A Colored Path Problem and Its Applications*

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Given a set of obstacles and two points in the plane, is there a path between the two points that does not cross more than k different obstacles? Equivalently, can we remove k obstacles so that there is an obstacle-free path between the two designated points? This is a fundamental problem that has undergone a tremendous amount of work by researchers in various areas, including computational geometry, graph theory, wireless computing, and motion planning. It is known to be NP-hard, even when the obstacles are very simple geometric shapes (*e.g.*, unit-length line segments). The problem can be formulated and generalized into the following graph problem: Given a planar graph G whose vertices are colored by color sets, two designated vertices $s, t \in V(G)$, and $k \in \mathbb{N}$, is there an s-t path in G that uses at most k colors? If each obstacle is connected, the resulting graph satisfies the color-connectivity property, namely that each color induces a connected subgraph.

We study the complexity and design algorithms for the above graph problem with an eye on its geometric applications. We prove a set of hardness results, among which a result showing that the color-connectivity property is crucial for any hope for fixed-parameter tractable (FPT) algorithms (even for various restrictions and parameterizations of the problem), as without it, the problem is W[SAT]-hard parameterized by k. Previous results only implied that the problem is W[2]-hard. A corollary of the aforementioned result is that, unless W[2] = FPT, the problem cannot be approximated in FPT time to within a factor that is a function of k. By describing a generic plane embedding of the graph instances, we show that our hardness results translate to the geometric instances of the problem.

We then focus on graphs satisfying the color-connectivity property. By exploiting the planarity of the graph and the connectivity of the colors, we develop topological results that reveal rich structural properties of the problem. These results allow us to prove that, for any vertex v in the graph, there exists a set of paths whose cardinality is upper bounded by some function of k, that "represents" the valid s-t paths containing subsets of colors from v. We employ these structural results to design an FPT algorithm for the problem parameterized by both k and the treewidth of the graph, and extend this result further to obtain an FPT algorithm for the parameterization by both k and the length of the path. The latter result directly implies and explains previous FPT results for various obstacle shapes, such as unit disks and fat regions.

CCS Concepts: • **Theory of computation** → **Parameterized complexity and exact algorithms**; **Computational geometry**; • **Mathematics of computing** → **Graph algorithms**;

Additional Key Words and Phrases: parameterized complexity and algorithms, motion planning, barrier coverage, barrier resilience, colored path, minimum constraint removal, planar graphs

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1 INTRODUCTION

We consider the following problem: Given a set of obstacles and two designated points in the plane, is there a path between the two points that does not cross more than k obstacles? Equivalently, can we remove k obstacles so that there is an obstacle-free path between the two designated points? We refer to this problem as Obstacle Removal, and to its restriction to instances in which each obstacle is connected as Connected Obstacle Removal.

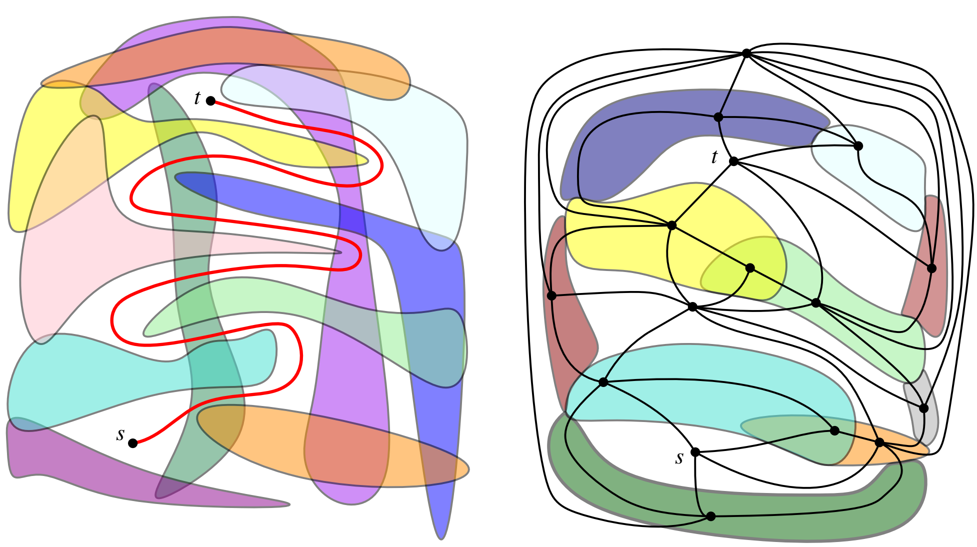
By considering the auxiliary plane graph that is the dual of the plane subdivision determined by the obstacles, Obstacle Removal was formulated and generalized into the following graph problem, referred to as Colored Path (see Figure 1 for illustrations):

Colored Path

Given: A planar graph G; a set of colors C; $\chi : V \longrightarrow 2^C$; two designated vertices $s, t \in V(G)$; and $k \in \mathbb{N}$

Question: Does there exist an s-t path in G that uses at most k colors?

Denote by Colored Path-Con the restriction of Colored Path to instances in which each color induces a connected subgraph of G.



(a) An instance in which the optimal path crosses two obstacles, zigzagging between the other obstacles.

(b) An instance and its auxiliary plane graph.

Fig. 1. Illustration of instances of the problem under consideration.

As we discuss next, Connected Obstacle Removal and Colored Path are fundamental problems that have undergone a tremendous amount of work, albeit under different names and contexts, by researchers in various areas, including computational geometry, graph theory, wireless computing, and motion planning.

Remark 1.1. *An obstacle may or may not contain its interior. We assume that the regions formed by the obstacles can be computed in polynomial time. We also assume that each obstacle is a 2-D region (or the union of 2-D regions), as if an obstacle is not, then we can "thicken" its borders properly without changing the sets of obstacles it intersects. Clearly, this can be done in polynomial time. (More formally,* *we can replace each point p of an obstacle with a small disk that is contained in the same regions as p, and define the obstacle to be the union of all these disks.)*

1.1 Related Work

In motion planning, the goal is generally to move a robot from a starting position to a final position, while avoiding collision with a set of obstacles. This is usually referred to as the *piano-mover*'s problem. Obstacle Removal is a variant of the piano-mover's problem, in which the obstacles are in the plane and the robot is represented as a point. Since determining if there is an obstacle-free path for the robot in this case is solvable in polynomial time, if no such path exists, it is natural to seek a path that intersects as few obstacles as possible. Motivated by planning applications, Connected Obstacle Removal and Colored Path were studied under the name Minimum Constraint Removal [13, 14, 17, 20]. Connected Obstacle Removal has also been studied extensively, motivated by applications in wireless computing, under the name Barrier Coverage or Barrier Resilience [1, 2, 27, 28, 32, 34]. In such applications, we are given a field covered by sensors (usually simple shapes such as unit disks), and the goal is to compute a minimum set of sensors that need to fail before an entity can move undetected between two given sites.

Kumar *et al.* [28] were the first to study Connected Obstacle Removal. They showed that for unit-disk obstacles in some restricted setting, the problem can be solved in polynomial time. The complexity of the general case for unit-disk obstacles remains open. Several works showed the NP-hardness of the problem, even when the obstacles are very simple geometric shapes such as line segments (*e.g.*, see [1, 32, 34]). The complexity of the problem when each obstacle intersects a constant number of other obstacles is open [14, 20].

Bereg and Kirkpatrick [2] designed approximation algorithms when the obstacles are unit disks by showing that the length, referred to as the *thickness* [2] (*i.e.*, number of regions visited), of a shortest path that crosses k disks is at most $3k$; this follows from the fact that a shortest path does not cross a disk more than a constant number of times.

Korman *et al.* [27] showed that Connected Obstacle Removal is FPT parameterized by k for unit-disk obstacles, and extended this result to similar-size fat-region obstacles with a constant *overlapping number*, which is the maximum number of obstacles having nonempty intersection. Their result draws the observation, which was also used in [2], that for unit-disk (and fat-region) obstacles, the length of an optimal path can be upper bounded by a linear function of the number of obstacles crossed (*i.e.*, the parameter). This observation was then exploited by a branching phase that decomposes the path into subpaths in (simpler) restricted regions, enabling a similar approach to that of Kumar *et al.* [28].

Motivated by its applications to networking, among other areas, the problem of computing a minimum-colored path in a graph received considerable attention (*e.g.*, see [3, 35]). In particular, the problem of computing a minimum-color path in mesh networks was studied in [35], motivated by its applications for finding a "reliable" path in the network. (Note that since a mesh is planar, this problem is a special case of Colored Path.) The problem of finding a minimum-color path in a graph was shown to be NP-hard in several works [3, 4, 20, 35].[1] Most of the NP-hardness reductions start from Set Cover, and result in instances of Colored Path (*i.e.*, planar graphs), as was also observed by [2]. These reductions are FPT-reductions, implying the W[2]-hardness of Colored Path. Moreover, these reductions imply that, unless P = NP, the minimization version of Colored Path cannot be approximated to within a factor of $c \lg n$, for any constant $c < 1$. Hauser [20], and Gorbenko and Popov [17], implemented exact and heuristic algorithms for the problem on general

[1]We note that some works consider the edge-colored version of the problem, but for all purposes considered in this paper the two versions are equivalent.

graphs. Very recently, Eiben *et al.* [13] designed exact and heuristic algorithms for Colored Path and Obstacle Removal, and proved computational lower bounds on their subexponential-time complexity, assuming the Exponential Time Hypothesis.

The Colored Path problem also falls into the category of many computationally-hard problems on colored graphs, where the objective is to compute a graph structure (satisfying certain desired properties) that uses the minimum number of colors. These structures have applications in telecommunication/transportation networks, where they can be used as backbones that utilize few communication/transportation media (*e.g.*, see [5, 33] for more information).

Finally, we mention that there is a related problem that is solvable in polynomial time, which has received considerable attention [6, 21, 22], where the goal is to find a shortest path w.r.t. the Euclidean length between two given points in the plane that intersects at most k obstacles.

1.2 Our Results and Techniques

We study the complexity and parameterized complexity of Colored Path and Colored Path-Con, eyeing the implications on their geometric counterparts Obstacle Removal and Connected Obstacle Removal, respectively. We do not treat the problem on general graphs because, as we point out in Remark 6.14, this problem is computationally very hard, even when restricted to graphs satisfying the color-connectivity property.

Our first set of hardness results shows that both problems are NP-hard, even when restricted to graphs of small outerplanarity and pathwidth, and that it is unlikely that they can be solved in subexponential time:

(i) Colored Path is NP-complete, even for outerplanar graphs of pathwidth at most 2 and in which every vertex contains at most one color (Theorem 6.1).
(ii) Colored Path-Con is NP-complete even for 2-outerplanar graphs of pathwidth at most 3 (Corollary 6.2).
(iii) Unless ETH fails, Colored Path-Con (and hence Colored Path) is not solvable in subexponential time, even for 2-outerplanar graphs of pathwidth at most 3 and in which each color appears at most 4 times (Corollary 6.3).

The reduction used to prove (i) produces instances of Colored Path that can be realized as geometric instances of Obstacle Removal whose overlapping number is at most 2. Thus, this hardness result extends to the aforementioned restriction of Obstacle Removal. The same reduction is then modified to yield (ii) and (iii) for Colored Path-Con; this reduction produces instances of Colored Path that can be realized as geometric instances of Connected Obstacle Removal whose overlapping number is at most 4, again showing that the hardness results extend to these restrictions of Connected Obstacle Removal.

With respect to the parameterized complexity of Colored Path and Colored Path-Con, clearly, Colored Path is in the parameterized class XP. We show that the color-connectivity property is crucial for any hope for an FPT-algorithm, since even very restricted instances and combined parameterizations of Colored Path are W[1]-complete:

(iv) Colored Path, restricted to instances of pathwidth at most 4, and in which each vertex contains at most one color and each color appears on at most 2 vertices, is W[1]-complete parameterized by k (Theorem 6.8).
(v) Colored Path, parameterized by both k and the length of the sought path ℓ, is W[1]-complete (Theorem 6.7).

Without restrictions, the problem sits high in the parameterized complexity hierarchy:

(vi) Colored Path, parameterized by k, is W[SAT]-hard (Theorem 6.10) and is in W[P] (Theorem 6.9).

A corollary of (vi) is that, unless W[2] = FPT, Colored Path cannot be approximated in FPT time to within a factor that is a function of k (Corollary 6.13).

By producing a generic construction (Remark 6.4) that can be used to realize any graph instance of Colored Path as a geometric instance of Obstacle Removal, the hardness results in (iv)–(vi), and the inapproximability result discussed above, translate to Obstacle Removal. This geometric realization may slightly increase the overlapping number by at most 2. Previously, Colored Path was only known to be W[2]-hard, via the standard reduction from Set Cover [3, 20, 35]. Our results refine the parameterized complexity and approximability of Colored Path and Obstacle Removal.

As noted in Remark 6.14, the color-connectivity property without planarity is hopeless: We can tradeoff planarity for color-connectivity by adding a single vertex that serves as a color-connector, thus establishing the W[SAT]-hardness of the problem on apex graphs.

The above hardness results show that we can focus our attention on Colored Path-Con. We show the following algorithmic result:

(vii) Colored Path-Con, parameterized by both k and the treewidth ω of the input graph, is FPT (Theorem 4.12).

We remark that bounding the treewidth does not make Colored Path-Con much easier, as we show in this paper that Colored Path-Con is NP-hard even for 2-outerplanar graphs of pathwidth at most 3 (Corollary 6.2).

The folklore dynamic programming approach based on tree decomposition, used for the Hamiltonian Path/Cycle problems, does not work for Colored Path-Con to prove the result in (vii) for the following reasons. As opposed to the Hamiltonian Path/Cycle problems, where it is sufficient to keep track of how the path/cycle interacts with each bag in the tree decomposition, this is not sufficient in the case of Colored Path-Con because we also need to keep track of which color sets are used on both sides of the bag. Although (by color connectivity) any subset of colors appearing on both sides of a bag must appear on vertices in the bag as well, there can be too many such subsets (up to $|C|^k$, where C is the set of colors), and certainly we cannot afford to enumerate all of them if we seek an FPT algorithm. To overcome this issue, we develop in Section 3 topological structural results that exploit the planarity of the graph and the connectivity of the colors to show the following. For any vertex $w \in V(G)$, and for any pair of vertices $u, v \in V(G)$, the set of (valid) u-v paths in $G - w$ that use colors appearing on vertices in the face of $G - w$ containing w can be "represented" by a minimal set of paths whose cardinality is a function of k.

In Section 4, we extend the notion of a minimal set of paths w.r.t. a single vertex to a "representative set" of paths w.r.t. a specific bag, and a specific enumerated configuration for the bag, in a tree decomposition of the graph. This enables us to use the upper bound on the size of a minimal set of paths, derived in Section 3, to upper bound the size of a representative set of paths w.r.t. a bag and a configuration. This, in turn, yields an upper bound on the size of the table stored at a bag, in the dynamic programming algorithm, by a function of both k and the treewidth of the graph, thus yielding the desired result.

In Section 5, we extend the FPT result for Colored Path-Con in (vii) w.r.t. the parameters k and ω, to the parameterization by both k and the length ℓ of the path:

(viii) Colored Path-Con, and hence Connected Obstacle Removal, parameterized by both k and ℓ is FPT (Theorem 5.15).

The dependency on both ℓ and k is *essential* for the result in (viii): If we parameterize only by k, or only by ℓ, then the problem becomes W[1]-hard (Theorem 5.1 and Theorem 5.2). By Remark 6.4, these two results translate to Connected Obstacle Removal.

The result in (viii) generalizes and explains Korman *et al.*'s results [27] showing that Connected Obstacle Removal is FPT parameterized by k for unit-disk obstacles, which they also generalized to similar-size fat-region obstacles with bounded overlapping number. Their results exploit the obstacle shape to upper bound the length of the path by a linear function of k, and then use branching to reduce the problems to a simpler setting. Our result directly implies that, regardless of the (connected) obstacle shapes, as long as the path length is upper bounded by some function of k (Corollary 5.16), the problem is FPT. The FPT result in (viii) also implies that:

(ix) For any computable function h, Colored Path-Con restricted to instances in which each color appears on at most $h(k)$ vertices, is FPT parameterized by k (Corollary 5.18).

Result (ix) has applications to Connected Obstacle Removal, in particular, to the interesting case when the obstacles are convex polygons, each intersecting a constant number of other polygons. The question about the complexity of this problem was posed in [14, 20], and remains open. The result in (ix) implies that this problem is FPT (Theorem 5.19).

We finally mention that it remains open whether Colored Path-Con and Connected Obstacle Removal are FPT parameterized by k only.

The paper is organized as follows. Section 2 presents some definitions and terminologies, in addition to a basic operation (Lemma 2.4), that are used throughout the paper. Section 3 presents the structural results needed to show that Colored Path-Con is FPT (Theorem 4.12). Section 4 presents the FPT algorithm for Colored Path-Con, and Section 5 presents extensions and applications of this algorithm. Section 6 presents hardness results for Colored Path, Colored Path-Con and their geometric counterparts Obstacle Removal and Connected Obstacle Removal, respectively. We conclude in Section 7 with some remarks and open questions.

2 PRELIMINARIES

We assume familiarity with the basic notations and terminologies in graph theory and parameterized complexity. We refer the reader to the standard books [11, 12] for more information on these subjects.

Graphs. All graphs in this paper are simple (*i.e.*, loop-less and with no multiple edges).

Let $G = (V(G), E(G))$ be an undirected graph. For a subset $S \subseteq V(G)$ of vertices, we write $G[S]$ for the subgraph of G induced by S. We write $G - S$ for $G[V(G) \setminus S]$. If $S = \{v\}$ is a singleton, we write $G - v$ instead of $G - \{v\}$. For a subgraph H of G and a vertex $v \in V(G) \setminus V(H)$, we write $H + v$ for the subgraph of G whose vertex-set is $V(H) \cup \{v\}$ and edge-set is $E(H) \cup \{uv \mid u \in V(H) \wedge uv \in E(G)\}$. For a subset $E' \subseteq E(G)$ of edges, we write $G - E'$ for the subgraph $(V(G), E(G) \setminus E')$. If $E' = \{e\}$ is a singleton, we write $G - e$ instead of $G - \{e\}$. For $E' \subseteq V(G) \times V(G)$, we write $G + E'$ for the graph $(V(G), E(G) \cup E')$; as above, if $E' = \{e\}$, we write $G + e$ instead of $G + \{e\}$. For a subset of edges $E' \subseteq E(G)$, the subgraph of G induced by E' is the graph whose vertex-set is the set of endpoints of the edges in E', and whose edge-set is E'.

For an edge $e = uv$ in G, *contracting* e means removing the two vertices u and v from G, replacing them with a new vertex w, and for every vertex y in the neighborhood of v or u in G, adding an edge wy in the new graph, not allowing multiple edges. Given a vertex-set $S \subseteq V(G)$, *contracting* S means contracting the edges between the vertices in S; if S induces a connected subgraph of G, then contracting S results in a single vertex. A graph is *planar* if it can be drawn in the plane without edge intersections (except at the endpoints). An *apex graph* is a graph in which the removal of a single vertex results in a planar graph. A *plane graph* has a fixed drawing. Each maximal connected region of the plane minus the drawing is an open set; these are the *faces*. One is unbounded, called the *outer face*. A *1-outerplane graph*, or simply an *outerplane graph*, is a plane graph for which every vertex is incident to the outer face; and a *1-outerplanar graph*, or simply an *outerplanar graph*, is a graph that has such a plane embedding. An i-*outerplane graph* (resp. i-*outerplanar graph*), for $i > 1$, is defined inductively as a graph such that the removal of its outer face results in an $(i-1)$-*outerplane graph* (resp. $(i-1)$-*outerplanar graph*) graph.

Let S be a set of points in the plane, and let C_1, C_2 be two non self-intersecting curves that meet S in precisely their common endpoints a and b. We say that C_1 and C_2 are *isotopic* w.r.t. S (also known as *homotopic rel. boundary*) if there is a continuous deformation from C_1 to C_2 through curves between a and b such that no intermediate curve in this deformation meets a vertex of S in its interior.

Let $W_1 = (u_1, \ldots, u_p)$ and $W_2 = (v_1, \ldots, v_q)$, $p, q \in \mathbb{N}$, be two walks such that $u_p = v_1$. Define the *gluing* operation $\circ$ that when applied to W_1 and W_2 produces that walk $W_1 \circ W_2 = (u_1, \ldots, u_p, v_2, \ldots, v_q)$.

For a graph G and two vertices $u, v \in V(G)$, we denote by $d_G(u, v)$ the *distance* between u and v in G, which the length of a shortest path between u and v in G. For a graph G and two vertices $u, v \in V(G)$, Menger's theorem states that the minimum number of vertices separating u from v in G is equal to the maximum number of vertex-disjoint u-v paths in G (see [11] for a proof).

Treewidth, Pathwidth and Tree Decomposition.

Definition 2.1. Let $G = (V, E)$ be a graph. A *tree decomposition* of G is a pair $(\mathcal{V}, \mathcal{T})$ where $\mathcal{V}$ is a collection of subsets of V such that $\bigcup_{X_i \in \mathcal{V}} = V$, and $\mathcal{T}$ is a rooted tree whose node set is $\mathcal{V}$, such that:

1. For every edge $\{u, v\} \in E$, there is an $X_i \in \mathcal{V}$, such that $\{u, v\} \subseteq X_i$; and
2. for all $X_i, X_j, X_k \in \mathcal{V}$, if the node X_j lies on the path between the nodes X_i and X_k in the tree $\mathcal{T}$, then $X_i \cap X_k \subseteq X_j$.

   The *width* of the tree decomposition $(\mathcal{V}, \mathcal{T})$ is defined to be $\max\{|X_i| \mid X_i \in \mathcal{V}\} - 1$. The *treewidth* of the graph G is the minimum width over all tree decompositions of G.

A *path decomposition* of a graph G is a tree decomposition $(\mathcal{V}, \mathcal{T})$ of G, where $\mathcal{T}$ is a path. The *pathwidth* of a graph G is the minimum width over all path decompositions of G.

A tree decomposition $(\mathcal{V}, \mathcal{T})$ is *nice* if it satisfies the following conditions:

1. Each node in the tree $\mathcal{T}$ has at most two children.
2. If a node X_i has two children X_j and X_k in the tree $\mathcal{T}$, then $X_i = X_j = X_k$; in this case node X_i is called a *join node*.
3. If a node X_i has only one child X_j in the tree $\mathcal{T}$, then either $|X_i| = |X_j| + 1$ and $X_j \subset X_i$, and in this case X_i is called an *insert node*; or $|X_i| = |X_j| - 1$ and $X_i \subset X_j$, and in this case X_i is called a *forget node*.
4. If X_i is a leaf node or the root, then $X_i = \emptyset$.

Boolean Circuits and Parameterized Complexity. A *circuit* is a directed acyclic graph. The vertices of indegree 0 are called the (input) *variables*, and are labeled either by *positive literals* x_i or by *negative literals* $\overline{x}_i$. The vertices of indegree larger than 0 are called the *gates* and are labeled with Boolean operators AND or OR. A special gate of outdegree 0 is designated as the *output* gate. We do not allow NOT gates in the above circuit model, since by De Morgan's laws, a general circuit can be effectively converted into the above circuit model. A circuit is said to be *monotone* if all its input literals are positive. The *depth* of a circuit is the maximum distance from an input variable to the output gate of the circuit. A circuit represents a Boolean function in a natural way. The size of a circuit C, denoted $|C|$, is the size of the underlying graph (*i.e.*, number of vertices and edges). An *occurrence* of a literal in C is an edge from the literal to a gate in C. Therefore, the total number of occurrences of the literals in C is the number of outgoing edges from the literals in C to its gates.

We say that a truth assignment τ to the variables of a circuit C *satisfies* a gate g in C if τ makes the gate g have value 1, and that τ *satisfies the circuit* C if τ satisfies the output gate of C. A circuit

C is *satisfiable* if there is a truth assignment to the input variables of C that satisfies C. The *weight* of an assignment τ is the number of variables assigned value 1 by τ.

A *parameterized problem* Q is a subset of $\Omega^* \times \mathbb{N}$, where Ω is a fixed alphabet. Each instance of the parameterized problem Q is a pair (x, k), where $k \in \mathbb{N}$ is called the *parameter*. We say that the parameterized problem Q is *fixed-parameter tractable* (FPT) [12], if there is a (parameterized) algorithm, also called an FPT-*algorithm*, that decides whether an input (x, k) is a member of Q in time $f(k) \cdot |x|^{O(1)}$, where f is a computable function. Let FPT denote the class of all fixed-parameter tractable parameterized problems.

A parameterized problem Q is FPT-*reducible* to a parameterized problem Q' if there is an algorithm, called an FPT-*reduction*, that transforms each instance (x, k) of Q into an instance (x', k') of Q' in time $f(k) \cdot |x|^{O(1)}$, such that $k' \leq g(k)$ and $(x, k) \in Q$ if and only if $(x', k') \in Q'$, where f and g are computable functions. By FPT-*time* we denote time of the form $f(k) \cdot |x|^{O(1)}$, where f is a computable function and $|x|$ is the input instance size. Based on the notion of FPT-reducibility, a hierarchy of parameterized complexity, *the* W-*hierarchy* $\bigcup_{t \geq 0} \mathrm{W}[t]$, where $\mathrm{W}[t] \subseteq \mathrm{W}[t+1]$ for all $t \geq 0$, has been introduced, in which the 0-th level W[0] is the class FPT. The hardness and completeness have been defined for each level $\mathrm{W}[i]$ of the W-hierarchy for $i \geq 1$ [12]. It is commonly believed that $\mathrm{W}[1] \neq \mathrm{FPT}$ (see [12]). The W[1]-hardness has served as the main working hypothesis of fixed-parameter intractability.

The class W[SAT] contains all parameterized problems that are FPT-reducible to the weighted satisfiability of Boolean formulas. It contains the classes W[t], for every $t \geq 0$. Boolean formulas can be represented (in polynomial time) by Boolean circuits that are in the *normalized* form (see [12]). In the normalized form every (nonvariable) gate has outdegree at most 1, and the gates are structured into alternating levels of ORS-of-ANDS-of-ORS.... Therefore, the underlying undirected graph of the circuit with the input variables removed is a tree; the input variables can be connected to any gate in the circuit, including the output gate. The class W[P] contains all parameterized problems that are FPT-reducible to the weighted satisfiability of Boolean circuits of polynomial size, and contains the class W[SAT].

Let O be a parameterized minimization problem, and $\rho : \mathbb{N} \longrightarrow \mathbb{R}_{\geq 1}$ a computable function such that $\rho(k) \geq 1$ for every $k \geq 1$. A decision algorithm $\mathbb{A}$ is an FPT *cost approximation algorithm* for O with approximation ratio ρ [8], if for every input $(x, k) \in \Sigma^* \times \mathbb{N}$, its output satisfies the following:

- If $k \leq OPT(x)$, then $\mathbb{A}$ rejects (x, k), and
- if $k \geq \rho(OPT(x)) \cdot OPT(x)$, then $\mathbb{A}$ accepts (x, k).

Furthermore, $\mathbb{A}$ runs in FPT-time.

