

2012/61



Equilibrium in secure strategies

Mikhail Iskakov and Alexey Iskakov



CORE

DISCUSSION PAPER

Center for Operations Research
and Econometrics

Voie du Roman Pays, 34
B-1348 Louvain-la-Neuve
Belgium

<http://www.uclouvain.be/core>

CORE DISCUSSION PAPER
2012/61

Equilibrium in secure strategies

Mikhail ISKAKOV¹ and Alexey ISKAKOV²

December 2012

Abstract

A new concept of equilibrium in secure strategies (EinSS) in non-cooperative games is presented. The EinSS coincides with the Nash-Cournot Equilibrium when Nash-Cournot Equilibrium exists and postulates the incentive of players to maximize their profit under the condition of security against actions of other players. The new concept is illustrated by a number of matrix game examples and compared with other closely related theoretical models. We prove the existence of equilibrium in secure strategies in four classic games that fail to have Nash-Cournot equilibria. On an infinite line we obtain the solution in secure strategies of the classic Hotelling's price game (1929) with a restricted reservation price and linear transportation costs. New type of monopolistic solution in secure strategies is discovered in the Tullock Contest (1967, 1980) of two players. For the model of insurance market we prove that the contract pair found by Rothschild, Stiglitz and Wilson (1976) is always an equilibrium in secure strategies. We characterize all equilibria in secure prices in the Bertrand-Edgeworth duopoly model with capacity constraints.

Keywords: equilibrium in secure strategies, Hotelling model, Tullock contest, insurance market, Bertrand-Edgeworth duopoly.

JEL classification: C72, D03, D43, D72, L12, L13

¹ Institute of Control Sciences RAS, Moscow.

² Institute of Control Sciences RAS, Moscow.

We are deeply indebted to Claude d'Aspremont who helped us to find an elegant and intuitively clear formulation of the EinSS concept and inspired us to write this paper (partly during our stay at CORE). The results on the Tullock Contest were obtained with participation of Alexey Zakharov.

1. Introduction

Applications of Generalized Nash-Cournot equilibria concepts is increasing steadily in the past 50 years. In general the objects of research are real-world applications or games with some additional mathematical structure (Facchinei and Kanzow, 2007). In this paper we propose new generalization of Nash-Cournot equilibrium which introduce an additional criteria of security. We define the notion of threat and the notion of secure strategy which are the basis of the model of cautious behavior. In many practical situations the security considerations are indeed no less important than increasing profit. For example rational players do not break the rules if the expected penalty exceeds the profit from breaking the rules. In a similar way we assume that cautious player refuses from increasing his profit if it creates threat to lose more. He would rather prefer the greatest possible *secure* profit at a given strategies of other players. Taking into account this logic of behavior one can discover equilibrium positions which sometimes can not be revealed by standard logic of best responses. These equilibria positions we call *Equilibria in Secure Strategies (EinSS)*. The first formulation of EinSS was published in (Iskakov, 2005). In this paper we present a reformulation of the concept and discuss all its aspects in detail. Generally speaking, the EinSS is realized when all players maximize their profits under the condition to avoid all threats from other players. We prove that any Nash-Cournot equilibrium is an Equilibrium in Secure Strategies. However, the EinSS can exist in games that fail to have Nash-Cournot equilibria as will be demonstrated by examples in this paper.

Our following step is to investigate the concept of *the best secure response (BSR)*. The Nash-Cournot equilibrium is the profile in which the strategy of each player is the best response. In a similar way the strategy of each player in the EinSS turns out to be the best secure response. However, the set of profiles of best secure responses (or BSR-profiles) may be larger than the set of EinSS. An additional condition defined as *stability* makes the two coincide. Thereby we prove that a BSR-profile is an Equilibrium in Secure Strategies if and only if it is stable. This property provides a practical method for finding the EinSS. First, all BSR-profiles are to be found, then the unstable BSR-profiles are to be excluded.

The concept of the EinSS assumes that players make conjectures about the threats of other players. Implicitly this implies that players may choose their actions non-simultaneously. Therefore it would be interesting to find the game with minimal elements of dynamics which would mimic the reasoning of the players in a similar way to the EinSS. This investigation resulted in the concept of *a game with an uncertain insider*. Briefly it can be formulated in the following way. All players simultaneously choose their strategies in the original game and after that an "insider" is chosen randomly among them and has an opportunity to change his strategy. Nobody knows beforehand who is going to be the insider (even the insider himself). We prove that the EinSS in the game is the Nash-Cournot equilibrium of the corresponding game with an uncertain insider, if all players resolve the uncertainty by the maximin criterion. However the set of equilibria in the game with an uncertain insider is wider than the set of Equilibria in Secure Strategies.

In order to illustrate the practical value and adequacy of the proposed concept we consider in this paper four classic games that fail to have Nash-Cournot equilibria without using mixed strategies as was suggested by Dasgupta and Maskin (1986). The first one

is the Hotelling's price game with a restricted reservation price and linear transportation costs (1929) on an infinite line. We obtain solution in secure strategies for arbitrary distance between two players. The second model is the Tullock Contest (Tullock 1967, 1980) of two players. The EinSS for arbitrary values of the power parameter can be found. Depending on the power parameter there are three types of equilibria. Either it coincides with the Nash-Cournot equilibrium found by Tullock (1980) or it corresponds to the newly discovered monopolistic solution or it represents an intermediate case of equilibrium of unequal or limited access. As the third example we consider the model of insurance market suggested by Rothschild and Stiglitz (1976) and Wilson (1977). We prove that the so-called "Rothschild-Stiglitz-Wilson" contract pair is always an equilibrium in secure strategies even when it is not a Nash-Cournot equilibrium. And finally we characterize all equilibria in secure prices in the Bertrand-Edgeworth duopoly model. We find that in those cases when Nash-Cournot equilibrium does not exist EinSS prices are lower than the monopoly price.

The organization of the paper is as follows. In the next section the definitions of EinSS are given. In Section 3 we introduce the concept of the best secure response profile and investigate its relation to the EinSS. In Section 4 we discuss different ways of weakening of the EinSS concept. Finally in Sections 5, 6, 7 and 8 we consider the Hotelling's price game on an infinite line, the Tullock Contest of two players, the model of insurance market of Rothschild, Stiglitz and Wilson and the Bertrand-Edgeworth duopoly model.

2. Equilibrium in Secure Strategies

We consider n-person non-cooperative game in the normal form $G = (i \in N, s_i \in S_i, u_i \in R)$. The concept of equilibria is based on the notion of threat and on the notion of secure strategy.

Definition 1. *A threat of player j to player i at strategy profile s is a pair of strategy profiles $\{s, (s'_j, s_{-j})\}$ such that $u_j(s'_j, s_{-j}) > u_j(s)$ and $u_i(s'_j, s_{-j}) < u_i(s)$. The strategy profile s is said to pose a threat from player j to player i .*

Definition 2. *A strategy s_i of player i is a secure strategy for player i at given strategies s_{-i} of all other players if profile s poses no threats to player i . A strategy profile s is a secure profile if all strategies are secure.*

In other words a threat means that it is profitable for one player to worsen the situation of another. A secure profile is one where no one gains from worsening the situation of other players.

Definition 3. *A secure deviation of player i with respect to s is a strategy s'_i such that $u_i(s'_i, s_{-i}) > u_i(s)$ and $u_i(s'_i, s'_j, s_{-ij}) \geq u_i(s)$ for any threat $\{(s'_i, s_{-i}), (s'_i, s'_j, s_{-ij})\}$ of player $j \neq i$ to player i .*

There are two conditions in the definition. In the first place a secure deviation increases the profit of the player. In the second place his gain at a secure deviation covers losses which could appear from retaliatory threats of other players. It is important to

note that secure deviation does not necessarily mean deviation into secure profile. After the deviation the profile (s'_i, s_{-i}) can pose threats to player i . However these threats can not make his or her profit less than in the initial profile s . We assume that the player with incentive to maximize his or her profit securely will look for secure deviations.

Definition 4. *A secure strategy profile is an **Equilibrium in Secure Strategies (EinSS)** if no player has a secure deviation.*

There are two conditions in the definition of EinSS. There are no threats in the profile and there are no profitable secure deviations². The second condition implicitly implies maximization over the set of secure strategies.

Let us now formulate the first important property of the EinSS concept.

Proposition 1. *Any Nash equilibrium is an Equilibrium in Secure Strategies.*

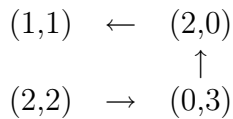
Proof. Since Nash equilibrium poses no threats so it is a secure profile. And no player in Nash equilibrium can improve his or her profit using whatever deviation. Both conditions of the EinSS are fulfilled. \square

First this means that a Nash equilibrium is always secure profile in terms of the proposed definitions. Second, the existence results can not be worse for EinSS than for Nash equilibrium. Whenever a Nash equilibrium exists an EinSS also exists. However for some practically important problems without Nash equilibrium (such as Hotelling's model and Tullock contest which will be considered in this paper) the EinSS exists and provides an interesting interpretation.

Let us now consider a simple matrix game example having no Nash equilibrium in order to illustrate the definitions introduced above:

	t_1	t_2
s_1	(1,1)	(2,0)
s_2	(2,2)	(0,3)

One can find all threats in the game. First, the strategy profile (s_2, t_1) poses a threat to player 1 as we move from payoffs (2, 2) to payoffs (0, 3). Second, the strategy profile (s_1, t_2) poses a threat to player 1 as we move from payoffs (2, 0) to payoffs (1, 1). And finally the profile (s_2, t_2) poses a threat from player 1 to player 2 as we move from payoffs (0, 3) to payoffs (2, 0). In all three cases one player can make himself better off and another player worse off. These threats in the game can be visualized graphically in the following way:



²Some details on previously used form of definitions are included in the Appendix A for interested readers.

The only secure profile in the game (which is secure for both players) is the profile (s_1, t_1) with payoffs $(1, 1)$. If players were choosing best responses sequentially in the game they would end up in an infinite cycle so that there is no Nash equilibrium. This situation can change if we take into account the considerations of security. The profiles with payoffs $(2, 2)$, $(0, 3)$ and $(2, 0)$ can not be an equilibrium in secure strategies because they pose threats. The profile (s_1, t_1) is the only secure profile in the game. The second player can not increase his profit by any deviation from it. There is a profitable deviation for the first player from this profile into the profile (s_2, t_1) with payoffs $(2, 2)$. However it is not a secure deviation since the first player can lose more after the response deviation of the second player from the profile (s_2, t_1) with payoffs $(2, 2)$ into the profile (s_2, t_2) with payoffs $(0, 3)$. Therefore no player has in the profile (s_1, t_1) a secure deviation and this profile is an EinSS. This means that a cautious player would prefer the guaranteed payment of 1 in the (s_1, t_1) to the possibility of gaining 2 in (s_2, t_1) accompanied by a high-risk to get zero in (s_2, t_2) .

Let us now add additional row and column to the matrix of the previous game.

	t_1	t_2	t_3
s_1	$(1,1)$	$(2,0)$	$(-1,-1)$
s_2	$(2,2)$	$(0,3)$	$(-1,-1)$
s_3	$(-1,-1)$	$(-1,-1)$	$(0,0)$

Now we have a Nash equilibrium with payoffs $(0, 0)$ (perhaps, not a very good one!). It is also an EinSS according to Proposition 1. Threats in this game are the same as in the previous example. All newly added profiles are secure. Profiles with payoffs $(-1, -1)$ are secure only because they are the worst for both players, so they can not be EinSS. Equilibrium (s_3, t_3) with payoffs $(0, 0)$ is Pareto dominated by all other profiles of the game. However there is a profile (s_1, t_1) which is still an EinSS and dominates (s_3, t_3) . This example shows that there may be games with Nash equilibrium which nevertheless have another more reasonable solution given by an Equilibrium in Secure Strategies.

For some games the reverse of Proposition 1 is true. For instance for strictly competitive games.

Proposition 2. *Any Equilibrium in Secure Strategies in a strictly competitive game is a Nash equilibrium.*

Proof. Suppose there is an EinSS in a strictly competitive game which is not a Nash equilibrium. Then there is at least one player who can increase his profit and there is at least one player who will decrease his profit. Therefore the profile is not secure and can not be EinSS. \square

However this property may not hold if the condition of strict competitiveness is weakened. For instance it does not hold for almost strictly competitive games introduced by Aumann (1961) on the basis of the concept of twisted equilibrium. A twisted equilibrium point in the two-player game is a pair of strategies at which neither player can decrease the other player's payoff by a unilateral change in strategy. A game is called almost strictly competitive if (i) the set of payoffs vectors to Nash equilibrium strategy

profiles is equal to the set of payoffs vectors to twisted equilibrium strategy profiles and (ii) if the set of Nash equilibrium strategy profiles and the set of twisted equilibrium strategy profiles intersect. Let us modify the previous matrix game in the following way:

	t_1	t_2	t_3
s_1	(1,1)	(2,0)	(-1,1)
s_2	(2,2)	(0,3)	(-1,1)
s_3	(1,-1)	(1,-1)	(0,0)

It is an almost strictly competitive game. There is a unique Nash equilibrium profile (s_3, t_3) with payoffs (0,0) which at the same time is a unique twisted equilibrium. The profile (s_1, t_1) with payoffs (1, 1) is still an EinSS. However it is not a Nash equilibrium.

3. Best Secure Response

The definition of EinSS implicitly implies maximization of payoff functions over the set of secure strategies. We can therefore expect that the EinSS is an analogue of the Nash equilibrium on a narrower set of strategies (the secure ones). In this case the EinSS would be a profile in which the "secure strategy" of each player is the best one in the same way as the Nash equilibrium is a profile in which strategy of each player is the best response. In order to clarify this question let us start with the rigorous definition of the best secure response. Denote by $V_i(s_{-i})$ the set of secure strategies of player i at given strategies s_{-i} of all other players. Notice that $V_i(s_{-i})$ can be empty if all strategies of player i are insecure at s_{-i} .

Definition 5. *A strategy s_i^* of player i is a Best Secure Response to strategies s_{-i}^* of all other players if*

$$s_i^* \in V_i(s_{-i}^*) \quad \text{and} \quad u_i(s^*) = \max_{s_i \in V_i(s_{-i}^*)} u_i(s_i, s_{-i}^*).$$

A profile s^ is the Best Secure Response profile (BSR-profile) if strategies of all players are Best Secure Responses.*

Let us now consider the following matrix game example.

	t_1	t_2	t_3
s_1	(0,0)	(2,2)	(2,2)
s_2	(2,2)	(1,3)	(3,1)
s_3	(2,2)	(3,1)	(1,3)

There are no Nash equilibria and no EinSS in the game. The profiles (s_2, t_1) , (s_2, t_3) , (s_3, t_1) , (s_3, t_2) are insecure for the first player. The profiles (s_1, t_2) , (s_1, t_3) , (s_2, t_2) , (s_3, t_3) are insecure for the second player. Therefore (s_1, t_1) is the only and the best secure profile in the game. Are there any secure deviations from it? If the first player for example deviates from the profile (s_1, t_1) with payoffs (0,0) into profile (s_2, t_1) or (s_3, t_1) with payoffs (2, 2) his new position will be subjected to threat of the second player. However the expected loss from these threats (equal to 1) does not exceed the gain obtained at deviation from (s_1, t_1) (equal to 2). Therefore deviations from (s_1, t_1) are secure for players since no threats can make their payoffs less than zero payoffs in (s_1, t_1) . Hence the profile

(s_1, t_1) can not be stable situation in the game and it is not the EinSS. Based on this example one can establish the following relationship between the BSR-profile and the EinSS.

