"Exact tracer age from two passive tracer concentrations and ventilation timescales: analytical solutions from a highly-idealised water-column model"

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Exact tracer age from two passive tracer concentrations and ventilation timescales: analytical solutions from a highly-idealised water-column model

Eric Deleersnijder, March 26, 2014

Deleersnijder (2011) established a relation between the age of a tracer injected into the flow under study by a point source with a constant release rate, the concentration of this tracer and the concentration of another tracer injected by a source located at the same point whose rate of release increases linearly in time. Let t and $\mathbf{x}$ represent the time and the position vector. The first tracer, whose concentration is denoted $C_1(t,\mathbf{x})$, is injected at the constant rate Q. Next, the injection rate of the second tracer is Qt/T, where the constant T is a relevant timescale. In other words, the amount of the first and second tracers injected during the time interval $[t, t+\delta t]$ tends to $Q\delta t$ and $(Qt/T)\delta t$, respectively, as $\delta t \rightarrow 0$. The age of the first tracer (i.e. the time elapsed since entering the flow) was seen to be (Deleersnijder 2011)

$$a_1(t,\mathbf{x}) = t - T\frac{C_2(t,\mathbf{x})}{C_1(t,\mathbf{x})} \tag{1}$$

This formula[1] holds true at any time and position, and is equal to the age that would be derived from the equations of the Constituent-oriented Age and Residence time Theory (CART, www.climate.be/cart). Fortunately, the aforementioned age is independent of the release rate Q and the timescale T.

The objectives of the present working note are to:

- apply the general theory of Deleersnijder (2011) to a higly-idealised water column model, allowing for exact analytical solution to be derived;
- compare the age of the first tracer to the CART ages usually calculated in ventilation studies (e.g. Deleersnijder 2009).

Although a number of analytical solutions are derived and discussed, the present study is rather inconclusive, since, in the water column model under consideration, the age of the first tracer appears to be an approximation of none of the ventilation timescales.

[1] The author of the present working note has been told on several occasions that expression (1) has been established a long time ago in atmospheric studies. However, the relevant references, if any, are still unknown to him.

Exact age from two tracer concentrations

The domain of interest is a water column of depth h. All variables are assumed to be horizontally homogeneous. Let z represent the vertical coordinate, increasing upward. Then, the sea surface is located at $z = 0$ while $z = -h$ refers to the seabed (Figure 1).

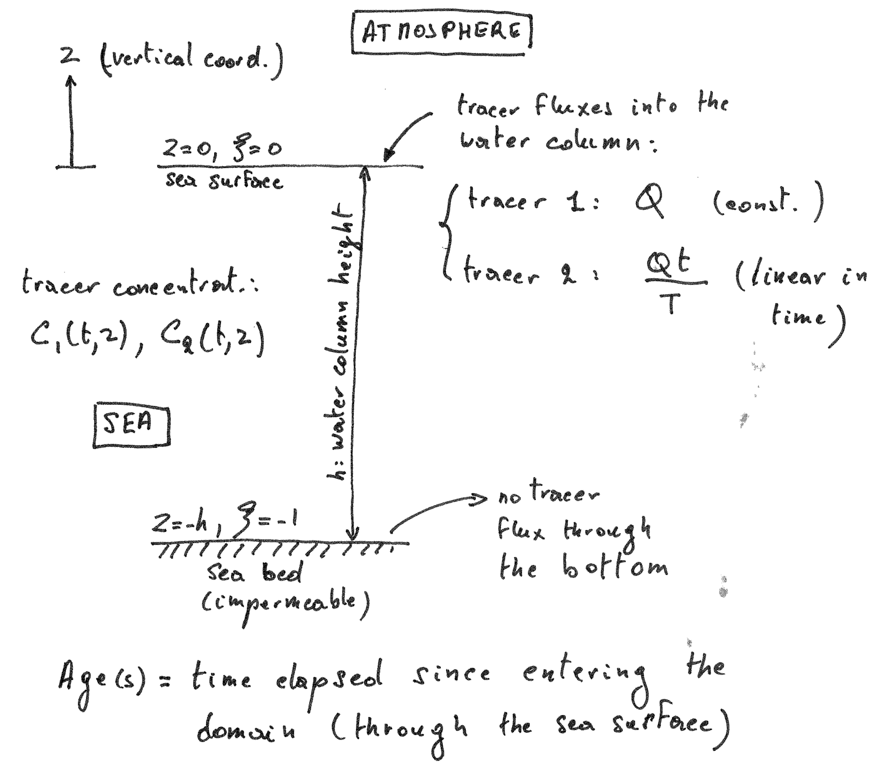

Figure 1. Geometry and boundary conditions for studying the dynamics of the passive tracers considered in this Section. All variables are assumed to be horizontally homogeneous. The tracers are injected into the domain through the sea surface. The injection rate Q of the first tracer is constant, while that of the other if proportional to the elapsed time (Qt/T). The age is defined as the time elapsed since entering the domain.

