"Tight MIP Formulations for Bounded Up/ Down Times and Interval-Dependent Start-Ups"

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CORE

DISCUSSION PAPER

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Maurice Queyranne[1] and Laurence A. Wolsey[2]

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JEL Classification: C44, C61

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[1] CORE, Université catholique de Louvain, and Sauder School of Business at the University of British Columbia.
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CORE DISCUSSION PAPER
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1. Introduction

Switching machines on and off is an important aspect of unit commitment and production planning problems, among others. For unit commitment problems four of the most commonly cited features are (i) nonlinear production costs, (ii) start-up costs that are a function of the machine down time, (iii) minimum up/down times and (iv) ramp rates; whereas in production planning minimum run times may be necessary because it takes time for the product to stabilize or for other economic reasons, and maximum run times may be imposed because of machine deterioration, etc. Both these classes of problems are often formulated as mixed integer programs. As the successful solution of such problems often depends on the quality of the formulation, providing stronger formulations of different aspects of the problem may help significantly in obtaining good or optimal solutions. In particular a *best possible* (*tight*) formulation of a mixed integer set $X \subset \mathbb{Z}^n \times \mathbb{R}^p$ is provided by an explicit or implicit (with additional variables) description of the convex hull of X.

Here we describe briefly some earlier work. There is a significant literature on MIP formulations of different aspects of unit commitment problems, see in particular articles in the journal, *IEEE Transactions on Power Systems*, and on MIP formulations of production planning problems, see Pochet and Wolsey [9]. Among others Frangioni et al. [3] and Wu [15] discuss strong formulations of nonlinear production costs (i), and Damcı-Kurt et al. [1] formulations of ramping constraints (iv).

Here we consider the two other aspects: minimum/maximum up-down times (iii) and time-varying (hot/warm/cold) start-ups (ii). For lower bounds on the length of on- and off-intervals, necessary inequalities can be found in Wolsey [14]. Malkin [5] showed that these inequalities describe the convex hull of solutions in the space of machine-on/set-up and start-up variables, and Lee et al. [4] describe the convex hull in the space of the machine-on/set-up variables. Van den Bergh et al. [12] present a very general formulation. Hedman et al. [2] discuss different formulations for minimum up/down times and show how their strength can be compared. For time independent start-ups Morales-España et al. [7, 8] and Viana and Pedroso [13] present basic MIP formulations.

In Section 2 we consider first the joint problem with both lower and upper bounds on the length of the on-intervals and interval-dependent switch-ons. We present a simple shortest path network formulation that provides a tight extended formulation with $O(n^2)$ constraints and variables and an $O(n^2)$ optimisation algorithm for an n-period instance. Then in the following sections we examine cases in which it is possible to obtain a tight formulation with only $O(n)$ variables. In Section 3 we present a new tight network dual formulation for the problem with both lower and upper bounds on the length of the on- and off-intervals. This allows us to generalize and simplify earlier results (treating just lower bounds) of Malkin [5] and Lee et al. [4]. In Section 4 we turn to the problem of interval-dependent start-ups. Based on a different path formulation, we obtain via projection a description of the convex hull of solutions in the space of the machine-on and start-up variables. In Section 5 we discuss possible extensions.

It is perhaps of interest that three different proof techniques are used in Sections 3 and 4 to obtain the various polyhedral descriptions: integer "event count" variables in Section 3.1; dynamic programming functionals providing feasible solutions for an extended formulation in 3.2 as in [10] and Hoffman cuts for node flows in Section 4.

Notation.

$[p, q]$ denotes the set of elements lying in the interval from p to q, i.e. the set $\{p, p+1, \cdots, q\}$. For a set S, $x(S) \equiv \sum_{u \in S} x_u$.

2. The Two Subproblems

2.1 The Problem with On/Off Interval Bounds

First we consider the problem with interval bounds. Given a discrete time horizon of n periods, the state of the system consists first of a number of periods (possibly zero) in which the machine is off, followed by intervals in which it is then on, then off, then on, etc. The first period of an on-interval is called a *switch-on period* and the first period of an off-interval (apart from the initial off-interval) is called a *switch-off period.* Various constraints are considered. In particular one can have bounds on the lengths of the on- and off-intervals. Thus one has parameters :
$\alpha_t \geq 1$ is a lower bound on the length of an on-interval starting in period t;
$\beta_t \geq \alpha_t$ is an upper bound on the length of an on-interval starting in period t;
$\gamma_t \geq 1$ is a lower bound on the length of an off-interval starting in period t;
$\delta_t \geq \gamma_t$ is an upper bound on the length of an off-interval starting in period t.

We define the following decision variables:
$y_t = 1$ if period t is an on-period and $y_t = 0$ otherwise;
$z_t = 1$ if period t is a switch-on period ($y_{t-1} = 0$ and $y_t = 1$);
$w_t = 1$ if period t is a switch off period[3] ($y_{t-1} = 1$ and $y_t = 0$).
Note that the equation $y_t - y_{t-1} = z_t - w_t$ always links the on and switch-on, switch-off variables. This allows us to eliminate either the w_t or the z_t variables.

A simple formulation is as follows:

$$\begin{aligned}
z_t &\leq y_t \quad \forall\, t && (1)\\
z_t &\geq y_t - y_{t-1} \quad \forall\, t && (2)\\
z_t &\leq 1 - y_{t-1} \quad \forall\, t && (3)\\
y_{t+k} &\geq z_t \quad \text{for } k = 0, \ldots, \alpha_t - 1,\ \forall\, t && (4)\\
z_t &\leq \sum_{j=t+1}^{t+\beta_t} (1 - y_j) \quad \forall\, t \leq n - \beta_t && (5)\\
1 - y_{t+k} &\geq w_t \quad \text{for } k = 0, \ldots, \gamma_t - 1,\ \forall\, t && (6)\\
w_t &\leq \sum_{j=t+1}^{t+\delta_t} y_j \quad \forall\, t \leq n - \delta_t && (7)\\
y_t - y_{t-1} &= z_t - w_t \quad \forall\, t && (8)\\
y,\ z,\ w &\in \{0,1\}^T. && (9)
\end{aligned}$$

Here (1)–(3) and (9) model the link between start-ups and on-periods; (4) ensures that the machine is on for α_t periods after a start-up in t; (5) that the machine cannot remain on for $\beta_t + 1$ periods after a start-up in t; etc. Note that (1), (3) and (8) imply $z_t + w_t \leq 1$ for all t.

[3]Note that this definition of w_t differs by one period from that used in [9].

