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APPLICATIONS OF BOURGAIN-BREZIS INEQUALITIES TO FLUID MECHANICS AND MAGNETISM

SAGUN CHANILLO, JEAN VAN SCHAFTINGEN, AND PO-LAM YUNG

Abstract. As a consequence of inequalities due to Bourgain-Brezis, we obtain local in time well-posedness for the two dimensional Navier–Stokes equation with velocity bounded in spacetime and initial vorticity in bounded variation. We also obtain spacetime estimates for the magnetic field vector through improved Strichartz inequalities.

1. Incompressible Navier–Stokes flow

Let $\mathbf{v}(x,t) \in \mathbb{R}^2$ be the velocity and $p(x,t)$ be the pressure of a fluid of viscosity $\nu > 0$ at position $x \in \mathbb{R}^2$ and time $t \in \mathbb{R}$, governed by the incompressible two-dimensional Navier–Stokes equation:

$$\begin{cases} \mathbf{v}_t + (\mathbf{v}\cdot\nabla)\mathbf{v} = \nu\Delta\mathbf{v} - \nabla p, \\ \nabla\cdot\mathbf{v} = 0, \end{cases} \tag{1}$$

When the viscosity coefficient ν degenerates to zero, (1) becomes the Euler equation. In two spatial dimensions, the vorticity of the flow is a scalar, defined by

$$\omega = \partial_{x_1} v_2 - \partial_{x_2} v_1$$

where we wrote $\mathbf{v} = (v_1, v_2)$. In the sequel, when we consider the Navier-Stokes equation, without loss of generality we set the viscosity coefficient $\nu = 1$.

The vorticity associated to the incompressible Navier-Stokes flow in two dimensions propagates according to the equation

$$\omega_t - \Delta\omega = -\nabla\cdot(\mathbf{v}\omega). \tag{2}$$

This follows from (1) by taking the curl of both sides. We express the velocity $\mathbf{v}$ in the Navier-Stokes equation in terms of the vorticity through the Biot-Savart relation

$$\mathbf{v} = (-\Delta)^{-1}(\partial_{x_2}\omega, -\partial_{x_1}\omega). \tag{3}$$

This follows formally by differentiating $\omega = \partial_{x_1} v_2 - \partial_{x_2} v_1$, and using that $\nabla\cdot\mathbf{v} = 0$.

Our theorem states:

Theorem 1. *Consider the two-dimensional vorticity equation* (2) *and an initial vorticity* $\omega_0 \in W^{1,1}(\mathbb{R}^2)$ *at time* $t = 0$. *If*

$$\|\omega_0\|_{W^{1,1}(\mathbb{R}^2)} \le A_0,$$

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then there exists a unique solution to the vorticity equation (2) *for all time* $t \le t_0 = C/A_0^2$, *such that*

$$\sup_{t\le t_0} \|\omega(\cdot,t)\|_{W^{1,1}(\mathbb{R}^2)} \le cA_0.$$

Moreover, the solution ω *depends continuously on the initial data* ω_0, *in the sense that if* $\omega_0^{(i)}$ *is a sequence of initial data converging in* $W^{1,1}(\mathbb{R}^2)$ *to* ω_0, *then the corresponding solutions* $\omega^{(i)}$ *to the vorticity equation* (2) *satisfies*

$$\sup_{t\le t_0} \|\omega^{(i)}(\cdot,t) - \omega(\cdot,t)\|_{W^{1,1}(\mathbb{R}^2)} \to 0$$

as $i \to \infty$.

Finally, the velocity vector $\mathbf{v}$ *defined by the Biot-Savart relation (3) solves the 2-dimensional incompressible Navier-Stokes (1), and satisfies*

$$\sup_{t\le t_0} \|\mathbf{v}(\cdot,t)\|_{L^\infty(\mathbb{R}^2)} + \sup_{t\le t_0} \|\nabla\mathbf{v}(\cdot,t)\|_{L^2(\mathbb{R}^2)} \le cA_0.$$

Via the Gagliardo-Nirenberg inequality we note that we can conclude from our theorem that

$$\sup_{0\le t\le t_0} \|\omega(\cdot,t)\|_{L^p(\mathbb{R}^2)} \le C, \quad 1 \le p \le 2.$$

In particular this is enough to apply Theorem II of Kato [8] to express the velocity vector in the Navier-Stokes equation (1) in terms of the vorticity via the Biot-Savart relation displayed above.

