cap \gamma_t} d\mathscr{H}^1 \leq \int_{-\delta}^{\delta} |\gamma'(s)| ds \leq 2\delta C_1.$$

6.1.3 Long-Time Existence

Using the mean curvature flow equation (6.3), and the C^1 bounds we just derived, we can now prove long-time existence.

Lemma 6.1.5. *A finite-time singularity for a solution of (6.1) cannot occur.*

Proof. By (6.6), the mean curvature flow equation (6.3) is uniformly parabolic on any annulus centred at the origin. Therefore, Schauder estimates give a bound on all curvatures for as long as the flow exists, and so a singularity cannot occur away from the origin.

Unfortunately, the equation (6.3) degenerates at the origin, so this case must be dealt with separately. Assume that a singularity occurs at the origin at time 0, and let L_t^i, γ_t^i be the Type I rescalings of the rotated flow and their profile curves around this singularity with factor λ^i, defined by

$$L_t^i := \lambda^i L_{(\lambda^i)^{-2}t}.$$

We will show that the density of γ_t^i converges to 1, and then White's local regularity theorem will imply that the curvatures are bounded, contradicting the assumption of a singularity at $(O,0)$.

Lemma 6.1.6. *Let L_t^i be a sequence of rescalings of an equivariant LMCF $L_t \subset \mathbb{C}^2$ around the space-time point $(O,0)$. Assume that $\partial L_t^i \to \infty$ as $i \to \infty$, uniformly on the time interval $[t_0,0)$, and assume also that the flow is uniformly bounded in C^3 on ∂L_t.*

Then for any $a < b < 0$ and $R > 0$,

$$\lim_{i\to\infty} \int_a^b \int_{L_t^i \cap B_R} \left(|H|^2 + |x^\perp|^2\right) d\mathscr{H}^2 = 0.$$

Proof. We need the following version of Huisken's monotonicity formula, which holds for flows M_t^n with boundary. For a space-time point $X := (x_0, t_0)$,

$$\frac{\partial}{\partial t}\int_{M_t} f\Phi_X \, d\mathscr{H}^n = \int_{M_t} \Phi_X \left(\frac{\partial f}{\partial t} - \Delta^M f - f\left| \vec{H} + \frac{(x-x_0)^\perp}{2(t_0-t)} \right|^2 \right) d\mathscr{H}^n + \int_{\partial M_t} \Phi_X \left\langle f\frac{x-x_0}{2(t_0-t)} + \nabla^M f, \nu \right\rangle d\mathscr{H}^{n-1}. \quad (6.7)$$

This formula is derived the same way as the standard monotonicity formula, but there are extra boundary terms from use of the divergence theorem. Using (6.4),(6.7) and denoting by Φ the monotonicity kernel centred at $(O,0)$,

$$\frac{\partial}{\partial t}\int_{L_t^i} \Phi d\mathscr{H}^2 = -\int_{L_t^i} \Phi\left| \vec{H} - \frac{x^\perp}{2t} \right|^2 d\mathscr{H}^2 + \int_{\partial L_t^i} \Phi\left\langle -\frac{x}{2t}, \nu \right\rangle d\mathscr{H}^1, \quad (6.8)$$

$$\frac{\partial}{\partial t}\int_{L_t^i} (\theta_t^i)^2 \Phi d\mathscr{H}^2 = \int_{L_t^i} \Phi\left(-2|H|^2 - (\theta_t^i)^2\left| \vec{H} - \frac{x^\perp}{2t} \right|^2 \right) d\mathscr{H}^2 + \int_{\partial L_t^i} \Phi\left\langle -(\theta_t^i)^2\frac{x}{2t} + \nabla^M(\theta_t^i)^2, \nu \right\rangle d\mathscr{H}^1.$$

Therefore,

$$\lim_{i\to\infty} 2\int_a^b \int_{L_t^i} |H|^2 \Phi d\mathscr{H}^2 dt \le \lim_{i\to\infty}\left(\int_{L_a^i} (\theta_a^i)^2 \Phi d\mathscr{H}^2 - \int_{L_b^i} (\theta_b^i)^2 \Phi d\mathscr{H}^2 + \int_a^b \int_{\partial L_t^i} \Phi\left\langle -(\theta_t^i)^2\frac{x^\perp}{2t} + \nabla^M(\theta_t^i)^2, \nu \right\rangle d\mathscr{H}^1 dt \right).$$

The boundary ∂L_t^i is a circle, radius $d^i(t) > \mu^i$ for $\mu^i \to \infty$ independent of t, and circumference $2\pi d^i(t)$. Additionally, the Lagrangian angle and its derivative are bounded on ∂L_t^i by the assumed C^3 bound, so we can estimate the last integral using a constant C depending only on this bound. Using this, and relating the first two integrals to the original flow by scaling invariance of the heat kernel,

$$\begin{aligned}
&\lim_{i\to\infty} 2\int_a^b \int_{L_t^i} |H|^2 \Phi d\mathscr{H}^2 dt \\
&\le \lim_{i\to\infty}\left(\int_{L_{(\lambda^i)^{-2}a}} (\theta_{(\lambda^i)^{-2}a})^2 \Phi d\mathscr{H}^2 - \int_{L_{(\lambda^i)^{-2}b}} (\theta_{(\lambda^i)^{-2}b})^2 \Phi d\mathscr{H}^2 \right. \\
&\qquad \left. + C\int_a^b 2\pi d^i(t) e^{\frac{-d^i(t)^2}{2t}} (d^i(t)+1) dt \right).
\end{aligned}$$

This limit is equal to 0, since by Huisken monotonicity with boundary (6.7) the first two terms cancel in the limit and by assumption $d^i(t) \to \infty$. It can similarly be shown using (6.8) that

$$\lim_{i\to\infty} \int_a^b \int_{L_t^i} \left| H - \frac{x^\perp}{2t} \right|^2 \Phi d\mathscr{H}^2 dt = 0,$$

and since on $B_R \times [a,b]$ we can estimate Φ from below, these together imply the result.