Let $Z(\alpha,\beta,\gamma,\delta)$ denote the set of vectors (y,z) for which there exists (y,z,w) satisfying (1)–(9). Similarly define $Z(\alpha,\gamma)$ and $Z(\beta,\delta)$ as the sets of (y,z)-vectors arising from (1)–(4), (6), (8)–(9) and (1)–(3), (5), (7)–(9) respectively. Denote the corresponding projections onto the y variables by $Y(\alpha,\beta,\gamma,\delta)$, $Y(\alpha,\gamma)$, and $Y(\beta,\delta)$ respectively. Our goal in Section 3 will be to describe the convex hulls: $conv(Z(\alpha,\beta,\gamma,\delta))$, $conv(Y(\alpha,\gamma))$, etc.

2.2 The Problem with Interval-Dependent Start-Ups

The costs of a start-up may depend on the time that the machine has been idle – in particular one talks of hot, warm and cold start-ups [7, 8, 13]. Assuming $\gamma_t = \gamma$ and $\delta_t = \delta$ for all t, a type p start-up occurs if the machine has been idle for between $\theta^{p-1}+1$ and θ^p periods where $p=1,\dots,P$, and the given parameters $\theta=(\theta^1,\dots,\theta^P)$ satisfy $0 \le \gamma-1=\theta^0 \le \theta^1 < \theta^2 < \cdots < \theta^P = \delta$. The variable $z_t^p = 1$ if there is a type p start-up in t and $z_t^p = 0$ otherwise. Obviously $z_t = \sum_{p=1}^P z_t^p$. We assume for simplicity that the machine is on in period 0 (i.e., $y_0 = 1$). The general case will be discussed later.

A basic formulation $H(\theta)$ is:

$$\sum_{p=1}^{P} z_t^p \le y_t \quad \forall\, t \tag{10}$$

$$\sum_{p=1}^{P} z_t^p \ge y_t - y_{t-1} \quad \forall\, t \tag{11}$$

$$z_t^p \le 1 - y_{t-1-j} \quad \forall\, j=0,\dots,\theta^{p-1},\ \forall\, t,p \tag{12}$$

$$z_t^p \le \sum_{j=\theta^{p-1}+1}^{\theta^p} y_{t-1-j} \quad \forall\, t,p \tag{13}$$

$$y \in \{0,1\}^n;\quad z_t^p \in \{0,1\} \quad \forall\, t \ge \theta^{p-1}+2,\ \forall\, p. \tag{14}$$

Here (12) implies that if there is a type p switch-on in t, the machine was not on in the previous θ^{p-1} periods, whereas (13) implies that it was on sometime in the previous θ^p periods. In Section 4 we will describe $conv(H(\theta))$.

2.3 A Tight Extended Formulation and Optimization over $Y(\alpha,\beta,\gamma,\delta) \cap H(\theta)$

Here we consider the problem with bounds $\alpha,\beta,\gamma,\delta$ (constant over time for simplicity) as well as time dependent start-ups. Feasible solutions are represented by paths in an acyclic digraph $D=(V,A)$ that we now describe. The nodes are $V=\{0\}\cup\{1,\dots,n\}\cup\{1',\dots,n'\}\cup\{n+1\}$ and the arcs are of two types, $A = A_1 \cup A_2$: an arc $(i',j) \in A_1$ represents a switch-off in i followed by a start-up in j, and an arc $(i,j') \in A_2$ represents a start-up in i followed by a switch-off in j. More precisely, because of the bounds, $\{(i',j) : \gamma \le j-i \le \delta\} \subseteq A_1$ and $\{(i,j') : \alpha \le j-i \le \beta\} \subseteq A_2$. The initial and end conditions define some additional arcs leaving node 0 and arriving at node $n+1$ respectively. If the machine is off in 0 and was last on in $-\rho$ where $1 \le \rho \le \delta$, one includes the arcs $(0,t) \in A_1$ for $t \in [\max(1,\rho+\gamma+1), -\rho+\delta+1]$, while if the machine is on in 0 and was last on in $-\sigma$ where $1 \le \sigma \le \beta$, one includes the arcs $(0,t') \in A_2$ for $t \in [\max(1,-\sigma+\alpha+1), \sigma+\beta+1]$. If $j+\beta > n$, there is an arc $(j,n+1) \in A_1$ and if $j+\delta > n$ there is an arc $(j',n+1) \in A_2$.

It is well-known that the corresponding flow polyhedron in which 1 unit enters at node 0 and leaves at node $n+1$ has integral extreme points corresponding to the paths P from 0 to $n+1$. To describe this flow polyhedron, set $x_{ij} = 1$ if $(i', j) \in A_1 \cap P$ with $x_{ij} = 0$ otherwise, and $w_{ij} = 1$ if $(i', j) \in A_2 \cap P$ with $w_{ij} = 0$ otherwise.

Theorem 1 *i) The polyhedron Q:*

$$\begin{aligned}
\sum_{(0,j)\in A_1} x_{0j} + \sum_{(0,j)\in A_2} w_{0j} &= 1 && (15)\\
\sum_{(i,t)\in A_1} x_{it} - \sum_{(t,j)\in A_2} w_{tj} &= 0 \quad \text{for } t = 1,\dots,n && (16)\\
\sum_{(i,t)\in A_2} w_{it} - \sum_{(t,j)\in A_1} x_{tj} &= 0 \quad \text{for } t = 1,\dots,n && (17)\\
\sum_{i\le t, j>t} w_{ij} &= y_t \quad \text{for } t = 1,\dots,n && (18)\\
\sum_{t-\theta^p \le i < t-\theta^{p-1}} x_{it} &= z_t^p \quad \text{for } t = 1,\dots,n,\ \forall\, p && (19)\\
x,\ w &\in \mathbb{R}_+^A && (20)
\end{aligned}$$

is integral.
ii) $proj_{y,z}(Q) = conv(Z(\alpha,\beta,\gamma,\delta) \cap H(\theta))$.
iii) For any objective $(f, c^1, \dots, c^P)$, *the linear program* $\max\{fy - \sum_{p,t} c_t^p z_t^p : (y,z,x,w) \in Q\}$ *solves the optimization problem* $\max\{fy - \sum_{p,t} c_t^p z_t^p : (y,z,x,w) \in Z(\alpha,\beta,\gamma,\delta) \cap H(\theta)\}$.

Proof. Constraints (15), (16), and (17) are flow conservation constraints at nodes 0, $\{1,\dots,n\}$ and $\{1',\dots,n'\}$ respectively. (18) and (19) express y_t and z_t^p as linear functions of the x, w variables. Thus one can rewrite the objective function as a linear function in the x, w variables leaving a linear program over the path polytope (15)-(17), (20). □

Corollary 2 *The optimization problem over* $Z(\alpha,\beta,\gamma,\delta) \cap H(\theta)$ *can be solved as a longest path problem in an acyclic digraph with* $O(n^2)$ *arcs in* $O(n^2)$ *time.*

3. Polyhedral Results for On/Off Interval Bounds

3.1 Convex Hull in the (y, z)-Space

In this section we add the fairly natural assumption that each bound $\epsilon_t \in \{\alpha_t, \beta_t, \gamma_t, \delta_t\}$, satisfies $|\epsilon_t - \epsilon_{t+1}| \le 1$ for all t. Essentially this means that by waiting one period, one cannot be forced to switch on or off earlier.

