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Linear prices for non-convex electricity markets: models and algorithms

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Abstract

Strict Linear Pricing in non-convex markets is a mathematical impossibility. In the context of electricity markets, two different classes of solutions have been proposed to this conundrum on both sides of the Atlantic. We formally describe these two approaches in a common framework, review and analyze their main properties, and discuss their shortcomings.

In US, some orders are not settled at the market price, but at their bidding price, deviating from uniform pricing (all orders are financially settled at the same prices). This creates a disincentive to bid one's own true cost, and creates a missing money problem for the clearing house of the market. In Europe, all accepted orders are in-the-money are settled at the uniform market price. This implies that the welfare-maximizing solution is considered infeasible and also that the optimization problem is much less convex and more difficult to solve. This also creates fairness issues for orders of small volume, and the solution obtained does not implement a Walrasian equilibrium.

Based on this analysis we propose a new model that draws on both approaches and retains their best theoretical properties. We also show how the different approaches compare on classical toy problem.

Keywords: electricity markets, non-convexities, pricing.

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1 Introduction

Electricity is nowadays traded on organized exchanges in many parts of the world. These markets are usually non-convex, reflecting inherent non-convexities of the electrical system in general and the power plants in particular (minimum power output, start-up costs). The spirit of these markets is to implement a pay-as-cleared or linear pricing scheme: all electricity traded on the exchange at the same location and at the same time period are traded at the same market-clearing price.

Because of the non-convexities, this linear pricing scheme cannot be strictly implemented for the welfare-maximizing solution: such prices simply do not exist in most cases. Many different ways have been implemented to solve this issue around the world, and some more have been proposed. The main dividing line has been between markets implementing the true welfare-maximizing solution but somewhat deviating from linear pricing, and those implementing true linear pricing but executing a suboptimal solution in terms of welfare. The former approach is essentially the american one (as in PJM, New York, New England, Mid-West SO, ERCOT but also New-Zealand), and the latter is mostly found across Europe (as in Nordool, the Central-Western Europe Region, OMEL).

The US markets typically work as follows. First a welfare-maximizing non-convex optimization program is formed and ideally solved to optimality (if not, this would only be for keeping computational time low). This program is usually a Mixed-Integer Linear Program. This first step determines which orders will be executed and at what level, while ensuring that the implied flows on the electrical network are feasible. Then market prices are determined in a second step. If the market was convex, these prices would simply be the optimal dual variables associated to the equality constraint enforcing the balance between production, consumption, imports and exports at each location and time period. Because of the non-convexities, any fixed set of prices will usually leave some executed orders out-of-the-money and/or some rejected orders in-the-money. To achieve a Walrasian equilibrium where each participant is perfectly happy to execute the computed program at these prices, these problematic orders are financially compensated to make them at least indifferent to the computed solution based on the given market prices. This financial compensation is also called the *uplift*, and it should be clear that this uplift is uniquely determined by a set of market prices.

Many ways of determining the market prices and therefore the uplifts has been proposed or implemented. PJM determines the market prices as the dual variables of the linear relaxation of the welfare-maximizing MIP solved in the first step. Bjorndal and Jornsten [5] propose to determine prices based on the generation of a separating valid inequality that supports the optimal resource allocation (whatever that means). O'Neil and co-authors [8, 18] propose to take the dual variables of the restriction of the same program where the integer variables are fixed at their values computed at the first step. Hogan and Ring [12] propose to choose the sets of prices that minimize the total associated uplift. Gribik et al. [11] subsequently show a very elegant result: these prices correspond to the optimal Lagrangean multipliers when dualizing the market balance constraints, or equivalently to the local slope of the best convex dual function of the welfare-maximizing MIP. Araoz-Catillo and Jornsten [3] propose to compute prices based on the semi-Lagrangean approach proposed by Beltran et al. [4] for the p-median problem.

In Europe, the global consensus has been to reject the idea of uplift entirely, and to implement strict linear pricing: all electricity traded at the same location and at the same time period are traded at the same price. Indeed the uplift has the consequence that some orders will not pay or receive the market price but their own bid price. To make that possible, it is allowed in these european markets to have in-the-money orders with a non-convex component that are entirely rejected [16, 6, 17]. Note that these orders are not financially compensated, so that this solution does not constitute a Walrasian equilibrium. However it has other virtues that are discussed in Sections 5.4 and 4.3.

The european approach shares with the US approach the same welfare-maximizing MIP model, but adds some more constraints to forbid accepted out-of-the money orders. Thus these additional constraints relate prices and physical quantities. Moreover, market prices are defined as optimal dual variables of the LP/QP obtained by fixing the integer variables at their values (similar to the approach suggested by O'Neil et al. [18]). This setting makes the underlying optimization problem even more non-convex than its US counterpart, with the consequence that it is also computationally more difficult to solve.

2 Contribution and Outline

This paper is the first of its kind to seriously discuss the european power market model, describe existing algorithms for computing solutions, and discuss its most important properties. Second, it is also the first time that european and US markets are discussed within the same framework, enabling us to compare their respective merit. Finally, we also propose ways to improve the US way of dealing with non-convexities, sometimes drawing from good properties of the european market model.

In Section 3 we review basic concepts of electricity markets seen from the point of view of a market operator and introduce necessary notation. We give a very abstract and general formulation for this problem.

We also indicate how to extend it to encompass specificities of each market in Sections 5.1 and 4.2.

Section 4 is devoted to the US approach of dealing with non-convexities: welfare-maximization, definition of uplifts to implement a Walrasian equilibrium, and uplift minimization. We open this section by discussing the various proposals that have been made to determine prices, and uplifts. Section 4.1 is a neat review of the nice theoretical results of Gibrik et al. [11] about the connection between Lagrangean relaxation, duality gap and uplift minimization. This enables us to point out an alternative way to compute uplift-minimizing prices, and make a new connection with the "dispatchable model" used by the New-York System Operator. We also show in Section 4.2 how to properly model complex non-convex products as linear programs. This implies that indeed, computing uplift-minimizing prices is equivalent to solve a linear program. Finally, and most importantly, we discuss in Section 4.3 two issues associated to the US model:

- *Truthtelling* : the US market model does not incentivize market participants to bid their true costs, because some orders are essentially pay-as-bid because of the uplift. Moreover, and perhaps more importantly, insisting that the solution constitutes a Walrasian equilibrium implies that it is possible to make money by submitting orders guaranteed to be rejected.

- the *Missing Money* problem: the uplift, even if minimized, is substantial. But the money necessary to finance this uplift is not present in the market as it can possibly be larger than the welfare generated.

We then turn in Section 5 to the european market models. We first give a generic "core" model and discuss in Section 5.1 how it relates to the problem introduced in Section 3 and practical implementations in Europe. We then formally describe in Section 5.2 how strict linear pricing, but not welfare-maximizing, is implemented in Europe. We formulate the problem as a mixed-integer complementarity problem. We then give a second formulation as a bilevel problem that does not involve prices. This alternative formulation is new and give new insight into this model. We also discuss in Section 5.3 the various algorithmic approaches that has been proposed and/or implemented. We discuss in Section 5.4 four issues associated to the european model:

- the european market model might be unfair to small non-convex bids compared to large ones,
- it does not implement a Walrasian Equilibrium for rejected in-the-money orders,
- it is suboptimal in terms of welfare,
- it gives rise to a very difficult non-convex optimization problem.

In Section 6 we propose a new market model based on the discussions of the previous section. The new proposal is of the US flavour since it follows the two-step approach: first solve the welfare optimizing a primal non-convex (usually MIP) problem, then determine prices by solving a linear program. This second step is however different from the uplift minimization. We solve the missing money problem by directly incorporating in the model the financing of the uplift by in-the-money order. We propose to solve the truthtelling issue by having uplift guarantee that weaker condition that no order is loss-making, instead of insisting to guarantee a Walrasian equilibrium. This is in the same spirit as the european model. This already also partially solves the missing money problem as the uplift necessary are smaller in total. Our new proposal also directly includes in the determination of prices the financing of the remaining uplift by in-the-money accepted orders. For fairness reasons, we propose to minimize the maximum such contribution in \$/MWh or €/MWh across all orders instead of the total uplift.