□

We now continue with the proof. Note that Schauder estimates applied to the graph equation (6.3) imply that our flow has uniformly bounded curvatures at the boundary, and since the Lawlor neck is static, it diverges to infinity under any sequence of rescalings – therefore Lemma 6.1.6 may be applied. Consider the set

$$K := \{(s\cosh(v), s\sinh(v)) | s \in [-R,R], v \in [-V,V]\};$$

where V is the constant from Lemma 6.1.4. K must contain $\gamma_t \cap (B_R \backslash B_\delta)$ for any t and δ. The set K is itself contained in a larger ball, $B_{\tilde{R}}$, and on this ball we can apply Lemma 6.1.6 to show that, for almost all t,

$$\int_{\gamma_t^i \cap B_{\tilde{R}}} |\gamma^\perp|^2 d\mathscr{H}^1 \to 0$$

as $i \to \infty$ (where we suppress the superscript i for readability). Therefore,

$$\int_{\gamma_t^i \cap B_{\tilde{R}}} |\gamma^\perp|^2 d\mathscr{H}^1 \geq 2\int_\delta^R \frac{s^4 (v_t')^2}{|\gamma_t'|^2} ds \geq \frac{2\delta^4}{C_1^2} \int_\delta^R (v_t')^2 ds \to 0.$$

It follows by Hölder's inequality that $v \to \bar{v} \in \mathbb{R}$ uniformly as $i \to \infty$, and that $v^{-1}(\gamma_t^i \cap B_R) \to \left[-\frac{R}{\sqrt{\cosh(2\bar{v})}}, \frac{R}{\sqrt{\cosh(2\bar{v})}} \right]$. Now fixing $r > 0$ and using a localised heat kernel Φ^ρ supported in B_R and centred at the origin 0, we use this L^2 estimate and the co-area formula to calculate the localised Gaussian density ratio:

$$\begin{aligned}
\lim_{i\to\infty}\Theta^\rho(L^i,0,r) &= \lim_{i\to\infty}\int_{L^i_{-r^2}}\Phi^\rho d\mathscr{H}^2 \\
&= \lim_{i\to\infty}\int_\delta^{\frac{R}{\sqrt{\cosh(2\bar{v})}}}\Phi^\rho(\gamma^i,-r^2)2\pi|\gamma||\gamma'|ds + \lim_{i\to\infty}\int_0^\delta \Phi^\rho(\gamma^i(s),-r^2)2\pi|\gamma||\gamma'|ds \\
&\leq \lim_{i\to\infty}\int_\delta^{\frac{R}{\sqrt{\cosh(2\bar{v})}}}\Phi^\rho(\gamma^i,-r^2)2\pi s\sqrt{\cosh(2v)} \\
&\qquad\qquad \times\sqrt{(1+s^2(v')^2)\cosh(2v)+2s(v')\sinh(2v)}ds + C\delta \\
&\leq \int_0^{\frac{R}{\sqrt{\cosh(2\bar{v})}}} 2\pi s\cosh(2\bar{v})\Phi^\rho(s\cosh(\bar{v})+is\sinh(\bar{v}),-r^2)\,ds + C\delta \\
&= \int_0^R 2\pi\sigma\Phi^\rho(\sigma,-r^2)\,d\sigma + C\delta \\
&= \int_{D_R}\Phi^\rho(\cdot,-r^2)d\mathscr{H}^2 + C\delta \quad = \quad 1+C\delta,
\end{aligned}$$

for $D_R := \{(s\cos(\psi), s\sin(\psi)) \in \mathbb{C}^2 \mid s < R,\ \psi \in [0,2\pi]\}$, where the last line follows from the fact that Φ^ρ is normalised to integrate to 1 over a plane. $\Theta^\rho(L^i,0,r)$ can therefore be made as close to 1 as desired, by choosing δ sufficiently small and i sufficiently large.

More generally, we are able to bound the density $\Theta^\rho\left(L^i, X, \frac{1}{\sqrt{2}}r\right)$ for all $(x_0,r_0) \in P\left(O,\frac{1}{\sqrt{2}}r\right) = B_r(O)\times\left(-\frac{1}{2}r^2,0\right]$. Using the monotonicity formula (6.7),

$$\begin{aligned}
\Theta^\rho\left(L^i,(x_0,r_0),\frac{1}{\sqrt{2}}r\right) &= \int_{L^i_{r_0-\frac{1}{2}r^2}}\Phi^\rho_{(x_0,r_0)}(\cdot,r_0-\tfrac{1}{2}r^2)d\mathscr{H}^2 \\
&\leq \int_{L^i_{-r^2}}\Phi^\rho_{(x_0,r_0)}(\cdot,-r^2)d\mathscr{H}^2,
\end{aligned}$$

and by a very similar calculation to the above we can choose i large so that this is less than $1+\varepsilon$. It follows by White's local regularity theorem (Theorem 2.2.10) that $|A|$ and its derivatives are bounded uniformly in the parabolic ball $P(O,\frac{r}{8})$. This is a contradiction, and so no singularity can occur. □

6.1.4 Smooth Convergence to the Disc

We now prove that the profile curve γ converges smoothly in infinite time to the real axis.

Theorem 6.1.7. *Any solution to (6.1) is immortal, and converges smoothly in infinite time to the real axis.*

Proof. The C^1 bound (6.6) implies that our graphical mean curvature flow equation (6.3) is uniformly parabolic, and so Schauder estimates give bounds on all curvatures outside a ball B_ε. By the bound on $|H|$, it follows from Lemma 4.4.5 that there exists c,T such that for $t > T$, $|H| < e^{-\frac{c}{4}t}$. As in the proof of Proposition 4.4.1, it now follows that $L_t \cap B_\varepsilon^c$ converges smoothly to a special Lagrangian. This special Lagrangian cannot be a Lawlor neck, as we may take B_ε smaller than the waist of a candidate Lawlor neck, and then the flow could not reach B_ε in the limit. It follows that $L_t \cap B_\varepsilon$ converges smoothly to a plane as $t \to \infty$.