We first present a valid formulation P_Z for the set $Z(\alpha,\beta,\gamma,\delta)$, namely a polytope such that $Z(\alpha,\beta,\gamma,\delta) = P_Z \cap (\mathbb{Z}^n \times \mathbb{Z}^n)$. We then show that P_Z is integral by showing that there is a one-to-one unimodular transformation between P_Z and a polytope Q_{UV} that is integral.

Proposition 1 *The polytope* P_Z

$$0 \leq y_t \leq 1 \quad \forall\, t \tag{21}$$
$$0 \leq z_t \leq 1 \quad \forall\, t \tag{22}$$
$$y_t \leq y_{t-1} + z_t \quad \forall\, t \tag{23}$$
$$z_t + \cdots + z_{t+\alpha_t-1} \leq y_{t+\alpha_t-1} \quad \forall\, t \tag{24}$$
$$z_{t+1} + \cdots + z_{t+\beta_t} \geq y_{t+\beta_t} \quad \forall\, t \tag{25}$$
$$y_t + z_{t+1} + \cdots + z_{t+\gamma_t} \leq 1 \quad \forall\, t \tag{26}$$
$$y_t + z_{t+1} + \cdots + z_{t+\delta_t} \geq 1 \quad \forall\, t \leq n - \delta_t \tag{27}$$

is a valid IP formulation for $Z(\alpha, \beta, \gamma, \delta)$.

Note that the constraints (24) imply the inequality (1) $z_t \leq y_t$ and the constraints (26) imply the constraints (3) $z_t \leq 1 - y_{t-1}$ used in defining switch-on variables.

Proof. Consider first the inequalities (24). If a point does not satisfy the definition of α_t, namely $z_t = 1$ and $\sum_{j=t}^{t+\alpha_t-1} w_j \geq 1$. Using $w_t = z_t + y_{t-1} - y_t$, the latter can be rewritten as $\sum_{j=t}^{t+\alpha_t-1} z_j + y_{t-1} - y_{t+\alpha_t-1} \geq 1$. As $y_{t-1} = 0$, the point is cut off. To see that the inequality is valid, observe that by definition $z_t \leq y_{t+\alpha_t-1}$. Also from the condition $\alpha_{t+j} \geq \alpha_t - j$, it follows that $z_{t+j} \leq y_{t+\alpha_t-1}$. Finally as $\sum_{j=t}^{t+\alpha_t-1} z_j \leq 1$, the inequality is valid.

Now consider inequality (25). Again using the equality $w_t = z_t + y_{t-1} - y_t = 0$, $\sum_{j=t+1}^{t+\beta_t} w_j = \sum_{j=t+1}^{t+\beta_t} z_j - y_{t-1} + y_{t+\beta_t}$ and thus the inequality can be rewritten as $y_t \leq \sum_{j=t+1}^{t+\beta_t} w_j$. Clearly a point that does not satisfy the definition of β_t satisfies $z_t = 1$ and $\sum_{j=t+1}^{t+\beta_t} w_j = 0$. As this implies $y_t = 1$, such points are cut off. To see that the inequality is valid, suppose that $y_t = 1$, and the corresponding on-interval starts in $t - j$ for some $j \geq 0$. Thus $w_{t-j} = \cdots = w_t = 0$. Also by definition of β, $z_{t-j} = 1$ implies $\sum_{i=t-j+1}^{t-j+\beta_{t-j}} w_i \geq 1$. However $t - j + \beta_{t-j} \leq t + \beta$, and thus the inequality is again valid.

The cases (26) and (27) are similar. □

We now introduce integer (not binary) variables:
$v_t \in \mathbb{Z}^1_+$ denotes the number of switch-ons in the interval $[1, t]$, i.e., $v_t = \sum_{j=1}^{t} z_j$
$u_t \in \mathbb{Z}^1_+$ denotes the number of switch-offs in the interval $[1, t]$, i.e., $u_t = \sum_{j=1}^{t} w_j$.

Since on and off intervals alternate, v_t and u_t differ by at most one unit. More precisely:

Observation 1 *There is a one-to-one unimodular transformation between the variables* (u, v) *and* (y, z) *given by:*

$$z_t = v_t - v_{t-1} \quad \forall\, t \tag{28}$$
$$y_t = v_t - u_t \quad \forall\, t. \tag{29}$$

In addition one has the link to the switch-off variable given by:

$$w_t = u_t - u_{t-1} = z_t + y_{t-1} - y_t \quad \forall\, t.$$

Proposition 2 *Let* $Q_{UV} \subset \mathbb{R}^n \times \mathbb{R}^n$ *be the polytope:*

$$
\begin{array}{rcll}
0 \;\le\; v_t - u_t & \le & 1 & \forall\, t \quad (30)\\
0 \;\le\; v_t - v_{t-1} & \le & 1 & \forall\, t \quad (31)\\
0 \;\le\; u_t - u_{t-1} & \le & 1 & \forall\, t \quad (32)\\
v_{t-1} - u_{t+\alpha_t-1} & \ge & 0 & \forall\, t \quad (33)\\
v_t - u_{t+\beta_t} & \le & 0 & \forall\, t+\beta_t \le n \quad (34)\\
v_{t+\gamma_t-1} - u_{t-1} & \le & 1 & \forall\, t \quad (35)\\
v_{t+\delta_t} - u_t & \ge & 1 & \forall\, t \le n-\delta_t. \quad (36)
\end{array}
$$

Q_{UV} with the linking equations (28)–(29) defines an extended formulation for conv($Z(\alpha,\beta,\gamma,\delta)$) under the unimodular transformation of Observation 1.

Proof. The constraints (30)–(36) are obtained from (21)–(27) by substitution.

Theorem 3 *P_Z and Q_{UV} are integral polytopes and P_Z = conv($Z(\alpha,\beta,\gamma,\delta)$).*

Proof. To see that the polyhedron Q_{UV} is integral, we observe that each constraint in (30)–(36) has one +1 and one -1 coefficient. Thus the corresponding matrix is the dual of a network matrix and is totally unimodular. As the right hand-side is integer, the extreme point solutions are integer. (28) and (29) are just equations defining z and y respectively. Combined with Proposition 2, the claim follows. □

Note that Malkin [5] proved the integrality of the polytope $P_{Z(\alpha,\gamma)}$, namely the case with lower bounds on the interval lengths. The corresponding formulation was tested computationally by Rajan and Takriti [11].

