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Uncapacitated Lot-Sizing with Stock Upper Bounds, Stock Fixed Costs, Stock Overloads and Backlogging: A Tight Formulation

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Uncapacitated Lot-Sizing with Stock Upper Bounds, Stock Fixed Costs, Stock Overloads and Backlogging: A Tight Formulation

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Abstract

For an n-period uncapacitated lot-sizing problem with stock upper bounds, stock fixed costs, stock overload and backlogging, we present a tight extended shortest path formulation of the convex hull of solutions with $O(n^2)$ variables and constraints, also giving an $O(n^2)$ algorithm for the problem. This corrects and extends a formulation in [11] for the problem with just stock upper bounds.

Keywords: lot-sizing, stock upper bounds, stock fixed costs, mixed integer programming, convex hull
Mathematics Subject Classification: 90C11, 90C27, 90B05, 90B30.

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1 Introduction

It was shown in Wolsey [11] that single item lot-sizing with upper bounds on the stock levels is equivalent to a lot-sizing problem with production time windows. Each order k consists of an order size D^k and a time window $[b^k, e^k]$ indicating that normal production should take place within the interval $[b^k, e^k]$ and delivery of the order should be in period e^k. In addition the orders should have non-inclusive time windows, i.e. after possible reordering $b^k \leq b^{k+1}$ and $e^k \leq e^{k+1}$ for all k. For an n-period uncapacitated lot-sizing problem this led to a $O(n^2)$ dynamic program. Taking its dual provided a shortest path extended formulation that contained errors in the indexing of the constraints and the explanation of the variables. Here we derive this (corrected) shortest path formulation directly, and show how it can be used to treat not just stock upper bounds, but also i) stock fixed costs, ii) stock overloads and iii) backlogging. It also provides an $O(n^2)$ algorithm for the problem.

Lot-sizing with stock upper bounds was first treated by Love [8]. Van Vyve and Ortega [10] describe the convex hull of solutions for lot-sizing with stock fixed costs. For lot-sizing with both stock upper bounds and stock fixed costs Atamtürk and Küçükyavuz [1] present several valid inequalities as well as computational experience, and in [2] they present an $O(n^2)$ algorithm. Fast algorithms for stock upper bounds and backlogging are presented in Hwang and van den Heuvel [7]. Lot-sizing with production time windows was examined by Brahimi [3], treating both inclusive and non-inclusive cases. Fast algorithms for the non-inclusive case are presented in Hwang [6]. [4, 5] contain computational studies. The equivalence of lot-sizing with stock upper bounds and lot-sizing with non-inclusive time windows was shown in Wolsey [11]. Further equivalences are examined in [9].

Below in Section 2 we first present a formulation of the problem to be solved, denoted LS-U-SUB*, and then we recall some of the basic results linking lot-sizing with stock upper bounds and lot-sizing with non-inclusive time windows. In Section 3 we present the shortest path reformulation and show that it provides a tight formulation for LS-U-SUB*.

2 The Problem and Background

2.1 An MIP Formulation

First we present the lot-sizing problem LS-U-SUB*. One has demands $d_t \geq 0$ in period t and an upper bound $\bar{S}_t \geq 0$ on the stock at the end of period t. There are unit production costs p_t, production set-up costs q_t, unit storage costs h_t, unit backlog costs c_t, unit storage overload costs o_t and a storage fixed cost g_t in each period t. The problem is to satisfy all the demands at minimum cost.

Using the notation $d_{uv} \equiv \sum_{t=u}^{v} d_t$, we now formulate LS-U-SUB* as a mixed integer program. We introduce the variables:
x_t is the production in period t
y_t is a binary set-up variable taking value 1 if $x_t > 0$
s_t is the stock at the end of period t
σ_t is a binary stock set-up variable taking value 1 if $s_t > 0$
r_t is the backlog in period t
δ_t is the stock overload at the end of period t.
The problem can now be written as follows:

