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Half and quarter BPS operators in $\mathcal{N} = 4$ super Yang-Mills

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Abstract

In this thesis we perform calculations on the CFT side of the duality between $\mathcal{N} = 4$ supersymmetric Yang-Mills theory and type IIB string theory on $AdS_5 \times S^5$. The results are used to study quantum gravity on AdS.

Chapters 3 and 4 explore the structure and combinatorics of the quarter BPS sector with gauge groups $U(N)$, $SO(N)$ and $Sp(N)$ in the planar free field limit. For $U(N)$, we identify the multi-traces with a word monoid, with aperiodic single traces corresponding to Lyndon words. For $SO(N)$ and $Sp(N)$ we generalise Lyndon words using minimally periodic conditions. We present the quarter-BPS generating function for $SO(N)/Sp(N)$ gauge groups.

Chapter 5 examines the permutation algebras behind operator construction in the free field theory with $SO(N)$ and $Sp(N)$ gauge groups. There is a rich group independent structure, including formulae for correlators expressed purely in terms of permutations. We introduce Schur and restricted Schur bases for the baryonic sector of the $SO(N)$ theory, derive covariant bases for the quarter-BPS sectors of $SO(N)$ and $Sp(N)$ theories, and calculate their correlators.

Chapter 6 studies the projection of the half-BPS sector from the $U(N)$ theory to the $SO(N)/Sp(N)$ theory, dual to an orientifold projection of S^5 to $\mathbb{RP}^5$. This is characterised by a plethystic refinement of Littlewood-Richardson coefficients, expressible in terms of the combinatorics of domino diagrams. A second expression for the projection is derived in terms of a product of $SO(N)/Sp(N)$ giant graviton states.

Chapter 7 looks at the quarter-BPS sector of the $U(N)$ theory at weak coupling. Multi-symmetric functions allow systematic study of the finite N properties, involving combinatorics of set partitions. We construct a quarter-BPS, finite N-compatible, $U(2)$ covariant, orthogonal basis, labelled by a $U(N)$ Young diagram and a multiplicity, for which we derive precise counting results. These are interpreted as quarter-BPS deformations of the half-BPS giant graviton states.

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Declaration

I, Christopher Harry Lewis-Brown, confirm that the research included within this thesis is my own work or that where it has been carried out in collaboration with, or supported by others, that this is duly acknowledged below and my contribution indicated. Previously published material is also acknowledged below.

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Details of collaboration and publications:
This thesis describes research carried out with my supervisor Sanjaye Ramgoolam, which was published in [1, 2]. It also contains some unpublished material.

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Chapter 1

Introduction

The AdS/CFT correspondence [3–5] has revolutionised theoretical physics over the last twenty years. It is a conjectural identification between a string theory on $d+1$ dimensional Anti-de-Sitter space (times a compact manifold) and a conformal theory living on the d dimensional boundary manifold. What makes this conjecture both incredibly interesting and difficult to prove, is the strong-weak nature of the duality. The weakly coupled gauge theory, accessible to study via perturbation theory, is dual to strongly interacting stringy physics, for which we have no good mathematical description, and vice versa. Consequently, assuming the validity of the conjecture, new features of strongly coupled conformal field theory and gravity can be investigated through the dual description. For a thorough review of AdS/CFT and its varied applications see [6].

The most studied example of the correspondence, and the one explored in this thesis, is in $d = 4$ dimensions with the maximal amount of supersymmetry. On the AdS side, this is type IIB string theory on $AdS_5 \times S^5$, while on the CFT side, we have the $\mathcal{N} = 4$ super Yang-Mills theory with $U(N)$ gauge group.

There are two parameters needed to define the Yang-Mills theory, the coupling g_{YM} and the rank N of the gauge group. In the dual theory, the two parameters are the string coupling g_s and the ratio $\frac{R}{l_s}$ between the radius of AdS and the string length. These are identified via

$$\frac{R^4}{4\pi l_s^4} = g_{YM}^2 N = \lambda \qquad\qquad g_s = g_{YM}^2 = \frac{\lambda}{N} \tag{1.0.1}$$

where λ is the 't Hooft coupling.

It was shown by 't Hooft [7] that in a $U(N)$ gauge theory with fields in the adjoint, one can take the large N limit while keeping λ fixed and obtain a perturbative expansion in powers of $\frac{1}{N}$ with the gth term corresponding to double line Feynman diagrams on surfaces of genus g. This matches the genus expansion of string theory, and was an early indication of string-gauge duality.

A result of particular importance coming from the 't Hooft expansion is exactly when the perturbative description of a $U(N)$ gauge theory is valid. Naively, one would expect this to be when $g_{YM} \ll 1$, but if N is large then contributions from loops in the diagrams can outweigh the coupling constant, rendering the perturbative description invalid. In fact it is when $\lambda \ll 1$ that one can use the Feynman diagram computations.

The strong-weak nature of the duality is apparent from (1.0.1). Perturbative gauge theory is valid at small λ, meaning the radius of curvature R of the AdS space is comparable to the string length l_s. In this regime, stringy effects become important, and the supergravity description can no longer be trusted. On the other hand, at large λ, inaccessible to gauge theory calculations, the curvature is much greater than the string length scale, and the supergravity description is valid.

In principle, the matching between the two sides of the AdS/CFT duality depends on two factors. Firstly, the Hilbert space must fit into the same representations of the global symmetry group $PSU(2,2|4)$. In particular the energy of an AdS state must match the scaling dimension of a CFT local operator (by the operator-state correspondence, we use CFT local operators rather than states). Secondly, the correlators of the AdS states and CFT local operators must agree. The conjectural identification between states, operators and their respective correlators was given in [4, 5].

In general, these correlators are very hard to calculate precisely, and many can only be given order by order in perturbation theory. An important and influential exception to this rule is the *planar limit* of the $\mathcal{N} = 4$ super Yang-Mills theory. This refers to the $N \to \infty$ limit with λ fixed, so only the leading term in the 't Hooft expansion survives, and therefore only planar Feynman diagrams contribute. It was proved in [8] that this theory is integrable, allowing the application of powerful mathematical techniques that provide concrete results to all orders in λ. This has become a vast and extremely fruitful area of research. A review of this huge topic can be found in [9].

A more difficult problem is to study the correspondence while including sub-leading terms in the $\frac{1}{N}$ expansion, or explicitly at finite N. While many important results have been found, the understanding is not as complete as for the planar limit.

In this thesis, we focus on half- and quarter-BPS local primary operators in $\mathcal{N} = 4$ super Yang-Mills. These are annihilated by, respectively, a half and a quarter of the 16 Poincaré supercharges in the theory. As a result, they live in short or semi-short representations and their conformal dimension is determined by their charges under the $SU(4)$ R-symmetry and the Lorentz group. This restriction means concrete results can be found even at finite N.

1.1 Half-BPS sector

In $\mathcal{N} = 4$ super Yang-Mills, there are 6 real scalar fields ϕ_a, filling out the six dimensional representation of the R-symmetry $SU(4)$, which is a double cover for $SO(6)$. On the AdS side of the duality, the $SO(6)$ symmetry corresponds to rotations of the S^5 factor. The 6 scalar fields are combined into 3 complex scalar fields $X_i = \phi_i + i\phi_{i+3}$, hiding some of the R-symmetry and leaving only an $SU(3) \times U(1)$ subgroup apparent. Half-BPS operators in the the CFT are exactly the multi-traces of one of the complex scalars, which we will refer to as X.

From arguments based purely on the representation theory of $PSU(2, 2|4)$, the spectrum of half-BPS multiplets remains unchanged for any value of the coupling constant [10]. Additionally, there are strong non-renormalisation theorems on correlators [11–15], so calculations can be performed on either side of the duality at any value of the coupling and compared directly.

The half-BPS states have been well studied, and much is known about them. The spectrum of single trace operators corresponds to Kaluza-Klein gravitons compactified on the S^5 factor of $AdS_5 \times S^5$ [5]. The operator $\mathrm{Tr}X^n$ has charge n under a $U(1)$ subgroup of the $SU(4)$ R-symmetry, and the dual AdS state has n units of angular momentum around the S^5. More generally, multi-trace operators are dual to multi-graviton states.

This identification between operators and Kaluza-Klein gravitons is valid if $n \sim O(1)$ compared to N. However, if n is taken to grow in size comparable to N, then the large energy of the state causes a backreaction and the supergravity approximation to the full string theory no longer holds. The behaviour of the BPS states as n scales with respect to N is an important problem that has illuminated many interesting physical aspects of the two theories.

The first qualitatively different behaviour is observed when $n \sim \sqrt{N}$. It was found in [16] that operators of the form $\mathrm{Tr}X^J$, where $J \sim \sqrt{N}$, can be understood as strings with large angular momentum on the S^5. These *BMN states* can be identified with strings in the pp-wave background, which are an unusual case of strings that can be quantised exactly, with the spectrum exactly matching that of the super Yang-Mills operators.

Continuing to scale n, it was demonstrated in [17] that gravitons with angular momentum $J \sim N$ expand into a D3-brane wrapped around a 3-cycle in the S^5. These *giant gravitons* also explained the string theoretical origins of the *stringy exclusion principle* [18], a cut-off in the spectrum when the angular momentum of a single graviton exceeds N. The radius of the S^3 wrapped by a giant graviton increases with the angular momentum, with an upper bound given by the radius of the S^5. This upper bound corresponds exactly to $J \leq N$.

The gauge theory origins of the stringy exclusion principle are much simpler, emerging from relations between traces at finite N. The Cayley-Hamilton theorem for a matrix implies that a trace $\mathrm{Tr}X^n$ with $n > N$ can be re-written in terms of products of traces of size $\leq N$.

Another family of brane states, called *dual giants* or *AdS giants* were found in [19]. These are D3-branes wrapping a S^3 within the AdS_5 factor, with similar properties to sphere giants except they do not suffer from a stringy exclusion principle.

After early identifications of sphere giants as sub-determinant operators [20, 21], a complete basis for half-BPS operators in the gauge theory was found in [22], valid for all n and N. These are operators $\mathcal{O}_R = \chi_R(X)$ labelled by a Young diagram R with n boxes and constrained to have length no greater than N. They can be simply understood as the analytic continuation of the $U(N)$ character $\chi_R(U)$ to a generic complex matrix X.

The basis $\mathcal{O}_R$ allowed a simple identification of all giant graviton states, dual giants, and a smooth interpolation between the two. Take R to be a single column of length $J \sim N$. This is dual to a giant graviton of angular momentum J, and the cut-off on the length of the Young diagrams corresponds to the stringy exclusion principle. Similarly, a Young diagram with a single row of length $J \sim N$ is dual to an AdS giant of angular momentum J, with no cut-off on the length of the row. Young diagrams with several long columns or rows correspond to multi-giant states. This correspondence between Young diagram operators and giants has been confirmed from a number of directions: holographic comparison of correlators of two Young diagrams with a trace [23–26], moduli space quantisation [27, 28] and strings attached to giants [29–35].

Beyond the identification of $\mathcal{O}_R$ with giant gravitons, [22] also demonstrated that the half-BPS sector of $\mathcal{N} = 4$ SYM is dual to N non-interacting fermions in a 1-dimensional harmonic oscillator. The correspondence between free fermions, half-BPS operators and giant gravitons was further developed in [36].

A complete classification of half-BPS excitations of $AdS_5 \times S^5$ was given in [37]. These are referred to as LLM geometries, and are specified by a colouring of the LLM plane, a 2-dimensional plane of boundary conditions split into coloured regions and empty regions. Since this is a complete characterisation, all scaling behaviour of half-BPS states can be seen in LLM solutions. The simple $AdS_5 \times S^5$ appears as a circle. Small perturbations of this correspond to Kaluza-Klein gravitons. A donut with large radius corresponds to a sphere giant, while a central circle with a distant ring around it is an AdS giant. When the energy of the BPS state is $O(N^2)$, we find different topological or geometrical spaces. These correspond to states where the energy has grown sufficiently large that the backreaction has changed the geometry of the space.

The colouring of the LLM plane is interpreted as the 2-dimensional phase space of N fermions. Coloured areas of the plane, of total area N, correspond to occupation by fermions. This gives a natural correspondence between Young diagram operators and LLM geometries via the free fermion interpretation. The detailed matching is described in [38].

This scaling behaviour of the half-BPS states on both sides of the duality represents a thorough understanding of this sub-sector of the $U(N)$ $\mathcal{N} = 4$ super Yang-Mills theory and its $AdS_5 \times S^5$ dual.

1.2 Quarter-BPS sector

There has been much work done on developing an understanding of the quarter-BPS sector that can compare with the half-BPS equivalent. While many important results have been found, this project is still incomplete.

We begin with the AdS side of the duality, as more is known here than for the dual gauge theory. Giant gravitons were generalised to the quarter and eighth-BPS sectors in [39], with worldvolumes given by the intersection of holomorphic surfaces in $\mathbb{C}^3$ with a 5-sphere. These represent the entire quarter-BPS sector (for states with energy of the appropriate order), but only a subsector of the eighth-BPS sector, since world-volume fermions or gauge fields are set to zero [27]. These giant gravitons were quantised in [27], where the authors proved that the space of quarter/eighth-BPS states correspond to a system of N non-interacting bosons moving in a 2/3-dimensional harmonic oscillator. Dual eighth-BPS giants were quantised in [40] and the same result was obtained for the Hilbert space.

While giant gravitons have been generalised to the quarter and eighth-BPS sector, these branes have not been studied as much as the half-BPS equivalents, and their properties are not as well understood.

The quarter and eighth-BPS equivalent of LLM geometries were derived in [41]. Like the half-BPS case, solutions are expressed in terms of boundary conditions on 4 and 6 dimensional spaces. However the procedure to generate the metric from the boundary conditions is more difficult, involving non-linear differential equations, and the relation to free fermion dynamics no longer holds.

On the gauge theory side of the correspondence, a lack of non-renormalisation theorems means quarter-BPS operators at generic coupling are difficult to find. Representation theory only protects the spectrum of very special quarter-BPS multiplets (those in the $SU(4)$ R-symmetry representation with Young diagram $[n-1, 1]$). Generic ones are not protected.

More is known in the free field theory with $\lambda = 0$. In this limit, quarter-BPS operators are exactly the multi-traces of two of the complex scalar fields, which we will write as X and Y. Various Young diagram bases, generalisations of the half-BPS operators $\mathcal{O}_R$, have been developed for this space [42–46].

The spectrum changes discontinuously when interactions are turned on, as some of the semi-short multiplets recombine into generic long multiplets which acquire anomalous dimensions. From index calculations [47], it is expected that the spectrum then remains fixed from $\lambda \ll 1$ to $\lambda \gg 1$. There are also non-renormalisation results for correlators of quarter-BPS operators [15,48], and it is believed that 2 and 3-point functions of quarter-BPS operators do not get renormalised as we travel from weak to strong coupling.

Finding quarter-BPS operators at weak coupling is a difficult problem, and has not yet been solved in full generality. A systematic method to find them was developed in [49] and applied at low dimension. This approach gave candidate states and then found the BPS operator by orthogonalising to descendent states.

In [50], the dilatation operator was shown to be an effective method for finding quarter-BPS operators at weak coupling. After diagonalising the dilatation operator, the quarter-BPS states are exactly those in the zero eigenspace. A lot of work has been done to diagonalise the one-loop dilatation operator on the free-field bases. In [51], one of the Young diagram bases called the *covariant basis* was investigated, and a method was developed to find quarter-BPS states. An alternative basis, called the *restricted Schur* basis, was explored in [31, 35, 52]. This latter approach succeeded in finding quarter and eighth-BPS operators for Young diagrams in the distant corner limit. Finally, a special class of quarter BPS operators at weak coupling was found to be related to Brauer algebra constructions [53].

1.3 $SO(N)$ and $Sp(N)$ gauge theories

In a different direction, one can look at $\mathcal{N} = 4$ super Yang-Mills with $SO(N)$ and $Sp(N)$ gauge groups and ask whether they have a dual string theory description.

In the formalism of 't Hooft, two line Feynman diagrams are used to describe the perturbative expansion of a $U(N)$ gauge theory with fields in the adjoint representation. These two lines have arrows pointing in opposite directions, as the adjoint is composed of a product between the fundamental and anti-fundamental representations. For $SO(N)$ and $Sp(N)$, the fundamental representation is real, and isomorphic to its complex conjugate. It follows that the lines no longer have a preferred direction, and consequently, in addition to the normal genus expansion of the $U(N)$ theory, the perturbative description admits Feynman graphs that live on non-orientable surfaces such as $\mathbb{RP}^2$ [54]. This implies not only that there is a string theory description of $SO(N)$ and $Sp(N)$ gauge theories, but that it is given in terms of non-orientable strings.

The AdS/CFT dual of $\mathcal{N} = 4$ super Yang-Mills with these gauge groups was found in [55]. The S^5 factor in the standard AdS/CFT correspondence is replaced by a $\mathbb{RP}^5$ factor by identifying $x \sim -x$ for $x \in S^5$. At the same time, the string worldsheet has its orientation reversed. This is called the orientifold quotient, and maps the $U(N)$ gauge theory to $SO(N)$ or $Sp(N)$ theories. The gauge group that emerges depends on the topological class of the two-form fields coupled to fundamental strings and D-strings.

Indeed, the $SO(N)$ theory emerges quite naturally from this description. One way of arriving at the $U(N)$ correspondence is to place N D3-branes at the origin of Minkowski space and note that these deform the near horizon geometry from flat space to $AdS_5 \times S^5$. In this picture, the six real scalar fields of the Yang-Mills theory emerge as the transverse displacement of the branes from the origin, which appear in the near-horizon geometry as the radial direction of AdS times the S^5. Correspondingly, the identification $x \sim -x$ on S^5 sets $X_i \sim -X_i$, where X_i $(i = 1, 2, 3)$ are the three complex scalar fields.

In tandem, the (a, b)th component of the gauge field corresponds to the amplitude of a string stretching from the ath brane to the bth. Reversing the orientation of a string, this instead stretches from the bth to the ath. Therefore the gauge indices are transposed. It follows that the orientifold quotient sets $X_i \sim -X_i^T$, which is exactly the map from the complexified adjoint of $\mathfrak{u}(N)$ to the complexified adjoint of $\mathfrak{so}(N)$. In order for the $Sp(N)$ theory to emerge from the orientifold quotient, one needs to consider the subtle topological factors, and a more detailed description is needed.

For half-BPS states of conformal dimension $n = O(1)$, the picture for $SO(N)/Sp(N)$ gauge theory and its dual string theory is similar to that of $U(N)$. Kaluza-Klein gravitons, compactified on the $\mathbb{RP}^5$, correspond to single trace states $\mathrm{Tr}X^n$. From the gauge theory, n must be even due to the constraints on X, while in the string theory, this is because the $U(1)$ R-symmetry charge of X is twice the quantised angular momentum on the $\mathbb{RP}^5$. Multi-trace operators then correspond to multi-particle graviton states.

The allowed brane wrappings on $AdS_5 \times \mathbb{RP}^5$ were investigated in [55]. For both $SO(N)$ and $Sp(N)$, there are the standard giant graviton branes wrapped on 3-spheres within either the AdS_5 or $\mathbb{RP}^5$ factors. However, for $SO(N)$ with N even, there is an additional brane state wrapped around a $\mathbb{RP}^3$ within the $\mathbb{RP}^5$. In the gauge theory, this is a Pfaffian operator, consisting of $\frac{N}{2}$ copies of the scalar fields X_i contracted using the $SO(N)$ invariant tensor $\varepsilon_{a_1 \dots a_N}$. Evidence that Pfaffian operators should be considered as D-brane states was presented in [21], which demonstrated that the 't Hooft expansion of such states included string worldsheets with boundary.

A Young diagram basis for half-BPS $SO(N)$ and $Sp(N)$ multi-trace operators was developed in [56, 57]. For $SO(N)$ with N even, a basis for Pfaffian operators was also derived, labelled by Young diagrams with first column of length N. It is expected that these bases are dual to giant gravitons with the appropriate brane wrapping, in much the same way as for the $U(N)$ Young diagram basis operators.

Further, these Young diagram bases can be interpreted as $\frac{N}{2}$ non-interacting fermions in a harmonic oscillator potential. This relates to the bubbling orientifold description of $SO(N)$ and $Sp(N)$ theory, derived in [58]. This replaces the LLM plane with a half-plane, corresponding to fermions moving on a half-line. The free fermion picture nicely summarises the correspondence between bubbling orientifold geometries of $AdS_5 \times \mathbb{RP}^5$ and half-BPS operators in the $SO(N)$ and $Sp(N)$ theories.

There are few studies of the quarter-BPS sector for the $SO(N)$ and $Sp(N)$ theories. A Young diagram basis was constructed for the free field theories in [59, 60], though this did not include Pfaffian operators. Little is known about the quarter-BPS sector at weak coupling.

1.4 Outline of thesis

We begin this thesis by giving some of the necessary mathematical background for the following chapters. Chapter 2 introduces the permutation group S_n, and how it can be used to construct gauge invariant operators in the $U(N)$ theory taking account of finite N effects.

In chapter 3 we study the quarter-BPS sector of $\mathcal{N} = 4$ super Yang-Mills with $U(N)$ gauge group in the free field limit. This can be split into two distinct halves. The first part examines the combinatorics of the matrix words that appear in the multi-trace operators in the planar limit. The generating function for the planar free field quarter-BPS sector was derived in [61], and is also the generating function for a graded, non-commutative monoid. We investigate the correspondence between these two systems, the structure this entails, and how this structure is reflected in the large N generating function. The second half summarises the role of permutations in the quarter-BPS sector and the different bases that can be used to describe it.

Chapter 4 studies the word combinatorics of the quarter-BPS sector for the $SO(N)$ and $Sp(N)$ gauge theories, again in the planar free field theory. The anti-symmetry of the matrix fields induces relations between traces of different matrix words. We study these relations, examine the structure of the space of quarter-BPS sector, derive the large N generating function and describe how different expressions for this function reflect the structure of the space.

Chapter 5 studies the same sector as chapter 4 but with N finite. We describe the rich structure of permutation algebras lying behind the construction of BPS operators and give gauge group independent characterisations of operators and correlators.

Chapter 6 analyses in detail the dual description of the orientifold quotient taking $AdS_5 \times S^5$ to $AdS_5 \times \mathbb{RP}^5$. We focus on the half-BPS $U(N)$ multi-trace operators and map this to an $SO(N)/Sp(N)$ operator by replacing the generic complex matrix X in the complexified adjoint of $\mathfrak{u}(N)$ with an anti-symmetric matrix in the complexified adjoint of $\mathfrak{so}(N)/\mathfrak{sp}(N)$. By using the Young diagram bases in both gauge groups, we obtain a description of the orieintifold quotient of the dual giant gravitons.

Finally, chapter 7 looks at weak coupling quarter-BPS operators in the $U(N)$ gauge theory. We use the mathematics of multi-symmetric functions to derive the finite N behaviour of this sector and give a construction algorithm for a Young diagram basis.

Chapter 2

Mathematical Preliminaries: Permutations, Traces and Partitions

Permutations, and the wider symmetric group algebra $\mathbb{C}(S_n)$, have proved an important tool in theoretical physics. For an overview of their varied applications see [62]. The connection between symmetric and unitary groups established by Schur-Weyl duality has played a major role in studying the BPS sector of $\mathcal{N}=4$ super Yang-Mills. The use of permutations has been the key technical step in allowing the explicit construction of operator bases, calculation of correlators and understanding the restrictions imposed by finite N [22, 29, 32, 42–46, 56, 57, 59, 60]. They have also, through the lens of AdS/CFT provided a new viewpoint on the stringy exclusion principle and have enabled further study into giant gravitons and the BPS sector of strongly coupled type IIB string theory on $AdS_5 \times S^5$.

We begin this chapter by reviewing how one constructs multi-traces from permutations in S_n acting on the n-fold tensor product of the fundamental of $U(N)$. We describe how invariances on $\mathbb{C}(S_n)$ result in different permutation algebras controlling this construction, and in particular define the algebras relevant for the half and quarter-BPS sectors of $\mathcal{N}=4$ super Yang-Mills. Partitions are then discussed along with their Young diagrams. Next we do a quick review of S_n representation theory, followed by an explanation of Schur-Weyl duality and the consequences for both operators and algebras when we allow $n<N$. Finally, we detail the uses of permutations in calculating correlators, before finishing with a description of symmetric functions and their relation to the half-BPS sector.

2.1 Constructing traces from permutation algebras

Let V be the carrier space for the N-dimensional fundamental representation of $U(N)$, and consider the n-fold tensor product $V^{\otimes n}$. S_n acts on this space by permutation of the factors. We can write this action in components as

$$\sigma^I_J = \sigma^{i_1 i_2 \dots i_n}_{j_1 j_2 \dots j_n} = \delta^{i_1}_{j_{\sigma(1)}} \delta^{i_2}_{j_{\sigma(2)}} \dots \delta^{i_n}_{j_{\sigma(n)}} = \delta^{i_{\sigma^{-1}(1)}}_{j_1} \delta^{i_{\sigma^{-1}(2)}}_{j_2} \dots \delta^{i_{\sigma^{-1}(n)}}_{j_n} \tag{2.1.1}$$

For permutations, we use the multiplication convention $(\sigma\tau)(i) = \tau(\sigma(i))$, or equivalently $(\sigma\tau)^I_K = \sigma^I_K \tau^K_J$.

By conjugating operators on the tensor space, S_n can act on these as well. Consider n matrices $Z_1, Z_2, \dots, Z_n$ in the adjoint of $U(N)$, and write $\mathbb{Z}$ for the tensor product $Z_1 \otimes Z_2 \otimes \dots \otimes Z_n$. Then

$$\begin{aligned}
\left(\sigma \mathbb{Z} \sigma^{-1}\right)^I_L &= \delta^{i_{\sigma^{-1}(1)}}_{j_1} \dots \delta^{i_{\sigma^{-1}(n)}}_{j_n} (Z_1)^{j_1}_{k_1} \dots (Z_n)^{j_n}_{k_n} \delta^{k_1}_{l_{\sigma^{-1}(1)}} \dots \delta^{k_n}_{l_{\sigma^{-1}(n)}} \\
&= (Z_1)^{i_{\sigma^{-1}(1)}}_{l_{\sigma^{-1}(1)}} (Z_2)^{i_{\sigma^{-1}(2)}}_{l_{\sigma^{-1}(2)}} \dots (Z_n)^{i_{\sigma^{-1}(n)}}_{l_{\sigma^{-1}(n)}} \\
&= \left(Z_{\sigma(1)}\right)^{i_1}_{l_1} \left(Z_{\sigma(2)}\right)^{i_2}_{l_2} \dots \left(Z_{\sigma(n)}\right)^{i_n}_{l_n} \\
&= \left[\sigma\left(\mathbb{Z}\right)\right]^I_L
\end{aligned} \tag{2.1.2}$$

where the last line defines $\sigma(\mathbb{Z})$.

In order to construct a $U(N)$ trace operator from this, we simply take the $V^{\otimes n}$ trace of a permutation σ multiplied by $\mathbb{Z}$.

$$\mathcal{O}_\sigma = \mathrm{Tr}\left(\sigma \mathbb{Z}\right) = (Z_1)^{i_1}_{i_{\sigma(1)}} (Z_2)^{i_2}_{i_{\sigma(2)}} \dots (Z_n)^{i_n}_{i_{\sigma(n)}} \tag{2.1.3}$$

The cycles of σ determine the structure of the trace. A single cycle $(a_1, a_2, \dots, a_k)$ in σ leads to the single trace $\mathrm{Tr} Z_{a_1} Z_{a_2} \dots Z_{a_k}$, while a permutation with several cycles leads to a multi-trace. For example

$$\sigma = (1,2,\dots,n) \qquad \mathrm{Tr}(\sigma\mathbb{Z}) = \mathrm{Tr} Z_1 Z_2 \dots Z_n \tag{2.1.4}$$

$$\sigma = (1,2,3,4)(5,6,7,8) \qquad \mathrm{Tr}\left(\sigma\mathbb{Z}\right) = \left(\mathrm{Tr} Z_1 Z_2 Z_3 Z_4\right)\left(\mathrm{Tr} Z_5 Z_6 Z_7 Z_8\right) \tag{2.1.5}$$

$$\sigma = (1,5,3)(2,6) \qquad \mathrm{Tr}(\sigma\mathbb{Z}) = \left(\mathrm{Tr} Z_1 Z_5 Z_3\right)\left(\mathrm{Tr} Z_2 Z_6\right)\left(\mathrm{Tr} Z_4\right) \tag{2.1.6}$$

We can reverse this relation. Given a multi-trace of $Z_1, Z_2, \dots, Z_n$ in which each matrix only appears once, we can identify the permutation $\sigma \in S_n$ which produces this trace. Therefore the full space of degree $(1,1,\dots,1)$ multi-traces is in correspondence with the group algebra $\mathbb{C}(S_n)$ via the identification $\mathcal{O}_\sigma \leftrightarrow \sigma$.