In order to apply Proposition 4.4.1 to our entire flow, it is left to show that we have uniform curvature bounds near the origin – for this we use White's regularity theorem. Fix $r > 0$, then for all δ,

$$\begin{aligned}\int_{L_{t-r^2}} \Phi^\rho_{(x,t)} d\mathscr{H}^2 &= \int_{L_{t-r^2}\cap B_\delta} \Phi^\rho_{(x,t)} d\mathscr{H}^2 + \int_{L_{t-r^2}\cap B^c_\delta} \Phi^\rho_{(x,t)} d\mathscr{H}^2 \\ &\leq \delta^2 C + \int_{L_{t-r^2}\cap B^c_\delta} \Phi^\rho_{(x,t)} d\mathscr{H}^2.\end{aligned}$$

Therefore for any ε, we may take δ sufficiently small such that

$$\int_{L_{t-r^2}} \Phi^\rho_{(x,t)} d\mathscr{H}^2 \leq \int_{L_{t-r^2}\cap B^c_\delta} \Phi^\rho_{(x,t)} d\mathscr{H}^2 + \varepsilon.$$

By smooth convergence to the disc outside B_ε, we may take t sufficiently large such that the integral in the last line is less than 1 (the localised kernel Φ^ρ has the property that it integrates to 1 on a hyperplane). In general then, for any ε we may take t sufficiently large such that

$$\int_{L_{t-r^2}} \Phi^\rho_{(x,t)} d\mathscr{H}^2 \leq 1 + \varepsilon,$$

locally uniformly in x and t. White's regularity theorem (Theorem 2.2.10) now gives a uniform bound on $|A|^2$ and its higher derivatives. This implies that our flow converges smoothly to a special Lagrangian by Proposition 4.4.1, that must be equivariant and pass through the origin. There is only one submanifold with these properties that also intersects the Lawlor neck, an equivariant disc, and so we are done. □

6.2 The Clifford Torus

Our second example concerns equivariant discs L (profile curve γ) with boundary on the Clifford torus. The Lagrangian angle of the Clifford torus Σ with profile curve σ is given by

$$\tilde{\theta} = \frac{\pi}{2} + 2\arg(\sigma),$$

and therefore the boundary condition of (4.1) becomes

$$\theta\big|_{\partial L} - 2\arg(\gamma) = -\alpha.$$

As before, we restrict to the $\alpha = 0$ case, which corresponds to the profile curves meeting orthogonally at the boundary.

The Clifford torus is slightly more complicated to work with than the Lawlor neck, as it is not a static solution to MCF. However it is a self-similarly shrinking solution, with profile curve

$$\sigma_t = \left(\sqrt{-4t}\cos(s), \sqrt{-4t}\sin(s)\right),$$

on the time interval $[t_0, 0)$. It is then natural to perform the rescaling

$$\begin{aligned} \overline{\Sigma}_\tau &:= \frac{1}{\sqrt{-t}}\Sigma_t\big|_{t=-e^{-\tau}} \\ \implies \overline{\sigma}_\tau &= (2\cos(s), 2\sin(s)) \end{aligned}$$

which is a static solution to the rescaled MCF equation

$$\left(\frac{\partial \overline{F}}{\partial \tau}\right)^{\perp} = H + \frac{\overline{F}^{\perp}}{2},$$

on the time interval $[\tau_0, \infty) = [-\log(-t_0), \infty)$. Applying this rescaling also to our LMCF with boundary means we are working with a static boundary manifold, albeit with a different PDE problem.

In this section, we will prove that the rescaled flow is immortal and converges in infinite time to a flat equivariant disc (see Figure 4.2a). In terms of the original flow, this means that no singularity occurs before the final time 0, and any sequence of parabolic rescalings centred at the singular space-time point $(O,0)$ converges to a flat equivariant disc. Throughout this section we will work with both the rescaled flow, denoted $\overline{L}_\tau$ with profile curve $\overline{\gamma}_\tau$, and the original flow, denoted L_t with profile curve γ_t. For reference, the rescaled flow for the profile curve is given by

$$\left(\frac{\partial \overline{\gamma}}{\partial \tau}\right)^{\perp} = k - \frac{\overline{\gamma}^{\perp}}{|\overline{\gamma}|^2} + \frac{\overline{\gamma}^{\perp}}{2}. \tag{6.9}$$

Theorem 6.2.1. *Let $\overline{L}_0 : D \to \mathbb{C}$ be an S^1-equivariant Lagrangian embedding of a disc D, with boundary on the Clifford torus*

$$\Sigma_{\mathrm{Cliff}} := \{2e^{i\phi}(\cos(\psi), \sin(\psi)) \in \mathbb{C}^2 : \phi, \psi \in [0, 2\pi)\},$$

and let $\overline{\gamma}_0 : [-2,2] : \mathbb{C}$ be its profile curve in $\mathbb{C}$. Assume that its Lagragian angle θ_0 satisfies

$$\theta_0(s) - 2\arg(\overline{\gamma}_0(s)) \in (-\tfrac{\pi}{2} + \varepsilon, \tfrac{\pi}{2} - \varepsilon)$$

for some $\varepsilon > 0$. Then there exists a unique, eternal solution to the rescaled LMCF problem:

$$\begin{cases} \left(\frac{\partial}{\partial \tau}\overline{F}(x,\tau)\right)^{NM} = H(x,\tau) + \frac{\overline{F}(x,\tau)^{\perp}}{2} & \text{for all } (x,\tau) \in D \times [\tau_0, \infty) \\ \overline{F}(x,\tau_0) = \overline{L}_0(x) & \text{for all } x \in D \\ \partial \overline{L}_\tau \subset \Sigma_{\mathrm{Cliff}} & \text{for all } \tau \in [\tau_0, \infty) \\ \theta_\tau|_{\partial \overline{L}_\tau} - 2\arg(\overline{\gamma}_0) = 0 & \text{for all } (x,\tau) \in \partial D \times [\tau_0, \infty), \end{cases} \tag{6.10}$$

which converges in smoothly in infinite time to a flat disc.

Remark 6.2.2. *Note that here, we demand the condition $\theta_0(s) - 2\arg(\gamma_0(s)) \in (-\frac{\pi}{2} + \varepsilon, \frac{\pi}{2} - \varepsilon)$ in place of the almost-calibrated condition of the Lawlor neck case. This is more natural, as not only is this always satisfied at the boundary, but it is also equivalent to graphicality in a radial parametrisation, as will be shown in the next section.*

If we work with a different boundary condition, $\alpha \neq 0$ (corresponding to a different fixed angle between the profile curves), numerical evidence suggests that we still have long-time existence, and the flow converges to a rotating soliton of the rescaled LMCF with boundary problem; see Figure 4.2b.