$$\begin{aligned}
\min \sum_{t=1}^{n} (p_t x_t + h_t s_t &+ c_t r_t + g_t \delta_t + f_t y_t + q_t \sigma_t) \\
s_{t-1} - r_{t-1} + x_t &= d_t + s_t - r_t \;\forall\, t && (1)\\
s_t &\leq \bar{S}_t + \delta_t \;\forall\, t && (2)\\
x, \delta &\in \mathbb{R}^n_+, s, r \in \mathbb{R}^{n+1}_+, && (3)\\
s_0 = r_0 &= s_n = r_n = 0 && (4)\\
x_t &\leq M y_t \;\forall\, t && (5)\\
s_t &\leq M \sigma_t \;\forall\, t && (6)\\
y, \sigma &\in \{0,1\}^n && (7)
\end{aligned}$$

where M is a large number with $M \geq d_{1n}$. Here (1) are the balance constraints, (2) ensures the stock upper bound and the overload, (3) and (4) fix the ranges of the continuous variables, (5) ensures that $y_t = 1$ if $x_t > 0$, (6) ensures that $\sigma_t = 1$ if $s_t > 0$ and (7) indicates that y_t and σ_t are binary variables.

In the lot-sizing with non-inclusive time windows problem, denoted LS-U-TWP*, the stocks, stock overload and backlog are similar. If order k is produced in period t with $1 \leq t < e^k$, it produces a stock of D^k in periods $t, \ldots, e^k - 1$, if k is produced in period t with $1 \leq t < b^k$, it produces in addition a stock overload of D^k in periods $t, \dots, b^k - 1$, while if $t > e^k$ it produces a backlog in periods $e^k, \dots, t-1$. The costs are the same.

2.2 Basic Results

The following results can be found in Brahimi [3] for lot-sizing with production time windows.
i) With non-inclusive time windows, there exists an optimal solution in which order k is produced before (or at the same time) as order $k+1$.
ii) In the uncapacitated case, there exists an optimal solution in which each order k is produced in a single period.
Using a standard exchange argument for the linear programming flow problem remaining once the 0-1 variables y, σ are fixed, it is easy to see that this property still holds in the presence of stock fixed costs, backlogging and storage overload.

The equivalence of the two problems is best seen by observing that, after elimination of the stock variables, the constraints linking the continuous variables in both problems can be rewritten as:

$$\begin{aligned} r_t + \sum_{\tau=1}^{t} x_\tau &\geq \Delta_{1t} \ \forall\ t \\ \sum_{\tau=1}^{t} x_\tau &\leq \Gamma_{1t} + \delta_t \ \forall\ t \\ x, r, \delta \in \mathbb{R}^n_+ & \end{aligned}$$

where (1)-(4) of LS-U-SUB* take this form with $\Delta_{1t} = d_{1t}$ and $\Gamma_{1t} = d_{1t} + \bar{S}_t$. On the other hand, for LS-U-TWP* $\Delta_{1t} = \sum_{k:e^k \leq t} D^k$ and $\Gamma_{1t} = \sum_{k:b^k \leq t} D^k$.

This allows us to reduce the lot-sizing with time-window problem to lot-sizing with stock upper bounds and vice versa.

Given a set of distinct non-inclusive orders $D^k, (b^k, e^k)$ for $k = 1, \dots, K$:
Set $\Delta_{1t} = \sum_{k:e^k \leq t} D^k$ and $\Gamma_{1t} = \sum_{k:b^k \leq t} D^k$.
Set $d_t = \Delta_{1t} - \Delta_{1,t-1}$ and $\bar{S}_t = \Gamma_{1t} - \Delta_{1t}$.

Conversely, given d_t and $\bar{S}_t$ for $t = 1, \dots, n$:
Set $\Delta_{1t} = d_{1t}$ and $\Gamma_{1t} = d_{1t} + \bar{S}_t$.

Set $L_t = \Gamma_{1t} - \Gamma_{1,t-1}$, $R_t = \Delta_{1t} - \Delta_{1,t-1} = d_t$ for all t and $k = 1$.
While $L, R \neq 0$
 Set $\sigma = \min\{t : L_t > 0\}, \tau = \min\{t : R_t > 0\}$.
 Set $D^k = \min\{L_\sigma, R_\tau\}$, $b^k = \sigma, e^k = \tau$.
 $L_\sigma \leftarrow L_\sigma - D^k, R_\tau \leftarrow R_\tau - D^k$
 $k \leftarrow k + 1$
end-While

3 The Shortest Path Extended Formulation

Here we use the fact that the production of the orders is determined by the time windows and that the production of an order takes place in a single period to derive a shortest path (unit flow) formulation of LS-U-TWP*.