The tracer concentrations, $C_i(t,z)$ with i=1,2, obey the equations

(2) $$\frac{\partial C_i}{\partial t} = \frac{\partial}{\partial z}\left(K\frac{\partial C_i}{\partial z}\right) \qquad (i=1,2) \ ,$$

where K denotes the eddy diffusivity, which, for the sake of simplicity, will be assumed to be a constant. The lower boundary is assumed to be impermeable, leading to the boundary condition

(3) $$\left[K\frac{\partial C_i}{\partial z}\right]_{z=-h} = 0 \qquad (i=1,2) \ .$$

A point source located at the sea surface injects tracer particles according to the following expressions

(4) $$\left[K\frac{\partial C_1}{\partial z}\right]_{z=-h} = Q$$

and

(5) $$\left[K\frac{\partial C_2}{\partial z}\right]_{z=0} = \frac{Qt}{T} \ .$$

At the initial instant, there is no tracer in the domain of interest:

(6) $$C_i(0,z) = 0 \qquad (i=1,2) \ .$$

The solution of the partial differential problem (2)-(6) reads

(7) $$C_1(t,z) = Q\int_0^t G(t',z)\,dt'$$

and

(8) $$C_2(t,z) = \frac{Q}{T}\int_0^t (t-t')G(t',z)\,dt' \ ,$$

where G is the Green's function of the problem under consideration. The latter is the solution of

$$(9) \qquad \begin{cases} \dfrac{\partial G}{\partial t} = \dfrac{\partial}{\partial z}\left(K\dfrac{\partial G}{\partial z}\right) \\ \left[K\dfrac{\partial G}{\partial z}\right]_{z=-h} = 0 \ , \quad \left[K\dfrac{\partial G}{\partial z}\right]_{z=0} = \delta(t-0) \ , \quad G(0,z) = 0 \end{cases}$$

where δ denotes the Dirac function. Upon using the dimensionless variables

$$(10) \qquad \tau = \frac{t}{h^2 / K} \quad , \quad \xi = \frac{z}{h}$$

the Green's function reads

$$(11) \qquad G(\tau,\xi) = \frac{1}{h} + \frac{2}{h}\sum_{n=1}^{\infty} e^{-\kappa_n^2 \tau}\cos(\kappa_n \xi) \ .$$

with $\kappa_n = n\pi$.

Combining (7) and (11), the concentration of the first tracer is obtained:

$$(12) \qquad C_1(\tau,\xi) = Q\int_0^t G(t',\xi)\,dt' = \frac{Qh^2}{K}\int_0^\tau G(\tau',\xi)\,d\tau' = \frac{Qh}{K}\left[\tau + 2\sum_{n=1}^{\infty}\frac{1}{\kappa_n^2}\cos(\kappa_n \xi) - 2\sum_{n=1}^{\infty}\frac{1}{\kappa_n^2}e^{-\kappa_n^2 \tau}\cos(\kappa_n \xi)\right] .$$

Next, upon using the Fourier series

$$(13) \qquad 3\xi^2 + 6\xi + 2 = 12\sum_{n=1}^{\infty}\frac{\cos(\kappa_n \xi)}{\kappa_n^2} \ ,$$

the expression (12) transforms to (Figure 1)

$$(14) \qquad C_1(\tau,\xi) = \frac{Qh}{K}\left[\tau + \frac{3\xi^2 + 6\xi + 2}{6} - 2\sum_{n=1}^{\infty}\frac{1}{\kappa_n^2}e^{-\kappa_n^2 \tau}\cos(\kappa_n \xi)\right]$$

In accordance with (8) and (11), the second tracer concentration is seen to be

$$C_2(\tau,\xi) = \int_0^t Q\frac{t-t'}{T}G(t',\xi)\,dt' = \frac{Qh^4}{TK^2}\int_0^\tau (\tau-\tau')G(\tau',\xi)\,d\tau' \tag{15}$$

$$= \frac{Qh^3}{TK^2}\left[\frac{\tau^2}{2} + 2\tau\sum_{n=1}^{\infty}\frac{1}{\kappa_n^2}\cos(\kappa_n\xi) - 2\sum_{n=1}^{\infty}\frac{1}{\kappa_n^4}\cos(\kappa_n\xi) + 2\sum_{n=1}^{\infty}\frac{1}{\kappa_n^4}e^{-\kappa_n^2\tau}\cos(\kappa_n\xi)\right]$$