In [7, 8], it was proved that under the hypothesis that the initial vorticity is a measure, there is a global solution that is well-posed to the vorticity and Navier–Stokes equation; see also an alternative approach in Ben-Artzi [1], and a stronger uniqueness result in Brezis [4]. The velocity constructed then satisfies the estimate [8, (0.5)]

$$\|\mathbf{v}(\cdot,t)\|_{L^\infty(\mathbb{R}^2)} \le Ct^{-\frac{1}{2}},\ t \to 0. \tag{4}$$

In contrast, in Theorem 1 we have $\mathbf{v} \in L_t^\infty L_x^\infty$, $x \in \mathbb{R}^2$, though we are assuming that the initial vorticity has bounded variation, that is, its gradient is a measure.

The estimate (4) is indeed sharp as can be seen by the famous example of the *Lamb–Oseen vortex* [9], which consists of an initial vorticity $\omega_0 = \alpha_0\delta_0$, a Dirac mass at the origin of $\mathbb{R}^2$ with strength α_0. The constant α_0 is called the total circulation of the vortex. A unique solution to the vorticity equation (2) can be obtained by setting

$$\omega(x,t) = \frac{\alpha_0}{4\pi t} e^{-\frac{|x|^2}{4t}}, \qquad \mathbf{v}(x,t) = \frac{\alpha_0}{2\pi}\frac{(-x_2,x_1)}{|x|^2}\Big(1 - e^{-\frac{|x|^2}{4t}}\Big).$$

It can be seen from the identities above that,

$$\|\omega(\cdot,t)\|_{W^{1,1}(\mathbb{R}^2)} \sim \|\mathbf{v}(\cdot,t)\|_{L^\infty(\mathbb{R}^2)} \sim ct^{-\frac{1}{2}},\ t \to 0.$$

Hence the assumption that the initial vorticity is a measure cannot yield an estimate like in Theorem 1. Thus to get uniform in time, L^∞ space bounds all the way to $t = 0$ we need a stronger hypothesis and one such is vorticity in BV (bounded variation).

It is also helpful to further compare our result with that of Kato [8] who establishes in (0.4) of his paper, that given the initial vorticity is a measure, one has for the vorticity at further time,

$$||\nabla\omega(\cdot,t)||_{L^q(\mathbb{R}^2)} \le ct^{\frac{1}{q}-\frac{3}{2}}, \quad 1 < q \le \infty.$$

In contrast we obtain uniform in time bounds for $q = 1$, as opposed to singular bounds for $q > 1$ when $t \to 0$.

It is an open question whether there is a global version of Theorem 1 of our paper.

In order to prove Theorem 1, we rely on a basic proposition that follows from the work of Bourgain and Brezis [2,3]. A part of this proposition also holds in three dimension. Recall that if $\mathbf{v}(x,t) \in \mathbb{R}^3$ be the velocity of a fluid at a point $x \in \mathbb{R}^3$ at time t, then the vorticity of $\mathbf{v}$ is defined by

$$\boldsymbol{\omega} = \nabla \times \mathbf{v}.$$

Under the assumption that the flow is incompressible, the Biot-Savart relation reads

$$\mathbf{v} = (-\Delta)^{-1}(\nabla \times \boldsymbol{\omega}). \tag{5}$$

Proposition 2. *(a) Consider the velocity $\mathbf{v}$ in 3 spatial dimensions. Assume that $\mathbf{v}$ satisfies the Biot-Savart relation* (5)*. Then at any fixed time t,*

$$\|\mathbf{v}(\cdot,t)\|_{L^3(\mathbb{R}^3)} + \|\nabla \mathbf{v}(\cdot,t)\|_{L^{3/2}(\mathbb{R}^3)} \leq C\|\nabla \times \boldsymbol{\omega}(\cdot,t)\|_{L^1(\mathbb{R}^3)}$$

where C is a constant independent of t, $\mathbf{v}$ and $\boldsymbol{\omega}$.