The *Exponential Time Hypothesis* (ETH) states that the satisfiability of k-CNF Boolean formulas, where $k \geq 3$, is not decidable in subexponential-time $O(2^{o(n)})$, where n is the number of variables in the formula. ETH has become a standard hypothesis in complexity theory for proving hardness results that is closely related to the computational intractability of a large class of well-known NP-hard problems, measured from a number of different angles, such as subexponential-time complexity, fixed-parameter tractability, and approximation.

The asymptotic notation O^* suppresses a polynomial factor in the input length.

COLORED PATH *and* COLORED PATH-CON. For a set S, we denote by 2^S the power set of S. Let $G = (V, E)$ be a graph, let $C \subset \mathbb{N}$ be a finite set of colors, and let $\chi : V \longrightarrow 2^C$. A vertex v in V is *empty* if $\chi(v) = \emptyset$. A color c *appears on*, or is *contained in*, a subset S of vertices if $c \in \bigcup_{v \in S} \chi(v)$. For two vertices $u, v \in V(G)$, $\ell \in \mathbb{N}$, a u-v path $P = (u = v_0, \ldots, v_r = v)$ in G is ℓ-*valid* if $|\bigcup_{i=0}^{r} \chi(v_i)| \leq \ell$; that is, if the total number of colors appearing on the vertices of P is at most ℓ. A color $c \in C$ is *connected* in G, or simply *connected*, if $\bigcup_{c \in \chi(v)} \{v\}$ induces a connected subgraph of G. The graph G is *color-connected*, if for every $c \in C$, c is connected in G.

For an instance (G, C, χ, s, t, k) of Colored Path or Colored Path-Con, if s and t are nonempty vertices, we can remove their colors and decrement k by $|\chi(s) \cup \chi(t)|$ because their colors appear on every s-t path. If afterwards k becomes negative, then there is no k-valid s-t path in G. Moreover, if s and t are adjacent, then the path (s, t) is a path with the minimum number of colors among all s-t paths in G. Therefore, we will assume:

Assumption 2.2. *For an instance (G, C, χ, s, t, k) of* Colored Path *or* Colored Path-Con, *we can assume that s and t are nonadjacent empty vertices.*

Definition 2.3. Let s, t be two designated vertices in G, and let x, y be two adjacent vertices in G such that $\chi(x) = \chi(y)$. We define the following operation to x and y, referred to as a *color contraction* operation, that results in a graph G', a color function χ', and two designated vertices s', t' in G', obtained as follows:

- G' is the graph obtained from G by contracting the edge xy, which results in a new vertex z;
- $s' = s$ (resp. $t' = t$) if $s \notin \{x, y\}$ (resp. $t \notin \{x, y\}$), and $s' = z$ (resp. $t' = z$) otherwise; and
- $\chi' : V(G') \longrightarrow 2^C$ is the function defined as $\chi'(w) = \chi(w)$ if $w \neq z$, and $\chi'(z) = \chi(x) = \chi(y)$.

G is *irreducible* if there does not exist two vertices in G to which the color contraction operation is applicable.

Lemma 2.4. *Let G be a color-connected plane graph, C a color set, $\chi : V \longrightarrow 2^C$, $s, t \in V(G)$, and $k \in \mathbb{N}$. Suppose that the color contraction operation is applied to two vertices in G to obtain G', χ', s', t', as described in Definition 2.3. Then G' is a color-connected plane graph, and there is a k-valid s-t path in G if and only if there is a k-valid s'-t' path in G'.*

Proof. Let x and y be the two adjacent vertices in G to which the color contraction operation is applied, and let z be the new vertex resulting from this contraction. It is clear that after the contraction operation the obtained graph G' is a plane color-connected graph.

Suppose that there is a k-valid s-t path in G, and let $P = (s = v_0, \ldots, v_r = t)$ be such a path. We can assume that P is an induced path. If no vertex in $\{x, y\}$ is on P, then $P' = P$ is a k-valid s'-t' path in G'. If exactly one vertex in $\{x, y\}$, say x, is on P, then since the color set of every vertex other than x on P is the same before and after the contraction operation, and since $\chi'(z) = \chi(x)$, the path P' obtained from P by replacing x with z is a k-valid s'-t' in G'. (Note that if $x = s$ then $s' = z$, and replacing x with z on P is obsolete in this case.) Finally, if both x and y are on P, then since P is induced, x and y must appear consecutively on P. Without loss of generality, assume $x = v_i$ and $y = v_{i+1}$, for some $i \in \{0, \ldots, r-1\}$. Since the color set of every vertex other than x and y on P is the same before and after the operation, and since $\chi'(z) = \chi(x) = \chi(y)$, the path $P' = (s' = v_0, \ldots, v_{i-1}, z, v_{i+1}, \ldots, t = v_r)$ is a k-valid s'-t' path in G'.

Conversely, suppose that there is a k-valid s'-t' path in G', and let $P' = (s' = v'_0, \ldots, v'_p = t')$, where $p > 0$, be such a path. If z does not appear on P' then P' is a k-valid s-t path in G. Otherwise, $z = v'_i$ for some $i \in \{0, \ldots, p\}$. If $i = 0$ and P' consists only of vertex z, then since $\chi(x) = \chi(y) = \chi'(z)$, either $s = t$, and in which case there is a trivial k-valid s-t path in G, or $s \neq t$, and in this case $P = (x, y)$ is a k-valid s-t path in G. Otherwise, when $i = 0$ we must have $s = x$ or $s = y$, $v'_i \in G$ for $i \in [p]$, and $t' = t$; without loss of generality, assume that $s = x$. Since z is adjacent to v'_1, either $x = s$ or y (or both) is adjacent to v'_1. Since $\chi(x) = \chi(y) = \chi'(z)$, if x is adjacent to v'_1 then $P = (s = x, v'_1, \ldots, v'_p = t')$ is a k-valid s-t path in G, and if y is adjacent to v'_1 then $P = (s = x, y, v'_1, \ldots, v'_p = t')$ is a k-valid s-t path in G. The case is similar if $i = p$. Suppose now that $i \neq 0$ and $i \neq p$. If x (resp. y) is adjacent to both v'_{i-1} and v'_{i-1}, then the path $P = (s, v'_1, \ldots, v'_{i-1}, x, v'_{i+1}, \ldots, v'_p = t)$ (resp. $P = (s, v'_1, \ldots, v'_{i-1}, y, v'_{i+1}, \ldots, v'_p = t)$) is a k-valid s-t-path in G; otherwise, one vertex in $\{x, y\}$, say x, must be adjacent to v'_{i-1}, and the other vertex y

must be adjacent to v'_{i+1}. In this case the path $P = (s, v'_1, \ldots, v'_{i-1}, x, y, v'_{i+1}, \ldots, v'_p = t)$ is a k-valid s-t-path in G. □

3 STRUCTURAL RESULTS

Let G be a color-connected plane graph, C a set of colors, and $\chi : V \longrightarrow 2^C$. In this section, we present structural results that are the cornerstone of the FPT-algorithm for Colored Path-Con presented in the next section. We start by giving an intuitive description of the plan for this section.

As mentioned in Section 1, the main issue facing a dynamic programming algorithm based on tree decomposition, is how to upper bound, by a function of k and the treewidth, the number of k-valid paths between (any) two vertices u and v that use color sets contained in a certain bag. As it turns out, this number cannot be upper bounded as desired. Instead, we "represent" those paths using a minimal set $\mathcal{P}$ of k-valid u-v paths, in the sense that any k-valid u-v path can be replaced by a path from $\mathcal{P}$ that is not "worse" than it. To do so, it suffices to represent the k-valid u-v paths that use color sets contained in a third vertex w, by a set whose cardinality is a function of k. This will enable us to extend the notion of a minimal set of k-valid u-v paths w.r.t. a single vertex to a representative set for the whole bag, which is the key ingredient of the dynamic programming FPT-algorithm—based on tree decomposition—in the next section.

As it turns out, the paths that matter are those that use "external" colors w.r.t. w (defined below), since those colors have the potential of appearing on both sides of a bag containing w. Therefore, the ultimate goal of this section is to define a notion of a minimal set $\mathcal{P}$ of k-valid u-v paths with respect to w (Definition 3.4), and to upper bound $|\mathcal{P}|$ by a function of k. Upper bounding $|\mathcal{P}|$ by a function of k turns out to be quite challenging, and requires ideas and topological results that will be discussed later in this section.

Throughout this section, we shall assume that G is color-connected. We start with the following simple observation:

Observation 3.1. *Let $x, y \in V(G)$ be such that there exists a color $c \in C$ that appears on both x and y. Then any x-y vertex-separator in G contains a vertex on which c appears.*

Proof. This follows because color c is connected. □

Observation 3.1 will be useful in the next section as well, in the dynamic programming algorithm based on tree decomposition, as every tree bag is a vertex-separator.

Let G' be a plane graph, let $w \in V(G')$, and let H be a subgraph of $G' - w$. Let f be the face in H such that w is interior to f; we call f the *external face* w.r.t. w in H, and the vertices incident to f *external vertices* w.r.t. w in H. A color $c \in C$ is an *external color* w.r.t. w in H, or simply *external* to w in H, if c appears on an external vertex w.r.t. w in H; otherwise, c is *internal* to w in H. See Figure 2 for illustration. The following observation is easy to see:

Observation 3.2. *Let G be a color-connected plane graph, and let $w \in V(G)$. Let H be any subgraph of $G - w$. If c is an external color to w in $G - w$ and c appears on some vertex in H, then c is an external color to w in H. This also implies that the set of internal colors to w in H is a subset of the set of internal colors to w in $G - w$.*

Proof. Since c is external to w in $G - w$, c appears on a vertex v incident to the external face f w.r.t. w in $G - w$. Let f_H be the external face w.r.t. w in H (*i.e.*, the face containing w). Let u be a vertex in H containing c. If either u or v is incident to f_H, then c is external to w in H. Otherwise, since v is incident to the external face f w.r.t. w in $G - w$, f_H separates u from v in G, and by Observation 3.1, there exists a vertex incident to f_H that contains c. □

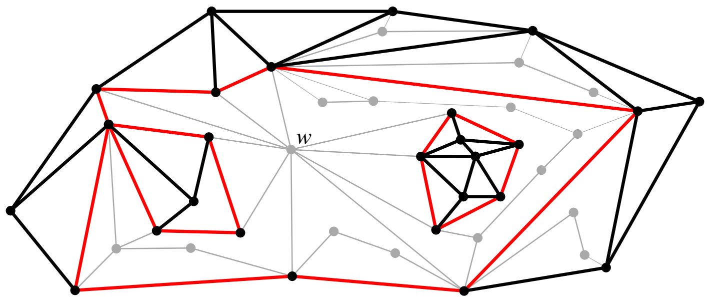


Fig. 2. A plane graph G', $w \in V(G')$, and a subgraph H of G' consisting of the black vertices and the black and red thick edges. The boundary of the external face f w.r.t. w in H is highlighted in red. The external vertices w.r.t. w in H are precisely the vertices incident to the highlighted (external) face.

Definition 3.3. Let $P = (w_1, \ldots, w_r)$ be a path in a graph G, and let $x, y \in V(G)$. Suppose that we apply the color contraction operation to x and y, and let z be the new vertex and G' the new graph resulting from this contraction, respectively. We define an operation, denoted Λ_{xy}, that when applied to path P in G results in another path $\Lambda_{xy}(P)$ in G' defined as follows:

1. If $\{x, y\} \cap \{w_1, \ldots, w_r\} = \emptyset$ then $\Lambda_{xy}(P) = P$.
2. If $\{x, y\} \cap \{w_1, \ldots, w_r\} = \{w_i\}$, where $i \in [r]$, then $\Lambda_{xy}(P) = (w_1, \ldots, w_{i-1}, z, w_{i+1}, \ldots, w_r)$.
3. If $\{x, y\} \cap \{w_1, \ldots, w_r\} = \{w_i, w_j\}$, where $i < j$, then $\Lambda_{xy}(P) = (w_1, \ldots, w_{i-1}, z, w_{j+1}, \ldots, w_r)$.

For a set of paths $\mathcal{P}$, we define $\Lambda_{xy}(\mathcal{P}) = \{\Lambda_{xy}(P) \mid P \in \mathcal{P}\}$.

Definition 3.4. Let $u, v, w \in V(G)$. A set $\mathcal{P}$ of k-valid u-v paths in $G - w$ is said to be *minimal* w.r.t. w if:

(i) There do not exist two paths $P_1, P_2 \in \mathcal{P}$ such that $\chi(P_1) \cap \chi(w) = \chi(P_2) \cap \chi(w)$;
(ii) there do not exist two paths $P_1, P_2 \in \mathcal{P}$ such that $\chi(P_1) \subseteq \chi(P_2)$; and
(iii) for any $P \in \mathcal{P}$, there does not exist a u-v path P' in $G - w$ such that $\chi(P') \subsetneq \chi(P)$.

Clearly, for any $u, v, w \in V(G)$, a minimal set of k-valid u-v paths in $G - w$ exists.

Observation 3.5. *Let $u, v, w \in V(G)$. Any set of u-v paths that is minimal w.r.t. w contains at most one path whose vertices contain only internal colors w.r.t. w in $G - w$.*

Proof. Since the external face f of w in $G - w$ is a Jordan curve that separates w from any vertex in $G - w$ that is not incident to f, by Observation 3.1, any color that appears both on w and on a vertex in $G - w$ must appear on a vertex incident to f, and hence, must be external to w by definition. Therefore, any path P containing only internal colors to w satisfies $\chi(P) \cap \chi(w) = \emptyset$. The observation now follows from property (i) in Definition 3.4. □

Lemma 3.6. *Let $u, v, w \in V(G)$, and let $\mathcal{P}$ be a minimal set of k-valid u-v paths in $G - w$. Suppose that we apply the color contraction operation to an edge $xy \in G - w$, and let G', χ' be the graph and color function obtained from the contraction operation, respectively. Let $\mathcal{P}' = \Lambda_{xy}(\mathcal{P})$. Then $\mathcal{P}'$ is a minimal set of k-valid paths w.r.t w in G'.*

Proof. Let H' be the subgraph of $G' - w$ induced by the edges of the paths in $\mathcal{P}'$, and denote by z the new vertex obtained from the contraction of the edge xy. We start by showing the following claim:

Claim 1. *For every $P \in \mathcal{P}$, it holds that $\chi(\Lambda_{xy}(P)) = \chi(P)$.*

Let $P = (u = w_1, \dots, w_r = v)$. Since $\chi'(z) = \chi(x) = \chi(y)$, it follows from Definition 3.3 that if $|\{x, y\} \cap \{w_1, \dots, w_r\}| \leq 1$, then $\chi(\Lambda_{xy}(P)) = \chi(P)$. Now assume that $\{x, y\} \cap \{w_1, \dots, w_r\} = \{w_i, w_j\}$, where $i < j$, and suppose to get a contradiction that $\chi(\Lambda_{xy}(P)) \neq \chi(P)$. Since $\Lambda_{xy}(P) = (w_1, \dots, w_{i-1}, z, w_{j+1}, \dots, w_r)$, it follows that $\chi(\Lambda_{xy}(P)) \subsetneq \chi(P)$. However, $G - w$ contains the u-v path $P' = (w_1, \dots, w_{i-1}, w_i, w_j, w_{j+1}, \dots, w_r)$, which satisfies $\chi(P') = \chi(\Lambda_{xy}(P)) \subsetneq \chi(P)$; this, together with $P \in \mathcal{P}$, contradicts the minimality of $\mathcal{P}$.

We now proceed to verify that $\mathcal{P}'$ is indeed minimal w.r.t. w. Properties (i) and (ii) in Definition 3.4 follow directly from Claim 1 and the minimality of $\mathcal{P}$. To prove that property (iii) holds, assume that there is a path $P' \in \mathcal{P}'$, and a path Q' in $G' - w$ between the endpoints of P' such that $\chi(Q') \subsetneq \chi(P')$. Let P be the path in $\mathcal{P}$ such that $\Lambda_{xy}(P) = P'$. It is straightforward to verify that $G - w$ contains a u-v path Q that is either identical to Q', or obtained from Q' by replacing z by either a single vertex x or y, or by the pair x, y. Clearly, $\chi(Q) = \chi(Q')$. Since $\chi(Q') \subsetneq \chi(P') = \chi(P)$ by Claim 1, it follows that $\chi(Q) \subsetneq \chi(P)$, contradicting the minimality of $\mathcal{P}$. It follows that Property (iii) holds, and the proof is complete. ◻

To derive an upper bound on the cardinality of a minimal set $\mathcal{P}$ of k-valid u-v paths w.r.t. a vertex w, we select a maximal set $\mathcal{M}$ of color-disjoint paths in $\mathcal{P}$. We first upper bound $|\mathcal{M}|$ by a function of k, which requires developing several results of topological nature. The key ingredient for upper bounding $|\mathcal{M}|$ is showing that the subgraph M induced by the paths in $\mathcal{M}$ has a u-v vertex-separator of cardinality $O(k)$ (Lemma 3.11), after a constant number of u-v paths in $\mathcal{M}$ have been removed. To show the existence of such a separator, we prove some structural lemmas (Lemmas 3.7 and 3.8) that essentially imply that, for any set of paths in $\mathcal{M}$ that each contains an external color w.r.t. w in M, there exist two paths P_1, P_2 in this set that induce a Jordan curve separating the neighbors of u on the paths in this set, except those neighbors on P_1, P_2, from v. The aforementioned result is subsequently used to show that for any set of paths in $\mathcal{M}$ that each contains an external color w.r.t. w in M, there exist two paths P_1, P_2 in this set such that any u-v path in M intersects (at least) one of P_1, P_2 at a vertex other than u or v. This result implies that there cannot be a set containing more than $O(k)$ k-valid vertex-disjoint u-v paths in M that each contains an external color w.r.t. w in $G - w$; otherwise, one of the two paths P_1, P_2 in this set, whose existence was alluded to earlier, would have to contain more than k colors, and hence, is not k-valid. The existence of the desired separator then follows by Menger's theorem. After establishing the existence of a small separator, we then upper bound $|\mathcal{M}|$ (Lemma 3.13) by upper bounding the number of different traces of the paths of $\mathcal{M}$ on this small separator, and inducting on both sides of the separator. Finally, we show (Theorem 3.14) that $|\mathcal{P}|$ is upper bounded by a function of $|\mathcal{M}|$, which proves the desired upper bound on $|\mathcal{P}|$. We proceed to the details.

Lemma 3.7. *Let G' be a plane graph, and let $x, y, z \in V(G')$. Let $x_1, \dots, x_r$, $r \geq 3$, be the neighbors of x in counterclockwise order. Suppose that, for each $i \in [r]$, there exists an x-y path P_i containing x_i such that P_i does not contain z and does not contain any x_j, $j \in [r]$ and $j \neq i$. Then there exist two paths P_i, P_j, $i, j \in [r]$ and $i \neq j$, such that the two paths P_i, P_j induce a Jordan curve separating $\{x_1, \dots, x_r\} \setminus \{x_i, x_j\}$ from z.*

Proof. We refer to Figure 3 for illustration of the lemma. The proof is by induction on $r \geq 3$. The base case is when $r = 3$, see Figure 4 for illustration of this case. Consider the faces induced by the two paths P_1 and P_2 in the embedding. If z and x_3 are in two separate faces, then clearly P_1 and P_2 induce a Jordan curve separating x_3 from z, and we are done. Therefore, we can assume that z and x_3 are in the same face induced by P_1 and P_2. Since P_1 does not contain x_2, we can continuously deform P_1 into an isotopic non self-intersecting curve P'_1 w.r.t. x_3, x_2, z, that contains the edge xx_1, intersects edges xx_2 and xx_3 only at x, and intersects P_2 only at x and y (see Subfigure 4b). Similarly,

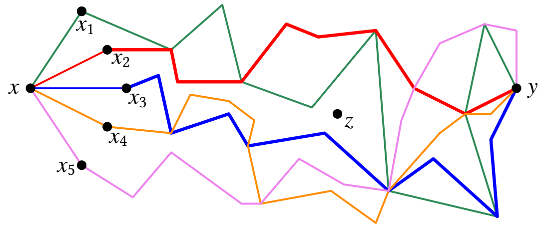
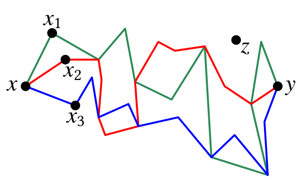
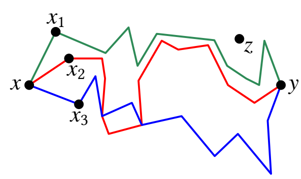


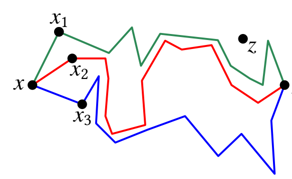
Fig. 3. Illustration of Lemma 3.7. The paths P_2 and P_3 (red and blue, respectively) separate z from x_1, x_4, and x_5.



(a) The paths P_1, P_2, and P_3 containing x_1, x_2, and x_3, respectively. The points x_3 and z are not separated by P_1 and P_2 and points x_1 and z are not separated by P_2 and P_3.



(b) Deformation of P_1 into an isotopic non self-intersecting curve P_1'.



(c) Deformation of P_3 into an isotopic non self-intersecting curve P_3'.

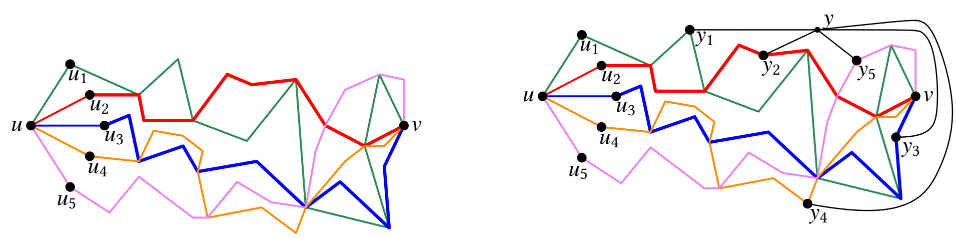
Fig. 4. Illustration of the proof of Lemma 3.7 for 3 paths.

if P_2 and P_3 do not separate z from x_1, then z and x_1 are in the same face induced by P_2 and P_3 and we can define a curve P_3' that is isotopic to P_3 w.r.t. x_2, x_1, z, and such that P_3' contains xx_3, intersects xx_2 and xx_1 only at x, and intersects P_2 only at x and y (see Subfigure 4c). Now if z and x_2 are in different faces induced by P_1' and P_3', then P_1' and P_3' separate z from x_2, and since P_1 is isotopic to P_1' w.r.t. z and x_2, and P_3' is isotopic to P_3 w.r.t. z and x_2, it follows that P_1 and P_3 induce a Jordan curve that separates x_2 from z. Assume now that z and x_2 are in the same face f induced by P_1' and P_3'. Since P_2 intersects with each of P_1' and P_3' precisely at x and y, it follows that P_2 splits f into two faces f_1, f_2, where xx_2, xx_1 are two consecutive edges on the boundary of f_1 and xx_2, xx_3 are two consecutive edges on the boundary of f_2. Then, z must be interior to exactly one of the two faces f_1, f_2. If z is interior to f_1, let f_1' be the face induced by P_1' and P_2 and containing z. Then f_1' contains f_1, and does not contain x_3 (because P_1' intersects xx_3 only at x). Therefore, f_1', and hence, P_1' and P_2 induce a Jordan curve that separates z from x_3. It follows that P_1, which is isotopic to P_1' w.r.t. x_2, x_3, z, and P_2 induce a Jordan curve that separates z from x_3. Similarly, if z is interior to f_2, then P_3', P_2 induce a Jordan curve that separates z from x_1, and hence, P_3 and P_2 induce a Jordan curve that separates z from x_1.

Assume inductively that the statement of the lemma is true for any $3 \leq \ell < r$. By the inductive hypothesis applied to $x_1, \ldots, x_{r-1}$, there exist two paths P_i, P_j, $i, j \in [r-1]$ and $i \neq j$, such that the two paths P_i, P_j induce a Jordan curve separating $\{x_1, \ldots, x_{r-1}\} \setminus \{x_i, x_j\}$ from z. If x_r and z are not in the same face induced by P_i, P_j, then P_i, P_j separate x_r from z as well, and we are done. Assume now that z and x_r are in the same face f induced by P_i, P_j. Since P_i, P_j separate z from $\{x_1, \ldots, x_{r-1}\} \setminus \{x_i, x_j\}$, none of $\{x_1, \ldots, x_{r-1}\} \setminus \{x_i, x_j\}$ is interior to f, and hence, x_r is the only neighbor of x between x_i and x_j w.r.t. the rotation system of G', which implies w.l.o.g. that

$x_1 = x_i$ and $x_{r-1} = x_j$. By the inductive hypothesis applied to x_1, x_{r-1}, x_r there are two paths in P_1, P_{r-1}, P_r that induce a Jordan curve that separates z from one of x_1, x_{r-1}, x_r. Since P_1 and P_{r-1} do not separate x_r from z, one of these two path must be P_r; assume, w.l.o.g., that the two paths are P_1 and P_r. Since x_1 and x_r are consecutive neighbors in the rotation system, and since P_1, P_r do not contain any of $x_2, \ldots, x_{r-1}$, it follows that $x_2, \ldots, x_{r-1}$ are in the same face induced by P_1, P_r, and this face does not contain z because P_1, P_r separate z from x_{r-1}. It follows that P_1, P_r induce a Jordan curve that separates z from $x_2, \ldots, x_{r-1}$. This completes the inductive proof. □

Lemma 3.8. *Let G' be a plane graph with a face f, and let $u, v \in V(G')$. Let $u_1, \ldots, u_r$, $r \geq 3$, be the neighbors of u. Suppose that, for each $i \in [r]$, there exists a u-v path P_i in G' containing u_i and a vertex incident to f different from v, and such that P_i does not contain any u_j, $j \in [r]$, $j \neq i$. Then there exist two paths P_i, P_j, $i, j \in [r]$, $i \neq j$, such that $V(P_i) \cup V(P_j) - \{v\}$ is a vertex-separator separating $\{u_1, \ldots, u_r\} \setminus \{u_i, u_j\}$ from v.*



(a) The u-v paths $P_1, \ldots, P_5$ containing $u_1, \ldots, u_5$, respectively. Each path has a vertex incident to the external face f.

(b) Adding a new vertex y inside f and making it adjacent to a vertex different from v on each P_i, results precisely in the situation described by Lemma 3.7, with u-y paths, none of which containing v.

Fig. 5. Illustration of the proof of Lemma 3.8.