We show in Section 7 how the main approaches discussed numerically compare on a toy example used in Gribik et al. [11].

We conclude in Section 8 by arguing that the proposed market model should be implemented on both sides of the Atlantic as it keeps best features of both models: computability, good incentives, welfare maximization, and fairness. We also discuss interesting questions that this work leaves open.

3 Basic Concepts

We consider the problem of a market operator receiving commitment (or orders) from producers, consumers and transmission operator about their willingness to produce, consume or transport electricity. His goal is to decide who should produce, consume or transport what amount of electricity, and what will the financial compensations be for these actions. We use the following notation:

x^i	is a vector of variables representing operating factors of order i,
$g_l^i(x^i)$	is the injection (if positive) or withdrawal (if negative) of power at node l associated with order i,
y_l	is a prespecified, external injection (if negative) or withdrawal (if positive) of power at node y_l,
r_l	is the net injection (if positive) or withdrawal (if negative) on the network at node l,
N	is the set of vectors r feasible for the transmission network,
X_i	is the set of vectors x^i satisfying the technical constraints of order i,
$f_i(x^i)$	is the contribution to the welfare of order i executed at level x^i.
$h(r)$	is the contribution to the welfare of transmitting r through the network.

Each node in the network is typically identified by its location and time period, so that the set N models both capacity types constraints and ramping types of constraints. Depending on the context, y_l might represent out-of-market commitments, or fixed power flows at the boundary with out-of-market network areas. $f_i(x^i)$ is typically negative to represent cost in case i is a production order and positive for consumption orders.

The natural optimization problem to solve by the market operator is then:

$$v(y) = \max_{x,e} \sum_i f_i(x^i) + h(r) \tag{1}$$

$$\sum_i g_l^i(x^i) = y_l + r_l \qquad \forall l \tag{2}$$

$$r \in N \tag{3}$$

$$x^i \in X_i \qquad \forall i \tag{4}$$

Assuming that the model represents true technical constraints, cost and revenues of market participants, then this model finds a feasible production / consumption / transmission schedule that maximizes social welfare from trade (if $y = 0$), minimize cost of meeting load y (if $y > 0$) or maximizes value of consumption (if $y < 0$). We will sometimes write the same problem using the more compact notation

$$v(y) = \max_{x} f(x) \tag{5}$$

$$g(x) = y \tag{6}$$

$$x \in X, \tag{7}$$

where $x = (x^1, \dots, x^{|I|}, e)$, $X = X^1 \times \dots \times X^{|I|} \times N)$ and $f(x) = \sum_i f_i(x^i) + h(e)$. At this point, we do not assume convexity X, concavity of f_i and h or linearity of g_l^i.

Formally, the role of the market operator is to determine x^* feasible in (5)–(7) together with a vector of prices p_l^* (one p_l^* for each constraint (2) or (6)). These values will trigger the following actions:

(a) The market participant having submitted order i injects power $g_l^i(x^{*i})$ at node l.

(b) The market participant having submitted order i receives $\sum_l p_l^* g_l^i(x^{*i})$ from the clearing house.

(c) The market operator pays $\sum_l p_l^* y_l$ to the clearing house.

(d) The transmission operator receives $\sum_l p_l^* r_l^*$ from the clearing house.

(e) The clearing house pays some additional compensation up_i to participants i (the uplift).

Why the last set of transactions (the uplift) is sometimes implemented in the US will be discussed later. One desirable property that is satisfied by this process if the uplifts are zero relates to the clearing house position:

Financial Balance: the clearing house is in balance and pays exactly as much money as it receives.

Another desirable property is related to the market participants:

No Loss: Each market participant i is not losing money: $f_i(x^{*i}) \leq \sum_l p_l^* g_l^i(x^{*i}) + up_i$.

Indeed, otherwise the market participant is losing money if its order represents its true cost/revenue. A stronger property is the following:

Walrasian Equilibrium: Each market participant i is perfectly happy: $\sum_l p_l^* g_l^i(x^{*i}) - f_i(x^{*i}) + up_i \geq \sum_l p_l^* g_l^i(z) - f_i(z)$ for any $z \in X_i$,

This property says that faced with markets with infinite depth at prices p^*, participant i could not make more money than the market result (again assuming its order represents its true cost/revenue). It is part of the folklore that satisfying all these properties simultaneously is possible when the underlying optimization problem is convex, and that in that case the uplift is zero. This will be reviewed among other things in Section 4.1.

Market organizations will differ through the following aspects:

- what type of orders are allowed on the market, or equivalently what type of functions f_i, g_l^i and sets X_i are allowed,
- what type of network model are allowed, or equivalently what type of function h and set N are allowed,
- how x^* and p^* are determined (algorithm) and what properties the solution obtained satisfies.

4 US Electricity Markets

US electricity markets typically adopt the following two-step approach to determine x^* and p^*:

1. solve the model (5)–(7) to obtain a solution x^*, ideally but not necessarily to optimality,
2. compute prices p^* (one p_i^* for each constraint (6)) and compensate market participants so that they are just perfectly happy (Walrasian Equilibrium) to commit to x^* at prices p^* (the *uplift*, a formal definition will be given in the next subsection) .

Different approaches have been proposed or implemented for the second step.

The proposition of Hogan and Ring [12] is to choose price p^* that minimize the total implied uplifts. This makes sense as uplifts can be seen as a measure of the deviation from linear pricing. If the optimal solution is zero, this means strict linear pricing is possible and non-convexities are actually not playing a role.

Bjorndal and Jornsten [5] propose to determine prices based on the generation of a separating valid inequality that supports the optimal resource allocation. The description of the scheme is not very precise, and no general argument is gven that would suggest that the methodology is superior to other proposals in some respect.

Araoz-Catillo and Jornsten [3] propose to compute prices based on the semi-Lagrangean approach proposed by Beltran et al. [4] for the p-median problem. However, the scheme seems to output very volatile market prices in the example discussed, and no general argument is given that would suggest that the methodology is superior to other proposals in some respect.

O'Neil et al. [18] propose to take the dual variables of the restriction of the same program where the primal integer variables are fixed at their (optimal) values. They interpret the dual variables of these fixing constraints as prices of additional contracts that together with market prices, give rise to a Walrasian equilibrium. However, the prices so obtained seem to be volatile compared to the uplift-minimizing approach [11]. Moreover, the proposal does not make sense when all orders are combinatorial. In this case, essentially all variables would be fixed in the program, and the proposed scheme is meaningless. This simple example shows that this proposal does not really take the non-convexities into account, it just relies on the convex part to determine prices. This is certainly not satisfactory.

PJM determines the market prices as the dual variables of the linear relaxation of the welfare-maximizing MIP (1) solved in the first step. In practice, this seems to give satisfactory results.

In Section 4.1, we give a neat and concise presentation of the main results of Gribik et al. [11] linking uplift minimization and duality gap of the Lagrangean of (5)–(7). We then use these results to show that the uplift-minimizing approach and the dispatchable model of PJM are in fact equivalent *if the non-convexities are properly modelled.* In Section 4.2 we then show that for even highly non-convex orders representing thermal power plants, it is not difficult to give a linear-programming formulation, yielding the desired equivalence. We finally discuss in Section 4.3 two issues with this US model. The first one is that the model does not give incentive to submit orders representing true costs and technical constraints, and the second is that the financing of the uplift is not specified. Even worse, the uplift is sometimes lower than the welfare, so that it is impossible to finance within the market model.