Proposition 3. *Any Equilibrium in Secure Strategies is a BSR-profile. A BSR-profile may not be an Equilibrium in Secure Strategies.*

Proof. An EinSS is a secure profile by definition. And it must be the best secure response for each player since otherwise there is a player who can increase his profit by secure deviation. Therefore an EinSS is a BSR-profile. The reverse is not true. In the above example the profile (s_1, t_1) is the BSR-profile. However it is not the EinSS. \square

Corollary. *Let M_{NE} , M_{EinSS} and M_{BSR} be the sets of Nash Equilibria, Equilibria in Secure Strategies and BSR-profiles respectively. Then $M_{NE} \subseteq M_{EinSS} \subseteq M_{BSR}$. The reverse inclusions generally do not hold.*

Let us now take the previous matrix game example and increase payoffs in the profile (s_1, t_1) and strengthen the threats:

	t_1	t_2	t_3
s_1	(1,1)	(2,2)	(2,2)
s_2	(2,2)	(0,3)	(3,0)
s_3	(2,2)	(3,0)	(0,3)

Now the BSR-profile (s_1, t_1) with payoffs (1, 1) is an EinSS since deviations from it pose threats to receive less payoffs than in the profile (s_1, t_1) . These examples demonstrate the property of BSR-profiles which makes the difference. In order to be EinSS the BSR-profile has to satisfy an additional condition which we can define as *stability*. More precisely,

Definition 6. *A BSR-profile is **stable** if there is no player i and deviation s'_i such that $u_i(s'_i, s_{-i}) > u_i(s)$ and $u_i(s'_i, s'_j, s_{-ij}) \geq u_i(s)$ for any threat $\{(s'_i, s_{-i}), (s'_i, s'_j, s_{-ij})\}$ of player $j \neq i$ to player i .*

In the unstable BSR-profile at least one player has non-secure alternatives with threats which in all cases are more profitable for him than staying in the initial BSR-profile. From the above definitions 4, 8 and 9 it follows:

Proposition 4. *A BSR-profile is an Equilibrium in Secure Strategies if and only if it is stable.*

Propositions 3 and 4 provide a practical method for finding EinSS. The BSR-profile is a Generalized Nash Equilibrium concept and all the corresponding results and algorithms of maximization can be applied to finding BSR-profiles. When all BSR-profiles are found, then unstable BSR-profiles are to be excluded.

4. Further Weakening of Equilibrium in Secure Strategies

4.1. Weak Equilibria in Secure Strategies

In the definition 3 we considered deviations with retaliatory threats bringing losses which exactly cover the gain obtained at deviation as secure deviations. However these deviations are intermediate as they lie on the boundary between secure and insecure deviations. One can define them as a special class of secure deviations.

Definition 7. *A secure deviation s'_i of player i with respect to s is **trivial** if there is a threat $\{(s'_i, s_{-i}), (s'_i, s'_j, s_{-ij})\}$ of player $j \neq i$ to player i such that $u_i(s'_i, s'_j, s_{-ij}) = u_i(s)$.*

One can argue that profiles in which players can only make trivial secure deviations could be still considered as weakly stable. If we consider these intermediate profiles as weak equilibria we come to the concept of weak EinSS.

Definition 8. *A secure strategy profile is a **weak Equilibrium in Secure Strategies** if no player has a non-trivial secure deviation.*

Obviously all EinSS are also weak EinSS. The following three matrix games illustrate the concept of weak EinSS.

	t_1	t_2	t_3		t_1	t_2	t_3		t_1	t_2	t_3
s_1	(0,0)	(2,2)	(2,2)	s_1	(1,1)	(2,2)	(2,2)	s_1	(1.5,1.5)	(2,2)	(2,2)
s_2	(2,2)	(1,3)	(3,1)	s_2	(2,2)	(1,3)	(3,1)	s_2	(2,2)	(1,3)	(3,1)
s_3	(2,2)	(3,1)	(1,3)	s_3	(2,2)	(3,1)	(1,3)	s_3	(2,2)	(3,1)	(1,3)

The matrix game on the left has been already used as an example of the game without EinSS. There is no weak EinSS either. In the matrix game on the right the payoffs for the profile (s_1, t_1) are increased up to (1.5, 1.5) so that this profile becomes an EinSS. Let us consider the matrix game in the middle which represents an intermediate case. Player 2 can deviate from profile (s_1, t_1) with payoffs (1, 1) into profile (s_1, t_2) with payoffs (2, 2). However there is a retaliatory threat of player 1 to deviate into profile (s_3, t_2) with payoffs (3, 1) which reduces payoff of player 2 back to 1, i.e. back to the same level as he had in (s_1, t_1) . Therefore the deviation of player 2 from (s_1, t_1) into (s_1, t_2) is a trivial secure deviation according to Definition 7. One can see in a similar way that all deviations from (s_1, t_1) are trivial secure deviations. Hence the profile (s_1, t_1) is not an EinSS because there are secure deviations. At the same time it is a weak EinSS according to Definition 8 because all these secure deviations are trivial.

4.2. Game with Uncertain Insider

Players in the EinSS make conjectures about threats of other players. Implicitly it implies that the players may choose their actions non-simultaneously. In Economics the assumption of simultaneous and independent decision making by players is indeed a very strong one. This raises natural questions about the relationship of the EinSS concept with dynamic games (especially if there are more than two players). Let us try to find a game with minimal elements of dynamics which would reproduce the reasoning of players in a

similar way as in the EinSS. Let us suppose that after the players simultaneously choose their strategies an "insider" is chosen randomly among them and has an opportunity to change his strategy. Nobody knows beforehand who is going to be the insider (even the insider himself). Let us suppose also that players adopt a cautious behavior with respect to the actions of the insider.

Let us provide now a rigorous formulation. Take a non-cooperative game in the normal form $G = \{N = \{1, \dots, n\}, s_i \in S_i, u_i(s) \in R\}$. We define an associated sequential game. During the first stage all players select simultaneously their strategies $s = (s_1, \dots, s_n)$. At the second stage Nature chooses randomly the insider player number $j_0 \in N$. Then finally player j_0 either keeps the same strategy s_{j_0} or choose another one that would increase his profit: $\tilde{s}_{j_0}(s) \in \Theta_{j_0}(s) = \{s_{j_0}\} \cup \{s'_{j_0} \in S_{j_0} : u_{j_0}(s'_{j_0}, s_{-j_0}) > u_{j_0}(s)\}$. The final payoffs of players are $u_i(\tilde{s}_{j_0}, s_{-j_0})$.

We assume further that all players adopt at the beginning a cautious behavior and resolve uncertainty by the maximin criterion so that their payoff functions can be written as $\hat{u}_i(s) = \min_{j \in N, j \neq i, \tilde{s}_j \in \Theta_j(s)} u_i(\tilde{s}_j, s_{-j}) = \min_{j \in N, \tilde{s}_j \in \Theta_j(s)} u_i(\tilde{s}_j, s_{-j})$. This defines a game $\hat{G} = \{N, s_i \in S_i, \hat{u}_i \in R\}$ that we call *the game with an uncertain insider*. The following proposition establishes basic relationship between an EinSS and the corresponding game with uncertain insider.

Proposition 5. *An Equilibrium in Secure Strategies of the game G is a Nash equilibrium of the corresponding game \hat{G} with an uncertain insider.*

Proof. Let s^* be the EinSS of the game G . By the definition of EinSS there are no threats in the profile s^* . Thus no deviation $\tilde{s}_j \in \Theta_j(s^*)$ of player j can decrease the profit of other players, i.e. $\min_{j \in N, \tilde{s}_j \in \Theta_j(s^*)} u_i(\tilde{s}_j, s^*_{-j}) \geq u_i(s^*)$. Besides $s^*_j \in \Theta_j(s^*)$ and

$\min_{j \in N, \tilde{s}_j \in \Theta_j(s^*)} u_i(\tilde{s}_j, s^*_{-j}) \leq u_i(s^*)$. Therefore for all i we have $u_i(s^*) = \hat{u}_i(s^*)$.

Assume there is a deviation s'_i of player i such that $\hat{u}_i(s'_i, s^*_{-i}) > \hat{u}_i(s^*)$, i.e. $\min_{j \in N, j \neq i, s'_j \in \Theta_j} u_i(s'_i, s'_j, s^*_{-ij}) > \hat{u}_i(s^*) = u_i(s^*)$. In particular this implies that $u_i(s'_i, s^*_{-i}) > u_i(s^*)$ and $u_i(s'_i, s'_j, s^*_{-ij}) > u_i(s^*)$ for any threat $\{(s'_i, s^*_{-i}), (s'_i, s'_j, s^*_{-ij})\}$ of player $j \neq i$ to player i . According to the definitions 3 and 4 the player i can increase his profit by secure deviation s'_i and s^* is not the EinSS. This is a contradiction, and therefore our assumption was wrong. $\hat{u}_i(s'_i, s^*_{-i}) \leq \hat{u}_i(s^*)$ for all i and deviations s'_i , i.e. s^* is the Nash equilibrium of the game \hat{G} with an uncertain insider. \square

However the set of Nash equilibria in \hat{G} is wider than the set of EinSS in G . This can be seen if we come back to the matrix game from the previous section (without EinSS):

	t_1	t_2	t_3
s_1	(0,0)	(2,2)	(2,2)
s_2	(2,2)	(1,3)	(3,1)
s_3	(2,2)	(3,1)	(1,3)

The profiles (s_1, t_2) , (s_1, t_3) , (s_2, t_1) , (s_3, t_1) are Nash Equilibria in the corresponding game \hat{G} with uncertain insider but are not even secure in the original game G . The following proposition characterize the set of Nash equilibria in \hat{G} in case of secure profiles.

Proposition 6. *A secure profile in the game G is a Nash equilibrium of the game \hat{G} if and only if it is a weak Equilibrium in Secure Strategies in the original game G .*

Proof. Let s^* be a weak EinSS. By definition s^* is a secure profile. Like in the previous proposition for the secure profile s^* we can prove that for all i : $u_i(s^*) = \hat{u}_i(s^*)$. Let us consider arbitrary player i and his arbitrary change of strategy s'_i . Either $u_i(s'_i, s^*_{-i}) \leq u_i(s^*)$ and $\hat{u}_i(s'_i, s^*_{-i}) \leq u_i(s'_i, s^*_{-i}) \leq u_i(s^*) = \hat{u}_i(s^*)$. Or $u_i(s'_i, s^*_{-i}) > u_i(s^*)$ and there is a threat $(s'_i, s^*_{-i}) \rightarrow (s'_j, s'_i, s^*_{-ij})$ of player $j \neq i$ such that $u_i(s'_j, s'_i, s^*_{-ij}) \leq u_i(s^*)$. Notice that $s'_j \in \Theta_j(s'_i, s^*_{-i})$ and $\hat{u}_i(s'_i, s^*_{-i}) = \min_{j \in N, j \neq i, s'_j \in \Theta_j(s'_i, s^*_{-i})} u_i(s'_j, s'_i, s^*_{-ij}) \leq u_i(s'_j, s'_i, s^*_{-ij}) \leq u_i(s^*) = \hat{u}_i(s^*)$. We have proved that in both cases $\hat{u}_i(s'_i, s^*_{-i}) \leq \hat{u}_i(s^*)$, i.e. s^* is a Nash equilibrium of the game \hat{G} .

Let now s^* be a Nash equilibrium of the game \hat{G} . If s^* is secure profile in the game G we have for all i : $u_i(s^*) = \hat{u}_i(s^*)$. Let us assume that player i has a non-trivial secure deviation s'_i . This implies that for any $s'_j \in \Theta_j(s'_i, s^*_{-i})$: $u_i(s'_j, s'_i, s^*_{-ij}) > u_i(s^*)$. Therefore $\hat{u}_i(s'_i, s^*_{-i}) = \min_{j \in N, j \neq i, s'_j \in \Theta_j(s'_i, s^*_{-i})} u_i(s'_j, s'_i, s^*_{-ij}) > u_i(s^*) = \hat{u}_i(s^*)$ and profile s^* can not be Nash equilibrium of the game \hat{G} . This is a contradiction, and therefore our assumption was wrong. No player has a non-trivial secure deviation in s^* . Profile s^* is a weak Equilibrium in Secure Strategies. \square

The model of the game with uncertain insider gives another (dynamic) approach to the concept of EinSS. An EinSS is an equilibrium in the game of cautious players with the minimal dynamics introduced by the possibility to change strategy of one random player, unknown in advance.

4.3. Games with more than two players

Definitions 1-4 of EinSS are based on the analysis of pairwise interaction of players. Nevertheless the threats in the game may take more complicated forms. This raises the question if the EinSS can adequately describe the collective behavior of many players. In this section we show how our concept can be weakened in order to take into account simultaneous and independent actions of several players.

Let us first consider the following illustrative matrix game example. The first player chooses the matrix (r_1 or r_2), the second chooses the row (s_1 or s_2) and the third chooses the column (t_1 or t_2).

$$\begin{array}{c}
 r_1 : \\
 \begin{array}{c|cc}
 & t_1 & t_2 \\
 \hline
 s_1 & (1,0,0) & (1,0,0) \\
 s_2 & (1,0,0) & (1,0,0)
 \end{array}
 \end{array}
 \quad
 \begin{array}{c}
 r_2 : \\
 \begin{array}{c|cc}
 & t_1 & t_2 \\
 \hline
 s_1 & (2,0,0) & (2,0,1) \\
 s_2 & (2,1,0) & (-1,-1,-1)
 \end{array}
 \end{array}$$

To make this example more obvious we will call it "the game with rescue boat at shipwreck". The first player (the boat captain) has two strategies: r_1 - keep the boat for himself and r_2 - provide place in the rescue boat for other players. Players 2 and 3 have two strategies: s_1, t_1 - avoid the rescue boat and s_2, t_2 - try to get place in the rescue boat. The rescue boat sinks with all players at profile (r_2, s_2, t_2) when all players get place in the boat. There are no threats in the game according to an formal definition. There are three EinSS: (r_1, s_2, t_2) , (r_2, s_1, t_2) , (r_2, s_2, t_1) which are also Nash equilibria. If we

assume that players 2 and 3 take their actions sequentially then the game will be set either in the equilibrium profile (r_2, s_1, t_2) or in the equilibrium profile (r_2, s_2, t_1) which implies two players on board and one player trying to save his life by himself. Let us now consider deviation of the first player from (r_1, s_1, t_1) into (r_2, s_1, t_1) as we move from payoffs $(1, 0, 0)$ to payoffs $(2, 0, 0)$. Formally it is a secure deviation for the first player since there are no individual threats of other players to him in his new position. The profile (r_1, s_1, t_1) is not formally an EinSS. However if we assume that players 2 and 3 would take their actions simultaneously and independently (which is probably the case at the shipwreck) they will end up in the profile (r_2, s_2, t_2) and would not only do harm to player 1 but also to themselves. Therefore if player 1 takes into account the possibility of simultaneous and independent actions of players 2 and 3 then he would not consider deviation $(r_1, s_1, t_1) \rightarrow (r_2, s_1, t_1)$ as a secure one. He would rather consider the profile (r_1, s_1, t_1) as an equilibrium in the generalized sense.