Proof. We refer to Figure 5 for illustration. Create a new vertex y interior to f. Each path P_i, $i \in [r]$, contains a vertex y_i incident to f and different from v; we define a new path P'_i from u to y, consisting of the prefix of P_i up to y_i, and extending this prefix by adding a new edge between y_i and the new vertex y. Note that we can extend the rotation system of G' in a straightforward manner to obtain a rotation system for the plane graph resulting from adding y and the edges $y_i y$ to G', $i \in [r]$. Since v is the endpoint of P_i and $v \neq y_i$, it follows that v is not contained in P'_i, for $i \in [r]$. By Lemma 3.7, there exist two paths P'_i, P'_j, $i, j \in [r]$, and $i \neq j$, such that the two paths P'_i, P'_j induce a Jordan curve separating $\{u_1, \ldots, u_r\} \setminus \{u_i, u_j\}$ from v in $G' + y$. It follows that $V(P'_i) \cup V(P'_j)$ is a vertex-separator separating $\{u_1, \ldots, u_r\} \setminus \{u_i, u_j\}$ from v in $G' + y$, and hence, $V(P'_i) \cup V(P'_j) - \{y\} \subseteq V(P_i) \cup V(P_j) - \{v\}$ is a vertex-separator separating $\{u_1, \ldots, u_r\} \setminus \{u_i, u_j\}$ from v in G'. □

Lemma 3.9. *Let x, y be two vertices in an irreducible subgraph G' of G, and let f be a face in G'. Then there are at most two color-disjoint x-y paths in G' that contain only colors that appear on f.*

Proof. We refer to Figure 6 for illustration. Suppose, to get a contradiction, that there are three color-disjoint x-y paths P_1, P_2, P_3 in G' that contain only colors that appear on f. We create a new vertex z interior to f and add edges between z and each vertex incident to f. Note that we can extend the rotation system of G' in a straightforward manner to obtain a rotation system for the

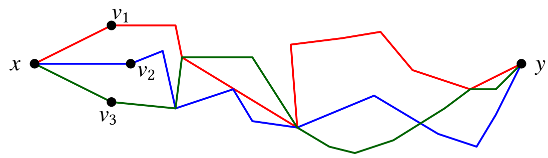


Fig. 6. Illustration of Lemma 3.9. Each vertex v_i, for $i \in [3]$, has a color that is unique to P_i. Hence, the vertex v_i, where $i \in \{1, 2, 3\}$, that is separated from f by the two paths not containing v_i, has a color that does not appear on f.

plane graph resulting from adding z and the edges incident to it to G'. Clearly, none of P_1, P_2, P_3 contains z. Because the paths P_1, P_2, P_3 are color-disjoint, both x and y must be empty vertices. Let v_1, v_2, v_3 be the neighbors of x on P_1, P_2, P_3, respectively. Since x is an empty vertex and G' is irreducible, none of v_1, v_2, v_3 is an empty vertex, and hence each v_i, $i \in [3]$, must contain a color c_i that appears on f. Since P_1, P_2, P_3 are pairwise color-disjoint, it follows that no vertex in $\{v_1, v_2, v_3\} \setminus \{v_i\}$ is contained in P_i, for $i \in [3]$. By Lemma 3.7, there is a v_i, $i \in [3]$, such that the two paths in $\{P_1, P_2, P_3\} - P_i$ induce a Jordan curve in $G' + z$ separating v_i and z, and hence separating v_i from each vertex incident to f. Since c_i appears on both v_i and a vertex incident to f, by Observation 3.1, it follows that c_i must appear on a vertex in $V(P_1) \cup V(P_2) \cup V(P_3) - V(P_i)$. This is a contradiction since c_i appears on P_i and the paths P_1, P_2, P_3 are pairwise color-disjoint. □

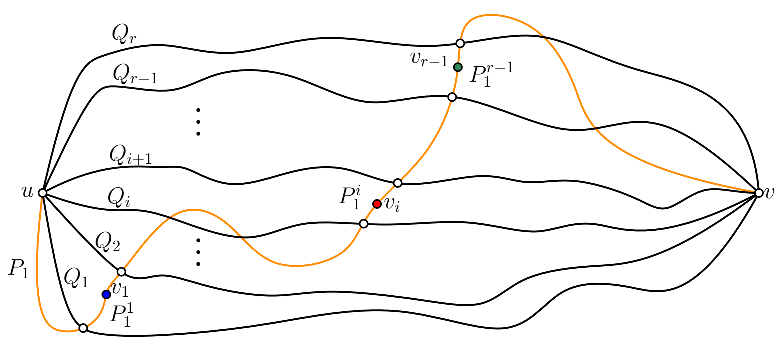


Fig. 7. Illustration for the proof of Lemma 3.11. Each of the vertices $v_1, \ldots, v_i, \ldots, v_{r-1}$ contains at least one color. Moreover, these vertices are pairwise separated by paths $Q_1, \ldots, Q_r$ that do not contain any color on P_1 and $\chi(v_i) \cap \chi(v_j) = \emptyset$ for all $i \neq j$.

For the rest of this section, let $u, v, w \in V(G)$, and let $\mathcal{P}$ be a set of minimal k-valid u-v paths in $G - w$. Let $\mathcal{M}$ be a set of minimal k-valid color-disjoint u-v paths in $G - w$, and let M be the subgraph of $G - w$ induced by the edges of the paths in $\mathcal{M}$.

Observation 3.10. *If $P \in \mathcal{M}$ contains a color c that is external to w in M, then c appears on a vertex in P that is incident to the external face to w in M.*

Proof. By definition, c appears on a vertex x incident to the external face w.r.t. w in M. Since the paths in $\mathcal{M}$ are pairwise color-disjoint and c appears on P, it follows that x is a vertex of P. □

Lemma 3.11. *Suppose that M is irreducible, then there exist paths $P_1, P_2, P_3 \in \mathcal{M}$ such that $M - (E(P_1) \cup E(P_2) \cup E(P_3))$ has a u-v vertex-separator of cardinality at most $2k + 3$.*

Proof. By Observation 3.5 and Observation 3.2, $\mathcal{M}$ contains at most one path that contains only internal colors w.r.t. w in M. Therefore, it suffices to show that $\mathcal{M}$ contains two paths P_1, P_2 such that $M - (E(P_1) \cup E(P_2))$ has a u-v vertex-separator of cardinality at most $2k + 3$, assuming that every path in $\mathcal{M}$ contains an external color w.r.t. w in M.

By Observation 3.10, every path in $\mathcal{M}$ passes through an external vertex w.r.t. w in M that contains an external color to w in M. Because the paths in $\mathcal{M}$ are pairwise color-disjoint and u and v are empty vertices, every path in $\mathcal{M}$ passes through a vertex on the external face of w in M that is different from u and v. Let $u_1, \ldots, u_q$ be the neighbors of u in M, and note that since u is empty and M is irreducible, each u_i, $i \in [q]$, contains a color. Let $P_1, \ldots, P_q$ be the paths in $\mathcal{M}$ containing $u_1, \ldots, u_q$, respectively, and note that since the paths in $\mathcal{M}$ are color-disjoint, no P_i passes through u_j, for $j \neq i$. By Lemma 3.8, there are two paths in $P_1, \ldots, P_q$, say P_1, P_2 without loss of generality, such that $V_{12} = V(P_1) \cup V(P_2) - \{v\}$ is a vertex-separator that separates $\{u_3, \ldots, u_q\}$ from v.

We proceed by contradiction and assume that $M^- = M - (E(P_1) \cup E(P_2))$ does not have a u-v vertex-separator of cardinality $2k+3$. By Menger's theorem [11], there exists a set $\mathcal{D}$ of $r' \geq 2k+3$ vertex-disjoint u-v paths in M^-. Since V_{12} separates $\{u_3, \ldots, u_q\}$ from v in M, every u-v path in M^- intersects at least one of P_1, P_2 at a vertex other than v. It follows that there exists a path in $\{P_1, P_2\}$, say P_1, that intersects at least $k+2$ paths in $\mathcal{D}$ at vertices other than v. Since the paths in $\mathcal{D}$ are vertex-disjoint and incident to u, we can order the paths in $\mathcal{D}$ that intersect P_1 around u (in counterclockwise order) as $\langle Q_1, \ldots, Q_r \rangle$, where $r \geq k+2$, and Q_{i+1} is counterclockwise from Q_i, for $i \in [r-1]$. P_1 intersects each path Q_i, $i \in [r]$, possibly multiple times. Moreover, since the paths in $\mathcal{M}$ are pairwise color-disjoint, each intersection between P_1 and a path Q_i, $i \in [r]$, must occur at an empty vertex. We choose $r-1$ subpaths, $P_1^1, \ldots, P_1^{r-1}$, of P_1 satisfying the property that the endpoints of P_1^i are on Q_i and Q_{i+1}, for $i = 1, \ldots, r-1$, and the endpoints of P_1^i are the only vertices on P_1^i that appear on a path Q_j, for $j \in [r]$. It is easy to verify that the subpaths $P_1^1, \ldots, P_1^{r-1}$ of P_1 can be formed by following the intersection of P_1 with the sequence of (ordered) paths $Q_1, \ldots, Q_r$. See Figure 7 for illustration.

Recall that the endpoints of $P_1^1, \ldots, P_1^{r-1}$ are empty vertices. Since M is irreducible, no two empty vertices are adjacent, and hence, each subpath P_1^i must contain an internal vertex v_i that contains at least one color. We claim that no two vertices v_i, v_j, $1 \leq i < j \leq r-1$, contain the same color. Suppose not, and let v_i, v_j, $i < j$, be two vertices containing a color c. Since v_i, v_j are internal to P_1^i and P_1^j, respectively, $Q_1, \ldots, Q_r$ are vertex-disjoint u-v paths, and by the choice of the subpaths $P_1^1, \ldots, P_1^{r-1}$, the paths Q_i and Q_{i+1} form a Jordan curve, and hence a vertex-separator in G, separating v_i from v_j. By Observation 3.1, color c must appear on a vertex in Q_p, $p \in \{i, i+1\}$, and this vertex is clearly not in P_1 since P_1 intersects Q_p at empty vertices. Since every vertex in M appears on a path in $\mathcal{M}$, and c appears on $P_1 \in \mathcal{M}$ and on a vertex not in P_1, this contradicts that the paths in $\mathcal{M}$ are pairwise color-disjoint, and proves the claim.

Since no two vertices v_i, v_j, $1 \leq i < j \leq r$, contain the same color, the number $r-1$ of subpaths $P_1^1, \ldots, P_1^{r-1}$ is upper bounded by the number of distinct colors that appear on P_1, which is at most k. It follows that r is at most $k+1$, contradicting our assumption above and proving the lemma. □

Lemma 3.12. *Let S be a minimal u-v vertex-separator in M. Let M_u, M_v be a partition of $M - S$ containing u and v, respectively, and such that there is no edge between M_u and M_v. For any vertex $x \in S$, M_u is contained in a single face of $M_v + x$.*

Proof. Let $x \in S$. It suffices to show that the subgraph F of M induced by $V(M_u) \cup (S \setminus \{x\})$ is connected. This suffices because $V(F)$ and $V(M_v + x)$ are disjoint, and hence every face in $M_v + x$ separates the vertices in $V(F)$ inside the face from those outside of it. We will show that F is connected by showing that there is a path in F from each vertex in F to $u \in V(F)$. Let $z \in V(F)$. If

$z \in S$, then by minimality of S, there is a path from u to z whose internal vertices are all in M_u, and hence this path is in F. If $z \notin S$, let P be a u-v path containing z. If P passes through z before passing through any vertex in S, then clearly there is a path from u to z in F. Otherwise, P passes through a vertex $y \in S$ before passing through z. In this case, there exists a vertex $y' \in S$, such that $y' \neq y$ and P passes through y' after passing through z. Either y or y', say y', is different from x. From the above discussion, there is a path P' from u to y' in F, which when combined with the subpath of P between y' and z yields a path from u to z in F. □

Lemma 3.13. *$|\mathcal{M}| \leq g(k)$, where $g(k) = O(c^k k^{2k})$, for some constant $c > 1$.*

Proof. By Observation 3.5, there can be at most one path in $\mathcal{M}$ that contains only internal colors w.r.t. w in $G - w$. Therefore, it suffices to upper bound the number of paths in $\mathcal{M}$ that contain at least one external color to w in $G - w$. Without loss of generality, in the rest of the proof, we shall assume that $\mathcal{M}$ does not include a path that contains only internal colors w.r.t. w in $G - w$, and upper bound $|\mathcal{M}|$ by $g(k)$; adding 1 to $g(k)$ we obtain an upper bound on $|\mathcal{M}|$ with this assumption lifted. Note that by Observation 3.2, the previous assumption implies that every path in $\mathcal{M}$ contains a color that is external to w in M.

The proof is by induction on k, over every color-connected plane graph G, every triplet of vertices u, v, w in G, and every minimal set $\mathcal{M}$ w.r.t. w of k-valid pairwise color-disjoint u-v paths in $G - w$. If $k = 1$, then any path in $\mathcal{M}$ contains exactly one external color w.r.t. w in M. By Lemma 3.9, at most two paths in $\mathcal{M}$ contain only external colors. It follows that for $k = 1$, $|\mathcal{M}| \leq 2 \leq g(1)$, if we choose the hidden constant in the O asymptotic notation to be at least 2.

Suppose by the inductive hypothesis that for any $1 \leq i < k$, we have $|\mathcal{M}| \leq g(i)$. We can assume that M is irreducible; otherwise, we apply the color contraction operation to any edge xy in $\mathcal{M}$ to which the operation is applicable, and replace $\mathcal{M}$ with the set of paths $\Lambda_{xy}(\mathcal{M})$, which is pairwise color-disjoint, contains the same number of paths as $\mathcal{M}$, and is minimal w.r.t. w by Lemma 3.6.

By Lemma 3.11, there are at most 3 paths in $\mathcal{M}$, such that the subgraph of M induced by the remaining paths of $\mathcal{M}$ has a u-v vertex-separator S satisfying $|S| \leq 2k+3$. To simplify the argument, in what follows, we assume that we already removed these 3 paths from $\mathcal{M}$ and that M already has a u-v vertex-separator S satisfying $|S| \leq 2k + 3$. We will add 3 to the upper bound of $|\mathcal{M}|$ at the end to account for these removed paths. We can assume, without loss of generality, that S is minimal (w.r.t. containment). S separates M into two subgraphs M_u and M_v such that $u \in V(M_u)$, $v \in V(M_v)$, and there is no edge between M_u and M_v. We partition $\mathcal{M}$ into the following groups, where each group excludes the paths satisfying the properties of the groups defined before it: (1) The set of paths in $\mathcal{M}$ that contain a nonempty vertex in S; (2) the set of paths $\mathcal{M}_u^k$ consisting of each path P in $\mathcal{M}$ such that all colors on P appear on vertices in M_u (these colors could still appear on vertices in M_v as well); (3) the set of paths $\mathcal{M}_v^k$ consisting of each path P in $\mathcal{M}$ such that all colors on P appear on vertices in M_v; and (4) the set $\mathcal{M}^{<k}$ of remaining paths in $\mathcal{M}$, satisfying that each path contains a nonempty external vertex to w in M and contains less than k colors from each of M_u and M_v. Note that by Observation 3.10, each path in $\mathcal{M}$ belongs to one of the 4 groups above. For the remainder of the proof, we refer the reader to Figure 8 for illustration.

Since the paths in $\mathcal{M}$ are pairwise color-disjoint, no nonempty vertex in S can appear on two distinct paths from group (1). Therefore, the number of paths in group (1) is at most $|S| \leq 2k + 3$. Observe, that the vertices in S contained in any path in groups (2)-(4) are empty vertices.

To upper bound the number of paths in group (2), for each path P, there is a last vertex x_P (*i.e.*, farthest from u) in P that is in S. Fix a vertex $x \in S$, and let us upper bound the number of paths P in group (2) for which $x = x_P$. Let P_v be the subpath of P from x to v. Note that since v is empty and all the vertices in S that are contained in paths in group (2) are empty, and since M is irreducible, P_v must contain at least one color. Since all colors appearing on P appear on vertices in M_u, all colors appearing on P_v appear in M_u. By Lemma 3.12, M_u is contained in a single face f of $M_v + x$. Since f is a vertex-separator that separates $V(M_u)$ from $V(P_v)$ in G, by Observation 3.1, every color that appears on P_v appears on f. By Lemma 3.9, there are at most two x-v paths that contain only colors that appear on f. This shows that there are at most two paths in group (2) for which x is the last vertex in S. Since $|S| \le 2k+3$, this upper bounds the number of paths in group (2) by $2(2k+3) = 4k+6$. By symmetry, the number of paths in group (3) is upper bounded by $4k+6$.

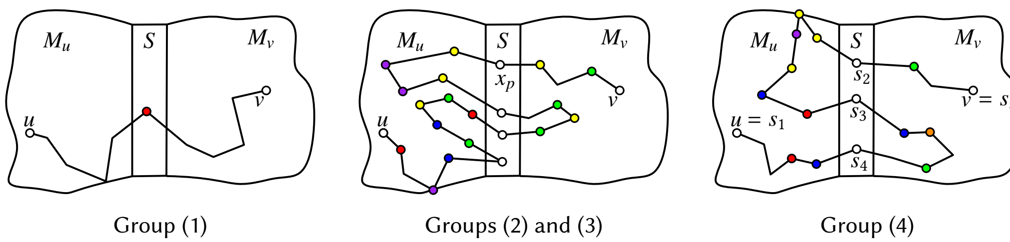


Fig. 8. Illustration for bounding the number of paths in the different groups in the proof of Lemma 3.13.

Finally, we upper bound the number of paths in group (4). Let $S = \{s_2, \ldots, s_{r-1}\}$, where $r \le 2k+5$, and extend S by adding the two vertices $s_1 = u$ and $s_r = v$ to form the set $A = \{s_1, s_2, \ldots, s_r\}$. For every two (distinct) vertices $s_j, s_{j'} \in A$, $j, j' \in [r], j < j'$, we define a set of paths $\mathcal{P}_{jj'}$ in $G - w$ whose endpoints are s_j and $s_{j'}$ as follows. For each path P in group (4), partition (the edges in) P into subpaths $P_1, \ldots, P_q$ satisfying the property that the endpoints of each P_i, $i \in [q]$, are in A, and no internal vertex to P_i is in A. Since each P is a u-v path, clearly, P can be partitioned as such. For each P_i, $i \in [q]$, such that P_i contains a vertex that contains an external color to w in $G - w$, let P'_i (possibly equal to P_i) be a subpath in $G - w$ between the endpoints of P_i satisfying that $\chi(P'_i) \subseteq \chi(P_i)$ and $\chi(P'_i)$ is minimal w.r.t. containment (*i.e.*, there does not exist a path P''_i in $G - w$ between the endpoints of P_i satisfying $\chi(P''_i) \subsetneq \chi(P'_i)$). Since P contains a vertex that contains an external color to w in $G - w$, there exists an $i \in [q]$ such that P'_i contains a vertex that contains an external color to w in $G - w$; otherwise, by concatenating (in the right sequence) the P_i's that do not contain an external color to w (in $G - w$), with the P'_i's (instead of the P_i's) of the P_i's that contain an external color to w (in $G - w$), we would obtain a u-v path P' in $G - w$ satisfying $\chi(P') \subsetneq \chi(P)$ (since $\chi(P') \subseteq \chi(P)$ and P contains an external color to w and P' does not), thus contradicting the minimality of $\mathcal{M}$. Pick any $i \in [q]$ satisfying that P'_i contains a vertex that contains an external color to w in $G - w$, associate P with P'_i, and assign P'_i to the set of paths $\mathcal{P}_{jj'}$ such that s_j and $s_{j'}$ are the endpoints of P'_i. Since each P'_i contains an external color that appears on P and the paths in $\mathcal{M}$ are pairwise-color disjoint, it follows that the map that maps each P to its P'_i is a bijection.

Therefore, to upper bound the number of paths in group (4), it suffices to upper bound the number of paths assigned to the sets $\mathcal{P}_{jj'}$, where $j, j' \in [r], j < j'$. Fix a set $\mathcal{P}_{jj'}$. The paths in $\mathcal{P}_{jj'}$ have $s_j, s_{j'}$ as endpoints, and are pairwise color-disjoint. Moreover, each path in $\mathcal{P}_{jj'}$ contains a vertex that contains an external color to w in $G - w$. It follows from the previous statements that $\mathcal{P}_{jj'}$ satisfies properties (i) and (ii) of Definition 3.4 w.r.t. G and w. Moreover, from the definition of each path in $\mathcal{P}_{jj'}$, $\mathcal{P}_{jj'}$ satisfies property (iii) of Definition 3.4 as well. Finally, observe that each path $P'_i \in \mathcal{P}_{jj'}$ was constructed based on a subpath P_i of a path P in group 4, and satisfying that P_i has endpoints $s_j, s_{j'}$ and no internal vertex on P_i is in A. Since P is a u-v path in $\mathcal{M}$ and S is a vertex-separator of M, $V(P_i)$ is either contained in $V(M_u) \cup S$ or in $V(M_v) \cup S$. Since P is in group (4), P contains at most $k-1$ colors from each of M_u and M_v. Since the vertices in S are empty, we deduce that P_i contains at most $k-1$ colors. Since $\chi(P'_i) \subseteq \chi(P_i)$, P'_i contains at most $k-1$ colors as well, and hence, every path in $\mathcal{P}_{jj'}$ contains at most $k-1$ colors. It follows that $\mathcal{P}_{jj'}$ is a minimal set of $(k-1)$-valid color-disjoint s_j-$s_{j'}$ paths in $G - w$ w.r.t. w. By the inductive hypothesis, we have $|\mathcal{P}_{jj'}| \le g(k-1)$. Since the number of sets $\mathcal{P}_{jj'}$ is at most $\binom{2k+5}{2}$, the number of paths in group (4) is $O(k^2) \cdot g(k-1)$.

It follows from the above that $|\mathcal{M}| \le g(k)$, where $g(k)$ satisfies the recurrence relation $g(k) \le 3 + (2k+3) + 2(4k+6) + O(k^2) \cdot g(k-1) = O(k^2) \cdot g(k-1)$, where 3 accounts for the 3 paths we removed from $\mathcal{M}$ at the beginning of the proof to get a small u-v vertex-separator. Solving the aforementioned recurrence relation gives $g(k) = O(c^k k^{2k})$, where $c > 1$ is a constant. Adding 1 to $g(k)$ to account for the single path in $\mathcal{M}$ containing only internal colors w.r.t. w in M yields the same asymptotic upper bound. □

Theorem 3.14. *Let G be a plane color-connected graph, let $u, v, w \in V(G)$, and let $\mathcal{P}$ be a set of minimal k-valid u-v paths w.r.t. w in $G - w$. Then $|\mathcal{P}| \le h(k)$, where $h(k) = O(c^{k^2} k^{2k^2+k})$, for some constant $c > 1$.*

Proof. The proof is by induction on k. If $k = 1$, then by minimality of $\mathcal{P}$, we have $\mathcal{P} = \mathcal{M}$. Lemma 3.13 gives an upper bound of $O(c^k k^{2k}) = O(c^{k^2} k^{2k^2+k})$ on $|\mathcal{P}|$.

Assume by the inductive hypothesis that the statement of the lemma is true for $1 \le i < k$. Let $\mathcal{M}$ be a maximal set of pairwise color-disjoint paths in $\mathcal{P}$. By Lemma 3.13, $|\mathcal{M}| \le g(k) = O(c^k k^{2k})$. The number of colors contained in vertices of $\mathcal{M}$ is at most $r \le k \cdot g(k)$. We group the paths in $\mathcal{P}$ into r groups $\mathcal{P}_1, \ldots, \mathcal{P}_r$, such that all the paths in $\mathcal{P}_i$, $i \in [r]$, share the same color c_i, where $i \in [r]$, that is distinct from each color c_j shared by the paths in $\mathcal{P}_j$, for $j \ne i$. We upper bound the number of paths in each $\mathcal{P}_i$, $i \in [r]$, to obtain an upper bound on $|\mathcal{P}|$.

Let G_i be the graph obtained by removing color c_i from each vertex in G that c_i appears on, and let $\mathcal{P}'_i$ be the set of paths obtained from $\mathcal{P}_i$ by removing color c_i from each vertex in $\mathcal{P}_i$ that c_i appears on. Clearly, every path in $\mathcal{P}'_i$ is a $(k-1)$-valid u-v path. Moreover, it is easy to verify that $\mathcal{P}'_i$ satisfies properties (i)-(iii) in Definition 3.4, and hence, $\mathcal{P}'_i$ is minimal w.r.t. w in $G_i - w$. By the inductive hypothesis, we have $|\mathcal{P}'_i| \le h(k-1)$. It follows that the total number of paths in $\mathcal{P}$ is at most $h(k)$, where $h(k)$ satisfies the recurrence relation $h(k) \le r \cdot h(k-1) \le k \cdot g(k) \cdot h(k-1)$. Solving the aforementioned recurrence relations yields $h(k) = O((k \cdot g(k))^k) = O(c^{k^2} k^{2k^2+k})$. □

The result of Theorem 3.14 will be employed in the next section in the form presented in the following corollary:

Corollary 3.15. *Let G be a plane color-connected graph, and let $w \in V(G)$. Let G' be a subgraph of $G - w$, and let $u, v \in V(G')$. Every set $\mathcal{P}$ of minimal k-valid u-v paths in G' w.r.t. w satisfies $|\mathcal{P}| \le h(k)$, where $h(k) = O(c^{k^2} k^{2k^2+k})$, for some constant $c > 1$.*

Proof. Contract every connected component of $(G - w) - G'$ into a single vertex containing the union of the color-sets of the vertices in the component, and add $k + 1$ new distinct colors to the resulting vertex. Denote the resulting graph by G''. Observe that the resulting graph is color-connected, and that every k-valid u-v path in G' w.r.t. w is a k-valid u-v path in G'' w.r.t. w, and vice versa. Therefore, every set $\mathcal{P}$ of minimal k-valid u-v paths in G' w.r.t. w is also a set of minimal k-valid u-v paths in G'' w.r.t. w. For any set $\mathcal{P}$ of minimal k-valid u-v paths w.r.t. w in G', by applying Theorem 3.14 to $\mathcal{P}$ in $G'' - w$, the corollary follows. □

4 THE ALGORITHM

In this section, we present an FPT algorithm for Colored Path-Con, parameterized by both k and the treewidth of the input graph. As pointed out in Section 3, there can be too many (*i.e.*, more than FPT-many) subsets of colors that appear in a bag, and hence, that the algorithm may need to store/remember. To overcome this issue, we extend the notion of a minimal set of k-valid u-v paths w.r.t. a vertex—from the previous section—to a "representative set" of paths w.r.t. a specific bag and a specific enumerated configuration for the bag. This allows us to upper bound the size of the table, in the dynamic programming algorithm, stored at a bag by a function of both k and the treewidth of the graph.

4.1 Representative sets of paths

Let (G, C, χ, s, t, k) be an instance of Colored Path-Con. The algorithm is a dynamic programming algorithm based on a tree decomposition of G. Let $(\mathcal{V}, \mathcal{T})$ be a nice tree decomposition of G. By Assumption 2.2, we can assume that s and t are nonadjacent empty vertices. We add s and t to every bag in $\mathcal{T}$, and now we have $\{s, t\} \subseteq X_i$, for every bag $X_i \in \mathcal{T}$. For a bag X_i, we say that $v \in X_i$ is *useful* if $|\chi(v)| \leq k$. Let U_i be the set of all useful vertices in X_i and let $\overline{U_i} = X_i \setminus U_i$. We denote by V_i the set of vertices in the bags of the subtree of $\mathcal{T}$ rooted at X_i. For any two vertices $u, v \in X_i$, let $G^i_{uv} = G[(V_i \setminus X_i) \cup \{u, v\}]$. We extend the notion of a minimal set of k-valid u-v paths w.r.t. a vertex, developed in the previous section, to the set of vertices in a bag of $\mathcal{T}$.