Then, substituting into this expression the Fourier series

$$15\xi^4+60\xi^3+60\xi^2-8 = -720\sum_{n=1}^{\infty}\frac{\cos(\kappa_n\xi)}{\kappa_n^4} \tag{16}$$

yields (Figure 2)

$$C_2(\tau,\xi) = \frac{Qh^3}{TK^2}\left[\frac{\tau^2}{2} + \tau\frac{3\xi^2+6\xi+2}{6} + \frac{15\xi^4+60\xi^3+60\xi^2-8}{360} + 2\sum_{n=1}^{\infty}\frac{1}{\kappa_n^4}e^{-\kappa_n^2\tau}\cos(\kappa_n\xi)\right] \tag{17}$$

By combining (1), (14) and (17), the age of the first tracer is readily obtained (Figure 3), i.e.

$$a_1(\tau,\xi) = \frac{1 - \dfrac{15\xi^4+60\xi^3+60\xi^2-8}{180\tau^2} - \dfrac{4}{\tau^2}\displaystyle\sum_{n=1}^{\infty}\frac{1+\tau\kappa_n^2}{\kappa_n^4}e^{-\kappa_n^2\tau}\cos(\kappa_n\xi)}{1 + \dfrac{3\xi^2+6\xi+2}{6\tau} - \dfrac{2}{\tau}\displaystyle\sum_{n=1}^{\infty}\frac{1}{\kappa_n^2}e^{-\kappa_n^2\tau}\cos(\kappa_n\xi)} \times \frac{t}{2} \tag{18}$$

The tracers are passive and are injected into the domain through the upper boundary, while the lower boundary is impermeable. Therefore, the depth mean of each tracer concentration must be equal to the ratio the amount of tracer injected during the time interval $[0,t]$ to the water column height, i.e. Qt/h for the first tracer and $Qt^2/(2Th)$ for the second one. Since

$$\int_{-1}^{0}\cos(\kappa_n\xi)\,d\xi = 0\ , \tag{19}$$

the depth mean of (14) and that of (17) are readily seen to satisfy the aforementioned property:

$$(20) \qquad \bar{C}_1(\tau) \equiv \int_{-1}^{0} C_1(\tau,\xi)\, d\xi = \frac{Qh}{K}\int_{-1}^{0}\left[\tau + \frac{3\xi^2+6\xi+2}{6} - 2\sum_{n=1}^{\infty}\frac{1}{\kappa_n^2}e^{-\kappa_n^2\tau}\cos(\kappa_n\xi)\right]d\xi = \frac{Qh\tau}{K} = \frac{Qt}{h}$$

and

$$(21) \qquad \bar{C}_2(\tau) \equiv \int_{-1}^{0} C_2(\tau,\xi)\, d\xi = \frac{Qh^3}{TK^2}\int_{-1}^{0}\left[\frac{\tau^2}{2} + \tau\frac{3\xi^2+6\xi+2}{6} + \frac{15\xi^4+60\xi^3+60\xi^2-8}{360} + 2\sum_{n=1}^{\infty}\frac{1}{\kappa_n^4}e^{-\kappa_n^2\tau}\cos(\kappa_n\xi)\right]d\xi = \frac{Qt^2/(2T)}{h} .$$