The concept of EinSS in our current formulation takes into account only individual deviations and hence can not treat this effect properly. Perhaps in its current formulation it also can not describe properly games with multiple players creating small threats which can be ignored individually but taken together become crucial. However these threats which arise as the result of simultaneous and independent actions of many players could be taken into account by an appropriate extension of the concept of secure deviation.

Definition 3'. *A secure deviation of player i with respect to s is a strategy s'_i such that $u_i(s'_i, s_{-i}) > u_i(s)$ and, whenever $u_l(s'_i, s'_l, s_{-il}) > u_l(s'_i, s_{-i})$ for all l in some set $N' = \{j, \dots, k\}$, $i \notin N'$, then $u_i(s'_i, s'_j, \dots, s'_k, s_{-ij\dots k}) \geq u_i(s)$.*

The Definition 3' sets more restrictive conditions for the secure deviation and corresponds to a more cautious behavior. This modification reduces the number of secure deviations for a given profile. All EinSS according to the Definitions 1 – 4 would still be EinSS after changing the Definition 3 for the Definition 3'. However some "new" EinSS appear which correspond to the possibility of threats from simultaneous and independent actions of other players. It is important to notice that these other players according to Definition 3' do not take into account the behavior of each other. Therefore their behavior represents rather the behavior of a crowd than a collusion in a group of players. Indeed the crowd behavior plays an important role in many practical situations when analyzing security which justifies our modification.

5. EinSS in the Hotelling's Model

To illustrate the concept of EinSS we examine the classic model of spatial competition between two players formulated by Hotelling (1929). The principal theoretical problem of this model is that for a great variety of transportation cost functions no price equilibrium exists. In particular, D'Aspremont et al. (1979) showed that in the original Hotelling's game with linear transportation costs there is no price equilibrium when duopolists choose locations too close to each other. The following matrix game example can be considered as an illustration for the Hotelling's game:

	t_1	t_2	t_3
s_1	(2,2)	(3,3)	(8,0)
s_2	(3,3)	(4,4)	(5,5)
s_3	(0,8)	(5,5)	(6,6)

There is no Nash equilibrium in this matrix game as well. The incentive to maximize profits impels players to choose strategies with a higher number. However if at least one player chooses the third strategy it makes profitable for his competitor to choose the first strategy and leave the first player with zero profit. The first player can in turn choose the strategy with a lower number and get positive profit again. The highest possible secure profits are reached at the profile with payoffs (4, 4) which is the EinSS.

This corresponds to the situation in the Hotelling's price game when one player can undercut his rival's price and take away his entire business with profit to himself. However the player pressed out of the market can decrease his price and regain some positive profit. Although the Hotelling's game has no Nash price equilibrium when players choose locations too close to each other it does have just like the above matrix game an equilibrium in secure strategies. The solution of the Hotelling's price game in secure strategies in the original setting was presented in M.Iskakov and A.Iskakov (2012). In the particular case of the discrete Hotelling's problem an equilibrium concept which coincides with the EinSS was published in Shy (2002). In this paper we provide solution in secure strategies of the Hotelling's price game with a restricted reservation price on an infinite line.

On an infinite line two sellers of a homogeneous product with zero production cost are located at the distance δ from each other. The sellers maximize profits by setting prices p_1, p_2 noncooperatively. Customers are evenly distributed with a unit density along the line. When buying from one of the sellers the consumer bears a transportation cost which is linear in the distance. The transportation rate is t . A customer purchases from the seller who quotes the lower full price (including transportation). In contrast to the original version of Hotelling's model we assume that the customer refrains from buying if the full price exceeds his reservation price v . The sold quantities are equal respectively to the lengths of intervals with the customers choosing the corresponding seller. Therefore the profit functions of the firms are:

$$(1) \quad \tilde{u}_1(\tilde{p}_1, \tilde{p}_2) = \begin{cases} \tilde{u}_1^I = 2\tilde{p}_1(v - \tilde{p}_1)/t, & \tilde{p}_1 < \tilde{p}_2 - \delta t \\ \tilde{u}_1^{II} = \tilde{p}_1(v - \tilde{p}_1 + \\ + \min\{v - \tilde{p}_1, \frac{\delta t + \tilde{p}_2 - \tilde{p}_1}{2}\})/t, & |\tilde{p}_1 - \tilde{p}_2| \leq \delta t \\ 0, & \tilde{p}_1 > \tilde{p}_2 + \delta t \end{cases}$$

$$\tilde{u}_2(\tilde{p}_1, \tilde{p}_2) = \begin{cases} \tilde{u}_2^I = 2\tilde{p}_2(v - \tilde{p}_2)/t, & \tilde{p}_2 < \tilde{p}_1 - \delta t \\ \tilde{u}_2^{II} = \tilde{p}_2(v - \tilde{p}_2 + \\ + \min\{v - \tilde{p}_2, \frac{\delta t + \tilde{p}_1 - \tilde{p}_2}{2}\})/t, & |\tilde{p}_1 - \tilde{p}_2| \leq \delta t \\ 0, & \tilde{p}_2 > \tilde{p}_1 + \delta t \end{cases}$$

These expressions can be simplified if we make the following change of variables:

$$(2) \quad u = \tilde{u}t/v^2, \quad p = \tilde{p}/v, \quad d = \delta t/v$$

In this dimensionless form profit functions (1) can be written as:

$$(3) \quad \begin{aligned} u_1(p_1, p_2) &= \begin{cases} u_1^I = 2p_1(1-p_1), & p_1 < p_2 - d \\ u_1^{II} = p_1(1-p_1 + \min\{1-p_1, \frac{d+p_2-p_1}{2}\}), & |p_1 - p_2| \leq d \\ 0, & p_1 > p_2 + d \end{cases} \\ u_2(p_1, p_2) &= \begin{cases} u_2^I = 2p_2(1-p_2), & p_2 < p_1 - d \\ u_2^{II} = p_2(1-p_2 + \min\{1-p_2, \frac{d+p_1-p_2}{2}\}), & |p_1 - p_2| \leq d \\ 0, & p_2 > p_1 + d \end{cases} \end{aligned}$$

Dimensionless prices and payoffs depend upon only one free parameter d instead of three parameters δ, v, t . In order to find equilibria in secure strategies in the dimensionless Hotelling's game (3) one can first analyze the threats and identify the secure profiles. Then one can find the Best Secure Response functions, identify BSR-profiles and select the stable ones. According to Propositions 3 and 4 they will correspond to the Equilibria in Secure Strategies. The obtained result is summarized in the following proposition.

Proposition 7. *The dimensionless Hotelling's price-setting game $\{i \in \{1, 2\}, p_i \in [0, 1], u_i(p_1, p_2) \in R\}$ on an infinite line with a restricted reservation price and the profit functions (3) has the following solution in secure strategies depending on the distance d between the sellers:*

$$(4a) \quad \text{when } d \in \left[0, \frac{10\sqrt{10} - 14}{67}\right] \approx [0, 0.263]:$$

$$p_1^* = p_2^* = p^* = \frac{2 + 7d - \sqrt{17d^2 - 4d + 4}}{4},$$

$$u_1^* = u_2^* = 2(p^* - d)(1 - p^* + d);$$

$$(4b) \quad \text{when } d \in \left[\frac{10\sqrt{10} - 14}{67}, \frac{6}{7}\right]:$$

$$p_1^* = p_2^* = p^* = \frac{2 + d}{5}, \quad u_1^* = u_2^* = \frac{3}{2}p^{*2};$$

$$(4c) \quad \text{when } d \in \left[\frac{6}{7}, 1\right] - \text{multiple solutions:}$$

$$\max\left\{\frac{1}{2}, \frac{10}{7} - d\right\} \leq p_1^* \leq \min\left\{\frac{4}{7}, \frac{3}{2} - d\right\},$$

$$p_2^* = 2 - d - p_1^*, \quad u_i^* = 2p_i^*(1 - p_i^*), \quad i \in \{1, 2\};$$

$$(4d) \quad \text{when } d \geq 1: \quad p_i^* = u_i^* = 0.5, \quad i \in \{1, 2\}.$$

Proof. See Appendix B. \square

Corollary. *The Hotelling's price-setting game $\{i \in \{1, 2\}, \tilde{p}_i \in [0, v], \tilde{u}_i(\tilde{p}_1, \tilde{p}_2) \in R\}$ on an infinite line with a restricted reservation price v and the profit functions (1) has the*

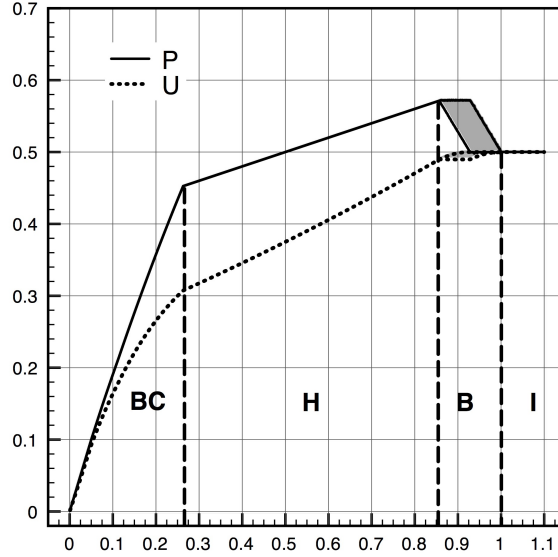


Fig. 1: The equilibrium secure prices (P) and profits (U) in the price Hotelling's game on an infinite line depending on the distance d between the stores.

following solution in secure strategies depending on δ, v, t :

$$(5) \quad \tilde{u}(\delta, v, t) = \frac{v^2}{t} u\left(\frac{\delta t}{v}\right), \quad \tilde{p}(\delta, v, t) = vp\left(\frac{\delta t}{v}\right)$$

where $u(d)$ and $p(d)$ is given by (4).

Proof is given by inverse change of variables in relation to (2). \square

The dependence (4) of the equilibrium prices and profits from the distance between the stores is shown in Fig.1 for dimensionless price game. The shaded area corresponds to the multiple solutions. The analysis of the price competition on a line in secure strategies allows to distinguish four qualitative cases of interaction between competitors. When they are situated very close (the area BC in Fig.1) both players are limited by the threat of mill-price undercutting. The corresponding Equilibrium in Secure Strategies (4a) can be interpreted as *the Bilateral Containment equilibrium (or BC-Equilibrium)*. Under the threat of being driven out of the market by undercutting the equilibrium prices in the BC area are much lower as compared Hotelling price equilibrium. In the second area (the area H in Fig.1) the Nash equilibrium (4b) found by Hotelling is realized. One can call it *the Hotelling Equilibrium (or H-Equilibrium)*. In the third area (the area B in Fig.1) the competition reaches the multiple Nash equilibria (4c). They can be called *Borderline Equilibria or B-Equilibria* and interpreted as a division of spheres of influence on the border of trade zones of players. And finally in the fourth area when $d > 1$ the local monopoly (4d) is realized when trade zones of players are not intersected. One can call it *the Independent Price Equilibrium (or I-Equilibrium)*.

Let us consider multiple price equilibria in the area B . The full price on the border of trade zones of players reaches in B-equilibrium exactly the reservation price. The

equilibrium payoff function can be calculated through equilibrium price according to (4c) as $u^* = 2p^*(1 - p^*)$. It is not profitable for player to raise the price and break away from trade zone of the rival when $\frac{\partial u}{\partial p}|_{p=p^*+0} = 2(1 - 2p^*) < 0$, i.e. when $p^* \geq \frac{1}{2}$. It is not profitable for player to lower the price and take market share from the rival when $\frac{\partial u}{\partial p}|_{p=p^*-0} = \frac{4-7p}{2} > 0$, i.e. when $p^* \leq \frac{4}{7}$. Hence inside the price interval $\frac{1}{2} \leq p^* \leq \frac{4}{7}$ we obtain multiple price equilibrium solutions for both players.

6. EinSS in the Tullock Contest³

In the Tullock Contest n players compete for a prize and each player exerts effort x_i so as to increase his probability of winning $x_i / \sum_{j=1}^n x_j$ (Tullock, 1967, 1980). Scaperdas (1996) suggested a more generalized form of the game with the expected profits of players $x_i^\alpha / \sum_{j=1}^n x_j^\alpha - x_i$, $\alpha > 0$. The detailed analysis of the game in terms of secure strategies will be provided in our new publication (M.Iskakov, A.Iskakov, A.Zaharov, 2012). Here we consider the Tullock Contest of two players to illustrate the EinSS concept. The players exert efforts x_1 and x_2 . The contest is supposed to be fair and the payoff functions of players are taken as:

$$(6) \quad u_1 = \frac{x_1^\alpha}{x_1^\alpha + x_2^\alpha} - x_1, \quad u_2 = \frac{x_2^\alpha}{x_1^\alpha + x_2^\alpha} - x_2, \quad \alpha > 0$$

This game reaches the unique Nash equilibrium $(\alpha/4, \alpha/4)$ when $\alpha \leq 2$ and there is no equilibrium when $\alpha > 2$.

The following matrix game example can be considered as an illustration for the Tullock Contest of two players.

	t_1	t_2	t_3
s_1	(0,0)	(0,4)	(0,3)
s_2	(4,0)	(2,2)	(-1,-1)
s_3	(3,0)	(-1,-1)	(-2,-2)

There is Nash equilibrium (s_2, t_2) in this game when players get equal payoffs (2, 2). There are also two EinSS (s_1, t_3) and (s_3, t_1) in which one player gains 3 and the other player has to be content with zero payoff to avoid losses. Formally the first player could deviate from (s_3, t_1) into (s_2, t_1) increasing his payoff from 3 to 4. But he would prefer not to do it since it is not secure deviation and the other player would in turn bring the game into the Nash equilibrium (s_2, t_2) with equal payoffs (2, 2). This example shows that even if there is a unique Nash equilibrium (which seems to complete the study of the game) there may be additional equilibria in secure strategies which significantly alter the overall picture. In the given case there are three stable profiles which have different values for players. Which of them will be realized in the game is not predetermined and each player is interested in the profile favorable to him (like in the game of battle of the sexes).