4.1 Welfare Maximization, Duality Gap, Uplift and Continuous Relaxation

Consider the following Lagrangean of the problem (5)-(7) obtained by dualizing the balance equations (6):

$$\begin{aligned} L(y,p) &= \max_{x\in X} \left\{ f(x) + p^T(y - g(x)) \right\} \\ &= p^T y + \sup_z \{ -p^T z + \max_{x\in X} (f(x) | g(x) = z) \} \\ &= p^T y + \sup_z \{ v(z) - p^T z \} \end{aligned}$$

The duality gap at prices p is therefore:

$$L(y,p) - v(y) = \sup_z \left[v(z) - p^T z\right] - \left[v(y) - p^T y\right] \tag{8}$$

For given prices p, $\sup_z \left[v(z) - p^T z\right]$ can be rewritten as

$$\begin{aligned} \max_{x,z}\ & f(x) - p^T z \\ & g(x) - z = 0 \\ & x \in X \end{aligned}$$

This is a problem similar to (5)-(7) but with the right handside of balance constraint equal to 0, and with a market with infinite depth at price p_i in node i (i.e. unlimited amount of power can be bought or sold at that price). Put slightly differently, $\sup_z \left[v(z) - p^T z\right]$ is the problem of maximizing the revenue the system can extract when faced with market prices p.

Using the same argument $[v(y) - p^T y]$ is the revenue the system extracts *if forced to produce/consume* y when faced with market prices p. Therefore $L(y,p) - v(y)$ is exactly the amount of money the market must be compensated to be just perfectly happy to commit to y when faced with prices p (Walrasian Equilibrium). This is exactly the definition of the "uplift" in the US electricity markets.

Prices minimizing the uplift, as proposed by Hogan and Ring [12], are therefore precisely the optimal Lagrangean multipliers of problem (5)-(7). Moreover, the duality gap of the Lagrangean problem is exactly the uplift. When the duality gap is zero, the solution implies no uplift, and therefore satisfies the Financial Balance and the Walrasian properties simultaneously. These are the main results of Gibrik et a. [11].

Let us be now slightly more specific and let us assume that (i) $f(x)$ is linear (this can be done without loss of generality as any optimization problem $\max_{x \in X} f(x)$ can be equivalently written as $\max_{(t,x):t \leq f(x), x \in X} t$), (ii) $g(x)$ is linear and (iii) X is non-convex but is expressed as the intersection as the convex set X' and Z^n. In the same work, Gribik et al. [11] contrast this uplift-minimizing approach with the approach of determining prices as optimal dual variables of the continuous relaxation (so called "dispatchable model" as implemented by the New-York System Operator):

$$\max_x\ f(x) \tag{9}$$

$$g(x) = y \tag{10}$$

$$x \in X' \tag{11}$$

Indeed, this problem is convex and optimal dual variables of (10) are computable. However, we have the following classical result:

Theorem 1 *[9, 10] If f and g are linear functions and X is a mixed-integer set, then*

$$\begin{aligned} \min_p \max_x\ & f(x) - p^T(y - g(x)) \geq & \max_x\ & f(x) \\ & x \in X & & g(x) = y \\ & & & x \in conv(X) \end{aligned}$$

The inequality holds if the set X is bounded (which is essentially always the case for physical systems). Moreover, in this case, the set optimal vectors p of the first problem is equal to the set of optimal dual variables of the second problem.

This theorem implies that the uplift-minimizing approach and the dispatchable model approach are equivalent, provided that $conv(X' \cap \mathbb{Z}^n) = X'$, or equivalently that the formulation of the set X is provably good, in the sense that the linear programming formulation X' provides the convex hull of the solution set $X = X' \cap Z^n$.

Observe that the set X is always separable into the different products traded on the market and the network model. Therefore, a good formulation for X is equivalent to a good formulation for each product separately, and also the network model. Because in all currently implemented markets, the network model is linear, the question boils down to finding good formulations for (complex) products traded on these electricity markets. We show in the next section 4.2 that even fairly complicated product admit tight formulation. This means in particular that to compute uplift-minimizing prices, there is an alternative to solve the Lagrangean of Theorem 1: to solve the equivalent linear program of the same theorem.

Even if only a good approximation of the convex hull of X is known, it can be expected that the duality gap will be small, and therefore that the dispatchable model approach will produce prices implying small uplift. This might very well explain why this approach implemented by the New York System Operator works well in practice.

4.2 Good formulation of a complex non-convex products

The products available on the US markets are typically more complex than in Europe, with start-up costs, minimum output rate, ramping, etc...In this section, we describe a complicated product modelling most constraints of a thermal power plant and show how to write a tight (extended) linear-programming formulation of this non-convex product.

To describe this product, we need the following data:

$h \in H$	set of thermal orders,
L_h	location of thermal order h,
G_h	startup cost of thermal order h,
V_h	variable cost of thermal order h,
L_h	Minimum output of thermal order h,
U_h	Maximum output of thermal order h,
RU_h	Ramp-up limit on thermal order h,
RD_h	Ramp-down limit on thermal order h.

The schedule of such a thermal plant is defined by an alternating succession of start-ups and shut-downs such that (i) between two consecutive start-up and shut-down, the plant output varies between it minimum and maximum outputs L_h and U_h while respecting the ramp-up and ramp-down limits RU_h and RD_h and (ii) between two consecutive shutdown and startup the production level is null. To model this, we need the following variables:

$u^h_{t',t''}$	=1 if the schedule of thermal order h includes consecutive start-up in period t' and shutdown in t''
$y^{h,t',t''}_t$	output level of thermal order h in period t between a start-up in t' and shut-down in t'',
x^h_t	output level of thermal order h in period t,
w^h_t	=1 there is a startup of thermal order h in period t.

The model for thermal order h is then expressed by the following set of constraints:

$$\sum_{t''>t'} u_{t''+1,t'} - \sum_{t'\leq t''} u^h_{t',t''} = \begin{cases} 1 & \text{if } t''=0 \\ 0 & \text{otherwise} \end{cases} \qquad \forall t'' \tag{12}$$

$$L_h u^h_{t',t''} \leq y^{h,t',t''}_t \leq U_h u^h_{t',t''} \qquad \forall t' \leq t < t'', \tag{13}$$

$$-RD_h u^h_{t',t''} \leq y^{h,t',t''}_t - y^{h,t',t''}_{t-1} \leq RU_h u^h_{t',t''} \qquad \forall t' < t < t'', \tag{14}$$

$$x^h_t = \sum_{t'\leq t''} y^{h,t',t''}_t \qquad \forall t, \tag{15}$$

$$w^h_t = \sum_{t'>t} u^h_{t,t'} \qquad \forall t, \tag{16}$$

$$0 \leq u^h_{t',t''} \leq 1 \qquad \forall t' \leq t'', \tag{17}$$

$$u^h_{t',t''} \in \mathbb{Z} \qquad \forall t' \leq t'', \tag{18}$$

$$\tag{19}$$

Constraints (12) are network flow constraints modelling the succession of start-ups and shutdowns. Note that $u^h_{t',t'} = 1$ means that the order h is kept shutdown throughout period t'. Constraints (13) model minimum and maximum output between two successive start-up and shutdown. Constraints (14) model minimum and maximum ramping limits between two successive start-up and shutdown. Constraints (15) link the natural output level variables x^h_t to the decomposed output level variables $y^{h,t',t''}_t$. Constraints (16) link the natural start-up variables w^h_t to the decomposed start-up/shut-down variables $u^h_{t',t''}$.

Even if already complex, they are several important characteristics that cannot be modeled by (12)–(18). But in general, these can simply be taken into account by making the input data more specific as follows:

- **Ramp-Up Profile.** Sometimes it takes several time periods to reach the minimum output level. In general, the plant follows then a fixed ramp-up profile. By letting the minimum and maximum output levels L_h andU_h be redefined as $L_{h,t,t',t''}$ and$U_{h,t,t',t''}$, we can easily force the variable $y^{h,t',t''}_t$ to follow the prescribed profile through constraints (13). If the ramping limits are violated on the profile, then the corresponding constraints can be dropped

- **Ramp-Down Profile.** Similar to Ramp-Up Profile.