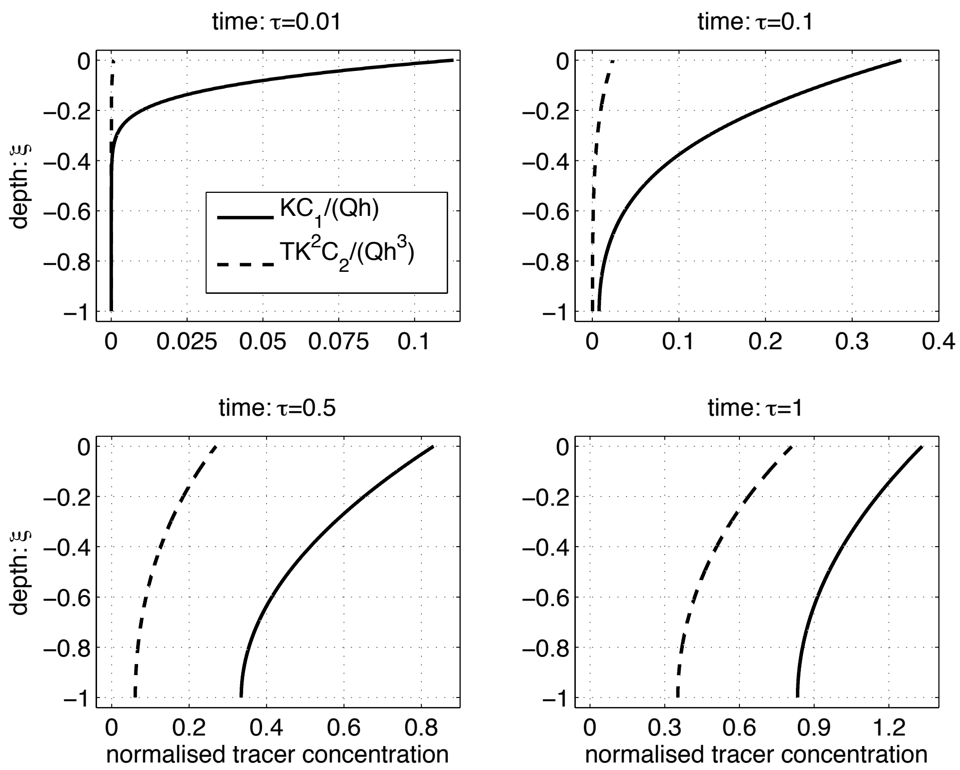

Figure 2. Vertical profiles at various instants of normalised tracer concentrations, i.e. $KC_1/(Qh)$ and $TK^2C_2/(Qh^3)$.

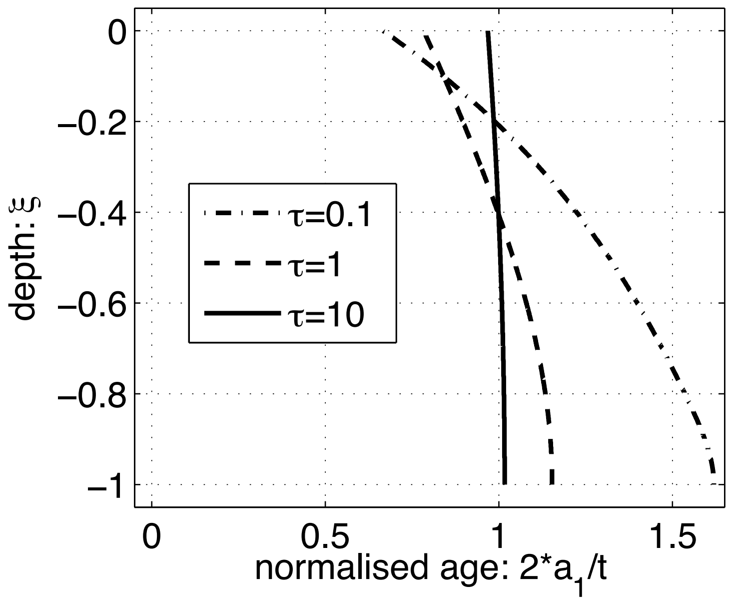

Figure 3. Profile of the normalised age of the first tracer ($2a_1/t$) at various instants. The vertical constrasts of the age of the first tracer are relatively smaller than those of its concentration.

The age of the first tracer particles that were injected into the domain during the time interval $[t-t', t-t'+\delta t']$ tends to t' as $\delta t' \to 0$. Therefore, at time t, the mean of the first tracer parcel present in the domain must be

$$(22) \qquad \bar{a}_1(t) = \frac{\int_0^t Qt'\,dt'}{\int_0^t Q\,dt'} = \frac{t^2/2}{t} = \frac{t}{2} \, .$$

This result may also be obtained from (1), (20) and (21):

$$(23)\qquad \bar{a}_1(t) = \frac{\int_{-1}^{0} C_1 a_1 \, d\xi}{\int_{-1}^{0} C_1 \, d\xi} = \frac{\int_{-1}^{0} C_1 \overbrace{(t - TC_2 / C_1)}^{=a_1,\text{ see (1)}} d\xi}{\bar{C}_1} = \frac{\bar{C}_1 t - T\bar{C}_2}{\bar{C}_1} = t - T\frac{\bar{C}_2}{\bar{C}_1} = t - T\frac{\overbrace{\frac{Qt^2}{2Th}}^{\substack{\text{see (21)}\\ =\bar{C}_2}}}{\underbrace{\frac{Qt}{h}}_{\substack{=\bar{C}_1\\ \text{see (20)}}}} = t - \frac{t}{2} = \frac{t}{2} \; .$$

All this is in accordance with elementary physical intuition.

It is worth demonstrating that formula (1) is in accordance with CART. According to this theory, the age of a constituent is to be estimated as the ratio of the age concentration to the concentration (e.g. Delhez et al. 1999; Deleersnijder et al. 2001). Accordingly, upon denoting $\alpha_1(t,z)$, the age concentration of the first tracer, the age of the latter is $a_1(t,z) = \alpha_1(t,z) / C_1(t,z)$. Therefore, for expression (1) to be consistent with CART, it remaisn to be seen that the age concentration of the first tracer is

$$(24)\qquad \alpha_1(t,z) = t\,C_1(t,z) - T\,C_2(t,z) \; .$$

Given that the age of the tracer particles injected into the domain is assumed to be zero at the instant they enter the water column and that the bottom of the sea is impermeable, the age concentration of the first tracer is the solution of the following partial differential problem:

$$(25)\qquad \begin{cases} \dfrac{\partial \alpha_1}{\partial t} = C_1 + \dfrac{\partial}{\partial z}\left(K\dfrac{\partial \alpha_1}{\partial z}\right) \\ \left[K\dfrac{\partial \alpha_1}{\partial z}\right]_{z=-h} = 0 \; , \quad \left[K\dfrac{\partial \alpha_1}{\partial z}\right]_{z=0} = 0 \; , \quad \alpha_1(0,z) = 0 \end{cases}$$