In the Tullock Contest the equilibria from the above matrix game correspond to the EinSS of the two possible types. One of them coincides with the Nash equilibrium found by Tullock (1980). The other two equilibria correspond to the monopolistic EinSS. In these equilibria the winning monopolist fixes high enough payment for the rent to create

³The results of this section were obtained with participation of Alexey Zakharov.

the entrance barrier for the other player making him unprofitable to participate in the competition.

The general algorithm of finding solution in secure strategies is following. First the set of secure profiles is found as well as the best secure responses of players. Then the BSR-profiles are found as an intersection of the best secure responses of players plotted in the plane of strategies (x_1, x_2) . And finally the conditions of the EinSS are checked for these BSR-profiles.

Secure profiles and BSR-profiles for two players in Tullock Contest are shown in the plane of strategies (x_1, x_2) in Fig.2. The shaded (gray) area corresponds to secure profiles. The solid points and curves represent BSR-profiles. The analysis of the Tullock Contest of two players in terms of secure strategies can be summarized by the following proposition.

Proposition 8. *When $0 < \alpha < 1$ the Tullock Contest (6) of two players reaches the following unique equilibrium in secure strategies (which is also Nash equilibrium):*

$$(7) \quad \{(\alpha/4, \alpha/4)\}.$$

When $1 \leq \alpha \leq 2$ the Tullock Contest (6) reaches the following equilibria in secure strategies (the first profile is a Nash equilibrium):

$$(8) \quad \{(\alpha/4, \alpha/4)\} \cup \{(0, \bar{x})\} \cup \{(\bar{x}, 0)\},$$

where $\bar{x} = \frac{1}{\alpha}(\alpha - 1)^{\frac{\alpha-1}{\alpha}}$, $\alpha > 1$ and $\bar{x} = 1$, $\alpha = 1$

and all other equilibria in secure strategies lie on the curve:

$$(9) \quad \left\{ (x_1, \xi^+(x_1)) : \frac{\alpha-1}{\alpha} \leq x_1 \leq \frac{\alpha}{4} \right\} \cup \left\{ (\xi^+(x_2), x_2) : \frac{\alpha-1}{\alpha} \leq x_2 \leq \frac{\alpha}{4} \right\},$$

where $\xi^+(x_i) \equiv \left(\frac{x_i^{\alpha-1}}{2} \left(\alpha - 2x_i + \sqrt{\alpha^2 - 4\alpha x_i} \right) \right)^{1/\alpha}$, $\max \left\{ 0, \frac{\alpha^2 - 1}{4\alpha} \right\} \leq x_i \leq \alpha/4$.

When $\alpha > 2$ the Tullock Contest (6) reaches only two monopolistic equilibria in secure strategies (which are not Nash equilibria):

$$(10) \quad \{(0, \bar{x})\} \cup \{(\bar{x}, 0)\}.$$

Proof. For a sketch of the proof see Appendix C. Full proof see in (M.Iskakov, A.Iskakov, A.Zaharov, 2012). \square

Remark. Our numerical computations showed that all points on the curve (9) are in fact multiple equilibria in secure strategies.

One can easily check that when $1 \leq \alpha \lesssim 1.08$ all other EinSS are Pareto dominated by the Nash equilibrium $(\alpha/4, \alpha/4)$. When $1.08 \lesssim \alpha < 2$ the two monopolistic EinSS coexist with (but are not dominated by) the symmetric Nash equilibrium in a similar way as they coexist in the matrix game example considered before. The difference however with the matrix game is the intermediate EinSS lying on the curve (9). One can easily verify that when $\alpha \lesssim 1.22$ all these equilibria are Pareto dominated by the Nash equilibrium

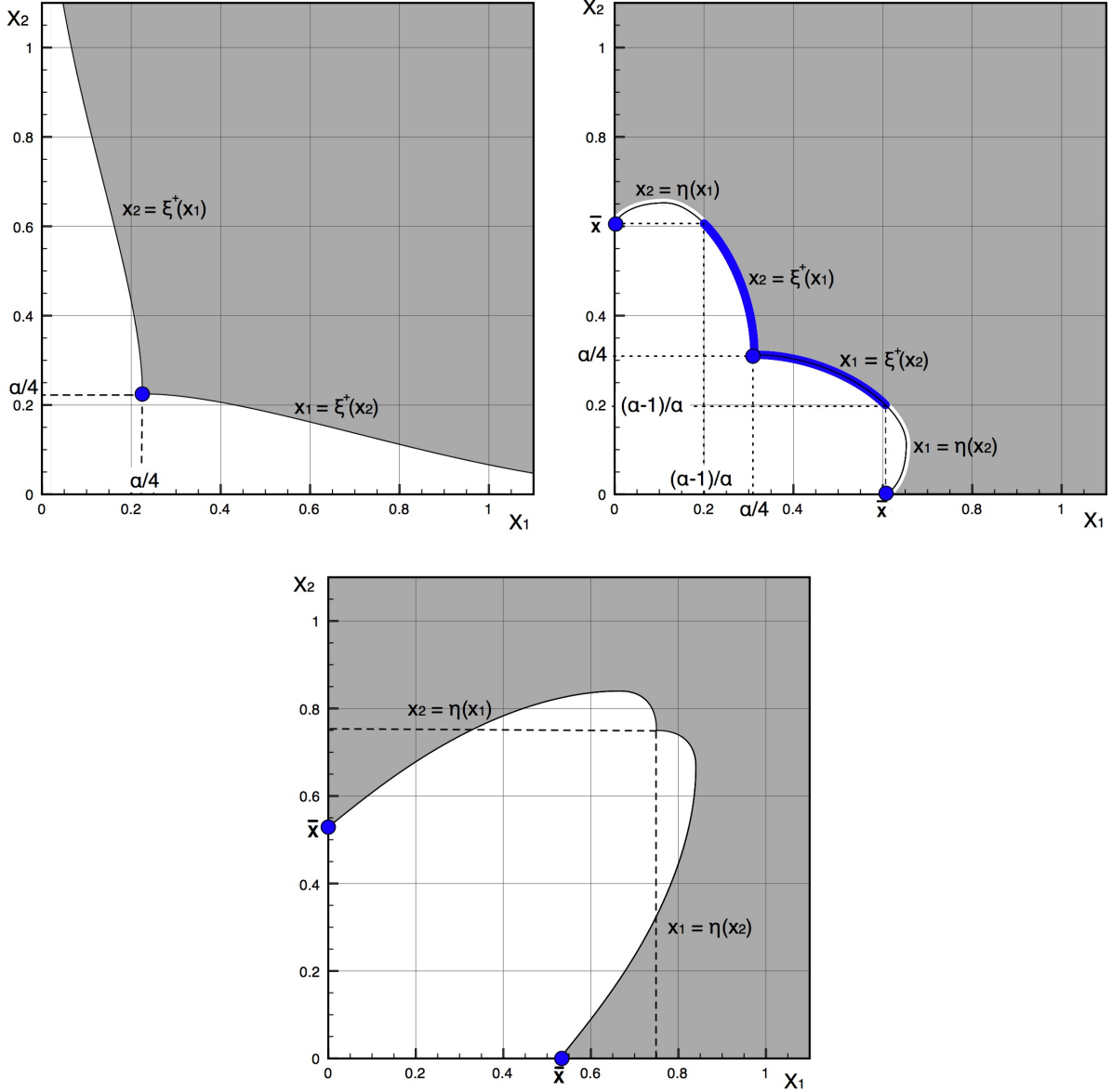


Fig. 2: Secure profiles (gray area) and BSR-profiles (solid points and curves) for two players in Tullock Contest depending on the parameter α : $\alpha < 1$ (left), $1 \leq \alpha \leq 2$ (right) and $\alpha > 2$ (center). $\bar{x} \equiv \frac{1}{\alpha} (\alpha - 1)^{\frac{\alpha-1}{\alpha}}$, $\alpha > 1$ and $\bar{x} = 1$, $\alpha = 1$.

$(\alpha/4, \alpha/4)$. However when $1.22 \lesssim \alpha \leq 2$ they can be interpreted as an intermediate type of solutions when players participate in the contest non-symmetrically. One (the "stronger") player with larger level of effort chooses his strategy x and another (the "weaker") player adjust his strategy by choosing his best response $(\xi^+)^{-1}(x)$ at a given x . The weaker player always gains less than the stronger player and less than he would gain in the symmetric Nash equilibrium. The payoff of weaker player monotonically decrease from his payoff in the Nash equilibrium to zero with the increase of the effort of stronger player. One can show that if $\alpha \geq \sqrt{2} \approx 1.41$ the payoff of stronger player monotonically increases along the curve (9) with the increase of his effort. Therefore the intermediate EinSS lying on the curve (9) can be considered as positions which in terms of profitability are in between the Nash equilibrium and the monopolistic EinSS. The stronger player continuously increases his payoff and weak player continuously decreases his payoff up to the point $(\frac{\alpha-1}{\alpha}, \frac{1}{\alpha}(\alpha-1)^{(\alpha-1)/\alpha})$ in which the weak player leaves the contest and the strong player settles himself in the monopolistic EinSS.

The monopolistic EinSS is a new type of equilibria in the Tullock game of the rent-seeking. In these equilibria the player prefer to fix his or her secure monopolistic position rather than to participate in the competition. Moreover when power parameter $\alpha > 2$ the monopolistic situation is the only stable position in the game in terms of secure strategies. The logic of the best responses can not reveal the possibility of such kind of equilibria since it does not take into account the security considerations and assumes the player would choose the most profitable but insecure and possibly eventually not-profitable for him strategy.

The power parameter α can be interpreted as stiffness of competition in the rent-seeking game. There is an egalitarian distribution of rent at $\alpha \leq 1$, i.e. the probability to win for the player paying less is more than proportional to his contribution. Following the classification of North, Wallis and Weingast (2009) we can interpret the corresponding Nash equilibrium as *an equilibrium of an open access*. If $\alpha \geq 2$ then the competition rules are strongly differentiating. The chances to win for players contributing small payments are much less than proportional to their contributions. The only possible equilibrium in this case can be interpreted as an *equilibrium of the privileged monopoly* which fixes access to resources or institutions to one player. In the intermediate case of $1 \leq \alpha \leq 2$ the rules of competition are weakly differentiating and there are possibilities both for the equilibrium of an open access and for the equilibrium of the privileged monopoly. Furthermore there are also intermediate equilibria which could be interpreted as *equilibria of unequal or limited access*.

7. EinSS in the model of insurance market

In this section we consider a model of insurance market suggested by Rothschild and Stiglitz (1976) and Wilson (1977) and show that it always has an equilibrium in secure strategies.

Two insurance companies sell insurance contracts to consumers which fall into two classes: n_H high risk consumers and n_L low risk consumers. High risk consumers have accidents with probability p_H and low risk consumers with probability $p_L < p_H$. All consumers have the same strictly positive initial endowment $w = (w_1, w_2) \in R^2$ representing their income in the two states of nature: that of having an accident (w_2) and that of not (w_1). Preferences of all consumers are represented by the same strictly

concave utility function u . Each insurance contract is a vector $c = (c_1, c_2) \in R^2$, where c_1 is the insurance premium and c_2 is the accident benefit net of premium. The endowment of consumer with insurance contract becomes $(w_1 - c_1, w_2 + c_2)$. Consumers of a given risk class j buy at most one insurance contract c (if they prefer it to their initial endowment w) which maximizes their expected utility:

$$(11) \quad V_j(c) = p_j u(w_2 + c_2) + (1 - p_j) u(w_1 - c_1), \quad \text{where } j = H \text{ or } L$$

Each insurance company offers a pair of contracts (c^H, c^L) , where without loss of generality one can assume that high risk consumers find c^H at least as desirable as c^L and low risk consumers find c^L at least as desirable as c^H . The expected profit of the company from the contract $c^j = (c_1^j, c_2^j)$ sold to a customer of class j ,

$$(12) \quad \pi_j(c^j) = -p_j c_2^j + (1 - p_j) c_1^j, \quad \text{where } j = H \text{ or } L$$

Suppose that company 1 offers contracts $(c^H(1), c^L(1))$ and company 2 $(c^H(2), c^L(2))$. Then the expected profit of company 1 is

$$(13) \quad U_1 = \sum_{j=H,L} \begin{cases} n_j \pi_j(c^j(1)), & \text{if } V_j(c^j(1)) > V_j(c^j(2)) \\ \frac{1}{2} n_j \pi_j(c^j(1)), & \text{if } V_j(c^j(1)) = V_j(c^j(2)) \\ 0, & \text{otherwise} \end{cases}$$

And expected profit of company 2 is defined symmetrically.

The detailed interpretation and investigation of this model can be found in Rothschild and Stiglitz (1976) and Wilson (1977). In particular it was proven that, if a pure strategy equilibrium exists, both companies must offer the same contract pair $c^* = (c^{*H}, c^{*L})$ satisfying $w_2 + c_2^{*H} = w_1 - c_1^{*H}$ (i.e. high risks are perfectly insured), $\pi_H(c^{*H}) = \pi_L(c^{*L}) = 0$ (i.e. customers of each risk class generate zero expected profits for companies) and $V_H(c^{*H}) = V_H(c^{*L})$ (i.e. high risk customers are indifferent between the low risk contract and their own). Following Dasgupta and Maskin (1986) we will call contract pair c^* a "Rothschild-Stiglitz-Wilson" or RSW contract pair. However if there is a sufficiently high proportion of low risk customers one company can deviate from c^* and earn positive profit. It can offer a "pooling" contract c^{**} that both high and low risk customers prefer to c^* . It was proven that there is no Nash equilibrium in this case. We can show however that contract pair c^* in this situation is still an equilibrium in terms of secure strategies.

Proposition 9. *A RSW contract pair c^* is always an Equilibrium in Secure Strategies in the insurance market game.*

Proof. In our proof we will follow the graphical procedure introduced in Rothschild and Stiglitz (1976). In Fig.3 the horizontal and vertical axis represent income of customers in the states of no accident and accident respectively. The point E with coordinates $w = (w_1, w_2)$ is the uninsured state of customer. Purchasing the insurance contract $c = (c_1, c_2)$ moves the individual from E to the point $(w_1 - c_1, w_2 + c_2)$. The set of insurance contracts for low-risk customers that break even in the conditions of free entry and perfect competition lies on the line EL . The set of contracts for high-risk customers lies on the line EH respectively. If company offers a "pooling" contract which is the same for both groups (such that $c^H = c^L$) in case of equilibrium it shall lie on the market odds line EF .