An intuitive way to think about the index contractions in (2.1.3) is to use a contraction diagram like the one given in figure 2.1. The permutation σ connects the index

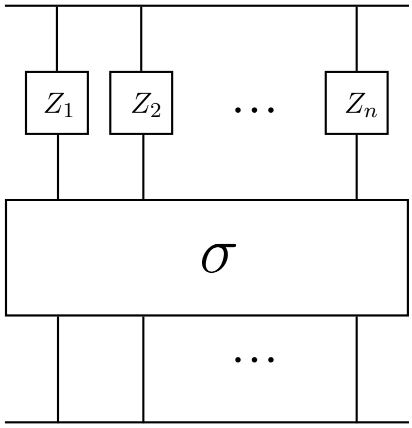


Figure 2.1: Diagrammatic representation of the index contraction of n matrices with a permutation. Each vertical line represents an index, while the horizontal lines at the top and bottom indicate we trace over these indices.

lines of the different Z_i. By connecting the corresponding index lines at the top and bottom, one acquires a set of loops, each corresponding to a single trace factor.

A simple consequence of this identification between permutations and traces is obtained by setting $Z_1 = Z_2 = \cdots = Z_n = I$ the identity matrix. Then each trace factor contributes N, and we get

$$\operatorname{Tr}\sigma = \sigma^{i_1 i_2 \ldots i_n}_{i_1 i_2 \ldots i_n} = N^{c(\sigma)} \tag{2.1.7}$$

where $c(\sigma)$ is the number of cycles in σ.

Similarly, to study an M-matrix system $X_1, X_2, \ldots, X_M$, we set each of the Z_i to be equal to one of the X_j. To study the degree $(n_1, n_2, \ldots, n_M)$ subspace of the matrix system, we set $Z_1, Z_2, \ldots, Z_{n_1}$ equal to X_1; $Z_{n_1+1}, \ldots, Z_{n_1+n_2}$ equal to X_2 and so on. Looking at $\mathbb{X}$, we see that there is a subgroup of S_n that leaves this invariant under the action defined in (2.1.2). For any $\tau \in S_{n_1} \times S_{n_2} \times \cdots \times S_{n_M}$, we have $\tau(\mathbb{X}) = \mathbb{X}$. Therefore

$$\mathcal{O}_\sigma = \mathcal{O}_{\tau\sigma\tau^{-1}} \tag{2.1.8}$$

where the notation $\mathcal{O}_\sigma = \operatorname{Tr}(\sigma\mathbb{X})$ is used to denote both the n and M-matrix operator. It will be clear from context which is under discussion.

It follows that the degree $(n_1, n_2, \ldots, n_M)$ subspace of the M-matrix system corresponds to the subalgebra of $\mathbb{C}(S_n)$ invariant under conjugation by the subgroup $H = S_{n_1} \times S_{n_2} \times \cdots \times S_{n_M}$. This is denoted

$$\mathcal{A}_{n_1, n_2, \ldots, n_M} = \mathcal{A}_H = \left\{\alpha : \alpha = \sigma\alpha\sigma^{-1} \text{ for all } \sigma \in H\right\} \tag{2.1.9}$$

The case $M = 1$ describes the half BPS sector of $\mathcal{N} = 4$ SYM. The conjugating subgroup is the entirety of S_n, so the subalgebra of interest is invariant under

$$\alpha \to \sigma\alpha\sigma^{-1} \qquad\qquad \sigma \in S_n \tag{2.1.10}$$

The invariant algebra $\mathcal{A}_n$ is the centre of $\mathbb{C}(S_n)$, and can be described in terms of the standard conjugacy classes of S_n or the irreducible representations, which are both reviewed in section 2.3. The consequences of understanding this algebra in the setting of the half BPS sector were first explored in [22].

When considering just the single complex matrix X, a permutation σ of cycle type $p = [\lambda_1, \lambda_2, \dots, \lambda_k] = [1^{p_1}, 2^{p_2}, \dots]$ (p is a partition, introduced in section 2.2) produces the trace

$$\mathrm{Tr}(\sigma\mathbb{X}) = T_p = \prod_{i=1}^{k} \mathrm{Tr}X^{\lambda_i} = \prod_i \left(\mathrm{Tr}X^i\right)^{p_i} \tag{2.1.11}$$

Setting $M = 2$, we obtain the quarter BPS sector of $\mathcal{N} = 4$ SYM. The invariant algebra $\mathcal{A}_{n_1,n_2}$ is invariant under the action

$$\alpha \to \sigma\alpha\sigma^{-1} \qquad\qquad \sigma \in S_{n_1} \times S_{n_2} \tag{2.1.12}$$

$\mathcal{A}_{n_1,n_2}$ has been studied in [63] and we will review the conjugacy class and Fourier description in sections 3.5.3 and 3.6.1 respectively.

For the 2-matrix case we call our matrices X and Y rather than X_1 and X_2, and will use the notation $X^{\otimes n_1}Y^{\otimes n_2} = \mathbb{X}$. The form of the traces arising from $\mathcal{A}_{n_1,n_2}$ is harder to describe, involving partitions labelled by the Lyndon words of a monoid on two letters. We give this description in section 3.5.2.

The eighth BPS sector of $\mathcal{N} = 4$ SYM is larger than just multi-traces of three matrices, since it also includes fermion contractions [50]. However, the scalar component can be found by setting $M = 3$. Many of the techniques used for the $M = 2$ case are directly applicable here, and the results can be generalised very simply, so we do not study this in detail.

Other subalgebras of $\mathbb{C}(S_n)$ become relevant when we consider the traces with $SO(N)$ or $Sp(N)$ gauge group in chapter 5 or symmetrised traces in chapter 7.

2.2 Partitions

The conjugacy classes and irreducible representations of S_n are labelled by integer partitions p of n, for which we use the standard notation $p \vdash n$.

We write partitions in two ways, either in components: $p = [\lambda_1, \lambda_2, \dots]$, where the λ_i are weakly decreasing, or in terms of the multiplicities of i as a component of p: $p = [1^{p_1}, 2^{p_2}, 3^{p_3}, \dots]$. To interchange between the two, p_1 is the number of λs equal to 1, p_2 is the number of λs equal to 2 etc.

The sum of a partition p is denoted by

$$|p| = \sum_i \lambda_i = \sum_i i p_i = n \tag{2.2.1}$$

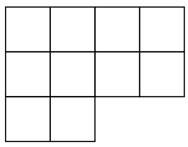
and the number of components by

$$l(p) = \#\text{ of non-zero } \lambda_i = \sum_i p_i \tag{2.2.2}$$

Since all permutations of cycle type p have the same sign, we define the sign of a partition to be the sign of any permutation with that cycle type. In terms of components and multiplicities

$$(-1)^p = \prod_i (-1)^{\lambda_i + 1} = \prod_{i \text{ even}} (-1)^{p_i} \tag{2.2.3}$$

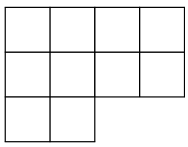
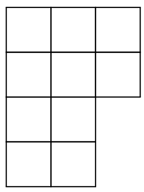
The Young diagram of a partition p is a visual representation, formed by arranging boxes such that the number of boxes in each row corresponds to the components of the partition. So for example the partition $p = [4, 4, 2]$ has the Young diagram

$Y(p) =$  (2.2.4)

We will not distinguish between a partition and its Young diagram, and will use the terms interchangeably. We denote Young diagrams in the same way as a partition, $p \vdash n$.

At various points in this thesis, we will place numbers within the boxes of a Young diagram. This is called a Young tableau, and there are many different varieties that we will come across. Rather than introduce them all here, we will wait until each type is relevant and define them then.

For a partition $p \vdash n$, we denote the conjugate (transposed) partition by p^c. Visually, this reflects the Young diagrams in the diagonal, for example

$p =$  $\longleftrightarrow$ $p^c =$  (2.2.5)

The relation in terms of components is

$$\lambda_i(p^c) = (\#\text{ components of } p \text{ that are } \geq i) \tag{2.2.6}$$

Two partitions $p = [\lambda_1, \lambda_2, \ldots, \lambda_k] \vdash n$ and $q = [\mu_1, \mu_2, \ldots, \mu_l] \vdash m$ can be combined into a partition $p + q \vdash n + m$ by adding together the components

$$p + q = [\lambda_1 + \mu_1, \lambda_2 + \mu_2, \ldots] \tag{2.2.7}$$

If one partition has length greater than the other (e.g. $k \geq l$), then a suitable number of zeros is appended to the shorter partition in order to define the components needed for the addition (e.g. $\mu_{l+1} = \cdots = \mu_k = 0$). Addition of partitions can be thought of intuitively as concatenating their two Young diagrams horizontally.

For $p \vdash n$, we define the partition $2p \vdash 2n$ by

$$2p = p + p \tag{2.2.8}$$

which has components double that of p. For any positive integer k, we define the partition $kp \vdash kn$ similarly.

We could also combine Young diagrams by concatenating them vertically. This can be formalised by considering $p = [1^{p_1}, 2^{p_2}, \ldots]$ and $q = [1^{q_1}, 2^{q_2}, \ldots]$ in terms of their multiplicities. We define

$$p \cup q = \left[1^{p_1+q_1}, 2^{p_2+q_2}, \ldots\right] \tag{2.2.9}$$

This notation for the two ways of 'adding' permutations was used in [64].

2.3 S_n representation theory

2.3.1 Conjugacy classes

The conjugacy classes in S_n are labelled by a partition $p \vdash n$, where the members of the p conjugacy class are just the permutations with cycle type p. For $\sigma \in S_n$ of cycle type p the centraliser of σ is define to be the subgroup of S_n that commutes with σ. The form of this group is given in section 3.5. For now, we only need the size of the centraliser, which is

$$z_p = \prod_i i^{p_i} p_i! \tag{2.3.1}$$

Using the orbit-stabiliser theorem [65] then tells us that the size of the conjugacy class (number of elements in S_n with cycle type p) is $\frac{n!}{z_p}$.

2.3.2 Representations

The irreducible representations of S_n are also labelled by partitions. The dimension d_R of the representation $R \vdash n$ is given in terms of the 'hook lengths' of the boxes in the Young diagram of R. The hook length of a box $b \in R$ is the number of boxes contained in the 'hook' of b, consisting of b, all boxes to the right of b in its row, and all boxes below b in its column. For example, using the same Young diagram as in (2.2.4), the hook lengths of each box are

$$\begin{array}{|c|c|c|c|}\hline 6 & 5 & 3 & 2 \\ \hline 5 & 4 & 2 & 1 \\ \hline 2 & 1 \\ \cline{1-2}\end{array} \tag{2.3.2}$$

Then H_R is defined to be the product of the hook lengths of each box in R. So for (2.3.2) we have $H_R = 14400$.

The dimension d_R is given by the hook length formula

$$d_R = \frac{n!}{H_R} \tag{2.3.3}$$

This dimension can be interpreted combinatorially as the number of standard Young tableaux of shape R. The Young basis given in appendix A.1 is an explicit basis for R that demonstrates this dimensionality.

The two simplest representations of S_n are the two 1-dimensional reps: the trivial (symmetric), with $R = [n]$ and the sign (anti-symmetric), with $R = [1^n]$.

It is well known that S_n representations are real and that the representation space can be given an inner product so as to make them orthogonal. The matrix representatives of group or group algebra elements are denoted by $D^R(\sigma)$. These matrices satisfy the orthogonality relations

$$\sum_{\sigma \in S_n} D^R_{ij}(\sigma) D^S_{kl}(\sigma^{-1}) = \frac{n!}{d_R} \delta^{RS} \delta_{il} \delta_{jk} \tag{2.3.4}$$

We write $\chi_R(\sigma) = \mathrm{Tr} D^R(\sigma)$ for the character of a permutation. The orthogonality relations for characters are

$$\frac{1}{n!} \sum_{\sigma \in S_n} \chi_R(\sigma) \chi_S(\sigma) = \delta_{RS} \qquad\qquad \sum_{R \vdash n} \chi_R(\sigma) \chi_R(\tau) = z_{p_\sigma} \delta_{p_\sigma p_\tau} \tag{2.3.5}$$

where p_σ is the cycle type of σ. Setting $\tau = 1$ in the right hand equation, we obtain the resolution of the identity

$$\frac{1}{n!} \sum_{R \vdash n} d_R \chi_R(\sigma) = \delta(\sigma) \tag{2.3.6}$$

where δ is a function defined on S_n by

$$\delta(\sigma) = \begin{cases} 1 & \sigma = 1 \\ 0 & \text{otherwsie} \end{cases} \tag{2.3.7}$$

The character of a permutation in an irrep R depends only on its cycle type, so taking $\sigma \in S_n$ to be of cycle type $p \vdash n$ we define

$$\chi_R(p) = \chi_R(\sigma) \tag{2.3.8}$$

This notation neatens the orthogonality relations (2.3.5)

$$\sum_{p \vdash n} \frac{1}{z_p} \chi_R(p)\chi_S(p) = \delta_{RS} \qquad \sum_{R \vdash n} \chi_R(p)\chi_R(q) = z_p \delta_{pq} \tag{2.3.9}$$

For a representation $R \vdash n$, the conjugate representation R^c is isomorphic to the tensor product of R with the sign representation, so

$$R^c = [1^n] \otimes R \qquad D^{R^c}_{ij}(\sigma) = (-1)^\sigma D^R_{ij}(\sigma) \qquad \chi_{R^c}(p) = (-1)^p \chi_R(p) \tag{2.3.10}$$

2.3.3 Centre of $\mathbb{C}(S_n)$

The centre of $\mathbb{C}(S_n)$ is the sub-algebra that commutes with everything in $\mathbb{C}(S_n)$. A basis for the algebra can be found by summing over the conjugacy classes. Using any $\tau \in S_n$ of cycle type p, we define

$$\alpha_p = \frac{1}{n!} \sum_{\sigma \in S_n} \sigma \tau \sigma^{-1} \tag{2.3.11}$$

By reparameterising the sum to run over $\sigma' = \pi\sigma$ instead of σ, it follows that α_p commutes with any permutation $\pi \in S_n$.

In the 1-matrix system, when the centre is the algebra of interest, multiplying α_p by $\mathbb{X}$ and tracing over $V^{\otimes n}$ produces a simple multi-trace, just as seen in (2.1.11)

$$\mathrm{Tr}\left(\alpha_p \mathbb{X}\right) = T_p = \prod_i \left(\mathrm{Tr} X^i\right)^{p_i} \tag{2.3.12}$$

There is another important basis for the centre, constructed using the irreducible representations $R \vdash n$

$$P_R = \frac{d_R}{n!} \sum_{\sigma \in S_n} \chi_R(\sigma)\sigma \tag{2.3.13}$$

It follows from the conjugation invariance of χ_R that P_R commutes with the whole of S_n. Multiplying P_R by $\mathbb{X}$ and tracing over $V^{\otimes n}$, we obtain the Schur operators

$$\mathcal{O}_R = \frac{1}{d_R} \mathrm{Tr}\left(P_R \mathbb{X}\right) = \frac{1}{n!} \sum_{\sigma \in S_n} \chi_R(\sigma) \mathrm{Tr}\left(\sigma \mathbb{X}\right) \tag{2.3.14}$$

where we have normalised the operators with respect to the two-point functions introduced later in section 2.6.

These were first defined in [22] and are named Schur operators since when written as a function of the eigenvalues of X, they are exactly the symmetric Schur functions s_R. For more on symmetric functions and the relation they have to the half-BPS sector see section 2.7.

It is a simple consequence of Schur's lemma and the orthogonality relations (2.3.5) that P_R is represented by the identity matrix in R and by the zero matrix in all other irreps

$$D^S(P_R) = \delta_{RS} \tag{2.3.15}$$

It also follows from the orthogonality relations (2.3.5) that the P_R satisfy the multiplication identity

$$P_R P_S = P_S P_R = \delta_{RS} P_R \tag{2.3.16}$$

This means that in any representation of S_n, P_R acts as a projector onto the R subspace.

There is a particular element of the centre that will be especially important

$$\Omega = \sum_{\sigma \in S_n} (\mathrm{Tr}\, \sigma)\, \sigma = \sum_{\sigma \in S_n} N^{c(\sigma)} \sigma \tag{2.3.17}$$

where $c(\sigma) = l(p(\sigma))$ is the number of cycles in σ, and we have used (2.1.7) to evaluate the trace of σ over $V^{\otimes n}$. Ω commutes with S_n because conjugate permutations have the same $c(\sigma)$

$$\alpha \Omega \alpha^{-1} = \sum_{\sigma \in S_n} N^{c(\sigma)} \alpha \sigma \alpha^{-1} = \sum_{\sigma \in S_n} N^{c(\alpha^{-1}\sigma\alpha)} \sigma = \Omega \tag{2.3.18}$$

Since Ω commutes with S_n, by Schur's lemma it must act proportionally to the identity on any irrep $R \vdash n$. The constant of proportionality is given in terms of the *contents* of the boxes of the Young diagram of R. Let (r, c) label the box of R in the rth row and the cth column. Then the content of box (r, c) is $r - c$. So for example for $R = [4, 4, 2]$, the contents of each box are

0	1	2	3
−1	0	1	2
−2	−1		

(2.3.19)

For a box $b \in R$, the contents of b is written c_b. Then in rep $R \vdash n$, Ω has representative

$$D^R(\Omega) = \prod_{b \in R} (N + c_b) := f_R(N) \tag{2.3.20}$$

where this serves as a definition for the polynomial $f_R(N)$. This result is explained in more detail using Jucys-Murphy elements in appendix A.

Since P_R is the projector onto the R representation of S_n, it follows from (2.3.20) that

$$\Omega = \sum_R f_R(N) P_R \tag{2.3.21}$$

In a Young diagram of length $l(R) > N$, the $(N+1)$th box in the first column has content $-N$, and therefore the factor $(N + c_b)$ in (2.3.20) vanishes. Consequently,

$$f_R(N) = 0 \qquad\qquad \text{if} \;\; l(R) > N \tag{2.3.22}$$

So $\Omega = 0$ in representations with $l(R) > N$. The significance of this is discussed in section 2.6.

In Ω we see our first example of an N-dependent element of $\mathbb{C}(S_n)$. There are two possible interpretations of N in this context. Firstly, we can view N as a number given by the rank of the gauge group $U(N)$. This is more useful when working with matrices. Secondly, we can view N as a formal variable, in which case our correlators and other gauge invariants are formal power series in this variable. This latter interpretation is more helpful when working with permutation algebras.

2.4 Schur-Weyl duality

Schur-Weyl duality connects the representation theory of S_n with that of $U(N)$, and is the key mathematical link that allows us to talk about $U(N)$ matrix invariants using permutations.

2.4.1 $U(N)$ representations

Irreducible representations of $U(N)$ are labelled by partitions p with $l(p) \leq N$, with the sum $|p|$ unrestricted. The dimension of the representation R is given by

$$d_R^{U(N)} = \frac{f_R}{H_R} \tag{2.4.1}$$

where f_R and H_R are defined in (2.3.20) and (2.3.2) respectively.

Using the Hook length formula (2.3.3) and the expression for Ω in an S_n representation in (2.3.20), we can write

$$d_R^{U(N)} = \frac{\chi_R(\Omega)}{n!} \tag{2.4.2}$$

Similarly to the S_n representation dimension formula (2.3.3), we can interpret this dimensionality combinatorially as the number of semi-standard Young tableaux one can construct using the letters $1, 2, \ldots, N$. These semi-standard tableaux are described in section 3.6.2, along with a corresponding Young basis that exhibits this dimensionality.

2.4.2 Schur-Weyl duality and the double centraliser theorem

Consider the tensor product space $V^{\otimes n}$. As discussed in section 2.1, there is an action of S_n on this space defined by permuting the factors. Since V carries the fundamental of $U(N)$, $V^{\otimes n}$ also carries the tensor product representation of $U(N)$. This representation is the diagonal action of a $U(N)$ matrix on each of the tensor factors. Since the action is the same on each factor, this will commute with any permutation of the factors. Therefore $V^{\otimes n}$ is a representation of the direct product group $S_n \times U(N)$.

We give two statements of Schur-Weyl duality, the first of these is the nature of the decomposition of $V^{\otimes n}$ into representations of $S_n \times U(N)$

$$V^{\otimes n} = \bigoplus_{\substack{R \vdash n \\ l(R) \leq N}} V_R^{S_n} \otimes V_R^{U(N)} \tag{2.4.3}$$

We already mentioned that the S_n and $U(N)$ actions on $V^{\otimes n}$ commute. More formally, we can say that S_n and $U(N)$ both embed into the endomorphism algebra $\mathrm{End}\left(V^{\otimes n}\right)$, and that these two sub-algebras commute with each other. The second statement of Schur-Weyl duality is that these two sub-algebras are each other's centraliser within $\mathrm{End}\left(V^{\otimes n}\right)$. In simpler terms, if any $A \in \mathrm{End}\left(V^{\otimes n}\right)$ commutes with the $U(N)$ action, it must be (the endomorphism corresponding to) an element of $\mathbb{C}(S_n)$. Similarly if A commutes with S_n, it must be part of the algebra generated by the $U(N)$ action.

The relation between these two forms of Schur-Weyl duality can be understood by noting the form of the sum in (2.4.3). For a generic representation W of a direct product group $G \times H$, we can decompose W as

$$W = \bigoplus_{R_G, R_H} V_{R_G} \otimes V_{R_H} \otimes V_{R_G, R_H}^{mult} \tag{2.4.4}$$

where R_G and R_H run over the irreducible representations of G and H respectively, while V_{R_G, R_H}^{mult} is the multiplicity space whose dimension is just the multiplicity of the representation $R_G \otimes R_H$. If this multiplicity space has a dimension greater than 1, then there are endomorphisms of W that act only on V_{R_G, R_H}^{mult} while keeping V_{R_G}, V_{R_H} fixed. These endomorphisms must commute with both G and H, so they are not each other's centraliser within $\mathrm{End}\,(W)$.

There is a second condition before we can conclude G and H are each other's centraliser. For a given representation R_G of G that appears in the decomposition (2.4.4), there must be a unique representation R_H of H that pairs with R_G. If this were not the case, we could construct an endomorphism of W that commuted with both G and H by swapping the R_G components of $R_G \otimes R_{H,1}$ and $R_G \otimes R_{H,2}$.

In the case of Schur-Weyl duality, the multiplicity spaces are one-dimensional, and each R on the left corresponds to the same R on the right, and therefore the two algebras are each others' full centraliser. This is related to the *double centraliser theorem*, an important result of abstract algebra.

In appendices D.3 and F.4, we see examples where we do have multiplicity spaces, and have to consider the sub-algebras of $\mathrm{End}\,(W)$ that commute with both G and H.

2.5 Finite N constraints

The Schur-Weyl decomposition (2.4.3) hides an important detail that we emphasise here. In section 2.1 we explained how the relevant algebra for constructing multi-traces in the half-BPS sector is the centre of $\mathbb{C}(S_n)$. In equation (2.3.13) we defined P_R, the elements of this algebra that we contract with $\mathbb{X}$ to give the Schur operators of [22]. However (2.4.3) states that when considering permutations acting on $V^{\otimes n}$, only those representations R with $l(R) \leq N$ contribute. Those P_R with $l(R) > N$ are represented by the zero operator on $V^{\otimes n}$, and thus the Schur operators with $l(R) > N$ must vanish.

From an algebra point of view, this restricts us further to a subalgebra of the centre, spanned by those P_R with $l(R) \leq N$. Since $R \vdash n$, this restriction only has any effect when $n > N$, and we will use the term 'large N' to refer to any $N \geq n$, and 'finite N' for any $N < n$.

There are similar cut-offs in the full $\mathbb{C}(S_n)$ algebra. This can be expressed using the Fourier basis for $\mathbb{C}(S_n)$, defined by

$$\beta^R_{ij} = \sum_{\sigma \in S_n} D^R_{ij}(\sigma)\sigma \tag{2.5.1}$$

where $R \vdash n$ and i, j are basis indices for the representation R. Then at finite N, the Fourier basis is restricted to those R with $l(R) \leq N$, and those R with $l(R) > N$ are removed from the algebra.

In more abstract terms, the Fourier basis is an explicit identification of the decomposition of $\mathbb{C}(S_n)$ as a representation of S_n

$$\mathbb{C}(S_n) = \bigoplus_{R \vdash n} V^{left}_R \otimes V^{right}_R \tag{2.5.2}$$

where V^{left}_R indicates that this space is in the R representation of S_n under left multiplication of $\mathbb{C}(S_n)$, and in the trivial representation under right multiplication, while for V^{right}_R it is vice versa. At finite N, we consider the smaller algebra obtained by removing the terms with $l(R) > N$ from the sum (2.5.2).

The quarter-BPS algebra $\mathcal{A}_{n_1,n_2}$ also gets reduced when we consider $N < n$. This is described in more detail in section 3.6.1 when we define the restricted Schur basis.

Schur-Weyl duality is a very nice way of introducing this finite N cut-off, but there are other ways of understanding this result. We can see the vanishing of these operators as a consequence of relations between multi-traces of degree $> N$ when considering $N \times N$ matrices. These relations come, for example, from the Cayley-Hamilton theorem. We could also understand it as a result of the expansion of the projectors P_R in terms of Young symmetrisers, which involve anti-symmetrisation down the columns of R. Therefore if $l(R) > N$, we have an anti-symmetrisation on more than N indices, which must vanish for an object whose indices can only take N values.

The final interpretation of this result is in terms of the AdS/CFT correspondence. Half-BPS operators are dual to giant gravitons, D3-branes that wrap an S^3 in the S^5 factor of $AdS_5 \times S^5$ [17]. The length of a column in a Young diagram corresponds to the angular momentum of the D3, which in turn determines the radius of the S^3. This radius is bounded from above by the radius of the S^5, and this restriction limits the column to length at most N. From this viewpoint, the restriction $l(R) \leq N$ is ensuring that the S^3 doesn't expand beyond the S^5 which contains it. This is called the stringy exclusion principle [18], and finding the dual interpretation in terms of Young diagrams was one of the major results of [22].

2.5.1 SEP-compatible bases

Throughout this thesis, we will define and use various different bases of degree n operators; these will (nearly) all come with a partition label $p \vdash n$. They will also have other labels that will depend on which space we are considering at the time; for now we bundle these together in u.

For $N < n$, only a subset of the operators $\mathcal{O}_{p,u}$ will be needed to form a basis, specifically those with $l(p) \leq N$. This means the operators with $l(p) > N$ can be written as a linear combination of the shortened basis

$$\mathcal{O}_{p,u} = \sum_{\substack{q,v \\ l(q) \leq N}} c_{p,u}^{q,v} \, \mathcal{O}_{q,v} \tag{2.5.3}$$

We call the basis *SEP-compatible* (where SEP stands for Stringy Exclusion Principle) if $c_{p,u}^{q,v} = 0$ for all p, u, q, v, i.e. if the operators with $l(p) > N$ vanish identically. Intuitively, these bases are aligned along the direction of the finite N quotient, and in this sense better capture the finite N behaviour of the space.

In general, multi-trace bases such as (2.3.12) are not SEP-compatible, while Young diagram bases such as (2.3.14) are.

2.6 Correlators

Starting with the generic n-matrix system, the correlator of two $U(N)$ matrix fields is

$$\left\langle (Z_s)^k_l \,|\, (Z_r)^i_j \right\rangle = \delta_{rs}\delta^i_l\delta^k_j \tag{2.6.1}$$

For two operators $\mathcal{O}_\sigma, \mathcal{O}_\tau$ of the form given in (2.1.3), this translates to

$$\langle \mathcal{O}_\tau | \mathcal{O}_\sigma \rangle = \delta(\Omega\sigma\tau^{-1}) \tag{2.6.2}$$

where Ω is defined in (2.3.17).

The properties of Ω encode the finite N cut-off into the inner product (2.6.2). In (2.3.22) we explained how Ω vanishes in a representation R if $l(R) > N$. Therefore (2.6.2) is identically zero for any operators that disappear after imposing the finite N restrictions, or equivalently for any permutations in the $l(R) > N$ sector of $\mathbb{C}(S_n)$.

Using the correspondence between $\mathcal{O}_\sigma$ and σ, (2.6.2) defines the physical inner product on $\mathbb{C}(S_n)$. This means for large N we can work completely in $\mathbb{C}(S_n)$, and largely forget about the matrices $Z_1, \dots, Z_n$. At finite N, if we wish to work purely with permutation algebras, we must be careful to incorporate the restrictions imposed by removing elements of the Fourier basis (2.5.1) with $l(R) > N$.

There is also the standard inner product on $\mathbb{C}(S_n)$, given by

$$\langle \tau | \sigma \rangle_{S_n} = \delta(\sigma\tau^{-1}) = \delta_{\sigma\tau} \tag{2.6.3}$$

Reversing the correspondence between $\mathcal{O}_\sigma$ and σ, this defines the S_n inner product on $U(N)$ gauge-invariant operators. We denote this with a S_n subscript on the brackets. After incorporating the finite N cut-off, the S_n inner product is

$$\langle \mathcal{O}_\tau | \mathcal{O}_\sigma \rangle_{S_n} = \delta_N(\sigma\tau^{-1}) \tag{2.6.4}$$

where

$$\delta_N(\sigma) = \frac{1}{n!} \sum_{\substack{R \vdash n \\ l(R) \leq N}} d_R \chi_R(\sigma) \tag{2.6.5}$$

From the resolution of the identity (2.3.6), this reduces to $\delta(\sigma)$ if $N \geq n$.