3.2 Convex Hull in the y-Space

We now consider the question of describing the convex hulls of $Y(\alpha,\beta,\gamma,\delta)$, $Y(\alpha,\gamma)$ and $Y(\beta,\delta)$, i.e., of the projections of the corresponding Z sets in the original y-space. Lee et al. [4] describe conv($Y(\alpha,\gamma)$) when α and γ are constant over time.

To describe the most important family of valid inequalities, we introduce some notation.

Definition 1 *Let $S = \{j_1, \cdots, j_p\}$ with $1 \le j_1 < \cdots < j_p \le T$.*
If $p = |S|$ is odd, $Odd(S,y) \equiv y_{j_1} - y_{j_2} + \cdots - y_{j_{p-1}} + y_{j_p}$ and
if $p = |S|$ is even, $Even(S,y) \equiv y_{j_1} - y_{j_2} + \cdots - y_{j_p}$.
$Length(S) \equiv j_p - j_1$.
An inequality of the form

$$Odd(S,y) \le\ge \mu$$

is called an alternating inequality.

Observation 2 *Let $S = \{j_1, \ldots, j_k\} \subseteq [t,\tau]$. With k even, there exists $T \subseteq [t, \tau-1]$ such that*

$$Even(S,y) = \sum_{j\in T}(y_j - y_{j+1}),$$

and conversely; and with k odd there exists $U \subseteq [t, \tau - 1]$ such that

$$Odd(S, y) = \sum_{j \in U} (y_j - y_{j+1}) + y_\tau,$$

and conversely.

Proof. With k even, it suffices to take $T = [j_1, j_2 - 1] \cup \cdots \cup [j_{k-1}, j_k - 1]$. For the converse, if T consist of intervals with $T = [p_1, q_1] \cup \cdots \cup [p_r, q_r]$, then it suffices to take $S = \{p_1, q_1 + 1, \ldots, p_r, q_r + 1\}$. The case with k odd follows. $\square$.

Proposition 3 *The following inequalities are valid for Y:*
All alternating inequalities $Odd(S, y) \geq 0$ with $S \subseteq [t - 1, t - 1 + \alpha_t]$.
All alternating inequalities $Odd(S, y) \leq 1$ with $S \subseteq [t - 1, t + \gamma_t - 1]$.
The inequalities $\sum_{j=t}^{t+\beta_t} y_j \leq \beta_t$ for all t.
The inequalities $\sum_{j=t}^{t+\delta_t} y_j \geq 1$ for all $t \leq n - \delta_t$.

Proof. The validity of the alternating inequalities is simple. Namely, replacing each z_t variable by either lower bound 0 or $y_t - y_{t-1}$ in (24) and (26) respectively gives the first two sets of inequalities. The validity of the last two inequalities is immediate from the definitions of β_t and δ_t.$\square$

We now examine the polytope described by the alternating inequalities as well as considering the separation problem for these inequalities. Given $y^* \in [0, 1]^T$, define

$$F(t) = \max_{S \subseteq [1,t]} Odd(S, y^*) \quad \text{and} \quad G(t) = \max_{S \subseteq [1,t]} Even(S, y^*).$$

F and G are easy to compute and have interesting properties. To compute them in linear time, one has the recursions:

$$F(t) = \max\{F(t - 1), G(t - 1) + y_t^*\} \quad \text{and} \quad G(t) = \max\{F(t - 1) - y_t^*, G(t - 1)\}$$

with $F(1) = y_1^*$ and $G(1) = 0$.

Lemma 1 *i)* $0 \leq F(t) - F(t - 1) \leq 1 \ \forall\ t$;
ii) $0 \leq G(t) - G(t - 1) \leq 1 \ \forall\ t$;
iii) $0 \leq F(t) - G(t) = y_t^* \leq 1 \ \forall\ t$;
iv) $F(t) - G(\tau) = \max_{S \subseteq [\tau,t]} Odd(S, y^*) \ \forall\ \tau \leq t$;
v) $F(\tau) - G(t) = \min_{S \subseteq [\tau,t]} Odd(S, y^*) \ \forall\ \tau \leq t$.

Proof. The first two range inequalities follow directly from the recursion and $y^* \in [0, 1]^T$. $F(t) - G(t) = y_t^*$ is immediate from the definitions of $F(t)$ and $G(t)$. To establish *iv)*, note that $G(k + 1) - G(k) = \max(\{G(k), F(k) - y_{k+1}^*\} - G(k) = \max\{0, y_k^* - y_{k+1}^*\}$. Therefore for $\tau \leq t$, $F(\tau) - G(t) = \sum_{j=\tau}^{t-1} (G(j + 1) - G(j)) + F(t) - G(t) = \sum_{j=\tau}^{t-1} \max\{0, y_j^* - y_{j+1}^*\} + y_t^* = \max_{S \subseteq [\tau,t]} Odd(S, y^*)$, where the last equation follows from Observation 2. The proof of *v)* is similar. $\square$

This leads to a simple proof of the structure of $conv(Y(\alpha, \gamma))$.

Theorem 4 *[4]. conv($Y(\alpha,\gamma)$) is given by the two families of alternating inequalities in Proposition 3.*

Proof. Let P be the polytope obtained as the intersection of the two families of alternating inequalities with $[0,1]^T$ and $P_Z(\alpha,\gamma)$ the polytope P_Z without constraints (25) and (27) . From Proposition 3 and Theorem 3, $proj_y(P_Z(\alpha,\gamma)) \subseteq P$. To show the converse, suppose that $y^* \in P \subseteq [0,1]^T$. We will show that $y^* \in proj_y(P_Z(\alpha,\gamma))$. Calculate F and G with respect to y^*. Set $v_t = F(t), u_t = G(t)$ for all t. Lemma 1 already shows that u, v satisfy (30)–(32). As $y^* \in P$, it follows that all the alternating inequalities are satisfied and thus $v_{t-1} - u_{t+\alpha_t-1} = F(t-1) - G(t+\alpha_t-1) = \min_{S\subseteq[t-1,t+\alpha_t-1]} Odd(S,y^*) \geq 0$ satisfying (33). Similarly the fact that $\max_{S\subseteq[t-1,t+\gamma_t-1]} Odd(S,y^*) \leq 1$ implies that (35) is satisfied. So $(u,v) \in Q_{UV}$ and the claim follows. □

Corollary 5 *There is a linear time separation algorithm for conv($Y(\alpha,\gamma)$). Given a point y^*, it suffices to calculate F and G, set $v = F, u = G$ and verify if the point (u,v) lies in Q_{UV}.*

Proposition 4 *conv($Y(\beta,\gamma)$) is given by:*

$$\begin{aligned} \sum_{j=t}^{t+\beta_t} y_t &\leq \beta_t \quad \forall\, t \\ \sum_{j=t}^{t+\delta_t} y_t &\geq 1 \quad \forall\, t \\ y &\in [0,1]^n. \end{aligned}$$