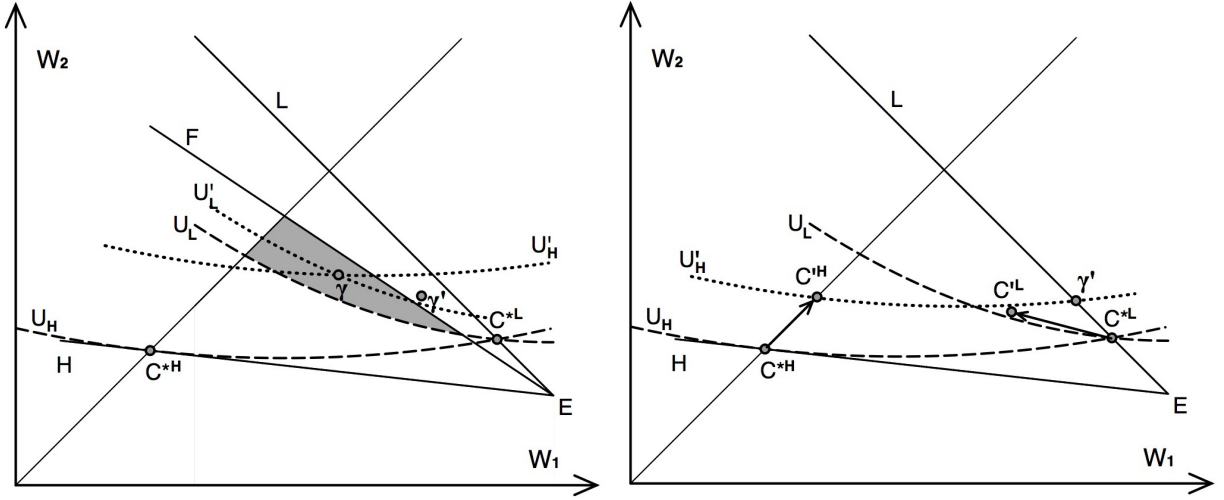


Fig. 3: Deviations from RSW solution (c^{*H}, c^{*L}) are not secure. Both "pooling" deviation γ (on the left) and separating deviation (c^H, c^L) (on the right) pose a threat γ' to receive negative payoff.

A pair of contracts (c^{*H}, c^{*L}) in Fig.3 represents a RSW solution of the insurance market game. The indifference curves through c^{*H} and c^{*L} for high-risk and low-risk customers U_H and U_L are shown by broken lines.

Let us consider position when both insurance companies offer the RSW contract c^* and obtain zero payoffs. If it is a Nash equilibrium it is also an EinSS according to Proposition 1. Consider the case when c^* is not a Nash equilibrium. It is still a secure profile since any change in the insurance policies of one company will not bring losses to the other company. Its payoffs will still remain zero. Suppose one company can deviate by offering a new insurance policy. It is either **(A)** a "pooling" insurance contract or **(B)** a separating insurance contract.

(A) If it is a pooling contract γ it must lie above the low-risk indifference curve U_L through c^{*L} in order to be profitable for both low- and high-risk customers (see Fig.3 on the left). A deviating company can make a positive profit only if γ lies below the market odds line EF in the shaded area. Let us draw the indifference curves U'_H and U'_L for low-risk and high-risk customers through γ . Then the second company as a response to γ can offer a pooling contract γ' between curves U'_H and U'_L somewhere to the right from the γ and below low-risk line EL . In this case all low-risk customers would choose γ' and the second company could make a profit. All high-risk customers would stay with γ contract and the first company would lose money. Hence there is a retaliatory threat to deviate into γ' such that the deviating company loses more money than it gains at deviation γ . Therefore offering pooling contract γ is not a secure deviation.

(B) Let us now assume that a deviating company offers a new separating contract (c^H, c^L) which is more profitable than (c^{*H}, c^{*L}) (see Fig.3 on the right). If it is more profitable for low-risk customers it also must be more profitable for high-risk customers (since in this case they always prefer c^L to c^H). In order to be more profitable for high-risk customers c^H must lie above the high-risk indifference curve U_H through c^{*H} . Therefore c^H also lies above high-risk line EH and makes a loss for a deviating company. Consequently c^L must lie below low-risk line EL and make a profit for a

deviating company. In this case profits from c^L subsidize the losses of c^H and (c^H, c^L) can be more profitable than the RSW solution (c^{*H}, c^{*L}) . Let us draw the indifference curves U'_H through c^H . Then the second company as a response to (c^H, c^L) can offer a pooling contract γ' at the intersection of U'_H with low-risk line EL . In this case all low-risk customers would choose γ' and the second company could make a profit. All high-risk customers would stay with (c^H, c^L) contract and the deviating company would lose money. Hence there is a retaliatory threat to deviate into γ' such that the deviating company loses more money than it gains at deviation into (c^H, c^L) . Therefore offering separating contract (c^H, c^L) is not a secure deviation either. No company can make a secure deviation from (c^{*H}, c^{*L}) . Therefore it is an EinSS. \square

For the described model Wilson (1976) introduced and analyzed an equilibrium concept based on the following assumption. Each insurance company believes that after offering its contract, the other company would immediately withdraw any unprofitable contract. Under this assumption an equilibrium in the insurance market game always exists. In contrast to the Wilson approach we assume that companies take into account *all threats existing in the game*, i.e. companies take into account the possibility of any change of policy by the rival company and not only the possibility to withdraw the unprofitable insurance contract. As a result our concept provides a different equilibrium solution which corresponds to more cautious behavior.

8. EinSS in the Bertrand-Edgeworth duopoly model

In this section we consider a model of price setting duopolists with capacity constraints originated in papers of Bertrand (1883) and Edgeworth (1925). We consider the market for some homogeneous product with a continuous strictly decreasing consumer's demand function $D(p)$. There are two firms in the industry $i = 1, 2$, each with a limited amount of productive capacity S_i such that $D(0) \geq S_1 + S_2$. Firms choose prices p_i and play non-cooperatively. The firm quoting the lower price serves the entire market up to its capacity and the residual demand is met by the other firm. All consumers are identical and choose the lower available price on a first-come-first-serve basis. Following Shubik (1959) and Beckmann (1965) we assume in our analysis that the residual demand to the firm quoting the higher price is a proportion of total demand at that price. If duopolists set the same prices firms share the market in proportion to their capacities. Formally we define the payoff functions of players to be:

$$(14) \quad \begin{aligned} u_1(p_1, p_2) &= \begin{cases} p_1 \min\{S_1, D(p_1)\}, & p_1 < p_2 \\ p_1 \min\{S_1, \frac{S_1}{S_1+S_2} D(p_1)\}, & p_1 = p_2 \\ p_1 \min\{S_1, \frac{D(p_1)}{D(p_2)} \max\{0, D(p_2) - S_2\}\}, & p_1 > p_2 \end{cases} \\ u_2(p_1, p_2) &= \begin{cases} p_2 \min\{S_2, D(p_2)\}, & p_2 < p_1 \\ p_2 \min\{S_2, \frac{S_2}{S_1+S_2} D(p_2)\}, & p_2 = p_1 \\ p_2 \min\{S_2, \frac{D(p_2)}{D(p_1)} \max\{0, D(p_1) - S_1\}\}, & p_2 > p_1 \end{cases} \end{aligned}$$

It is well known that the model of Bertrand-Edgeworth may not possess a Nash equilibrium (see e.g. d'Aspremont and Gabszewicz (1980)). We will show that in some of these cases there is an Equilibrium in secure strategies. However for some (big enough)

capacities EinSS does not exist either.

Proposition 10. *Let the receipt function $pD(p)$ be strictly concave and reach its maximum at p_M . Then in the game of Bertrand-Edgeworth with payoff functions (14) there is an EinSS (p^*, p^*) where $D(p^*) = S_1 + S_2$ if and only if*

$$(15) \quad \begin{cases} \arg \max_{p>0} \{p(D(p) - S_1)\} \leq p^* \\ \arg \max_{p>0} \{p(D(p) - S_2)\} \leq p^* \end{cases}$$

If $p^* \geq p_M$ it is a Nash equilibrium. There are no other EinSS in the game.

Proof. Since the receipt function $pD(p)$ is strictly concave then the function $p(D(p) - S)$ at a given S is also strictly concave in p and reaches the unique maximum at $p > 0$. Therefore $\arg \max_{p>0} \{p(D(p) - S)\}$ can be considered as a function of S . The proof see in Appendix D. \square

Corollary. *If function $pD(p)$ is differentiable the condition (15) is equivalent to*

$$(15') \quad \left. \frac{d}{dp} (pD(p)) \right|_{p=p^*} \leq \min\{S_1, S_2\}$$

Proof. One can easily check that $\hat{p} = \arg \max_{p>0} \{p(D(p) - S)\} \iff \left. \frac{d}{dp} (pD(p)) \right|_{p=\hat{p}} = S$. Besides $\left. \frac{d}{dp} (pD(p)) \right|_{p=\hat{p}}$ is strictly decreasing. Therefore $\hat{p} \leq p^* \iff \left. \frac{d}{dp} (pD(p)) \right|_{p=p^*} \leq \left. \frac{d}{dp} (pD(p)) \right|_{p=\hat{p}} = S$. Hence the equivalence of (15) and (15'). \square

As an example let us consider the demand function $D(p) = 1 - p$. Then $p^* = 1 - S_1 - S_2$, $\arg \max_{p>0} \{p(D(p) - S)\} = \frac{1-S}{2}$ and conditions (15) takes the form:

$$(16) \quad S_1 + 2S_2 \leq 1 \quad \text{and} \quad S_2 + 2S_1 \leq 1$$

Equilibria in secure prices in the space of capacity parameters (S_1, S_2) are shown in Fig.4. The profile (p^*, p^*) is a Nash equilibrium if $S_1 + S_2 \leq \frac{1}{2}$ (dark gray area). Under the weaker conditions (16) it is an EinSS. The area of EinSS which are not Nash equilibria are shaded by light gray in Fig.4. If conditions (16) do not hold this profile is no longer an EinSS and corresponds to an unstable BSR-profile. The found solution can be compared with the price which would maximize the joint profits in the industry $p_M = \max\{1 - S_1 - S_2, \frac{1}{2}\}$. If Nash equilibrium exists (i.e. if $S_1 + S_2 \leq \frac{1}{2}$) then both equilibrium prices coincide. However if EinSS exists and Nash equilibrium does not exist (i.e. if $S_1 + S_2 > \frac{1}{2}$ and (16) holds) both EinSS prices $p^* = 1 - S_1 - S_2$ are lower than the monopoly price $p_M = \frac{1}{2}$. One can interpret the price difference $S_1 + S_2 - \frac{1}{2}$ as an additional payment for the preservation of security in the situation when players take into account mutual threats and behave cautiously.

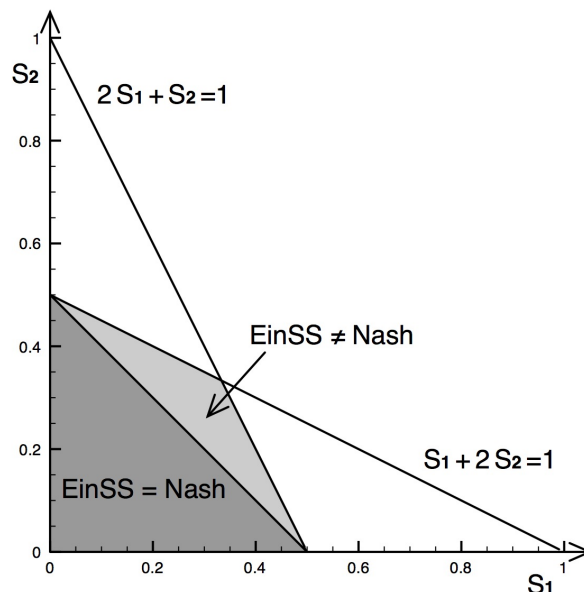


Fig. 4: Equilibria in secure prices in the Bertrand-Edgeworth duopoly model with $D(p) = 1 - p$ in the space of capacity parameters (S_1, S_2) . Dark gray area: EinSS which coincide with Nash equilibria. Light gray area: EinSS which are not Nash equilibria.

Conclusion

The article presents a new concept of equilibrium, that of Equilibrium in Secure Strategies which provides a model of cautious behavior for non-cooperative games. It is suitable for games in which threats of other players is an important factor in the decision making. In the EinSS players refrain from using some strategies if they anticipate retaliation threats. Generally speaking, it is defined by two conditions: (i) no one can increase payoff by worsening the situation of other players and (ii) no one can increase payoff without creating a threat to lose more than he gains. From one side our concept is a generalization of Nash-Cournot equilibrium, i.e. any Nash-Cournot equilibrium is an EinSS. From the other side the EinSS logic allows to reveal new equilibrium positions which sometimes can not be revealed by standard logic of best responses. Therefore an EinSS exists in many discontinuous games that fail to have Nash-Cournot equilibrium.

The basic concept of our model is the notion of a secure strategy. A strategy of player is a secure strategy at given strategies of other players if none of them can decrease his payoff by unilateral deviation. The most profitable secure strategy of a player at a given strategies of other players is a best secure response of that player. In these terms EinSS has an intuitive interpretation as a BSR-profile (i.e. a profile in which all strategies are best secure responses). A BSR-profile is a Generalized Nash-Cournot Equilibrium in which each player's strategy set is the set of his secure strategies at given strategies of all other players. The set of Nash-Cournot equilibria is a subset of EinSS which in turn is a subset of BSR-profiles. This double inclusion of EinSS between Nash-Cournot equilibria and Generalized Nash-Cournot Equilibria provides a reliable algorithm of finding EinSS. First all BSR-profiles can be found as a solution of the corresponding maximization problem. Then the definition of EinSS shall be checked for these BSR-profiles.

An EinSS concept can be extended in two practically important ways. One weakening

concept of EinSS is the game with "uncertain insider". Let us suppose that after the players simultaneously choose their strategies an "insider" is chosen randomly among them and has an opportunity to change his strategy. In the game with uncertain insider all players minimize the worst potential threats which may appear as a result of the insider move. If players choose their strategies only among secure strategies then EinSS in a slightly weakened modification (namely weak EinSS) coincides with Nash-Cournot equilibrium in the game with uncertain insider. Hence a weak EinSS can be defined as a secure strategy profile which is a solution of the corresponding maximin problem. This provides another practical approach to finding EinSS. The second weakening of the EinSS concept allows to take into account threats from simultaneous and independent actions of several players. In particular it allows to take into account such effects as panic and herd behavior which play an important role in many practical situations when analyzing security.

In their seminal paper Dasgupta and Maskin (1986) obtained existence results for mixed strategy Nash equilibrium in a family of games with discontinuous payoff functions. Very often these discontinuities generate specific threats between players. Therefore it is not surprising that proper consideration of these threats enables us to find an EinSS in many discontinuous games. In contrast to the equilibrium in mixed strategies however our approach provides an explicit solution which is easy to interpret in terms of cautious behavior.