Consider a digraph $D = (V, A)$ with nodes $(k, t, 1)$ and $(k, t, 2)$ for $1 \leq t \leq n$ and all k. There are four types of arcs:
z-arcs with flow z_{kt} from $(k-1, t-1, 2)$ to $(k, t, 1)$ for $1 \leq t \leq n$ and all k.
w-arcs with flow w_{kt} from $(k, t, 1)$ to $(k, t, 2)$ for $1 \leq t \leq n-1$ and all k, and for $k = K, t = n$.
u-arcs with flow u_{kt} from $(k-1, t, 1)$ to $(k, t, 1)$ for $1 \leq t \leq n$ and all $k > 1$.
v-arcs with flow v_{kt} from $(k, t-1, 2)$ to $(k, t, 2)$ for $1 \leq t \leq n-1$ and all k, and for $k = K, t = n$.
$z_{kt} = 1$ if and only if k is the first order produced in period t.
$w_{kt} = 1$ if and only if k is the last order produced in period t.
$u_{kt} = 1$ if and only if k is produced in period t, but it is not the first order produced in t.
$v_{kt} = 1$ if and only if k is produced between 1 and $t-1$, and that $k+1$ is produced in $t+1$ or later.

We now present the formulation:

$$\min \sum_{t=1}^{n} (p_t x_t + h_t s_t + c_t r_t + g_t \delta_t + f_t y_t + q_t \sigma_t)$$

$$\sum_{t=1}^{n} z_{1t} = 1 \tag{8}$$

$$z_{kt} + u_{kt} = w_{kt} + u_{k+1,t} \ \forall\ k, t \tag{9}$$

$$w_{kt} + v_{kt} = z_{k+1,t+1} + v_{k,t+1} \ \forall\ k, t,\ k < K \text{ or } t < n \tag{10}$$

$$z, w, u, v \geq 0, \tag{11}$$

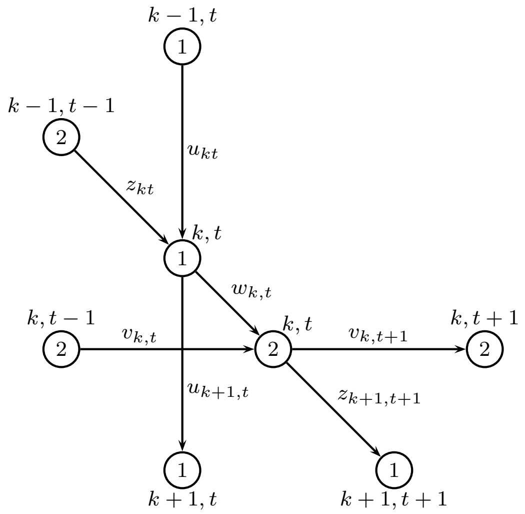

Figure 1: Arcs entering and leaving nodes $(k, t, 1)$ and $(k, t, 2)$

$$x_t = \sum_k D^k (z_{kt} + u_{kt}) \ \forall \ t \tag{12}$$

$$s_t = \sum_{k:t<e^k} \sum_{\tau=1}^{t} D^k (z_{k\tau} + u_{k\tau}) \ \forall \ t \tag{13}$$

$$r_t = \sum_{k:t\geq e^k} \sum_{\tau=t+1}^{n} D^k (z_{k\tau} + u_{k\tau}) \ \forall \ t \tag{14}$$

$$\delta_t = \sum_{k:t<b^k} \sum_{\tau=1}^{t} D^k (z_{k\tau} + u_{k\tau}) \ \forall \ t \tag{15}$$