From the definition (2.3.17), the large N expansion of Ω is

$$\Omega = N^n \left[1 + O\left(\frac{1}{N}\right) \right] \tag{2.6.6}$$

Therefore in the leading N limit, the physical inner product differs from the S_n inner product only by a factor of N^n. This large N limit is called the planar limit, and we therefore often use the S_n inner product to give planar results. However the S_n inner product is not the same as the planar inner product, since in the planar inner product we would also take the leading N limit in any operator coefficients, while in the S_n inner product we retain these N-dependencies. When all coefficients are N-independent, we can use the S_n inner product to derive planar results.

From the definition (2.6.4), we see that $\mathcal{O}_\sigma$ and $\mathcal{O}_\tau$ are orthogonal if $\sigma \neq \tau$ and $N \geq n$. So in the large N limit, different multi-traces are orthogonal.

For all correlators here, and in subsequent chapters, we have suppressed the position dependence as this is purely determined by conformal invariance. For a thorough description of why we can do this see [22].

2.6.1 Half-BPS sector

Consider the 1-matrix system, as described in section 2.1, in which all the Z_i become X and the appropriate algebra is the centre of $\mathbb{C}(S_n)$. When calculating correlators we must now consider Wick contractions between the copies of X. This leads to the physical and S_n inner products

$$\langle \mathcal{O}_\tau | \mathcal{O}_\sigma \rangle = \sum_{\alpha \in S_n} \delta\left(\Omega \alpha \sigma \alpha^{-1} \tau^{-1}\right) \tag{2.6.7}$$

$$\langle \mathcal{O}_\tau | \mathcal{O}_\sigma \rangle_{S_n} = \sum_{\alpha \in S_n} \delta_N\left(\alpha \sigma \alpha^{-1} \tau^{-1}\right) \tag{2.6.8}$$

We can see that in the S_n inner product, $\mathcal{O}_\sigma$ and $\mathcal{O}_\tau$ are orthogonal if σ and τ are in different conjugacy classes and $N \geq n$. So as in the general n-matrix case, different multi-traces are orthogonal to each other in the planar limit. Using the notation T_p defined in (2.1.11) the normalisation is

$$\langle T_p | T_q \rangle_{S_n} = z_p \delta_{pq} \tag{2.6.9}$$

where we have used the result from section 2.3.1 that the size of the centraliser for a permutation of cycle type p is z_p. Therefore z_p has a physical interpretation in the planar limit as the norm of multi-trace operators.

The physical inner product mixes traces of different types and does not admit a nice formula on multi-traces.

For the Schur operators defined in (2.3.14), we can use orthogonality of characters, (2.3.5), and the representation of Ω in a representation R, (2.3.20), to show

$$\langle \mathcal{O}_R | \mathcal{O}_S \rangle = \delta_{RS} f_R \tag{2.6.10}$$

$$\langle \mathcal{O}_R | \mathcal{O}_S \rangle_{S_n} = \begin{cases} \delta_{RS} & l(R) \leq N \\ 0 & l(R) > N \end{cases} \tag{2.6.11}$$

So the Schur basis is exactly orthogonal to all orders in N using both inner products.

Three and higher point function can be calculated using product rules involving Littlewood-Richardson coefficients. For a description of these rules, and associated subtleties relating to the position dependence of the scalar field involved, see [22].

2.6.2 Quarter-BPS sector

In the 2-matrix system we set n_1 of the Z_i to be X and n_2 of the Z_i to be Y. This leads to the physical and S_n inner products

$$\langle \mathcal{O}_\tau | \mathcal{O}_\sigma \rangle = \sum_{\alpha \in S_{n_1} \times S_{n_2}} \delta\left(\Omega \alpha \sigma \alpha^{-1} \tau^{-1}\right) \tag{2.6.12}$$

$$\langle \mathcal{O}_\tau | \mathcal{O}_\sigma \rangle_{S_n} = \sum_{\alpha \in S_{n_1} \times S_{n_2}} \delta_N\left(\alpha \sigma \alpha^{-1} \tau^{-1}\right) \tag{2.6.13}$$

We study these further in section 3.5.3 for quarter-BPS multi-trace operators and section 3.6 for the restricted Schur and covariant bases, generalisations of the half-BPS Schur basis to the quarter-BPS sector.

2.7 Symmetric functions

The half-BPS sector is composed of multi-traces of a single complex matrix X. Diagonalising in terms of its eigenvalues, we have

$$X = \begin{pmatrix} x_1 & 0 & 0 & \dots & 0 \\ 0 & x_2 & 0 & \dots & 0 \\ 0 & 0 & x_3 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & x_N \end{pmatrix} \tag{2.7.1}$$

Thus any multi-trace of X can instead be written as a function of the eigenvalues $x_1, x_2, \dots, x_N$. These functions must be completely symmetric in the N variables, and are called *symmetric functions*. The theory of symmetric functions is well studied in mathematics, and they have many interesting properties [64]. In this section we review some basic concepts from this field.

Symmetric functions are defined as polynomials in the N variables $x_1, x_2, x_3, \dots, x_N$ that are invariant under all permutations of the x_i. More explicitly, given a polynomial $f(x_1, x_2, \dots, x_N)$, f is a symmetric function if

$$f(x_1, x_2, \dots, x_N) = f(x_{\sigma(1)}, x_{\sigma(2)}, \dots, x_{\sigma(N)}) \tag{2.7.2}$$

for all $\sigma \in S_N$.

We can take the infinite N limit of this definition by defining symmetric functions as formal power series in infinitely many variables $x_1, x_2, \ldots$. To return to the finite N case (or to reduce a symmetric function in $M > N$ variables to one in N variables), we can set $x_{N+1} = 0, x_{N+2} = 0, \ldots$.

There are many different bases for the ring of symmetric functions, of which we will look at three. In each of these bases, each basis element consists of polynomials of a single degree, n, and the basis for the degree n subspace is labelled by the partitions of n.

2.7.1 Monomial basis

We start with the monomial basis. Given a partition $p = [\lambda_1, \lambda_2, \ldots, \lambda_k]$ of n, take the monomial

$$x_1^{\lambda_1} x_2^{\lambda_2} \ldots x_k^{\lambda_k} \tag{2.7.3}$$

and then add all distinct permutations of the lower indices to form a symmetric function. So for example if we take $p = [3, 1, 1]$ (and use $N = 3$ for simplicity), the associated monomial basis element is

$$m_{[3,1,1]} = x_1^3 x_2 x_3 + x_1 x_2^3 x_3 + x_1 x_2 x_3^3 \tag{2.7.4}$$

For $p = [1^{p_1}, 2^{p_2}, \ldots]$, we can define the monomial functions more formally by

$$m_p = \left(\prod_i \frac{1}{p_i!} \right) \sum_{\sigma \in S_N} x_{\sigma(1)}^{\lambda_1} x_{\sigma(2)}^{\lambda_2} \ldots x_{\sigma(l(p))}^{\lambda_{l(p)}} \tag{2.7.5}$$

where $N \geq l(p)$ and the normalisation in front accounts for non-trivial coefficients introduced by redundancies in the components of p.

For future convenience, we also define the rescaled monomial function M_p to be (2.7.5) without the normalisation factor.

$$M_p = \sum_{\sigma \in S_N} x_{\sigma(1)}^{\lambda_1} x_{\sigma(2)}^{\lambda_2} \ldots x_{\sigma(l(p))}^{\lambda_{l(p)}} \tag{2.7.6}$$

The notion of SEP-compatibility defined in (2.5.1) applies equally well to symmetric functions as it does to gauge-invariant operators. Since the definitions (2.7.5) and (2.7.6) involve $l(p)$ distinct xs, if $N < l(p)$, some of these are zero, and therefore $m_p = M_p = 0$. Hence the monomial symmetric functions are an SEP-compatible basis.

2.7.2 Power-sum basis

The second basis we consider is the power-sum basis. This is constructed from polynomials of the form

$$T_k = \sum_{i=1}^{N} x_i^k \tag{2.7.7}$$

Note we use a T rather than the more standard P to denote the power-sum functions. This is to emphasise the relations between these symmetric functions and the multi-trace operators of the half-BPS sector.

For a partition $p = [\lambda_1, \lambda_2, \dots \lambda_k] = [1^{p_1}, 2^{p_2}, \dots]$, the power-sum symmetric function is

$$T_p = \prod_{i=1}^{k} T_{\lambda_i} = \prod_i (T_i)^{p_i} \tag{2.7.8}$$

Consider a $N \times N$ diagonal matrix X with entries x_i, as in (2.7.1). Then (2.7.7) can be written in terms of X as $T_k = \mathrm{Tr} X^k$. For the general symmetric function (2.7.8) we have

$$T_p = \prod_i \left(\mathrm{Tr} X^i\right)^{p_i} \tag{2.7.9}$$

Identifying this with the multi-trace operators (2.1.11) provides the link between symmetric functions and the half-BPS sector of $\mathcal{N} = 4$ super Yang-Mills with gauge group $U(N)$.

2.7.3 Schur basis

Finally, we look at the Schur basis. These are labelled by partitions $R \vdash n$, thought of as representations of the symmetric group S_n.

$$s_R = \sum_{p \vdash n} \frac{1}{z_p} \chi_R(p) T_p \tag{2.7.10}$$

where z_p is defined in (2.3.1). Through the identification of T_p with the multi-trace operator (2.1.11) and noting that the size of the p conjugacy class in S_n is $\frac{n!}{z_p}$, we see s_R are exactly the Schur basis (2.3.14) for half-BPS states

$$\mathcal{O}_R = s_R(x_1, x_2, \dots, x_N) \tag{2.7.11}$$

Since the Schur and monomial functions form a basis for the degree n symmetric functions, there is a basis change matrix transforming between them. This is given by the Kostka numbers K_{Rp}

$$s_R = \sum_{p \vdash n} K_{Rp} m_p \tag{2.7.12}$$

The Kostka numbers have a combinatoric interpretation in terms of the number of semi-standard Young tableaux of shape R and evaluation p. These Young tableaux are defined in section 3.6.2.

As discussed in section 2.5 for the equivalent Schur operators, the Schur basis is SEP-compatible.

The Schur functions are also connected to the characters of $U(N)$ representations. Consider a matrix $U \in U(N)$ and the projector $P_R \in \mathbb{C}(S_n)$, defined in (2.3.13), acting on the tensor product $V^{\otimes n}$. Using the Schur-Weyl decomposition (2.4.3), we have

$$\mathrm{Tr}_{V^{\otimes n}} P_R U = \left(\mathrm{Tr}_{V_R^{S_n}} P_R \right) \left(\mathrm{Tr}_{V_R^{U(N)}} U \right) = d_R \, \chi_R^{U(N)}(U) \tag{2.7.13}$$

where $\chi_R^{U(N)}$ is the character of the $U(N)$ representation R. We can also write $\mathrm{Tr}_{V^{\otimes n}} P_R U$ as a sum over permutations

$$d_R \chi_R^{U(N)}(U) = \frac{d_R}{n!} \sum_{\sigma \in S_n} \chi_R^{S_n}(\sigma) \mathrm{Tr}_{V^{\otimes n}}(\sigma U) \tag{2.7.14}$$

which is d_R times the Schur operators defined in (2.3.14), with X replaced by U.

Since the Schur operators are Schur functions in the eigenvalues of X, it follows that

$$\chi_R^{U(N)}(U) = s_R(u_1, u_2, \ldots, u_N) \tag{2.7.15}$$

where $u_1, u_2, \ldots, u_N$ are the eigenvalues of U.

A formula for the multiplication of Schur function in terms of Littlewood-Richardson coefficients is given in appendix D.1.

Chapter 3

Structure, combinatorics and correlators of the free field quarter-BPS sector with $U(N)$ gauge group

All fields in the $\mathcal{N} = 4$ super Yang-Mills theory lie in the adjoint representation of the gauge group. Therefore gauge invariant operators are constructed by taking a trace over words on the alphabet of local operators. This forms a strong connection between the classification of gauge invariant operators and the combinatorics of words. For more on the links between these two topics see [61, 66, 67].

The counting of quarter-BPS operators in the planar free field limit was given in terms of an infinite product generating function in [61].

$$F_{U(N)}(x, y) = \prod_{k=1}^{\infty} \frac{1}{1 - x^k - y^k} \tag{3.0.1}$$

The individual factors in the product are obtained by substituting x^k, y^k into a *root function*, given by $(1 - x - y)^{-1}$. In [68], this root function was found to be a generic feature in free quiver gauge theories with $U(N)$ gauge groups, and an interpretation of the root function in terms of word counting was given in [69]. This combinatorics of gauge invariants is closely related to paths on graphs, which have interesting number theoretic aspects studied recently [70].

Consider the root function

$$\frac{1}{1 - x - y} \tag{3.0.2}$$

The coefficient of $x^{n_1} y^{n_2}$ is $\binom{n}{n_1}$, which counts the number of different ways of ordering

n_1 xs and n_2 ys, or equivalently the number of different words that can be made from n_1 $\hat{x}$s and n_2 $\hat{y}$s, in the space of words generated freely by two generators $\hat{x}, \hat{y}$. This space of words form a monoid, where the product is given by concatenation.

Consider the implications of interpreting the whole infinite product $F_{U(N)}(x, y)$ in terms of words. The coefficient of $x^{2n_1}y^{2n_2}$ in $(1 - x^2 - y^2)^{-1}$ counts the number of words formed from n_1 $\hat{x}$s and n_2 $\hat{y}$s, so now the letters have weight 2. We denote the $\hat{x}$s and $\hat{y}$s with weight one by $\hat{x}_1$ and $\hat{y}_1$ and those with weight 2 by $\hat{x}_2$ and $\hat{y}_2$. Multiplying the two generating functions then counts words made from all four available letters, where the weight 1 letters commute with weight 2 letters. So the coefficient of $x^{n_1}y^{n_2}$ in

$$\frac{1}{(1 - x - y)(1 - x^2 - y^2)} \tag{3.0.3}$$

counts words constructed from $n_{1,1}$ $\hat{x}_1$s, $n_{2,1}$ $\hat{y}_1$s, $n_{1,2}$ $\hat{x}_2$s and $n_{2,2}$ $\hat{y}_2$s such that $n_{1,1} + 2n_{1,2} = n_1$ and $n_{2,1}+2n_{2,2} = n_2$, subject to an equivalence between two words when they are obtained from each other by commuting letters of different weights (equivalently, we could say all weight 1 letters precede all weight 2 letters).

Repeating this process, we see that $F_{U(N)}(x, y)$ counts words constructed from $\hat{x}$s and $\hat{y}$s of all weights (i.e. $\hat{x}_k, \hat{y}_k$ with k any positive integer), where within each level, $\hat{x}_k$ and $\hat{y}_k$ are non-commutative, but different levels commute with each other. We will refer to this kind of word counting problem as an integrally-graded word combinatorics. A natural problem is to give a bijection between the words in this counting and the traces of two matrices X, Y in the large N limit. This is given in section 3.3.

The first part of this chapter, section 3.1, is devoted to understanding the structure of the multi-traces in the large N quarter-BPS sector, and in particular how the entire space is related to the much smaller set of aperiodic single traces. Along the way we derive the Hilbert series for various intermediate structures, which are quarter BPS generating functions. In section 3.3, we show the integrally-graded words exhibit the exact same structure, with Lyndon words playing the role of aperiodic traces. The bijection is then constructed by matching Lyndon words with aperiodic traces. This bijection allows transferring the concatenation product on words to a non-commutative product on traces.

In the second half of the chapter, we use the theory of permutations developed in chapter 2 to describe the different bases one can use for the quarter-BPS sector. Section 3.5 starts by considering the multi-trace basis, corresponding to a conjugacy class description of the permutation algebra. This has a labelling set consisting of a partition for each Lyndon word, generalising of the partition label of the half-BPS sector to the quarter-BPS.

In section 3.6, we define two different orthogonal SEP-compatible bases for the free field quarter-BPS sector, the restricted Schur basis first defined in [44, 45] and the covariant basis introduced in [43, 46]. Both give different combinatoric expressions for the dimension of the degree (n_1, n_2) subspace. We give the physical two-point functions for both bases, expressed in terms of the Young diagram label R that they share. We use these two bases in later chapters to study the orientifold quotient to $SO(N)/Sp(N)$ gauge theory and the $U(N)$ theory at weak coupling.

This chapter consists of work originally presented in [1].

3.1 Structure of the space of $U(N)$ multi-traces of two matrices

We consider the global structure of the set of multi-traces, as well as how this structure is reflected in (3.0.1). We find it is simplest to express this in the language of vector spaces, so we consider T, the space spanned by the $U(N)$ multi-traces.

The generating function (3.0.1) is the Hilbert series of T, where T is graded by how many Xs and Ys appear in each multi-trace. More explicitly, we can split T into a direct sum of subspaces $T_{(n_1,n_2)}$ spanned by the degree (n_1, n_2) multi-traces. Then the Hilbert series is defined by

$$H_T(x, y) = \sum_{n_1,n_2} x^{n_1} y^{n_2} \operatorname{Dim} T_{(n_1,n_2)} \tag{3.1.1}$$

Note that we use the term 'Hilbert Series' only with reference to graded vector spaces. When the vector space also has the structure of an algebra, the Hilbert series imparts information about the relations between the generating elements of the algebra. While some of the vector spaces we consider do have an algebra structure, indeed $T_{(n_1,n_2)}$ is isomorphic (as a vector space) to the algebra $\mathcal{A}_{n_1,n_2}$ defined in section 2.1, we will not focus on this aspect.

To describe the factorisation of multi-traces into single traces, the full space T is divided into subspaces T_r spanned by multi-traces formed from r single traces.

$$T = \bigoplus_{r=0}^{\infty} T_r \tag{3.1.2}$$

T_0 is the one-dimensional space spanned by 1, thought of as the trivial multi-trace (the multi-trace containing no single traces). We define T_{ST} to be the space spanned by the single traces, so that T_1 is just T_{ST}. T_2 contains multi-traces with two single traces in their factorisation. Naively this space appears to be $T_{ST} \otimes T_{ST}$, but in this space there is a distinction between $t_1 \otimes t_2$ and $t_2 \otimes t_1$, whereas given the two traces t_1 and t_2, there is a unique multi-trace formed from their product. Instead we have $T_2 = \operatorname{Sym}^2(T_{ST})$, defined to be the symmetric part of $T_{ST} \otimes T_{ST}$. Similarly, $T_r = \operatorname{Sym}^r(T_{ST})$, defined to be the completely symmetric part of $(T_{ST})^{\otimes r}$. So we have

$$\begin{aligned} T &= \mathbb{C} \oplus T_{ST} \oplus \mathrm{Sym}^2\left(T_{ST}\right) \oplus \dots \\ &= \bigoplus_{r=0}^{\infty} \mathrm{Sym}^r\left(T_{ST}\right) \\ &:= \mathrm{Sym}\left(T_{ST}\right) \end{aligned} \tag{3.1.3}$$

where (3.1.3) is the definition of the Sym operator on vector spaces.

The Hilbert series for T_{ST} is the generating function for the counting of single traces, which is related to the counting of multi-traces via the plethystic exponential. Given the generating function

$$f(x,y) = \sum_{n_1,n_2} A_{n_1,n_2} x^{n_1} y^{n_2} \tag{3.1.4}$$

for the single traces, the generating function for the multi-traces is given by

$$\begin{aligned} F(x,y) = \mathrm{PExp}(f)(x,y) &= \exp\left(\sum_{k=1}^{\infty} \frac{f(x^k,y^k)}{k}\right) \\ &= \prod_{n_1,n_2} \frac{1}{(1-x^{n_1}y^{n_2})^{A_{n_1,n_2}}} \end{aligned} \tag{3.1.5}$$

We can see this diverges if $f(0,0) = A_{0,0} \neq 0$. This is expected, since a single trace operator of weight 0 would lead to an infinite number of multi-trace operators of weight 0. Since there is no single trace operator containing no matrices, this is not a problem.

For an explanation of why the plethystic exponential takes the single trace counting to the multi-trace counting, and for more details on the interesting properties of the plethystic exponential, see [71, 72].

The plethystic exponential can be inverted using the plethystic logarithm

$$f(x,y) = \mathrm{PLog}(F)(x,y) = \sum_{k=1}^{\infty} \frac{\mu(k)}{k} \log F(x^k,y^k) \tag{3.1.6}$$

where μ is the Möbius function defined in (B.0.3). The proof that (3.1.5) and (3.1.6) are inverses of each other comes from the identity (B.0.4). See appendix B for a detailed description of the useful properties of the Möbius function.

The Hilbert series for T and T_{ST} are related by

$$H_T = \mathrm{PExp}\left(H_{T_{ST}}\right) \qquad\qquad H_{T_{ST}} = \mathrm{PLog}\left(H_T\right) \tag{3.1.7}$$

Now consider the structure of T_{ST}. A single trace can be written as

$$\mathrm{Tr} W^k \tag{3.1.8}$$

where W is an aperiodic matrix word, and k is the number of periods. The number of periods and the aperiodic matrix word (which is only defined up to cyclic rotations) identify the trace. Therefore we can write

$$T_{ST} = K \otimes T_{ST}^{(1)} \tag{3.1.9}$$

where $T_{ST}^{(1)}$ is spanned by the aperiodic single traces and K is spanned by the positive integers. Consider an element $k \otimes w$, where w is an aperiodic single trace of weight (n_1, n_2), then the weight of $k \otimes w$ is (kn_1, kn_2). So the tensor factors interact non-trivially with respect to the weightings. Taking account of this, the Hilbert series of T_{ST} and $T_{ST}^{(1)}$ are related by

$$H_{T_{ST}}(x, y) = \sum_{k=1}^{\infty} H_{T_{ST}^{(1)}}(x^k, y^k) \tag{3.1.10}$$

where the kth term in the sum corresponds to the subspace $k \otimes T_{ST}^{(1)}$ of T_{ST}. Defining the coefficients of the two Hilbert series by

$$H_{T_{ST}}(x, y) = \sum_{n_1, n_2} A_{n_1,n_2} x^{n_1} y^{n_2} \qquad\qquad H_{T_{ST}^{(1)}}(x, y) = \sum_{n_1, n_2} a_{n_1,n_2} x^{n_1} y^{n_2} \tag{3.1.11}$$

the relation (3.1.10) becomes

$$A_{n_1,n_2} = \sum_{d|n_1,n_2} a_{\frac{n_1}{d}, \frac{n_2}{d}} \tag{3.1.12}$$

where $d|n_1, n_2$ means d is a divisor of both n_1 and n_2.

This relation can be inverted using the Möbius inversion formula (B.0.8) to get

$$a_{n_1,n_2} = \sum_{d|n_1,n_2} \mu(d) A_{\frac{n_1}{d}, \frac{n_2}{d}} \tag{3.1.13}$$

In terms of the Hilbert series, this becomes

$$H_{T_{ST}^{(1)}}(x, y) = \sum_{k=1}^{\infty} \mu(k) H_{T_{ST}}(x^k, y^k) \tag{3.1.14}$$

We call $H_{T_{ST}^{(1)}}$ the Möbius transform of $H_{T_{ST}}$

$$H_{T_{ST}^{(1)}} = \mathcal{M}\left(H_{T_{ST}}\right) \qquad\qquad H_{T_{ST}} = \mathcal{M}^{-1}\left(H_{T_{ST}^{(1)}}\right) \tag{3.1.15}$$

In full, T can be decomposed as

$$T = \mathrm{Sym}\left(K \otimes T^{(1)}_{ST}\right) \tag{3.1.16}$$

and the corresponding decomposition in the generating function is

$$\begin{aligned} H_T &= \mathrm{PExp}\left[\mathcal{M}^{-1}\left(H_{T^{(1)}_{ST}}\right)\right] \\ &= \mathrm{PExp}\left[\sum_{k=1}^{\infty} H_{T^{(1)}_{ST}}(x^k, y^k)\right] \\ &= \prod_{k=1}^{\infty} \mathrm{PExp}\left[H_{T^{(1)}_{ST}}\right](x^k, y^k) \end{aligned} \tag{3.1.17}$$

The expressions (3.1.16) and (3.1.17) reflect splitting the multi-traces into single traces, and then decomposing the single traces by the number of periods. This can be done the other way round. A multi-trace can be split into factors, where each factor consists only of single traces with a specified number of periods. These factors can then be decomposed into those single traces. Doing things in this order gives the structure

$$T = \left[T^{(1)}\right]^{\otimes K} := \left[\mathrm{Sym}\left(T^{(1)}_{ST}\right)\right]^{\otimes K} \tag{3.1.18}$$

where by $V^{\otimes K}$, we mean

$$V^{\otimes K} = V_1 \otimes V_2 \otimes V_3 \otimes \ldots = \bigotimes_{k=1}^{\infty} V_k \tag{3.1.19}$$

and each V_k is a copy of V but with all weights multiplied by k. The Hilbert series of $V^{\otimes K}$ is then given by

$$H_{V^{\otimes K}}(x, y) = \prod_{k=1}^{\infty} H_V(x^k, y^k) \tag{3.1.20}$$

Just as for the sum (3.1.10), we can invert this

$$H_V(x, y) = \prod_{k=1}^{\infty}\left[H_{V^{\otimes K}}(x^k, y^k)\right]^{\mu(k)} \tag{3.1.21}$$

The proof of this inversion relies on the multiplicative version of the Möbius inversion formula, (B.0.6). We say H_V is the multiplicative Möbius transform of $H_{V^{\otimes K}}$

$$H_V = \mathcal{M}_{mult}\left(H_{V^{\otimes K}}\right) \qquad H_{V^{\otimes K}} = \mathcal{M}^{-1}_{mult}\left(H_V\right) \tag{3.1.22}$$

Using this notation, the generating function version of (3.1.18) is

$$\begin{aligned} H_T &= \mathcal{M}^{-1}_{mult}\left(H_{T^{(1)}}\right) = \mathcal{M}^{-1}_{mult}\left[\mathrm{PExp}\left(H_{T^{(1)}_{ST}}\right)\right] \\ H_T(x,y) &= \prod_{k=1}^{\infty} H_{T^{(1)}}(x^k,y^k) = \prod_{k=1}^{\infty} \mathrm{PExp}\left[H_{T^{(1)}_{ST}}\right](x^k,y^k) \end{aligned} \tag{3.1.23}$$

which matches (3.1.17). From that the (not immediately obvious) result

$$\mathrm{Sym}\,(K \otimes V) = (\mathrm{Sym}\, V)^{\otimes K} \tag{3.1.24}$$

corresponds to the trivial result

$$\mathrm{PExp}\left(\sum_{k=1}^{\infty} H_V(x^k,y^k)\right) = \prod_{k=1}^{\infty} \mathrm{PExp}\left[H_V\right](x^k,y^k) \tag{3.1.25}$$

Comparing (3.1.23) with (3.0.1) we see that $H_{T^{(1)}}$ is what we called the root function. Additionally, we find the root function is not the most fundamental object. It is the plethystic exponential of $H_{T^{(1)}_{ST}}$, and we should think of this Hilbert series as the fundamental object of interest. It would be interesting to see whether this additional structure of the root function has an analogue in the general quiver theory explored in [69].

The structure described above, both for the vector spaces and their associated Hilbert series, is summarised in figure 3.1.