As a first example we consider the classic Hotelling's model (1929) with the linear transport costs. There is no price Nash-Cournot equilibrium in this game when duopolists choose locations too close to each other (d'Aspremont et al., 1979). In these cases one player can undercut his rival's price and take away his entire business with profit to himself. In order to restore the existence of a pseudo equilibrium in this situation Eaton and Lipsey (1978) proposed that the players while choosing their prices assume that they can never drive the competitors out of the market. In fact this assumption effectively rules out the threat of pressing out of the market by undercutting and allows to find the corresponding 'zero conjectural variation equilibrium' solution. Another approach is an equilibrium in mixed strategies which existence in the Hotelling's price game was proved by Dasgupta and Maskin (1986). These equilibria were studied in detail by Osborne and Pitchik (1987). They found that at certain locations the support of the subgame equilibrium price strategy is the union of two short intervals with most probability weight in the upper interval. However they were unable to provide a complete characterization of equilibria. The obtained results in terms of mixed strategies are difficult to interpret. In this paper we provide solution in secure strategies of the Hotelling's price game with a restricted reservation price on an infinite line. It coincides with Nash-Cournot equilibrium price whenever it exists. And there is a unique EinSS price if Nash-Cournot equilibrium does not exist. In contrast to Eaton and Lipsey approach the EinSS concept takes into account the mill-price undercutting as an essential factor of the game. In the EinSS players do assume that they can be driven out of the market and therefore keep their prices sufficiently low to secure themselves against such undercutting. In practice this assumption results in lower equilibrium prices as compared with the solution based on the assumption of Eaton and Lipsey (see M.Iskakov and A.Iskakov, 2012). In contrast to the price solution in mixed strategies EinSS solution is obtained in explicit form and can be easily interpreted as an equilibrium of bilateral containment.

As a second example we consider a canonical example of a contest described by Tullock (1969,1980). It is well known that a pure-strategy Nash-Cournot equilibrium does not exist for a two-player contest when the contest success function parameter α is greater than two. For a symmetric, two-player contest Baye, Kovenock and de Vries (1993) have shown that for $\alpha > 2$ rent dissipation in a mixed strategy equilibrium is complete. However the equilibrium was not characterized analytically. We show however that EinSS always exist in the contest of two-players. Our concept allows to discover a new type of equilibria in the rent-seeking game, those for which one player prefer to fix his or her secure monopolistic position rather than to participate in the competition. Moreover when power parameter $\alpha > 2$ the monopolistic situation is the only stable position in the game in terms of secure strategies. The efficiency of equilibrium in the contest is characterized by rent dissipation which is equal to the ratio of total effort of both players to the value of the prize. The higher is the degree of rent dissipation, the lower is the efficiency of the equilibrium. For $\alpha > 2$ rent dissipation in mixed-strategy equilibria is equal to one. However for the monopolistic EinSS it is significantly less than one. Hence the concept of EinSS provides more efficient solution than the mixed-strategy Nash equilibrium. The obtained solution has a straightforward interpretation for the rent-seeking game.

As the next example we consider a model of insurance market suggested by Rothschild and Stiglitz (1976) and Wilson (1977). It was proven that if a Nash-Cournot equilibrium exists both companies must offer the same "Rothschild-Stiglitz-Wilson" (RSW) contract pair. However if there is a sufficiently high proportion of low risk customers a single pooling contract or a pair of cross-subsidizing contracts may be preferred by everyone and will therefore upset the RSW equilibrium contract. One way to resolve this problem of the non-existence of equilibrium in pure strategies is to allow insurance companies to use mixed strategies. While equilibria in mixed strategies always exist (Dasgupta and Maskin, 1986) to our knowledge they have not been characterized. Therefore the economic interpretation of this solution is not clear. Alternatively several ad hoc equilibrium concept have been proposed. For instance Wilson (1977) and Riley (1979) suggested two different equilibrium concepts. A set of contracts is a Wilson equilibrium if no company has a profitable deviation that remains profitable once existing contracts that lose money after the deviation are withdrawn. A set of contracts is a Riley equilibrium, if no company has a profitable deviation that remains profitable once new contracts that make money after the deviation are added. Under either definitions equilibria always exist. However they are different. In contrast to these concepts all companies in EinSS take into account all threats existing in the game, i.e. any profitable change of policy by the rival company. This would include both threats of "Wilson-type" and threats of "Riley-type" as well as any combination of them. Therefore EinSS is a more cautious concept. We show that RSW contract is always an EinSS. Even if RSW is not a Nash-Cournot equilibrium it is still an equilibrium in our terms of security.

And finally we consider the Bertrand-Edgeworth duopoly model with capacity constraints which may not possess a Nash-Cournot equilibrium. D'Aspremont and Gabzewicz (1980) proposed the concept of quasi-monopoly which restores the existence of pseudo equilibrium in some of these cases when one capacity is quite small compared to the other. The existence of mixed-strategy equilibrium was demonstrated by Dasgupta and Maskin (1986) and Huw Dixon (1984). However it proved not to be easy to characterize what the equilibrium actually looks like. Allen and Hellwig (1986) were able to show that

in a large market with many firms, the average price set would tend to the competitive price. We show that in some cases when Nash-Cournot equilibrium does not exist there is an EinSS with equilibrium prices lower than the monopoly price. The corresponding difference in prices can be interpreted as an additional payment for the preservation of security when duopolists behave cautiously and avoid mutual threats.

All of the above games without Nash-Cournot equilibrium can be approached in two ways. Either one can use ad hoc equilibrium concepts developed within the framework of the given game and obtain an explicit solution with specific interpretation. Or one can look for equilibria in mixed strategies which are usually not explicit and difficult to interpret. In this paper we propose a third alternative, a general concept of equilibrium which exists in many games without Nash-Cournot equilibrium. This concept provides an explicit solution easy to interpret. The obtained results confirm the practical value and adequacy of the proposed approach and lay a firm ground for the future research.

Acknowledgements

We are deeply indebted to C.d'Aspremont who helped us to find an elegant and intuitively clear formulation of the EinSS concept and inspired us to write this paper. We are especially thankful to F.Aleskerov and D.Novikov for regular discussions on the subject during seminars at Moscow High School of Economics and at V.A.Trapeznikov Institute of Control Sciences RAS. The results on the Tullock Contest were obtained with participation of Alexey Zakharov. We wish to thank leaders of the seminars V.Burkov, V.Danilov, V.Polterovich and J.-F.Thisse. We wish to thank A.Chkhartishvili, M.Gubko, S.Izmalkov, N.Korgin, S.Kokovin, N.Kukushkin, S.Mishin, P.Pavlov, A.Savvateev, K.Sonin, A.Vasin and E.Zhelobodko for the fruitful discussions. Our work was supported by the grant of the RFBR (No.10-07-00063-a).

APPENDIX A

Comment on the definitions of EinSS

In order to keep consistency with the previously used form of definitions of EinSS we prove here that *the definitions of EinSS 1-4 are equivalent to the definitions published in (M.Iskakov and A.Iskakov, 2012)*.

Proof. Below we provide the definitions of the EinSS published in M.Iskakov and A.Iskakov (2012). The definition of threat and the definition of secure profile are the same. The definitions 3 and 4 were formulated in the following form:

Definition 3*. A set $W_i(s)$ of preferable strategies secured against threats is a set of strategies s'_i of player i at a given s such that $u_i(s'_i, s_{-i}) \geq u_i(s)$ and provided that $u_i(s'_i, s'_j, s_{-ij}) \geq u_i(s)$ for any threat $\{(s'_i, s_{-i}), (s'_j, s_{-ij})\}$ of player $j \neq i$ to player i .

Definition 4*. A strategy profile s^* is an *Equilibrium in Secure Strategies* (EinSS) if and

only if for all i we have that

$$W_i(s^*) \neq \emptyset, \quad s_i^* \in \arg \max_{s_i \in W_i(s^*)} u_i(s_i, s_{-i}^*).$$

Any strategy from the set $W_i(s)$ of preferable strategies secured against threats according to the definition 3* is either **(a)** a secure strategy s'_i such that $u_i(s'_i, s_{-i}) = u_i(s)$ or **(b)** *secure deviation* according to the Definition 3. If s^* is an EinSS according to 4* then for all i $s_i^* \in W_i(s^*)$. s_i^* can not be **(b)** *secure deviation* according to the Definition 3. Therefore s_i^* must be **(a)**, i.e a secure strategy in the profile s^* . Since all strategies s_i^* are secure then the profile s^* is a secure profile. If some player can increase his profit by secure deviation then $s_i^* \notin \arg \max_{s_i \in W_i(s^*)} u_i(s_i, s_{-i}^*)$. Therefore no player in s^* can make a *secure deviation* (according to the Definition 3). s^* is an EinSS according to the Definition 4. Now let s^* is the EinSS according to the Definition 4. As s^* is a secure profile so for all i $s_i^* \in W_i(s^*)$ and $W_i(s^*) \neq \emptyset$. As no player in s^* can increase his profit by secure deviation so $s_i^* \in \arg \max_{s_i \in W_i(s^*)} u_i(s_i, s_{-i}^*)$. And s^* is an EinSS according to the definition 4*. \square

APPENDIX B

Proof of the Proposition 7

We will use below the following notation according to (3). As $u_i^I(p_i)$ we denote the payoff function of player i in the domain $p_i < p_{-i} - d$ where player i captures the whole market and his payoff function depends only upon his own price. As $u_i^{II}(p_i, p_{-i})$ we denote the payoff function of player i in the domain $|p_i - p_{-i}| \leq d$ where price competition between two players takes place. In order to find equilibria in secure strategies in the dimensionless Hotelling's game (3) let us first identify the secure profiles and prove the following Lemma.

Lemma. *The profile (p_1, p_2) in the dimensionless price-setting game $\{i \in \{1, 2\}, p_i \in [0, 1], u_i(p_1, p_2)\}$ with the profit functions (3) is a secure strategy profile if and only if*

1) when $d < 1$:

$$\begin{aligned} (\text{II.1a}) \quad & \left\{ |p_1 - p_2| \leq d \right. \\ (\text{II.1b}) \quad & \left. p_i \leq \arg \max_p u_i^{II}(p, p_{-i}), \quad i \in \{1, 2\} \right. \\ (\text{II.1c}) \quad & \left. \text{if } p_{-i} > d, \quad u_i^I(p_{-i} - d) \leq u_i^{II}(p_i, p_{-i}), \quad i \in \{1, 2\} \right\} \end{aligned}$$

2) when $d \geq 1$:

$$(\text{II.2}) \quad \left\{ |p_1 - p_2| \leq d \right. \quad \cup \quad \left\{ |p_1 - p_2| \leq d - 1 \right. \\ \left. p_i \leq \arg \max_p u_i^{II}(p, p_{-i}), \quad i \in \{1, 2\} \right. \quad \left. p_1 + p_2 > 2 - d \right\}$$

Proof of Lemma. When $p_1 < p_2 - d$ player 2 gets zero profit and there is always a threat to player 1 that player 2 will decrease his price till $\hat{p}_2 < p_1$ and will get positive profit. Since the trade zones of players are in contact then the market share and the profit of player 1 will decrease. Therefore the profile (p_1, p_2) is not secure for player 1. In a similar way when $p_2 < p_1 - d$ the profile (p_1, p_2) is not secure for player 2. Hence all

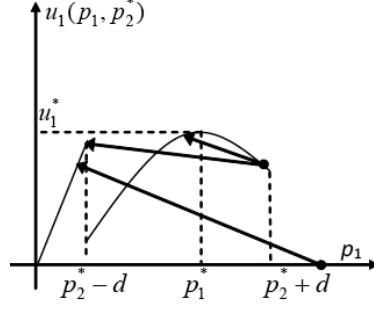


Fig. 5: The first player increases profit either by shifting price in area I or in area II .

secure profiles lie in the area $|p_1 - p_2| \leq d$. The condition (II.1a) and the first condition in (II.2) are proven.

Let us consider the existing threats to player 2 when $|p_1 - p_2| \leq d$. According to (3) the payoff function of the first player $u_1(p_1)$ in each price area I ($p_1 < p_2 - d$) and II ($|p_1 - p_2| \leq d$) is concave and one-picked (see Fig.5). Player 1 can increase his profit only in two ways: either by shifting price in the price area I or by moving closer to the pick in the price area II . The first situation is possible when $\max_{p \in (0, p_2 - d)} u_1^I(p) > u_1^{II}(p_1, p_2)$ and always produces a threat to player 2 to be driven out of the market. The corresponding security condition is $\max_{p \in (0, p_2 - d)} u_1^I(p) \leq u_1^{II}(p_1, p_2)$. In the second situation player 2 keeps his security in two cases. Either the first player can not increase his profit by reducing price, i.e. $p_1 \leq \arg \max_p u_1^{II}(p, p_2)$. Or he can increase his profit by reducing price but even at maximum reduction of his price profitable to him his trade zone will not get in contact with the trade zone of player 2. For the profit functions (3) the last condition can be written as $1/2 < p_1 \leq d - 1 + p_2$. Therefore the security condition of player 2 against both types of threats can be written as:

$$\begin{cases} \text{if } p_2 > d, & \max_{p \in (0, p_2 - d)} u_1^I(p) \leq u_1^{II}(p_1, p_2) & (1*) \\ p_1 \leq \arg \max_p u_1^{II}(p, p_2) \text{ or } 1/2 < p_1 \leq d - 1 + p_2 & & (2*) \end{cases}$$

According to (1) at $|p_1 - p_2| \leq d$ we have $u_1^{II}(p_1, p_2) \leq u_1^I(p_1)$ for all p_2 . Then it follows from the (1*) that $\max_{p \in (0, p_2 - d)} u_1^I(p) \leq u_1^I(p_1)$, i.e. the concave function $u_1^I(p)$ reaches maximum at $p > p_2 - d$ and therefore $\max_{p \in (0, p_2 - d)} u_1^I(p) = u_1^I(p_2 - d)$. Then the first condition (1*) can be written in a more convenient form as $u_1^I(p_2 - d) \leq u_1^{II}(p_1, p_2)$. The security condition for the profile (p_1, p_2) can be written then as:

$$\begin{cases} |p_1 - p_2| \leq d \\ p_i \leq \arg \max_p u_i^{II}(p, p_{-i}) \text{ or } 1/2 < p_i \leq d - 1 + p_{-i}, \quad i \in \{1, 2\} \\ \text{if } p_{-i} > d, \quad u_i^I(p_{-i} - d) \leq u_i^{II}(p_i, p_{-i}), \quad i \in \{1, 2\} & (*) \end{cases}$$

Now let us assume that for the secure profile (p_1, p_2) at least one of the conditions $1/2 < p_i \leq d - 1 + p_{-i}$ is satisfied. For example $1/2 < p_1 \leq d - 1 + p_2$ which implies

$p_2 > 3/2 - d$ and $p_1 + p_2 > p_2 + (1 - p_1) \geq 2 - d$.

If $p_2 \leq 1/2$ then $3/2 - d < p_2 \leq 1/2 \Rightarrow d > 1$.

If $p_2 > 1/2$ and $p_1 + p_2 > 2 - d \Rightarrow p_2$ must be on the descending part of the function $u_2^{II}(p_1, p)$, i.e. $p_2 > \arg \max_p u_2^{II}(p_1, p) \Rightarrow$ the condition $p_2 \leq d - 1 + p_1$ must be satisfied

$\Rightarrow p_1 + p_2 \leq 2d - 2 + p_1 + p_2 \Rightarrow d \geq 1$.