- **Minimum Shut-Down Time**: this can be implemented by defining a ramp-down profile that stays at zero for the desired number of time periods.

It is clear that all subsequent derivations and conclusions hold when these generalizations are part of the model.

Now let P^h and X^h be the set of solutions to (12)–(17) and (12)–(18) respectively. X^h is a mixed-integer set, and the feasible set of its linear programming relaxation is the polyhedron P^h. The next result will be useful later and is similar to or a special case of earlier results [14, 19, 13]:

Proposition 2 *The convex hull of X^h is P^h.*

Proof. Let any linear objective function $\max f(u, y, w, x)$ be given. We prove that there is an optimal solution to $\max f(u, y, w, x)$ $s.t.$ $(u, w, x, y) \in P^h$ that is integral in u. Observe first that we can assume that the objective function is independent of w and x as these variables are linearly dependent on y and u through constraints (15)–(16). Therefore we can write it as $f(u, y)$.

Let u be fixed to an arbitrary value. Then the rest of the problem is (13)–(14) and is separable in (t', t''). Moreover, each subproblem (t', t'') is of the form $Ay^{h,t',t''} \leq u_{t',t''}b$. Therefore if $y^{*h,t',t''}$ is an optimal solution to $Ay^{h,t',t''} \leq b$, then $u_{t',t''}y^{*h,t',t''}$ is an optimal solution to $Ay^{h,t',t''} \leq u_{t',t''}b$. Hence the problem $\max f(u, y)$ $s.t.$ $(u, w, x, y) \in P^h$ can be rewritten as $\max g(u)$ $s.t.$ u satisfies $(12), (17)$ with $g(u)$ linear. This last problem is a network flow problem and therefore admits an optimal solution that is integral. ■

It should be clear that specific orders could be similarly defined for hydro units, etc.... In general, any product that can be optimized against given market prices p^* by dynamic programming can be modelled similarly with a linear-programming formulation using the techniques described in [14, 19, 13]. Therefore, the approach suggested by Theorem 1 to compute uplift-minimizing prices is feasible.

4.3 Two Issues with the US model

The approached advocated in the US for the determination of market prices in the presence of non-convexities, as reviewed in Section 4.1, is a nice theoretical construct: the prices have a nice theoretical interpretation, and are the closest valid linear (dual) prices that can be defined for this non-convex problem. However this approach suffers from two problems that we review here, and that we consider conceptually fatal, even if they can be mitigated in practice.

4.3.1 Truthtelling

Ideally, a market model should induce agents to bid their true costs and physical constraints. We call here this property "truthtelling". Truthtelling is a desirable property, because if it does not hold, it is impossible to guarantee that the market optimally coordinates production, consumption and transportation of power. When truthtelling is not guaranteed through incentives, this makes the existence of a strong regulator necessary.

The US model does not induce truthtelling. Indeed, it is easy to make money by submitting, for example, arbitrarily large (in volume) sell orders at minimum (e.g. zero) price with the non-convex requirement that the entire volume must be accepted, if any. If the order is sufficiently large, there cannot exist a counterparty (assuming all other agents bid truthfully), and the order will be rejected but compensated (so that the solution form a Walrasian equilibrium).

Consider the following simplistic example. There is only one time period and one location, there is no pre-specified external injection or withdrawal of power (y=0), and four orders submitted:

A: a large buy order of 100MW at 100$, with the non-convexity that it must be entirely accepted or rejected,

B: a large sell order of 200MW at 10$, with the non-convexity that it must be entirely accepted or rejected,

C: a small convex buy order: 10MW at 50$,

D: a small convex sell order: 15MW at 40$.

The optimal solution is: reject orders A and B (no counterpart can be found because of their non-convexity), accept entirely order C, and 10MW of order D. The uplift minimizing price p^* is 10$ and the following financial transfers are executed: A receives 9000$ (uplift), C pays 100$ (payment at the market price) and D receives 400$ (revenue of 100$ at market price + uplift of 300$). This is indeed the uplift-minimizing price because no uplift is paid to the very large non-convex order B.

One thing that is striking in the example is that A makes a lot of money by doing nothing. In particular, if A is sure that his order cannot be matched, he does not have to own a corresponding electricity producing or consuming asset, while always making a non-negative amount of money.

This example is extreme, but in general, it is possible to make money in the US model by submitting fake orders in the hope that they will be paradoxiacally rejected, and collect the possible uplift. In general, this will lead agents to submit orders which are more non-convex than in reality, to increase the likelihood that they will be rejected. This increase of non-convexity has unwanted consequences, both from a computational and a modelling point of view.

Market participants are strongly regulated in the US, and this solves the issue in practice. It might even be that the regulation is strong partly to sove the issue. But even if in practice the issue is not too damaging, this is certainly a conceptual weakness of this market model.

4.3.2 Missing Money

A second issue with the US model is the financing of the compensation offered for paradoxically accepted or rejected orders. Indeed, in the US model as outlined in Section 4, it is not true that the total amount of money paid by buyers is equal to the total amount of money collected by sellers plus money collected by network operators (congestion revenue). This missing money has to be financed somehow, but this is not part of the formal model.

The previous example illustrates the problem. In the solution proposed by the uplift-minimizing approach, 100$ is collected and 9300$ is distributed. Obviously, some money is missing and has to be financed somehow. The problem is even worse that it might seem at first sight for two reasons.

Firstly, even if the 9000$ was not paid to A (see the Truthtelling problem above), there is still 200$ missing to pay the uplift to D. Therefore, these truthtelling and missing money problems are two different issues. Secondly and more importantly, the amount of money missing (9300$ or even 200$) is larger than the welfare generated from trade (100$). So there is truly no way to finance the uplift (Financial Balance property of Section 3) while still satisfying even the weak and arguably necessary No Loss property.

In practice, it can be expected that the financing of the uplift is not too problematic because it remains small compared to the financial transfers derived from trade at market prices, and also compared to the welfare generated. Typically the uplifts are financed by subscription fees to the exchange by market participants. Thus, this financing is shared in some ad hoc manner by market participants. In principle, this might mean that some market participants simply obtain a negative welfare from their participation to the market, as they might contribute more to the financing of compensation than the welfare generated by their participation to the market. This is an undesirable property.

5 European Markets

In this section we first describe a typical "core" european power market and in particular what type of non-convexities it allows. We then discuss how this core model compare with practical european day-ahead electricity markets. We then discuss how pricing is done in these non-convex markets. The solution implemented is very different from the US approach discussed in Section 4. It does not suffer from the Truthtelling and Missing Money discussed above, but suffers from other types of issues.

European power markets are typically simpler than their US counterparts. They differ somewhat from Scandinavia to the Mediterranean, but the following is a core model that forms the skeleton of all of them. Let us define the following notation:

$i \in I$	set of orders,
$j \in J \subseteq I$	set of non-convex orders,
$l \in L$	set of locations,
$t \in T$	set of time periods,
$Q_{i,t,l}$	volume of order i in period t and location l, negative if supply and positive if demand,
P_i	price of order i,
A	network constraint matrix,
C	network limits vector.

For notational convenience, we will generally omit the indexing set and write $\sum_j$ for $\sum_{j \in J}$, $\forall t$ for $\forall t \in T$, etc...

The following are the decision variables

x_i	proportion of order i that is executed/accepted,
$r_{l,t}$	net injection at location l in period t,
$p_{l,t}$	prices at location l and period t,
s_i	surplus associated to order i,
z	dual variables of the network constraints.