6.2.1 Radial Parametrisation

We will work throughout with the radial parametrisation of the rescaled profile curve:

$$\begin{aligned}\overline{\gamma} : [-2,2] \to \mathbb{C}, \quad \overline{\gamma}(r) &:= re^{i\phi(r)} \\ \implies \overline{\gamma}'(r) &= (1 + ir\phi')e^{i\phi} \\ \implies \overline{\gamma}''(r) &= (-r(\phi')^2 + i(2\phi' + r\phi''))e^{i\phi}. \end{aligned} \tag{6.11}$$

Writing $\nu := \frac{i\overline{\gamma}'}{|\overline{\gamma}'|}$, the mean curvature is given by:

$$\begin{aligned}\vec{H} &= k - \frac{\overline{\gamma}^\perp}{|\overline{\gamma}|^2} = \frac{(\overline{\gamma}'')^\perp}{|\overline{\gamma}'|^2} - \frac{\overline{\gamma}^\perp}{|\overline{\gamma}|^2} \\ &= \left(\frac{r\phi'' + r^2(\phi')^3 + 2\phi'}{|\overline{\gamma}'|^3} + \frac{\phi'}{|\overline{\gamma}'|} \right) \nu, \end{aligned}$$

and therefore in this parametrisation, the problem (6.10) becomes

$$\begin{cases} r\frac{\partial \phi}{\partial t} = \frac{r\phi'' + r^2(\phi')^3 + 2\phi'}{1 + r^2(\phi')^2} + \phi' - \frac{r^2\phi'}{2} & \text{for } r \in [-2,2],\ \tau \geq \tau_0, \\ \phi(r,\tau) = \phi_0 & \text{for } r \in [-2,2], \\ \phi'(r,\tau) = 0 & \text{for } r \in \{-2,2\},\ \tau \geq \tau_0. \end{cases} \tag{6.12}$$

Lemma 6.2.3. *In the above parametrisation, the only static solutions to the rescaled LMCF with boundary (6.10) are straight lines through the origin, with $\phi = \phi_0$.*

Proof. Using (6.12),

$$\begin{aligned} H + \frac{\overline{F}^\perp}{2} = 0 &\iff r\phi'' + 3\phi' + 2r^2(\phi')^3 - \frac{r^2\phi'}{2} - \frac{r^4(\phi')^3}{2} = 0 \\ &\iff \frac{d\lambda}{dr} + (\lambda + \lambda^3)\left(\frac{2}{r} - \frac{r}{2}\right) = 0 \end{aligned}$$

away from $r = 0$, for $\lambda = r\phi'$. This ODE, along with the boundary condition $\lambda = 0$, has the unique solution $\lambda = 0$, which implies that our static solution is a straight line. $\square$

6.2.2 C^1-Bounds on the Graph Function

The chosen parametrisation is special in that our assumed condition on the Lagrangian angle corresponds to graphicality and gradient bounds for ϕ.

Lemma 6.2.4. *Assume that $\overline{F}_\tau$ is a solution to (6.10) on $[\tau_0, T)$, such that at time τ_0,*

$$\theta_t - 2\arg(\overline{\gamma}_\tau) \in \left(-\frac{\pi}{2} + \varepsilon, \frac{\pi}{2} - \varepsilon\right). \tag{6.13}$$

Then for all $\tau \in [\tau_0, T)$:

- *The condition (6.13) holds,*
- *The flow can be radially parametrised as $\overline{\gamma}_\tau(r) = re^{i\phi_\tau(r)}$,*
- *In this parametrisation, there exists a constant C_2 such that $|r\phi'_\tau| \leq C_2$. Therefore $|\overline{\gamma}'|$ is uniformly bounded, and ϕ'_τ is uniformly bounded on any annulus centred at the origin.*

Proof. If we parametrise the initial profile curve $\overline{\gamma}_0$ by arclength, then it may be written in polar coordinates as

$$\overline{\gamma}_0(s) = r(s)e^{i\phi(s)}, \qquad \overline{\gamma}'_0(s) = (r' + ir\phi')e^{i\phi}. \tag{6.14}$$

Therefore the Lagrangian angle of γ_0 may be expressed as

$$\theta(s) = 2\phi + \tan^{-1}\left(\frac{r\phi'}{r'}\right).$$

Note that at the origin, we must have $r' > 0$. Using $|\overline{\gamma}'| = \sqrt{(r')^2 + r^2(\phi')^2} = 1$ and

(6.13), there exists N such that if $r' < 1$:

$$\begin{aligned}\left|\frac{r\phi'}{r'}\right| &\leq N \\ \implies 1-(r')^2 &\leq N^2(r')^2 \\ \implies (r')^2 &\geq \frac{1}{N^2+1} \\ \implies r' &\geq \frac{1}{\sqrt{N^2+1}}.\end{aligned}$$

This lower bound allows us to reparametrise as $\overline{\gamma}(r) = re^{i\phi(r)}$, and in this parametrisation,

$$\theta(r) = 2\phi + \tan^{-1}(r\phi'),$$

therefore the condition (6.13) corresponds to a uniform upper bound on $|r\phi'|$.

It is left to prove that (6.13) is preserved; we start by calculating the evolution equation of $\theta - 2\phi$. Working with the arclength parametrisation of the original unrescaled flow, $\gamma(s) = r(s)e^{i\phi(s)}$, the metric and Laplacian on the manifold are given by

$$\begin{aligned} g &= ds\otimes ds + r^2 d\beta \otimes d\beta, \\ \Delta f &= \frac{1}{|g|}\partial_i\left(|g|g^{ij}\partial_j f\right) = \frac{\partial^2 f}{\partial s^2} + \frac{1}{r^2}\frac{\partial^2 f}{\partial \beta^2} + \frac{\langle \gamma', \gamma\rangle}{r^2}\frac{\partial f}{\partial s},\end{aligned}$$

where β is the coordinate of the S^1-equivariance. If f is an equivariant function, as θ and ϕ both are, then the middle term vanishes. Now, writing $\nu := i\gamma'$, it follows from (6.14) that