Proof. The formulation guarantees that the machine is not on for $\beta_t + 1$ consecutive periods and not off for $\delta_t + 1$ consecutive periods. So it provides a valid formulation for Y. The resulting constraint matrix has the consecutive 1's property. Thus it is totally unimodular. As the right-hand side vector is integer, the claim follows. □

To terminate this section, we provide an example showing that the facets of *conv*(Y) are more complex in the presence of *both* lower (α,γ) and upper bounds (β,δ):

Example 1 *For an instance of $Y(\alpha,\beta)$ with $\alpha = 2$ and $\beta = 3$, one obtains facet-defining inequalities of the form:*

$$\begin{aligned} -y_t + y_{t+1} + y_{t+4} - y_{t+5} &\leq 1 \\ y_t + y_{t+1} + y_{t+3} - y_{t+4} &\leq 2 \\ -y_t + y_{t+1} + y_{t+3} + y_{t+4} &\leq 2 \\ y_t + y_{t+1} + y_{t+3} + y_{t+5} + y_{t+6} &\leq 4 \end{aligned}$$

as well as alternating inequalities and on-interval upper bound (Prop. 3 (iii)) inequalities

$$\begin{aligned} y_t - y_{t+1} + y_{t+2} &\geq 0 \\ y_t + y_{t+1} + y_{t+2} + y_{t+3} &\leq 3. \end{aligned}$$

4. Interval-Dependent Start-Ups

Here we consider the problem in which the start-up costs depend on the number of periods during which the machine has been off. We suppose without loss of generality that $y_0 = 1$. The case in which $y_0 = y_{-1} = \cdots = y_{-\tau+1} = 0$, $y_{-\tau} = 1$ for some $\tau > 0$ can be treated by adding τ periods at the beginning of the n-period horizon and then setting $y_0 = 1$ and $y_1 = \cdots = y_\tau = 0$ in the augmented problem. One obtains the following:

Theorem 6 *$conv(H(\theta))$ is described by the polytope*

$$\sum_p z_t^p \leq y_t \quad \forall\, t \tag{37}$$

$$y_t + \sum_{p:\theta^p=\theta^{p-1}+1, t+\theta^p \leq n} z_{t+\theta^p}^p \leq y_{t-1} + \sum_p z_{t+1}^p \quad \forall\, t \geq 2 \tag{38}$$

$$y_t + \sum_p \sum_{u=t+1}^{\min[n,t+\theta^{p-1}+1]} z_u^p \leq 1 \quad \forall\, t \tag{39}$$

$$y_k + \sum_p \sum_{u=k+1}^{k+\theta^p} z_u^p \geq 1 \quad \forall\, k \leq n-\delta \tag{40}$$

$$y_t + \sum_p \sum_{u=\max[t+1,k+\theta^p+1]}^{\min[n,t+\theta^{p-1}+1]} z_u^p \leq y_k + \sum_p \sum_{u=k+1}^{\min[k+\theta^p,t]} z_u^p \quad \forall\, k,t \;\; \text{with } n-\delta < k \leq t-2,\; t < n \tag{41}$$

$$y \in \mathbb{R}_+^n; \quad z_t^p \in \mathbb{R}_+^1 \quad \forall t \geq \theta^{p-1}+2,\; \forall\, p. \tag{42}$$

Observe that when $P = 1$, constraints (37)–(40), (42) are precisely the constraints of $conv(Z(\gamma,\delta))$. When $k = t-1$, (41) reduces to (38).

Lemma 7 *i) Inequality (41) can be rewritten as*

$$y_t + \sum_p \sum_{u=k+\theta^p+1}^{\min\{n,t+\theta^{p-1}+1\}} z_u^p \leq y_k + \sum_p \sum_{u=k+1}^{t} z_u^p. \tag{43}$$

ii) If $k+\delta \leq n$, $t \geq k+2$ and $t < k+\theta^p \leq t+\theta^{p-1}$ for all $p \in P$, inequality (41) can be rewritten as

$$y_t + \sum_p \sum_{u=t+1}^{\min\{n,t+\theta^{p-1}+1\}} z_u^p \leq y_k + \sum_p \sum_{u=k+1}^{k+\theta^p} z_u^p. \tag{44}$$

It is then the sum of the inequalities (39) and (40).

Proof.
i) For p with $k+\theta^p \geq t$, the z^p terms in (41) and (43) are identical. If $k+\theta^p < t$, it suffices to add the term $\sum_{u=k+\theta^p+1}^{t} z_u^p$ to both sides.
ii) Add $\sum_p \sum_{u=t+1}^{t+\theta^p} z_u^p$ to both sides. □

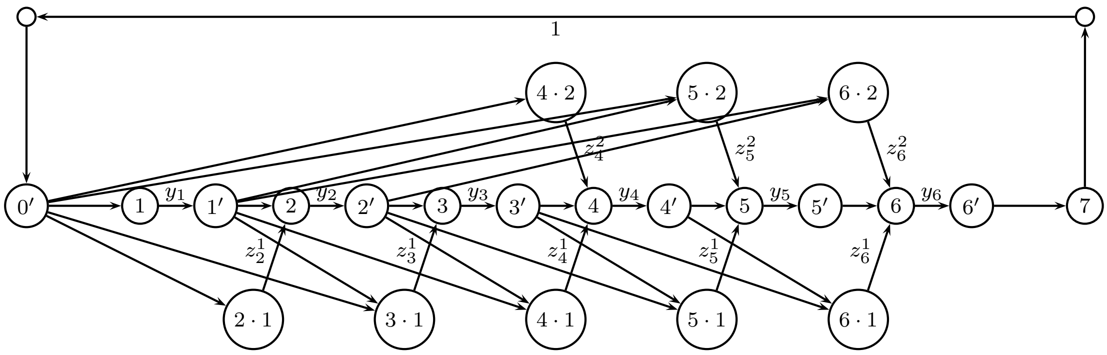

Figure 1: $n = 6, P = 2$, $\theta = (0, 2, 4)$ and $y_0 = 1$. Type-5 arcs $(t', 7)$ with $t = 2 \dots, 5$ not drawn.

To prove Theorem 6 we show how the set $conv(H(\theta))$ can be viewed as the solution set of a network flow (path) model and then use the projection of this network onto the arcs corresponding to the y_t and z_t^p variables to obtain $conv(H(\theta))$.

First we describe the flow model. The digraph $D = (V, A)$ has nodes t for all $t = 1, \dots, n+1$; t' for all $t = 0, \dots, n$; and $t \cdot p$ for all $p = 1, \dots, P$ and $t = 1 \dots, n$. The arcs are of the following types, with specified lower and upper bounds l_a and u_a on the flow on each arc a:

1. Arcs (t, t') for $t = 1, \dots, n$ with flow bounds $l_{t,t'} = u_{t,t'} = y_t$. Flow on this arc indicates whether the machine is on in period t.