We define the Core Model as follows:

$$\max \sum_{i,l,t} Q_{i,l,t} P_i x_i \tag{20}$$

$$r_{l,t} = \sum_i Q_{i,l,t} x_i \qquad \forall l, t, \qquad (p_{l,t}) \tag{21}$$

$$Ar \leq C, \qquad (z) \tag{22}$$

$$0 \leq x_i \leq 1 \qquad \forall i, \qquad (s_i) \tag{23}$$

$$x_j \in \mathbb{Z} \qquad \forall j, \tag{24}$$

The objective function is to maximize welfare: executed buy orders contribute positively in proportion to their volume and price, and executed sell orders contribute negatively. Constraints (21) are the balance constraints for each location and time period: net injection must equal difference between executed buy and sell orders. Constraints (22) may model any restriction to the power flows on the network: capacities of elements, ramping, losses,... as long as they are linear. Note that this general model is sufficiently general to describe a capacitated network flow model, also called ATC model within the power industry. Constraints (23) define x_i as proportions, and constraints (24) force non-convex orders to be entirely accepted or rejected. We also indicate between parenthesis the dual variable associated to each constraint that will be used in Section 5.2.

Clearly, the model (20)-(24) is a particular case of model (1)-(4). In particular, f and g are linear, and the non-convexities are very simple. Also, Theorem 1 trvially applies. The reason why the non-convexities are very limited in european markets will be discussed later in Section 5.4.

5.1 The core model and implementations in Europe

The Core Model is already more general than what is currently implemented in the Central-Western Europe area, and captures nearly all available products and constraints currently implemented in Nordpool. In european terminology, the non-convex orders are usually called *block orders* or simply *blocks*, due to their on-off nature on the one hand, and that they typically have positive volume on several (consecutive) time periods on the other hand.

Two features of Nordpool are not handled by our Core Model: linked block orders and flexible block orders. If two orders j and j' are linked, then order j can only be accepted if order j' is accepted. This condition can easily be modeled by the following constraint:

$$x_j \leq x_{j'}.$$

The set K of block orders is said to be flexible if at most one of them can be accepted. This is modeled as

$$\sum_{i \in K} x_i \leq 1$$

All subsequent derivations are easy to generalize if those consraints were part of the model. Therefore we omit those for notational convenience.

OMEL in Spain implements a different condition called Minimum Income Condition. GME in Italy implements a Prezzo Unico Nationale (PUN) that forces the price paid for specific demand orders in several locations to be identical. We will not model these features because they are very specific and somewhat questionable from an economic point of view.

European markets typically have few locations as these locations can be as large as a whole country as in France and Germany. Obviously a nodal model is by definition more precise and therefore preferrable, but this 'zonal vs nodal price' issue is irrelevant here because on both sides of the Atlantic the network models used are linear and therefore convex models. Moreover, our core model handles both of them nicely.

All models described in this Section are non-convex as they include integer variables. Therefore, there is *stricto sensu* no dual variables associated to constraints of these models. From a theoretical perspective, for mixed-integer programming, dual variables are replaced by the more general concept of dual functions (see for example [22]). From this perspective, classical dual variables are linear dual functions. But this concept of non-linear dual function does not seem to be useful for pricing, or equivalently for deciding which financial transactions should be executed as counterparts of physical exchanges of power resulting from solving the models described here. Moreover, in practice, all markets in the US and in Europe publish market prices, i.e. single prices for each market and location, and certainly not functions.

5.2 Strict Linear Pricing

In this Section, we formally describe the approach taken in Europe to deal with the non-convexities appearing in the core model (20)-(24).

To formally give a complete formulation of the model, we first need to write down the dual of (20)-(23):

$$\min \sum_i s_i + z^T C$$

$$s_i \geq \sum_{l,t} Q_{i,l,t}(P_i - p_{l,t}) \qquad \forall i, \qquad (x_i) \tag{25}$$

$$p = z^T A, \qquad (r) \tag{26}$$

$$s_i \geq 0 \qquad \forall i, \tag{27}$$

and complementarity conditions

$$z \perp (C - Ar) = 0 \tag{28}$$

$$s_i(1 - x_i) = 0 \qquad \forall i, \tag{29}$$

$$x_i(s_i - \sum_{l,t} Q_{i,l,t}(P_i - p_{l,t})) = 0 \qquad \forall i, \tag{30}$$

Still neglecting the integrality requirements (24), it is clear that at optimality, s_i is the surplus generated by order i. Also, from duality theory, it is clear that the solutions to the constraint system (21)–(23),(25)–(30) is exactly the set of welfare-maximizing solutions and their supporting prices.

Now, to model the block orders, we would like to add the constraints $x_j \in \mathbb{Z}, \quad \forall j \in J$ to this last constraint system. However, this would typically make the solution set empty.

The european solution is based on realizing that accepting out-of-the-money orders and rejecting in-the-money orders are both problematic, but not equally problematic. Indeed, accepting an out-of-the-money orders implies a negative-welfare transaction for one participant, while rejecting an in-the-money order does not. True, in the second case, there is a foregone opportunity, but only in theory : the reality is non-convex, so it is not necessarily the case that electricity can actually be bought or sold at the market price in sufficient quantity. In the first case, the transaction is directly and actually unacceptable for the market participant. This is reflected in practice: market participants can accept to have rejected in-the-money orders, but find it unacceptable to have accepted out-of-the-money orders.

This suggest the following formal solution: drop constraints (29) for block orders. That is we write

$$s_i(1 - x_i) = 0, \qquad \forall i \in I \setminus J, \tag{31}$$

instead of (29). Indeed, this exactly allows block orders to be rejected ($x_i < 1$) even when in-the-money ($s_i > 0$). To summarize, the European Market Model (EMM) is the following mixed-integer complementarity problem: maximize (20) such that (21)–(28),(30)–(31) are satisfied. Let us call this formulation the 'primal-dual formulation' and denote its feasible set by X^{EMM} and

Because of the presence of complementarity constraint Problem EMM is certainly much more difficult to solve than the MIP formed by the primal problem (20)–(23) augmented with integrality constraints (24).

We now give an alternative formulation for EMM, yielding some new insight on the problem. For any disjoint subsets $J_0, J_1 \in J$ of block orders, let us form the restricted problem $\mathcal{P}(J_0, J_1)$ where blocks in J_0 are forced to be rejected and blocks in J_1 are forced to be accepted:

$$\begin{aligned} \max\ & (20) \\ & (21) - (23) \\ & x_j = 0 \qquad \forall j \in J_0, \\ & x_j = 1 \qquad \forall j \in J_1. \end{aligned}$$

Proposition 3 *If problem $\mathcal{P}(J_0, \emptyset)$ admits an optimal solution (x, r) that is integral in $x_j, j \in J$, then there exists (p, z, s) such that $(x, r, p, z, s) \in X^{EMM}$.*

Proof. It suffices to take as (p, z, s) any optimal dual solution of $\mathcal{P}(J_0, \emptyset)$, and complete it with $s_j = \infty$ for $j = J_0$. This works in particular because (31) is not part of EMM for $i \in J_0$. ■

This suggests to formulate EMM as follows: find subsets of blocks that, if removed, make the problem (20)-(23) admit an integral optimal solution. Among all these solutions, take the one that maximizes welfare. Formally, let $\mathcal{J} \in \{0,1\}^J)$ be the set of sets J_0's for which Proposition 3 applies. Then EMM can be formulated as

$$\max_{J_0 \in \mathcal{J}, x, r} (20) \tag{32}$$

$$(21) - (23) \tag{33}$$

$$x_j = 0 \qquad \forall j \in J_0. \tag{34}$$

Let us denote by Y^{EMM} the set of vectors (x, r) feasible in (32)–(34). The following proposition formalizes the fact that this "primal formulation" is also a correct formulation of EMM.