Definition 4.1. A set of k-valid u-v paths $\mathcal{P}_{uv}$ in G^i_{uv} is *minimal* w.r.t. X_i if it satisfies the following properties:

(i) There do not exist two paths $P_1, P_2 \in \mathcal{P}_{uv}$ such that $\chi(P_1) \cap \chi(X_i) = \chi(P_2) \cap \chi(X_i)$;
(ii) there do not exist two paths $P_1, P_2 \in \mathcal{P}_{uv}$ such that $\chi(P_1) \subseteq \chi(P_2)$; and
(iii) for any $P \in \mathcal{P}_{uv}$ there does not exist a u-v path P' in G^i_{uv} such that $\chi(P') \subsetneq \chi(P)$.

The following lemma uses the upper bound on the cardinality of a minimal set of k-valid u-v paths w.r.t. a vertex, derived in Corollary 3.15 in the previous section, to obtain an upper bound on the cardinality of a minimal set of k-valid u-v paths w.r.t. a bag of $\mathcal{T}$:

Lemma 4.2. *Let X_i be bag, $u, v \in X_i$, and $\mathcal{P}_{uv}$ a set of k-valid u-v paths in G^i_{uv} that is minimal w.r.t. X_i. Then the number of paths in $\mathcal{P}_{uv}$ is at most $h(k)^{|X_i|}$, where $h(k) = O(c^{k^2} k^{2k^2+k})$, for some constant $c > 1$.*

Proof. Let $X_i \setminus \{u, v\} = \{w_1, \ldots, w_r\}$, where $r = |X_i| - 2$. For each $w_j \in X_i$, $j \in [r]$, let $\mathcal{P}_j$ be a minimal set of k-valid u-v paths w.r.t. w_j in G^i_{uv}. Without loss of generality, we can pick $\mathcal{P}_j$ such that there is no k-valid u-v path P in G^i_{uv} such that $\mathcal{P}_j \cup \{P\}$ is minimal. From Corollary 3.15, we have $|\mathcal{P}_j| \leq h(k) = O(c^{k^2} k^{2k^2+k})$, for some constant $c > 1$. For each $P \in \mathcal{P}_{uv}$, and each $j \in [r]$, define $C_j = \chi(P) \cap \chi(w_j)$. Define the *signature* of P (w.r.t. the colors of $w_1, \ldots, w_r$) to be the tuple $(C_1, \ldots, C_r)$. Observe that no two (distinct) paths $P_1, P_2 \in \mathcal{P}_{uv}$ have the same signature; otherwise, since u and v appear on both P_1, P_2, $\chi(P_1) \cap \chi(X_i) = \chi(P_2) \cap \chi(X_i)$, which contradicts condition (i) of the minimality of $\mathcal{P}_{uv}$. For each $P \in \mathcal{P}_{uv}$, and each $j \in [r]$, there is a path $P' \in \mathcal{P}_j$ such that $\chi(P') \cap \chi(w_j) = C_j$. If this were not true, then P would have been added to $\mathcal{P}_j$ for the following reasons. Clearly, P does not contradict conditions (i) and (iii) of the minimality of $\mathcal{P}_j$. It cannot contradict (ii) either, because otherwise, and since P does not contradict (i), there would be a path $P'' \in \mathcal{P}_j$ such that either $\chi(P'') \subsetneq \chi(P)$ or $\chi(P) \subsetneq \chi(P'')$, contradicting the minimality of $\mathcal{P}_{uv}$ or $\mathcal{P}_j$, respectively. It follows that the number of signatures of paths in $\mathcal{P}_{uv}$ is at most $\prod_{j=1}^{r} |\mathcal{P}_j| \leq h(k)^{|X_i|}$. Since no two distinct paths in $\mathcal{P}_{uv}$ have the same signature, it follows that $|\mathcal{P}_{uv}| \leq h(k)^{|X_i|}$. □

Definition 4.3. Let X_i be a bag in $\mathcal{T}$. A *pattern* π for X_i is a sequence $(v_1 = s, \sigma_1, v_2, \sigma_2, \ldots, \sigma_{r-1}, v_r = t)$, where $\sigma_i \in \{0, 1\}$ and $v_i \in U_i$. For a bag X_i, and a pattern $(v_1 = s, \sigma_1, v_2, \sigma_2, \ldots, \sigma_{r-1}, v_r = t)$ for X_i, we say that a sequence of walks $\mathcal{S} = (W_1, \ldots, W_{r-1})$ *conforms* to (X_i, π) if:

- For each $j \in [r-1]$, $\sigma_j = 1$ implies that W_j is a walk from v_j to v_{j+1} whose internal vertices are contained in $V_i \setminus X_i$ and W_j is empty otherwise; and

- $|\chi(\mathcal{S})| = |\bigcup_{j\in[r-1]} \chi(W_j)| \leq k$.

Definition 4.4. Let X_i be a bag, π a pattern for X_i, and $\mathcal{S}_1, \mathcal{S}_2$ two sequences of walks that conform to (X_i, π). We write $\mathcal{S}_1 \preceq_i \mathcal{S}_2$ if $|\chi(\mathcal{S}_1) \cup (\chi(\mathcal{S}_2) \cap \chi(X_i))| \leq |\chi(\mathcal{S}_2)|$.

Lemma 4.5. *Let X_i be a bag and π a pattern for X_i. The relation $\preceq_i$ is a transitive relation on the set of all sequences of walks that conform to (X_i, π).*

Proof. Let $\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3$ be three sequences that conform to (X_i, π). Suppose that $\mathcal{S}_1 \preceq_i \mathcal{S}_2$ and $\mathcal{S}_2 \preceq_i \mathcal{S}_3$. We need to show that $\mathcal{S}_1 \preceq_i \mathcal{S}_3$. To simplify the notation in the proof, let $A = \chi(S_1), B = \chi(S_2), C = \chi(S_3), X = X_i$. Since $\mathcal{S}_1 \preceq_i \mathcal{S}_2$, we have

$$\begin{aligned} |A \cup B \cap X| &\leq |B| \\ |A| + |B \cap X| - |A \cap B \cap X| &\leq |B|, \end{aligned} \tag{1}$$

and since $\mathcal{S}_2 \preceq_i \mathcal{S}_3$ we have:

$$\begin{aligned} |B \cup C \cap X| &\leq |C| \\ |B| + |C \cap X| - |B \cap C \cap X| &\leq |C|. \end{aligned} \tag{2}$$

From Inequalities (1) and (2) we get:

$$\begin{aligned} |A| + |B \cap X| + |C \cap X| - |A \cap B \cap X| - |B \cap C \cap X| &\leq |C| \\ |A| + |B \cap X| + |C \cap X| - (|A \cap B \cap X| + |B \cap C \cap X| - |A \cap B \cap C \cap X| + |A \cap B \cap C \cap X|) &\leq |C| \\ |A| + |B \cap X| + |C \cap X| - (|A \cap B \cap X \cup B \cap C \cap X| + |A \cap B \cap C \cap X|) &\leq |C| \\ |A| + |B \cap X| + |C \cap X| - (|(A \cup C) \cap (B \cap X)| + |A \cap B \cap C \cap X|) &\leq |C| \\ |A| + |B \cap X| + |C \cap X| - (|B \cap X| + |A \cap B \cap C \cap X|) &\leq |C| \\ |A| + |C \cap X| - |A \cap B \cap C \cap X| &\leq |C| \\ |A| + |C \cap X| - |A \cap C \cap X| &\leq |C|. \end{aligned}$$

The last inequality proves that $\mathcal{S}_1 \preceq_i \mathcal{S}_3$.

□

Using the relation $\preceq_i$ on the set of sequences that conform to (X_i, π), we are now ready to define the key notion of representative sets that makes the dynamic programming approach work:

Definition 4.6. Let X_i be a bag and $\pi = (v_1, \sigma_1, v_2 \ldots, \sigma_{r-1}, v_r)$ a pattern for X_i. A set $\mathcal{R}_\pi$ of sequences of *paths*[2] that conform to (X_i, π) is a *representative* set for (X_i, π) if:

(i) For every sequence $\mathcal{S}_1 \in \mathcal{R}_\pi$, and for every sequence $\mathcal{S}_2 \neq \mathcal{S}_1$ that conforms to (X_i, π), if $\mathcal{S}_1 \preceq_i \mathcal{S}_2$ then $\mathcal{S}_2 \notin \mathcal{R}_\pi$;
(ii) for every sequence $\mathcal{S} \in \mathcal{R}_\pi$, and for every path $P \in \mathcal{S}$ between v_j and v_{j+1}, $j \in [r-1]$, there does not exist a v_j-v_{j+1} path P' in $G^i_{v_j v_{j+1}}$ such that $\chi(P') \subsetneq \chi(P)$; and
(iii) for every sequence of paths $\mathcal{S} \notin \mathcal{R}_\pi$ that conforms to (X_i, π) and satisfies that no two paths in $\mathcal{S}$ share a vertex that is not in X_i, there is a sequence $\mathcal{W} \in \mathcal{R}_\pi$ such that $\mathcal{W} \preceq_i \mathcal{S}$.

Observation 4.7. *Let X_i and X_j be two bags such that $X_i \subseteq X_j$, let π be a pattern for both X_i and X_j, and let $\mathcal{S}, \mathcal{S}'$ be two sequences of walks that conform to both (X_i, π) and (X_j, π). If $\mathcal{S} \preceq_j \mathcal{S}'$ then $\mathcal{S} \preceq_i \mathcal{S}'$.*

[2]Note that in the definition of representative sets, we consider sequences of paths as opposed to sequences of walks. During the computation, the dynamic programming algorithm computes a sequence of walks that conforms to (X_i, π), but then it refines it into a sequence of paths that represents (X_i, π).

Proof. Since $X_i \subseteq X_j$, we have $|\chi(\mathcal{S}) \cup \chi(\mathcal{S}') \cap \chi(X_i)| \leq |\chi(\mathcal{S}) \cup \chi(\mathcal{S}') \cap \chi(X_j)|$. □

Lemma 4.8. *Let X_i be a bag, π a pattern for X_i, and $\mathcal{S}_1, \mathcal{S}_1', \mathcal{S}_2, \mathcal{S}_2', \mathcal{S}, \mathcal{S}'$ sequences of walks that conform to (X_i, π) and that satisfy the following: $\mathcal{S}_1' \preceq_i \mathcal{S}_1$, $\mathcal{S}_2' \preceq_i \mathcal{S}_2$, $\chi(\mathcal{S}_1) \cup \chi(\mathcal{S}_2) = \chi(\mathcal{S})$, $\chi(\mathcal{S}_1') \cup \chi(\mathcal{S}_2') = \chi(\mathcal{S}')$, and $\chi(\mathcal{S}_1) \cap \chi(\mathcal{S}_2) \subseteq \chi(X_i)$. Then $\mathcal{S}' \preceq_i \mathcal{S}$.*

Proof. Let $A = \chi(\mathcal{S}_1), B = \chi(\mathcal{S}_2), C = \chi(\mathcal{S}), A' = \chi(\mathcal{S}_1'), B' = \chi(\mathcal{S}_2'), C' = \chi(\mathcal{S}')$, and $X = \chi(X_i)$. Since $\mathcal{S}_1' \preceq_i \mathcal{S}_1$ we have:

$$\begin{aligned} |A' \cup A \cap X| &\leq |A| \\ |A'| + |A \cap X| - |A' \cap A \cap X| &\leq |A|. \end{aligned} \tag{3}$$

Since $\mathcal{S}_2' \preceq_i \mathcal{S}_2$ we have:

$$\begin{aligned} |B' \cup B \cap X| &\leq |B| \\ |B'| + |B \cap X| - |B' \cap B \cap X| &\leq |B|. \end{aligned} \tag{4}$$

Adding Inequality (3) to (4) and subtracting $|A \cap B|$ from each side of the resulting inequality, we obtain:

$$|A'| + |B'| + |A \cap X| + |B \cap X| - |A' \cap A \cap X| - |B' \cap B \cap X| - |A \cap B| \leq |A \cup B|. \tag{5}$$

Replacing in the last Inequality (5) $|A'| + |B'|$ by $|A' \cup B'| + |A' \cap B'|$, and $|A \cap X| + |B \cap X|$ by $|(A \cup B) \cap X| + |A \cap B \cap X|$, observing that $A \cap B \cap X = A \cap B$ (because $A \cap B \subseteq X$), and simplifying, we get:

$$\begin{aligned} |A' \cup B'| + |(A \cup B) \cap X| + |A' \cap B'| - |A' \cap A \cap X| - |B' \cap B \cap X| &\leq |A \cup B| \\ |(A' \cup B') \cup (A \cup B) \cap X| + |(A' \cup B') \cap (A \cup B) \cap X| + |A' \cap B'| - |A' \cap A \cap X| - |B' \cap B \cap X| &\leq |A \cup B|. \end{aligned}$$

Replacing $-|A' \cap A \cap X| - |B' \cap B \cap X|$ in the last inequality with $-(|(A' \cap A \cup B' \cap B) \cap X| + |A' \cap A \cap B' \cap B \cap X|) = -(|(A' \cap A \cup B' \cap B) \cap X| + |A' \cap A \cap B' \cap B|)$ (because $A \cap B \subseteq X$), and observing that $|(A' \cap A \cup B' \cap B) \cap X| \leq |(A' \cup B') \cap (A \cup B) \cap X|$, and $|A' \cap A \cap B' \cap B| \leq |A' \cap B'|$, we conclude that:

$$|(A' \cup B') \cup (A \cup B) \cap X| \leq |A \cup B|. \tag{6}$$

Inequality (6) establishes that $\mathcal{S}' \preceq_i \mathcal{S}$. □

Lemma 4.9. *Let X_i be bag, π a pattern for X_i, and $\mathcal{R}_\pi$ be a representative set for (X_i, π). Then the number of sequences in $\mathcal{R}_\pi$ is at most $h(k)^{|X_i|^2}$, where $h(k) = O(c^{k^2} k^{2k^2+k})$, for some constant $c > 1$.*

Proof. Let $\pi = (v_1 = s, \sigma_1, v_2, \sigma_2, \ldots, \sigma_{r-1}, v_r = t)$ and let v_j and v_{j+1} be two consecutive vertices in π such that $\sigma_j = 1$. For each $j \in [r-1]$ such that $\sigma_j = 1$, let $\mathcal{P}_j$ be a minimal set of k-valid v_j-v_{j+1} paths w.r.t. X_i. Without loss of generality, we can pick $\mathcal{P}_j$ such that there is no k-valid u-v path P in $G^i_{v_j v_{j+1}}$ such that $\mathcal{P}_j \cup \{P\}$ is minimal w.r.t. X_i. From Lemma 4.2, it follows that $|\mathcal{P}_j| \leq h(k)^{|X_i|}$, where $h(k) = O(c^{k^2} k^{2k^2+k})$, for some constant $c > 1$. For a sequence $\mathcal{S} = (P_1, \ldots, P_{r-1})$ in $\mathcal{R}_\pi$, we define the *signature* of $\mathcal{S}$ (w.r.t. X_i) to be the tuple $(\chi(P_1) \cap \chi(X_i), \ldots, \chi(P_{r-1}) \cap \chi(X_i))$. Observe that if $\mathcal{S}_1$ and $\mathcal{S}_2$ have the same signature w.r.t. X_i, then $\chi(\mathcal{S}_1) \cup (\chi(\mathcal{S}_2) \cap \chi(X_i)) = \chi(\mathcal{S}_1)$ and $\chi(\mathcal{S}_2) \cup (\chi(\mathcal{S}_1) \cap \chi(X_i)) = \chi(\mathcal{S}_2)$; hence, either $\mathcal{S}_1 \preceq_i \mathcal{S}_2$ or $\mathcal{S}_2 \preceq_i \mathcal{S}_1$. It follows from property (i) of representative sets that no two sequences in $\mathcal{R}_\pi$ have the same signature w.r.t. X_i. Now let $\mathcal{S} = (P_1, \ldots, P_{r-1})$ be a sequence in $\mathcal{R}_\pi$ with a signature $(C_1, \ldots, C_{r-1})$. Note that if $C_j \neq \emptyset$, then P_j

is not the empty path, and hence $\sigma_j = 1$. We show that for each $j \in [r-1]$ such that $C_j \neq \emptyset$, there is a path $P \in \mathcal{P}_j$ such that $\chi(P) \cap \chi(X_i) = C_j$. Suppose, for a contradiction, that this is not the case. Then for some $j \in [r-1]$ such that $C_j \neq \emptyset$, there is no path $P \in \mathcal{P}_j$ such that $\chi(P) \cap \chi(X_i) = C_j$. Clearly, $P_j \notin \mathcal{P}_j$, and therefore, by our choice of $\mathcal{P}_j$, the set $\mathcal{P}_j \cup \{P_j\}$ is not a minimal set w.r.t. X_i. By assumption, $\mathcal{P}_j \cup \{P_j\}$ does not contradict property (i) in the definition of minimal set of paths w.r.t. X_i. Moreover, since $\mathcal{S} \in \mathcal{R}_\pi$, it follows from property (ii) of representative sets that P_j, and hence $\mathcal{P}_j \cup \{P_j\}$, satisfies property (iii) of minimal set of paths w.r.t. X_i. Therefore, $\mathcal{P}_j \cup \{P_j\}$ has to contradict property (ii) in the definition of minimal set of paths w.r.t. X_i, and there are two paths $Q_1, Q_2 \in \mathcal{P}_j \cup \{P_j\}$ such that $\chi(Q_1) \subseteq \chi(Q_2)$. However, if $\chi(Q_1) = \chi(Q_2)$, then Q_1 and Q_2 contradict property (i) of a minimal set of paths w.r.t. X_i, and if $\chi(Q_1) \subsetneq \chi(Q_2)$, then Q_2 contradicts property (iii), and we already established that $\mathcal{P}_j \cup \{P_j\}$ satisfies properties (i) and (iii). Therefore, $\mathcal{P}_j \cup \{P_j\}$ is a set of minimal paths w.r.t. X_i, which is a contradiction. We conclude that, for each $j \in [r-1]$ such that $C_j \neq \emptyset$, there is a path $P \in \mathcal{P}_j$ such that $\chi(P) \cap \chi(X_i) = C_j$. It follows that the number of signatures of paths in $\mathcal{P}_{uv}$ is at most $\prod_{j=1}^{r-1} |\mathcal{P}_j| \leq h(k)^{|X_i|^2}$. Since no two distinct sequences in $\mathcal{R}_\pi$ have the same signature, it follows that $|\mathcal{R}_\pi| \leq h(k)^{|X_i|^2}$. □

For two vertices $u, v \in X_i$ and two u-v walks W, W' in G^i_{uv}, we say that W' *refines* W if $\chi(W') \subseteq \chi(W)$. For two sequences $\mathcal{S} = (W_1, \ldots, W_{r-1})$ and $\mathcal{S}' = (W'_1, \ldots, W'_{r-1})$ of walks that conform to (X_i, π), we say that $\mathcal{S}'$ *refines* $\mathcal{S}$ if each walk W'_j refines W_j, for $j \in [r-1]$.

Lemma 4.10. *Let X_i be a bag, $\pi = (v_1 = s, \sigma_1, v_2, \sigma_2, \ldots, \sigma_{r-1}, v_r = t)$ a pattern for X_i, and $\mathcal{W} = (W_1, \ldots, W_{r-1})$ a sequence of walks, where each W_j is a walk between vertices v_j and v_{j+1} in $G^i_{v_j v_{j+1}}$ satisfying $\chi(W_j) \leq k$. Then in time $O^*(2^k)$ we can compute a sequence $\mathcal{S} = (P_1, \ldots, P_{r-1})$ of induced paths, where each P_j is an induced path between vertices v_j and v_{j+1} in $G^i_{v_j v_{j+1}}$ such that $\chi(P_j) \subseteq \chi(W_j)$, for $j \in [r-1]$, and such that $\mathcal{S}$ satisfies property (ii) of representative sets.*

Proof. For each walk W_j, $j \in [r-1]$, we do the following. For each subset $C' \subseteq \chi(W_j)$ considered in a nondecreasing order of cardinality, we form the subgraph G' from $G^i_{v_j v_{j+1}}$ by removing every vertex x in $G^i_{v_j v_{j+1}}$ that does not satisfy $\chi(x) \subseteq C'$. We then check if there is a v_j-v_{j+1} induced path in G', and set P_j to this path if it exists. It is clear that the path P_j satisfies $\chi(P_j) \subseteq \chi(W_j)$ and that the sequence $\mathcal{S}' = (P_1, \ldots, P_{r-1})$ conforms to π w.r.t. X_i and satisfies property (ii) of representative sets. Since each W_j satisfies $\chi(W_j) \leq k$, we can enumerate all subsets of $\chi(W_j)$ in time $O^*(2^k)$. Since checking if there is an induced v_j-v_{j+1} path in G' takes polynomial time, it follows that computing P_j from W_j takes $O^*(2^k)$, and so does the computation of $\mathcal{S}$. □

For a bag X_i, pattern π for X_i, and a set of sequences $\mathcal{R}$ that conform to (X_i, π), we define the procedure **Refine()**, given below, that takes a set $\mathcal{W}$ of sequences of walks and outputs a set

$\mathcal{R}'$ of sequences of paths that conform to (X_i, π), and does not violate properties (i) and (ii) of representative sets:

ALGORITHM 1: The procedure **Refine()**

Data: A set $\mathcal{W}$ of sequences of walks
Result: a set $\mathcal{R}'$ of sequences of paths
1. For each sequence $\mathcal{S}$ in $\mathcal{W}$, compute a sequence $\mathcal{S}'$ that refines $\mathcal{S}$ and satisfies property (ii) of representative sets (using Lemma 4.10), and replace $\mathcal{S}$ with $\mathcal{S}'$ in $\mathcal{W}$;
2. $\mathcal{R}' := \emptyset$;
3. Order the sequences in $\mathcal{W}$ w.r.t. the relation $\preceq_i$, where ties are broken arbitrarily[3];
4. **for** *each sequence* $\mathcal{S}_p$ *in* $\mathcal{W}$ *considered in order* **do**
    Add $\mathcal{S}_p$ to $\mathcal{R}'$ if there is no sequence $\mathcal{S}$ already in $\mathcal{R}'$ such that $\mathcal{S} \preceq_i \mathcal{S}_p$;
end
5. Output $\mathcal{R}'$.

Lemma 4.11. *Let X_i be a bag, $\pi = (v_1 = s, \sigma_1, v_2, \sigma_2, \ldots, \sigma_{r-1}, v_r = t)$ a pattern for X_i, and $\mathcal{W} = (W_1, \ldots, W_{r-1})$ a sequence of walks, where each W_j, for $j \in [r-1]$, is a walk between vertices v_j and v_{j+1} in $G^i_{v_j v_{j+1}}$ satisfying $\chi(W_j) \leq k$. The procedure* **Refine()** *on input $\mathcal{W}$ produces a set $\mathcal{R}'$ of sequences of paths that conform to (X_i, π) satisfying properties (i) and (ii) of representative sets, and such that, for each sequence $\mathcal{S} \in \mathcal{W}$, there is a sequence $\mathcal{S}' \in \mathcal{R}'$ satisfying $\mathcal{S}' \preceq_i \mathcal{S}$. Moreover, the procedure runs in time $O^*(2^k|\mathcal{W}| + |\mathcal{W}|^2)$.*

Proof. From Lemma 4.10, it follows that the step 1 can be implemented in overall time of $O^*(2^k|\mathcal{W}|)$ and afterwards, every sequence in $\mathcal{W}$ satisfies property (ii) of representative sets. To prove that property (i) of representative sets is satisfied, assume to get a contradiction that there are two sequences $\mathcal{S}_1$ and $\mathcal{S}_2$ in $\mathcal{W}$ such that $\mathcal{S}_1 \preceq_i \mathcal{S}_2$ and $\mathcal{S}_1$ and $\mathcal{S}_2$ are both in $\mathcal{R}'$. From step 4 in the algorithm, since $\mathcal{S}_1 \preceq_i \mathcal{S}_2$, $\mathcal{S}_2$ must appear before $\mathcal{S}_1$ in the ordering, which implies that $\mathcal{S}_2 \preceq_i \mathcal{S}_1$. This is a contradiction, since in this case $\mathcal{S}_1$ would not be added to $\mathcal{R}'$. Now if a sequence $\mathcal{S}$ is not added to $\mathcal{R}'$, then from step 4 in the algorithm, $\mathcal{R}'$ already contains a sequence $\mathcal{S}'$ such that $\mathcal{S}' \preceq_i \mathcal{S}$. Finally, it is easy to see that we can implement step 4 in $O^*(|\mathcal{W}|^2)$ time, and the lemma follows. □

4.2 The dynamic programming algorithm

For each bag X_i, the algorithm maintains a table Γ_i that contains, for each pattern π for X_i, a representative set $\Gamma_i[\pi]$ for (X_i, π). We describe next how to update the table stored at a bag X_i, based on the tables stored at its children in $\mathcal{T}$. We distinguish the following cases based on the type of bag X_i.

If a bag X_i is a leaf in $\mathcal{T}$, then $X_i = V_i = \{s, t\}$, and there are only two patterns $(s, 0, t)$ and $(s, 1, t)$ for X_i. Clearly, the only sequence that conforms to $(s, 0, t)$ is the sequence $(())$ containing exactly one empty path. Moreover, there is no edge $st \in E(G)$. Therefore, there is no sequence that conforms to $(s, 1, t)$, and the following claim holds:

Claim 2. *If a bag X_i is a leaf in $\mathcal{T}$, then $\Gamma_i = \{((s, 0, t), \{(())\}), ((s, 1, t), \emptyset)\}$ contains, for each pattern for X_i, a representative set for (X_i, π).*

Case 1. X_i is an introduce node with child X_j. Let $X_i = X_j \cup \{v\}$.

For every pattern π for X_i that does not contain v, we set $\Gamma_i[\pi] = \Gamma_j[\pi]$.

[3]That is, order the sequences in $\mathcal{W}$ so that if two sequences $\mathcal{S}_p$, $\mathcal{S}_q$ are such that $\mathcal{S}_p \preceq_i \mathcal{S}_q$ but $\mathcal{S}_q \not\preceq_i \mathcal{S}_p$ then $\mathcal{S}_p$ appears before $\mathcal{S}_q$ in the ordering.

Now let $\pi = (v_1 = s, \sigma_1, v_2, \sigma_2, \ldots, \sigma_{r-1}, v_r = t)$ be a pattern such that $v_q = v$, $q \in \{2, \ldots, r-1\}$, and let $\pi' = (v_1, \sigma_1, \ldots v_{q-1}, 0, v_{q+1}, \sigma_{q+1}, \ldots, \sigma_{r-1}, v_r)$. Note that since X_j is a separator between v and V_j, the only possibility for a path from v to a different vertex in X_i to have all internal vertices in $V_i \setminus X_i$ is if it is a direct edge. Therefore, if $\sigma_{q-1} = 1$ (resp. $\sigma_q = 1$) then $v_{q-1}v$ (resp. $v_q v$) is an edge in G. Otherwise, there is no sequence that conforms to (X_i, π).