Therefore it is proven that if $d < 1$ then neither of the conditions $1/2 < p_i \leq d - 1 + p_{-i}$, $i \in \{1, 2\}$ is satisfied for the secure profile (p_1, p_2) . Therefore the formula (II.1) is proven.

Let us prove that the (*) is equivalent to (II.2) when $d \geq 1$. The conditions (II.1c) and (1*) together with the threat of mill-price undercutting disappear when $d \geq 1$ since in this case we have $p_i \leq 1 \leq d$, $i \in \{1, 2\}$.

For the profiles (p_1, p_2) which satisfy the condition $p_1 + p_2 \leq 2 - d$ the conditions (II.2) and (*) are obviously equivalent (since for these profiles the second conditions in (2*) are not satisfied). For the profiles (p_1, p_2) which satisfy the condition $p_1 + p_2 > 2 - d$ we obtain $\arg \max_p u_i^{II}(p, p_{-i}) = \min \left\{ \frac{2+d+p_{-i}}{6}, \max \{2 - d - p_{-i}, 1/2\} \right\} = \frac{1}{2}$ and the conditions (II.2) and (*) take the following forms:

$$\{(p_1, p_2) : p_1 \leq 1/2, p_2 \leq 1/2\} \cup \{(p_1, p_2) : |p_1 - p_2| \leq d - 1\}$$

$$\text{and} \quad \begin{cases} |p_1 - p_2| \leq d \\ p_1 \leq 1/2 \text{ or } 1/2 < p_1 \leq d - 1 + p_2 \\ p_2 \leq 1/2 \text{ or } 1/2 < p_2 \leq d - 1 + p_1 \end{cases}$$

The equivalence of these conditions at $d \geq 1$ for the profiles satisfying $p_1 + p_2 > 2 - d$ can be proven by straightforward verification. \square

Now we are ready to prove the Proposition.

According to Lemma the set of secure strategies in the price Hotelling's game at $d \geq 1$ is given by (II.2). Substituting into this system the expressions (3) for the payoff functions we obtain:

$$\begin{cases} \text{if } p_1 + p_2 \leq 2 - d, & |p_1 - p_2| \leq d \\ \text{if } p_1 + p_2 > 2 - d, & |p_1 - p_2| \leq d - 1 \\ p_{-i} \leq \min \left\{ \frac{2+d+p_i}{6}, \max \{2 - d - p_i, 1/2\} \right\}, & i \in \{1, 2\} \end{cases}$$

The best secure responses of players at $|p_1 - p_2| \leq d$ take the following form ($i \in \{1, 2\}$):

$$p_{-i} = \max \left\{ 1 - d + p_i, \min \left\{ \frac{2 + d + p_i}{6}, \max \{2 - d - p_i, 1/2\} \right\} \right\}.$$

which has at $d \geq 1$ the unique solution (4d).

The set of secure strategies in the price Hotelling's game at $d < 1$ according to Lemma is given by the system (II.1). Substituting into this system the expressions (1) for the

payoff functions we obtain:

$$\begin{cases} |p_1 - p_2| \leq d & (*a) \\ p_{-i} \leq \min \left\{ \frac{2+d+p_i}{6}, \max \{2-d-p_i, 1/2\} \right\}, i \in \{1, 2\} & (*b) \\ p_{-i} \leq \max \left\{ d, d + \frac{4-p_i}{8} - \sqrt{\left(d + \frac{4-p_i}{8}\right)^2 - \frac{p_i}{4}(2+d-3p_i) - d(d+1)} \right\} & (*c) \end{cases}$$

In the last inequality we take into account that $(*b) \Rightarrow 5p_{-i} \leq 2+d+p_i - p_{-i} \leq 2+2d \Rightarrow p_{-i} \leq \frac{2+2d}{5} < \frac{4+7d}{9} \Rightarrow 8p_{-i} < 4+8d-d-p_{-i} \leq 4+8d-p_i \Rightarrow p_{-i} < d + \frac{4-p_i}{8} \Rightarrow$ the second branch of the solution of the quadratic inequality $(*c)$ is not realized.

Under the found conditions $(*a, *b, *c)$ the function $u_i^{II}(p_i, p_{-i})$ increases by p_i and hence the Best Secure Response (BSR) of players at $|p_1 - p_2| < d$ takes the following form ($i \in \{1, 2\}$):

$$(*) \quad p_{-i} = \min \left\{ \frac{2+d+p_i}{6}, \max \{2-d-p_i, 1/2\}, \max \left\{ d, d + \frac{4-p_i}{8} - \sqrt{\left(d + \frac{4-p_i}{8}\right)^2 - \frac{p_i}{4}(2+d-3p_i) - d(d+1)} \right\} \right\}.$$

These equations define the plots of the best secure responses of players in the plain (p_1, p_2) at $|p_1 - p_2| < d$. The intersection of these plots is the point of the BSR-profile. According to the Proposition 3 any EinSS is the BSR-profile, i.e. it must satisfy $(*)$. From the other side any solution of $(*)$ is the EinSS. Indeed any deviation of player from $(*)$ in the direction of lower price decreases his profit. And any deviation from $(*)$ in the direction of higher price either decreases his profit or creates the threat of being undercut throughout the whole market, i.e. no player can increase his profit by secure deviation.

The solution of the system $(*)$ corresponds to the first three cases in the Proposition 7. Indeed this solution shall be symmetric about a line $p_1 = p_2$ and shall be of two types. The multiple solutions of $(*)$ lie in the interval of the line $p_1 + p_2 = 2 - d$ which is the common place for the BSR of both players. Checking the limit conditions provides the solution (4c). The solutions of $(*)$ of another type are located on line $p_1 = p_2 \equiv p$. When $d \geq 6/7$ the solution is defined by the following equation:

$$p = \min \left\{ \frac{2+d+p}{6}, \max \{2-d-p, 1/2\} \right\},$$

When $6/7 \leq d \leq 1$ this solution takes the form $p_1 = p_2 = 1 - d/2$ which is a special case of the multiple solution (4c). Finally the solution at $d \leq 6/7$ is defined by the equation:

$$p = \min \left\{ \frac{2+d+p}{6}, \max \left\{ d, d + \frac{4-p}{8} - \sqrt{\left(d + \frac{4-p}{8}\right)^2 - \frac{p}{4}(2+d-3p) - d(d+1)} \right\} \right\},$$

which gives solutions (4a) and (4b). The Proposition is proven. \square

Sketch of the proof of the Proposition 8

Proof. (1). The profile $\{(\alpha/4, \alpha/4)\}$ is an EinSS at $\alpha \leq 2$ according to Proposition 1 because it is a Nash equilibrium in the game (Tullock, 1980).

(2). Let us prove that profiles $\left\{ \left(0, \frac{1}{\alpha} (\alpha - 1)^{\frac{\alpha-1}{\alpha}} \right) \right\}$ and $\left\{ \left(\frac{1}{\alpha} (\alpha - 1)^{\frac{\alpha-1}{\alpha}}, 0 \right) \right\}$ are EinSS at $\alpha > 1$. Consider for example the first one. Player 1 can not increase his payoff in it by whatever deviation. Therefore the profile $\left\{ \left(0, \frac{1}{\alpha} (\alpha - 1)^{\frac{\alpha-1}{\alpha}} \right) \right\}$ satisfies the definition of EinSS for player 1. Consider player 2. Any deviation into $x_2 > \frac{1}{\alpha} (\alpha - 1)^{\frac{\alpha-1}{\alpha}}$ is not profitable for him. Let us prove that deviation of player 2 into $x_2 : 0 < x_2 < x_2^m \equiv \frac{1}{\alpha} (\alpha - 1)^{\frac{\alpha-1}{\alpha}} < 1$ is not a secure deviation either. Indeed, player 1 in response can deviate into $x_1 > 0 : U_1(x_1, x_2) = 0$. Expressing x_2 through x_1 one gets $x_2 = x_1 \left(\frac{1-x_1}{x_1} \right)^{1/\alpha}$. Let us prove that in this case $U_2(x_1, x_2) - U_2^m = x_2^m - x_1 - x_2 < 0$ for all $x_2 \in (0, x_2^m)$, or

$$\frac{1}{\alpha} (\alpha - 1)^{\frac{\alpha-1}{\alpha}} < x_1 \left(1 + \left(\frac{1-x_1}{x_1} \right)^{1/\alpha} \right) \equiv f(x_1) \quad \text{for all} \quad \frac{\alpha-1}{\alpha} < x_1 < 1 \quad (*)$$

One can easily check that $f''(x_1) = \frac{(1-\alpha)}{\alpha^2 x_1 (1-x_1)} \left(\frac{1-x_1}{x_1} \right)^{1/\alpha} < 0$ at $\alpha > 1$. Therefore $\min_{\frac{\alpha-1}{\alpha} < x_1 < 1} f(x_1) = \min\{f(\frac{\alpha-1}{\alpha}), f(1)\} = \min\{\frac{\alpha-1}{\alpha} + \frac{1}{\alpha} (\alpha - 1)^{\frac{\alpha-1}{\alpha}}, 1\}$ and estimation (*) is true. The deviation of player 2 into $0 < x_2 < x_2^m$ is not a secure deviation. Thus no player can make secure deviation in the profile $(0, \frac{1}{\alpha} (\alpha - 1)^{\frac{\alpha-1}{\alpha}})$ and it is an EinSS by definition. By symmetry the profile $(\frac{1}{\alpha} (\alpha - 1)^{\frac{\alpha-1}{\alpha}}, 0)$ is either an EinSS.

(3). Let us first consider the case of $0 < \alpha \leq 1$. A pair of strategies is secure in the rent-seeking contest if and only if no player can be made better off by increasing his or her effort (which would always reduce the payoff of another player). The payoff functions of players U_i at $0 < \alpha \leq 1$ are concave and one peak in their strategies x_i . Therefore no one can increase his profit by increasing his effort if and only if $x_1 \geq BR_1(x_2), x_2 \geq BR_2(x_1)$, where $BR_i(x_{-i})$ are best responses of players. One can show that this set can be written as

$$\left\{ x_1 \geq \frac{\alpha}{4}, x_2 \geq \frac{\alpha}{4} \right\} \cup \left\{ x_1 < \frac{\alpha}{4}, x_2 \geq \xi^+(x_1) \right\} \cup \left\{ x_2 < \frac{\alpha}{4}, x_1 \geq \xi^+(x_2) \right\}.$$

All EinSS profiles must lie on the boundary of this set of secure profiles (otherwise any player can securely increase his payoff by arbitrarily small deviation). Let us choose any profile on this boundary other than $(\alpha/4, \alpha/4)$. It must be either $(x_1 < \alpha/4, \xi^+(x_1))$ or $(\xi^+(x_2), x_2 < \alpha/4)$. Consider for example the first case. Then there is a secure deviation of player 1 into the profile $(x_1, \xi^-(x_1))$, where $\xi^-(x_i) \equiv \left(\frac{x_i^{\alpha-1}}{2} (\alpha - 2x_i - \sqrt{\alpha^2 - 4\alpha x_i}) \right)^{1/\alpha}$. Indeed one can prove that $U_1(x_1, \xi^-(x_1)) > U_1(x_1, \xi^+(x_1))$ at $\alpha < 1$ and profile $(x_1, \xi^-(x_1))$ is secure for player 1 (since player 2 get in this profile his maximum payoff and pose no threat to player 1). Therefore the profile $(x_1 < \alpha/4, \xi^+(x_1))$ is not an EinSS. By symmetry the profile $(\xi^+(x_2), x_2 < \alpha/4)$ is not an EinSS either. There are no EinSS

profiles at $\alpha < 1$ other than $(\alpha/4, \alpha/4)$.

(4). When $\alpha \geq 1$ all EinSS profiles must also lie on the boundary of the set of secure profiles (otherwise any player can securely increase his payoff by arbitrarily small deviation). From the other hand all EinSS must lie in the set $\{(x_1, x_2) : \max(x_1, x_2) \leq \frac{1}{\alpha}(\alpha - 1)^{\frac{\alpha-1}{\alpha}}\}$ (otherwise at least one player gets negative payoff and can make a secure deviation into zero strategy). From these two conditions it follows that at $\alpha \geq 1$ all EinSS other than $\left\{0, \frac{1}{\alpha}(\alpha - 1)^{\frac{\alpha-1}{\alpha}}\right\}$ and $\left\{\frac{1}{\alpha}(\alpha - 1)^{\frac{\alpha-1}{\alpha}}, 0\right\}$ must lie on the curve (9). In particular it implies that there are no EinSS other than two monopolistic equilibria at $\alpha > 2$. \square

APPENDIX D

Proof of the Proposition 10

Proof. (1). Any EinSS in the game must be a BSR-profile with positive payoffs (since any profile with zero payoffs always possesses a secure deviation into profiles with positive payoffs). First, let us find all secure profiles in the game with positive payoffs. Consider the case $p^* < p_1 < p_2$. If $D(p_1) > S_1$ player 1 always threatens player 2 by slight increasing his price p_1 . If $D(p_1) \leq S_1$ then according to (14): $u_2(p_1, p_2) = 0$. Symmetrically, if $p^* < p_2 < p_1$ either player 2 threatens player 1 or $u_1(p_1, p_2) = 0$. If $p^* < p_2 = p_1$ there is always threat of undercutting. If $p_1 \leq p^* < p_2$ player 1 always threatens player 2 by increasing his price till $p^* + 0$ which exceeds p^* by an arbitrarily small amount. Indeed in this case $D(p_1) > D(p^* + 0) \geq S_1$ and $u_1(p_1, p_2) = p_1 S_1 < (p^* + 0) S_1 = u_1(p^* + 0, p_2)$. On the other hand, $u_2(p^* + 0, p_2) = p_2 \frac{D(p_2)}{S_1 + S_2} S_2 < p_2 S_2$ and $u_2(p^* + 0, p_2) = p_2 D(p_2) \left(1 - \frac{S_1}{D(p^* + 0)}\right) < p_2 D(p_2) \left(1 - \frac{S_1}{D(p_1)}\right) \Rightarrow u_2(p^* + 0, p_2) < u_2(p_1, p_2)$. Symmetrically, if $p_2 \leq p^* < p_1$ player 2 always threatens player 1. Therefore all secure profiles with positive payoffs must lie in the set $\{(p_1, p_2) : 0 < p_i \leq p^*, i = 1, 2\}$. From the other hand if $p_1 \leq p^*$: $u_1(p_1, p_2) = S_1 p_1$ linearly increases in p_1 and does not depend on p_2 . Hence there are no threats for player 1. Symmetrically, if $p_2 \leq p^*$ there are no threats for player 2. Therefore (p_1, p_2) is a secure profile with positive payoffs in the game (14) if and only if it lies in the set $M = \{(p_1, p_2) : 0 < p_i \leq p^*, i = 1, 2\}$.