Proposition 4 *$Y^{EMM} = Proj_{x,r}(X^{EMM})$.*

Proof. Proposition 3 shows $Y^{EMM} \subseteq Proj_{x,r}(X^{EMM})$. We now show the converse. Let $(x, r, p, z, s) \in X^{EMM}$ be given. Let $J_0 = \{j \in B : x_j = 0\}$. By definition of X^{EMM}, this solution satisfies all primal, dual, and complementarity constraints of Problem $\mathcal{P}(J_0, \emptyset)$. Therefore it is an optimal solution to $\mathcal{P}(J_0, \emptyset)$. This solution is also integral. Therefore $J_0 \in \mathcal{J}_I$ and $(x, r) \in Y^{EMM}$. ■

Corollary 5 *The primal-dual formulation* (20)–(28),(30)–(31) *and the primal formulation* (32)–(34) *are both correct formulations of EMM.*

The main advantage of this alternative formulation of EMM is that it uses primal variables only. In essence, we have modeled all non-convexities of EMM in the set $\mathcal{J}_I$. This shows that the "difficult" part of EMM is finding which set of block orders to remove. This is a truly combinatorial problem.

5.3 Algorithms

Unfortunately the set system $\mathcal{J}_I$ is not a matroid, and even not an independence system. So it is not surprising that a greedy algorithm will not ouput an optimal solution to EMM. Several algorithmic approaches have been proposed and/or implemented for solving EMM.

Algorithms like Sapri (old algorithms developed by Nordpool, and used until 2007 by Powernext and EEX, the french and german power exchanges), TLC/MLC [2] (algorithm developed by APX, and used for France-Belgium-The Netherlands market coupling until September 2009) are iterative heuristics. At each iteration these algorithms partition the set of block orders as $J = J_0 \cup J_1$ and solve $\mathcal{P}(J_0, J_1)$. Based on prices obtained as optimal dual variables of this last problem, they guess another partition, etc.... The process stops when a feasible solution is found. Cycling is avoided and termination guaranteed by ad-hoc rules that guarantee that at some point blocks will not be re-inserted.

Tergsteen et al. [21] model EMM as one large MIP model by defining, for each complementarity constraint (28),(30),(31), one binary additional variable and two big-M constraints. It is however part of Integer Programming folklore that this is not an efficient way to solve complementarity problems.

Sesam [7] (current algorithm developed and used at Nordpool) is a branch-and-bound algorithm that solves $\mathcal{P}(J_0, \emptyset) \cap \{x_j \in \mathbb{Z}\}$ for some J_0 at each node (so a MIP at each node).

Martin et al. [15] propose a more traditional branch-and-bound applied to (20)–(24), and therefore solves the LP $\mathcal{P}(J_0, J_1)$ for some $J_0, J_1 \subseteq J$ at each node. The missing dual and complementarity constraints (25)–(28),(30)–(31) are enforced by adding an inequality rejecting the current block selection at integer nodes that do not support linear prices. Because this algorithm is slow, they also implement a much aggressive version of the inequality, which results in a fast but heuristic algorithm.

COSMOS [1] is the algorithm used in the Central-Western European integrated market since November 2010. It is similar to the algorithm described by Martin et al. [15]. However the cuts added at integer nodes that do not support linear prices are stronger while still guaranteed not to cut any optimal solution.To the best of our knowledge, this is the only efficient algorithm that can solve EMM to optimality.

5.4 Issues with the European Model

Before discussing issues with the european market model just outlined, let us quickly point out its virtues. Because strict linear pricing is applied, no uplifts are necessary, and the financial balance of the clearing house is always guaranteed. Therefore there is no missing money problem. Also, because rejected orders are not financially compensated through uplift, it is not possible for a market participant to make money with rejected orders. Therefore, the incentives to bid true costs and technical constraints is higher than in the US market model discussed in Section 4.

5.4.1 Fairness

The fairness issue was actually identified recently because only recently was an algorithm (COSMOS) able to compute optimal solutions to EMM. These solutions were just not discovered by earlier algorithms. It was indeed observed that, for some instances, the welfare-maximizing solution included rejected blocks with small volume that were surprisingly very deep in-the-money. The reason why this block is not included in the solution is *not* that including it would modify prices and make the block itself out-of-the money, but rather that it would make some other, typically large and contributing much to the welfare, block out-of-the money.

Example 1 *Consider the following stylized example with one market and one time period. There are two supply block orders: block A has 2MW at 0 and block B has 100MW at 50. On the demand side there is one order C of 101MW at 60 and one order D of 100MW at 49.*

If block orders A and B were curtainable, we would solve (20)–(23) *which gives* $x_A = x_C = 1$, $x_B = \frac{99}{100}$ *and* $x_D = 0$, *with* $p = 50$ *and a welfare of 1110. Because of the combinatorial constraints, the optimal solution to EMM is* $x_B = 1$, $x_A = x_D = 0$ *and* $x_C = \frac{100}{101}$ *with a price of 60 and a welfare of 1000.*

It is impossible to accept order A because it would make the market price p to drop to 49, which would make the accepted block B out-of-the-money. Note that at this price of 49, order A is indeed in-the-money by a wide margin, and the participant having submitted this order might not be happy with the outcome.

The results are summarized in the next table

x_A	x_B	x_C	x_D	*price p*	*welfare*	*EMM feasible*	*US feasible*
1	$\frac{99}{100}$	1	0	50	1110	no	no
1	1	1	$\frac{1}{100}$	49	1109	no	yes
0	1	$\frac{100}{101}$	0	60	1000	$\boxed{\text{yes}}$	yes
1	0	$\frac{2}{101}$	0	60	120	yes	yes
0	0	0	0	n.a.	0	yes	yes

The previous example is a bit artificial and it could be argued that order A must be accepted before order B, because they are at the same location and period, and A has a lower price than B. They are two problems with this idea. First, this will decrease welfare, and it is not clear that this is desirable. Second, in a multi-hour setting, orders cannot be *a priori* ordered by price. So this idea is in general not applicable.

What this example shows is that block orders are more likely to be rejected if they have small volume, even if they have a more attractive price than some other. This is obvious from the formulation (32)–(34) of EMM: we want to remove block orders so as to make the solution feasible (in terms of prices) but that do not decrease the welfare too much. Since welfare is essentially a product price*volume, removing a block with small volume is more likely to decrease welfare less. And if a small block is enough, it might very well be a better candidate for removal than another order block with a better price but a larger volume.

A possible more precise definition of fairness in this context is as follows: a solution is fair if each rejected in-the-money block, if included in the solution, would be out-of-the-money. It is an open question whether there always exists a fair solution to EMM.

In practice however, most algorithms for EMM will output solutions that are not too unfair. Indeed iterative heuristics like Sapri and TLC/MLC remove accepted block orders at one iteration only if they are out-the-money. Sesam does only create a branch removing a block if this block is paradoxically accepted at the parent node. A parameter in COMSOS prevents blocks that are too deep in-the-money to be included in J_0 at any node of the branch-and-bound tree. So in practice, no algorithm proposed actually strongly suffers from this issue.

However, it is clear that ensuring fairness can only be attained at the cost of welfare, and the tradeoff between the two has to be resolved. The European Market Model already sacrifices welfare to get linear pricing. Therefore it would be somewhat logical to also sacrifice welfare to strictly guarantee fairness as well. It is also the subjective perception of the authors that this is the (maybe unconscious) position of most european power exchanges.

In the US model, this fairness issue is much less present by construction: a small non-convex order can only be rejected because of a global criterion (global welfare), and not because it would make some precise order out-of-the-money (as in the european model).

5.4.2 Walrasian Equilibrium

In a Walrasian equilibrium, each agent cannot afford any combination of goods superior to the one allocated to him. The EMM solution does not constitute a Walrasian equilibrium as agents owning rejected in-the-money orders would prefer these orders to be accepted at current market prices. Note already that this only potentially applies to rejected orders. On the other hand, The US solution constitutes a Walrasian equilibrium as orders, if necessary, are financially compensated through the uplifts. Note that this is true both for rejected in-the-money orders and accepted out-of-the-money orders.