The expression (24) satisfies the differential equation governing the evolution of the age concentration:

$$(26)\qquad \frac{\partial \alpha_1}{\partial t} = \frac{\partial}{\partial t}\overbrace{(tC_1 - TC_2)}^{=\alpha_1,\text{ see (24)}} = C_1 + t\frac{\partial C_1}{\partial t} - T\frac{\partial C_2}{\partial t} = C_1 + t\overbrace{\frac{\partial}{\partial z}\left(K\frac{\partial C_1}{\partial z}\right)}^{=\frac{\partial C_1}{\partial t},\text{ see (2)}} - T\overbrace{\frac{\partial}{\partial z}\left(K\frac{\partial C_2}{\partial z}\right)}^{=\frac{\partial C_2}{\partial t},\text{ see (2)}}$$

$$= C_1 + \frac{\partial}{\partial z}\underbrace{(tC_1 - TC_2)}_{=\alpha_1,\text{ see (24)}} = C_1 + \frac{\partial}{\partial z}\left(K\frac{\partial \alpha_1}{\partial z}\right)$$

It is trivial to see that the age concentration (24) satisfies the impermeability condition of the bottom and the initial condition that is part of (25). The surface boundary condition in (25) is also satisfied:

$$(27) \qquad \left[K\frac{\partial\alpha_1}{\partial z}\right]_{z=-h} = [K\frac{\partial}{\partial z}\underbrace{(tC_1-TC_2)}_{=\alpha_1,\ \text{see (24)}}]_{z=-h} = t\underbrace{\left[K\frac{\partial C_1}{\partial z}\right]_{z=-h}}_{=Q,\ \text{see (4)}} - T\underbrace{\left[K\frac{\partial C_2}{\partial z}\right]_{z=-h}}_{=Qt/T,\ \text{see (5)}} = Qt - T\frac{Qt}{T} = Qt - Qt = 0\ .$$

QED.

Ventilation timescales

According to England (1995), the "World Ocean circulation at its largest scale can be thought of as a gradual renewal or ventilation of the deep ocean by water that was once at the sea surface". Thus, estimating the age as the time elapsed since leaving the ocean surface layers is likely to provide useful insight into the ventilation processes of the World Ocean. This is why the age is a popular diagnostic tool in this domain of interest.

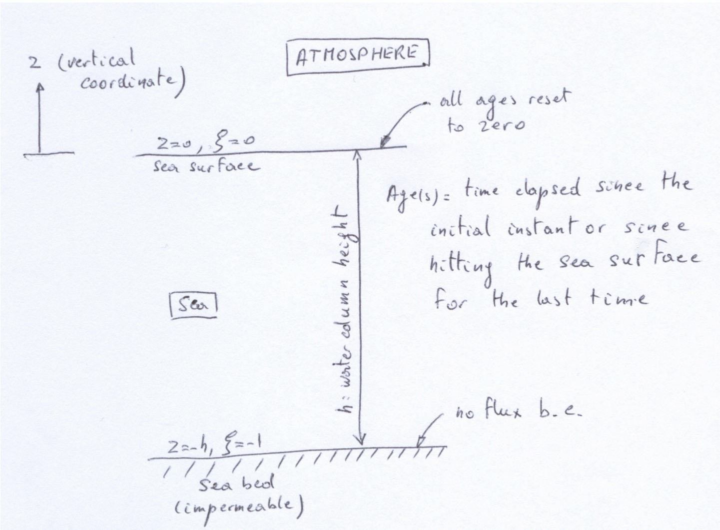

Figure 4. Geometry and boundary conditions for studying the dynamics of water types and related ventilation timescales. The seabed is impermeable and all ages are reset to zero at the ocean surface.

CART provides a coherent set of partial differential equations from which ventilation-related timescales may be derived. As was suggested in Deleersnijder (2009), the water may be regarded as a passive tracer, which will be identified hereinafter by the subscript "w". Next, the water may be split into two categories, i.e. the surface water and the water of the interior of the ocean, or interior water for short. These passive tracers will be identified by the subscripts "sw" and "iw", respectively. The surface water refers to all the water particles that have touched the ocean surface at least once, while the interior water is made up of those water particles that have not yet touched the ocean-atmosphere interface. Once an interior water particle touches the ocean surface for the first time, it turns into a surface water particle. This approach will be applied to a horizontally-homogeneous water column similar to that dealt with in the previous Section (Figure 4).