$$\frac{\partial \phi}{\partial s} = -\frac{\langle \gamma, \nu\rangle}{r^2}, \qquad \frac{\partial^2 \phi}{\partial s^2} = -\frac{\langle \gamma, i\gamma''\rangle}{r^2} + 2\frac{\langle \gamma', \gamma\rangle \langle \gamma, \nu\rangle}{r^4} = \frac{\left\langle i\gamma, \vec{k}\right\rangle}{r^2} + 2\frac{\langle \gamma', \gamma\rangle}{r^2}\frac{\langle \gamma, \nu\rangle}{r^2},$$

where $\vec{k}$ is the curvature of the profile curve, and using the standard equivariant MCF equation (5.6),

$$\frac{\partial \gamma}{\partial t} = \vec{k} - \frac{\gamma^\perp}{r^2}, \quad \implies \quad \frac{\partial \phi}{\partial t} = \left\langle \frac{i\gamma}{r^2}, \frac{\partial \gamma}{\partial t}\right\rangle = \frac{\left\langle i\gamma, \vec{k}\right\rangle}{r^2} - \frac{\langle \gamma', \gamma\rangle}{r^2}\frac{\langle \gamma, \nu\rangle}{r^2}.$$

Additionally, under this flow the Lagrangian angle satisfies the heat equation

$$\left(\frac{\partial}{\partial t} - \Delta\right)\theta = 0.$$

Putting this all together, we arrive at the evolution equation:

$$\left(\frac{\partial}{\partial t} - \Delta\right)(\theta - 2\phi) = 2\left(-\frac{\partial}{\partial t} + \frac{\partial^2}{\partial s^2} + \frac{\langle\gamma',\gamma\rangle}{r^2}\frac{\partial}{\partial s}\right)\phi = 4\frac{\langle\gamma',\gamma\rangle}{r^2}\frac{\langle\gamma,\nu\rangle}{r^2}.$$

Now, since $\theta - 2\phi = \arg(\gamma') - \arg(\gamma)$, it follows that

$$\begin{aligned}
\cos(\theta - 2\phi) &= \cos(\arg(\gamma') - \arg(\gamma)) = \frac{\langle\gamma',\gamma\rangle}{r}, \\
\sin(\theta - 2\phi) &= \cos\left(\arg(\gamma') - \arg(\gamma) - \frac{\pi}{2}\right) = -\frac{\langle\gamma,\nu\rangle}{r}, \\
\implies \left(\frac{\partial}{\partial t} - \Delta\right)(\theta - 2\phi) &= -2\frac{\sin(2(\theta - 2\phi))}{r^2}. \qquad (6.15)
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\left(\frac{\partial}{\partial t} - \Delta\right)\sin(\theta - 2\phi) \\
&= \cos(\theta - 2\phi)\left(\frac{\partial}{\partial t} - \Delta\right)(\theta - 2\phi) + \sin(\theta - 2\phi)\left\langle\frac{\partial(\theta - 2\phi)}{\partial s}, \frac{\partial(\theta - 2\phi)}{\partial s}\right\rangle \\
&= -\frac{4}{r^2}\sin(\theta - 2\phi)\cos^2(\theta - 2\phi) + \frac{\sin(\theta - 2\phi)}{\cos^2(\theta - 2\phi)}|\nabla\sin(\theta - 2\phi)|^2. \qquad (6.16)
\end{aligned}$$

Now for a contradiction, assume that at some point $p \in \gamma_t$ after the initial time, we have an increasing maximum of $\theta - 2\phi$ (and therefore of $\sin(\theta - 2\phi)$). Since this function is zero on the boundary and at the origin, it must occur at some interior point away from the origin. Then at this point, it is valid to parametrise by arclength and use standard (normal) mean curvature flow, so that the above calculation is valid. The weak maximum principle applied to (6.16) then provides a contradiction, and so the function $\theta - 2\alpha$ is bounded by its initial values. □

Finally, using simple barriers we also obtain uniform C^0 estimates on the function ϕ.

Lemma 6.2.5. *Let $\overline{\gamma}$ be a radially parametrised solution to (6.10) on the time interval $[t_0,T)$, which satisfies $\phi_{t_1} \in [\phi_-,\phi_+]$, $\theta_{t_1} \in [\theta_-,\theta_+]$ for some $t_1 \in [t_0,T)$. Define*

$$A_- := \min\left\{\frac{\theta_-}{2},\phi_-\right\}, \qquad A_+ := \max\left\{\frac{\theta_+}{2},\phi_+\right\}.$$

Then for all $t \in [t_1,T)$, $\phi_t \in [A_-,A_+]$.

Proof. We only prove that $\phi_t \leq A_+$, since the A_- case is identical. For a contradiction, assume that there exist δ and a first time $t_\delta \in (t_1,T)$ such that

$$\max_{L_{t_\delta}} = A_\delta := A_+ + \delta.$$

Then using the radial parametrisation, if this maximum is achieved on $[-2,2]\setminus\{0\}$, we may use the strong parabolic maximum principle applied to the boundary value problem (6.12), comparing with the static solution $\tilde{\phi} \equiv A_\delta$. This implies that locally in space and time $\phi \equiv A_\delta$, which is a contradiction.

On the other hand, if this maximum is achieved at the origin $r = 0$, then since $\theta - 2\phi = 0$ at this point, $\theta_{t_\delta}(0) = 2\phi_{t_\delta}(0) = 2A_\delta$, which is larger than the maximum of θ_{t_1}. Since θ satisfies a heat equation on the abstract disc, it follows by the parabolic maximum principle and the fact that $\theta - 2\phi = 0$ on the boundary that we must have

$$\theta_{t_\delta}(-2) = \theta_{t_\delta}(2) = 2A_\delta \implies \phi_{t_\delta}(-2) = \phi_{t_\delta}(2) = A_\delta.$$

But now as before we may apply the maximum principle at the boundary to ϕ to derive a contradiction. □

6.2.3 Long-Time Existence

We now prove long-time existence for our rescaled flow, in a very similar way to the Lawlor neck case.

Lemma 6.2.6. *A finite-time singularity for a solution of (6.10) cannot occur.*

Proof. Note that a finite-time singularity of (6.10) corresponds to a singularity of the unrescaled flow before time 0.