(b) Consider the velocity $\mathbf{v}$ in 2 spatial dimensions. Assume that $\mathbf{v}$ satisfies the Biot-Savart relation (3). Then at any fixed time t,

$$\|\mathbf{v}(\cdot,t)\|_{L^\infty(\mathbb{R}^2)} + \|\nabla \mathbf{v}(\cdot,t)\|_{L^2(\mathbb{R}^2)} \leq C\|\nabla \omega(\cdot,t)\|_{L^1(\mathbb{R}^2)}.$$

where C is a constant independent of t, $\mathbf{v}$ and ω.

We remark that in 2 dimensions, by the Poincaré inequality, it follows from $\|\nabla \mathbf{v}\|_{L^2(\mathbb{R}^2)} < \infty$, that $\mathbf{v}$ lies in $VMO(\mathbb{R}^2)$, i.e. has vanishing mean oscillation.

Proof of Proposition 2. Note that

$$\nabla \cdot (\nabla \times \boldsymbol{\omega}) = 0.$$

Thus we can immediately apply the result of Bourgain-Brezis [3] (see also [2,5,10]), to the Biot-Savart formula (5) and get the desired conclusions in part (a).

To consider the 2-dimensional flow, note that $(-\partial_{x_2}\omega, \partial_{x_1}\omega)$ is a vector field in $\mathbb{R}^2$ with vanishing divergence. In view of the two-dimensional Biot-Savart relation (3), we can then use the two-dimensional Bourgain–Brezis result [3], and we obtain (b). $\square$

We note further that the proposition applies to both the Euler (inviscid) or the Navier–Stokes (viscous) flow.

Proof of Theorem 1. Now set K_t for the heat kernel in 2-dimensions, i.e.

$$K_t(x) = \frac{1}{4\pi t} e^{-\frac{|x|^2}{4t}}.$$

Rewriting (2) as an integral equation for ω using Duhamel's theorem, where ω_0 is the initial vorticity, we have,

$$\omega(x,t) = K_t \star \omega_0(x) + \int_0^t \partial_x K_{t-s} \star [\mathbf{v}\omega(x,s)] ds \tag{6}$$

where $\mathbf{v}$ is given by (3).

We apply a Banach fixed point argument to the operator T given by

$$T\omega(x,t) = K_t \star \omega_0(x) + \int_0^t \partial_x K_{t-s} \star [\mathbf{v}\omega(x,s)]ds, \tag{7}$$

where again $\mathbf{v}$ is given by (3). Let us set

$$E = \Big\{ g \mid \sup_{0<t<t_0} \|g(\cdot,t)\|_{W^{1,1}(\mathbb{R}^2)} \le A \Big\}.$$

We will first show that T maps E into itself, for t_0 chosen as in the theorem.

Differentiating (7) in the space variable once, we get

$$(T\omega(x,t))_x = K_t \star f_0(x) + \int_0^t \partial_x K_{t-s} \star \Big(\mathbf{v}_x\omega\Big)ds + \int_0^t \partial_x K_{t-s} \star \Big(\mathbf{v}\omega_x\Big)ds.$$

Here we denote by f_0 the spatial derivative of the initial vorticity ω_0. Using Young's convolution inequality, we have

$$\|(T\omega(\cdot,t))_x\|_{L^1(\mathbb{R}^2)} \le \|f_0\|_{L^1(\mathbb{R}^2)} + C\int_0^t (t-s)^{-1/2}(\|\mathbf{v}_x\omega\|_{L^1(\mathbb{R}^2)} + \|\mathbf{v}\omega_x\|_{L^1(\mathbb{R}^2)})ds.$$