The water type concentrations, age concentrations and ages are denoted $C_\varsigma(t,z)$, $\alpha_\varsigma(t,z)$ and $a_\varsigma(t,z)$, respectively, with

$$(28) \qquad a_\varsigma(t,z) = \frac{\alpha_\varsigma(t,z)}{C_\varsigma(t,z)}$$

and $\varsigma = iw, sw, w$. As the water is the aggregate made up of the interior water and the surface water, the water concentration must satisfy the constraints

$$(29) \qquad C_w(t,z) = C_{iw}(t,z) + C_{sw}(t,z) = 1 \ .$$

Clearly, the age of the interior water must be equal to the elapsed time

$$(30) \qquad a_{iw}(t,w) = t \ .$$

On the other hand, CART's age-averaging hypothesis[2] implies that the following constraint must be satisfied at any time and position

$$(32) \qquad \underbrace{C_w\, a_w}_{=\alpha_w} = \underbrace{C_{iw}\, a_{iw}}_{=\alpha_{iw}} + \underbrace{C_{sw}\, a_{sw}}_{=\alpha_{sw}} \ .$$

In other words, the mean age of a mixture is the mass-weighted average of the ages of its constituents. Then, combining (29)-(32) yields

[2] Consider, for instance, two particles that are identified by means of superscripts "A" and "B". Their mass and age are denoted m^X and a^X, respectively, with X=A,B. The mass of the system "A+B" obvioulsy is $m^{A+B} = m^A + m^B$. This is in agreement with basic physical principles that stipulate that mass is an additive quantity. No such principle exists for the age. Therefore, to obtain the mean age of the system "A+B", an arbitrary decision is to be made. The latter was formulated as the so-called age-averaging hypothesis (Deleersnijder et al., 2001), which is the only arbitrary element of CART. It stipulates that the mean age of a system made up of various particles is the mass-weighted average of the ages of the particles. Accordingly, the mean age of the system "A+B", a^{A+B}, satisfies the follow relation: $m^{A+B}a^{A+B} = m^A a^A + m^B a^B$. Clearly, the age is not an additive quantity, but the age content is — the age content of a particle being defined as the product of its mass and its age. Ages other than CART's, such as the carbon-14 age, implicitly or explicitly rely on an age-averaging hypothesis (Deleersnijder et al., 2001). However, it is believed that CART's age-averaging hypothesis is the simplest one could think of.

(33) $a_w = (1 - C_{sw})t + C_{sw}\, a_{sw}$,

which implies that the age of the water must satisfy the inequalities

(34) $a_{sw}(t,z) \leq a_w(t,z) \leq a_{iw}(t,z) = t$.

Table 1. Initial and boundary conditions for solving the equations (35)-(36) governing the concentration and age concentration of the water types under considered herein, i.e. the interior water, the surface water and the water, which is the aggregate of the former two water types.

	initial condition	surface boundary condition	bottom boundary condition
interior water	$C_{iw}(0,z)=1$ $\alpha_{iw}(0,z)=0$	$C_{iw}(t,0)=0$ $\alpha_{iw}(t,0)=0$	$\left[K\dfrac{\partial C_{iw}}{\partial z}\right]_{z=-h}=0$ $\left[K\dfrac{\partial \alpha_{iw}}{\partial z}\right]_{z=-h}=0$
surface water	$C_{sw}(0,z)=0$ $\alpha_{sw}(0,z)=0$	$C_{sw}(t,0)=1$ $\alpha_{sw}(t,0)=0$	$\left[K\dfrac{\partial C_{sw}}{\partial z}\right]_{z=-h}=0$ $\left[K\dfrac{\partial \alpha_{sw}}{\partial z}\right]_{z=-h}=0$
water	$C_w(0,z)=1$ $\alpha_w(0,z)=0$	$C_w(t,0)=1$ $\alpha_w(t,0)=0$	$\left[K\dfrac{\partial C_w}{\partial z}\right]_{z=-h}=0$ $\left[K\dfrac{\partial \alpha_w}{\partial z}\right]_{z=-h}=0$

The concentrations and age concentrations of the water types are governed by the following partial differential equations

$$(35) \qquad \frac{\partial C_\varsigma}{\partial t} = \frac{\partial}{\partial z}\left(K\frac{\partial C_\varsigma}{\partial z}\right) \ , \quad \varsigma = iw, sw, w$$

and

$$(36) \qquad \frac{\partial \alpha_\varsigma}{\partial t} = C_\varsigma + \frac{\partial}{\partial z}\left(K\frac{\partial \alpha_\varsigma}{\partial z}\right) \ , \quad \varsigma = iw, sw, w \ .$$

The initial and boundary conditions to be applied to theses variables are listed in Table 1, while the analytical solutions are provided in Table 2 and illustrated in Figures 5 and 6.