Working with the rescaled flow, we have shown that it is graphical and that the graph function ϕ satisfies the equation (6.12), which is uniformly parabolic away from the origin by the C^1 bounds of the last section. Therefore we have uniform bounds on all derivatives by parabolic Schauder estimates, and no singularity can occur away from the origin.

Just as before, we must deal with the origin separately. Assuming that a singularity of the original flow L_t occurs before the final time 0, the image of ∂L_t under any sequence of rescalings around this singularity will diverge to infinity, just as with the Lawlor neck (since at the time of the singularity, the Clifford torus is outside a neighbourhood of the origin). Therefore Lemma 6.1.6 applies, and it follows that

$$\int_{\gamma_t^i \cap (B_R \setminus B_\delta)} |\gamma^\perp|^2 d\mathscr{H}^1 = 2\int_\delta^R \frac{r^4(\phi')^2}{|\gamma'|^2} dr \geq \frac{2\delta^4}{C_2}\int_\delta^R (\phi')^2 dr$$
$$\implies \int_\delta^R (\phi')^2 dr \to 0.$$

In exactly the same way as in the proof of Lemma 6.1.5, this estimate gives us bounds on the densities, and White regularity implies smooth convergence of the rescalings. This is a contradiction to the assumption of singularity formation at $(O,0)$. □

6.2.4 Subsequential Convergence to the Disc

We now prove subsequential convergence to the disc, working with the original flow throughout. Take a sequence of rescalings L_t^i around the space-time point $(O,0)$ with factors $\lambda_i \to \infty$. We may use the graphicality and smooth estimates from Schauder theory away from the origin to conclude that, subsequentially, the profile curves γ_t^i converge to a limiting smooth graph on $A \times [a,b]$, where A is any annulus centred at the origin. A diagonal argument gives a subsequence converging locally smoothly away from the origin to a limiting flow γ_t^∞, with limiting angle function ϕ_t^∞ well defined everywhere but the origin.

Using the boundary version of Huisken's monotonicity formula (6.7) with $f = (\theta - 2\phi)^2$, using the evolution equation (6.15) and noting that $f = 0$ and $\nabla f = 0$ on the boundary gives the monotonicity formula:

$$\begin{aligned}
\left(\frac{\partial}{\partial t}-\Delta\right) f &= 2(\theta-2\phi)\left(\frac{\partial}{\partial t}-\Delta\right)(\theta-2\phi) - 2\left(\frac{\partial}{\partial s}(\theta-2\phi)\right)^2 \\
&= -\frac{4}{r^2}(\theta-2\phi)\sin(2(\theta-2\phi)) - 2\left(\frac{\partial}{\partial s}(\theta-2\phi)\right)^2, \qquad (6.17)\\
\frac{\partial}{\partial t}\int_{L_t^i} f\Phi d\mathscr{H}^2 &= \int_{L_t^i}\Phi\left(\left(\frac{\partial}{\partial t}-\Delta\right)f - f\left|H-\frac{x^\perp}{2t}\right|^2\right)d\mathscr{H}^2 \\
&\quad + \int_{\partial L_t^i}\Phi\left\langle f\frac{x}{2t}+\nabla f,\nu\right\rangle d\mathscr{H}^1 \\
&= \int_{L_t^i}\Phi\Big(-\frac{4}{r^2}(\theta-2\phi)\sin(2(\theta-2\phi)) - 2\left(\frac{\partial}{\partial s}(\theta-2\phi)\right)^2 \\
&\quad - (\theta-2\phi)^2\left|H-\frac{x^\perp}{2t}\right|^2\Big)d\mathscr{H}^2. \qquad (6.18)
\end{aligned}$$

Therefore, choosing $0<a<b$,

$$\begin{aligned}
\lim_{i\to\infty}\int_a^b\int_{L_t^i}\Phi(\theta-2\phi)^2\left|H-\frac{x^\perp}{2t}\right|^2 d\mathscr{H}^2 dt
&\leq \lim_{i\to\infty}\left(\int_{L_a^i} f\Phi d\mathscr{H}^n - \int_{L_b^i} f\Phi d\mathscr{H}^2\right) \\
&= \lim_{i\to\infty}\left(\int_{L_{(\lambda^i)^{-2}a}} f\Phi d\mathscr{H}^n - \int_{L_{(\lambda^i)^{-2}b}} f\Phi d\mathscr{H}^2\right) \\
&= 0.
\end{aligned}$$

This implies (by the locally smooth convergence) that $(\theta-2\phi)^2\left|H-\frac{x^\perp}{2t}\right|^2 \equiv 0$ for the limiting manifold L_t^∞, for any $t\in\mathbb{R}$. But if on an open subset we have $\theta-2\phi\equiv 0$, then the subset must be a part of a straight line through the origin. Therefore on this subset we also have $\left|H-\frac{x^\perp}{2t}\right|\equiv 0$, and so γ^∞ is a self-shrinker. By Lemma 6.2.3 the only option is a straight line through the origin; therefore $\phi^\infty = A$ for some constant $A\in\mathbb{R}$. Additionally, since we have smooth convergence on any annulus, we have the integral estimate

$$\int_{L_t^i}\Phi(\theta-2\phi)^2 d\mathscr{H}^2 \to 0. \qquad \text{as} \quad i\to\infty. \qquad (6.19)$$

This convergence of the rescalings corresponds to subsequential convergence in the rescaled flow. Taking any sequence τ_i, and choosing $\lambda_i := e^{\frac{\tau_i}{2}}$:

$$\overline{L}_{\tau_i} = e^{\frac{\tau_i}{2}} L_{-e^{-\tau_i}} = \lambda_i L_{-\lambda_i^{-2}} = L^i_{-1}.$$

By the work above we know that, up to a subsequence, this converges smoothly away from the origin to a disc.

6.2.5 Smooth Convergence to the Disc

We have proven smooth subsequential convergence to the disc, but we could still have different subsequences converging to different discs, and we also haven't shown that the curvature remains bounded at the origin. To solve these problems, we will demonstrate uniform curvature estimates via a Type II blowup argument.