Now we apply Proposition 2(b) to each of the terms on the right. For the first term we have, by Cauchy-Schwartz,

$$\|\mathbf{v}_x\omega\|_{L^1(\mathbb{R}^2)} \le C\|\nabla\mathbf{v}\|_{L^2(\mathbb{R}^2)}\|\omega\|_{L^2(\mathbb{R}^2)}$$

The Gagliardo-Nirenberg inequality applies as $\omega \in E$ and so $\omega(\cdot,t) \in L^1(\mathbb{R}^2)$ and so,

$$\|\omega\|_{L^2(\mathbb{R}^2)} \le C\|\nabla\omega\|_{L^1(\mathbb{R}^2)},$$

and to $\|\nabla\mathbf{v}\|_{L^2(\mathbb{R}^2)}$ we apply Proposition 2(b). Similarly

$$\|\mathbf{v}\omega_x\|_{L^1(\mathbb{R}^2)} \le \|\mathbf{v}\|_{L^\infty(\mathbb{R}^2)}\|\omega_x\|_{L^1(\mathbb{R}^2)}.$$

Again we apply Proposition 2(b) to $\|\mathbf{v}\|_{L^\infty(\mathbb{R}^2)}$. Hence in all we have,

$$\|(T\omega)_x\|_{L^1(\mathbb{R}^2)} \le \|f_0\|_{L^1(\mathbb{R}^2)} + C\int_0^t (t-s)^{-1/2}\|\nabla\omega\|^2_{L^1(\mathbb{R}^2)}ds.$$

Thus setting $\|f_0\|_{L^1(\mathbb{R}^2)} = \|\omega_0\|_{\dot{W}^{1,1}(\mathbb{R}^2)} \le A_0$, we get for $t \le t_0$ and since $\omega \in E$,

$$\|\nabla(T\omega)(\cdot,t)\|_{L^1(\mathbb{R}^2)} \le A_0 + Ct_0^{1/2}A^2.$$

Next from Young's convolution inequality it follows from (7) that,

$$\|T\omega(\cdot,t)\|_{L^1(\mathbb{R}^2)} \le A_0 + \int_0^t (t-s)^{-1/2}\|\mathbf{v}\omega(\cdot,s)\|_1 ds$$

But by Proposition 2(b) again,

$$||\mathbf{v}\omega||_1 \le ||\mathbf{v}||_\infty ||\omega||_1 \le cA^2.$$

Thus,

$$||T\omega(\cdot,t)||_1 \le A_0 + ct^{1/2}A^2.$$

So adding the estimates for $T\omega$ and $\nabla(T\omega)$ we have,

$$\sup_{t\le t_0} ||T\omega(\cdot,t)||_{W^{1,1}(\mathbb{R}^2)} \le 2A_0 + ct_0^{1/2}A^2.$$

By choosing A so that $A_0 = A/8$ and $t < t_0 = C/A_0^2$ we can assure that if $\omega \in E$, then

$$\sup_{t \le t_0} \|(T\omega)(\cdot, t)\|_{W^{1,1}(\mathbb{R}^2)} \le \frac{A}{2}.$$

Thus $T\omega \in E$, if $\omega \in E$. If we establish that T is a contraction then we are done.

Next we observe that the estimates in Proposition 2(b) are linear estimates. That is

$$\|\mathbf{v}_1 - \mathbf{v}_2\|_\infty + \|\nabla \mathbf{v}_1 - \nabla \mathbf{v}_2\|_2 \le C\|\omega_1 - \omega_2\|_{W^{1,1}(\mathbb{R}^2)}.$$

We easily can see from the computations above, that we have

$$\sup_{t \le t_0} \|T\omega_1 - T\omega_2\|_{W^{1,1}(\mathbb{R}^2)} \le CAt_0^{1/2} \sup_{t \le t_0} \|\omega_1 - \omega_2\|_{W^{1,1}(\mathbb{R}^2)}.$$

By the choice of t_0, it is seen that T is a contraction. Thus using the Banach fixed point theorem on E, we obtain our operator T has a fixed point and so the integral equation (6) has a solution in E. The remaining part of our theorem follows easily from Proposition 2(b). $\square$