$$1 \geq y_t \geq \sum_k z_{kt} \ \forall \ t \tag{16}$$

$$1 \geq \sigma_t \geq \sum_{k:t<e^k} (v_{kt} + w_{kt}) \ \forall \ t \tag{17}$$

$$y, \sigma \in \mathbb{Z}^n \tag{18}$$

Constraints (8)-(11) describe the shortest path/unit flow problem in the z, w, u, v variables. (8) indicates that one unit of flow enters, namely that order 1 must be the first order produced in one of the periods $1, \ldots, n$.
(9) is the flow conservation constraint at node $(k, t, 1)$. There is a flow of one unit through the node if and only if order k is produced in period t.
(10) is the flow conservation constraint at node $(k, t, 2)$. There is a flow of

one unit through this node if and only if order k is produced in the interval $[1, t]$ and order $k + 1$ in the interval $[t + 1, n]$. See Figure 1.

Constraints (12)-(17) provide the link between the flow variables and the original lot-sizing variables. (12)-(15) calculate the value of the corresponding production, stock and backlog and stock overload variables respectively. (16) forces $y_t = 1$ if there is a first order produced in t. Finally we prove below that $\sum_{k:t<e^k}(v_{kt} + w_{kt})$ takes value 1 if and only if there is positive stock at the end of period t, thereby forcing $\sigma_t = 1$.

Lemma 1 *In any feasible solution of (8)-(18),* $\sum_{k:t<e^k}(v_{kt} + w_{kt}) = 1$ *if* $s_t > 0$ *and takes value 0 otherwise.*

Proof Consider period t. Either all orders are produced before t, or all after t, or there exists an order κ such that κ is produced in or before t, and order $\kappa + 1$ is produced after t.
i) If all orders are produced before , or all after t, $v_{kt} = w_{kt} = 0$ for all k.
ii) Order κ is produced before period t. Thus there is no production in t and $w_{kt} = 0$ for all k. Also $v_{\kappa t} = 1$ and $v_{kt} = 0$ for all $k \neq \kappa$. If $t \geq e^\kappa$, there is no stock in t as $e^k \leq e^\kappa$ for all $k < \kappa$. On the other hand $\sum_{k:t<e^k}(v_{kt}+w_{kt}) = 0$ as required. If $t < e^\kappa$, there is stock of order κ in t and $\sum_{k:t<e^k}(v_{kt}+w_{kt}) = 1$ as required.
iii) Order κ is produced in period t. Here $w_{\kappa t} = 1$, $w_{kt} = 0$ for $k \neq \kappa$ and $v_{kt} = 0$ for all k. Again if $t \geq e^\kappa$, there is no stock in t and $\sum_{k:t<e^k}(v_{kt} + w_{kt}) = 0$, while if $t < e^\kappa$, there is stock of order κ in t and $\sum_{k:t<e^k}(v_{kt} + w_{kt}) = 1$. □

Let Q denote the polyhedron obtained by taking (8)-(17).

Theorem 2 *i)* $proj_{x,s,r,\delta,y,\sigma} Q = \text{conv}(X^{LS-U-SUB^*})$
ii) The linear program

$$\min\{px + hs + cr + g\delta + fy + q\sigma : (x, s, r, \delta, y, \sigma, z, u, v, w) \in Q\}$$

solves both problems $LS - U - SUB^*$ *and* $LS - U - TWP^*$.

Proof. The unit flow polyhedron (8)-(11) is totally unimodular and hence its extreme points are binary. (12)-(15) are equations representing the variables s, r, δ in terms of the the unit flow variables and thus do not affect integrality. If $q_t \leq 0$ for some t, then there exists an optimal solution with $y_t = 1$. Otherwise if $q_t \geq 0$, (16) holds as an equality in every optimal solution. The same argument holds for σ_t. Therefore for every objective function there exists an optimal solution with y and σ binary.

□

Note that if no stock overload is allowed, the path variables z_{kt}, u_{kt}, v_{kt} and w_{kt} can be set to zero for all $t < b^k$, while if backlogging is not allowed they can be set to zero for $t > e_k$.

Finally as the number K of non-inclusive time windows is at most $2n-1$ and as there are $4nK$ variables (arcs) in the shortest path formulation, one has:

Theorem 3 *There is an $O(n^2)$ algorithm to solve $LS-U-SUB^*$.*

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