Assume that the curvature of the rescaled flow $|A|$ diverges to infinity as $\tau \to \infty$. Then we may find a sequence τ_i such that $\max_{\overline{L}_{\tau_i}} |\overline{A}_{\tau_i}| \to \infty$ as $i \to \infty$. In the unrescaled flow, this sequence corresponds to a sequence of times $t_i = -e^{-\tau_i}$, such that

$$\sqrt{-2t_i} \max_{L_{t_i}} |A_{t_i}| \to \infty;$$

i.e. the singularity is a Type II singularity.

Passing to a subsequence we may ensure that the manifolds $\overline{L}_{\tau_i}$ converge smoothly to a disc on an annulus by the work of the previous section – therefore the curvature blowup must be uniformly away from the boundary. By standard theory of Type II blowups, we also know that we may choose a sequence of points x_i such that the sequence

$$\hat{L}_t^{(x_i,t_i)} := A_i \left(L_{t_i + A_i^{-2} t} - x_i \right)$$

converges locally smoothly to a limiting flow $\hat{L}_t^{\infty}$, where $A_i := \max_{L_{t_i}} |A_{t_i}|$. We may pick these points in $\mathbb{C} \times \{0\}$, and define the rescaled profile curve $\hat{\gamma}_t^i$ in the same say as above by considering x_i to be an element of $\mathbb{C}$.

We now prove locally uniform convergence of $\theta - 2\phi$ to 0 for the Type II rescalings $\hat{L}_t^{(x_i,t_i)}$. The argument is identical to the proof of Theorem 5.2.13, and so we suppress some of the details.

Lemma 6.2.7. *Consider the sequence of Type II blowups as defined above. For any bounded parabolic region $\Omega \times I \subset \mathbb{C}^2 \times \mathbb{R}$,*

$$\theta - 2\phi \to 0 \quad as \quad i \to \infty, \quad uniformly\ in\ \Omega \times I. \tag{6.20}$$

Explicitly, for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for any $t \in I$, and any sequence $\chi_i \in \Omega \cap \hat{L}_t^{(x_i,t_i)}$,

$$\theta_t^i - 2\phi_t^i \leq \varepsilon,$$

where $\phi_t^i(p)$ is the angle of the point $\hat{\gamma}_t^i(p)$ in the rescaled profile curve, relative to the image of the origin under the rescaling, $-A_i x_i$.

Proof. Choosing

$$\lambda_i := \frac{1}{2}\min\left\{\frac{1}{\sqrt[4]{-t_i}}, \frac{1}{\sqrt{x_i}}, \sqrt{A_i}\right\},$$

it is then possible to pick an N such that for any $i > N$ and $\tau \in I$,

$$(-t_i A_i^2)(1 + t_i^{-1}\lambda_i^{-2}) \geq \tau.$$

Since $(\theta - 2\phi)^2$ is a subsolution to the heat equation by (6.17), it follows that

$$\begin{aligned}
|\theta_\tau^i(\chi_i) - 2\phi_\tau^i|^2 &= \int_{\hat{L}_\tau^{(x_i,t_i)}} (\theta - 2\phi)^2 \Phi_{(\chi_i,\tau)} d\mathscr{H}^2 \\
&\leq \int_{\hat{L}^{(x_i,t_i)}_{(-t_i A_i^2)(1+t_i^{-1}\lambda_i^{-2})}} (\theta - 2\phi)^2 \Phi_{(\chi_i,\tau)} d\mathscr{H}^2 \\
&= \int_{L_{-1}^i} (\theta - 2\phi)^2 \Phi_{\left(\lambda_i\left(A_i^{-1}\chi_i + x_i\right), \lambda_i^2\left(A_i^{-2}\tau + t_i\right)\right)} d\mathscr{H}^2,
\end{aligned}$$

where for the first inequality we use Huisken monotonicity (6.18), and in the second we use invariance of the kernel Φ (Lemma 2.2.8) to equate the integral over the Type II rescaling with an integral over the Type I rescaling L_{-1}^i, centred at $(O,0)$ and with rescaling factor λ_i. Then, since $\left(\lambda_i\left(A_i^{-1}\chi_i + x_i\right), \lambda_i^2\left(A_i^{-2}\tau + t_i\right)\right) \to (O,0)$ uniformly in $\Omega \times I$, and by the L^2 convergence (6.19), we may find $\tilde{N} \geq N$ such that for $i \geq \tilde{N}$,

$$|\theta_\tau^i(\chi_i) - 2\phi_\tau^i|^2 \leq \int_{L_{-1}^i} (\theta - 2\phi)^2 \Phi_{(0,0)} d\mathscr{H}^2 + \tfrac{\varepsilon}{2} \leq \varepsilon.$$

□

This lemma implies that the limiting profile curve $\hat{\gamma}_t^\infty$ is a straight line, since $\theta - 2\phi = 0 \iff \arg(\gamma) = \arg(\gamma')$. However, this is a contradiction, as the Type II blowup satisfies $\max|\hat{A}| = 1$ by construction.

Therefore, the rescaled flow $\overline{L}_\tau$ satisfies uniform curvature bounds, and so the subsequential convergence of $\overline{L}_{\tau_i}$ to a disc is in fact everywhere smooth. In particular, on passing to a subsequence their Lagrangian angles converge smoothly to a constant, as do their angle functions ϕ. We may now apply Lemma 6.2.5 to conclude that the flow converges smoothly in τ to a Lagrangian disc, which proves Theorem 6.2.1.

Chapter 7

Conclusions and Future Research

This work has focused on two main themes – extending Lagrangian mean curvature flow to include flows with boundary, and investigating the long-time behaviour of Lagrangian mean curvature flow and the nature of singularities. Here, we summarise the results of the thesis, and consider future extensions of our work.

In Chapter 4, we demonstrated that there is a suitable boundary condition for a submanifold L_0 on a Lagrangian mean curvature flow Σ_t, such that if L_0 is Lagrangian, the mean curvature flow with initial condition L_0 is Lagrangian until the maximal time of existence. As well as augmenting the rich subject of Lagrangian mean curvature flow, this theorem has potential applications in producing compactly supported Lagrangian deformations. To take one example, a long-standing conjecture of symplectic geometry suggests that every compactly supported symplectomorphism from $\mathbb{R}^{2n}$ to itself is smoothly isotopic to the identity via compactly-supported symplectomorphisms. Since the graph of a symplectomorphism may be considered as a Lagrangian submanifold, LMCF with boundary could be a useful tool for this problem.