2. Arcs $(t', t+1)$ for $t = 1, \dots, n$ with flow bounds $l_{t',t+1} = 0$ and $u_{t',t+1} = \infty$. This arc is used if the machine stays on from period t to $t+1$.

3. Start-up Arcs $(t \cdot p, t)$ with flow bounds $l_{t \cdot p,t} = u_{t \cdot p,t} = z_t^p$ for $t = 1, \dots, n$ and $p = 1, \dots, P$, used if a type-p start-up occurs in period t.

4. Arcs $(t', (t+k) \cdot p)$ with $k \in [\theta^{p-1} + 2, \theta^p + 1]$, for $t = 1, \dots, n$ and $p = 1, \dots, P$, with flow bounds $l_{t',(t+k) \cdot p} = 0$ and $u_{t',(t+k) \cdot p} = \infty$. Such an arc is used if a switch-off in $t+1$ is followed by a type-p start-up in period $t+k$.

5. Arcs $(t', n+1)$ for $t \geq n - \delta$ with flow bounds $l_{t',n+1} = 0, u_{t',n+1} = \infty$, used if there is a switch-off in $t+1$ and the machine then remains off.

6. Return Arc $(n+1, 0')$ with flow bounds $l_{n+1,0'} = u_{n+1,0'} = 1$.

7. Initial Arcs $(0', t \cdot p)$ for $t \in [\theta^{p-1} + 2), \theta^p + 1]$ with flow bounds $l_{0',t \cdot p} = 0$ and $u_{0',t \cdot p} = \infty$. Terminal Arc $(0', n+1)$ with flow bounds $l_{0',n+1} = 0$ and $u_{0',n+1} = \infty$ if $\delta \geq n$.

An instance of the digraph with $n = 6$, $P = 2$, $(\theta^0, \theta^1, \theta^2) = (0, 2, 4)$ (hence $\gamma = 0$ and $\delta = 4$) and $y_0 = 1$ is shown in Figure 1.

Given $X \subset V$, let $\bar{X} = V \setminus X$,

$$l(X, \bar{X}) = \sum_{(i,j) \in A \,:\, i \in X,\ j \in \bar{X}} l_{ij} \quad \text{and} \quad u(\bar{X}, X) = \sum_{(i,j) \in A \,:\, i \in \bar{X},\ j \in X} u_{ij}.$$

Observation 3 *To describe conv$(H(\theta))$, it is necessary and sufficient to consider valid inequalities of the form*

$$(X, \bar{X}) - u(\bar{X}, X) \leq 0 \tag{45}$$

where $u(\bar{X}, X) < +\infty$.

This follows from the Hoffman circulation theorem, see in particular Martens et al. [6]. We shall call inequalities of the above form "cut-inequalities". Note that such inequalities in the variables y_t and z_t^p have coefficients $0, +1$ or -1.

Using this Observation, we now show that the five families of inequalities (37)–(41) are valid.

Proposition 5 *The inequalities (37)-(41) are valid for* $H(\theta)$.

Proof. In each case we specify the set X that is used in Observation 5 to obtain the inequality.
Taking $X = V \setminus \{t\}$ gives the valid inequality (37).
Taking $X = \{(t-1)', t\}$ gives the valid inequality (38).
Taking $X = [1, t] \cup [0', (t-1)'] \cup \bigcup_{p=1}^{P} [1 \cdot p, (t + \theta^{p-1} + 1) \cdot p]$ gives the valid inequality (39).
Taking $X = [t+1, n+1] \cup [t', n'] \cup \bigcup_p [(t + \theta^p + 1) \cdot p, n \cdot p]$ gives the valid inequality (40).
Taking $X = [k+1, t] \cup [k', (t-1)'] \cup \bigcup_p [k + \theta^{p-1} + 1) \cdot p, (t+1) \cdot p]$ gives the valid inequality (41).
□

The proof of Theorem 6 is given in a series of Observations and Propositions below.

Observation 4 *Finiteness of* $u(\bar{X}, X)$ *implies:*
(a) if $t' \in \bar{X}$*, then* $t+1 \in \bar{X}$ *from arc type 2;*
(b) if $t' \in \bar{X}$*, then* $(t+k) \cdot p \in \bar{X}$ *for* $k \in [\theta^{p-1} + 2, \theta^p + 1]$ *from arc type 4;*
(c) if $t' \in \bar{X}$*, then* $n+1 \in \bar{X}$ *if* $t \geq n - \theta^p$ *from arc type 5;*
(d) if $0' \in \bar{X}$*, then* $t \cdot p \in \bar{X}$ *for* $t \in [\max(1, \tau + \theta^{p-1} + 2), \tau + \theta^p + 1]$ *from arc type 7. Also if* $0' \in \bar{X}$ *and* $\delta > n$*, then* $n+1 \in \bar{X}$.

Our approach will be to consider all possible X-*assignments* of the path $PA = \{0', 1, 1', \ldots, n, n', n+1\}$, i.e. assignments of its nodes to X or $\bar{X}$. The following Proposition shows that facet-defining inequalities (45) are uniquely defined by the X-assignment of the path PA.

Proposition 6 *Consider an* X*-assignment of the path* PA*. If any node* $t \cdot p$*, other than those assigned to* $\bar{X}$ *by the cases in Observation 4 (a), is assigned to* $\bar{X}$*, then any resulting inequality is not facet-defining.*

Proof. If $t \cdot p$ is assigned to X, the contribution to the violation $l(X, \bar{X}) - u(\bar{X}, X)$ of the arcs incident to node $t \cdot p$ is 0 if $t \in X$, and z_t^p if $t \in \bar{X}$. If $t \cdot p$ is assigned to $\bar{X}$, the contribution is $-z_t^p$ if $t \in X$, and 0 if $t \in \bar{X}$. Therefore an inequality with $t \cdot p \in \bar{X}$ is the sum of the inequality with $t \cdot p$ moved to X and the inequality $z_t^p \geq 0$. □

Proposition 7 *For any valid inequality (45) other than (37) with an* X*-assignment of the path* PA*, the inequality is not facet-defining if* $(t-1)'$ *and* t *are on opposite sides of the cut, i.e., are assigned differently to* X *and* $\bar{X}$.