We obtain $\Gamma_i[\pi]$ from $\Gamma_j[\pi']$ as follows. For every $\mathcal{S}' = (P'_1, P'_2, \ldots, P'_{r-2}) \in \Gamma_j[\pi']$, we replace the empty path corresponding to 0 between v_{q-1} and v_{q+1} in π' by two paths P_{q-1}, P_q such that $P_{q-1} = ()$ (resp. $P_q = ()$) if $\sigma_{q-1} = 0$ (resp. $\sigma_q = 0$) and $P_{q-1} = (v_{q-1}, v)$ (resp. $P_{q-1} = (v, v_q)$) otherwise and we obtain $\mathcal{S} = (P'_1, \ldots, P'_{q-2}, P_{q-1}, P_q, P'_q, \ldots, P'_{r-2})$. Denote by $\mathcal{R}_\pi$ the set of all formed sequences $\mathcal{S}$. Finally, we set $\Gamma_i[\pi] = \textbf{Refine}(\mathcal{R}_\pi)$. We claim that $\Gamma_i[\pi]$ is a representative set for (X_i, π).

Claim 3. *If X_i is an introduce node with child X_j, and Γ_j contains for each pattern π' for X_j a representative set for (X_j, π'), then $\Gamma_i[\pi]$ defined above is a representative set for (X_i, π).*

Proof. If the pattern π does not contain v, then $\Gamma_j[\pi]$ is a representative set for (X_i, π) for the following reasons: (i) follows because every color in $\chi(X_i) \setminus \chi(X_j)$ does not appear in V_j, since X_j is a vertex-separator in G separating v and V_j and colors are connected. Hence, if two sequences in $\Gamma_j[\pi]$ that conform to (X_i, π) contradicted (i), then they would contradict (i) w.r.t. (X_j, π) as well, but we have that $\Gamma_j[\pi]$ is a representative set for (X_j, π). Furthermore, since v does not appear on any path between two vertices in π having internal vertices in $V_i \setminus X_i$, properties (ii) and (iii) are inherited from the child node X_j.

For the rest of the proof, we assume that the pattern π contains v. It is clear from the application of **Refine()** that $\Gamma_i[\pi]$ does not contradict properties (i) and (ii) of the definition of representative sets. Assume now that there exists a sequence $\mathcal{S} \notin \Gamma_i[\pi]$ that conforms to (X_i, π) such that $\mathcal{S}$ violates property (iii). We define the sequence $\mathcal{S}'$ that conforms to π' and contains all the paths in $\mathcal{S}$ whose endpoints do not contain v. Since no two paths in $\mathcal{S}$ share a vertex that is not in X_i (since $\mathcal{S}$ violates (iii)), and all paths in $\mathcal{S}'$ are also in $\mathcal{S}$, it follows that no two paths in $\mathcal{S}'$ share a vertex that is not in X_j. Since $\Gamma_j[\pi']$ is a representative set for (X_j, π'), it follows that there exists $\mathcal{S}'_1 \in \Gamma_j[\pi']$ such that $\mathcal{S}'_1 \preceq_j \mathcal{S}'$. Let $\mathcal{S}_1$ be the sequence obtained from $\mathcal{S}'_1$, in the manner described above Claim 3, and conforming to (X_i, π). Then $\mathcal{S}_1 \in \mathcal{R}_\pi$, and hence by Lemma 4.11, there is a sequence $\mathcal{S}_2 \in \Gamma_i[\pi]$ such that $\mathcal{S}_2 \preceq_i \mathcal{S}_1$. Since both $\mathcal{S}_1$ and $\mathcal{S}$ contain v, we have $\mathcal{S}_1 \preceq_i \mathcal{S}$. By transitivity of $\preceq_i$ (Lemma 4.5), it follows that $\mathcal{S}_2 \preceq_i \mathcal{S}$. This contradicts the assumption that $\mathcal{S}$ violates property (iii). □

Case 2. X_i is a forget node with child X_j. Let $X_i = X_j \setminus \{v\}$.

Let $\pi = (s = v_1, \sigma_1, \ldots, \sigma_{r-1}, v_r = t)$ be a pattern for the vertices in X_i. For $q \in [r-1]$, such that $\sigma_q = 1$, we define $\pi^q = (s = v'_1, \sigma'_1, \ldots, \sigma'_r, v'_{r+1} = t)$ to be the pattern obtained from π by inserting v between v_q and v_{q+1} and setting $\sigma'_q = \sigma'_{q+1} = 1$. More precisely, we set $v'_p = v_p$ and $\sigma'_p = \sigma_p$ for $1 \le p \le q$, $v'_{q+1} = v$ and $\sigma'_{q+1} = 1$, and finally $v'_p = v_{p-1}$ and $\sigma'_p = \sigma_{p-1}$ for $q+2 \le p \le r$. We define $\mathcal{R}_\pi$ as follows:

$$\mathcal{R}_\pi = \Gamma_j[\pi] \cup \{\mathcal{S} = (P_1, \ldots, P_{q-1}, P_q \circ P_{q+1}, P_{q+2}, \ldots, P_r) \mid (P_1, \ldots, P_r) \in \Gamma_j[\pi^q], q \in [r-1] \wedge \sigma_q = 1\}.$$

Finally, we set $\Gamma_i[\pi] = \textbf{Refine}(\mathcal{R}_\pi)$ and we claim that $\Gamma_i[\pi]$ is a representative set for (X_i, π).

Claim 4. *If X_i is a forget node with child X_j, and Γ_j contains for each pattern π' for X_j a representative set for (X_j, π'), then $\Gamma_i[\pi]$ defined above is a representative set for (X_i, π).*

Proof. It is straightforward to see that $\Gamma_i[\pi]$ satisfies properties (i) and (ii) due to the way procedure **Refine()** works. Assume for a contradiction that there exists a sequence $\mathcal{S}$ that violates property (iii). We distinguish two cases.

First, suppose that no path in $\mathcal{S}$ contains the vertex v. Then this path conforms to the pattern π in X_j. Since no two paths in $\mathcal{S}$ share a vertex that is not in X_i, and since $\Gamma_j[\pi]$ is a representative set, there exists $\mathcal{S}_1 \in \Gamma_j[\pi]$ such that $\mathcal{S}_1 \preceq_j \mathcal{S}$. Then $\mathcal{S}_1 \in \mathcal{R}_\pi$, and hence by Lemma 4.11, $\Gamma_i[\pi]$ contains a sequence $\mathcal{S}_2$ such that $\mathcal{S}_2 \preceq_i \mathcal{S}_1$. Since $\mathcal{S}_1 \preceq_j \mathcal{S}$ and $X_i \subsetneq X_j$, it follows from Observation 4.7 that $\mathcal{S}_1 \preceq_i \mathcal{S}$. By transitivity of $\preceq_i$, it follows that $\mathcal{S}_2 \preceq_i \mathcal{S}$, which is a contradiction to the assumption that $\mathcal{S}$ violates property (iii).

Second, suppose that there is a path P_q in $\mathcal{S}$ between v_q and v_{q+1} that contains v. We form a sequence $\mathcal{S}'$ from $\mathcal{S}$ by keeping every path $P \neq P_q$ in $\mathcal{S}$, and replacing P_q in the sequence by the two subpaths of P_q, $P'_q = (v_q, \ldots, v)$ and $P'_{q+1} = (v, \ldots, v_{q+1})$. The sequence $\mathcal{S}'$ conforms to (X_j, π^q), and since no two paths in $\mathcal{S}$ share a vertex that is not in X_i, no two paths in $\mathcal{S}'$ share a vertex that is not in X_j. Since $\Gamma_j[\pi^q]$ is a representative set for (X_j, π^q), it follows that there exists a sequence $\mathcal{S}'_1 \in \Gamma_j[\pi^q]$ such that $\mathcal{S}'_1 \preceq_j \mathcal{S}'$. Let $\mathcal{S}_1$ be the sequence conforming to (X_i, π) obtained from $\mathcal{S}'_1$ by applying the operation $\circ$ to the two paths in $\mathcal{S}'_1$ that share v. Then $\mathcal{S}_1 \in \mathcal{R}_\pi$. Therefore, by Lemma 4.11, $\Gamma_i[\pi]$ contains a sequence $\mathcal{S}_2$ such that $\mathcal{S}_2 \preceq_i \mathcal{S}_1$. Since $\mathcal{S}'_1 \preceq_j \mathcal{S}'$, $\chi(\mathcal{S}') = \chi(\mathcal{S})$, $\chi(\mathcal{S}'_1) = \chi(\mathcal{S}_1)$, and $X_i \subsetneq X_j$, it follows that $\mathcal{S}_1 \preceq_i \mathcal{S}$. By transitivity of $\preceq_i$, it follows that $\mathcal{S}_2 \preceq_i \mathcal{S}$, which is a contradiction to the assumption that $\mathcal{S}$ violates property (iii). ◻

Case 3. X_i is a join node with children $X_j, X_{j'}$.

Let $\pi = (s = v_1, \sigma_1, \ldots, \sigma_{r-1}, v_r = t)$ be a pattern for X_i. Initialize $\mathcal{R}_\pi = \emptyset$. For every two patterns $\pi_1 = (s = v_1, \tau_1, \ldots, \tau_{r-1}, v_r = t)$ and $\pi_2 = (s = v_1, \mu_1, \ldots, \mu_{r-1}, v_r = t)$ such that $\sigma_q = \tau_q + \mu_q$, and for every two sequences $\mathcal{S}_1 = (P_1^1, \ldots, P_1^{r-1}) \in \Gamma_j[\pi_1]$ and $\mathcal{S}_2 = (P_2^1, \ldots, P_2^{r-1}) \in \Gamma_{j'}[\pi_2]$, we add the sequence $\mathcal{S} = (P_1, \ldots, P_{r-1})$ to $\mathcal{R}_\pi$, where $P_q = P_1^q$ if P_2^q is the empty path, otherwise, $P_q = P_2^q$, for $q \in [r-1]$. We set $\Gamma_i[\pi] = \mathbf{Refine}(\mathcal{R}_\pi)$, and we claim that $\Gamma_i[\pi]$ is a representative set for (X_i, π).

Claim 5. *If X_i is a join node with children $X_j, X_{j'}$, and Γ_j (resp. $\Gamma_{j'}$) contains for each pattern π' for $X_j = X_{j'} = X_i$ a representative set for (X_j, π') (resp. $(\pi', X_{j'})$), then $\Gamma_i[\pi]$ defined above is a representative set for (X_i, π).*

Proof. Clearly $\Gamma_i[\pi]$ satisfies properties (i) and (ii) due to the application of the procedure **Refine()**. To argue that $\Gamma_i[\pi]$ satisfies property (iii), suppose not, and let $\mathcal{S} = (P_1, \ldots, P_{r-1})$ be a sequence that violates property (iii). Notice that every path P_q, $q \in [r-1]$, is either an edge between two vertices in X_i, or a path between two vertices in X_i such that its internal vertices are either all in $V_j \setminus X_i$ or in $V_{j'} \setminus X_i$; this is true because X_i is a vertex-separator separating $V_j \setminus X_i$ from $V_{j'} \setminus X_i$ in G. Define the two sequences $\mathcal{S}_1 = (P_1^1, \ldots, P_1^{r-1})$ and $\mathcal{S}_2 = (P_2^1, \ldots, P_2^{r-1})$ as follows. For $q \in [r-1]$, if P_q is empty then set both P_1^q and P_2^q to the empty path; if P_q is an edge then set $P_1^q = P_q$ and P_2^q to the empty path. Otherwise, P_q is either a path in $G[V_j]$ or in $G[V_{j'}]$; in the former case set $P_1^q = P_q$ and P_2^q to the empty path, and in the latter case set $P_2^q = P_q$ and P_1^q to the empty path. Since no two paths in $\mathcal{S}$ share a vertex that is not in X_i, and $X_i = X_j = X_{j'}$, no two paths in $\mathcal{S}_1$ (resp. $\mathcal{S}_2$) share a vertex that is not in X_j (resp. $X_{j'}$). Let $\pi_1 = (s = v_1, \tau_1, \ldots, \tau_{r-1}, v_r = t)$ and $\pi_2 = (s = v_1, \mu_1, \ldots, \mu_{r-1}, v_r = t)$ be the two patterns that $\mathcal{S}_1$ and $\mathcal{S}_2$ conform to, respectively, and observe that, for every $q \in [r-1]$, we have $\sigma_q = \tau_q + \mu_q$. Since $\Gamma_j[\pi_1]$ and $\Gamma_{j'}[\pi_2]$ are representative sets, it follows that there exist $\mathcal{S}'_1 = (Y'_1, \ldots, Y'_{r-1})$ in $\Gamma_j[\pi_1]$ and $\mathcal{S}'_2 = (Z'_1, \ldots, Z'_{r-1})$ in $\Gamma_{j'}[\pi_2]$ such that $\mathcal{S}'_1 \preceq_j \mathcal{S}_1$ and $\mathcal{S}'_2 \preceq_{j'} \mathcal{S}_2$. Let $\mathcal{S}' = (P'_1, \ldots, P'_{r-1})$, where $P'_q = Y'_q$ if Z'_q is the empty path, otherwise, $P_q = Z'_q$, for $q \in [r-1]$. The sequence $\mathcal{S}'$ conforms to π and is in $\mathcal{R}_\pi$. By Lemma 4.11, $\Gamma_i[\pi]$ contains a sequence $\mathcal{S}''$ such that $\mathcal{S}'' \preceq_i \mathcal{S}'$. From Observation 4.7, since $X_i = X_j = X_{j'}$, from

$\mathcal{S}'_1 \preceq_j \mathcal{S}_1$ and $\mathcal{S}'_2 \preceq_{j'} \mathcal{S}_2$ it follows that $\mathcal{S}'_1 \preceq_i \mathcal{S}_1$ and $\mathcal{S}'_2 \preceq_i \mathcal{S}_2$. Since $\chi(\mathcal{S}_1) \cup \chi(\mathcal{S}_2) = \chi(\mathcal{S})$ and $\chi(\mathcal{S}'_1) \cup \chi(\mathcal{S}'_2) = \chi(\mathcal{S}')$, and since $\chi(\mathcal{S}_1) \cap \chi(\mathcal{S}_2) \subseteq \chi(X_i)$, by Lemma 4.8, it follows that $\mathcal{S}' \preceq_i \mathcal{S}$. Since $\mathcal{S}'' \preceq_i \mathcal{S}'$, by transitivity of $\preceq_i$, it follows that $\mathcal{S}'' \preceq_i \mathcal{S}$, which concludes the proof. □

We can now conclude with the following theorem:

Theorem 4.12. *There is an algorithm that on input (G, C, χ, s, t, k) of* Colored Path-Con, *either outputs a k-valid s-t path in G or decides that no such path exists, in time $O^\star(f(k)^{6\omega^2})$, where ω is the treewidth of G and $f(k) = O(c^{k^2} k^{2k^2+k})$, for some constant $c > 1$. Therefore,* Colored Path-Con *parameterized by both k and the treewidth of the input graph is in* FPT.

Proof. First, in time $O(|V(G)|^4)$, we can compute a branch decomposition of G, and hence a tree decomposition, of width at most $3\omega/2$, where ω is the treewidth of G [23, 30, 31]. From this tree decomposition, in polynomial time we can compute a nice tree decomposition $(\mathcal{V}, \mathcal{T})$ of G whose width is at most $3\omega/2$ and satisfying $|\mathcal{V}| = O(|V(G)|)$ [26]. The algorithm starts by removing the colors of s and t from G, and decrements k by $|\chi(s) \cup \chi(t)|$ (see Assumption 2.2). Afterwards, if $k < 0$, the algorithm concludes that there is no k-valid s-t path in G. If $st \in E(G)$ and $k \geq 0$, the algorithm outputs the path (s, t). Now we know that s and t are not adjacent, and that $\chi(s) = \chi(t) = \emptyset$. The algorithm then adds s and t to every bag in $\mathcal{T}$, and executes the dynamic programming algorithm based on $(\mathcal{V}, \mathcal{T})$, described in this section, to compute a table Γ_i for each bag X_i in $\mathcal{T}$, that contains, for each pattern π for X_i, a representative set $\mathcal{R}_\pi$ for (X_i, π).

From Claims 2, 3, 4, 5, it follows, by induction on the height of the tree-decomposition $(\mathcal{V}, \mathcal{T})$ (the base case corresponds to the leaves), that the root node X_r contains a representative set $\Gamma_r[\pi]$ for the sequence $\pi = (s, 1, t)$. If $\Gamma_r[\pi]$ is empty, the algorithm concludes that there is no k-valid s-t path in G. Otherwise, noting that there is only one sequence $\mathcal{S}$ in the representative set $\Gamma_r[\pi]$ since $X_r = \{s, t\}$ and s and t are empty, the algorithm outputs the k-valid s-t path P formed by $\mathcal{S}$. The correctness follows from the following argument, which shows that if there is a k-valid s-t path in G, then the algorithm outputs such a path. Suppose that P' is a k-valid induced s-t path such that there does not exist an s-t path P'' in G satisfying $\chi(P'') \subsetneq \chi(P')$, and let $\mathcal{S}' = (P')$. Since $G^r_{st} = G$, it follows that $\mathcal{S}'$ conforms to (X_r, π). Since $\mathcal{S}'$ contains exactly one path that is induced, no two paths in $\mathcal{S}'$ share a vertex. Therefore, by property (iii) of representative sets, there exists a sequence $\mathcal{S}$ in $\Gamma_r[\pi]$ satisfying $\mathcal{S} \preceq_r \mathcal{S}'$. Noting that a sequence in $\Gamma_r[\pi]$ must consist of a single k-valid s-t path, it follows that the algorithm correctly outputs such a path.

Next, we analyze the running time of the algorithm. We observe that among the three types of bags in $\mathcal{T}$, the worst running time is for a join bag. Therefore, it suffices to upper bound the running time for a join bag, and since $|\mathcal{V}| = O(n)$, the upper bound on the overall running time would follow.

Consider a join bag X_i with children $X_j, X_{j'}$. Let ω' be the width of $\mathcal{T}$ plus 1, which serves as an upper bound on the bag size in $\mathcal{T}$, and note that $\omega' \leq 3\omega/2 + 3$, where the (additional) plus 2 is to account for the vertices s and t that were added to each bag. The algorithm starts by enumerating each pattern π for X_i. The number of such patterns is at most $2^{\omega'} \cdot \omega' \cdot \omega'! = O^*(2^{\omega'} \cdot \omega'!)$, where $\omega' \cdot \omega'!$ is an upper bound on the number of ordered selections of a subset of vertices from the bag, and $2^{\omega'}$ is an upper bound on the number of combinations for the σ_i's in the selected pattern. Fix a pattern π for X_i. To compute $\Gamma_i[\pi]$, the algorithm enumerates all ways of partitioning π into pairs of patterns π_1, π_2 for the children bags; there are $2^{\omega'}$ ways of partitioning π into such pairs, because for each $\sigma_i = 1$ in π, the path between v_i and v_{i+1} is either reflected in π_1 or in π_2. For a fixed pair π_1, π_2, the algorithm iterates through all pairs of sequences in the two tables $\Gamma_j[\pi_1]$ and $\Gamma_{j'}[\pi_2]$. Since each table contains a representative set, by Lemma 4.9, the size of each table is $O(h_1(k)^{\omega'^2})$, where $h_1(k) = O(c_1^{k^2} k^{2k^2+k})$, for some constant $c_1 > 1$, and hence iterating over all pairs of sequences in the two tables can be done in $O(h_1(k)^{2\omega'^2})$ time. From the above, it follows that the set $\mathcal{R}_\pi$ can be computed in time $2^{\omega'} \cdot O(h_1(k)^{2\omega'^2}) = O(h_2(k)^{2\omega'^2})$, where $h_2(k) = O(c_2^{k^2} k^{2k^2+k})$, for some constant $c_2 > 1$, which is also an upper bound on the size of $\mathcal{R}_\pi$. By Lemma 4.11, applying **Refine()** to $\mathcal{R}_\pi$ takes time $O^*(2^k h_2(k)^{2\omega'^2} + h_2(k)^{4\omega'^2}) = O^*(h_3(k)^{4\omega'^2})$, where $h_3(k) = O(c_3^{k^2} k^{2k^2+k})$, for some constant $c_3 > 1$. It follows from all the above that the running time taken by the algorithm to compute Γ_i is $O^*(h_3(k)^{4\omega'^2} \cdot 2^{\omega'} \cdot \omega'!) = O^*(h_4(k)^{4\omega'^2})$, where $h_4(k) = O(c_4^{k^2} k^{2k^2+k})$, for some constant $c_4 > 1$, and hence the running time of the algorithm is $O^\star(f(k)^{6\omega^2})$, where $f(k) = O(c^{k^2} k^{2k^2+k})$, for some constant $c > 1$. □

5 EXTENSIONS AND APPLICATIONS

In this section, we extend the FPT results for Colored Path-Con w.r.t. the combined parameters k and ω—the treewidth of the input graph, to show that Colored Path-Con parameterized by both k and the length ℓ of the sought path is FPT. We also present applications of these FPT results. We formally define the problem Bounded-length Colored Path-Con:

Bounded-length Colored Path-Con

Given: A planar graph G; a set of colors C; $\chi : V \longrightarrow 2^C$; two designated vertices $s, t \in V(G)$; and $k, \ell \in \mathbb{N}$

Question: Does there exist a k-valid s-t path of length at most ℓ in G?

We first start by showing that if we parameterize only by one of ℓ, k then the problem is W[1]-hard.

Theorem 5.1. *Bounded-length Colored Path-Con is* W[1]*-hard parameterized by* k.

Proof. We reduce from the W[1]-hard problem Clique. The reduction is similar to that in the proof of Theorem 6.1. Let (G, k) be an instance of Clique, where $V(G) = \{v_1, \ldots, v_n\}$ and $E(G) = \{e_1, \ldots, e_m\}$. We assume that the edges in $E(G)$ are incident to at least $k + 3$ different vertices. This assumption is safe because Clique is trivially FPT for instances where the edges in $E(G)$ are incident to at most $k + 2$ different vertices. Similarly to the proof of Theorem 6.1, we start by describing the instance I of Connected Obstacle Removal whose associated graph is the desired instance of Bounded-length Colored Path-Con.

The regions of I are $O \cup \{Z_0, \ldots, Z_m\} \cup \bigcup_{i=1}^m \{O_i^1, O_i^2, O_i^3\}$, depicted in Figure 9. The obstacles of I are defined as follows. For each vertex $v_j \in V(G)$, the obstacle V_j corresponding to v_j is the polygon whose boundary is the boundary of the region formed by the union of O, and each O_i^3 such that v_j is incident to e_i. More formally, the obstacle corresponding to v_j is $\partial(O \cup \bigcup_{v_j \in e_i} O_i^3)$. Besides the obstacles corresponding to the vertices of G, there are two auxiliary obstacles $A_1 = \partial(O \cup \bigcup_{i \in [m]} O_i^1)$ and $A_2 = \partial(O \cup \bigcup_{i \in [m]} O_i^2)$. Finally, we place s in Z_0, t in Z_m, and ask whether there is a path from s to t that intersects at most $k + 2$ obstacles and visits at most $3m - \binom{k}{2} + 1$ regions (including Z_0 and Z_m).

Suppose that H is a complete subgraph of G with exactly k vertices. We define an s-t path P as follows. P starts at s in Z_0. If for $i \in [m]$, P enters Z_{i-1} and $e_i \in E(H)$, then P goes to Z_i through O_i^3; otherwise, P goes to Z_i through O_i^1 and O_i^2. Since P does not enter the region O, and enters region O_i^3 if and only if edge e_i is part of the clique H, P intersects an obstacle V_j if and only if $v_j \in V(H)$. Hence, together with A_1 and A_2, P intersects at most $k + 2$ obstacles. Moreover, P visits regions $Z_0, \ldots, Z_m$, regions O_i^3 for $e_i \in E(H)$ ($\binom{k}{2}$ times), and regions O_i^1, O_i^2 for $e_i \notin E(H)$ ($m - \binom{k}{2}$ times). Therefore, P visits exactly $3m - \binom{k}{2} + 1$ regions.

Conversely, suppose that there is an s-t path P that visits at most $3m - \binom{k}{2} + 1$ regions and intersects at most $k + 2$ obstacles. It is easy to see that P does not visit region O. Furthermore, by our assumption, the edges of G are incident to at least $k+3$ vertices, and hence P cannot intersect all the O_i^3's, for $i \in [m]$. Therefore, P visits O_i^1 and O_i^2 for some $i \in [m]$, and hence intersects A_1 and A_2. Furthermore, since P visits at most $3m - \binom{k}{2} + 1$ regions, P visits at least $\binom{k}{2}$ different O_i^3's. Since each O_i^3 contains a pair of two obstacles V_{j_1}, V_{j_2} such that $e_i = v_{j_1}v_{j_2}$, there are no multiple edges in G (and hence, no two O_i^3's have the same set of obstacles), and P intersects at most k obstacles that corresponds to vertices of G, it follows that P intersects exactly k obstacles that correspond to vertices, and visits exactly $\binom{k}{2}$ different O_i^3's. Moreover, for each pair of obstacles V_{j_1}, V_{j_2} that P intersects, there must be an O_i^3 that contains exactly these two obstacles. This means that there is an edge $v_{j_1}v_{j_2}$ in G for each such pair, and hence a k-clique in G. □

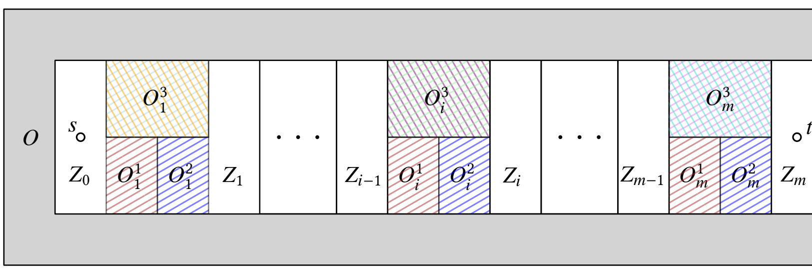


Fig. 9. Illustration for the proof of Theorem 5.1.

Theorem 5.2. Bounded-length Colored Path-Con *is* W[1]*-hard parameterized only by* ℓ.

Proof. The proof is similar to the proof of Lemma 6.5. We will describe the reduction from Multi-Colored Clique and point out the differences.

Let (G, k) be an instance of Multi-Colored Clique, where $V(G)$ is partitioned into the color classes $C_1, \ldots, C_k$. We assume that all color classes have the same cardinality N. Let $C_j = \{u_i^j \mid i \in [N]\}$. We describe how to construct an instance $(G', C', \chi', s, t, k', \ell)$ of Bounded-length Colored Path-Con. For an edge $e \in G$, associate a distinct color c_e, and define $C' = \{c_e \mid e \in E(G)\}$. Moreover, for convenience, we denote by $C'_{i,j}$ the set $\{c_e \mid e = uv \in E(G) \wedge u \in C_i \wedge v \in C_j\}$. To simplify the description of the construction, we start by defining a gadget that will serve as a building block for this construction.