3.2 Generating functions at large N

In figure 3.1 and the work leading to it, we showed the relations between the vector spaces T, T_{ST}, $T^{(1)}$ and $T^{(1)}_{ST}$ and their associated Hilbert series, which are the quarter BPS generating functions. Since these relations are invertible, we can find all the Hilbert series from just one of them. We already know from (3.0.1) that

$$H_T(x,y) = F(x,y) = \prod_{k=1}^{\infty} \frac{1}{1-x^k-y^k} \tag{3.2.1}$$

which counts the full set of 2-matrix $U(N)$ multi-traces. Comparing with (3.1.23), we see that

$$H_{T^{(1)}}(x,y) = \frac{1}{1-x-y} \tag{3.2.2}$$

which counts aperiodic multi-traces. This allows us to interpret the product in (3.2.1). The factor $(1-x-y)^{-1}$ counts multi-traces constructed only from aperiodic single traces, while the factor $(1-x^k-y^k)^{-1}$ counts multi-traces constructed only from single

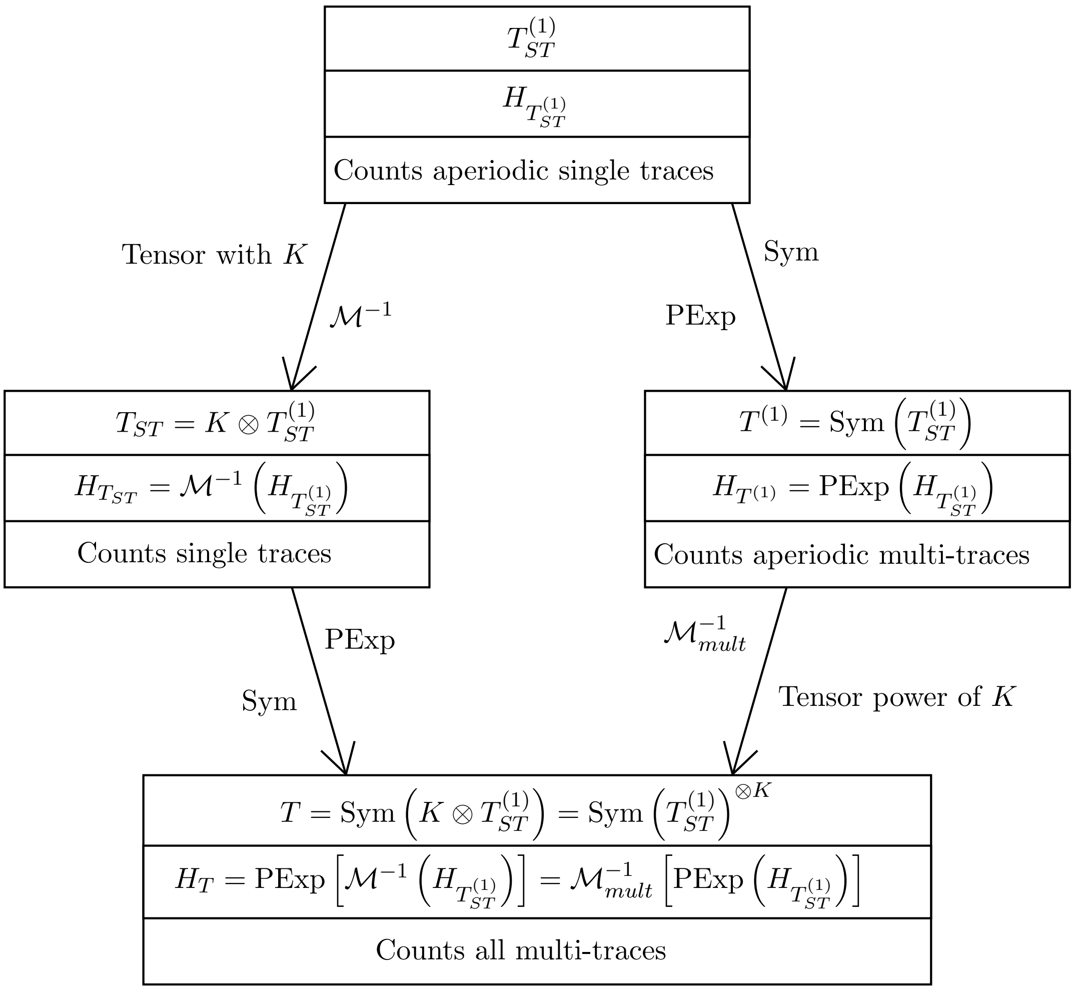


Figure 3.1: Diagram summarising the structure of T, the space of $U(N)$ 2-matrix multi-traces, and its relation to $T^{(1)}_{ST}$, the space of $U(N)$ aperiodic single traces. Each box contains the vector space in question, the corresponding Hilbert series and the trace description of what these are counting. The outer labels on the arrows show the vector space operations to travel between the boxes, while the inner labels show the equivalent Hilbert series operations.

traces with k periods.

Applying the plethystic logarithm to (3.2.1) and (3.2.2) gives

$$\begin{aligned} H_{T_{ST}}(x,y) &= \sum_{l=1}^{\infty} \frac{\mu(l)}{l} \log\left(\prod_{k=1}^{\infty} \frac{1}{1-x^{kl}-y^{kl}} \right) \\ &= -\sum_{k,l=1}^{\infty} \frac{\mu(l)}{l} \log\left(1-x^{kl}-y^{kl}\right) \\ &= -\sum_{d=1}^{\infty} \log\left(1-x^d-y^d\right) \sum_{l|d} \frac{\mu(l)}{l} \\ &= -\sum_{d=1}^{\infty} \frac{\phi(d)}{d} \log\left(1-x^d-y^d\right) \qquad (3.2.3) \\ H_{T_{ST}^{(1)}}(x,y) &= -\sum_{d=1}^{\infty} \frac{\mu(d)}{d} \log\left(1-x^d-y^d\right) \qquad (3.2.4) \end{aligned}$$

where in the first calculation we have changed variables from $(k,l : 1 \leq k,l \leq \infty)$ to $(d,l : 1 \leq l \leq \infty, l|d)$ by setting $d = kl$. We have also used the identity (B.0.15), and $\phi(d)$ is the Euler totient function defined in (B.0.14).

These two series (3.2.3) and (3.2.4) count single traces and aperiodic single traces respectively. Later it will be important to have explicit formulae for the coefficients A_{n_1,n_2}, a_{n_1,n_2} of these series as defined in (3.1.11). Expanding the logarithm in (3.2.3) and reparameterising

$$\begin{aligned} H_{T_{ST}}(x,y) &= \sum_{d,k=1}^{\infty} \frac{\phi(d)}{dk} (x^d+y^d)^k \\ &= \sum_{d,k=1}^{\infty} \frac{\phi(d)}{dk} \sum_{r=0}^{k} \binom{k}{r} x^{dr} y^{d(k-r)} \\ &= \sum_{n_1,n_2} x^{n_1} y^{n_2} \frac{1}{n} \sum_{d|n_1,n_2} \phi(d) \binom{\frac{n}{d}}{\frac{n_1}{d}} \qquad (3.2.5) \end{aligned}$$

where the sum excludes $n = n_1 + n_2 = 0$. Similarly we find

$$H_{T_{ST}^{(1)}}(x,y) = \sum_{n_1,n_2} x^{n_1} y^{n_2} \frac{1}{n} \sum_{d|n_1,n_2} \mu(d) \binom{\frac{n}{d}}{\frac{n_1}{d}} \qquad (3.2.6)$$

Comparing with (3.1.11), we see that, for $n \neq 0$

$$A_{n_1,n_2} = \frac{1}{n} \sum_{d|n_1,n_2} \phi(d) \binom{\frac{n}{d}}{\frac{n_1}{d}} \qquad (3.2.7)$$

$$a_{n_1,n_2} = \frac{1}{n} \sum_{d|n_1,n_2} \mu(d) \binom{\frac{n}{d}}{\frac{n_1}{d}} \tag{3.2.8}$$

and $A_{0,0} = a_{0,0} = 0$. The combinatoric interpretation of these sequences is as follows: a_{n_1,n_2} is the number of aperiodic single traces that can be constructed from n_1 Xs and n_2 Ys, while A_{n_1,n_2} is the number of single traces (with any number of periods) that can be constructed from n_1 Xs and n_2 Ys. Tables of values for these sequences are given in appendix C. They are related by (3.1.12) and (3.1.13).

The generating functions (3.2.1) and (3.2.3) were first presented in [61], while the properties of (3.2.2) and generalisations were studied in [69]. We believe this is the first time (3.2.4) has been interpreted in the context of the quarter BPS sector of $\mathcal{N} = 4$ super Yang-Mills, though it has been found in other mathematical contexts [73].

3.3 Bijection between words and traces

The result of section 3.1 is to identify the aperiodic single traces as the fundamental building block from which we can construct the space of all multi-traces. The equivalent objects in the integrally-graded monoid of words are Lyndon words. We define these, exhibit how these play the same role as aperiodic traces, and then give the bijection between aperiodic traces and Lyndon words.

3.3.1 Lyndon words

For legibility, we will use the alphabet $\{0, 1\}$ in the definition of Lyndon words, and then replace this with $\{\hat{x}, \hat{y}\}$ when constructing the bijection.

A Lyndon word is an aperiodic word which is smallest (for $\hat{x}$, $\hat{y}$, this is first alphabetically) among cyclic rotations of its letters. For example the word 000101 is aperiodic and is smaller than its cyclic rotations 001010, 010100, 101000, 010001,100010, and is therefore a Lyndon word. The Lyndon words of length ≤ 5 are

$$\begin{gathered} 0\ ,\ 1 \\ 01 \\ 001\ ,\ 011 \\ 0001\ ,\ 0011\ ,\ 0111 \\ 00001\ ,\ 00011\ ,\ 00101\ ,\ 00111\ ,\ 01011\ ,\ 01111 \end{gathered} \tag{3.3.1}$$

The utility of Lyndon words comes from the Chen-Fox-Lyndon theorem [74, Theorem 5.1.5] which states that all words can be uniquely factorised as a sequence of 'non-increasing' Lyndon words.

In this context, 'non-increasing' refers to the lexicographic ordering of words. View the strings as being the binary expansions of numbers between 0 and 1. Then the ordering is just the same as the ordinary ordering of numbers between 0 and 1. If two words would form the same number after the decimal point (for example 01, 010, 0100, etc), then the longer word is larger. When using letters $\hat{x}, \hat{y}$, this ordering is just the normal alphabetical order.

We provide some factorisations as an example

$$\begin{aligned} 100101 &= 1 \circ 00101 \\ 110010 &= 1 \circ 1 \circ 001 \circ 0 \\ 011010 &= 011 \circ 01 \circ 0 \end{aligned}$$

where we have used $\circ$ as the binary operation in the free monoid on 0 and 1. Note that we require the restriction to non-increasing sequences of Lyndon words, otherwise for example we could also factorise the first word as $1 \circ 001 \circ 01$, or even $1 \circ 0 \circ 0 \circ 1 \circ 0 \circ 1$.

Now consider words constructed not just from $\hat{x}_1, \hat{y}_1$, but also $\hat{x}_2, \hat{y}_2, \hat{x}_3, \hat{y}_3, \ldots$. To deal with this we use the set of Lyndon words for each level. The factorisation of a multi-level word then consists of the factorisation of its level one component, the factorisation of its level two component, etc.

3.3.2 Structure of the space of words

In section 3.1 we saw the structure of the vector space of traces. Clearly for a bijection to exist between traces and words, the vector space of words must also have the same structure. Define W to be the space spanned by the multi-level words. By repeating the arguments of 3.1, the factorisation of words into (multi-level) Lyndon words corresponds to

$$W = \mathrm{Sym}\left(W_{LW}\right) \tag{3.3.2}$$

where W_{LW} is the space spanned by the Lyndon words of all levels. Now a levelled Lyndon word is identified by its level and an un-levelled Lyndon word on just $\hat{x}, \hat{y}$. As in section 3.1, this corresponds to

$$W_{LW} = K \otimes W_{LW}^{(1)} \tag{3.3.3}$$

where the weight of a levelled Lyndon word $k \otimes l$ is given by k times the weight of the un-levelled Lyndon word l. This is exactly the structure we saw in T.

3.3.3 Bijection between Lyndon words and aperiodic single traces

An aperiodic trace is equivalent to an aperiodic matrix word constructed from X and Y up to cyclic rotations. In particular we can choose a representative from the orbit of cyclic rotations as that which is alphabetically smallest (this is equivalent to the ordering as defined in section 3.3.1 with X replaced by 0 and Y by 1). Then the aperiodic word, by definition, is just a Lyndon word on the two letters X and Y. Replacing those letters with $\hat{x}$ and $\hat{y}$ gives the bijection.

In order to reconstruct the full bijection, we just match the two factorisations (words into Lyndon words and multi-traces into single traces) and the two level structures (periodicities and word level).

3.3.4 Products across the bijection

This bijection allows the definition of an interesting and surprising non-commutative associative product on the quarter-BPS multi-traces. The integrally-graded word monoid has a non-commutative product given by concatenation, and through the bijection we can investigate this product on the other side. Due to the factorisation properties of Lyndon words, this has some unusual behaviours, very different from the standard product. We give some examples below, using only aperiodic traces as multiplying traces with different numbers of periods will just revert to ordinary trace multiplication.

$$(\mathrm{Tr}XY)^n \circ \mathrm{Tr}Y = \mathrm{Tr}(XY)^n Y \tag{3.3.4}$$

$$\mathrm{Tr}Y \circ (\mathrm{Tr}XY)^n = (\mathrm{Tr}XY)^n \, \mathrm{Tr}Y \tag{3.3.5}$$

$$(\mathrm{Tr}XY)^n \circ \mathrm{Tr}X = (\mathrm{Tr}XY)^n \, \mathrm{Tr}X \tag{3.3.6}$$

$$\mathrm{Tr}X \circ (\mathrm{Tr}XY)^n = \mathrm{Tr}X(XY)^n \tag{3.3.7}$$

$$\mathrm{Tr}XY \, \mathrm{Tr}X^3Y \circ \mathrm{Tr}X^2Y = \mathrm{Tr}XY \, \mathrm{Tr}X^3YX^2Y \tag{3.3.8}$$

$$\mathrm{Tr}X^2Y \circ \mathrm{Tr}XY \, \mathrm{Tr}X^3Y = \mathrm{Tr}X^2YXY \, \mathrm{Tr}X^3Y \tag{3.3.9}$$

We see that within traces of a certain periodicity, this product can concatenate constituent traces to form longer traces, though still of the same periodicity.

Since all words on $\hat{x}, \hat{y}$ can be generated by concatenating $\hat{x}$ and $\hat{y}$, all aperiodic multi-traces of X and Y can be written as a $\circ$-product of only the traces $\mathrm{Tr}X$ and $\mathrm{Tr}Y$.

We do not explore this product any further here, and leave investigations of its properties and significance as an interesting problem for the future.

3.4 $SO(2,1)$ representation

The structure found in section 3.1 carries a representation of the algebra $\mathfrak{so}(2,1)$. Let $\mathbf{e}_k$ ($k = 1, 2, 3, \ldots$) be the basis vectors for K. The generators for $\mathfrak{so}(2,1)$ are J_+, J_-, J_3. We define their action on K by

$$J_+ \mathbf{e}_k = k \, \mathbf{e}_{k+1}$$

$$J_3\mathbf{e}_k = k\,\mathbf{e}_k$$
$$J_-\mathbf{e}_k = \begin{cases} k\mathbf{e}_{k-1} & k > 1 \\ 0 & k = 1 \end{cases}$$

The commutation relations for these are

$$[J_3, J_+] = J_+$$
$$[J_3, J_-] = -J_-$$
$$[J_+, J_-] = -2J_3$$

Which are indeed the commutation relations for $\mathfrak{so}(2,1)$.

Using the standard rules of tensor product representations, $T_{ST} = K \otimes T_{ST}^{(1)}$ carries a representation of $\mathfrak{so}(2,1)$, where $T_{ST}^{(1)}$ is given the trivial representation.

Let V be the carrier space for an arbitrary representation of $\mathfrak{so}(2,1)$. We note that $\mathrm{Sym}^r(V)$ is an invariant subspace of $V^{\otimes r}$ with the standard tensor product representation. Therefore $\mathrm{Sym}^r(V)$ is also the carrier space for a representation of $\mathfrak{so}(2,1)$. Therefore $T = \mathrm{Sym}(T_{ST})$ carries a representation of $\mathfrak{so}(2,1)$.

Consider what this action looks like on a generic single trace with aperiodic matrix word W and k periods

$$J_+\mathrm{Tr}W^k = k\mathrm{Tr}W^{k+1} \tag{3.4.1}$$
$$J_3\mathrm{Tr}W^k = k\mathrm{Tr}W^k \tag{3.4.2}$$
$$J_-\mathrm{Tr}W^k = \begin{cases} k\mathrm{Tr}W^{k-1} & k > 1 \\ 0 & k = 1 \end{cases} \tag{3.4.3}$$

So this $\mathfrak{so}(2,1)$ produces traces with more periods from those with less, and in doing so mixes traces of different degree. However it does not change the total number of traces; a single trace remains a single trace, a double trace remains a double trace etc.

Each aperiodic trace forms the lowest weight state (with $J_3 = 1$) of an irreducible representation. More generally, any product of m aperiodic multi-traces is the lowest weight state (with $J_3 = m$) of an irreducible representation. There are many other irreducible representations, for example there is a lowest weight state (with $J_3 = 3$) of the form $\mathrm{Tr}W_1\mathrm{Tr}W_2^2 - \mathrm{Tr}W_1^2\mathrm{Tr}W_2$ for W_1, W_2 two different aperiodic matrix words.

This action plays a complementary role to the non-commutative product defined in section 3.3.4. There, the number of periods within a trace could not be changed, but different traces with the same periodicity could be combined into a longer trace. Here, we can change the number of periods, but the aperiodic matrix word inside the trace is fixed. Combining the $\circ$ product with the $\mathfrak{so}(2,1)$ action allows us to generate all multi-traces of X and Y.

It will be interesting to investigate whether this $\mathfrak{so}(2,1)$ can be interpreted geometrically in terms of spectrum generating algebras (SGAs) in the dual space-time. SGAs of the form $SO(p,1)$ were discussed in the context of AdS/CFT in [75].

3.5 Labelling of multi-traces and conjugacy classes

The conjugation action (2.1.10) splits S_n into orbits, called conjugacy classes, labelled by $p \vdash n$. In section 2.3 we explained that in the half-BPS sector, these conjugacy classes provide a basis (2.3.11) for the centre of S_n, the algebra responsible for operator construction. This basis produced the multi-trace basis of operators. In this section we extend that description to the quarter-BPS sector by considering the 2-matrix conjugation action (2.1.12).

We quickly review some facts on the half-BPS sector that were not included in section 2.3, before moving to the more general quarter-BPS case.

3.5.1 Half-BPS sector

As explained in (2.1.11), the half-BPS traces are labelled by a partition $p \vdash n$. Any permutation $\sigma \in S_n$ of cycle type p will, when contracted with $X^{\otimes n}$, produce the trace

$$T_p = \prod_i \left(\mathrm{Tr} X^i\right)^{p_i} \tag{3.5.1}$$

Similarly, the element α_p, defined in (2.3.11) as a sum over the conjugacy class, will produce T_p when contracted with $X^{\otimes n}$. To give the size of the conjugacy class, we use the orbit-stabiliser theorem on the action (2.1.10).

The stabiliser of a permutation σ of cycle type $p = [1^{p_1}, 2^{p_2}, \dots]$ is just those elements that commute with σ. This is composed of a semi-direct product. The first factor comes from powers of the cycles of σ, which generate a group of the form

$$G_1 = \bigtimes_i (\mathbb{Z}_i)^{p_i} \tag{3.5.2}$$

Intuitively, these rotate the cycles of σ. The second factor comes from permuting cycles of the same length. This has the form

$$G_2 = \bigtimes_i S_{p_i} \tag{3.5.3}$$

In the semi-direct product, the component S_{p_i} of G_2 acts on $(\mathbb{Z}_i)^{p_i}$ by permuting the factors. This is called the wreath product of S_{p_i} with $\mathbb{Z}_i$ and is denoted by $S_{p_i}[\mathbb{Z}_i]$. For more on the wreath product, see section 5.1.2.

Overall, we have the stabiliser

$$Stab(\sigma) \cong G_2 \ltimes G_1 = \bigtimes_i \left[S_{p_i} \ltimes (\mathbb{Z}_i)^{p_i}\right] = \bigtimes_i S_{p_i}\left[\mathbb{Z}_i\right] \tag{3.5.4}$$

which has size z_p, defined in (2.3.1). Applying the orbit-stabiliser theorem, the size of the conjugacy class is

$$\frac{n!}{z_p} \tag{3.5.5}$$

3.5.2 Quarter-BPS labelling

As discussed in section 3.1, a single trace is described by a Lyndon word w and the number of periods, while a multi-trace is defined by a collection of these single traces. Consider a generic multi-trace, and let the number of constituent single traces with Lyndon word w and number of periods i be $p_{w,i}$, then the multi-trace can be written

$$T_{\mathcal{P}} = \prod_{w,i} \left(\mathrm{Tr}W^i\right)^{p_{w,i}} \tag{3.5.6}$$

where W is the matrix word equivalent of the Lyndon word w. This trace is characterised by the set of numbers $\mathcal{P} = \{p_{w,i}\}$. A convenient way to package these numbers is to define a partition p_w for each Lyndon word

$$p_w = [1^{p_{w,1}}, 2^{p_{w,2}}, \ldots] \tag{3.5.7}$$

Then the label for a $U(N)$ multi-trace is

$$\mathcal{P} = \{p_w : w \text{ a Lyndon word}\} = \left\{p_x, p_y, p_{xy}, p_{x^2y}, p_{xy^2}, \ldots\right\} \tag{3.5.8}$$

The partition p_x is the partition used to label the half-BPS traces, and the remaining partitions have the same interpretation, replacing the matrix X with a matrix word. Consider just the p_w partition inside $\mathcal{P}$. Then the corresponding multi-trace is

$$T^w_{\mathcal{P}} = \prod_i \left(\mathrm{Tr}W^i\right)^{(p_w)_i} \tag{3.5.9}$$

Comparing with (3.5.1), we can see X has been substituted for W. A general $T_{\mathcal{P}}$ is a product of these for each w.

$$T_{\mathcal{P}} = \prod_w T^w_{\mathcal{P}} = \prod_{w,i} \left(\mathrm{Tr}W^i\right)^{(p_w)_i} \tag{3.5.10}$$

Define $l_x(w)$, $l_y(w)$ and $l(w)$ be the number of xs, the number of ys and the total length of w respectively. Then clearly $l(w) = l_x(w) + l_y(w)$, and the number of Xs and Ys in

$\mathcal{P}$	$T_{\mathcal{P}}$
$p_x = [1,1]\ ,\ p_y = [1,1]$	$(\mathrm{Tr}X)^2\,(\mathrm{Tr}Y)^2$
$p_x = [1,1]\ ,\ p_y = [2]$	$(\mathrm{Tr}X)^2\,(\mathrm{Tr}Y^2)$
$p_x = [2]\ ,\ p_y = [1,1]$	$(\mathrm{Tr}X^2)\,(\mathrm{Tr}Y)^2$
$p_x = [1]\ ,\ p_y = [1]\ ,\ p_{xy} = [1]$	$(\mathrm{Tr}X)\,(\mathrm{Tr}XY)\,(\mathrm{Tr}Y)$
$p_x = [2]\ ,\ p_y = [2]$	$(\mathrm{Tr}X^2)\,(\mathrm{Tr}Y^2)$
$p_{xy} = [1,1]$	$(\mathrm{Tr}XY)^2$
$p_{x^2y} = [1]\ ,\ p_y = [1]$	$(\mathrm{Tr}X^2Y)\,(\mathrm{Tr}Y)$
$p_{xy^2} = [1]\ ,\ p_x = [1]$	$(\mathrm{Tr}X)\,(\mathrm{Tr}XY^2)$
$p_{xy} = [2]$	$\mathrm{Tr}(XY)^2$
$p_{x^2y^2} = [1]$	$\mathrm{Tr}X^2Y^2$

Table 3.1: The 10 different $U(N)$ multi-traces at $n_1 = n_2 = 2$ along with their labels. Any constituent partitions of $\mathcal{P}$ that are not explicitly listed are set to zero.

a multi-trace is

$$n_1 = \sum_w l_x(w)|p_w| \qquad\qquad n_2 = \sum_w l_y(w)|p_w| \tag{3.5.11}$$

We summarise this with $\mathcal{P} \Vdash (n_1, n_2)$.

As an example of this labelling, table 3.1 lists the 10 different $\mathcal{P} \Vdash (2,2)$ and their associated multi-traces.

3.5.3 Quarter-BPS conjugacy classes

For the 2-matrix case, rather than conjugation by S_n in (2.1.10), we have conjugation by $S_{n_1} \times S_{n_2}$. We still call the orbits under this reduced conjugation action 'conjugacy classes'. Since permutations in a particular conjugacy class lead to the same multi-trace, and conversely each multi-trace corresponds to a conjugacy class, the labelling set for the conjugacy classes is exactly the same as that for the traces, the $\mathcal{P}$ defined in (3.5.8).

As in section 3.5.1, the size of these conjugacy classes is found using the orbit-stabiliser theorem. Take σ to be a representative member of the conjugacy class $\mathcal{P}$. From the examples given in (2.1.4-2.1.6), and the general case given above them, we can see how the cycles of σ produce the single trace components of $T_{\mathcal{P}}$: a number in $\{1, 2, \ldots, n_1\}$ corresponds to an X while a number in $\{n_1+1, n_1+2, \ldots, n\}$ corresponds to a Y.

The stabiliser of σ is composed of the elements of $S_{n_1} \times S_{n_2}$ that commute with σ. As in the half-BPS, each cycle has a rotation subgroup attached to it. However, conjugation by $S_{n_1} \times S_{n_2}$ rather than by S_n means we can only rotate the numbers $1, 2, \ldots, n_1$ amongst themselves (and similarly for $n_1+1, n_1+2, \ldots, n$). Therefore for a single cycle labelled by Lyndon word w and number of repetitions i (remember cycles correspond to single traces), the rotation group has size i (rather than $il(w)$, which is the length of the cycle). As in the half-BPS case, different cycles with the same labels can be permuted, and therefore the stabiliser is given by

$$Stab(\sigma) \cong \bigtimes_{w,i} S_{p_{w,i}}[\mathbb{Z}_i] \tag{3.5.12}$$

which has size

$$Z_{\mathcal{P}} = \prod_{w,i} i^{p_{w,i}} (p_{w,i})! = \prod_{w} z_{p_w} \tag{3.5.13}$$

So by the orbit-stabiliser theorem, the size of $S_{n_1} \times S_{n_2}$ conjugacy classes is

$$\frac{n_1!n_2!}{Z_{\mathcal{P}}} \tag{3.5.14}$$

Sums over the conjugacy classes span the invariant algebra $\mathcal{A}_{n_1,n_2}$. Let $\sigma_{\mathcal{P}}$ be any permutation in the conjugacy class labelled by $\mathcal{P}$. Then a basis for $\mathcal{A}_{n_1,n_2}$ is

$$\alpha_{\mathcal{P}} = \frac{1}{n_1!n_2!} \sum_{\tau \in S_{n_1} \times S_{n_2}} \tau \sigma_{\mathcal{P}} \tau^{-1} \tag{3.5.15}$$

These are the elements of $\mathcal{A}_{n_1,n_2}$ that produce the corresponding multi-trace operator $T_{\mathcal{P}}$.

We can use (3.5.13) to evaluate the S_n inner product of 2-matrix multi-traces using the formula (2.6.13)

$$\langle T_{\mathcal{P}} | T_{\mathcal{Q}} \rangle_{S_n} = Z_{\mathcal{P}} \delta_{\mathcal{P}\mathcal{Q}} \tag{3.5.16}$$

As expected, $Z_{\mathcal{P}}$ plays the same role as z_p in the half-BPS case, (2.6.9), and can be interpreted physically in the planar limit as the norm of a multi-trace operator.

3.6 Orthogonal Young diagram bases and correlators

For $N < n$, the multi-trace bases, at half and quarter-BPS, acquire highly non-trivial relations between the different elements, and the finite N behaviour is difficult to determine. In the language of section 2.5.1, they are not SEP-compatible.

For half-BPS operators, the Schur basis defined in (2.3.14) is exactly orthogonal at all order in N and is SEP-compatible. We now introduce two generalisations of this basis to the quarter-BPS sector, the restricted Schur basis and the covariant basis. Both have the same key properties, they are exactly orthogonal to all orders in N and are SEP-compatible.

3.6.1 Restricted Schur basis

Operator basis

The restricted Schur basis was first constructed in [44, 45] following earlier work in [29, 32]. It gives a basis for the space of degree (n_1, n_2) traces, and is labelled by $R \vdash n, R_1 \vdash n_1, R_2 \vdash n_2$, along with two Littlewood-Richardson multiplicity indices μ, ν that satisfy $1 \leq \mu, \nu, \leq g_{R;R_1,R_2}$. The operators are given by

$$\mathcal{O}_{R,R_1,R_2,\mu,\nu} = \sqrt{\frac{d_R}{d_{R_1} d_{R_2} n!(n_1)!(n_2)!}} \sum_{\sigma \in S_n} \chi_{R,R_1,R_2,\mu,\nu}(\sigma) \mathrm{Tr}\left(\sigma X^{\otimes n_1} \otimes Y^{\otimes n_2}\right) \quad (3.6.1)$$

The first expression inside the sum is called the restricted character of σ, and is defined by

$$\chi_{R,R_1,R_2,\mu,\nu}(\sigma) = \mathrm{Tr}_R \left[P^R_{R_1,R_2;\mu\to\nu} D^R(\sigma)\right] \quad (3.6.2)$$

where Tr_R is the trace over the representation R of S_n, and $P^R_{R_1,R_2;\mu\to\nu}$ is an intertwiner that takes the μth copy of $R_1 \otimes R_2$ (a representation of $S_{n_1} \times S_{n_2}$) inside R to the νth copy, and is zero on everything else. For more on the Littlewood-Richardson multiplicity indices μ and ν, including a systematic way of choosing a basis for the multiplicity space, see appendix D.

At finite $N < n$, R is restricted to have at most N rows. Since R_1, R_2 are such that there is a non-zero multiplicity of $R_1 \otimes R_2$ inside R, the same restriction applies to R_1 and R_2.

When $n_1 = n$ and $n_2 = 0$, the restricted character reverts to just the ordinary character χ_R, and (3.6.1) become exactly the Schur operators defined in (2.3.14).