Proof. The case in which $(t-1)' \in \bar{X}$ and $t \in X$ is excluded by Observation 4. Thus assume that $(t-1)' \in X$. Here there are two possibilities.
i) $t' \in X$. We consider the contribution to the violation of the arcs incident to node t. If $t \in \bar{X}$, the contribution is $-y_t + \sum_{p:t\cdot p \in X} z_t^p$, while if $t \in X$, the contribution is $-\sum_{p:t\cdot p\in \bar{X}} z_t^p$.
ii) $t' \in \bar{X}$. If $t \in \bar{X}$ the contribution is $\sum_{p:t\cdot p\in X} z_t^p$, while if $t \in X$, the contribution is $y_t - \sum_{p:t\cdot p\in\bar{X}} z_t^p$.
Thus an inequality (45) with $(t-1)' \in X$ and $t \in \bar{X}$ is the sum of the inequality with $(t-1)'$ and $t \in X$ and the inequality $\sum_p z_t^p \leq y_t$. □

It follows that the remaining candidates to provide facet-defining inequalities are determined by the *flip* periods $\sigma(1) < \ldots < \sigma(K)$ (where $1 \leq \sigma(1)$ and $\sigma(K) \leq n$) in which nodes $\sigma(i)$ and $\sigma(i)'$ on the path PA lie on opposite sides of the cut $(X, \bar{X})$.

We now complete the proof of Theorem 5. There are four different possible X-assignments of nodes $0'$ and $n+1$.
Case 1. $0', n+1 \in X$. Then K is even, say $K = 2I$. The flips $\sigma(i)$ on the path PA lead to a partition $S_0, S_1, \cdots, S_I$ of the nodes $X \cap \{1,\ldots,n\}$ with $S_i = [\sigma(2i)+1, \sigma(2i+1)]$ where $\sigma(0) = 0$ and $\sigma(2I+1) = n$. The complement is a partition $\tilde{S}_0, \cdots, \tilde{S}_{I-1}$ of $\bar{X} \cap \{1,\ldots,n\}$ with $\tilde{S}_i = [\sigma(2i+1)+1, \sigma(2i+2)]$.

For each p, Proposition 6 then gives a collection of possibly overlapping sets $(\tilde{T}_0^p, \cdots, \tilde{T}_{I-1}^p)$ such that $\left(\cup_{j=0}^{I-1} \tilde{T}_j^p\right)\cdot p = [1\cdot p, n\cdot p] \cap \bar{X}$ where $\tilde{T}_j^p = [\sigma(2j+1)+\theta^{p-1}+2, \sigma(2j+2)+\theta^p] \cap [1,n]$.

Now the complement is a collection of disjoint sets $(T_0^p, \cdots, T_I^p)$ such that $\left(\cup_{j=0}^{I} T_j^p\right)\cdot p = [1\cdot p, n\cdot p] \cap X$ with $T_j^p = [\sigma(2j)+\theta^p+1, \sigma(2j+1)+\theta^{p-1}+1] \cap [1,n]$. Some of the T_j^p may be empty. The resulting cut inequality is

$$\begin{aligned} &\sum_{i=0}^{I-1}\left(y_{\sigma(2i+1)} + \sum_{p=1}^{P}\sum_{j=0}^{I} z^p(T_j^p \cap \tilde{S}_i)\right) \\ \leq\ &\sum_{i=1}^{I}\left(y_{\sigma(2i)} + \sum_{p=1}^{P}\sum_{j=0}^{I-1} z^p(S_i \cap \tilde{T}_j^p)\right). \end{aligned}$$

Adding

$$\sum_{p=1}^{P} z^p(T_j^p \cap S_i)$$

to each side for all pairs (i,j) except $(0,0)$ and (I,I), and using the following identities,

$$\begin{aligned} \tilde{S}_0 \cup S_1 \cdots \cup \tilde{S}_{I-1} \cup S_I &= [\sigma(1)+1, n]; \\ S_0 \cup \tilde{S}_0 \cup S_1 \cdots \cap \tilde{S}_{I-1} \cup S_I &= [1,n]; \\ T_I^p \cap (S_0 \cup \tilde{S}_0 \cup S_1 \cdots \cap \tilde{S}_{I-1}) &= \emptyset; \\ S_0 \cap (\tilde{T}_0^p \cup T_1^p \cup \cdots \tilde{T}_{I-1}^p \cup T_I^p) &= \emptyset; \\ T_0^p \cup \tilde{T}_0^p \cup T_1^p \cup \cdots \tilde{T}_{I-1}^p \cup T_I^p &= [1,n]; \quad \text{and} \\ T_0^p \cup \tilde{T}_0^p \cup T_1^p \cup \cdots \tilde{T}_{I-1}^p &= [1, \sigma(2I)+\theta^p], \end{aligned}$$

this simplifies to:

$$\sum_{i=0}^{I-1} y_{\sigma(2i+1)} + \sum_{p=0}^{P} z^p(T_0^p \cap [\sigma(1)+1, n]) + \sum_{p=1}^{P}\sum_{j=1}^{I-1} z^p(T_j^p) + 0$$
$$\leq \sum_{i=1}^{I} y_{\sigma(2i)} + 0 + \sum_{p=1}^{P}\sum_{i=1}^{I-1} z^p(S_i) + \sum_{p=1}^{P} z^p(S_I \cap [1, \sigma(2I)+\theta^p]).$$

This is the sum of the inequalities:

$$y_{\sigma(1)} + \sum_p z^p([\sigma(1)+1, \sigma(1)+\theta^{p-1}+1]) \leq 1;$$
$$y_{\sigma(2i+1)} + \sum_p z^p(T_i^p) \leq y_{\sigma(2i)} + \sum_p z^p(S_i) \qquad \text{for } i = 1, \dots I-1; \text{ and}$$
$$-y_{\sigma(2I)} - \sum_p z^p([\sigma(2I)+1, \sigma(2I)+\theta^p]) \leq -1,$$

namely: inequality (39); $I-1$ inequalities (43) (equivalent to (41)); and inequality (40) (since $n+1 \in X$ implies $\sigma(2I)+\theta^P \leq n$).

Thus we have shown that every valid inequality with $0'$ and $n+1 \in X$, other than (37), is a nonnegative combination of the inequalities (38)–(41).

Case 2, $0' \in X, n+1 \in \bar{X}$. Here the flips occur at $\sigma(1), \dots, \sigma(2I+1)$. The sets $S_i, \tilde{S}_i, \tilde{T}_i^p$ and T_i^p are unchanged, and sets $\tilde{S}_I$ and $\tilde{T}_I^p$ are added for all p. The resulting cut inequality is

$$\sum_{i=0}^{I} \big(y_{\sigma(2i+1)} + \sum_p \sum_{j=0}^{I} z^p(T_j^p \cap \tilde{S}_i)\big)$$
$$\leq \sum_{i=1}^{I} \big(y_{\sigma(2i)} + \sum_p \sum_{j=0}^{I} z^p(S_i \cap \tilde{T}_j^p)\big) + 1$$

where the 1 appears because of the arc $(n+1, 0')$ with $n+1 \in \bar{X}$ and $0' \in X$. After adding the terms $\sum_p z^p(S_i \cap T_j^p)$ as before, the resulting inequality is the sum of

$$y_{\sigma(1)} + \sum_p z^p([\sigma(1)+1, \sigma(1)+\theta^{p-1}+1]) \leq 1$$

and the I inequalities

$$y_{\sigma(2i+1)} + \sum_p z^p(T_i^p) \leq y_{\sigma(2i)} + \sum_p z^p(S_i)$$

of the form (43).