In what respect is it important that the market model implements a Walrasian equilibrium? The reason is that without this property, some agents have a potential reason to leave the market and sell or buy (or not) their goods in another market (e.g. over-the-counter). This will usually imply a loss of welfare and market efficiency.

Note again that the european solution is assymetric: it compensates for loss-making paradoxically accepted orders, but do not compensate for loss-of-opportunity paradoxically rejected orders. This is justified because the situation itself is assymetric. Indeed the owner of a paradoxically accepted order has an immediate out-of-the-market improving alternative: if he had just not submitted his order on the market, he would be better off. This is in contrast with the owner of a paradoxically rejected order: the loss of opportunity is only theoretical. To realize it, he has to find a counterparty for his non-convex order at market prices, which is always costly and not necessarily possible, especially because the solution is already welfare-maximizing. Therefore this assymetric treatment of paradoxical orders on european markets actually makes sense.

Moreover, in practice, this does not seem to be a real problem in european markets. These markets are not mandatory and do not implement a Walrasian equilibrium, but they attract a substantial fraction of all trades done at day-ahead. We can therefore conclude that the fact that the European solution does not implement a true Walrasian equilibrium is not truly problematic.

5.4.3 Welfare Sub-Optimality

One obvious drawback of the european model compared to the US one is that it produces solutions that generate less welfare. From the example above, it is clear that this loss of welfare can be large in principle. Another question is whether this loss is large in practice. It is the author's impression that in the current situation, the loss is rather small (a few hundreds to a few thousands of euros for CWE coupling). However, this just reflects the fact that the non-convex orders in these markets represents a rather small proportion (in volume) of the market. The problem being close to convex, it is not surprising that different ways of dealing with the small non-convex aspect does not make a large global difference.

However, it should be mentioned that the size and number of block (non-convex) orders in the european markets are limited ex ante. The fact that this limit is imposed, shows that the market would be willing to submit more and larger non-convex orders. One possible explanation is that this limit is imposed precisely to avoid a large loss of welfare. This is anyway not desirable from a market architecture point pof view, as the (close-to-convex) market cannot represents the (substantially non-convex) reality well.

5.4.4 Algorithmic Complexity and Modelling Capabilities

The US model is a MIP, while the European model is a similar MIP with additional complementarity constraints. Therefore it is certainly much more difficult to solve in practice than the US model. This has several consequences:

1. It is less likely to find a provably optimal (or in general good) solution in the European model. This might raise fairness issues (on top of the ones discussed in Section 5.4.1).

2. To guarantee tractability of the resulting optimization problem, this has led the european markets to limit the number, the size and the type of non-convex orders. This prevents in the European model the agents to submit orders accurately representing their physical assets.

This is definitely a major issue in the european model, especially for small players that do not own a large portfolio of assets to allow them to "smooth" their non-convexities.

6 A New Proposal

Based on the discussion in the two last sections, we propose now a new market model that retains as much as possible the good properties of the two models discussed. This new proposal is essentially based on the US model, but modified in the spirit of the european model so as to solve the two issues described in Section 4.3.

In the spirit of the US model, the new proposal

1. is welfare-maximizing,
2. implements uplift to loss-making accepted orders, and
3. roughly minimizes uplift.

But in the spirit of the EU model, we do not aim at a walrasian equilibrium. Instead the uplift is sufficient to guarantee the No Loss property of Section 3. In practice, uplifts are just enough so that no order is money-losing, even though at the published market prices, some orders could be executed more profitably. The fact that non-executed orders are never financially compensated solves the Truthtelling issue in the same way as it is solved by the European market model.

Further, we internalize in the model the financing of uplifts. By doing so, we deal with the missing money issue discussed in Section 4.3.2 by having all in-the-money orders contribute in proportion to their accepted volume and with a maximum of their surplus. By construction, under our proposition, the necessary uplift is always lower than the welfare, and therefore that there always exists such a solution.

The new proposal follows the two-step approach found in the US:

1. solve the non-convex (MIP) model (1)–(4), ideally but not necessarily to optimality,
2. Based on the outcome of 1., compute prices.

Step 1. is obvious and need not be discussed further. For Step 2, we introduce one variable π_i for each order i that roughly represents the difference between the market price and the actual price at which an order is settled. If negative, it represents a compensation (or uplift) for an order that is money-losing at market prices. If positive, it represents a positive contribution to the financing of uplifts. Finally we define a single variable λ that represents the maximum contribution in €/Mwh to the financing of the uplifts (missing money). The overall goal is to minimize λ, leading to the following linear program.

$$\min \quad \lambda \tag{35}$$

$$\sum_{i,l} |g_l^i(x^{*i})|\pi_i = 0 \tag{36}$$

$$\sum_{l} |g_l^i(x^{*i})|\pi_i \leq f_i(x^{*i}) - \sum_{l} g_l^i(x^{*i})p_l \qquad \forall i \tag{37}$$

$$\pi_i \leq \lambda \qquad \forall i \tag{38}$$

Constraint (36) represents the financial balance of the clearing house, guaranteeing financial balance. It balances the uplifts with the contribution-to-uplifts of the in-the-money orders. If an order i is out-of-the money at the choosen market prices p_l, then the righthand side of constraint (37) is negative, forcing the corresponding term in (36) to be negative as well. This must then be compensated by allowing some π_i of in-the-money orders to be positive, forcing λ to be positive by constraint (38). In particular, if there is an optimal solution with $\lambda = 0$, this implies that we can find market prices p_l for which all accepted orders are in-the-money.

The financial transaction linked with an accepted order i is then $\sum_l g_l^i(x^{*i})p_l + \sum_l |g_l^i(x^{*i})|\pi_i$ (positive for a payment to the clearing house). The first term is the settlement of the power produced/consumed at market prices. The second term is negative for out-of-the money orders (uplift) and positive for in-the-money orders (uplift financing). Again, constraint (37) ensures that the uplift is always just sufficient to make an out-of-the-money order just at-the-money, and not to make in-the-money orders out-of-the money.

The objective function is motivated by fairness: all in-the-money orders will contribute at the same level in €/MWh (or \$/MWh), up to their capacity to contribute. From a computational point of view, this linear program is much easier to solve than the optimization problem to be solved at the first step, as there are as many variables and constraints and there are orders.

Note that the proposed model does not strictly speaking minimize the total uplift, but goes roughly speaking in the same direction. Indeed, it minimizes the maximum contribution to the uplift financing of individual orders, and therefore will also roughly speaking minimizes the total contribution (which is equal to the total uplift).

The proposed model is also very flexible as it can accommodate additional requirements:

- adding linear constraints of type (25),(29) to guarantee that rejected out-of-the money convex orders are indeed out-of-the-money,
- adding linear constraints of the type (26),(28) to guarantee that price differences between nodes are always justified by congested network elements,
- having the transmission operator contribute to the uplift if the congestion rent is positive and receiving an uplift if it is negative. This can be done similarly to the orders: introduce a variable ζ representing the contribution to the uplift if positive and the uplift if negative, add the constraints $\zeta \leq \lambda$ and $\sum_l \zeta r_l^* \leq \sum_l p_l r_l^*$, and replace the financial balance constraint (36) by

  $$\sum_{i,l} |g_l^i(x^{*i})|\pi_i + \sum_r |r_l|\zeta = 0$$

  The last term represents the contribution of network operators to the financing of uplifts.