Algebra basis

In the equivalent algebra picture, the restricted Schur elements give a basis for $\mathcal{A}_{n_1,n_2}$

$$\beta_{R,R_1,R_2,\mu,\nu} = \frac{d_R}{n!} \sum_{\sigma \in S_n} \chi_{R,R_1,R_2,\mu,\nu}(\sigma)\sigma \quad (3.6.3)$$

where we have used a different normalisation to the operators (3.6.1). The normalisation for operators was chosen to give nice expressions for the correlators in (3.6.8), whereas the normalisation in (3.6.3) is chosen to give nice multiplicative properties in $\mathcal{A}_{n_1,n_2}$, given below.

This basis for $\mathcal{A}_{n_1,n_2}$ gives an explicit identification of the Wedderburn-Artin decomposition of the algebra, since they have the multiplication property

$$\beta_{R,R_1,R_2,\mu_1,\mu_2} \beta_{S,S_1,S_2,\nu_1,\nu_2} = \delta_{RS} \delta_{R_1 S_1} \delta_{R_2 S_2} \delta_{\mu_2 \nu_1} \beta_{R,R_1,R_2,\mu_1,\nu_2} \quad (3.6.4)$$

From this, we can think of the $\beta_{R,R_1,R_2,\mu,\nu}$ as block diagonal matrices with a block for each trio (R, R_1, R_2). This block has size $g_{R;R_1,R_2}$. The matrix for $\beta_{R,R_1,R_2,\mu,\nu}$ is the zero matrix in each block except the (R, R_1, R_2) block, in which there is a single 1 in the (μ, ν)th position.

Using this matrix picture, we can see that as a matrix algebra, $\mathcal{A}_{n_1,n_2}$ has the form

$$\mathcal{A}_{n_1,n_2} = \bigoplus_{\substack{R\vdash n \\ R_1\vdash n_1 \\ R_2\vdash n_2}} \mathcal{M}(g_{R;R_1,R_2}) \tag{3.6.5}$$

where $\mathcal{M}(k)$ is the algebra of $k \times k$ matrices. Representations of $\mathcal{M}(k)$ are the same as representations of $GL(k)$ and therefore the irreducible representations of $\mathcal{A}_{n_1,n_2}$ are labelled by the triple R, R_1, R_2 and a $GL(g_{R;R_1,R_2})$ Young diagram.

At finite N, we lose the basis elements with $l(R) > N$, and only those with $l(R) \leq N$ contribute to operator construction.

For more properties of this algebra see [63].

Combinatorics

From the labelling of the restricted Schur basis, we see the dimension of the degree (n_1, n_2) space can be written

$$N^N_{n_1,n_2} = \sum_{\substack{R\vdash n \\ R_1\vdash n_1 \\ R_2\vdash n_2 \\ l(R)\leq N}} g^2_{R;R_1,R_2} \tag{3.6.6}$$

From section 3.1, these dimensions (for infinite N) are generated by

$$\sum_{n_1,n_2} N^\infty_{n_1,n_2} x^{n_1} y^{n_2} = F(x, y) = \prod_{k=1}^{\infty} \frac{1}{1 - x^k - y^k} \tag{3.6.7}$$

Correlators

The two-point function of restricted Schur operators is given by

$$\langle \mathcal{O}_{R,R_1,R_2,\mu_1,\mu_2} | \mathcal{O}_{S,S_1,S_2,\nu_1,\nu_2} \rangle = \delta_{RS}\delta_{R_1S_1}\delta_{R_2S_2}\delta_{\mu_1\nu_1}\delta_{\mu_2\nu_2} f_R \tag{3.6.8}$$

$$\langle \mathcal{O}_{R,R_1,R_2,\mu_1,\mu_2} | \mathcal{O}_{S,S_1,S_2,\nu_1,\nu_2} \rangle_{S_n} = \begin{cases} \delta_{RS}\delta_{R_1S_1}\delta_{R_2S_2}\delta_{\mu_1\nu_1}\delta_{\mu_2\nu_2} & l(R) \leq N \\ 0 & l(R) > N \end{cases} \tag{3.6.9}$$

where f_R was defined in (2.3.20).

Three-point and higher functions can be obtained by using the product rule for restricted Schur operators explained in [45].

3.6.2 Covariant basis

The covariant basis was first introduced in [43,46]. In chapter 7 we use it to investigate quarter-BPS operators in the weakly coupled theory.

$U(2)$ action on traces

Let $X_1 = X$ and $X_2 = Y$. Then there is an action of $U(2)$ on the i index in X_i. By extension this acts on all traces and operators, so we can choose our basis to be $U(2)$ covariant. Since $U(2)$ turns Xs into Ys and vice versa, this basis mixes states with different numbers of Xs and Ys while keeping the total number of matrices, n, constant. We will say an operator has *field content* (n_1, n_2) if it contains n_1 Xs and n_2 Ys.

The $\mathfrak{u}(2)$ operators on traces are given by

$$R^i_j = \begin{pmatrix} \mathcal{X} & \mathcal{J}_+ \\ \mathcal{J}_- & \mathcal{Y} \end{pmatrix} = \begin{pmatrix} \mathrm{Tr} X \frac{\partial}{\partial X} & \mathrm{Tr} X \frac{\partial}{\partial Y} \\ \mathrm{Tr} Y \frac{\partial}{\partial X} & \mathrm{Tr} Y \frac{\partial}{\partial Y} \end{pmatrix} = \begin{pmatrix} X^i_j \frac{\partial}{\partial X^i_j} & X^i_j \frac{\partial}{\partial Y^i_j} \\ Y^i_j \frac{\partial}{\partial X^i_j} & Y^i_j \frac{\partial}{\partial Y^i_j} \end{pmatrix} \tag{3.6.10}$$

The operator $\mathcal{X}$ counts the number of X matrices in a trace, similarly for $\mathcal{Y}$. The lowering operator $\mathcal{J}_-$ 'lowers' a trace by turning an X into a Y, and the raising operator $\mathcal{J}_+$ 'raises' a trace by turning a Y into an X.

Acting on the matrices X_i with a $U(2)$ index

$$R^i_j X_k = \delta^i_k X_j \tag{3.6.11}$$

Define new operators

$$\mathcal{J}_0 = \mathcal{X} + \mathcal{Y} \qquad\qquad \mathcal{J}_3 = \mathcal{X} - \mathcal{Y} \tag{3.6.12}$$

Then $\mathcal{J}_0$ counts the total number of matrices, while $\mathcal{J}_3$ counts the difference between the number of Xs and Ys. As the notation suggests, $\mathcal{J}_3, \mathcal{J}_\pm$ form an $\mathfrak{su}(2)$ subalgebra of $\mathfrak{u}(2)$, while $\mathcal{J}_0$ spans a $\mathfrak{u}(1)$ that commutes with the $\mathfrak{su}(2)$. This split decomposes $\mathfrak{u}(2)$ into a sum of $\mathfrak{su}(2)$ and $\mathfrak{u}(1)$.

The operators (3.6.10) obey standard hermiticity conditions $\left(R^i_j\right)^\dagger = R^j_i$ for R-symmetry generators

$$(\mathcal{J}_0)^\dagger = \mathcal{J}_0 \qquad\qquad (\mathcal{J}_3)^\dagger = \mathcal{J}_3 \qquad\qquad (\mathcal{J}_+)^\dagger = \mathcal{J}_- \tag{3.6.13}$$

It follows that operators with different $U(2)$ quantum numbers must be orthogonal.

U(2) representations

Semi-standard Young tableaux are defined to be Young tableaux in which the positive integers in the boxes increase weakly along the rows and strictly down the columns. For example if we take $R = [2, 1]$ and allow entries of 1,2 and 3, the possible semi-standard tableaux are:

$$\begin{ytableau} 1 & 1 \\ 2 \end{ytableau} \quad \begin{ytableau} 1 & 1 \\ 3 \end{ytableau} \quad \begin{ytableau} 1 & 2 \\ 2 \end{ytableau} \quad \begin{ytableau} 1 & 2 \\ 3 \end{ytableau} \quad \begin{ytableau} 1 & 3 \\ 2 \end{ytableau} \quad \begin{ytableau} 1 & 3 \\ 3 \end{ytableau} \quad \begin{ytableau} 2 & 2 \\ 3 \end{ytableau} \quad \begin{ytableau} 2 & 3 \\ 3 \end{ytableau} \tag{3.6.14}$$

The evaluation of a semi-standard tableau r is a sequence of numbers $\rho(r) = [\rho_1, \rho_2, \dots]$ where

$$\rho_i = (\# \text{ of occurences of the number } i \text{ in } r) \tag{3.6.15}$$

So for example the evaluations of the tableaux in (3.6.14) are respectively

$$[2,1,0] \quad [2,0,1] \quad [1,2,0] \quad [1,1,1] \quad [1,1,1] \quad [1,0,2] \quad [0,2,1] \quad [0,1,2] \tag{3.6.16}$$

When the evaluation $\rho(r)$ is a partition (i.e. $\rho_1 \geq \rho_2 \geq \dots$), these tableaux contribute to the Kostka number $K_{R\rho}$ seen in (2.7.12).

For a representation $\Lambda \vdash n$ of $U(2)$ with $l(\Lambda) \leq 2$, the basis vectors of Λ are labelled by the semi-standard Young tableaux of shape Λ containing only 1s and 2s. For $\Lambda = \left[\frac{n}{2} + j, \frac{n}{2} - j\right]$, there are $2j + 1$ possible tableaux, where j runs over the non-negative half-integers up to $\frac{n}{2}$. These possibilities are

$$\overbrace{\qquad\qquad}^{\frac{n}{2}-j}\overbrace{\qquad\qquad}^{k}\overbrace{\qquad\qquad}^{2j-k}$$
$$\begin{ytableau} 1 & 1 & \none[\cdots] & 1 & 1 & 1 & \none[\cdots] & 1 & 2 & 2 & \none[\cdots] & 2 \\ 2 & 2 & \none[\cdots] & 2 \end{ytableau} \tag{3.6.17}$$

where $0 \leq k \leq 2j$.

We can understand the representation $\Lambda = \left[\frac{n}{2} + j, \frac{n}{2} - j\right]$ in terms of the $\mathfrak{u}(1)$ spanned by $\mathcal{J}_0$ and the $\mathfrak{su}(2)$ spanned by $\mathcal{J}_3, \mathcal{J}_\pm$. All states in Λ have weight n under $\mathfrak{u}(1)$, and form a spin j of $\mathfrak{su}(2)$. The identification of basis vectors is

$$\overbrace{\qquad\qquad}^{\frac{n}{2}-j}\overbrace{\qquad\qquad}^{j+m_j}\overbrace{\qquad\qquad}^{j-m_j}$$
$$\begin{ytableau} 1 & 1 & \none[\cdots] & 1 & 1 & 1 & \none[\cdots] & 1 & 2 & 2 & \none[\cdots] & 2 \\ 2 & 2 & \none[\cdots] & 2 \end{ytableau} = |j, m_j\rangle \tag{3.6.18}$$

where $|j, m_j\rangle$ is the standard basis spanning the spin j representation of $\mathfrak{su}(2)$ with $-j \leq m_j \leq j$.

Operator basis

Consider $V_2^{\otimes n}$, where V_2 is the fundamental of $U(2)$, and in particular the basis vector $a = e_{a_1} \otimes e_{a_2} \otimes \cdots \otimes e_{a_n}$ of $V_2^{\otimes n}$ where $a_j \in \{1, 2\}$ for each j. Then we define $\mathbb{X}_a = X_{a_1} \otimes X_{a_2} \otimes \cdots \otimes X_{a_n}$. Combined with a permutation $\sigma \in S_n$, we write

$$\mathcal{O}_{a,\sigma} = \text{Tr}\left(\sigma \mathbb{X}_a\right) \tag{3.6.19}$$

The covariant basis is labelled by $\Lambda \vdash n$, a partition with at most 2 rows; M_Λ, a semi-standard tableau of shape Λ that indexes the basis vector of the Λ representation of $U(2)$; $R \vdash n$, a partition with at most N rows; and τ, a multiplicity index satisfying $1 \leq \tau \leq C(R, R, \Lambda)$. $C(R, R, \Lambda)$ is the multiplicity of Λ (as an S_n representation this time) within $R \otimes R$, or equivalently the multiplicity of the trivial representation within $R \otimes R \otimes \Lambda$.

Using these labels, the covariant basis operators are

$$\mathcal{O}_{\Lambda, M_\Lambda, R, \tau} = \frac{\sqrt{d_R}}{n!} \sum_{\sigma, a, i, j, m} S^{R\,R\,\Lambda,\,\tau}_{i\;j\;m} D^R_{ij}(\sigma) C^a_{\Lambda, M_\Lambda, m} \mathcal{O}_{a,\sigma} \tag{3.6.20}$$

where $C^a_{\Lambda, M_\Lambda, m}$ are the Clebsch-Gordon coefficients for the Schur-Weyl decomposition

$$V_2^{\otimes n} = \bigoplus_{\substack{\Lambda \vdash n \\ l(\Lambda) \leq 2}} V_\Lambda^{U(2)} \otimes V_\Lambda^{S_n} \tag{3.6.21}$$

and $S^{R\,R\,\Lambda\,\tau}_{i\;j\;m}$ are the Clebsch-Gordon coefficients for the τth copy of the trivial S_n representation inside $R \otimes R \otimes \Lambda$.

The field content (n_1, n_2) is the evaluation of the tableau M_Λ. The number of 1s in M_Λ is n_1, while the number of 2s is n_2.

When $\Lambda = [n]$ and M_Λ is the highest weight state, the operators (3.6.20) reduce to the standard Schur operators (2.3.14). Therefore the $\Lambda = [n]$ sector is (part of) an ultra-short multiplet and has the same properties as the half-BPS operators. The multiplicity index τ is trivial since $R \otimes R$ always contains a unique copy of the trivial representation

$$C(R, R, [n]) = 1 \tag{3.6.22}$$

The $\Lambda = [n-1, 1]$ sector also has special properties. In [10] it was proved that these multiplets cannot recombine to form long non-BPS multiplets and therefore must remain quarter-BPS at all values of the coupling. In these cases, the multiplicity τ in (3.6.20) runs over the number of corners of R minus 1.

$$C(R, R, [n-1, 1]) = (\#\text{ of corners in } R) - 1 \tag{3.6.23}$$

This is proved most simply by comparing the covariant basis with the combinatorics of the restricted Schur basis defined in (3.6.6).

Combinatorics

In the $\Lambda = [n-j, j]$ sector, there is a tableau M_Λ with field content (n_1, n_2) if and only if $n_1, n_2 \geq j$. So using the labelling of the covariant basis, we can write the dimension of the degree (n_1, n_2) space as

$$N^N_{n_1,n_2} = \sum_{\substack{R \vdash n \\ j \leq \min(n_1,n_2) \\ l(R) \leq N}} C(R, R, [n-j, j]) \tag{3.6.24}$$

Correlators

The physical and S_n correlators of generic $U(2)$ covariant multi-traces $\mathcal{O}_{a,\sigma}$ are

$$\langle \mathcal{O}_{b,\tau} | \mathcal{O}_{a,\sigma} \rangle = \sum_{\alpha \in S_n} \delta_{\alpha(a),b} \delta\left(\Omega \alpha \sigma \alpha^{-1} \tau^{-1}\right) \tag{3.6.25}$$

$$\langle \mathcal{O}_{b,\tau} | \mathcal{O}_{a,\sigma} \rangle_{S_n} = \sum_{\alpha \in S_n} \delta_{\alpha(a),b} \delta_N\left(\alpha \sigma \alpha^{-1} \tau^{-1}\right) \tag{3.6.26}$$

These are the same as the inner products (2.6.12) and (2.6.13), only with additional $U(2)$ covariant labels. The $\delta_{\alpha(a),b}$ factor enforces that $\alpha \in S_{n_1} \times S_{n_2}$ for the appropriate embedding of $S_{n_1} \times S_{n_2}$ into S_n.

Both inner products are diagonal on the covariant basis, and similarly to the Schurs and restricted Schurs, it is R that determines the physical norm [43].

$$\left\langle \mathcal{O}_{\Lambda, M_\Lambda, R, \tau} | \mathcal{O}_{\Lambda', M_{\Lambda'}, R', \tau'} \right\rangle = \delta_{\Lambda,\Lambda'} \delta_{M_\Lambda, M_{\Lambda'}} \delta_{R,R'} \delta_{\tau,\tau'} f_R \tag{3.6.27}$$

$$\left\langle \mathcal{O}_{\Lambda, M_\Lambda, R, \tau} | \mathcal{O}_{\Lambda', M_{\Lambda'}, R', \tau'} \right\rangle_{S_n} = \begin{cases} \delta_{\Lambda,\Lambda'} \delta_{M_\Lambda, M_{\Lambda'}} \delta_{R,R'} \delta_{\tau,\tau'} & l(R) \leq N \\ 0 & l(R) > N \end{cases} \tag{3.6.28}$$

Higher point (extremal) correlation functions can be obtained by using a product rule. For the covariant basis this was given, for the more general case of a quiver theory, in [68]. For the procedure to reinsert the position dependences into higher point correlation functions see [22].

In both the restricted Schur (3.6.1) and covariant, (3.6.20), bases we have a label $R \vdash n$, restricted to $l(R) \leq N$, that governs the S_n behaviour of the operator and also determines the norm. As might be expected, these labels coincide for the two bases, and the other labels are just different ways of parameterising the distinct R subspaces. In [76], the author gives an explicit basis change between the two bases for fixed field content (n_1, n_2), demonstrating that R is unchanged between the two.

Chapter 4

Structure and combinatorics of the planar free field quarter-BPS sector with $SO(N)$ and $Sp(N)$ gauge groups

Familiar AdS/CFT connects $\mathcal{N} = 4$ super Yang-Mills with $U(N)$ gauge group with string theory on $AdS_5 \times S^5$. In [55], the gravity dual of different gauge groups was considered. It was found that the appropriate space to consider was $AdS_5 \times \mathbb{RP}^5$, obtained from the standard space by taking an orientifold quotient that identifies opposite points of the S^5 while also reversing string worldsheet orientation. Depending on the cohomology class of the field strengths associated to the 2-form fields of the original S^5-based theory, this is dual to either an orthogonal or symplectic gauge group.

On the CFT side of the duality, this quotient is rather simpler to understand, we replace all fields in the complex adjoint of $\mathfrak{u}(N)$ with those in the complex adjoint of $\mathfrak{so}(N)$ or $\mathfrak{sp}(N)$ respectively. In the half and quarter-BPS sectors, this entails replacing the complex matrices X and Y with anti-symmetric matrices for the orthogonal group, or for the symplectic group, matrices satisfying the symplectic condition

$$X^T = -\Omega X \Omega^T \tag{4.0.1}$$

where

$$\Omega = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \tag{4.0.2}$$

and I is the $\frac{N}{2}$ by $\frac{N}{2}$ identity matrix (the symplectic group only exists for N even).

There are deep connections between the orthogonal group (for even N) and the symplectic group. It was proved in [77] that dimensions of $SO(N)$ and $Sp(N)$ irreducible representations (both labelled by a Young diagram R) are related by the transformation

$$R \to R^c \qquad\qquad N \to -N \tag{4.0.3}$$

This relation was extended to all (non-baryonic) gauge invariant quantities in [78]. This general pattern of anti-symmetrisation (conjugation) and $N \to -N$ was observed in [57, 60], and will occur repeatedly in chapter 5. For the purposes of this chapter, it is sufficient to note that the combinatorics of the two theories will match in the planar limit where finite N cut-offs on Young diagrams do not enter.

When we consider the gauge group $SO(N)$ (with N even), there is a qualitative difference in the spectrum of BPS primary operators compared to both $U(N)$ and $Sp(N)$. The latter two theories consist purely of multi-trace (mesonic) operators, while the orthogonal group also allows Pfaffian type (baryonic) operators. From the CFT perspective, this stems from the invariant tensor $\varepsilon_{i_1 i_2 \ldots i_N}$, and in [55] a $D3$-brane wrapped around a $\mathbb{RP}^3$ subspace of $\mathbb{RP}^5$ was presented as a candidate for the gravity dual. These Pfaffian operators were studied further in [21], where more evidence was provided that a wrapped $D3$-brane is the correct interpretation. These have conformal dimension $\frac{N}{2}$, so do not contribute to the planar quarter-BPS sector or generating function studied in this chapter. They will be considered in detail in chapter 5.

This chapter focuses on the planar structure of the $SO(N)$ and $Sp(N)$ theories, and in particular how the aperiodic matrix words of the $U(N)$ theory are replaced by 'minimally periodic' words for $SO(N)$ and $Sp(N)$ multi-traces. This is expressed in the equivalent combinatorics of words by defining *orthogonal Lyndon words* which play the role of Lyndon words from the $U(N)$ theory. We derive two distinct ways of decomposing the space of 2-matrix multi-traces at large N, one in terms of the minimally periodic words, the other in terms of the aperiodic words. The associated generating functions are then calculated.

This chapter consists of results originally presented in [1].

4.1 Constraints on single traces, orthogonal Lyndon words and labelling of multi-traces

We start by considering the $SO(N)$ theory, where we replace the generic complex matrices of the $U(N)$ theory with anti-symmetric complex matrices. We then move on to the symplectic case, where the new matrices satisfy the condition (4.0.1). This turns out to result in exactly the same set of traces.

In the $SO(N)$ half-BPS sector, $\mathrm{Tr}X^n$ vanishes for n odd, and hence there are no gauge invariant operators if n is odd. If n is even, they are labelled by a partition $p \vdash \frac{n}{2}$

with corresponding operator

$$T_p = \prod_i \left(\mathrm{Tr}X^{2i}\right)^{p_i} \tag{4.1.1}$$

To look at the quarter-BPS sector, begin by considering the trace of an arbitrary matrix word. It is specified by k, the number of periods, and an aperiodic matrix word W. Since a trace is invariant under transposition, we have

$$\mathrm{Tr}W^k = \mathrm{Tr}\left(W^T\right)^k \tag{4.1.2}$$

As X and Y are anti-symmetric, the relation (4.1.2) reduces the number of linearly independent single traces. The transpose reverses the matrix word - we call the reversed word $W^{(r)}$ - and introduces a factor of $(-1)^{k\,l(W)}$, where $l(W)$ is the length of W.

$$\mathrm{Tr}W^k = (-1)^{k\,l(W)}\mathrm{Tr}\left(W^{(r)}\right)^k \tag{4.1.3}$$

There are now two sets of two possibilities: either W and $W^{(r)}$ are the same (up to cyclic rotations), or they are not, and $k\,l(W)$ is either even or odd.

If $W \neq W^{(r)}$, then (4.1.3) tells us that two distinct traces that were previously unrelated are no longer independent. The parity of $k\,l(W)$ affects whether they are related with a positive or negative sign, but does not change the combinatorics.

If $W = W^{(r)}$, then the combinatorics is dependent on the parity of $k\,l(W)$. If $k\,l(W)$ is even, then (4.1.3) is trivial, and gives us no new information. If it is odd, then (4.1.3) implies that the trace vanishes. So for example, $\mathrm{Tr}X$, $\mathrm{Tr}Y^3$, $\mathrm{Tr}X^2Y$ and $\mathrm{Tr}(X^4Y)^5$ all vanish.

We can therefore write the linearly independent $SO(N)$ single traces as

$$\mathrm{Tr}\left(\widetilde{W}\right)^k \tag{4.1.4}$$

where $\widetilde{W}$, rather than being aperiodic, is instead minimally periodic. From the cases above, there are three different possibilities for $\widetilde{W}$, that we call types 1A, 1B and 2.

- Type 1A: $\widetilde{W}$ is a Lyndon word of even length which is invariant under reversal (up to cyclic rotations)
- Type 1B: $\widetilde{W}$ is the square of a Lyndon word of odd length that is invariant under reversal (up to cyclic rotations)
- Type 2: $\widetilde{W}$ is the first (lexicographically) of a pair of Lyndon words that transform into each other under reversal (up to cyclic rotations)

For type 2, there is nothing special about our choice of using the first of the pair, we merely need to choose a representative.

Type 1A	xy x^3y , x^2y^2 , xy^3 x^5y, x^4y^2, x^3yxy, x^3y^3, x^2y^4, $xyxy^3$, xy^5
Type 1B	x^2, y^2 x^2yx^2y, xy^2xy^2 x^4yx^4y, $x^3y^2x^3y^2$, x^2yxyx^2yxy, $x^2y^3x^2y^3$, $xyxy^2xyxy^2$, xy^4xy^4
Type 2	x^2yxy^2 x^3yxy^2, x^2yxy^3 $x^3yx^2y^2$, x^4yxy^2, x^3yxy^3, x^2yxyxy^2, x^2yxy^4, $x^2y^2xy^3$

Table 4.1: Lowest order examples of the three distinct types of orthogonal Lyndon words

We define type 1A, 1B and 2 *orthogonal Lyndon words* in the same manner as above, but on formal letters x, y rather than matrices X and Y. These play the same role for $SO(N)$ traces as the normal Lyndon words did for $U(N)$ traces. The lowest order examples of the three types of orthogonal Lyndon words are shown in table 4.1.

Although the orthogonal Lyndon words defined here play the same role in the labelling of traces for $SO(N)$ as the normal Lyndon words did for $U(N)$, there are two important differences. Firstly, the orthogonal words are not aperiodic; type 1B words contain two periods, hence the 'minimally periodic' condition. Secondly, the orthogonal Lyndon words do not form the factorisation units in a free monoid on two letters. This means we cannot define a product on the $SO(N)$ traces as we did for the $U(N)$ version in section 3.3.4.

4.1.1 Labelling of multi-traces

In section 3.5.2, we saw that to label $U(N)$ multi-traces, we gave each Lyndon word w a partition p_w. The analogous statement is true for $SO(N)$ with orthogonal Lyndon words.

We define $p_{\tilde{w},i}$ to be the number of single traces with i repetitions of the minimally periodic $\widetilde{W}$ that appear in a multi-trace. These combine into a partition $p_{\tilde{w}} = [1^{p_{\tilde{w},1}}, 2^{p_{\tilde{w},2}}, \dots]$ for each orthogonal Lyndon word. We define $\widetilde{\mathcal{P}}$ to be the set of these partitions

$$\begin{aligned} \widetilde{\mathcal{P}} &= \{p_{\tilde{w}} : \tilde{w} \text{ an orthogonal Lyndon word}\} \\ &= \left\{p_{x^2}, p_{xy}, p_{y^2}, p_{x^3y}, p_{x^2y^2}, p_{xy^3}, \dots, p_{x^2yxy^2}, \dots\right\} \end{aligned} \tag{4.1.5}$$

The multi-trace corresponding to $\widetilde{\mathcal{P}}$ is

$$T_{\widetilde{\mathcal{P}}} = \prod_{\tilde{w}} T^{\tilde{w}}_{\widetilde{\mathcal{P}}} = \prod_{\tilde{w},i} \left(\mathrm{Tr}\widetilde{W}^i\right)^{p_{\tilde{w},i}} \tag{4.1.6}$$

$\widetilde{\mathcal{P}}$	$T_{\widetilde{\mathcal{P}}}$
$p_{x^2} = [1]$, $p_{xy} = [1]$, $p_{y^2} = [1]$	$\left(\mathrm{Tr}X^2\right)\left(\mathrm{Tr}XY\right)\left(\mathrm{Tr}Y^2\right)$
$p_{xy} = [1,1,1]$	$\left(\mathrm{Tr}XY\right)^3$
$p_{x^3y} = [1]$, $p_{y^2} = [1]$	$\left(\mathrm{Tr}X^3Y\right)\left(\mathrm{Tr}Y^2\right)$
$p_{x^2y^2} = [1]$, $p_{xy} = [1]$	$\left(\mathrm{Tr}X^2Y^2\right)\left(\mathrm{Tr}XY\right)$
$p_{xy^3} = [1]$, $p_{x^2} = [1]$	$\left(\mathrm{Tr}XY^3\right)\left(\mathrm{Tr}X^2\right)$
$p_{xy} = [2,1]$	$\mathrm{Tr}\left(XY\right)^2\left(\mathrm{Tr}XY\right)$
$p_{x^3y^3} = [1]$	$\mathrm{Tr}X^3Y^3$
$p_{x^2yxy^2} = [1]$	$\mathrm{Tr}X^2YXY^2$
$p_{xy} = [3]$	$\mathrm{Tr}(XY)^3$

Table 4.2: The 9 different $SO(N)$ multi-traces at $n_1 = n_2 = 3$ along with their labels. Any constituent partitions of $\widetilde{\mathcal{P}}$ that are not explicitly listed are set to zero.

Let $l_x(\tilde{w})$, $l_y(\tilde{w})$ and $l(\tilde{w})$ be the number of xs, number of ys and total length of $\tilde{w}$ respectively. Then

$$n_1 = \sum_{\tilde{w}} l_x(\tilde{w})|p_{\tilde{w}}| \qquad\qquad n_2 = \sum_{\tilde{w}} l_y(\tilde{w})|p_{\tilde{w}}| \tag{4.1.7}$$

We use the same notation $\widetilde{\mathcal{P}} \Vdash (n_1, n_2)$ as for the $U(N)$ traces in section 3.5.2. It will always be clear whether we are referring to a $SO(N)$ or $U(N)$ trace.

As an example of the new notation, we give the 9 different $\widetilde{\mathcal{P}} \Vdash (3,3)$ in table 4.2.