Case 3, $0', n+1 \in \bar{X}$ can be treated as Case 2 by omitting sets S_0 and T_0^p.

Case 4, $0 \in \bar{X}$ and $n+1 \in X$ is then similar. □

Example 2 $n = 7,\ P = 3,\ \theta = (0, 2, 4, 5)$.

$$
\begin{array}{rcl}
 & (37) & \\
z_1^1 + z_1^2 + z_1^3 & \leq & y_1 \\
z_2^1 + z_2^2 + z_2^3 & \leq & y_2 \\
 & \vdots & \\
z_7^1 + z_7^2 + z_7^3 & \leq & y_7 \\
 & (38) & \\
y_2 & \leq & y_1 + z_2^1 + z_2^2 + z_2^3 \\
y_3 & \leq & y_2 + z_3^1 + z_3^2 + z_3^3 \\
 & \vdots & \\
y_7 & \leq & y_6 + z_7^1 + z_7^2 + z_7^3 \\
 & (39) & \\
y_1 + z_2^1 + z_{2,4}^2 + z_{2,6}^3 & \leq & 1 \\
y_2 + z_3^1 + z_{3,5}^2 + z_{3,7}^3 & \leq & 1 \\
 & \vdots & \\
y_6 + z_7^1 + z_7^2 + z_7^3 & \leq & 1 \\
 & (40) & \\
y_1 + z_{2,3}^1 + z_{2,5}^2 + z_{2,6}^3 & \geq & 1 \\
y_2 + z_{3,4}^1 + z_{3,6}^2 + z_{3,7}^3 & \geq & 1 \\
 & (41) & \\
y_5 + z_6^1 & \leq & y_3 + z_{4,5}^1 + z_{4,5}^2 + z_{4,5}^3 \\
y_6 + z_7^1 & \leq & y_3 + z_{4,5}^1 + z_{4,6}^2 + z_{4,6}^3 \\
y_6 + z_7^1 & \leq & y_4 + z_{5,6}^1 + z_{5,6}^2 + z_{5,6}^3 \\
y \in \mathbb{R}_+^7, z \in \mathbb{R}_+^{3\times 7}, & & z_1^1 = z_1^2 = z_2^2 = z_3^2 = z_1^3 = z_2^3 = z_3^3 = z_4^3 = z_5^3 = 0.
\end{array}
$$

5. Extensions

The results of Section 3 can be extended in different ways. One possibility is to include lower and/or upper bounds on the number of set-ups in a given interval. Bounds on $\sum_{t=\tau_1}^{\tau_2} z_t = v_{\tau_2} - v_{\tau_1 - 1}$ can be added to the Q_{UV} formulation without losing integrality. For $Y(\alpha, \gamma)$ with an upper bound on the number of setups, the constraint $\sum_{t=\tau_1}^{\tau_2} z_t \leq \Omega_{\tau_1,\tau_2}$ projects into another set of alternating inequalities $\max_{S \subseteq [\tau_1,\tau_2]} Odd(S, y) \leq \Omega_{\tau_1,\tau_2}$. These can be separated in linear time using the functions F and G and now the three sets of alternating inequalities give the convex hull. Another possibility is that the behaviour of the machine depends on the number of start-ups that have occurred. Thus $z_t^q = 1$ if the q^{th} start-up is in period t. Here the network dual formulation can be generalized based on the binary variables v_t^q and u_t^q taking the value 1 if the q^{th} start-up, respectively switch-off, occurs in or before t. Finally we note that one can generate inequalities for interval-dependent switch-offs, using a similar approach to that of Section 4.

References

[1] P. Damcı-Kurt, S. Küçükyavuz, D.Rajan and A. Atamtürk, A Polyhedral Study of Ramping in Unit Commitment, BCOL Research Report 13.02, Industrial Engineering & Operations Research, University of California, Berkeley, CA 94720–1777, October 2013.

[2] K. W. Hedman, R. P. O'Neill, and S. S. Oren, Analyzing Valid Inequalities of the Generation Unit Commitment Problem, Power Systems Conference and Exposition, 2009. PSCE '09. IEEE/PES, 2009, pp. 1–6.

[3] A. Frangioni, C. Gentile, and F. Lacalandra, Tighter approximated MILP formulations for unit commitment problems, IEEE Transactions on Power Systems, 24, 105–113, (2009).

[4] J. Lee, J. Leung and F. Margot, Min-up/min-down polytopes, Discrete Optimization 1, 77–85 (2004).

[5] P. Malkin, Minimum runtime and stoptime polyhedra, Working paper, Core, Université catholique de Louvain, July 2003.

[6] M. Martens, S. T. McCormick and M. Queyranne, Separation, dimension, and facet algorithms for node flow polyhedr,. Math. Program., Ser. B 24, 317–348 (2010).

[7] G. Morales-España, J. M. Latorre and A. Ramos, Tight and Compact MILP Formulation of Start-Up and Shut-Down Ramping in Unit Commitment, IEEE Transactions on Power Systems 28, 1288–1296 (2013).

[8] G. Morales-España, J. M. Latorre and A. Ramos, Tight and Compact MILP Formulation for the Thermal Unit Commitment Problem, IEEE Transactions on Power Systems 28, 4897–4908 (2013).

[9] Y. Pochet and L.A. Wolsey, Production Planning by Mixed Integer Programming, Springer, New York (2006).

[10] M. Queyranne and L.A. Wolsey, Modeling Poset Convex Subsets, Cologne-Twente Workshop, Istanbul, May 2015.

[11] D. Rajan and S. Takriti, Minimum up/down polytopes of the unit commitment problem with start-up costs, IBM Research Report RC23628 (W0506-050), June 2005.

[12] K. Van den Bergh, K. Bruninx, E. Delarue and W. D'haeseleer, A mixed-integer linear formulation of the unit commitment problem, TME Working Paper WP EN2014-07, University of Leuven (2014).

[13] A. Viana and J. P. Pedroso, A new MILP-based approach for unit commitment in power production planning, Electrical Power and Energy Systems 44, 997–1005 (2013).

[14] L.A. Wolsey, Integer Programming, Wiley-Interscience, New York (1998).

[15] L. Wu, A tighter piecewise linear approximation of quadratic cost curves for unit commitment problems, IEEE Transactions on Power Systems 26, 2581–2583 (2011).

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