For a vertex u_i^j in color class C_j, we define the gadget $G_{i,j}$, which is very similar to the one in Lemma 6.5, as follows. First, we create a copy of each color class $C_{j'}$, $j' \neq j$. Let the resulting copies of the color classes be $C'_1, \ldots, C'_{k-1}$. We define the color of a copy v' of a vertex $u_{i'}^{j'}$ as $\chi'(v') = C'_{j,j'} \setminus \{c_e\}$ if there is an edge $e = u_i^j u_{i'}^{j'}$, and define $\chi'(v') = C'_{j,j'}$ otherwise. Moreover, for each $i' \in [N-1]$ we connect the copies of vertices $u_{i'}^{j'}$ and $u_{i'+1}^{j'}$ by an edge.

Next, we introduce $k-2$ empty vertices y_r, $r \in [k-2]$. For $r \in [k-2]$, we connect all vertices in C'_r to y_r, and connect y_r to all vertices in C'_{r+1}. This completes the construction of gadget $G_{i,j}$; we refer to C'_1 and C'_{k-1} as the first and last color classes in gadget $G_{i,j}$, respectively. Observe that each color c_e for an edge $e = u_i^j u_{i'}^{j'}$ appears exactly on

- all vertices in the copies of $C_{j'}$ in the gadgets $G_{i^*,j}$ for $i^* \neq i$,
- all vertices in the copies of C_j in the gadgets $G_{i^*,j'}$ for $i^* \neq i'$,
- all vertices but the copy of $u_{i'}^{j'}$ in the copies of $C_{j'}$ in the gadget $G_{i,j}$, and
- all vertices but the copy of u_i^j in the copies of C_j in the gadget $G_{i',j'}$.

Furthermore, every path from a vertex in C'_1 to a vertex in C'_{k-1} of length at most $2k-2$ contains exactly one vertex from each C'_r, $r \in [k-1]$, and contains all vertices y_r, $r \in [k-2]$. Moreover, each such path contains all but at most one color from each $C_{j,j'}$, for all j' such that $j' \neq j$ and if the path does not contain a color $c_e \in C_{j,j'}$, then it contains a copy of a vertex $u_{i'}^{j'}$ such that there is an edge $e = u_i^j u_{i'}^{j'}$ in G.

We continue the construction similarly as in Lemma 6.5 by introducing $k+1$ new empty vertices $z_0, \ldots, z_k$, and connecting them as follows. For each color class C_j, $j \in [k]$, and each vertex $u_i^j \in C_j$, we create the gadget $G_{i,j}$, connect z_{j-1} to each vertex in the first color class of $G_{i,j}$, and connect each vertex in the last color class of $G_{i,j}$ to z_j. Let G' be the resulting graph. We now set $s = z_0$, $t = z_k$, $k' = |E(G)| - \binom{k}{2}$, and $\ell = 2k^2$. We are nearly finished with the reduction. However, we need to make the colors in G' connected. To achieve this, we first introduce a new vertex o such that $\chi(o) = C'$. Now for each $j \in [k]$, each $j' \neq j$, and each $i \in [N-1]$ we connect the copy of $u_N^{j'}$ in $G_{i,j}$ with the copy of $u_1^{j'}$ in $G_{i+1,j}$, and we connect the vertex o with the copies of $u_1^{j'}$ in $G_{1,j}$ and of $u_N^{j'}$ in $G_{N,j}$, respectively. It is not hard to see that every color is now connected. See Figure 10 for an illustration of the construction, and Figure 11, which highlights the subgraph induced by one color.

This completes the construction of the instance $(G', C', \chi', s, t, k', \ell)$ of Bounded-length Colored Path-Con.

Clearly, the reduction that takes an instance (G, k) of Multi-Colored Clique and produces the instance $(G', C', \chi', s, t, k', \ell)$ of Bounded-length Colored Path-Con is computable in FPT-time. To show its correctness, suppose that (G, k) is a yes-instance of Multi-Colored Clique, and let Q be a k-clique in G. Then Q contains a vertex from each C_j, for $j \in [k]$. For a vertex $u_i^j \in Q$, let $G_{i,j}$ be its gadget, and define the path P_j as follows. In each color class in $G_{i,j}$, pick the unique vertex that is a copy of a neighbor of u_i^j in Q; define P_j to be the path in $G_{i,j}$ induced by the picked vertices, plus the empty vertices y_r, $r \in [k-2]$, that appear in $G_{i,j}$. Finally, define P to be the s-t path in G' whose edges are: the (unique) edge between z_{r-1} and an endpoint of P_r, P_r, and the (unique) edge between an endpoint of P_r and z_r, for $r \in [k]$. Clearly, the length of P is exactly $(2k-2)k + 2k = 2k^2 = \ell$. To show that P is k'-valid, observe that none of the nonempty vertices in P contains a color of an edge between two vertices in Q. This shows that the number of colors that appear on P is at most $k' = |E(G)| - \binom{k}{2}$, and hence, P is k'-valid. It follows that $(G', C', \chi', s, t, k')$ is a yes-instance of Bounded-length Colored Path-Con.

Conversely, suppose that P is a k'-valid s-t path in G' of length at most $2k^2$. It is easy to see that P does not contain o, because $|\chi(o)| = |E(G)|$. Moreover, notice that the shortest path from s to t in $G - o$ has length $2k^2$ and each s-t path of length $2k^2$ must start at s, visit the gadgets of exactly k vertices $u_{i_j}^j \in C_j$, for $j \in [k]$, $i_j \in [N]$, and end at t. Furthermore, the subpath of the path in each gadget $G_{i,j}$ has length exactly $2k-2$. We claim that $Q = \{u_{i_j}^j \mid j \in [k]\}$ is a clique in G. Recall that the subpath of P that traverses a gadget $G_{i,j}$ contains all but at most one color from each $C_{j,j'}$, for all j' such that $j' \neq j$, and if such a path does not contain a color $c_e \in C_{j,j'}$, then it contains a copy of a vertex $u_{i'}^{j'}$ such that $e = u_i^j u_{i'}^{j'}$ is in G. It follows that if P does not contain the color c_e, for an edge $e = u_i^j u_{i'}^{j'}$, then it has to traverse the gadget for u_i^j and the gadget for $u_{i'}^{j'}$. Since P traverses at most k gadgets and it does not contain $\binom{k}{2}$ colors, it follows that there has to be an edge between every pair of vertices in G for which P traverses the corresponding gadgets, and Q is a clique in G. Since $|Q| = k$, it follows that Q is a k-clique in G, and that (G, k) is a yes-instance of Multi-Colored Clique. □

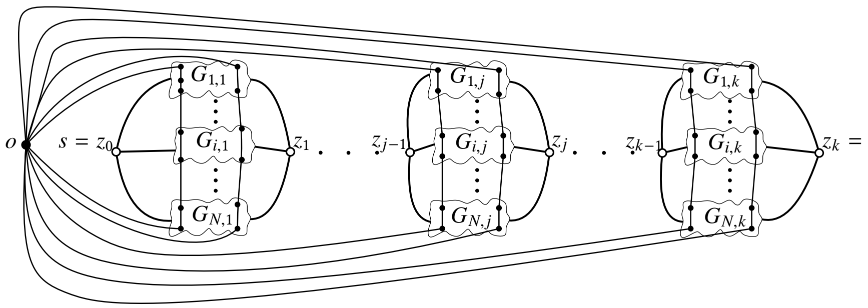


Fig. 10. Illustration of the reduced instance for the proof of Theorem 5.2.

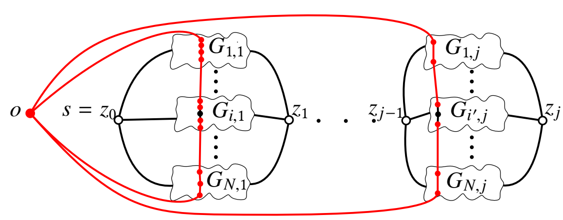


Fig. 11. Illustration for the proof of Theorem 5.2. The red highlights a color representing the edge between vertices u_i^1 in C_1 and $u_{i'}^j$ in C_j.

We now switch our attention to showing that Bounded-length Colored Path-Con parameterized by both k and ℓ is FPT. We start with the following lemma that enables us to upper bound the treewidth of the input graph by a function of the parameter ℓ:

Lemma 5.3. *Let $(G, C, \chi, s, t, k, \ell)$ be an instance of* Bounded-length Colored Path-Con*, and let v be a vertex in G such that $d_G(s, v) > \ell + 1$. Let G' be the graph obtained from G by contracting any edge uv that is incident to v, and let $\chi'(x) = \chi(u) \cup \chi(v)$, where x is the new vertex resulting from contracting uv, and $\chi'(w) = \chi(w)$ for every $w \in V(G) \setminus \{u, v\}$. Then $(G', C, \chi', s, t, k, \ell)$ is a yes-instance of* Bounded-length Colored Path-Con *if and only if $(G, C, \chi, s, t, k, \ell)$ is.*

Proof. Since G is color-connected and $\chi(x) = \chi(u) \cup \chi(v)$, it is easy to see that G' is color-connected as well. Because $d_G(s, v) > \ell + 1$, any solution to $(G, C, \chi, s, t, k, \ell)$ does not contain any of u, v, and hence, is a solution to $(G', C, \chi', s, t, k, \ell)$. Conversely, because $d_G(s, v) > \ell + 1$, any solution to $(G', C, \chi', s, t, k, \ell)$ does not contain x, and hence is a solution to $(G, C, \chi, s, t, k, \ell)$. □

By Lemma 5.3, we may assume w.l.o.g. that in an instance $(G, C, \chi, s, t, k, \ell)$ of Bounded-length Colored Path-Con, every vertex $v \in V(G)$ satisfies $d_G(s, v) \leq \ell + 1$. Therefore, we may assume that G has radius at most $\ell + 1$, and hence G has treewidth at most $3 \cdot (\ell + 1) + 1 = 3\ell + 4$ [30].

At this point we draw the following observation. Although the treewidth of G is bounded by a function of ℓ, we cannot use the FPT algorithm for Colored Path, parameterized by k and the treewidth of G, to solve Bounded-length Colored Path-Con because the k-valid path returned by the algorithm for Colored Path-Con may have length exceeding the desired upper bound ℓ. In fact, extending the FPT results for Colored Path to Bounded-length Colored Path-Con turns out to be a nontrivial task that necessitates a nontrivial extension of the structural results in Section 3, as well as the dynamic programming algorithm in Section 4. In particular, the color contraction operation, on which the structural results developed in Section 3 hinge, is no longer applicable since contracting an edge may decrease the distance between s and t in the resulting instance, and hence, may not result in an equivalent instance of the problem. However, we will show in the next subsection that we can extend the notion of a minimal set of k-valid paths between two vertices to incorporate the length of these paths, while still be able to upper bound the size of such a set by a function of both k and the length of these paths. We omit the proofs of some of the extended results that are very similar to those in the previous section to avoid repetition.

5.1 Extended Structural Results

We start with the following definition:

Definition 5.4. Let $u, v, w \in V(G)$, and let $\lambda \in [\ell]$. Let $\mathcal{P}$ be a set of k-valid u-v paths in $G-w$, each of length λ. The set $\mathcal{P}$ is said to be λ-*minimal* w.r.t. w if there does not exist two paths $P_1, P_2 \in \mathcal{P}$ such that $\chi(P_1) \cap \chi(w) = \chi(P_2) \cap \chi(w)$.

Let $u, v, w \in V(G)$, $\lambda \in [\ell]$, and let $\mathcal{P}$ be a set of λ-minimal k-valid u-v paths in $G - w$. Let $\mathcal{M}$ be a set of λ-minimal k-valid color-disjoint u-v paths in $G - w$. Let H be the subgraph of $G - w$ induced by the edges of the paths in $\mathcal{P}$, and let M be that induced by the edges of the paths in $\mathcal{M}$.

Lemma 5.5. *M has a u-v vertex-separator of cardinality at most $2(\lambda + 1)$.*

Proof. We proceed by contradiction, and assume that M does not have a u-v vertex-separator of cardinality at most $2(\lambda + 1)$. By Menger's theorem [11], there exists a set $\mathcal{D} = \{P_1, \ldots, P_r\}$, where $r \geq 2\lambda+3$, of vertex-disjoint u-v paths in M. Let $u_1, \ldots, u_r$ be the neighbors of u in counterclockwise order such that P_i contains u_i, $i \in [r]$, and let Q_i be a path in $\mathcal{M}$ containing u_i.

Since all paths in $\mathcal{M}$ have the same length λ, Definition 5.4 implies that Observation 3.5 holds. Therefore, at most one path in $\mathcal{M}$ contains only internal colors w.r.t. w in M. By Observation 3.10, any vertex on a path in $\mathcal{M}$ such that the vertex contains an external color w.r.t. w in M must be incident to the external face to w in M. Choose $r' \in [r]$ such that $|r' - \lfloor \frac{r}{2} \rfloor|$ is minimum and $Q_{r'}$ contains an external color w.r.t. w in M. Since $Q_{r'}$ contains an external color w.r.t. w in M, $Q_{r'}$ contains a vertex incident to the external face to w in M. Since all paths in $\mathcal{D}$ are u-v vertex-disjoint paths, $Q_{r'}$ contains vertices other than u and v from at least $\lfloor r/2 \rfloor - 1$ distinct paths (including itself) in $\mathcal{D}$. Since the paths in $\mathcal{D}$ are all vertex disjoint, it follows that $|Q_{r'}| \geq r/2 - 1$, and hence $\lambda \geq r/2 - 1$, which implies that $r \leq 2(\lambda + 1)$. This contradicts our assumption that $r \geq 2\lambda + 3$. □

Lemma 5.6. *$|\mathcal{M}| \leq g(\lambda)$, where $g(\lambda) = O(c^{\lambda}\lambda^{3\lambda})$, for some constant $c > 1$.*

Proof. As in the proof of Lemma 5.5, Definition 5.4 implies that Observation 3.5 holds, and hence, at most one path in $\mathcal{M}$ contains only internal colors w.r.t. w in $G - w$. Therefore, we upper bound the number of paths in $\mathcal{M}$ that each contains at least one external color to w in $G - w$, and add 1 to $g(\lambda)$ at the end. Henceforth, we shall assume that every path in $\mathcal{M}$ contains a color that is external to w in M.

The proof is by induction on λ, over every color-connected plane graph G, every triplet of vertices u, v, w in G, and every λ-minimal set $\mathcal{M}$ w.r.t. w in $G - w$ of k-valid pairwise color-disjoint u-v paths. If $\lambda = 1$, then $|\mathcal{M}| \leq 1 \leq g(1)$, if we choose $g(1)$ to be at least 1.

Suppose, by the inductive hypothesis, that for any $1 \leq i < \lambda$, we have $|\mathcal{M}| \leq g(i)$. By Lemma 5.5, M has a u-v vertex-separator S satisfying $|S| \leq 2\lambda + 2$. S separates M into two subgraphs M_u and M_v such that $u \in V(M_u)$, $v \in V(M_v)$, and there is no edge between M_u and M_v. We partition $\mathcal{M}$ into two groups: (1) The set of paths in $\mathcal{M}$ that each contains a nonempty vertex in S; and (2) the set of remaining paths $\mathcal{M}_{\emptyset}$, which contains each path in $\mathcal{M}$ whose intersection with S consists of only empty vertices. Since the paths in $\mathcal{M}$ are pairwise color-disjoint, no nonempty vertex in S can appear on two distinct paths from group (1). Therefore, the number of paths in group (1) is at most $|S| \leq 2\lambda + 2$.

To upper bound the number of paths in group (2), suppose that $S = \{s_2, \ldots, s_{r-1}\}$, where $r \leq 2\lambda+4$, and extend S by adding the two vertices $s_1 = u$ and $s_r = v$ to form the set $A = \{s_1, s_2, \ldots, s_r\}$. For every two (distinct) vertices $s_j, s_{j'} \in A$, $j, j' \in [r]$, $j < j'$, we define a set of paths $\mathcal{P}_{jj'}$ in $G - w$ whose endpoints are s_j and $s_{j'}$ as follows. For each path P in group (2), partition P into subpaths $P_1, \ldots, P_q$ satisfying the property that the endpoints of each P_i, $i \in [q]$, are in A, and no internal vertex to P_i is in A. Since P contains a vertex that contains an external color to w in $G - w$, there exists an $i \in [q]$ such that P_i contains a vertex that contains an external color to w in $G - w$; pick any such $i \in [q]$, and assign P_i to the set of paths $\mathcal{P}_{jj'}$ such that s_j and $s_{j'}$ are the endpoints of P_i. Since each P_i contains an external color that appears on P and the paths in $\mathcal{M}$ are pairwise-color disjoint, it follows that the map that maps each P to its P_i is a bijection. Moreover, since each path P must intersect $S \setminus \{u, v\}$, the length of each path in $\mathcal{P}_{jj'}$, for $j, j' \in [r]$, $j < j'$, is strictly smaller than λ.

Fix a set $\mathcal{P}_{jj'}$. For any fixed length $i' \in [\lambda - 1]$, the subset of paths in $\mathcal{P}_{jj'}$ of length i', $\mathcal{P}^{i'}_{jj'}$, have $s_j, s_{j'}$ as endpoints, and are pairwise color-disjoint. Moreover, each path in $\mathcal{P}^{i'}_{jj'}$ contains a vertex that contains an external color to w in $G - w$. It follows from the previous statements that $\mathcal{P}^{i'}_{jj'}$ satisfies Definition 5.4 w.r.t. G and w, and hence $\mathcal{P}^{i'}_{jj'}$, $i' \in [\lambda - 1]$, is an i'-minimal set of k-valid s_j-$s_{j'}$ paths in G w.r.t. w. By the inductive hypothesis, we have $|\mathcal{P}^{i'}_{jj'}| \leq g(i')$. Since the number of sets $\mathcal{P}_{jj'}$ is at most $\binom{2\lambda+4}{2}$, $i' \leq \lambda - 1$, and assuming w.l.o.g. that g is an increasing function, the number of paths in group (2) is $O(\lambda^2) \cdot (\lambda - 1) \cdot g(\lambda - 1) = O(\lambda^3) \cdot g(\lambda - 1)$.

It follows from the above that $|\mathcal{M}| \leq g(\lambda)$, where $g(\lambda)$ satisfies the recurrence relation $g(\lambda) \leq (2\lambda + 2) + O(\lambda^3) \cdot g(\lambda - 1) = O(\lambda^3) \cdot g(\lambda - 1)$. Solving the aforementioned recurrence relation gives $g(\lambda) = O(c^{\lambda}\lambda^{3\lambda})$, where $c > 1$ is a constant. Adding 1 to $g(\lambda)$ to account for the single path in $\mathcal{M}$ containing only internal colors w.r.t. w in M yields the same asymptotic upper bound. □

Using Lemma 5.6, the theorem below follows in a very similar manner to that of Theorem 3.14:

Theorem 5.7. *Let G be a plane color-connected graph, let $u, v, w \in V(G)$, let $\lambda \in [\ell]$, and let $\mathcal{P}$ be a set of λ-minimal k-valid u-v paths w.r.t. w in $G - w$. Then $|\mathcal{P}| \leq h(k, \lambda)$, where $h(k, \lambda) = O(c^{\lambda k} \cdot k^k \cdot \lambda^{3\lambda k})$, for some constant $c > 1$.*

The result of Theorem 5.7 will be employed in the next section in the form presented in the following corollary, whose proof follows using the same arguments as in Corollary 3.15:

Corollary 5.8. *Let G be a plane color-connected graph, let $w \in V(G)$, and let $\lambda \in [\ell]$. Let G' be a subgraph of $G - w$, and let $u, v \in V(G')$. Every set $\mathcal{P}$ of λ-minimal k-valid u-v paths in G' w.r.t. w satisfies $|\mathcal{P}| \leq h(k, \lambda)$, where $h(k, \lambda) = O(c^{\lambda k} \cdot k^k \cdot \lambda^{3\lambda k})$, for some constant $c > 1$.*

5.2 The Extended Algorithm

Let $(G, C, \chi, s, t, k, \ell)$ be an instance of Bounded-length Colored Path-Con. The algorithm is a dynamic programming algorithm based on a tree decomposition of G. Let $(\mathcal{V}, \mathcal{T})$ be a nice tree decomposition of G. By Assumption 2.2, we can assume that s and t are nonadjacent empty vertices. We add s and t to every bag in $\mathcal{T}$, and from now on, we assume that $\{s, t\} \subseteq X_i$, for every bag $X_i \in \mathcal{T}$. For a bag X_i, we say that $v \in X_i$ is *useful* if $|\chi(v)| \leq k$. Let U_i be the set of all useful vertices in X_i and let $\overline{U_i} = X_i \setminus U_i$. We denote by V_i the set of vertices in the bags of the subtree of $\mathcal{T}$ rooted at X_i. For any two vertices $u, v \in X_i$, let $G^i_{uv} = G[(V_i \setminus X_i) \cup \{u, v\}]$. We extend the notion of a λ-minimal set of k-valid u-v paths w.r.t. a vertex, developed in the previous section, to the set of vertices in a bag of $\mathcal{T}$.

Definition 5.9. Let $\lambda \in [\ell]$. A set of k-valid u-v paths $\mathcal{P}_{uv}$ in G^i_{uv} is λ-*minimal* w.r.t. X_i if each path in $\mathcal{P}_{uv}$ has length exactly λ and there does not exist two paths $P_1, P_2 \in \mathcal{P}_{uv}$ such that $\chi(P_1) \cap \chi(X_i) = \chi(P_2) \cap \chi(X_i)$.

The lemma below follows from Corollary 5.8 using a similar proof to that of Lemma 4.2:

Lemma 5.10. *Let X_i be bag, $u, v \in X_i$, $\lambda \in [\ell]$ and $\mathcal{P}_{uv}$ a λ-minimal set of k-valid u-v paths w.r.t. X_i in G^i_{uv}. Then the number of paths in $\mathcal{P}_{uv}$ is at most $h(k, \lambda)^{|X_i|}$, where $h(k, \lambda) = O(c^{\lambda k} \cdot k^k \cdot \lambda^{3\lambda k})$, for some constant $c > 1$.*

We define the length of a sequence $\mathcal{S}$ of walks, denoted by $|\mathcal{S}|$, to be the sum of the lengths of the walks in $\mathcal{S}$.

Definition 5.11. Let X_i be a bag and $\pi = (v_1, \sigma_1, v_2 \dots, \sigma_{r-1}, v_r)$ a pattern for X_i. A set $\mathcal{R}_\pi$ of sequences of paths, each of length at most ℓ, that conform to (X_i, π) is a *representative* set for (X_i, π) if:

(i) For every sequence $\mathcal{S}_1 \in \mathcal{R}_\pi$, and for every sequence $\mathcal{S}_2 \neq \mathcal{S}_1$ that conforms to (X_i, π), if $\mathcal{S}_1 \preceq_i \mathcal{S}_2$ and $|\mathcal{S}_1| \leq |\mathcal{S}_2|$ then $\mathcal{S}_2 \notin \mathcal{R}_\pi$; and
(ii) for every sequence $\mathcal{S} \notin \mathcal{R}_\pi$, $|\mathcal{S}| \leq \ell$, that conforms to (X_i, π) and satisfies that no two paths in $\mathcal{S}$ share a vertex that is not in X_i, there is a sequence $\mathcal{W} \in \mathcal{R}_\pi$ such that $\mathcal{W} \preceq_i \mathcal{S}$ and $|\mathcal{W}| \leq |\mathcal{S}|$.

We mention that Lemma 4.5 and Observation 4.7 extend as they are to the current setting.

Lemma 5.12. *Let X_i be bag, π a pattern for X_i, and $\mathcal{R}_\pi$ a representative set for (X_i, π). Then the number of sequences in $\mathcal{R}_\pi$ is at most $h(k, \ell)^{|X_i|^2}$, where $h(k, \ell) = O(c^{\ell k} \cdot k^k \cdot \ell^{3\ell k})$, for some constant $c > 1$.*

Proof. Let $\pi = (v_1 = s, \sigma_1, v_2, \sigma_2, \dots, \sigma_{r-1}, v_r = t)$, and let $\Lambda = (\lambda_1, \dots, \lambda_{r-1})$ be such that, for each $j \in [r-1]$: (1) $\lambda_j = 0$ if $\sigma_j = 0$ and $\lambda_j \in [\ell]$ otherwise, and (2) $\sum_{j=1}^{r-1} \lambda_j \leq \ell$. For each $\lambda \in [\ell]$, the number of tuples $\Lambda = (\lambda_1, \dots, \lambda_{r-1})$ satisfying $\sum_{j=1}^{r-1} \lambda_j = \lambda$ is the number of *weak compositions* of λ into $r-1$ parts, which is $\binom{\lambda+r-2}{r-2}$. It follows that the number of tuples $\Lambda = (\lambda_1, \dots, \lambda_{r-1})$ satisfying $\sum_{j=1}^{r-1} \lambda_j \leq \ell$ is upper bounded by $\binom{\ell+r-1}{r-1}$ [18], which is at most $\binom{|X_i|+\ell}{r-2} \leq 2^{|X_i|+\ell}$. Therefore, if we upper bound the number of sequences in $\mathcal{R}_\pi$ corresponding to some fixed tuple Λ by $h_1(k, \ell)^{|X_i|^2} = O(c_1^{\ell k} \cdot k^k \cdot \ell^{3\ell k})$ for some constant $c_1 > 1$, then we obtain $\mathcal{R}_\pi \leq 2^{|X_i|+\ell} \cdot h_1(k, \ell)^{|X_i|^2} \leq h(k, \ell)^{|X_i|^2}$, where $h(k, \ell) = O(c^{\ell k} \cdot k^k \cdot \ell^{3\ell k})$, for some constant $c > 1$. For the rest of the proof, we fix $\Lambda = (\lambda_1, \dots, \lambda_{r-1})$ and we let $\mathcal{R}_\pi^\Lambda$ be the subset of $\mathcal{R}_\pi$ such that for each sequence $\mathcal{S} = (P_1, \dots, P_{r-1})$ in $\mathcal{R}_\pi^\Lambda$ it holds that the length of P_j is λ_j for each $j \in [r-1]$ such that $\sigma_j = 1$.