Table 2. The concentration and age concentration of the water types obtained by solving equations (35)-(36) under the initial and boundary conditions listed in Table 1. The following notation is used: $k_n = (n-1/2)\pi$.

	concentration	age concentration
interior water	$C_{iw}(\tau,\xi) = -2\sum_{n=1}^{\infty}\frac{1}{k_n}e^{-k_n^2\tau}\sin(k_n\xi)$	$\alpha_{iw}(\tau,\xi) = \frac{h^2}{K}\left[-2\sum_{n=1}^{\infty}\frac{\tau}{k_n}e^{-k_n^2\tau}\sin(k_n\xi)\right]$
surface water	$C_{sw}(\tau,\xi) = 1 + 2\sum_{n=1}^{\infty}\frac{1}{k_n}e^{-k_n^2\tau}\sin(k_n\xi)$	$\alpha_{sw}(\tau,\xi) = \frac{h^2}{K}\left[-\frac{\xi^2}{2} - \xi + 2\sum_{n=1}^{\infty}\frac{k_n^2\tau+1}{k_n^3}e^{-k_n^2\tau}\sin(k_n\xi)\right]$
water	$C_w(\tau,\xi) = 1$	$\alpha_w(\tau,\xi) = \frac{h^2}{K}\left[-\frac{\xi^2}{2} - \xi + 2\sum_{n=1}^{\infty}\frac{1}{k_n^3}e^{-k_n^2\tau}\sin(k_n\xi)\right]$

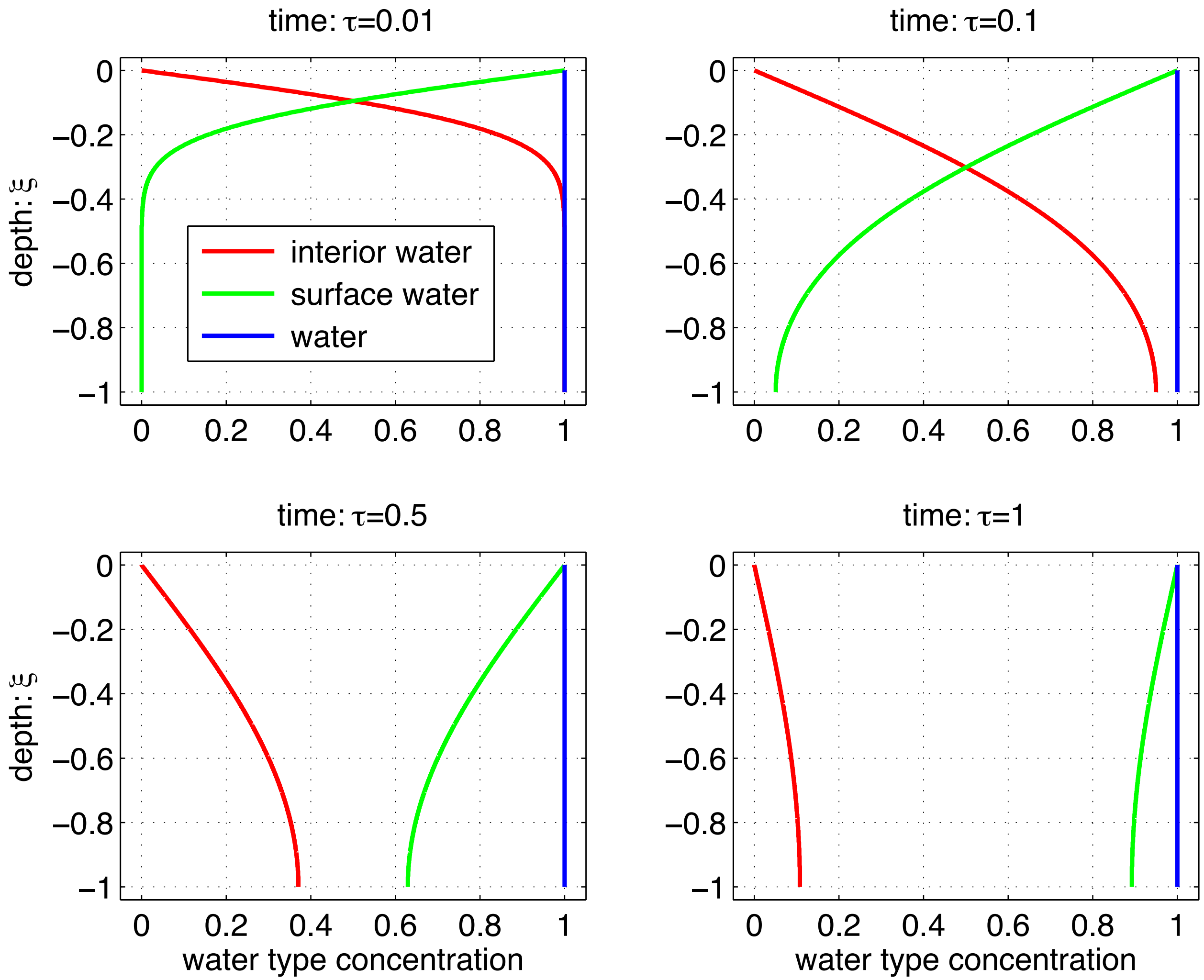

Figure 5. Concentration of the water types considered herein at various instants.