(2). The payoff functions (14) u_1 and u_2 increase in the set M linearly in p_1 and in p_2 respectively. Therefore there is only one BSR-profile (p^*, p^*) with positive payoffs in the set M (otherwise one player can securely slightly increase his price). According to Proposition 3 there are no other EinSS in the game except this profile.

(3). Let us consider profile (p^*, p^*) and prove the conditions (15). Suppose for example that $p^* < \hat{p}(S_2) \equiv \arg \max_{p>0} \{p(D(p) - S_2)\}$. Then player 1 can deviate $p_1^* \rightarrow \hat{p}$. His payoff will increase since $p^* < \hat{p} \leq p_M = \arg \max_{p>0} pD(p)$ and $u_1(p_1, p_2)$ is strictly increasing in p_1 if $p_1 \leq p_M$ according to (14). Any retaliatory threat of player 2 according to (14) can not make the payoff of player 1 less than $\min_{p_2} u_1(\hat{p}, p_2) = \min_{p_2 < \hat{p}} u_1(\hat{p}, p_2) = u_1(\hat{p}, p_2)|_{p_2=\hat{p}-0} = \hat{p} \min\{S_1, D(\hat{p}) - S_2\}$. The payoff of

player 1 in the initial profile does not exceed this value. Indeed $p(D(p) - S_2)$ is strictly increasing at $p < \hat{p}$ and we have $u_1(p^*, p^*) \leq p^*(D(p^*) - S_2) < \hat{p}(D(\hat{p}) - S_2)$ and $u_1(p^*, p^*) = p^*S_1 < \hat{p}S_1$. Therefore the deviation of player 1 into $\hat{p}(S_2)$ is always a secure deviation according to Definition 3. Hence profile (p^*, p^*) is not an EinSS. Symmetrically if $p^* < \hat{p}(S_1)$ then player 2 can make a secure deviation into $\hat{p}(S_1)$ and profile (p^*, p^*) is not an EinSS either. The necessity of (15) is proven.

(4). Let us now assume that (15) holds (i.e. $\hat{p}(S_1) \leq p^*$ and $\hat{p}(S_2) \leq p^*$). Consider an arbitrary deviation $p^* \rightarrow p_1$ of player 1. If $p_1 < p^*$ it can not be a profitable deviation for player 1. Therefore $p_1 > p^*$. Player 1 increases the payoff if and only if $u_1(p^*, p^*) = p^*S_1 = p^* \frac{D(p^*) - S_2}{D(p^*)} D(p^*) < u_1(p_1, p^*) = p_1 \frac{D(p^*) - S_2}{D(p^*)} D(p_1)$, i.e. there must be $p^*D(p^*) < p_1D(p_1)$. Then there is retaliatory threat of player 2 to deviate from profile (p_1, p_2^*) into profile arbitrarily close to $(p_1, p_1 - 0)$. From $p^*S_2 < p_1S_2$ and $p^*D(p^*) < p_1D(p_1)$ it follows that player 2 increases the payoff at this deviation. The payoff of player 1 in this profile is arbitrarily close to $u_1(p_1, p_1 - 0) = p_1 \min\{S_1, D(p_1) - S_2\}_{p^* < p_1} = p_1(D(p_1) - S_2)$. Since $p(D(p) - S_2)$ is strictly decreasing at $p \geq \hat{p}(S_2)$ and $p_1 > p^* \geq \hat{p}(S_2)$ then $u_1(p^*, p^*) = p^*(D(p^*) - S_2) > p_1(D(p_1) - S_2) = u_1(p_1, p_1 - 0)$. Therefore the deviation of player 1 into profile (p_1, p^*) is not a secure deviation. Symmetrically an arbitrary deviation of player 2 is not a secure deviation either. No player can make secure deviation in the profile (p^*, p^*) . By definition it is an EinSS. The sufficiency of (15) is proven.

(5). One can easily check that $p_M \leq p^*$ is the maximum condition of functions $u_1(p_1) = u_1(p_1, p^*)$ and $u_2(p_2) = u_2(p^*, p_2)$ in the points $p_1 = p^*$ and $p_2 = p^*$ respectively. In other words it is a condition of Nash equilibrium for the profile (p^*, p^*) . \square

REFERENCES

1. Allen B., Hellwig M. (1986), Bertrand-Edgeworth oligopoly in large markets, *Review of Economic Studies*, **53**, 175–204.
2. d'Aspremont C., Gabszewicz J., Thisse J.-F. (1979), On Hotelling's 'Stability in Competition', *Econometrica*, **47**(5), 1145–1150.
3. d'Aspremont C., Gabszewicz J. (1980), On Quasi-Monopolies, (*CORE Discussion Paper*, **8011**, Universite Catholique de Louvain).
4. Aumann R.J. (1961), Almost strictly competitive games, *J. Soc. Indust. Appl. Math.*, **9**, No.4, 544–550
5. Baye M.R., Kovenock D., de Vries C.G. (1993), The Solution of the Tullock Rent-Seeking Game when $R > 2$: Mixed-Strategy Equilibria and Mean Dissipation Rates, (*Discussion Paper*, **1993-68**, Tilburg University, Center for Economic Research).

6. Bertrand J. (1883), Review of Cournot's 'Rechercher sur la theoric mathematique de la richesse', *Journal des Savants*, 499-508.
7. Edgeworth F.M. (1925), *Papers Relating to Political Economy I*, (London: Macmillan).
8. Dasgupta P., Maskin E. (1986), The existence of equilibrium in discontinuous economic games, II: Applications, *Review of Economic Studies*, **LIII**, pp. 27-41
9. Dixon, H.D. (1984), The existence of mixed-strategy equilibria in a price-setting oligopoly with convex costs, *Economics Letters*, **16**, 205-212
10. Eaton B.C., Lipsey R. (1978), Freedom of Entry and the Existence of Pure Profits, *The Economic Journal*, **88** (351), 455-469.
11. Facchinei F., Kanzow C. (2007), Generalized Nash equilibrium problems. *4OR - A Quarterly Journal of Operations Research*, **5**, 173-210.
12. Hotelling H. (1929), Stability in Competition, *The Economic Journal*, **39** (153), 41-57.
13. Iskakov M.B. (2005), Equilibrium in Safe Strategies, *Automation and Remote Control*, **66**, 3, 465-478.
14. Iskakov M.B. (2008), Equilibrium in Safety Strategies and equilibriums in objections and counter objections in noncooperative games, *Automation and Remote Control*, **69**, 2, 278-298.
15. Iskakov M., Iskakov A. (2012), Solution of the Hotelling's game in secure strategies, *Economics Letters*, **117**, 115-118.
16. Iskakov M., Iskakov A., Zakharov A. (2012), Solution of the Tullock Contest in Secure Strategies, *Work Paper*, Moscow High School of Economics (in print).
17. North D.C., Wallis J.J., and Weingast B.R. (2009), *Violence and Social Orders: A Conceptual Framework for Interpreting Recorded Human History*, New York: Cambridge University Press.
18. Osborne M.J., Pitchik C. (1987), Equilibrium in Hotelling's Model of Spatial Competition, *Econometrica*, **55**, No.4, 911-922
19. Riley J.G. (1979), Informational Equilibrium, *Econometrica*, **47**, No.2, 331-359
20. Rothschild M. and Stiglitz J.E. (1976), Equilibrium in Competitive Insurance Markets: An Essay on the Economics of Imperfect Information, *Quarterly Journal of Economics*, **90**, 629-649
21. Shy O. (2002), A quick-and-easy method for estimating switching costs, *International Journal of Industrial Organization* **20**, 71-87.
22. Skaperdas S. (1994), Contest success functions, *Economic Theory*, **7**, 283-290.
23. Tullock G. (1967), The welfare costs of tariffs, monopoly and theft, *Western Economic Journal*, **5**, 224-232.

24. Tullock G. (1980), Efficient rent seeking. In James M. Buchanan, Robert D. Tollison, Gordon Tullock, (Eds.), *Toward a theory of the rent-seeking society*. College Station, TX: Texas A&M University Press, 97-112.
25. Wilson C. (1977), A model of Insurance Markets with Incomplete Information, *Journal of Economic Theory*, **16**, 167-207

Recent titles

CORE Discussion Papers

- 2012/24. Pilar GARCIA-GOMEZ, Erik SCHOKKAERT, Tom VAN OURTI and Teresa BAGO D'UVA. Inequity in the face of death.
- 2012/25. Christian HAEDO and Michel MOUCHART. A stochastic independence approach for different measures of concentration and specialization.
- 2012/26. Xavier RAMOS and Dirk VAN DE GAER. Empirical approaches to inequality of opportunity: principles, measures, and evidence.
- 2012/27. Jacques H. DRÈZE and Erik SCHOKKAERT. Arrow's theorem of the deductible : moral hazard and stop-loss in health insurance.
- 2012/28. Luc BAUWENS and Giuseppe STORTI. Computationally efficient inference procedures for vast dimensional realized covariance models.
- 2012/29. Pierre DEHEZ. Incomplete-markets economies: The seminar work of Diamond, Drèze and Radner.
- 2012/30. Helmuth CREMER, Pierre PESTIEAU and Grégory PONTIÈRE. The economics of long-term care: a survey.
- 2012/31. Mathieu VAN VYVE, Laurence A. WOLSEY and Hande YAMAN. Relaxations for two-level multi-item lot-sizing problem.
- 2012/32. Jean-François MERTENS and Anna RUBINCHIK. Equilibria in an overlapping generations model with transfer policies and exogenous growth.
- 2012/33. Jean-François MERTENS and Anna RUBINCHIK. Pareto optimality of the golden rule equilibrium in an overlapping generations model with production and transfers.
- 2012/34. Hélène LATZER and Florian MAYNERIS. Income distribution and vertical comparative advantage. Theory and evidence.
- 2012/35. Pascal MOSSAY and Takatoshi TABUCHI. Preferential trade agreements harm third countries.
- 2012/36. Aitor CALO-BLANCO. The compensation problem with fresh starts.
- 2012/37. Jean-François CARPANTIER and Arnaud DUFAYS. Commodities volatility and the theory of storage.
- 2012/38. Jean-François CARPANTIER and Christelle SAPATA. Unfair inequalities in France: A regional comparison.
- 2012/39. Shui-Ki WAN, Shin-Huei WANG and Chi-Keung WOO. Total tourist arrival forecast: aggregation vs. disaggregation.
- 2012/40. Thierry BRECHET, Yann MENIERE and Pierre M. PICARD. The clean development mechanism in a global carbon market.
- 2012/41. Nikolaos GEORGANTZIS, Rafael MONER-COLONQUES, Vicente ORTS and José J. SEMPERE-MONERRIS. Theoretical and experimental insights on firms' internationalization decisions under uncertainty.
- 2012/42. Berardino CESI and Dimitri PAOLINI. Peer group and distance: when widening university participation is better.
- 2012/43. Arnaud DUFAYS. Infinite-state Markov-switching for dynamic volatility and correlation models.
- 2012/44. Miguel JARA, Dimitri PAOLINI and J.D. TENA. Management efficiency in football: An empirical analysis of two extreme cases.
- 2012/45. Guillaume ROELS, Philippe CHEVALIER and Ying WEI. United we stand? Coordinating capacity investment and allocation in joint ventures.
- 2012/46. Mikel BEDAYO, Ana MAULEON and Vincent VANNETELBOSCH. Bargaining and delay in trading networks.
- 2012/47. Pierre M. PICARD and Ridwan D. RUSLI. State owned firms: Private debt, cost revelation and welfare.
- 2012/48. Shin-Huei WANG, Luc BAUWENS and Cheng HSIAO. Forecasting long memory processes subject to structural breaks.
- 2012/49. Thierry BRECHET, Carmen CAMACHO and Vladimir M. VELIOV. Adaptive model-predictive climate policies in a multi-country setting.

Recent titles

CORE Discussion Papers - continued

- 2012/50 Vladyslav NORA and Hiroshi UNO. Saddle functions and robust sets of equilibria.
- 2012/51 Thomas BAUDIN, David DE LA CROIX and Paula GOBBI. DINKs, DEWkS & Co. Marriage, fertility and childlessness in the United States.
- 2012/52 David DE LA CROIX and Omar LICANDRO. The longevity of famous people from Hammurabi to Einstein.
- 2012/53 Marijn VERSCHELDE, Jean HINDRIKS, Glenn RAYP and Koen SCHOORS. School staff autonomy and educational performance: within school type evidence.
- 2012/54 Thierry BRECHET and Susana PERALTA. Markets for tradable emission permits with fiscal competition.
- 2012/55 Sudipto BHATTACHARYA, Claude D'ASPREMONT, Sergei GURIEV, Debapriya SEN and Yair TAUMAN. Cooperation in R&D: patenting, licensing and contracting.
- 2012/56 Guillaume WUNSCH, Michel MOUCHART and Federica RUSSO. Functions and mechanisms in structural-modelling explanations.
- 2012/57 Simone MORICONI, Pierre M. PICARD and Skerdilajda ZANAJ. Commodity taxation and regulatory competition.
- 2012/58 Yurii NESTEROV and Arkadi NEMIROVSKI. Finding the stationary states of Markov chains by iterative methods.
- 2012/59 Tanguy ISAAC and Paolo PIACQUADIO. Equity and efficiency in an overlapping generation model.
- 2012/60 Luc BAUWENS, Giuseppe STORTI and Francesco VIOLANTE. Dynamic conditional correlation models for realized covariance matrices.
- 2012/61 Mikhail ESKAKOV and Alexey ISKAKOV. Equilibrium in secure strategies.

Books

- G. DURANTON, Ph. MARTIN, Th. MAYER and F. MAYNERIS (2010), *The economics of clusters – Lessons from the French experience*. Oxford University Press.
- J. HINDRIKS and I. VAN DE CLOOT (2011), *Notre pension en héritage*. Itinera Institute.
- M. FLEURBAEY and F. MANIQUET (2011), *A theory of fairness and social welfare*. Cambridge University Press.
- V. GINSBURGH and S. WEBER (2011), *How many languages make sense? The economics of linguistic diversity*. Princeton University Press.
- I. THOMAS, D. VANNESTE and X. QUERRIAU (2011), *Atlas de Belgique – Tome 4 Habitat*. Academia Press.
- W. GAERTNER and E. SCHOKKAERT (2012), *Empirical social choice*. Cambridge University Press.
- L. BAUWENS, Ch. HAFNER and S. LAURENT (2012), *Handbook of volatility models and their applications*. Wiley.
- J-C. PRAGER and J. THISSE (2012), *Economic geography and the unequal development of regions*. Routledge.
- M. FLEURBAEY and F. MANIQUET (2012), *Equality of opportunity: the economics of responsibility*. World Scientific.
- J. HINDRIKS (2012), *Gestion publique*. De Boeck.

CORE Lecture Series

- R. AMIR (2002), Supermodularity and complementarity in economics.
- R. WEISMANTEL (2006), Lectures on mixed nonlinear programming.
- A. SHAPIRO (2010), Stochastic programming: modeling and theory.