For each $j \in [r-1]$ such that $\sigma_j = 1$, let $\mathcal{P}_j$ be a λ_j-minimal set of k-valid v_j-v_{j+1} paths w.r.t. X_i. Without loss of generality, we can pick $\mathcal{P}_j$ such that there is no k-valid u-v path P of length λ_j in $G^i_{v_j v_{j+1}}$ such that $\mathcal{P}_j \cup \{P\}$ is λ_j-minimal w.r.t. X_i. From Lemma 5.10, it follows that $|\mathcal{P}_j| \leq h_1(k, \lambda_j)^{|X_i|}$, where $h_1(k, \lambda_j) = O(c_1^{\lambda_j k} \cdot k^k \cdot \lambda_j^{3\lambda_j k})$, for some constant $c_1 > 1$. From this point on, the proof proceeds in the same fashion as in Lemma 4.9. ◻

We recall the definition of a sequence of walks refining another from the previous section. We have the following lemma:

Lemma 5.13. *Let X_i be a bag, $\pi = (v_1 = s, \sigma_1, v_2, \sigma_2, \dots, \sigma_{r-1}, v_r = t)$ a pattern for X_i, and $\mathcal{W} = (W_1, \dots, W_{r-1})$ a sequence such that each W_j is a walk between vertices v_j and v_{j+1} in $G^i_{v_j v_{j+1}}$ satisfying $\chi(W_j) \leq k$. Then in time $O(r \cdot (|V(G)| + |V(E)|))$ we can compute a sequence $\mathcal{S} = (P_1, \dots, P_{r-1})$, where* *for each $j \in [r-1]$, P_j is an induced path between v_j and v_{j+1} in $G^i_{v_j v_{j+1}}$ such that $\chi(P_j) \subseteq \chi(W_j)$ and the length of P_j is at most the length of W_j.*

Proof. For each walk W_j, $j \in [r-1]$, we do the following. We form the subgraph G' from $G^i_{v_j v_{j+1}}$ by removing every vertex x in $G^i_{v_j v_{j+1}}$ that does not satisfy $\chi(x) \subseteq \chi(W_j)$. Clearly, W_j is a subgraph of G', and hence there exists a v_j-v_{j+1} path of length at most the length of W_j in G'. We find a shortest v_j-v_{j+1} path in G' in time $O(|V(G)| + |E(G)|)$ and set P_j to this path. Clearly, the computation of $\mathcal{S}$ takes time $O(r \cdot (|V(G)| + |E(G)|))$. □

For a bag X_i, pattern π for X_i, and a set of sequences (of walks) $\mathcal{R}$ that conform to (X_i, π), we define the procedure **Refine()** that takes the set $\mathcal{R}$ and outputs a set $\mathcal{R}'$ of sequences of length at most ℓ that conform to (X_i, π), and does not violate property (i) of Definition 5.11. First, for each sequence $\mathcal{S}$ in $\mathcal{R}$, we compute a sequence $\mathcal{S}'$ that refines $\mathcal{S}$ and has length at most the length of $\mathcal{S}$, and replace $\mathcal{S}$ with $\mathcal{S}'$ in $\mathcal{R}$. Afterwards, we initialize $\mathcal{R}' = \emptyset$, and order the sequences in $\mathcal{R}$ first w.r.t. $\preceq_i$ and, in case of ties, w.r.t. the lengths of the sequences afterwards (where ties w.r.t. both $\preceq_i$ and length are broken arbitrarily). We iterate through the sequences in $\mathcal{R}$ in order, and add a sequence $\mathcal{S}_p$ to $\mathcal{R}'$ if $|\mathcal{S}_p| \leq \ell$, there is no sequence $\mathcal{S}$ already in $\mathcal{R}'$ such that $\mathcal{S} \preceq_i \mathcal{S}_p$ and $|\mathcal{S}| \leq |\mathcal{S}_p|$. The proof of the following lemma is similar to that of Lemma 4.11:

Lemma 5.14. *Let X_i be a bag, π a pattern for X_i, and $\mathcal{W}$ be a set of sequences of walks that conforms to (X_i, π). The procedure* **Refine()***, on input $\mathcal{W}$, produces a set of sequences of induced paths $\mathcal{R}'$ that conform to (X_i, π) and satisfy property (i) of Definition 5.11, and such that for each sequence $\mathcal{S} \in \mathcal{W}$ with $|\mathcal{S}| \leq \ell$, there is a sequence $\mathcal{S}' \in \mathcal{R}'$ satisfying $\mathcal{S}' \preceq_i \mathcal{S}$ and $|\mathcal{S}'| \leq |\mathcal{S}|$. Moreover, the procedure runs in time $O^*(|\mathcal{W}|^2)$.*

Using the above procedure **Refine()**, the dynamic programming algorithm is the same as that in Section 4. For each bag X_i, it maintains a table Γ_i that contains, for each pattern π for X_i, a representative set $\Gamma_i[\pi]$ for (X_i, π). For illustration, we present the case of a join node, and omit the other cases to avoid repetition.

Case X_i is a join node with children $X_j, X_{j'}$.

Let $\pi = (s = v_1, \sigma_1, \ldots, \sigma_{r-1}, v_r = t)$ be a pattern for X_i. Initialize $\mathcal{R}_\pi = \emptyset$. For every two patterns $\pi_1 = (s = v_1, \tau_1, \ldots, \tau_{r-1}, v_r = t)$ and $\pi_2 = (s = v_1, \mu_1, \ldots, \mu_{r-1}, v_r = t)$ such that $\sigma_q = \tau_q + \mu_q$, and for every two sequences $\mathcal{S}_1 = (P_1^1, \ldots, P_1^{r-1}) \in \Gamma_j[\pi_1]$ and $\mathcal{S}_2 = (P_2^1, \ldots, P_2^{r-1}) \in \Gamma_{j'}[\pi_2]$, we add the sequence $\mathcal{S} = (P_1, \ldots, P_{r-1})$ to $\mathcal{R}_\pi$, where $P_q = P_1^q$ if P_2^q is the empty path, otherwise, $P_q = P_2^q$, for $q \in [r-1]$. We set $\Gamma_i[\pi] = \mathbf{Refine}(\mathcal{R}_\pi)$, and we claim that $\Gamma_i[\pi]$ is a representative set for (X_i, π).

Claim 6. *If X_i is a join node with children X_j, $X_{j'}$, and Γ_j (resp. $\Gamma_{j'}$) contains for each pattern π' for $X_j = X_{j'} = X_i$ a representative set for (X_j, π') (resp. $(\pi', X_{j'})$), then $\Gamma_i[\pi]$ defined above is a representative set for (X_i, π).*

Proof. Clearly $\Gamma_i[\pi]$ satisfies property (i) due to the application of the procedure **Refine()**. To argue that $\Gamma_i[\pi]$ satisfies properties (ii), suppose not, and let $\mathcal{S} = (P_1, \ldots, P_{r-1})$ be a sequence that violates property (ii). Notice that every path P_q, $q \in [r-1]$ is either an edge between two vertices in X_i, or is a path between two vertices in X_i such that its internal vertices are either all in $V_j \setminus X_i$ or in $V_{j'} \setminus X_i$; this is true because X_i is a vertex-separator separating $V_j \setminus X_i$ from $V_{j'} \setminus X_i$ in G. Define the two sequences $\mathcal{S}_1 = (P_1^1, \ldots, P_1^{r-1})$ and $\mathcal{S}_2 = (P_2^1, \ldots, P_2^{r-1})$ as follows. For $q \in [r-1]$, if P_q is empty then set both P_1^q and P_2^q to the empty path; if P_q is an edge then set $P_1^q = P_q$ and P_2^q to the empty path. Otherwise, P_q is either a path in $G[V_j]$ or in $G[V_{j'}]$; in the former case set $P_1^q = P_q$ and P_2^q to the empty path, and in the latter case set $P_2^q = P_q$ and P_1^q to the empty path. Since no two paths in $\mathcal{S}$ share a vertex that is not in X_i, and $X_i = X_j = X_{j'}$, no two paths in $\mathcal{S}_1$ (resp. $\mathcal{S}_2$)

share a vertex that is not in X_j (resp. $X_{j'}$). Moreover, it is easy to see that $|\mathcal{S}_1| + |\mathcal{S}_2| = |\mathcal{S}| \le \ell$. Let $\pi_1 = (s = v_1, \tau_1, \ldots, \tau_{r-1}, v_r = t)$ and $\pi_2 = (s = v_1, \mu_1, \ldots, \mu_{r-1}, v_r = t)$ be the two patterns that $\mathcal{S}_1$ and $\mathcal{S}_2$ conform to, respectively, and observe that, for every $q \in [r-1]$, we have $\sigma_q = \tau_q + \mu_q$. Since $\Gamma_j[\pi_1]$ and $\Gamma_{j'}[\pi_2]$ are representative sets, it follows that there exist $\mathcal{S}_1' = (Y_1', \ldots, Y_{r-1}')$ in $\Gamma_j[\pi_1]$ and $\mathcal{S}_2' = (Z_1', \ldots, Z_{r-1}')$ in $\Gamma_{j'}[\pi_2]$ such that $\mathcal{S}_1' \le_j \mathcal{S}_1$, $|\mathcal{S}_1'| \le |\mathcal{S}_1|$ and $\mathcal{S}_2' \le_{j'} \mathcal{S}_2$, $|\mathcal{S}_2'| \le |\mathcal{S}_2|$. Let $\mathcal{S}' = (P_1', \ldots, P_{r-1}')$, where $P_q' = Y_q'$ if Z_q' is the empty path, otherwise, $P_q = Z_q'$, for $q \in [r-1]$. The sequence $\mathcal{S}'$ conforms to π, is in $\mathcal{R}_\pi$, and $|\mathcal{S}'| = |\mathcal{S}_1'| + |\mathcal{S}_2'| \le |\mathcal{S}|$. By Lemma 5.14, $\Gamma_i[\pi]$ contains a sequence $\mathcal{S}''$ such that $\mathcal{S}'' \le_i \mathcal{S}'$ and $|\mathcal{S}''| \le |\mathcal{S}'|$. From Observation 4.7, since $X_i = X_j = X_{j'}$, from $\mathcal{S}_1' \le_j \mathcal{S}_1$ and $\mathcal{S}_2' \le_{j'} \mathcal{S}_2$ it follows that $\mathcal{S}_1' \le_i \mathcal{S}_1$ and $\mathcal{S}_2' \le_i \mathcal{S}_2$. Since $\chi(\mathcal{S}_1) \cup \chi(\mathcal{S}_2) = \chi(\mathcal{S})$ and $\chi(\mathcal{S}_1') \cup \chi(\mathcal{S}_2') = \chi(\mathcal{S}')$, and since $\chi(\mathcal{S}_1) \cap \chi(\mathcal{S}_2) \subseteq \chi(X_i)$, by Lemma 4.8, it follows that $\mathcal{S}' \le_i \mathcal{S}$. Since $\mathcal{S}'' \le_i \mathcal{S}'$, by transitivity of $\le_i$, it follows that $\mathcal{S}'' \le_i \mathcal{S}$. Moreover $|\mathcal{S}''| \le |\mathcal{S}|$, which concludes the proof. ◻

We can now conclude with the following theorem:

Theorem 5.15. *There is an algorithm that on input $(G, C, \chi, s, t, k, \ell)$ of* Bounded-length Colored Path-Con, *either outputs a k-valid s-t path of length at most ℓ in G, or decides that no such path exists, in time $O^\star(f(k,\ell)^{37\ell^2})$, where $f(k,\ell) = O(c^{\ell k} \cdot k^k \cdot \ell^{3\ell k})$, for some constant $c > 1$. Therefore,* Bounded-length Colored Path-Con *parameterized by both k and ℓ is* FPT.

Proof. If $d_G(s,t) > \ell$, then, by definition, there is no s-t path of length at most ℓ. Hence, we assume that $d_G(s,t) \le \ell$. By Lemma 5.3, if there exists a vertex v such that $d_G(s,v) > \ell + 1$, we can contract any edge incident to v and obtain an equivalent instance. The contraction of a single edge can be done in time polynomial in the size of the instance, and after applying Lemma 5.3 $|E|$ times, we would get a trivial instance. Moreover, from the proof of Lemma 5.3, it follows that we can obtain a solution for the original instance from a solution in the contracted instance in polynomial time. Therefore, we can assume for the rest of the proof that we applied Lemma 5.3 exhaustively, and hence G has radius at most $\ell + 1$ and treewidth ω that is at most $3\ell + 4$ [30]. Moreover, a tree decomposition of G of width ω can be computed in (polynomial) time $O(\ell \cdot n)$ [25]. From such a tree decomposition, in polynomial time we can compute a nice tree decomposition $(\mathcal{V}, \mathcal{T})$ of G whose width is at most $\omega \le 3\ell + 4$ and satisfying $|\mathcal{V}| = O(|V(G)|)$ [26].

The algorithm starts by removing the colors of s and t from G, and decrements k by $|\chi(s) \cup \chi(t)|$ (see Assumption 2.2). Afterwards, if $k < 0$, the algorithm concludes that there is no k-valid s-t path in G. If $st \in E(G)$ and $k \ge 0$, the algorithm outputs the path (s,t). Now we know that s and t are not adjacent, and that $\chi(s) = \chi(t) = \emptyset$. The algorithm then adds s and t to every bag in $\mathcal{T}$, and executes the dynamic programming algorithm based on $(\mathcal{V}, \mathcal{T})$ to compute a table Γ_i that contains, for each bag X_i in $\mathcal{T}$ and each pattern π for X_i, a representative set $\mathcal{R}_\pi$ for (X_i, π).

The correctness of the algorithm and the upper bound on its running time follow using similar arguments to those in the proof of Theorem 4.12. ◻

5.3 Applications

In this subsection, we describe some applications of Theorem 5.15. The first result is a direct consequence of Theorem 5.15.

Corollary 5.16. *For any computable function h, the restriction of* Colored Path-Con *to instances in which the length of the sought path is at most $h(k)$ is* FPT *parameterized by k.*

We note that the above restriction of Colored Path-Con is NP-hard, as a consequence of (the proof of) Corollary 6.3.

Corollary 5.16 directly implies Korman *et al.*'s results [27], showing that Obstacle Removal parameterized by k is FPT for unit-disk obstacles and for similar-size fat regions with constant overlapping number. Using Bereg and Kirkpatrick's result [2], the length of a shortest k-valid path for unit-disk obstacles is at most $3k$ (see also Lemma 3 in Korman *et al.* [27]). By Corollary 2 in [27], the length of a shortest k-valid path for similar-size fat-region obstacles with constant overlapping number is linear in k. Corollary 5.16 generalizes these FPT results, which required quite some effort, and provides an explanation to why the problem is FPT for such restrictions, namely because the path has length upper bounded by a function of the parameter. In particular, one may allow the connected obstacles to be of various shapes and sizes, provided that the length of the path is upper bounded by a function of the parameter.

The second application we describe is related to an open question posed in [14]. For an instance $I = (G, C, \chi, s, t, k)$ of Colored Path-Con, and a color $c \in C$, define the *intersection number* of c, denoted $\iota(c)$, to be the number of vertices in G on which c appears. Define the *intersection number* of G, $\iota(G)$, as $\max\{\iota(c) \mid c \in C\}$. Consider the following problem:

Bounded-intersection Colored Path-Con
Given: A planar graph G such that $\iota(G) \leq i$; a set of colors C; $\chi : V \longrightarrow 2^C$; two designated vertices $s, t \in V(G)$; and $k, i \in \mathbb{N}$
Question: Does there exist a k-valid s-t path in G?

Again, the above problem is NP-hard, as a consequence of (the proof of) Corollary 6.3.

Corollary 5.17. Bounded-intersection Colored Path-Con *is* FPT *parameterized by both k and i.*

Proof. Since the number of vertices in G on which any color $c \in C$ appears is at most $\iota(G)$, after applying the color contraction operation (see Lemma 2.4), the length of any k-valid s-t path is $O(k \cdot i)$. The result now follows from Theorem 5.15. □

The following corollary is a direct consequence of Corollary 5.17:

Corollary 5.18. *For any computable function h,* Bounded-intersection Colored Path-Con *restricted to instances (G, C, χ, s, t, k) satisfying $\iota(G) \leq h(k)$ is* FPT *parameterized by k.*

Corollary 5.17 has applications pertaining to instances of Connected Obstacle Removal whose auxiliary graph is an instance of Bounded-intersection Colored Path-Con. In particular, an interesting case that was studied corresponds to the case in which the obstacles are convex polygons, each intersecting at most a constant number of other polygons. The complexity of this problem was left as an open question in [14, 20], and remains unresolved. The result in Corollary 5.18 subsumes this case, and even the more general case in which the obstacles are arbitrary convex obstacles satisfying that each obstacle intersects a constant number of other obstacles, as it is easy to see that the auxiliary graph of such instances will have a constant intersection number[4]. In fact, we can even allow the intersection number to be any (computable) function of the parameter:

Theorem 5.19. *Let h be a computable function. The restriction of* Connected Obstacle Removal *to any set of convex obstacles in the plane satisfying that each obstacle intersects at most $h(k)$ other obstacles, is* FPT *parameterized by k.*

Whereas the complexity of the problem in the above theorem is open, the theorem settles its parameterized complexity by showing it to be in FPT.

[4]Note that convexity is essential here, as otherwise, the intersection number of the auxiliary graph may be unbounded.

6 HARDNESS RESULTS

In this section, we prove hardness results for Colored Path and Colored Path-Con, and their geometric counterparts Obstacle Removal and Connected Obstacle Removal. We start by showing that both problems are NP-hard, even when restricted to graphs of small outerplanarity and pathwidth.

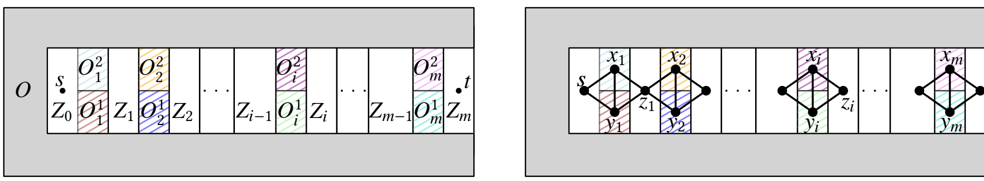


Fig. 12. Illustration of the construction in the proof of Theorem 6.1. The left figure shows the geometric instance of Colored Path, and the right figure the graph associated with it.

Theorem 6.1. Colored Path, *restricted to outerplanar graphs of pathwidth at most 2 and in which every vertex contains at most one color, is* NP*-complete.*

Proof. It is clear that Colored Path is in NP. To show its NP-hardness, we reduce from the NP-hard problem Vertex Cover [16]: Given an undirected graph G and $k \in \mathbb{N}$, decide if G contains a subset of at most k vertices such that every edge in G is incident to at least one vertex in this subset. Let (G, k) be an instance of Vertex Cover, where $V(G) = \{v_1, \ldots, v_n\}$ and $E(G) = \{e_1, \ldots, e_m\}$. In the rest of the proof, when we write $e = uw$ for an edge e in $E(G)$, we assume that $u = v_i$ and $w = v_j$ such that $i < j$ (*i.e.*, the vertex of smaller index always appears first). Although not necessary for the proof, we first describe an instance I of Obstacle Removal whose associated graph is the desired instance of Colored Path. The regions of I are $O \cup \{Z_0, \ldots, Z_m\} \cup \bigcup_{i=1}^{m} \{O_i^1, O_i^2\}$, depicted in Figure 12 (left figure). The obstacles of I are defined as follows. For each vertex $v_j \in V(G)$, the obstacle corresponding to v_j is the polygon whose boundary is the boundary of the region formed by the union of O, each O_i^1 such that $e_i = v_j v_q$, and each O_i^2 such $e_i = v_p v_j$. More formally, the obstacle corresponding to v_j is $\partial(O \cup \bigcup_{e_i = v_j v_q} O_i^1 \cup \bigcup_{e_i = v_p v_j} O_i^2)$. The graph associated with I, G_I, is defined as follows. Each (empty) region Z_i, $i = 0, \ldots, m$, corresponds to a vertex $z_i \in V(G_I)$, where Z_0 corresponds to s and Z_m to t. Each region O_i^1, $i \in [m]$, corresponds to a vertex y_i, and each region O_i^2, $i \in [m]$, corresponds to a vertex x_i. The set of edges $E(G_I)$ is $E(G_I) = \{z_{i-1}x_i, z_{i-1}y_i, x_i y_i, z_i x_i, z_i y_i \mid i \in [m]\}$. The color function $\chi : V(G_I) \longrightarrow 2^C$, where $C = [n]$, is defined as follows: $\chi(z_i) = \emptyset$, for $i = 0, \ldots, m$; $\chi(x_i) = \{j\}$ and $\chi(y_i) = \{p\}$, where $e_i = v_p v_j$, for $i \in [m]$. This completes the construction of G_I; see Figure 12 (right figure) for illustration. It is easy to see that G_I is outerplanar and has pathwidth at most 2.

Define the reduction from Vertex Cover to Colored Path that takes an instance (G, k) to the instance (G_I, C, χ, s, t, k). Clearly, this reduction is polynomial-time computable. Suppose that Q, where $|Q| = r \leq k$, is a vertex cover of G. Consider the s-t path $P = (s, w_1, z_1, , \ldots, w_m, z_m)$ in G_I, where $w_i = y_i$ if edge $e_i = v_p v_q$ is covered by v_p, and $w_i = x_i$ otherwise, for $i \in [m]$. Clearly this is a k-valid s-t path in G_I since each edge e_i is covered by a vertex in Q, each w_i is colored by the index of one of the vertices in Q, and each vertex in G_I (and hence each w_i) contains at most one color. Conversely, suppose that P is a k-valid s-t path in G_I. By construction of G_I, P has to contain at least one vertex from $\{x_i, y_i\}$, for each $i \in [m]$. If P contains both x_i and y_i, for some $i \in [m]$, then clearly, from the construction of P, P must contain either (z_{i-1}, x_i, y_i, z_i) or (z_{i-1}, y_i, x_i, z_i), as a subpath, and we can shortcut this subpath by removing one of x_i, y_i, to obtain another k-valid s-t path in G_I. Therefore, without loss of generality, we may assume that P contains exactly one vertex w_i from $\{x_i, y_i\}$, for $i \in [m]$. Now define the set of vertices Q in G as the vertices in G whose indices are the colors appearing on (the w_i's in) P. More formally, define $Q = \{v_p \mid w_i = x_i \in P \wedge e_i = v_q v_p\} \cup \{v_p \mid w_i = y_i \in P \wedge e_i = v_p v_q\}$. Since P is a k-valid path in G_I, the total number of colors appearing on $\{w_1, \ldots, w_m\}$ is at most k. Notice that the color of each of x_i, y_i is the index of a vertex in G that covers edge e_i. It follows that the set Q of vertices in G, that are the indices of the colors on P, form a k-vertex cover of G. □

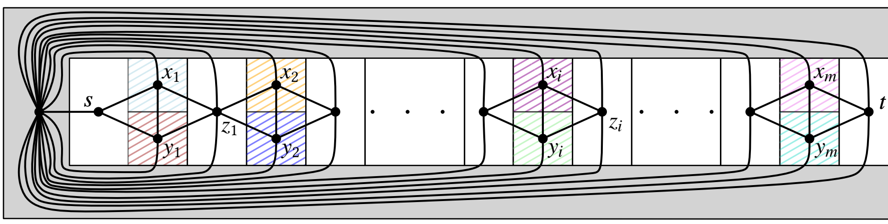


Fig. 13. Illustration of the proof of Corollary 6.2.

Corollary 6.2. Colored Path-Con, *restricted to 2-outerplanar graphs of pathwidth at most 3, is* NP-*complete.*

Proof. This follows directly from the NP-hardness reduction in the proof of Theorem 6.1 by observing the following. The graph G_I resulting from the reduction is outerplanar. We can add a new vertex to the outer face of G_I (see Figure 13) containing all colors that appear on G_I, and add edges between the new vertex and all vertices in G_I. The obtained graph is color-connected and has pathwidth at most 3. □

Assuming ETH, the following corollary rules out the existence of subexponential-time algorithms for Colored Path-Con (and hence for Colored Path), even for restrictions of the problem to graphs of small outerplanarity, pathwidth, and maximum number of occurrences of each color:

Corollary 6.3. *Unless ETH fails,* Colored Path-Con, *restricted to 2-outerplanar graphs of pathwidth at most 3 and in which each color appears at most 4 times, is not solvable in $O(2^{o(n)})$ time, where n is the number of vertices in the graph.*

Proof. It is well known, and follows from [24] and the standard reduction from Independent Set to Vertex Cover, that unless ETH fails, Vertex Cover, restricted to graphs of maximum degree at most 3, denoted VC-3, is not solvable in subexponential time. Starting from an instance of VC-3 with n vertices, and observing that the reduction in the proof of Theorem 6.1 results in an instance of Colored Path-Con whose number of vertices is $O(n)$, of pathwidth at most 3, and in which each color appears at most 4 times, proves the result. □

Next, we shift our attention to studying the parameterized complexity of Colored Path and Colored Path-Con. The reduction from Set Cover showing the NP-hardness of Colored Path, given in several works [3, 20, 35], is in fact an FPT-reduction implying the W[2]-hardness of Colored Path. We will strengthen this result, and show in the remainder of this section that Colored Path is W[SAT]-hard. We will also prove the membership of the problem in W[P], which adds a natural W[SAT]-hard problem to this class. The W[SAT]-hardness result shows that the problem is hopeless in terms of it having FPT-algorithms. We start by showing that the problem remains W[1]-hard, even when restricted to instances of small pathwidth (and hence small treewidth) and maximum number of occurrences of each color. We then show that the problem remains W[1]-hard even when parameterized by both k and the length of the sought path.

Remark 6.4. *Before we prove our parameterized hardness results for* Colored Path, *we remark that we can obtain equivalent hardness results for* Obstacle Removal *using the following generic realization of instances of* Colored Path *as instances of* Obstacle Removal. *Given an instance* $I = (G, C, \chi, s, t, k)$ *of* Colored Path, *we define an equivalent instance* I' *of* Obstacle Removal *as follows. We start by fixing a straight-line plane embedding* Π *of* G, *which always exists by Fáry's theorem [15]. Moreover, we can compute such an embedding in linear time [9]. We define the starting and finishing points in* I' *to be the images* s' *and* t' *of vertices* s *and* t *under* Π, *respectively. We correspond with every edge in* G *a "corridor" in* I' *as follows. We start by "thickening" the edges of* Π. *Then, inside each polygonal face* f *of* Π, *we nest* $k + 1$ *(distinct) disjoint polygonal obstacles, each excluding its interior (i.e., the interior is not part of the obstacle). For every color* $c \in C$ *and every vertex* $v \in V(G)$ *such that* $c \in \chi(v)$, *we place at the image of* v *(under* Π*) a rectangle that intersects exactly those polygonal obstacles nested within the faces incident to* v; *we define the obstacle corresponding to the color* c *in* I' *to be the union of the rectangles associated with color* c. *Finally, we remove the images under* Π *of all vertices and edges of* G, *except* s' *and* t', *thus creating corridors corresponding to the edges of* G *that are surrounded by nested polygonal obstacles created based on the faces of* G. *See Figure 14 for illustration. Observe that this geometric realization may increase the number of obstacles that overlap at a region, which corresponds to the number of colors on the vertex in the graph that corresponds to the region, by at most 1.*

Clearly, a k*-valid* s*-*t *path* P *in* G *corresponds to a* k*-valid* s'*-*t' *path in* I'; *this path would follow the corridors corresponding to the edges of* P, *and intersects only obstacles that correspond to the colors on* P. *Conversely, if there is a* k*-valid* s'*-* t' *path in* I', *then consider such a path that intersects the minimum number of obstacles. It is not difficult to see that such a path must stay strictly within the defined corridors, as deviating from these corridors would only increase the number of intersected obstacles. With such a path we can correspond a* k*-valid* s*-*t *walk in* G *that follows the edges corresponding to the corridors traversed by the path in* I'. *From such a walk a* k*-valid* s*-*t *path in* G *can be extracted.*

We mention that the above geometric realization of an instance $I = (G, C, \chi, s, t, k)$ *of* Colored Path *can be performed in polynomial time and using polynomial space. For instance, one can start from a straight-line embedding of the graph* G *on a grid of size* $O(n) \times O(n)$, *where* $n = |V(G)|$, *which always exists and can be computed in polynomial time (e.g., see [10]). One can then expand the grid size by a polynomial in* n, *for some properly-chosen polynomial, so that, for any face of the graph, one can nest polynomially-many faces within it (to represent the obstacles) whose incident vertices are on the grid, and hence have integer coordinates that are polynomially-bounded by* n. *Similarly, it is easy to see that by a proper choice of the polynomial used for the grid-expansion, the rectangular obstacles corresponding to the colors (on the vertices of* G*) can be placed so that their vertices are on the grid.*

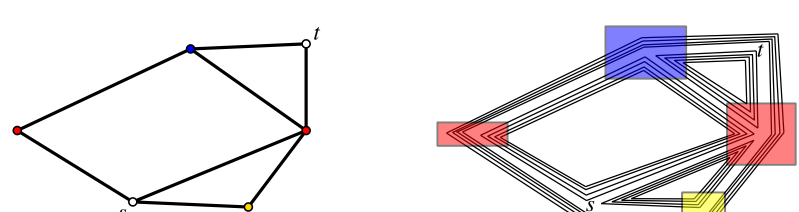


Fig. 14. Illustration of the realization of an instance of Colored Path as an instance of Obstacle Removal.