There is much potential for further extending LMCF with boundary, in particular to networks of Lagrangians. MCF of networks (see for example [47]) is a natural extension of curve shortening flow, where one allows triple-junctions of curves intersecting with an angle of $\frac{2\pi}{3}$. These conditions are analogous to the phenomenon that the walls of bubble complexes meet only at triple junctions with angles of $\frac{2\pi}{3}$. These *regular* network flows are in fact the 'stable' state of Brakke flow in the 1-dimensional case. F. Schulze, T. Ilmanen and A. Neves [34] have recently shown that, given a network which does not satisfy these conditions (a *non-regular* network), there exists a regular flow starting from that network. In contrast, LMCF with boundary introduced in Chapter 4 is *not* a Brakke flow when considered together with the boundary flow, as evidenced by the fact that Type I blowups of singularities are not necessarily self-shrinkers (see Figure 4.2b). A promising research direction is therefore to investigate Brakke flows of Lagrangian networks, in particular to find a boundary condition at triple junctions which preserves the class of Lagrangian submanifolds.

In Chapter 5, we succeeded in giving a near-complete picture of singularities of almost-calibrated equivariant Lagrangian mean curvature flow in $\mathbb{C}^n$. We showed that the Type I and Type II blowups are uniquely determined by the initial condition of the flow, and are given by a pair of planes and a Lawlor neck respectively. One thing that we did not focus on was the rate of blowup of the curvature, or of the formation of the Lawlor neck. If investigated, it may be possible to show that the mean curvature remains bounded for almost-calibrated equivariant mean curvature flow at a singularity, since the Lawlor neck is a minimal submanifold. Similar analysis has been performed in the hypersurface case by J. Velazquez [71], N. Sesum and S-H. Guo [25], and M. Stolarski [66], and resulted in the first example of a singularity with bounded mean curvature.

The work of Chapter 5 supports the conjecture that the Lawlor neck singularity is generic for Lagrangian mean curvature flow. Future work could investigate the singular behaviour of Lagrangian mean curvature flow when perturbed in non-equivariant directions, to discover whether non-equivariant Lawlor necks are possible as singularities of the flow. If only equivariant Lawlor necks arise, this may provide a strategy for proving genericity of Lawlor necks, by reducing a general singularity to the equivariant case in the limit. One limitation of our analysis is that it required the global *almost-calibrated* condition in order to exclude certain possibilities, despite the expectation that this singularity arises far more generally. For example, it is conjectured to arise from the unstable Hamiltonian perturbations of the Clifford torus considered by C. Evans, J. Lotay, and F. Schulze [21]. It would therefore be of use to localise our arguments so they can be used for a wider range of flows.

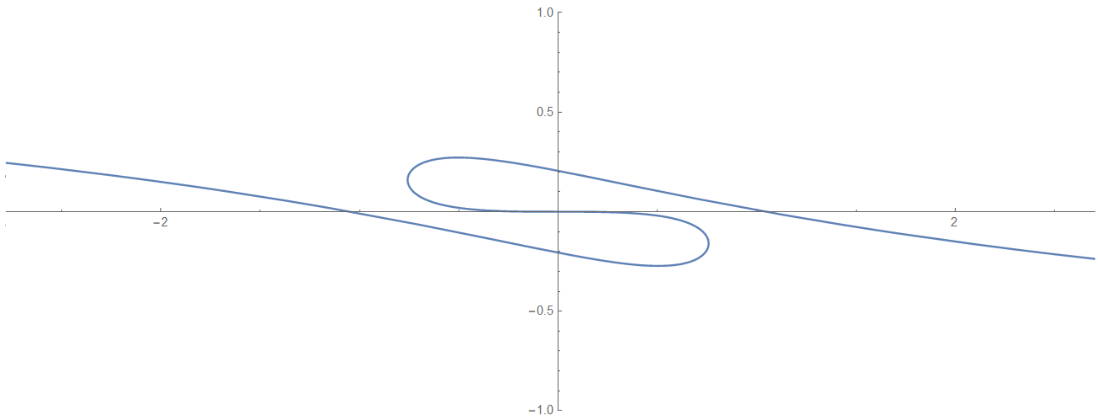


Figure 7.1: The profile curve of a Lagrangian submanifold, homeomorphic to a plane. It was shown by A. Neves [50] to form a Type II singularity at the origin.

There are interesting unstudied singularities, even in the equivariant case. One particularly intriguing example that warrants further study is one inspired by an example of A. Neves [50], depicted in Figure 7.1. This curve is not almost-calibrated so the theory of Chapter 5 does not apply, and indeed one can show that if the loops are large enough, a singularity forms at the origin whose Type I blowup is the union of three special Lagrangian planes. If the loops are small, however, the flow unravels without a singularity forming, and converges in infinite time to a single special Lagrangian plane. The behaviour at the boundary between these two schemes is unknown, but based on analysis of similar cases (for example [6]) it is expected that both the Lawlor neck singularity *and* a translating grim reaper singularity occur together, the former at the origin and the latter at the tip of the loops of the profile curve (compare with Figure 2.2).

Finally, in Chapter 6 we thoroughly investigated two examples of Lagrangian mean curvature flows with boundary, which showcased both long-time existence and singularity formation. These examples demonstrated that the flow is well-behaved; in particular the long-time existence result for the Lawlor neck highlights the potential uses of LMCF with boundary in finding minimal configurations and Lagrangian isotopies. In further work, it would be of interest to generalise the Lawlor neck example and prove a more general result on long-time existence of Lagrangian mean curvature flow with boundary on special Lagrangians.

Colophon

This document was set in the Times New Roman typeface using LaTeX and PDFLaTeX, and composed with the LaTeX text editor TeXstudio. The references were collected and exported using Mendeley, and compiled using BibTeX in TeXstudio.

Much of the mathematics was tested experimentally or derived using MatLab, Wolfram Mathematica, Python, and Brakke's surface evolver.

The non-diagrammatic images were produced with Wolfram Mathematica and Adobe Illustrator, and the hand-drawn diagrams were produced using Inkscape, Adobe Illustrator, Microsoft Paint and PaintTool SAI.

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