7 Computational Example

In this section, we illustrate the main approaches discussed on a simple example introduced by Scarf [20] and used in several subsequent papers. We use the variant discussed in Gibrik et al. [11]. There is only one time period and one location, three generators and no price-sensitive consumption. The three generators have the following characteristics

Generator	A	B	C
Fixed cost	0	6000	8000
Var cost up to 100MW	65	40	25
Var cost bt 100MW and 200MW	110	90	35

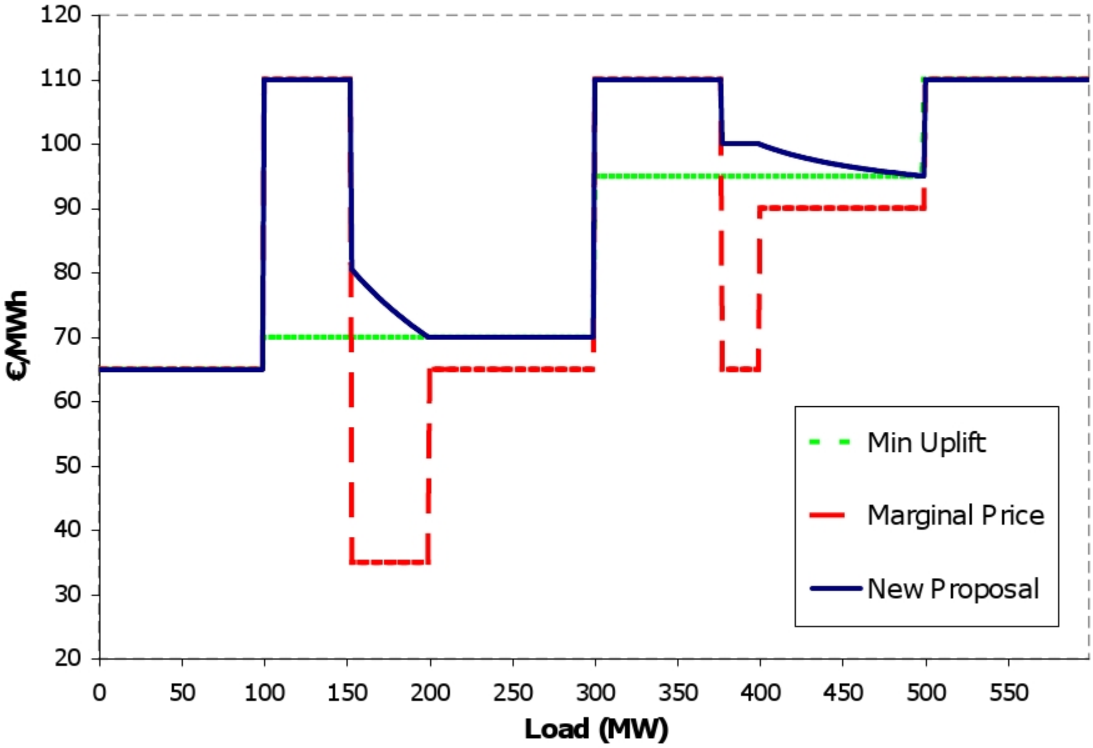

Figure 1: Clearing prices under different approaches

We let the pre-specified load y to be vary from 0 to 600 (maximum possible output in the system). Figure 7 gives marginal price, the uplift-minimizing price and the price coming from our new proposal in function of the load y.

The marginal price is the additional cost of meeting one more MW, and is equivalent to the optimal dual variable of the power balance constraint when the integer variables are fixed at their optimal value. It is not an increasing function, as the problem include fixed costs and is non-convex.

The uplift minimizing price is an increasing function of y as it is equivalent to the dual variable of a linear program with righthand side y. The uplift compensate for losses and foregone opportunities.

Under our new proposal, the market price is not a increasing function of the load. This is simply reflects the non-convexity of the underlying system, and there is no abstract virtue in having market price increase with load, a property of convex markets. In this example the uplift paid is null for all loads. This is because the demand is price-taking, and therefore, there is no upper limit on the market price. Indeed in problem (35)-(38), the only reason why a high price p_l will imply a larger objective function is when some scheduled demand order becomes out-of-the-money. In that case, uplift will need to be paid, increasing the objective function. In our example, the is no price-sensitive demand, and therefore no upper limit on the market price and no uplift. Of course, certain unscheduled orders will be in-the-money, but our model explicitly allows this.

Until a load of 150MW, the price matches the marginal cost of generator A, as it is the only generator scheduled. Starting from 150MW, only generator C is scheduled, the marginal price is 3580€/MWh and under our new proposal the price gradually decreases from 80€/MWh (just enough to have C recoup his costs) to 70€/MWh, matching average cost of generator C. From 200MW, generator A is again scheduled, and the price stays at 70€/MWh as this is the average cost of generator C when producing 200MW. From 300MW to 377MW, the market price is again 110€/MWh as Generator A output increases from 100MW to 177MW. From 378MW, Generator A output drops to 77MW and Generator B output is set at 100MW. Its average cost at this level is 100€/MWh and this is the new market price until the load reaches 400MW. From there the output of generator B starts increasing and the market price decrease with the average cost of generator B, reaching 95€/MWh at 500MW. The last 100MW of load can only be met at a pure variable cost of 110€/MWh and this is the market price under all approached until 600MW.

Note that this example is rather peculiar because of the lack of price-sensitive demand. This implies no uplift for our new proposal, but also that the market price will always be higher than the two other approaches analyzed. This will not be the case with price sensitive demand: the market price under our new proposal can very well then be under the uplift-minimizing and marginal prices. Note also that the lack of price-sensitive demand is unusual at least for european day-ahead markets, where many generators buy back large volumes of electricity already sold in longer term markets.

Another way in which the three proposals differ is the following. Marginal pricing is extremely myopic, as it sees only the next MW to be produced. Uplift-minimizing prices are very global because even producers with very high fixed cost can play a role in setting the price, well before they are economical (for example between 100MW and 150MW, where generator C is setting the price). Our proposal is a mix between the two: until a generator is not scheduled at all, it cannot play a role in setting the price. But if some volume is executed, then its fixed cost is fully taken into account in influencing the market price. This results in a middle position in this respect compared to the other approaches analyzed.

8 Conclusion

Strict Linear Pricing in non-convex markets is a mathematical impossibility. In the context of electricity markets, two different classes of solutions have been proposed to this conundrum. In US, some orders are not settled at the market price, but at their bidding price, deviating from uniform pricing (all orders are financially settled at the same prices). This creates a disincentive to bid one's own true cost, and creates a missing money problem for the clearing house of the market. In Europe, all accepted orders are in-the-money are settled at the uniform market price. This implies that the welfare-maximizing solution is considered infeasible and also that the optimization problem is much less convex and more difficult to solve. This also creates fairness issues for orders of small volume, and the solution obtained does not implement a Walrasian equilibrium.

Based on this analysis, we draw on the two existing solutions to propose a new market model. Compared to the european model, the new proposal :

(i) generates more welfare

(ii) does not create fairness issues for ordes of small volume

(iii) is less non-convex, and therefore computationally more easier to solve, allowing for more complex orders better representing physical assets

Compared the US model, the new proposal :

(i) finds the same primal production/consumption/transportation program,

(ii) does not give incentives to bid with the goal of having its order rejected, therefore giving less reasons to deviate from bidding one's own true costs and diminishing the need of regulation,

(iii) the uplifts are minimal, and explicitly financed within the model.

This analysis shows that this new proposal essentially improves on the implemented market models on both sides of the atlantic, and is therefore a good target model for both.

We have illustrated our approach on a toy example to highlight and make clear some aspects of our proposition. But the new proposal should be tested with much more realistic test instances directly coming from real-life, historical market data.

Eventhough we shortly discusses the algorithmic approaches available, this is not the main point of focus of this work. The mixed-integer model to be solved in the first step of the US model and our now proposal is still challenging, especially if more complex order types or stochastic aspects would be introduced. This indeed essentially makes the model close or equivalent to the unit commitment problem. Further research in that direction is still needed to be able to routinely solve this type of problem.

Other aspects of power markets seem only loosely related to the 'pricing under non-convexities" topic discussed here: zonal vs nodal pricing, relationships with forward and balancing markets, financial transmission rights and others. Analyzing whether these aspects are impacted by the modification proposed here is also important.

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