We note in passing an estimate in $\mathbb{R}^3$ from Proposition 2(a) above for the Navier–Stokes or the Euler flow:

$$\sup_{t>0} \|\mathbf{v}\|_{L^3(\mathbb{R}^3)} + \sup_{t>0} \|\nabla \mathbf{v}\|_{L^{3/2}(\mathbb{R}^3)} \le C \sup_{t>0} \|\nabla \times \boldsymbol{\omega}\|_{L^1(\mathbb{R}^3)}. \tag{8}$$

2. Magnetism

We next turn to our results on magnetism. We denote by $\mathbf{B}(x,t)$ and $\mathbf{E}(x,t)$ the magnetic and electric field vectors at $(x,t) \in \mathbb{R}^3 \times \mathbb{R}$. Let $\mathbf{j}(x,t)$ denote the current density vector. The Maxwell equations imply

$$\nabla \cdot \mathbf{B} = 0, \tag{9}$$

$$\partial_t \mathbf{B} + \nabla \times \mathbf{E} = 0, \tag{10}$$

$$\partial_t \mathbf{E} - \nabla \times \mathbf{B} = -\mathbf{j}. \tag{11}$$

Differentiating (10) in t and using (11), together with the vector identity $\nabla \times (\nabla \times \mathbf{B}) = \nabla(\nabla \cdot \mathbf{B}) - \Delta \mathbf{B}$ and (9), one obtains an inhomogeneous wave equation for $\mathbf{B}$:

$$\mathbf{B}_{tt} - \Delta \mathbf{B} = \nabla \times \mathbf{j}. \tag{12}$$

The right side of (12) satisfies the vanishing divergence condition

$$\nabla \cdot (\nabla \times \mathbf{j}) = 0$$

for any fixed time t. Thus an improved Strichartz estimate, namely Theorem 1 in [6] applies. We point out that the Bourgain-Brezis inequalities play a key role in the proof of Theorem 1 in [6]. We conclude easily:

Theorem 3. *Let $\mathbf{B}$ satisfy (12) and let $\mathbf{B}(x,0) = \mathbf{B}_0$, $\partial_t \mathbf{B}(x,0) = \mathbf{B_1}$ denote the initial data at time $t = 0$. Let $s, k \in \mathbb{R}$. Assume $2 \le q \le \infty$, $2 < \tilde{q} \le \infty$ and $2 \le r < \infty$. Let (q,r) satisfy the wave compatibility condition*

$$\frac{1}{q} + \frac{1}{r} \le \frac{1}{2},$$

and the following scale invariance condition is verified:

$$\frac{1}{q} + \frac{3}{r} = \frac{3}{2} - s = \frac{1}{\tilde{q}'} + 1 - k$$

Then, for $\frac{1}{\tilde{q}} + \frac{1}{\tilde{q}'} = 1$*, we have*

$$\|\mathbf{B}\|_{L^q_t L^r_x} + \|\mathbf{B}\|_{C^0_t \dot{H}^s_x} + ||\partial_t \mathbf{B}\|_{C^0_t \dot{H}^{s-1}_x} \leq C(||\mathbf{B}_0\|_{\dot{H}^s} + \|\mathbf{B}_1\|_{\dot{H}^{s-1}} + \|(-\Delta)^{k/2}(\nabla_x \mathbf{j})\|_{L^{\tilde{q}'}_t L^1_x}).$$

The main point in the theorem above is that we have L^1 norm in space on the right side.

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Department of Mathematics, Rutgers, the State University of New Jersey, NJ 08854
E-mail address: chanillo@math.rutgers.edu

Institut de Recherche en Mathématique et en Physique, Université catholique de Louvain, Chemin du Cyclotron 2 bte L7.01.01, 1348 Louvain-la-Neuve, Belgium
E-mail address: Jean.VanSchaftingen@uclouvain.be

Department of Mathematics, the Chinese University of Hong Kong, Shatin, Hong Kong
E-mail address: plyung@math.cuhk.edu.hk