It will also be helpful to consider traces of symmetric matrices X and Y. We use a tilde to refer to anti-symmetric matrix objects, while a bar is used for those related to symmetric matrices. A single trace of symmetric matrices is labelled by a Lyndon word up to reversal, $\bar{w}$, and the number of periods. The $\bar{w}$ can be split into two types; either it is a Lyndon word that is invariant under reversal (type 1), or it is the first (lexicographically) of a pair of Lyndon words that transform into each other under reversal (type 2). This differs from the $SO(N)$ case only in that there is no distinction between odd and even length words of type 1. We define $p_{\bar{w}}$, $\bar{\mathcal{P}}$, $\bar{W}$, $l_x(\bar{w})$, $l_y(\bar{w})$, $l(\bar{w})$ and $\Vdash$ in an analogous way to the $U(N)$ and $SO(N)$ traces.

4.1.2 Symplectic gauge group

We now study the single trace constraints and multi-trace labelling in the $Sp(N)$ setting. Rather than anti-symmetric X and Y, we have matrices satisfying (4.0.1). In the half-BPS sector, this implies

$$\begin{aligned}\mathrm{Tr}X^n &= \mathrm{Tr}(X^T)^n = \mathrm{Tr}\left(\Omega X \Omega\right)^n = \mathrm{Tr}\left(\Omega^2 X\right)^n = \mathrm{Tr}\left(-X\right)^n \\ &= (-1)^n \mathrm{Tr}X^n \end{aligned} \tag{4.1.8}$$

where we have used $\Omega^2 = -1$. So, just as in the $SO(N)$ case, the odd order 1-matrix single traces vanish while the even ones remain unchanged, and $Sp(N)$ of degree n are labelled by a partition $p \vdash \frac{n}{2}$.

Applying the same logic to the quarter-BPS case, we again find the $Sp(N)$ relations between traces are the same as those for $SO(N)$. For a trace with k periods and aperiodic matrix word W, we have

$$\mathrm{Tr} W^k = (-1)^{k\, l(W)} \mathrm{Tr} \left(W^{(r)} \right)^k \tag{4.1.9}$$

As claimed, this is identical to (4.1.3). Therefore symplectic single traces are specified by an orthogonal Lyndon word and a number of periods k, while multi-traces are labelled by $\widetilde{\mathcal{P}} \Vdash (n_1, n_2)$.

4.2 Structure of the space of $SO(N)$ multi-traces of two matrices

In section 3.1, we investigated the structure of T, the space of $U(N)$ gauge-invariant functions of two matrices in the large N limit. In particular we looked at the level structure corresponding to the number of periods in a trace, and the factorisation arising from the decomposition of multi-traces into their single trace constituents. These two processes were reflected in the generating functions by the inverse Möbius transform $\mathcal{M}^{-1}$ (3.1.10) and the plethystic exponential (3.1.5). This structure was deduced from the equation (3.1.8) for a generic $U(N)$ single trace. Since the equation (4.1.4) has the exact same form, we must have the same structure for $\widetilde{T}$, the space of $SO(N)$ multi-traces at large N. The only difference is to replace the aperiodic traces of $U(N)$ with the minimally periodic traces of $SO(N)$. This is shown in figure 4.1.

As indicated by the name, minimally periodic traces have either one or two periods. This leads to an alternate structure for $\widetilde{T}$ which respects the absolute number of periods, rather than the number of repetitions of the minimally periodic units. This second structure is summarised in figure 4.2.

Both these structures give relations between the Hilbert series for the relevant vectors spaces, and therefore they can all be determined from the generating function for $\widetilde{T}$. This function is not given in the literature, although similar results are presented in [79], in the context of $SO(N)$ superconformal indices, and [80], for free matrix models. We derive it in two distinct ways, firstly in section 4.3 using the structure built up from minimally periodic words and known results about cycle polynomials of dihedral groups. In appendix E we give an alternative method starting from the counting formula (5.6.74) giving the size of the large N quarter-BPS sector in terms of Littlewood-Richardson coefficients. This formula is derived independently of the structures given in this section, and instead comes from studying Fourier bases for the permutation algebras relevant for operator construction.

In order to calculate the various Hilbert series for the $SO(N)$ vector spaces of interest, we will relate them to the $U(N)$ equivalents defined in section 3.1. We use the notation $T \underset{\mathbb{Z}_2}{\longrightarrow} \widetilde{T}$ to denote that replacing generic matrices with anti-symmetric matrices sends the space T of $U(N)$ multi-traces to the space $\widetilde{T}$ of $SO(N)$ multi-traces. Similar notation will be used for subspaces of T.

As in chapter 3, we will consider various different vector spaces in addition to $\widetilde{T}$. In general, those relating to $SO(N)$ traces will have a tilde on top, whereas those primarily to do with $U(N)$ objects will not. Some vector spaces we define will be relevant to both, so the divide is not a sharp one. Similarly to the notation used in section 3.1, we use superscripts in brackets to refer to a space with a specified number of periods, and subscripts to add extra information on the type of traces being considered.

4.2.1 Structure from minimally periodic traces

Equation (4.1.3) gave new relations between traces of anti-symmetric matrices compared to unrestricted matrices, and we subsequently split the Lyndon words relevant for $U(N)$ traces into three categories. We can encode this structure into the $U(N)$ vector space of aperiodic single traces $T^{(1)}_{ST}$, defined in (3.1.9), by splitting it into three distinct subspaces

$$T^{(1)}_{ST} = T^{(1)}_{ST;inv;even} \oplus T^{(1)}_{ST;inv;odd} \oplus T^{(1)}_{ST;var} \tag{4.2.1}$$

The first space is spanned by those traces of even length with $W = W^{(r)}$ up to cyclic rotations ('inv' stands for invariant); the second space is spanned by traces of odd length with $W = W^{(r)}$; the third space is spanned by traces of any length with $W \neq W^{(r)}$ ('var' stands for variant). From (4.1.3), $T^{(1)}_{ST;inv;even}$ is unchanged under the $\mathbb{Z}_2$ quotient. This is spanned by orthogonal Lyndon words of type 1A. The other two spaces, corresponding to types 1B and 2, are more complex.

In section 4.1, we demonstrated that for reversal-invariant W of odd length, the discriminating factor determining whether the trace vanishes or not is whether k is odd or even respectively. If k is even, $T^{(1)}_{ST;inv;odd}$ is unchanged by the quotient, while if k is odd, it vanishes. So we have

$$K \otimes T^{(1)}_{ST;inv;odd} \underset{\mathbb{Z}_2}{\longrightarrow} K_{even} \otimes T^{(1)}_{ST;inv;odd} \tag{4.2.2}$$

where K_{even} is the space spanned by the even integers. This is isomorphic to K as a vector space but not as a graded vector space, since replacing K_{even} with K would lose information about the weight of a given trace. However, we can recover K as a tensor factor by doubling the weight of the space $T_{ST;inv;odd}$ to make up for halving the weight of the K_{even} factor. We have

$$K_{even} \otimes T^{(1)}_{ST;inv;odd} = K \otimes \left(T^{(2)}_{ST;inv;odd}\right) \tag{4.2.3}$$

Effectively this says rather than consider X (or Y, X^2Y, X^4Y, ...) as the aperiodic word identifying the trace, instead consider X^2 (or Y^2, $(X^2Y)^2$, $(X^4Y)^2$, ...) as the minimally periodic word. These are exactly the type 1B orthogonal Lyndon words.

Finally consider $T^{(1)}_{ST;var}$. It is spanned by aperiodic matrix words (up to cyclic rotations) which change under reversal. So we can split the spanning set into orbits (of size 2) under reversal. Define $\widetilde{T}^{(1)}_{ST;var}$ to be the space spanned by these orbits, or equivalently by orthogonal Lyndon words of type 2. Then

$$T^{(1)}_{ST;var} = \widetilde{T}^{(1)}_{ST;var} \oplus \widetilde{T}^{(1)}_{ST;var} \underset{\mathbb{Z}_2}{\longrightarrow} \widetilde{T}^{(1)}_{ST;var} \tag{4.2.4}$$

In full, the $\mathbb{Z}_2$ quotient of T_{ST} is

$$T_{ST} = K \otimes T^{(1)}_{ST} \underset{\mathbb{Z}_2}{\longrightarrow} \widetilde{T}_{ST} = K \otimes \widetilde{T}^{(min)}_{ST} = K \otimes \left(T^{(1)}_{ST;inv;even} \oplus T^{(2)}_{ST;inv;odd} \oplus \widetilde{T}^{(1)}_{ST;var}\right) \tag{4.2.5}$$

where the 'min' superscript refers to the words being minimally periodic as opposed to aperiodic. Extrapolating to the full space of multi-traces

$$T = \mathrm{Sym}\left(K \otimes T^{(1)}_{ST}\right) \underset{\mathbb{Z}_2}{\longrightarrow} \widetilde{T} = \mathrm{Sym}\left(K \otimes \widetilde{T}^{(min)}_{ST}\right) \tag{4.2.6}$$

We see this has the same structure as (3.1.16), but with a base space $\widetilde{T}^{(min)}_{ST}$. This allows us to reproduce figure 3.1, but with the new base space, shown in figure 4.1.

Furthermore, we saw in section 3.4 that the structure (3.1.16) allowed T to carry a representation of $\mathfrak{so}(2,1)$. By the same argument, $\widetilde{T}$ will also carry such a representation.

4.2.2 Structure from absolute periodicity

Briefly return to the description of the $U(N)$ single trace space T_{ST}. Breaking down the decomposition (3.1.9) further, we have

$$T_{ST} = K \otimes T^{(1)}_{ST} = \left(1 \otimes T^{(1)}_{ST}\right) \oplus \left(2 \otimes T^{(1)}_{ST}\right) \oplus \left(3 \otimes T^{(1)}_{ST}\right) \oplus \ldots \tag{4.2.7}$$

$\widetilde{T}_{ST}$ also has this structure, but there is a difference in interpretation. The subspace $k \otimes T^{(1)}_{ST}$ of T_{ST} corresponds to the traces with k periods, whereas the subspace $k \otimes \widetilde{T}^{(min)}_{ST}$ of $\widetilde{T}_{ST}$ does not, instead it contains traces with k repetitions of the minimally periodic words. Since these words can contain two periods (if they are of type 1B), $k \otimes \widetilde{T}^{(min)}_{ST}$

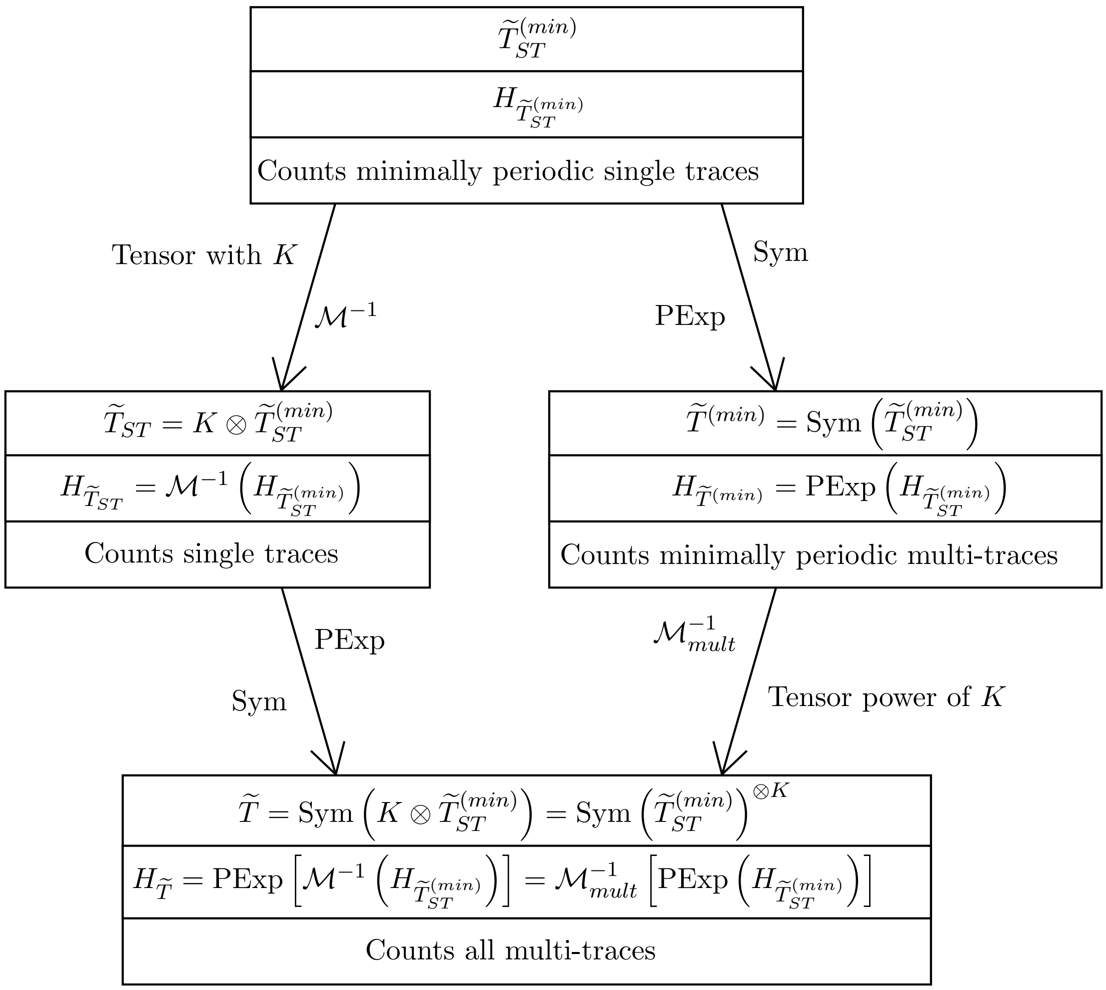


Figure 4.1: Diagram summarising the structure of $\widetilde{T}$, the space of $SO(N)$ multi-traces, and its relation to $\widetilde{T}_{ST}^{(min)}$, the space of $SO(N)$ minimally periodic single traces.

contains traces with k or $2k$ periods. We can instead decompose $\widetilde{T}_{ST}$ into subspaces corresponding to the number of periods rather than the number of repetitions.

$$\widetilde{T}_{ST} = (1 \otimes V_1) \oplus (2 \otimes V_2) \oplus (3 \otimes V_3) \oplus \dots \tag{4.2.8}$$

where $k \otimes V_k$ is the vector space of single traces with k periods.

From (4.1.3) it follows that for odd length, reversal invariant aperiodic matrix words, only the even periodicities survive the $\mathbb{Z}_2$ projection. For all other aperiodic matrix words, there is no distinction between even and odd periodicities. Therefore V_k will depend only on whether k is even or odd. From the discussions in section 4.1, we can write down the appropriate vector spaces. They are

$$\widetilde{T}^{(odd)}_{ST} = T^{(1)}_{ST;inv;even} \oplus \widetilde{T}^{(1)}_{ST;var} \tag{4.2.9}$$

$$\begin{aligned} \widetilde{T}^{(even)}_{ST} &= T^{(1)}_{ST;inv} \oplus \widetilde{T}^{(1)}_{ST;var} \\ &= T^{(1)}_{ST;inv;even} \oplus T^{(1)}_{ST;inv;odd} \oplus \widetilde{T}^{(1)}_{ST;var} \end{aligned} \tag{4.2.10}$$

Note that the odd and even superscripts refer to periodicities, while the odd and even subscripts refer to the length of the aperiodic trace/matrix word. Splitting $K = K_{odd} \oplus K_{even}$ in the obvious way, we have

$$\widetilde{T}_{ST} = \left(K_{odd} \otimes \widetilde{T}^{(odd)}_{ST}\right) \oplus \left(K_{even} \otimes \widetilde{T}^{(even)}_{ST}\right) \tag{4.2.11}$$

Now the combination of K_{odd} and K_{even} keeps track of the true periodicities of the traces.

Doing a analysis of the Hilbert series associated with these vector spaces, similar to that done in section 3.1, we arrive at the relations shown in figure 4.2. The transformations $\mathcal{S}$ and $\mathcal{S}_{mult}$ are defined by

$$\mathcal{S}\left[f,g\right](x,y) = \sum_{k \text{ odd}} f(x^k,y^k) + \sum_{k \text{ even}} g(x^k,y^k) \tag{4.2.12}$$

$$\mathcal{S}_{mult}\left[f,g\right](x,y) = \left(\prod_{k \text{ odd}} f(x^k,y^k)\right)\left(\prod_{k \text{ even}} g(x^k,y^k)\right) \tag{4.2.13}$$

Note that $\mathcal{S}$, while being similar to $\mathcal{M}^{-1}$, has a distinct disadvantage to it's analogue, namely it is not invertible. Given $\mathcal{S}[f,g]$, there are multiple f,g which would produce the same $\mathcal{S}$. This means we cannot instantly find the Hilbert series for $\widetilde{T}^{(odd)}_{ST}$ and $\widetilde{T}^{(even)}_{ST}$ just from the Hilbert series for $\widetilde{T}$. Instead we need to investigate the structures (4.2.9) and (4.2.10).

In order to do this, we introduce names for the coefficients of various Hilbert series. These are shown in table 4.3, along with a description of which set of traces these coef-

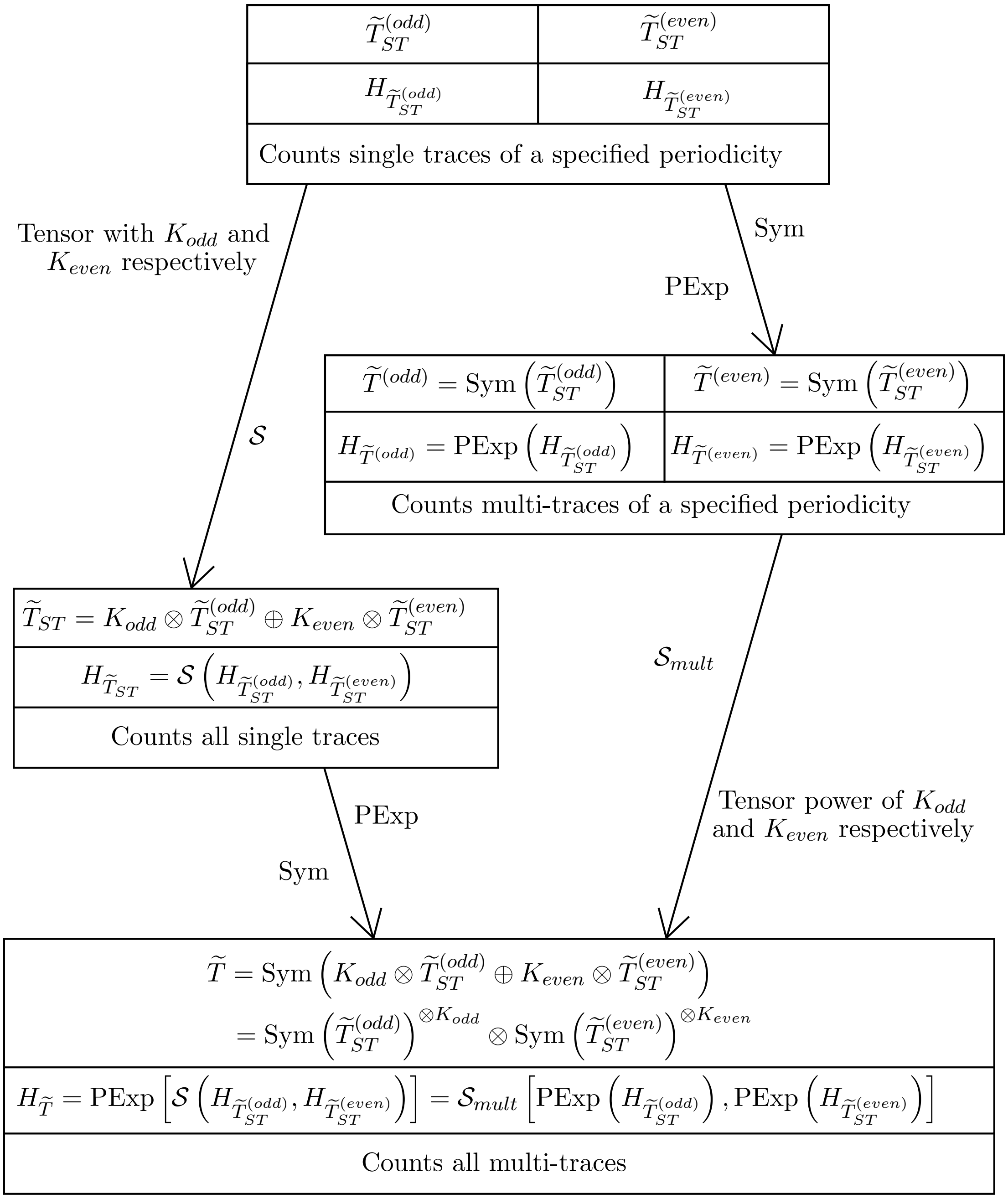


Figure 4.2: Diagram summarising the structure of $\widetilde{T}$, the space of $SO(N)$ multi-traces, and its relation to $\widetilde{T}_{ST}^{(odd)}$, the space of $SO(N)$ single traces with a specified odd number of periods, and $\widetilde{T}_{ST}^{(even)}$, the space of $SO(N)$ single traces with a specified even number of periods.

Hilbert series coefficients	Vector space	Counting interpretation
b_{n_1,n_2}	$\widetilde{T}_{ST}^{(min)}$	minimally periodic $SO(N)$ single traces
B_{n_1,n_2}	$\widetilde{T}_{ST} = K \otimes \widetilde{T}_{ST}^{(min)}$	all $SO(N)$ single traces
a_{n_1,n_2}^{inv}	$T_{ST;inv}^{(1)}$	aperiodic, reversal invariant $U(N)$ single traces
a_{n_1,n_2}^{var}	$\widetilde{T}_{ST;var}^{(1)}$	aperiodic pairs of $U(N)$ single traces that reverse into each other
A_{n_1,n_2}^{inv}	$T_{ST;inv} = K \otimes T_{ST;inv}^{(1)}$	all reversal invariant $U(N)$ single traces
A_{n_1,n_2}^{var}	$\widetilde{T}_{ST;var} = K \otimes \widetilde{T}_{ST;var}^{(1)}$	all pairs of $U(N)$ single traces that reverse into each other
$b_{n_1,n_2}^{(odd)}$	$\widetilde{T}_{ST}^{(odd)}$	$SO(N)$ single traces with a specified odd number of periods
$b_{n_1,n_2}^{(even)}$	$\widetilde{T}_{ST}^{(even)}$	$SO(N)$ single traces with a specified even number of periods

Table 4.3: Definition of various single trace counting sequences. Formally, they are defined as the coefficients of Hilbert series for certain vector spaces. We also give the counting interpretation.

ficients count. Tables of values are given in appendix C. Note that since the coefficients listed all count single traces, they all vanish when $n_1 = n_2 = 0$. Therefore in the later explicit expressions for these sequences, we implicitly set the $n_1 = n_2 = 0$ term to be 0.

Recall that a_{n_1,n_2} are the coefficients in the Hilbert series for $T_{ST}^{(1)}$, defined in (3.2.8). Then from definition (4.2.1), and recalling (4.2.4), we have

$$a_{n_1,n_2} = a_{n_1,n_2}^{inv} + 2a_{n_1,n_2}^{var} \tag{4.2.14}$$

The lower case sequences count aperiodic single traces, while the upper case ones count single traces of all periodicities. This leads to relations (3.1.12) and (3.1.13) between the as and As (although shown only for the undecorated versions, this is also true for both superscripts). Using these, we have

$$A_{n_1,n_2} = A_{n_1,n_2}^{inv} + 2A_{n_1,n_2}^{var} \tag{4.2.15}$$

From the definitions (4.2.9) and (4.2.10), we have the relations

$$b_{n_1,n_2}^{(even)} = a_{n_1,n_2}^{var} + a_{n_1,n_2}^{inv}$$

$$= \frac{1}{2}\left[a_{n_1,n_2} + a^{inv}_{n_1,n_2}\right] \tag{4.2.16}$$

$$b^{(odd)}_{n_1,n_2} = \begin{cases} a^{var}_{n_1,n_2} + a^{inv}_{n_1,n_2} & n \text{ even} \\ a^{var}_{n_1,n_2} & n \text{ odd} \end{cases}$$
$$= \frac{1}{2}\left[a_{n_1,n_2} + (-1)^n a^{inv}_{n_1,n_2}\right] \tag{4.2.17}$$

So to find the Hilbert series for $\widetilde{T}^{(odd)}_{ST}$ and $\widetilde{T}^{(even)}_{ST}$, we first need to find the generating function for the $a^{inv}_{n_1,n_2}$, or equivalently the $A^{inv}_{n_1,n_2}$, since they are related by (3.1.12) and (3.1.13).

In (4.2.1) we decomposed $T^{(1)}_{ST}$ into subspaces that were invariant or variant under reversal. We now do the same to T_{ST}.

$$T_{ST} = T_{ST;inv} \oplus T_{ST;var} = (T_{ST;inv;odd} \oplus T_{ST;inv;even}) \oplus T_{ST;var} \tag{4.2.18}$$

where the odd and even parts refer to the length of the entire single trace, not (as in (4.2.1)) the length of the aperiodic matrix word which, along with the number of periods, defined the single trace. We have

$$T_{ST;inv} = K \otimes T^{(1)}_{ST;inv} \qquad T_{ST;var} = K \otimes T^{(1)}_{ST;var} \tag{4.2.19}$$

but the split into odd and even parts does not respect the K tensor product. Instead, we have

$$T_{ST;inv;even} = \left(K \otimes T^{(1)}_{ST;inv;even}\right) \oplus \left(K_{even} \otimes T^{(1)}_{ST;inv;odd}\right) \tag{4.2.20}$$
$$T_{ST;inv;odd} = K_{odd} \otimes T^{(1)}_{ST;inv;odd} \tag{4.2.21}$$

By repeating the analysis from section 4.2.1, under the $\mathbb{Z}_2$ quotient $T_{ST;inv;odd}$ disappears, $T_{ST;inv;even}$ is unchanged, and $T_{ST;var}$ is 'halved' to $\widetilde{T}_{ST;var}$ as seen previously for the aperiodic version in (4.2.4). Therefore the quotient on the full set of single traces is

$$T_{ST} \underset{\mathbb{Z}_2}{\longrightarrow} \widetilde{T}_{ST} = T_{ST;inv;even} \oplus \widetilde{T}_{ST;var} \tag{4.2.22}$$

The coefficients of $H_{\widetilde{T}_{ST}}$ are B_{n_1,n_2}, so using (4.2.15) we find

$$B_{n_1,n_2} = \begin{cases} A^{var}_{n_1,n_2} + A^{inv}_{n_1,n_2} & n \text{ even} \\ A^{var}_{n_1,n_2} & n \text{ odd} \end{cases}$$
$$= \frac{1}{2}\left[A_{n_1,n_2} + (-1)^n A^{inv}_{n_1,n_2}\right] \tag{4.2.23}$$

We previously found a formula for A_{n_1,n_2}, (3.2.7), and in the next section we find an expression for B_{n_1,n_2}, (4.3.17). Comparing these with (4.2.23) allows us to find $A^{inv}_{n_1,n_2}$. Since $a^{inv}_{n_1,n_2}$ are related to $A^{inv}_{n_1,n_2}$ via the Möbius transform, we can then use (4.2.16) and (4.2.17) to find the Hilbert series for $\widetilde{T}^{(even)}_{ST}$ and $\widetilde{T}^{(odd)}_{ST}$.

4.2.3 Symplectic gauge group

In section 4.1.2 we observed that $Sp(N)$ and $SO(N)$ multi-traces were labelled by the same set, $\widetilde{\mathcal{P}} \Vdash (n_1, n_2)$, and consisted of the same matrix words. At large N, all distinct matrix words produce linearly independent gauge-invariant trace operators, and therefore the spaces of traces have the same structure.

From another point of view, the key relation on traces of $SO(N)$ matrix words is (4.1.3). From this equation, the entire structure of the space, exhibited in figures 4.1 and 4.2, was derived. The $Sp(N)$ relation, given in (4.1.9) is identical, hence we can follow the exact same process to get the same results.

Therefore, at large N, the structures of the $Sp(N)$ quarter-BPS sector is given in 4.1 in terms of orthogonal Lyndon words, the 'minimally periodic' matrix words of the theory, or 4.2 in terms of the true periodicity of aperiodic matrix words. All associated generating functions are given below in 4.3.

4.3 Generating functions at large N

Figures 4.1 and 4.2 give the structure of the large N space of multi-traces $\widetilde{T}$, the various sub-spaces that contribute, and the relations between the corresponding Hilbert series (generating functions).

As explained in section 3.2, any of the Hilbert series in figure 4.1 determines all others, and from the argument at the end of the previous section, we know finding the B_{n_1,n_2} (or equivalently $H_{\widetilde{T}_{ST}}$) will give all the series in figure 4.2. It is therefore sufficient to find just the series $H_{\widetilde{T}_{ST}}$.

This section gives a direct approach to finding $H_{\widetilde{T}_{ST}}$ that gives insight into its structure. In appendix E we present an independent argument that derives $H_{\widetilde{T}}$ from the combinatorics of the restricted Schur basis defined in section 5.6.3. This generating function is of interest to mathematicians [81], and we believe that our explicit evaluation of it is a new mathematical result.