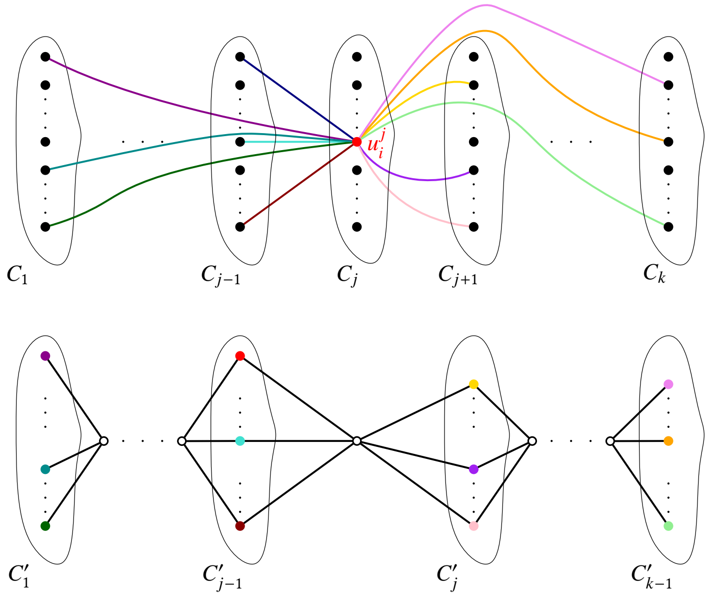


Fig. 15. Illustration of the construction of the gadget $G_{i,j}$ in the proof of Lemma 6.5.

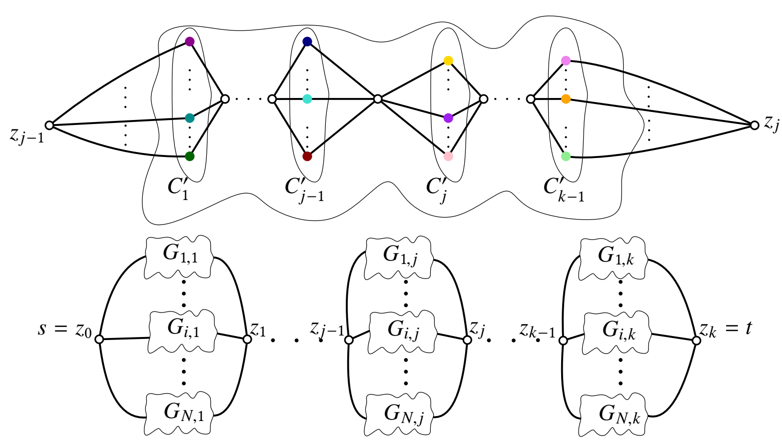


Fig. 16. Illustration of the construction of G' in the proof of Lemma 6.5.

Lemma 6.5. Colored Path, *restricted to instances of pathwidth at most 4 and in which each vertex contains at most one color and each color appears on at most 2 vertices, is* W[1]-hard parameterized by k.

Proof. We reduce from the W[1]-hard problem Multi-Colored Clique [19]: Given an undirected graph G, $k \in \mathbb{N}$, and a proper k-coloring of $V(G)$, decide if G contains a clique of k vertices. Let (G, k) be an instance of Multi-Colored Clique, where $V(G)$ is partitioned into the color classes $C_1, \ldots, C_k$. Let $C_j = \{u_i^j \mid i \in [|C_j|]\}$. We describe how to construct an instance $(G', C', \chi', s, t, k')$ of Colored Path. For an edge $e \in G$, associate a distinct color c_e, and define $C' = \{c_e \mid e \in E(G)\}$.

To simplify the description of the construction, we start by defining a gadget that will serve as a building block for this construction.

For a vertex u_i^j in color class C_j, we define the gadget $G_{i,j}$ as follows. Create a copy of each color class $C_{j'}$, $j' \neq j$, and remove from each $C_{j'}$ all copies of vertices that are not neighbors of u_i^j in G. Let the resulting copies of the color classes be $C'_1, \ldots, C'_{k-1}$. We define the color of a copy v' of a neighbor v of u_i^j as $\chi'(v') = \{c_e\}$, where $e = u_i^j v$. Next, we introduce $k-2$ empty vertices y_r, $r \in [k-2]$. For $r \in [k-2]$, we connect all vertices in C'_r to y_r, and connect y_r to all vertices in C'_{r+1}. This completes the construction of gadget $G_{i,j}$; we refer to C'_1 and C'_{k-1} as the first and last color classes in gadget $G_{i,j}$, respectively. See Figure 15 for illustration of $G_{i,j}$. Observe that every path from a vertex in C'_1 to a vertex in C'_{k-1} contains exactly one vertex from each C'_r, $r \in [k-1]$, and contains all vertices y_r, $r \in [k-2]$. Therefore, any such path contains the colors of exactly $k-1$ distinct edges that are incident to u_i^j.

We finish the construction of G' by introducing $k+1$ new empty vertices $z_0, \ldots, z_k$, and connecting them as follows. For each color class C_j, $j \in [k]$, and each vertex $u_i^j \in C_j$, we create the gadget $G_{i,j}$, connect z_{j-1} to each vertex in the first color class of $G_{i,j}$, and connect each vertex in the last color class of $G_{i,j}$ to z_j. Let G' be the resulting graph. Finally, we set $s = z_0$, $t = z_k$, and $k' = \binom{k}{2}$. See Figure 16 for illustration. This completes the construction of the instance $(G', C', \chi', s, t, k')$ of Colored Path. Observe that each vertex in G' contains at most one color, and that each color c_e of an edge $e = u_i^j u_{i'}^{j'}$ in G, appears on exactly two vertices in G': the copy of $u_{i'}^{j'}$ in the gadget $G_{i,j}$ of u_i^j, and the copy of u_i^j in the gadget $G_{i',j'}$ of $u_{i'}^{j'}$.

Clearly, the reduction that takes an instance (G, k) of Multi-Colored Clique and produces the instance $(G', C', \chi', s, t, k')$ of Colored Path is computable in FPT-time. To show its correctness, suppose that (G, k) is a yes-instance of Multi-Colored Clique, and let Q be a k-clique in G. Then Q contains a vertex from each C_j, for $j \in [k]$. For a vertex $u_i^j \in Q$, let $G_{i,j}$ be its gadget, and define the path P_j as follows. In each color class in $G_{i,j}$, pick the unique vertex that is a copy of a neighbor of u_i^j in Q; define P_j to be the path in $G_{i,j}$ induced by the picked vertices, plus the empty vertices y_r, $r \in [k-2]$, that appear in $G_{i,j}$. Finally, define P to be the s-t path in G' whose edges are: the (unique) edge between z_{r-1} and an endpoint of P_r, P_r, and the (unique) edge between an endpoint of P_r and z_r, for $r \in [k]$. To show that P is k'-valid, observe that all the nonempty vertices in P are vertices whose color is the color of an edge between two vertices in Q. This shows that the number of colors that appear on P is at most $k' = \binom{k}{2}$, and hence, P is k'-valid. It follows that $(G', C', \chi', s, t, k')$ is a yes-instance of Colored Path.

Conversely, suppose that P is a k'-valid s-t path in G'. Then P' must start at s, visit the gadgets of exactly k vertices $u_{i_j}^j \in C_j$, for $j \in [k]$, $i_j \in [|C_j|]$, and end at t. We claim that $Q = \{u_{i_j}^j \mid j \in [k]\}$ is a clique in G. Recall that the subpath of P that traverses a gadget $G_{i,j}$ of $u_{i_j}^j$ contains the colors of exactly $k-1$ edges that are incident to $u_{i_j}^j$. Therefore, the total number of occurrences of colors (counting multiplicities) on P is precisely $(k-1)k$. Since P is $\binom{k}{2}$-valid, and each color c_e of an edge e in G appears exactly twice in G', it follows that each color that appears on P appears exactly twice on P. This is only possible if the gadgets corresponding to the two endpoints of the edge are traversed by P, and hence, both endpoints of the edge are in Q. Therefore, P contains the colors of $k' = \binom{k}{2}$ edges, whose both endpoints are in Q. Since $|Q| = k$, it follows that Q is a k-clique in G, and that (G, k) is a yes-instance of Multi-Colored Clique. □

Lemma 6.6. Colored Path, *parameterized by both k and the length of the path ℓ, is in* W[1].

Proof. To prove membership in W[1], we use the characterization of the class W[1] given by Chen *et al.* [7]:

> A parameterized problem Q is in W[1] if and only if there is a computable function h and a nondeterministic FPT algorithm $\mathbb{A}$ for a nondeterministic-RAM machine deciding Q, such that, for each instance (x, k') of Q (k' is the parameter), all nondeterministic steps of $\mathbb{A}$ take place during the last $h(k')$ steps of the computation.

Therefore, to show that Colored Path is in W[1], it suffices to exhibit such a nondeterministic FPT algorithm $\mathbb{A}$. $\mathbb{A}$ works as follows: It guesses a set C' of k colors and guesses a sequence of $\ell - 1$ internal vertices $v_1, \ldots, v_{\ell-1}$ of the path. Then it verifies that $(s = v_0, v_1, \ldots, v_{\ell-1}, v_\ell = t)$ is a path in G, and that $\chi(v_i) \subseteq C'$, for $i = 0, \ldots, \ell$. It is not difficult to see that this verification can be implemented in $h(k, \ell)$ steps, where h is a computable function. □

By Lemma 2.4, we can assume that in an instance of Colored Path, no two adjacent vertices are empty. With this assumption in mind, if the instance satisfies that each vertex contains at most one color and that each color appears on at most 2 vertices, then any k-valid s-t path has length at most $4k + 1$. It follows from Lemma 6.5 and Lemma 6.6 that:

Theorem 6.7. Colored Path, *parameterized by both k and the length of the path ℓ, is* W[1]-complete.

Theorem 6.8. Colored Path, *restricted to instances of pathwidth at most 4 and in which each vertex contains at most one color and each color appears on at most 2 vertices, is* W[1]-complete parameterized by k.

Next, we show that Colored Path sits high up in the parameterized complexity hierarchy. We start by showing its membership in W[P]:

Theorem 6.9. Colored Path, *parameterized by k, is in* W[P].

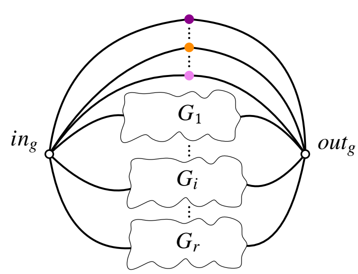
Proof. We give an FPT-reduction from Colored Path to Weighted Boolean Circuit Satisfiability (WBCS) on polynomial size (monotone) circuits. Given an instance (G, C, χ, s, t, k) of Colored Path, we construct an instance (B, k) of WBCS, where B is a circuit whose output gate is an or-gate, as follows. By Assumption 2.2, we can assume that s and t are nonadjacent empty vertices. By Lemma 2.4, we can also assume that no two adjacent vertices are empty. For each color $c \in C$, we create a variable x_c; those are the input variables to B. In addition to the output gate, B contains $n = |V(G)|$ layers of gates, where each layer, except the first, consists of two rows of gates, U_i, L_i, for $i = 2, \ldots, n$, and the first layer consists of one row L_1 of gates. The layers of B are defined as follows.

Each gate in L_1 is an and-gate g_v that corresponds to a neighbor v of s; the input to g_v is the set of input variables corresponding to the colors in $\chi(v)$. Suppose that row L_i in layer i, $i \geq 1$, has been defined, and we describe how U_{i+1} and L_{i+1} are defined. For every vertex $v \in V(G)$ with a neighbor u such that u has a corresponding and-gate g_u^2 in L_i, we create an or-gate g_v^1 in U_{i+1} and an and-gate g_v^2 in L_{i+1} corresponding to v; we connect the output of each and-gate g_u^2 in L_i corresponding to neighbor u of v to the input of or-gate g_v^1 in U_{i+1}, and connect the output of the or-gate g_v^1 and each input variable x_c such that $c \in \chi(v)$ to the and-gate g_v^2 in L_{i+1}. If $v = t$, then we connect the output of the and-gate g_v^2 to the output gate of the circuit. This completes the description of B. Clearly, the reduction that takes (G, C, χ, s, t, k) to (B, k) runs in polynomial time, and hence in FPT-time. Next, we prove its correctness.

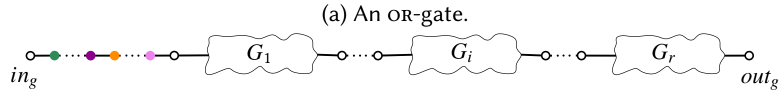
First observe that the only gates in B that are connected to its output gate are the and-gates that correspond to t. Second, every gate in B corresponds to a vertex that is reachable from s in

G. Moreover, for every AND-gate g corresponding to a vertex v, and every s-v path in G, the truth assignment that assigns 1 to the variables corresponding to the colors of this path satisfies g.

Suppose now that (G, C, χ, s, t, k) is a yes-instance of COLORED PATH. Then there is an s-t k-valid path P in G. Based on the above observations, the assignment that assigns $x_c = 1$ if and only if $c \in \chi(P)$ is a satisfying assignment to B of weight at most k. Conversely, suppose that B has a satisfying assignment τ of weight at most k. Then there is an AND-gate g corresponding to t that is satisfied by τ, and there is a path in B from a gate corresponding to neighbor of s in L_1 to g, all of whose gates are satisfied by τ. It is easy to verify that this path in B corresponds to an s-t path all of whose colors correspond to the input variables assigned 1 by τ, and hence this path is k-valid. □



(a) An OR-gate.



(b) An AND-gate.

Fig. 17. Illustrations of the construction of the gadgets in the proof of Theorem 6.10.

THEOREM 6.10. COLORED PATH, *parameterized by* k, *is* W[SAT]-hard.

To prove Theorem 6.10, we start from the W[SAT]-complete problem MONOTONE WEIGHTED BOOLEAN FORMULAS SATISFIABILITY (M-WSAT) [12] and show the following lemma that directly implies Theorem 6.10, and that will also be useful for proving Theorem 6.12.

LEMMA 6.11. *There is a polynomial time algorithm that takes an instance* (B, k) *of M-WSAT with* n *variables* $x_1, \ldots, x_n$ *and outputs a Colored Path instance* $(G, C = [n], \chi, s, t, k)$ *such that: For any assignment* τ *to* B *that assigns variables* $x_{i_1}, \ldots, x_{i_p}$ *the value 1 and all other variables the value 0,* τ *satisfies* B *if and only if there is an* s-t *path* P *in* G *that uses a subset of the colors* $\{i_1, \ldots, i_p\}$.

PROOF. Recall that a Boolean formula corresponds to a circuit in the normalized form. Therefore, we can assume that B is a monotone Boolean circuit in which each (non-variable) gate has fan-out at most 1, and the gates of B are structured into alternating levels of ORS-of-ANDS-of-ORS. We construct an instance (G, C, χ, s, t, k) of COLORED PATH as follows.

First, we let $C = [n]$, where color i will represent input variable x_i in B. We define G from B by defining a gadget for each gate in B recursively, starting the recursive definition at the output gate of B. For a gate g in B, its gadget is defined by distinguishing the type of g as follows.

If g is an AND-gate, let $g_1, \ldots, g_r$ be the OR-gates, and $x_{i_1}, \ldots, x_{i_p}$ be the input variables that feed into g. The gadget of g is defined as follows. First, create two empty vertices in_g and out_g, which will serve as the "entry" and "exit" vertices of the gadget for g, respectively. For each x_{i_j}, $j \in [p]$, create a vertex v_j colored with color i_j and an entry vertex v_0 and an exit vertex v_{p+1}; form a path

G_0 consisting of the vertices $v_0, v_1, \ldots, v_p, v_{p+1}$. For each OR-gate g_i, $i \in [r]$, recursively construct the gadget G_i for g_i. Connect all these gadgets $G_0, \ldots, G_r$ serially in arbitrary order, starting by identifying in_g with the entry vertex of the first gadget, the exit vertex of the first gadget with the entry of the second, ..., and the exit vertex of the last gadget with out_g. See Figure 17 (bottom) for illustration.

If g is an OR-gate, let $g_1, \ldots, g_r$ be the AND-gates, and $x_{i_1}, \ldots, x_{i_p}$ be the input variables that feed into g. The gadget of g is defined as follows. First, create two empty vertices in_g and out_g, which will serve as the "entry" and "exit" vertices of the gadget for g, respectively. For each x_{i_j}, $j \in [p]$, create a vertex v_j colored with color i_j, and connect each v_j to in_g and out_g. For each AND-gate g_i, $i \in [r]$, recursively construct the gadget G_i for g_i. Connect all these gadgets $G_1, \ldots, G_r$ in parallel by identifying all the entry vertices of $G_1, \ldots, G_r$ with in_g and all their exit vertices with out_g. This completes the description of G. It is not difficult to see that since B with its input variables removed is a tree, the above construction runs in polynomial time and results in a planar graph G. See Figure 17 (top) for illustration.

Finally, set s and t to be the entry and exit vertices of the gadget corresponding to the output gate of B. Clearly, the reduction that takes (B, k) and produces (G, C, χ, s, t, k) runs in time polynomial in the size of the input instance. Next, we prove its correctness.

We will prove the following statement: For any gate g in B, and any assignment τ to B that assigns variables $x_{i_1}, \ldots, x_{i_p}$ the value 1, and all other variables the value 0, τ satisfies g if and only if there is a path P in G from the entry vertex to the exit vertex of the gadget corresponding to g such that P uses a subset of the colors $\{i_1, \ldots, i_p\}$. Notice that the aforementioned statement applied to the output gate of B is precisely the statement of the lemma.

We prove the above statement by induction on the depth of the gate g in B. The base case is when g has depth 1. In this case the input to g consists only of input variables. Suppose first that g is an OR-gate, and let τ be an assignment that assigns exactly variables $x_{i_1}, \ldots, x_{i_p}$ the value 1. Then τ satisfies g if and only if x_{i_j} is an input variable to g, for some $j \in [p]$, which is true if and only if there is a path from the entry vertex of the gadget for g to its exit vertex that uses color i_j. Suppose now that g is an AND-gate, and let τ be an assignment that assigns exactly variables $x_{i_1}, \ldots, x_{i_p}$ the value 1. Then τ satisfies g if and only if the input variables to g form a subset S of $\{x_{i_1}, \ldots, x_{i_p}\}$; let $\eta(S)$ be the indices of the variables in S. Since the gadget for g consists of a path P between the entry and exit vertices of the gadget for g such that $\chi(P) = \eta(S)$, the statement follows.

Suppose, by the inductive hypothesis, that the statement we are proving is true for any gate g of depth $1 \leq a < \ell$, and let g be a gate of depth ℓ. Let $x_{j_1}, \ldots, x_{j_q}$ be the input variables to g, and $g_1, \ldots, g_r$ be the input gates to g. We again distinguish two cases based on the type of g.

Gate g is an OR-gate. Let τ be an assignment that assigns exactly variables $x_{i_1}, \ldots, x_{i_p}$ the value 1. Suppose first that τ satisfies g. Then either τ satisfies an input variable x_{j_z}, $z \in [q]$, or τ satisfies an input AND-gate g_y, $y \in [r]$. If τ satisfies x_{j_z} then there is a path between the entry and exit vertices of the gadget for g that uses color j_z. Otherwise, τ satisfies g_y, $y \in [r]$, and by the inductive hypothesis applied to g_y, there is a path P_y between the entry and exit vertices of the gadget for g_y such that $\chi(P_y) \subseteq \{i_1, \ldots, i_p\}$. From the way the gadget for g was constructed, it follows that P_y is also a path between the entry and exit vertices of the gadget for g. To prove the converse, suppose that there is a path P_g between the entry and exit vertices of the gadget for g that uses a subset of colors in $\{i_1, \ldots, i_p\}$. Either P_g is a path whose only internal vertex corresponds to an input variable, and in such case the input variable is in $\{x_{i_1}, \ldots, x_{i_p}\}$, and g is satisfied; or P_g is a path between the entry and exit vertices of the gadget for an AND-gate g_y that feeds into g, and by the inductive hypothesis, τ satisfies g_y and also g.

Gate g is an AND-gate. Let τ be an assignment that assigns exactly variables $x_{i_1}, \ldots, x_{i_p}$ the value 1. Suppose first that τ satisfies g. Then τ assigns 1 to every input variable x_{j_z} to g, $z \in [q]$. Hence, there is a path P between the entry and exit vertices of the gadget corresponding to $x_{j_1}, \ldots, x_{j_q}$ such that $\chi(P) \subseteq \{i_1, \ldots, i_p\}$. Assignment τ also satisfies each OR-gate g_y, where $y \in [r]$. By the inductive hypothesis, there is a path P_y between the entry and exit vertices of the gadget for g_y such that $\chi(P_y) \subseteq \{i_1, \ldots, i_p\}$. From the construction of g, it follows that the path between the entry and exit vertices of the gadget for g, which is $P_g = P \circ P_1 \circ \cdots \circ P_r$, satisfies $\chi(P_g) \subseteq \{i_1, \ldots, i_p\}$. Conversely, suppose that there is a path P_g between the entry and exit vertices of the gadget for g such that $\chi(P_g) \subseteq \{i_1, \ldots, i_p\}$. Then P_g can be decomposed into a subpath P that traverses the vertices corresponding to $x_{j_1}, \ldots, x_{j_q}$, and subpaths $P_1, \ldots, P_r$, where P_y is a subpath between the entry and exit vertices of the gadget for g_y. Since P traverses the vertices corresponding to $x_{j_1}, \ldots, x_{j_q}$, it follows that $\{x_{j_1}, \ldots, x_{j_q}\} \subseteq \{x_{i_1}, \ldots, x_{i_p}\}$. Since P_y, $y \in [r]$, is a subpath between the entry and exit vertices of the gadget for g_y, by the inductive hypothesis, it follows that τ satisfies g_y. It follows that τ assigns 1 to all input variables to g and satisfies all the input OR-gates to g, and hence, τ satisfies g. □

As it turns out, we can even exclude FPT cost approximation algorithms for Colored Path. We first need the following theorem:

Theorem 6.12 (Corollary 5 of [29]). *Unless* FPT= W[2], Monotone Weighted Boolean Circuit Satisfiability *for circuits with depth* 4 *is not* FPT *cost approximable.*

Corollary 6.13. *Unless* FPT= W[2], Colored Path *parameterized by k is not* FPT *cost approximable.*

Proof. Proceed by contradiction, and assume that there is an FPT cost approximation algorithm $\mathbb{A}$ for Colored Path with approximation ratio ρ. Using $\mathbb{A}$, we give an FPT cost approximation algorithm $\mathbb{B}$ for Monotone Weighted Boolean Circuit Satisfiability for circuits of depth 4 with the same approximation ratio ρ.

The algorithm $\mathbb{B}$ works as follows. Let (B, k) be an instance of Monotone Weighted Boolean Circuit Satisfiability such that B has depth at most 4. The algorithm first transforms B into a monotone circuit B' in the normalized form (see the proof of the Normalization Theorem in Section 23.2.2 of [12] for more details). Since B has depth at most 4, this procedure terminates in polynomial time, results in a polynomial blow-up of the instance size, and preserves precisely all the variables, the solutions and their sizes. Afterwards, $\mathbb{B}$ calls the algorithm of Lemma 6.11 on the instance (B', k) to produce a Colored Path instance (G, C, χ, s, t, k). Finally, $\mathbb{B}$ calls $\mathbb{A}$ on (G, C, χ, s, t, k).

From the construction of $\mathbb{B}$, it follows that any assignment τ to B that assigns variables $x_{i_1}, \ldots, x_{i_p}$ the value 1 and all other variables the value 0, satisfies the formula B if and only if there is an s-t path P in G that uses a subset of the colors $\{i_1, \ldots, i_p\}$. Therefore, $\text{OPT}(B) = \text{OPT}(G, C, \chi, s, t)$, and $\mathbb{B}$ is an FPT cost approximation algorithm for Monotone Weighted Boolean Circuit Satisfiability, for circuits of depth 4, with approximation ratio ρ. However, by Theorem 6.12 such an algorithm cannot exist unless FPT= W[2], and the corollary follows.

□

Remark 6.14. *A noteworthy remark that we close this section with, is to comment on the role that planarity plays in the parameterized complexity of* Colored Path-Con. *If one drops the planarity requirement on the instances of* Colored Path-Con (i.e., *considers* Colored Path-Con *on general graphs), then it follows from the proof of Theorem 6.10 that the resulting problem is* W[SAT]*-hard. This can be seen by adding a single vertex containing all colors, that serves as a "color-connector," to the instance of* Colored Path *produced by the* FPT*-reduction; this modification results in an instance* *of the connected obstacle removal problem on apex graphs, establishing the* W[SAT]*-hardness of this problem on apex graphs.*

7 CONCLUSION

In this paper, we studied the complexity and the parameterized complexity of the Colored Path problem and its geometric counterpart the Obstacle Removal problem. These problems have applications in several areas of computer science, in particular, in motion planning and wireless computing. Our work sheds light on structural parameters that play crucial roles in characterizing the complexity of the problems. Our results generalize and explain several results in the literature, as well as provide new insights into the structure of the problems. The following questions remain open:

- We showed in this paper that Colored Path-Con is FPT parameterized by k and the treewidth of the input graph, and is also FPT parameterized by k and the length of the path sought. The obvious—yet important—question is whether or not Colored Path-Con is FPT parameterized only by k.
- Although Connected Obstacle Removal was shown to be NP-hard even when the obstacles have very simple geometric shape, such as line segments, the complexity of the problem when the obstacles are unit disks remains a long-standing open problem that is worth investigating.
- Finally, as mentioned in the previous section, the complexity of the restriction of Connected Obstacle Removal to instances consisting of convex obstacles, each intersecting at most a constant number of other obstacles, is an interesting open question.

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