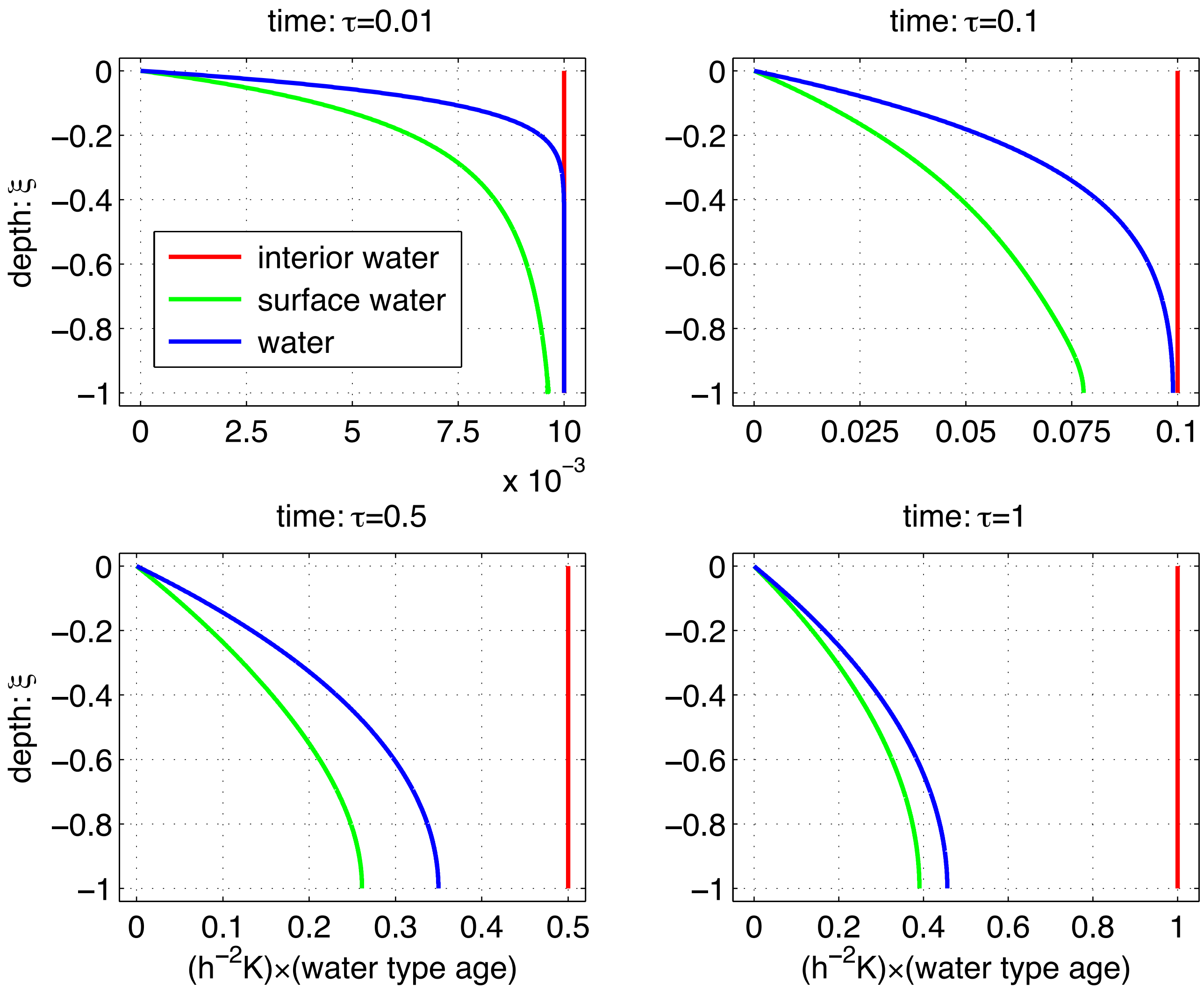

Figure 6. Normalised age of the water types considered herein at various instants.

Final remark

The age of the first tracer scales at $t/2$, while that of the surface water tends to a steady-state value of order h^2/K. Therefore, these two ages are intrinsically different.

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