To find B_{n_1,n_2}, consider the matrix words contained inside the traces. These words are constructed from n_1 Xs and n_2 Ys with $n_1 + n_2 = n$. In the $U(N)$ gauge theory, they are equivalent up to cyclic rotations only, but in the $SO(N)$ gauge theory, we also have to consider the effect of transposition. As seen in (4.1.3), this reverses the word and also multiplies by a factor of $(-1)^n$. The cyclic rotations and the reversal act as D_n on the matrix word.

To deal with the factor of $(-1)^n$, instead of considering D_n acting on the set of words, we consider D_n acting on the vector space spanned by the words. Let V_2 be the vector space spanned by two vectors, e_X and e_Y. A basis for $V_2^{\otimes n}$ is labelled by the set of words of length n constructed from X and Y. Define an operator Q on V_2 by

$$Qe_X = xe_X \qquad\qquad Qe_Y = ye_Y \tag{4.3.1}$$

Then the $x^{n_1}y^{n_2}$ eigenspace of $Q^{\otimes n}$ is spanned by words constructed from n_1 Xs and n_2 Ys.

Let σ be the generator of rotations in D_n and τ the reflection/transposition. These act on the basis vectors for $V_2^{\otimes n}$ as

$$\sigma\left[e_{i_1}\otimes e_{i_2}\otimes\ldots\otimes e_{i_n}\right] = e_{i_2}\otimes e_{i_3}\otimes\ldots\otimes e_{i_n}\otimes e_{i_1} \tag{4.3.2}$$

$$\tau\left[e_{i_1}\otimes e_{i_2}\otimes\ldots\otimes e_{i_n}\right] = (-1)^n e_{i_n}\otimes\ldots\otimes e_{i_2}\otimes e_{i_1} \tag{4.3.3}$$

where $i_j \in \{X,Y\}$. These commute with the action of $Q^{\otimes n}$. This action of the dihedral group on the space of matrix words was considered in [80].

To get the vector space spanned by traces of anti-symmetric matrices, we project down to those states which are invariant under the action of D_n using

$$P_{D_n} = \frac{1}{2n}\sum_{\rho\in D_n}\rho = \frac{1}{2n}\sum_{i=1}^{n}\sigma^i(1+\tau) \tag{4.3.4}$$

After projecting the $x^{n_1}y^{n_2}$ eigenspace of $Q^{\otimes n}$, the dimension of the reduced eigenspace is B_{n_1,n_2}. Therefore

$$\mathrm{Tr}\left(P_{D_n}Q^{\otimes n}\right) = \sum_{n_1+n_2=n} x^{n_1}y^{n_2}B_{n_1,n_2} \tag{4.3.5}$$

Forgetting the factor of $(-1)^n$ in (4.3.3), then σ,τ act as permutations on $V_2^{\otimes n}$. We can therefore use the techniques of section 2.1 to express (4.3.5) in terms of traces of Q. Since $\mathrm{Tr}\,Q^i = x^i+y^i$, this is

$$\sum_{n_1+n_2=n} x^{n_1}y^{n_2}B_{n_1,n_2} = \frac{1}{2n}\left[\sum_{i=1}^{n}(x+y)^{c_1(\sigma^i)}(x^2+y^2)^{c_2(\sigma^i)}(x^3+y^3)^{c_3(\sigma^i)}\ldots \right.$$
$$\left. +\,(-1)^n\sum_{i=1}^{n}(x+y)^{c_1(\sigma^i\tau)}(x^2+y^2)^{c_2(\sigma^i\tau)}(x^3+y^3)^{c_3(\sigma^i\tau)}\ldots\right] \tag{4.3.6}$$

where $c_j(\rho)$ is the number of cycles of length j in the permutation ρ.

We can evaluate (4.3.6) using the cycle index polynomial of D_n. For a subgroup H

of the symmetric group S_n, the cycle index polynomial of H is defined to be

$$\begin{aligned} Z^H(t_1, t_2, \ldots) &= \frac{1}{|H|} \sum_{\rho \in H} t_1^{c_1(\rho)} t_2^{c_2(\rho)} t_3^{c_3(\rho)} \ldots \\ &= \sum_{p \vdash n} Z_p^H \prod_i t_i^{p_i} \end{aligned} \tag{4.3.7}$$

where Z_p^H is the number of elements of H with cycle type p normalised by $|H|$.

The cycle polynomials of the dihedral group is well known

$$Z^{D_n}(t_1, t_2, \ldots) = \frac{1}{2n} \sum_{d|n} \phi(d) t_d^{\frac{n}{d}} + \begin{cases} \frac{1}{2} t_1 t_2^{\frac{M-1}{2}} & n \text{ odd} \\ \frac{1}{4} t_2^{\frac{M-2}{2}} \left(t_1^2 + t_2\right) & n \text{ even (and } \geq 2) \end{cases} \tag{4.3.8}$$

where $\phi(d)$ is the Euler totient function defined in (B.0.14). The first part of the polynomials is just half the cycle index polynomial of the cyclic group $\mathbb{Z}_n$. This corresponds to the rotations σ^i in D_n. The second part correspond to the reflections $\sigma^i \tau$. Therefore (4.3.6) is

$$\begin{aligned} \sum_{n_1+n_2=n} x^{n_1} y^{n_2} B_{n_1,n_2} &= \frac{1}{2n} \sum_{d|n} \phi(d) (x^d + y^d)^{\frac{M}{d}} \\ &+ \begin{cases} -\frac{1}{2}(x+y)(x^2+y^2)^{\frac{n-1}{2}} & n \text{ odd} \\ \frac{1}{4}(x^2+y^2)^{\frac{n-2}{2}} \left[(x+y)^2 + (x^2+y^2)\right] & n \text{ even} \end{cases} \end{aligned} \tag{4.3.9}$$

To find B_{n_1,n_2} explicitly we binomially expand the above. The first half of the expression was already expanded in (3.2.5), and is (half) the order n generating function for the A_{n_1,n_2}, so we focus on the second half. For n odd, we have

$$(x+y)\left(x^2+y^2\right)^{\frac{n-1}{2}} = \sum_{r=0}^{\frac{n-1}{2}} \binom{\frac{n-1}{2}}{r} \left(x^{2r+1} y^{n-2r-1} + x^{2r} y^{n-2r}\right) \tag{4.3.10}$$

and for n even

$$\left(x^2+xy+y^2\right)\left(x^2+y^2\right)^{\frac{n-2}{2}} = \sum_{r=0}^{\frac{n-2}{2}} \binom{\frac{n-2}{2}}{r} \left(x^{2r+2} y^{n-2r-2} + x^{2r+1} y^{n-2r-1} + x^{2r} y^{n-2r}\right) \tag{4.3.11}$$

Consider the coefficient of $x^{n_1} y^{n_2}$ if both n_1 and n_2 are even. Two of the three terms in (4.3.11) can contribute. Provided $n_1, n_2 \geq 2$, we get contributions from $r = \frac{n_1}{2}, \frac{n_1}{2} - 1$.

This leads to the coefficient

$$\binom{\frac{n}{2}-1}{\frac{n}{2}} + \binom{\frac{n}{2}-1}{\frac{n_1}{2}-1} = \binom{\frac{n_1}{2}+\frac{n_2}{2}}{\frac{n_1}{2}} \tag{4.3.12}$$

Checking the cases where $n_1 = 0$ or $n_2 = 0$, we get 1 as a coefficient, which agrees with (4.3.12).

Performing similar analyses for the other possible parity combinations leads to the coefficients

$$\binom{\frac{n_1}{2}+\frac{n_2-1}{2}}{\frac{n_1}{2}} \qquad n_1 \text{ even}, n_2 \text{ odd} \tag{4.3.13}$$

$$\binom{\frac{n_1-1}{2}+\frac{n_2}{2}}{\frac{n_1-1}{2}} \qquad n_1 \text{ odd}, n_2 \text{ even} \tag{4.3.14}$$

$$\binom{\frac{n-1}{2}+\frac{n_2-1}{2}}{\frac{n-1}{2}} \qquad n_1 \text{ odd}, n_2 \text{ odd} \tag{4.3.15}$$

All four cases can be summarised by the coefficient

$$\binom{\lfloor\frac{n_1}{2}\rfloor+\lfloor\frac{n_2}{2}\rfloor}{\lfloor\frac{n_1}{2}\rfloor} \tag{4.3.16}$$

Taking account of the signs and factors of a half in (4.3.9), we have

$$\begin{aligned} B_{n_1,n_2} &= \frac{1}{2}A_{n_1,n_2} + \frac{(-1)^n}{2}\binom{\lfloor\frac{n_1}{2}\rfloor+\lfloor\frac{n_2}{2}\rfloor}{\lfloor\frac{n_1}{2}\rfloor} \\ &= \frac{1}{2n}\sum_{d|n_1,n_2}\phi(d)\binom{\frac{n}{d}}{\frac{n}{d}} + \frac{(-1)^n}{2}\binom{\lfloor\frac{n_1}{2}\rfloor+\lfloor\frac{n_2}{2}\rfloor}{\lfloor\frac{n_1}{2}\rfloor} \end{aligned} \tag{4.3.17}$$

where we have used the expression for A_{n_1,n_2} from (3.2.7).

Comparing (4.2.23) with (4.3.17), we find

$$A^{inv}_{n_1,n_2} = \binom{\lfloor\frac{n_1}{2}\rfloor+\lfloor\frac{n_2}{2}\rfloor}{\lfloor\frac{n_1}{2}\rfloor} \tag{4.3.18}$$

which have generating function

$$H_{T_{ST;inv}}(x,y) = \frac{(1+x)(1+y)}{1-x^2-y^2} - 1 = \frac{x^2+xy+y^2+x+y}{1-x^2-y^2} \tag{4.3.19}$$

where the -1 comes from setting $A^{inv}_{0,0} = 0$.

To find the full generating function for B_{n_1,n_2}, we sum (4.3.9) from $n = 1$ to ∞. For the first half of this expression, this was already done in (3.2.3) (the generating function for A_{n_1,n_2}), and similarly for the second half in (4.3.19). Therefore

$$f_{SO(N)}(x,y) = H_{\widetilde{T}_{ST}}(x,y) = \frac{1}{2}\left[-\sum_{d=1}^{\infty}\frac{\phi(d)}{d}\log(1-x^d-y^d) + \frac{x^2+xy+y^2-x-y}{1-x^2-y^2}\right] \tag{4.3.20}$$

We can now take the plethystic exponential, given in (3.1.5), to get the multi-trace generating function

$$F_{SO(N)}(x,y) = H_{\widetilde{T}}(x,y) = \prod_{k=1}^{\infty}\frac{1}{\sqrt{1-x^k-y^k}}\exp\left[\frac{x^{2k}+x^ky^k+y^{2k}-x^k-y^k}{2k(1-x^{2k}-y^{2k})}\right] \tag{4.3.21}$$

where to evaluate the infinite products/sums we have used a change of variables similar to those in (3.2.4) and (3.2.5) as well as the identity (B.0.14).

Using the relations given in figure 4.1, we can find $H_{\widetilde{T}^{(min)}}$ and $H_{\widetilde{T}^{(min)}_{ST}}$. Taking the Möbius transform (see (3.1.14)) of (4.3.20) gives

$$H_{\widetilde{T}^{(min)}_{ST}}(x,y) = \frac{1}{2}\sum_{d=1}^{\infty}\mu(d)\left[-\frac{1}{d}\log(1-x^d-y^d) + \frac{x^{2d}+x^dy^d+y^{2d}-x^d-y^d}{1-x^{2d}-y^{2d}}\right] \tag{4.3.22}$$

where we have used the identity (B.0.16). Expanding to find the coefficients gives

$$b_{n_1,n_2} = \frac{1}{2}\sum_{d|n_1,n_2}\mu(d)\left[\frac{1}{n}\binom{\frac{n}{d}}{\frac{n_1}{d}} + (-1)^{\frac{n}{d}}\binom{\lfloor\frac{n_1}{2d}\rfloor+\lfloor\frac{n_2}{2d}\rfloor}{\lfloor\frac{n_1}{2d}\rfloor}\right] \tag{4.3.23}$$

Taking the plethystic exponential of (4.3.22), we get

$$H_{\widetilde{T}^{(min)}}(x,y) = \frac{1}{\sqrt{1-x-y}}\prod_{k=1}^{\infty}\exp\left[\frac{1}{2k}\frac{x^{2k}+x^ky^k+y^{2k}-x^k-y^k}{1-x^{2k}-y^{2k}}\sum_{d|k}d\mu(d)\right] \tag{4.3.24}$$

where we have used the identity (B.0.4). This is the *root function*, equivalent to (3.0.2) from the $U(N)$ theory.

The numbers appearing in the exponential here, $c_k = \sum_{d|k} d\mu(d)$, form an interesting mathematical sequence. It is sequence A023900 in the OEIS [82], and has the alternative expression

$$c_k = \prod_{\substack{p|k \\ p \text{ prime}}}(1-p) \tag{4.3.25}$$

This completes the description of the Hilbert series in 4.1.

To find the Hilbert series for the vector spaces shown in figure 4.2, we apply the

Möbius transform to (4.3.19) to find

$$H_{T^{(1)}_{ST;inv}}(x,y) = \mathcal{M}\left(H_{T_{ST;inv}}\right)(x,y) = \sum_{d=1}^{\infty} \mu(d) \frac{x^{2d} + x^d y^d + y^{2d} + x^d + y^d}{1 - x^{2d} - y^{2d}} \tag{4.3.26}$$

Then using the Hilbert series equivalents of the formulae (4.2.16) and (4.2.17), we have

$$H_{\widetilde{T}^{(odd)}_{ST}}(x,y) = \frac{1}{2}\left[H_{T^{(1)}_{ST}}(x,y) + H_{T^{(1)}_{ST;inv}}(-x,-y)\right] \tag{4.3.27}$$

$$= \frac{1}{2}\sum_{d=1}^{\infty} \mu(d)\left[-\frac{1}{d}\log(1-x^d-y^d) + \frac{x^{2d} + x^d y^d + y^{2d} + (-x)^d + (-y)^d}{1 - x^{2d} - y^{2d}}\right] \tag{4.3.28}$$

$$H_{\widetilde{T}^{(even)}_{ST}}(x,y) = \frac{1}{2}\left[H_{T^{(1)}_{ST}}(x,y) + H_{T^{(1)}_{ST;inv}}(x,y)\right] \tag{4.3.29}$$

$$= \frac{1}{2}\sum_{d=1}^{\infty} \mu(d)\left[-\frac{1}{d}\log(1-x^d-y^d) + \frac{x^{2d} + x^d y^d + y^{2d} + x^d + y^d}{1 - x^{2d} - y^{2d}}\right] \tag{4.3.30}$$

Note the similarities between these series, which count single traces with odd and even numbers of periods, and the minimally periodic version (4.3.22). The only difference between the three series is in the sign of the last two terms.

From these three Hilbert series we can derive explicit expressions for the coefficients $a^{inv}_{n_1,n_2}$, $b^{(odd)}_{n_1,n_2}$ and $b^{(even)}_{n_1,n_2}$. These are given in appendix C.

Taking the plethystic exponential of (4.3.28) and (4.3.30) gives

$$H_{\widetilde{T}^{(odd)}}(x,y) = \frac{1}{\sqrt{1-x-y}} \prod_{k=1}^{\infty} \exp\left[\sum_{d|k} \frac{d\mu(d)}{2k} \frac{x^{2k} + x^k y^k + y^{2k} + (-1)^d (x^k + y^k)}{1 - x^{2k} - y^{2k}}\right] \tag{4.3.31}$$

$$H_{\widetilde{T}^{(even)}}(x,y) = \frac{1}{\sqrt{1-x-y}} \prod_{k=1}^{\infty} \exp\left[\frac{x^{2k} + x^k y^k + y^{2k} + x^k + y^k}{2k(1 - x^{2k} - y^{2k})} \sum_{d|k} d\mu(d)\right] \tag{4.3.32}$$

where we have used the identity (B.0.4). This gives us all the Hilbert series featured in figure 4.2.

4.3.1 Half-BPS sector

The generating function for the half-BPS sector was given in [57]. By setting $y = 0$ in (4.3.21), we obtain the same results

$$F_{SO(N)}(x, 0) = \prod_{n=1}^{\infty} \frac{1}{(1 - x^n)^{B_{n,0}}} \tag{4.3.33}$$

Setting $n_2 = 0$ in (4.3.17) and using (B.0.14) we get

$$B_{n,0} = \frac{1}{2}\left(1 + (-1)^n\right) = \begin{cases} 1 & n \text{ even} \\ 0 & n \text{ odd} \end{cases} \tag{4.3.34}$$

Plugging this into (4.3.33)

$$F_{SO(N)}(x, 0) = \prod_{n=1}^{\infty} \frac{1}{1 - x^{2n}} \tag{4.3.35}$$

which matches the result found in [57].

Chapter 5

Algebraic structure of the free field $SO(N)$ and $Sp(N)$ gauge theories

In this chapter we study the permutation construction of $SO(N)$ and $Sp(N)$ operators at finite N. The major difference compared to the $U(N)$ theory is that we work with permutations in S_{2n} rather than S_n. This understanding was first used in [56] in order to construct a Young diagram basis for the half-BPS sector of the $SO(N)$ theory, shortly followed by the generalisation to $Sp(N)$ in [57]. This basis was used to calculate correlators to all orders in N, and is directly analogous to the original $U(N)$ Schur operators of [22]. These results included a Young diagram basis for half-BPS baryonic operators constructed from $\varepsilon_{i_1 \ldots i_N}$, though correlators for these operators were not found. An extension to the free field quarter-BPS sector was found in [59,60], giving a restricted Schur basis similar to the $U(N)$ version introduced in [44,45].

We expand upon this picture, introducing a gauge group independent way of looking at the permutation construction of operators, valid for $SO(N)$, $Sp(N)$ and $U(N)$. For each sector there is a permutation state space $\mathcal{A}$ that, when contracted with the appropriate invariant tensors, produces the gauge-invariant operators. $\mathcal{A}$ has two associated auxiliary algebras, $\mathcal{A}^L$ and $\mathcal{A}^R$,that act on the left and right respectively. Each of $\mathcal{A}$, $\mathcal{A}^L$ and $\mathcal{A}^R$ are defined in terms of an action $\alpha \to \sigma(\alpha)$ for $\sigma \in G$ where G is permutation group. This action splits $\mathbb{C}(S_{2n})$ into orbits that are called double cosets for $SO(N)/Sp(N)$ and conjugacy classes for $U(N)$. In the $SO(N)$ and $Sp(N)$ theories, the double cosets can be split into two categories depending on the sign associated with the action $\sigma(\alpha)$. The odd double cosets do not contribute to the algebras $\mathcal{A}, \mathcal{A}^L$ and $\mathcal{A}^R$, while the sums over the even double cosets form a basis. For the $SO(N)$ mesonic and $Sp(N)$ operators $\mathcal{A}$, this double coset basis constructs the multi-trace basis of operators.

By transforming to a Fourier basis of $\mathbb{C}(S_{2n})$ and studying the defining actions, we find Fourier bases of $\mathcal{A}, \mathcal{A}^L$ and $\mathcal{A}^R$ valid at finite N. These have nice multiplication properties within and between the algebras, allowing an identification of the Wedderburn-Artin decomposition of $\mathcal{A}^L$ and $\mathcal{A}^R$. Under the action of $\mathcal{A}^L \times \mathcal{A}^R$ on the state space, $\mathcal{A}$ forms representations which can be given explicitly.

In each theory there is a special theory dependent element $\Omega^{(G)}$ of $\mathbb{C}(S_{2n})$. This has nice eigenvalues on $\mathcal{A}^L$ and $\mathcal{A}$ and commutes with $\mathcal{A}^L$. However, its most important property is its role in correlators. For appropriately defined operators $\mathcal{O}_\alpha$ depending on a permutation $\alpha \in \mathcal{A}$, we find

$$\langle \mathcal{O}_\beta | \mathcal{O}_\alpha \rangle = \sum_{\sigma \in G} \delta\left(\Omega^{(G)} \sigma(\alpha) \beta^{-1}\right) \tag{5.0.1}$$

This is a theory independent correlator formula depending only on the permutation construction of operators.

For $SO(N)$ and $Sp(N)$ permutation state spaces, the Fourier bases construct orthogonal bases of operators labelled by Young diagrams. In the half-BPS sector, these are the Schur operators introduced in [56, 57]. For $SO(N)$ mesonic and symplectic operators in the quarter-BPS sectors, they are the restricted Schur basis first defined in [59, 60]. For quarter-BPS operators in the $SO(N)$ baryonic sector, they form a new baryonic restricted Schur basis.

In order to compare permutation algebras between the $SO(N)$ and $Sp(N)$ theories, we find that there is a slightly different relation from the link (4.0.3) between invariants. This is

$$R \to R^c \qquad\qquad \sigma \to (-1)^\sigma \sigma \tag{5.0.2}$$

We will generally start by deriving the results for the orthogonal group, and follow up with remarks on the generalisation to the symplectic case. The exception is when we deal with $SO(N)$ baryonic objects, which have no $Sp(N)$ equivalent.

In the final part of this chapter we define the $U(2)$ covariant basis for quarter-BPS mesonic, symplectic and baryonic operators. The $U(N)$ equivalent (3.6.20) has been used in [51] and chapter 7 of this thesis to construct quarter-BPS weak coupling operators.

A minority of the material in this chapter was originally presented in [1], while the majority is unpublished.

5.1 Technical differences from $U(N)$

In section 2.1 we considered V, the carrier space for the N-dimensional fundamental representation of $U(N)$. The matrices X^i_j and Y^i_j in the adjoint of $\mathfrak{u}(N)$ were given one up and one down index. This reflected the fact that V is a complex representation of $U(N)$ and the conjugate space V^* forms a non-isomorphic representation (the anti-fundamental). The upper index of X^i_j lives in V, while the lower index lives in V^*.

Compare this with $SO(N)$. V also carries the fundamental representation of $SO(N)$. This is a real representation, and therefore the conjugate representation is isomorphic to V. Therefore the two indices of matrices in the adjoint of $\mathfrak{so}(n)$ lie in the same space. Hence we use the index structure X^{ij} and Y^{ij}. Indeed, as X and Y are anti-symmetric, their indices must lie in the same space.

Consequently, for $SO(N)$ tensors, there is no difference between upper and lower indices. For our purposes, we will (in general) use downstairs indices for $SO(N)$ invariant tensors that we use to contract the indices of X and Y, while using upstairs indices for the operators. That being said, when it is convenient to break these conventions we will do so. We still use a combination of downstairs and upstairs indices for objects (such as permutations) acting on the tensor space $V^{\otimes 2n}$.

Similar statements hold true for $Sp(N)$, and we use the same index structure for these fields. The condition (4.0.1) is equivalent to saying that $(\Omega X)^{ij}$ is symmetric, which is an easier condition to work with, and we therefore use ΩX and ΩY in the construction of operators rather than the bare matrices.

5.1.1 Invariant tensors

The invariant tensors for $SO(N)$ are δ_{ij} and $\varepsilon_{i_1 i_2 \dots i_N}$, while for $Sp(N)$ we only have Ω_{ij}, where Ω is as defined in (4.0.2). $\varepsilon_{i_1 i_2 \dots i_N}$ is invariant in the symplectic theory, but is related to the $\frac{N}{2}$-fold tensor product (recall N is even in symplectic theories) of Ω_{ij}

$$\varepsilon_{i_1 i_2 \dots i_N} = \frac{1}{2^{\frac{N}{2}} \left(\frac{N}{2}\right)!} \sum_{\sigma \in S_N} \Omega_{i_{\sigma(1)} i_{\sigma(2)}} \Omega_{i_{\sigma(3)} i_{\sigma(4)}} \dots \Omega_{i_{\sigma(N-1)} i_{\sigma(N)}} \tag{5.1.1}$$

Therefore the baryonic operators in the symplectic theory are linearly dependent on the mesonic ones, and we will not consider them.

5.1.2 The wreath product $S_n[S_2]$

Gauge invariant operators will be constructed by contracting the indices of $X^{\otimes n_1} Y^{\otimes n_2}$, or $(\Omega X)^{\otimes n_1} (\Omega Y)^{\otimes n_2}$, which lie in the space $V^{\otimes 2n}$. It is therefore appropriate to consider permutations in S_{2n} rather than S_n. There is a particularly important subgroup of S_{2n} that will play a crucial role in this construction, the wreath product group $S_n[S_2]$.

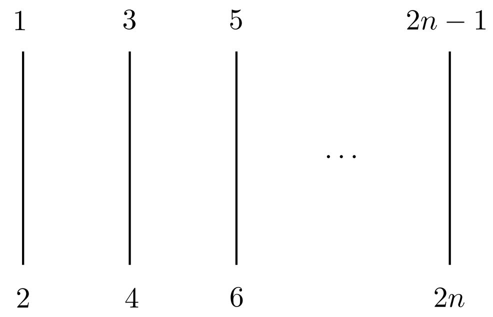


Figure 5.1: The set on which $S_n[S_2]$ acts. A group element can permute the n pairs, while switching or not switching each individual pair

Intuitively, $S_n[S_2]$ can be thought of as the permutations of n pairs of objects. Each pair can be individually switched, and the n pairs can be permuted among themselves, so we have $|S_n[S_2]| = 2^n n!$. By labelling the $2n$ objects as $\{1,2\},\{3,4\},\dots,\{2n-1,2n\}$, we see that $S_n[S_2]$ naturally lies within S_{2n}. It is simple to check that it is the centraliser of the permutation $(1,2)(3,4)\dots(2n-1,2n)$. Figure 5.1 shows the set on which $S_n[S_2]$ acts.

More formally, $S_n[S_2]$ is defined as the wreath product of S_n with S_2, or equivalently as the semi-direct product of S_n with $(S_2)^n$, where the S_n acts on $(S_2)^n$ by permutation of the factors.

Since $S_n[S_2]$ is a subgroup of S_{2n} it acts on $V^{\otimes 2n}$. The properties of this action are easiest to see if we label the indices slightly differently. Consider $A \in V^{\otimes 2n}$ with the indices labelled as follows

$$A^I = A^{i_{1,1}i_{1,2}i_{2,1}i_{2,2}\dots i_{n,1}i_{n,2}} \tag{5.1.2}$$

Then the S_n part of $S_n[S_2]$ act on the first index (j in $i_{j,k}$) while the n copies of S_2 acts on the second index (k). Therefore if M is a symmetric (anti-symmetric) matrix, $\left(M^{\otimes n}\right)^I$ will be invariant (anti-invariant) under the action of $S_n[S_2]$.

We can define projection operators onto irreducible representations of $S_n[S_2]$ just as we did with S_n representations in (2.3.13). There are two one-dimensional representations of $S_n[S_2]$ that are important for our analysis. The trivial (symmetric) representation takes σ to 1, and the anti-symmetric (sign) representation takes σ to $(-1)^\sigma$, defined by considering $\sigma \in S_n[S_2] \leq S_{2n}$. We denote these two representations by $[S]$ and $[A]$ respectively. The projectors of $[S]$ and $[A]$ are given by

$$P_{[S]} = \frac{1}{2^n n!}\sum_{\sigma\in S_n[S_2]}\sigma \qquad\qquad P_{[A]} = \frac{1}{2^n n!}\sum_{\sigma\in S_n[S_2]}(-1)^\sigma\sigma \tag{5.1.3}$$

For a description of a generic irreducible representation of $S_n[S_2]$ see appendix F

5.1.3 Invariant vectors

There are three distinct types of invariant vectors that are important in the construction of $SO(N)$ and $Sp(N)$ operators. The first two are relevant for $SO(N)$ mesonic and $Sp(N)$ operators, while the third is used in $SO(N)$ baryonic operators.

The vectors $|R,[S]\rangle$ and $|R,[A]\rangle$

Firstly, consider vectors in a representation $R \vdash 2n$ of S_{2n} that are invariant or anti-invariant under $S_n[S_2]$. Each invariant vector corresponds to a copy of $[S]$ when R is decomposed into irreducible representations of $S_n[S_2]$, and similarly the anti-invariant vectors correspond to the copies of $[A]$. It is proved in [64, Chapter VII.2] that $[S]$ appears in the decomposition of R if and only if R has an even number of boxes in each row, and then it appears with multiplicity 1. By conjugation of Young diagrams, $[A]$ appears in the decomposition if and only if R has an even number of boxes in each column, and then it appears with multiplicity 1. We denote the unit vectors in R that lie in the $[S]$ and $[A]$ representations (when they exist) as

$$|R,[S]\rangle \qquad\qquad |R,[A]\rangle \tag{5.1.4}$$

More detail is provided on how $|R,[S]\rangle$ and $|R,[A]\rangle$ embed into R in appendix A.

In a representation, $P_{[S]}$ and $P_{[A]}$ can be given in terms of these invariant vectors

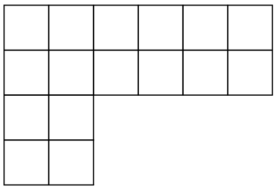
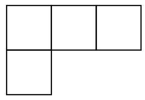
$$D^R\left(P_{[S]}\right) = \begin{cases} |R,[S]\rangle\langle R,[S]| & \text{if } R \text{ has even row lengths} \\ 0 & \text{otherwise} \end{cases} \tag{5.1.5}$$

$$D^R\left(P_{[A]}\right) = \begin{cases} |R,[A]\rangle\langle R,[A]| & \text{if } R \text{ has even column lengths} \\ 0 & \text{otherwise} \end{cases} \tag{5.1.6}$$

Consider an $R \vdash 2n$ with both even row lengths and even column lengths, so that it admits both an invariant and anti-invariant vector of $S_n[S_2]$. Such an R has n even and is made up of 2×2 blocks $\boxplus$. Define $\frac{R}{4} \vdash \frac{n}{2}$ to be the 'quartered' version of R where each 2×2 block is replaced by a single box. In terms of components

$$R = [R_1, R_1, R_2, R_2, \ldots, R_k, R_k] \quad \rightarrow \quad \frac{R}{4} = \left[\frac{R_1}{2}, \frac{R_2}{2}, \ldots, \frac{R_k}{2}\right] \tag{5.1.7}$$

For example

$$R = [6,6,2,2] =$$

$$\longleftrightarrow \quad \frac{R}{4} = [3,1] =$$

(5.1.8)

Define $S_n^{(odd)}$ to be the embedding of S_n into S_{2n} that acts on only the odd numbers. For $\sigma \in S_n$, we denote the equivalent permutation in $S_n^{(odd)}$ by $\sigma^{(odd)}$. In the paper [83], the author derives the matrix element of $\sigma^{(odd)}$ in a representation R with respect to the $S_n[S_2]$ invariant vector on the left and the $S_n[S_2]$ anti-invariant vector on the right.

Suppose the cycle type $p \vdash n$ of $\sigma \in S_n$ has an odd component. Then

$$\langle R, [S]| \, D^R \left(\sigma^{(odd)}\right) |R, [A]\rangle = 0 \tag{5.1.9}$$

If there is no odd component in p, then σ has cycle type $2p$, for $p \vdash \frac{n}{2}$. In this case

$$\langle R, [S]| \, D^R \left(\sigma^{(odd)}\right) |R, [A]\rangle = \frac{2^{l(p)}}{2^n n!} \sqrt{\frac{(2n)!}{d_R}} \chi_{\frac{R}{4}}(p) \tag{5.1.10}$$

Consider the behaviour of the vectors (5.1.4) under conjugation of R. If R has even row lengths and admits a vector $|R, [S]\rangle$, then the conjugate representation R^c has even column lengths and admits a vector $|R^c, [A]\rangle$.

Let V_R be the representation space for R. Then since R^c =sgn$\otimes R$, we have an orthogonal map ρ from V_R to V_{R^c} satisfying

$$D^{R^c}(\sigma) = (-1)^\sigma \rho D^R(\sigma) \rho^{-1} \tag{5.1.11}$$

Then for $\sigma \in S_n[S_2]$

$$D^{R^c}(\sigma)\rho|R, [S]\rangle = (-1)^\sigma \rho D^R(\sigma)|R, [S]\rangle = (-1)^\sigma \rho \, |R, [S]\rangle \tag{5.1.12}$$

and hence

$$|R^c, [A]\rangle = \rho \, |R, [S]\rangle \tag{5.1.13}$$

Similarly, if R had even column lengths, we have

$$|R^c, [S]\rangle = \rho \, |R, [A]\rangle \tag{5.1.14}$$

So the invariant and anti-invariant vectors switch places under conjugation of Young diagrams.

The vectors $|R_1, R_2, [S], \lambda\rangle$ and $|R_1, R_2, [A], \lambda\rangle$

We will also be interested in vectors invariant or anti-invariant under $S_{n_1}[S_2] \times S_{n_2}[S_2]$, for $n_1 + n_2 = n$. To understand these, consider the Littlewood-Richardson decomposition of S_{2n} representations into $S_{2n_1} \times S_{2n_2}$ representations

$$R = \bigotimes_{\substack{R_1 \vdash n_1 \\ R_2 \vdash n_2}} R_1 \otimes R_2 \otimes V^{mult}_{R;R_1,R_2} \tag{5.1.15}$$

where $V^{mult}_{R;R_1,R_2}$ is a Littlewood-Richardson multiplicity space of dimension $g_{R;R_1,R_2}$. For more detail on this decomposition, how $g_{R;R_1,R_2}$ is calculated and a way of choosing a basis for $V^{mult}_{R:R_1,R_2}$ see appendix D.

We can then look at the decompositions of the R_i representations of S_{2n_i} into $S_{n_i}[S_2]$ representations in exactly the same manner as discussed above for R. If both R_1 and R_2 have even row lengths, then R contains a corresponding $g_{R;R_1,R_2}$ copies of the $[S] \otimes [S]$ representation of $S_{n_1}[S_2] \times S_{n_2}[S_2]$. We denote the unit vectors spanning these by

$$|R_1, R_2, [S], \lambda\rangle \tag{5.1.16}$$

where λ indexes a basis element of $V^{mult}_{R;R_1,R_2}$.

Similarly, if both R_1 and R_2 have even column lengths, then R contains a corresponding $g_{R;R_1,R_2}$ copies of the $[A] \otimes [A]$ representation of $S_{n_1}[S_2] \times S_{n_2}[S_2]$, whose unit vectors are denoted by

$$|R_1, R_2, [A], \lambda\rangle \tag{5.1.17}$$

The projectors to the $[S] \otimes [S]$ and $[A] \otimes [A]$ representations of $S_{n_1}[S_2] \times S_{n_2}[S_2]$ are defined by

$$P_{[S]\otimes[S]} = \frac{1}{2^n n!} \sum_{\sigma \in S_{n_1}[S_2] \times S_{n_2}[S_2]} \sigma \qquad P_{[A]\otimes[A]} = \frac{1}{2^n n!} \sum_{\sigma \in S_{n_1}[S_2] \times S_{n_2}[S_2]} (-1)^\sigma \sigma \tag{5.1.18}$$

In a representation $R \vdash 2n$, these projectors have representatives in terms of the invariant vectors

$$D^R\left(P_{[S]\otimes[S]}\right) = \sum_{\substack{R_1 \vdash 2n_1 \\ R_2 \vdash 2n_2}} \sum_{\lambda=1}^{g_{R;R_1,R_2}} |R_1, R_2, [S], \lambda\rangle \langle R_1, R_2, [S], \lambda| \tag{5.1.19}$$

$$D^R\left(P_{[A]\otimes[A]}\right) = \sum_{\substack{R_1 \vdash 2n_1 \\ R_2 \vdash 2n_2}} \sum_{\lambda=1}^{g_{R;R_1,R_2}} |R_1, R_2, [A], \lambda\rangle \langle R_1, R_2, [A], \lambda| \tag{5.1.20}$$

where the sum runs over R_1, R_2 with even row lengths for $P_{[S]\otimes[S]}$ and even column lengths for $P_{[A]\otimes[A]}$.

Using the same notation as (5.1.13), the two vectors (5.1.16) and (5.1.17) transform into each other under conjugation of Young diagrams. If R contains a copy of $R_1 \otimes R_2$ where R_1, R_2 have even row lengths, then

$$|R_1^c, R_2^c, [A], \lambda\rangle = \rho\, |R_1, R_2, [S], \lambda\rangle \tag{5.1.21}$$

where the vector on the left lives in the representation R^c of S_{2n}. Similarly, if R contains a copy of $R_1 \otimes R_2$ where R_1, R_2 have even column lengths, then

$$|R_1^c, R_2^c, [S], \lambda\rangle = \rho\, |R_1, R_2, [A], \lambda\rangle \tag{5.1.22}$$

The vector $\left|[1^N]\right\rangle \otimes |\bar{R}, [S]\rangle$

The final vector of interest is (anti-)invariant under the subgroup $S_N \times S_q[S_2]$, where $2n = N + 2q$ and the S_N factor acts on $\{1, 2, \ldots, N\}$ while the $S_q[S_2]$ factor acts of the pairs $\{N+2i-1, N+2i\}$ for $1 \leq i \leq q$. This subgroup is relevant for baryonic operators in the $SO(N)$ theory. We are interested in vectors anti-invariant under the S_N factor and invariant under the $S_q[S_2]$ factor. Such a vector lives in a representation $[1^N] \otimes \bar{R}$ of $S_N \times S_{2q}$ where $\bar{R}$ has even row lengths. In appendix D.2.1, we characterise the R that admit such a representation, and prove that there is always a unique $\bar{R}$ associated to a given R, with Littlewood-Richardson coefficient $g_{R;[1^N],\bar{R}} = 1$. Therefore there is a unique unit vector (up to a minus sign) with the sought-after invariance. We write this as a tensor product in $[1^N] \otimes \bar{R}$

$$\left|[1^N]\right\rangle \otimes |\bar{R}, [S]\rangle \tag{5.1.23}$$

where $\left|[1^N]\right\rangle$ is the vector spanning the one-dimensional representation $[1^N]$ of S_N and $|\bar{R}, [S]\rangle$ is the unit $S_q[S_2]$-invariant vector in $\bar{R}$ as defined in (5.1.4).

If R is restricted to have $l(R) = N$ (note it must have $l(R) \geq N$ in order to contain a copy of $[1^N]$), then the relation between R and $\bar{R}$ simplifies to $R = [1^N] + \bar{R}$.

Define the projector onto the $[1^N] \otimes [S]$ representation of $S_N \times S_q[S_2]$ by

$$P_{[1^N]\otimes[S]} = \frac{1}{N! 2^q q!} \sum_{\sigma \in S_N \times S_q[S_2]} (-1)^{\sigma_1} \sigma \tag{5.1.24}$$

where for $\sigma \in S_N \times S_q[S_2]$, we define σ_1 to be the S_N component.

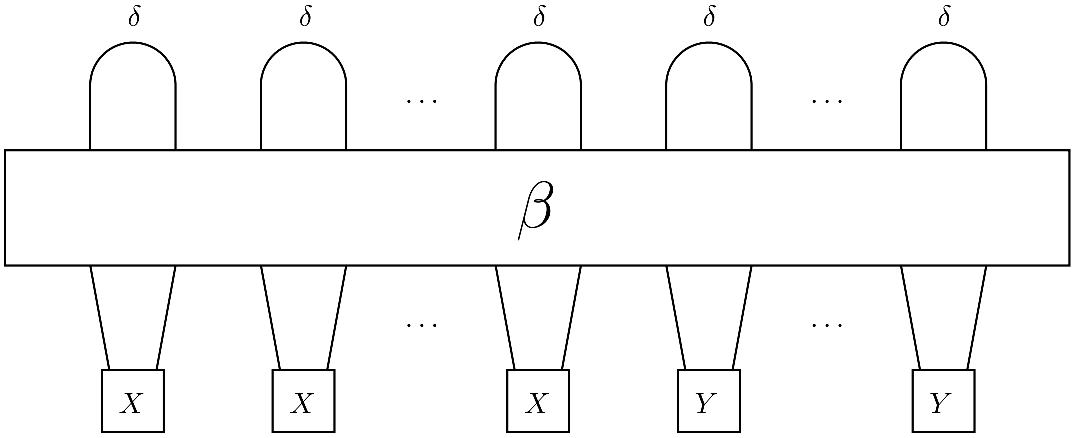


Figure 5.2: A diagrammatic representation of the index contraction in a mesonic operator, where each line represents an index. There are n_1 Xs and n_2 Ys, and $\beta \in \mathbb{C}(S_{2n})$.

In a representation, this projector is represented in terms of the invariant vector

$$D^R\left(P_{[1^N]\otimes[S]}\right) = \begin{cases} \left(\left|[1^N]\right\rangle \otimes |\bar{R},[S]\rangle\right)\left(\left\langle[1^N]\right| \otimes \langle\bar{R},[S]|\right) & \text{if } R \text{ of appropriate form} \\ 0 & \text{otherwise} \end{cases} \tag{5.1.25}$$

5.2 Permutation state spaces and auxiliary algebras

Consider permutations $\sigma \in S_{2n}$ acting on the $2n$ indices of the $SO(N)$ matrix tensor product $X^{\otimes n_1} Y^{\otimes n_2}$. As discussed beneath (5.1.2), this will be anti-invariant under $S_{n_1}[S_2] \times S_{n_2}[S_2]$ permutations. For the symplectic case, $(\Omega X)^{\otimes n_1} (\Omega Y)^{\otimes n_2}$ will be invariant under $S_{n_1}[S_2] \times S_{n_2}[S_2]$.

Now consider the possible contractions we can use. From the discussion of invariant tensors in section 5.1.1, there are two possibilities for the $SO(N)$ theory

$$C_I^{(\delta)} = \delta_{i_1 i_2}\delta_{i_3 i_4}\dots\delta_{i_{2n-1} i_{2n}} \tag{5.2.1}$$

$$C_I^{(\varepsilon)} = \varepsilon_{i_1 i_2 \dots i_N}\delta_{i_{N+1} i_{N+2}}\delta_{i_{N+3} i_{N+4}}\dots\delta_{i_{2n-1} i_{2n}} \tag{5.2.2}$$

where the baryonic contractor $C^{(\varepsilon)}$ is only available when N is even.

Looking at the action of permutations on these contractors, $C^{(\delta)}$ is invariant under $S_n[S_2]$ permutations. Defining $q = n - \frac{N}{2}$ the invariances of $C^{(\varepsilon)}$ are controlled by the group $S_N \times S_q[S_2]$. It is anti-invariant under the S_N factor of and invariant under the $S_q[S_2]$ factor.

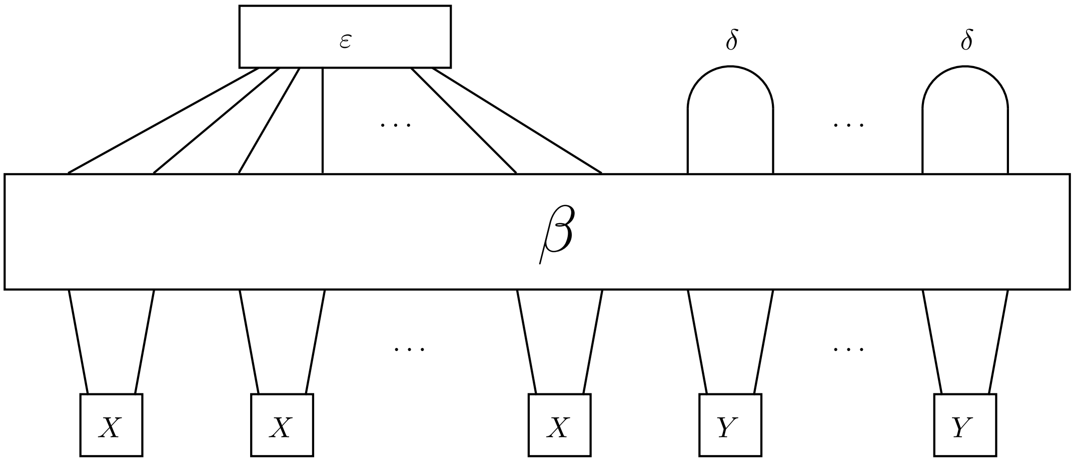


Figure 5.3: A diagrammatic representation of the index contraction in a baryonic operator. The ε vertex has N legs and there are n_1 Xs, n_2 Ys and q $\left(= n - \frac{N}{2}\right)$ δs. For convenience, this diagram shows $N = 2n_1$, but in general this does not have to be the case.

For the symplectic gauge group we have

$$C_I^{(\Omega)} = \Omega_{i_1 i_2} \Omega_{i_3 i_4} \ldots \Omega_{i_{2n-1} i_{2n}} \tag{5.2.3}$$

which is anti-invariant under $S_n[S_2]$.

For $\alpha \in \mathbb{C}(S_{2n})$, we define orthogonal mesonic and baryonic operators by

$$\mathcal{O}_\alpha^\delta = C_I^{(\delta)} \alpha_J^I \left(X^{\otimes n_1} Y^{\otimes n_2}\right)^J \tag{5.2.4}$$

$$\mathcal{O}_\alpha^\varepsilon = C_I^{(\varepsilon)} \alpha_J^I \left(X^{\otimes n_1} Y^{\otimes n_2}\right)^J \tag{5.2.5}$$

and symplectic operators by

$$\mathcal{O}_\alpha^\Omega = C_I^{(\Omega)} \alpha_J^I \left[(\Omega X)^{\otimes n_1} (\Omega Y)^{\otimes n_2}\right]^J \tag{5.2.6}$$

We refer to these as the $SO(N)$ mesonic, baryonic and $Sp(N)$ contraction patterns respectively, in contrast to the $U(N)$ contraction pattern (2.1.3). Figures 5.2, 5.3 and 5.4 show these three types of contractions diagrammatically.

Each of (5.2.4), (5.2.5) and (5.2.6) is invariant under left and right multiplication by different subgroups of S_{2n}. These lead to different sub-algebras of $\mathbb{C}(S_{2n})$ that control the construction of operators in each of the three cases. We first consider the $SO(N)$ mesonic and symplectic case, since they involve the action of the same groups, before moving on to the baryonic case.

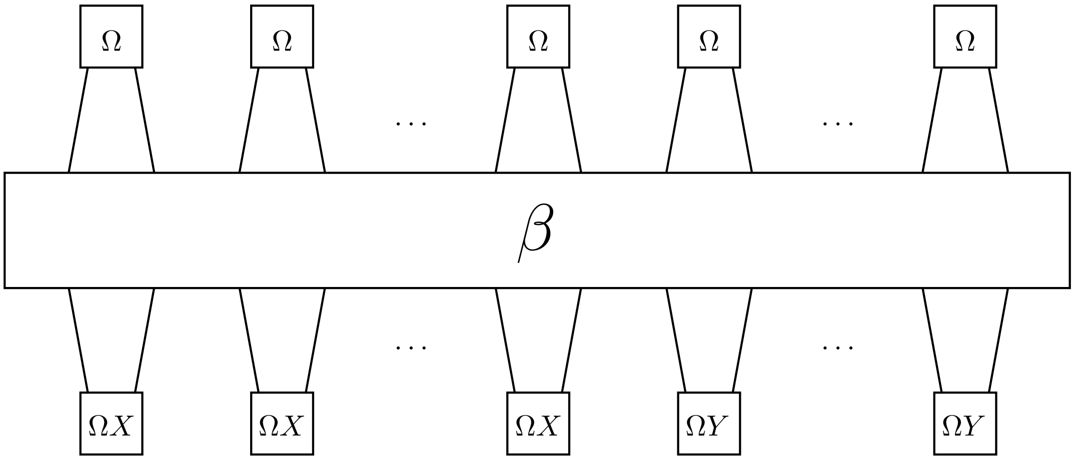


Figure 5.4: Diagrammatic representation of the contraction pattern for symplectic mesonic operators.

5.2.1 Mesonic and symplectic sectors

For $SO(N)$ mesonic operators, the invariances of the contraction (5.2.4) lead to $\mathcal{A}^{\delta}_{n_1,n_2}$, defined by invariance under

$$\mathcal{A}^{\delta}_{n_1,n_2}: \qquad \alpha \mapsto (-1)^{\tau}\, \sigma\alpha\tau^{-1} \qquad \sigma \in S_n[S_2]\ ,\ \tau \in S_{n_1}[S_2] \times S_{n_2}[S_2] \tag{5.2.7}$$

the equivalent for symplectic operators is $\mathcal{A}^{\Omega}_{n_1,n_2}$, defined by invariance under

$$\mathcal{A}^{\Omega}_{n_1,n_2}: \qquad \alpha \mapsto (-1)^{\sigma}\, \sigma\alpha\tau^{-1} \qquad \sigma \in S_n[S_2]\ ,\ \tau \in S_{n_1}[S_2] \times S_{n_2}[S_2] \tag{5.2.8}$$

which come from the contraction (5.2.6). Take two elements $\alpha, \beta \in \mathcal{A}^{\delta}_{n_1,n_2}$. Then using (5.2.7) first on β and then on α, we have

$$\alpha\beta = \alpha(1,2)\beta = -\alpha\beta = 0 \tag{5.2.9}$$

So the multiplication in $\mathcal{A}^{\delta}_{n_1,n_2}$ is trivial. We therefore do not call $\mathcal{A}^{\delta}_{n_1,n_2}$ a sub-algebra of $\mathbb{C}(S_{2n})$, instead it is a subspace, which we call a permutation state space. $\mathcal{A}^{\Omega}_{n_1,n_2}$ also has this property.

Despite (5.2.9), $\mathcal{A}^{\delta}_{n_1,n_2}$ and $\mathcal{A}^{\Omega}_{n_1,n_2}$ do have interesting multiplication properties when multiplied by other sub-algebras of $\mathbb{C}(S_{2n})$ that we call auxiliary algebras. Consider $\mathcal{A}^{+}_{n}$ and $\mathcal{A}^{-}_{n}$, defined respectively by invariance under

$$\mathcal{A}^{+}_{n}: \qquad \alpha \mapsto \sigma\alpha\tau^{-1} \qquad \sigma, \tau \in S_n[S_2] \tag{5.2.10}$$

$$\mathcal{A}^{-}_{n}: \qquad \alpha \mapsto (-1)^{\sigma}(-1)^{\tau}\sigma\alpha\tau^{-1} \qquad \sigma, \tau \in S_n[S_2] \tag{5.2.11}$$

These two definitions are similar to the half-BPS versions of (5.2.7) and (5.2.8) but have the same sign behaviour on the left and right, leading to a non-trivial multiplication. In section 5.4.1 we give a Fourier basis for $\mathcal{A}_n^{\pm}$, prove that they are Abelian algebras and give a description of the different one-dimensional irreducible representations.

The algebras $\mathcal{A}_n^{\pm}$ act by left multiplication on $\mathcal{A}_{n_1,n_2}^{\delta/\Omega}$ respectively.

$$\mathcal{A}_n^{+}\mathcal{A}_{n_1,n_2}^{\delta} \subseteq \mathcal{A}_{n_1,n_2}^{\delta} \qquad \mathcal{A}_n^{-}\mathcal{A}_{n_1,n_2}^{\Omega} \subseteq \mathcal{A}_{n_1,n_2}^{\Omega} \tag{5.2.12}$$

We could also consider other products between the auxiliary algebras and the permutation state space. By similar reasoning to (5.2.9), we have

$$\mathcal{A}_n^{-}\mathcal{A}_{n_1,n_2}^{\delta} = \mathcal{A}_{n_1,n_2}^{\delta}\mathcal{A}_n^{+} = 0 \tag{5.2.13}$$

$$\mathcal{A}_n^{+}\mathcal{A}_{n_1,n_2}^{\Omega} = \mathcal{A}_{n_1,n_2}^{\Omega}\mathcal{A}_n^{-} = 0 \tag{5.2.14}$$

The products $\mathcal{A}_{n_1,n_2}^{\delta}\mathcal{A}_n^{-}$ and $\mathcal{A}_{n_1,n_2}^{\Omega}\mathcal{A}_n^{+}$ are non-zero, but are in general quite complex and we will not study them here. Instead, we use a different pair of algebras to act on the right of $\mathcal{A}_{n_1,n_2}^{\delta/\Omega}$. These are called $\mathcal{A}_{n_1,n_2}^{\pm}$ and are defined respectively by invariance under

$$\mathcal{A}_{n_1,n_2}^{+}: \qquad \alpha \mapsto \sigma\alpha\tau^{-1} \qquad \sigma,\tau \in S_{n_1}[S_2] \times S_{n_2}[S_2] \tag{5.2.15}$$

$$\mathcal{A}_{n_1,n_2}^{-}: \qquad \alpha \mapsto (-1)^{\sigma}(-1)^{\tau}\sigma\alpha\tau^{-1} \qquad \sigma,\tau \in S_{n_1}[S_2] \times S_{n_2}[S_2] \tag{5.2.16}$$

Intuitively, $\mathcal{A}_n^{\pm}$ match the left-hand actions of (5.2.7) and (5.2.8) while $\mathcal{A}_{n_1,n_2}^{\pm}$ match the right-hand actions. As a result, $\mathcal{A}_{n_1,n_2}^{\pm}$ naturally act on the right of the permutation state spaces

$$\mathcal{A}_{n_1,n_2}^{\delta}\mathcal{A}_{n_1,n_2}^{-} \subseteq \mathcal{A}_{n_1,n_2}^{\delta} \qquad \mathcal{A}_{n_1,n_2}^{\Omega}\mathcal{A}_{n_1,n_2}^{+} \subseteq \mathcal{A}_{n_1,n_2}^{\Omega} \tag{5.2.17}$$

Fourier bases for $\mathcal{A}_{n_1,n_2}^{\pm}$ are given in section 5.4.2. These allow us to identify a matrix description of the auxiliary algebras and describe the different irreducible representations.

The actions of $\mathcal{A}_n^{\pm}$ and $\mathcal{A}_{n_1,n_2}^{\pm}$ on $\mathcal{A}_{n_1,n_2}^{\delta/\Omega}$ commute, since one acts by left multiplication and the other on the right. This means $\mathcal{A}_{n_1,n_2}^{\delta/\Omega}$ can be decomposed into representations of the product algebra $\mathcal{A}_n^{\pm} \times \mathcal{A}_{n_1,n_2}^{\mp}$. These decompositions are given in (5.6.63) and (5.6.64) respectively.

The description of the state spaces $\mathcal{A}_{n_1,n_2}^{\delta/\Omega}$ and the respective auxiliary algebras give our first concrete example of the anti-symmetrisation relation (5.0.2) between the $SO(N)$ mesonic sector and the $Sp(N)$ theory. The state spaces $\mathcal{A}_{n_1,n_2}^{\delta/\Omega}$ are anti-symmetrisations of each other, the left auxiliary algebras $\mathcal{A}_n^{\pm}$ are anti-symmetrisations of each other, and similarly for the right auxiliary algebras $\mathcal{A}_{n_1,n_2}^{\pm}$.

In the half-BPS sector, the state spaces $\mathcal{A}^{\delta/\Omega}_{n_1,n_2}$ reduce to $\mathcal{A}^{\delta/\Omega}_{n}$, defined by

$$\mathcal{A}^{\delta}_{n}: \qquad \alpha \mapsto (-1)^{\tau}\,\sigma\alpha\tau^{-1} \qquad \sigma,\tau \in S_n[S_2] \tag{5.2.18}$$

$$\mathcal{A}^{\Omega}_{n}: \qquad \alpha \mapsto (-1)^{\sigma}\,\sigma\alpha\tau^{-1} \qquad \sigma,\tau \in S_n[S_2] \tag{5.2.19}$$

while the right auxiliary algebras $\mathcal{A}^{\pm}_{n_1,n_2}$ reduce to the left auxiliary algebras $\mathcal{A}^{\pm}_{n}$, so for the half-BPS permutation state spaces we have the left and right actions

$$\mathcal{A}^{+}_{n}\mathcal{A}^{\delta}_{n} \subseteq \mathcal{A}^{\delta}_{n} \qquad \mathcal{A}^{\delta}_{n}\mathcal{A}^{-}_{n} \subseteq \mathcal{A}^{\delta}_{n} \tag{5.2.20}$$

$$\mathcal{A}^{-}_{n}\mathcal{A}^{\Omega}_{n} \subseteq \mathcal{A}^{\Omega}_{n} \qquad \mathcal{A}^{\Omega}_{n}\mathcal{A}^{+}_{n} \subseteq \mathcal{A}^{\Omega}_{n} \tag{5.2.21}$$

At large N, (5.2.7), (5.2.8), (5.2.10), (5.2.11), (5.2.15), (5.2.16), (5.2.18) and (5.2.19) define $\mathcal{A}^{\delta/\Omega}_{n_1,n_2}$, $\mathcal{A}^{\pm}_{n}$, $\mathcal{A}^{\pm}_{n_1,n_2}$, $\mathcal{A}^{\delta/\Omega}_{n}$ respectively, but for $N < n$ there are finite N cut-offs to consider. We denote the reduced algebras by adding '$;N$' (for example $\mathcal{A}^{\delta;N}_{n_1,n_2}$) to the upper index labels, and they are defined formally in section 5.6 in terms of their generators. Schematically, they are the intersection of the unrestricted versions with the N-restricted sub-algebra of $\mathbb{C}(S_{2n})$ as described in section 2.5.

In this section we have repeatedly referred to section 5.4 where we will introduce Fourier bases for each of the auxiliary algebras, give multiplication rules for them, describe their representations and give their finite N behaviour. For the permutation state spaces, the equivalent is done in section 5.6, where in addition we give the decompositions of the states spaces as representations of the auxiliary algebras and describe the operators constructed by the state spaces.

There is another basis described in section 5.3 for each of the spaces/algebras, obtained by summing over the orbits of the defining actions. These orbits are called double cosets. For the state spaces these bases correspond to the multi-trace operators.

The two types of bases, Fourier and double coset, are the $SO(N)/Sp(N)$ equivalent of the Fourier and conjugacy class bases for the $U(N)$ algebra $\mathcal{A}_{n_1,n_2}$ as described in sections 3.5.3 and 3.6.1.

5.2.2 Baryonic sector

The definition of baryonic operators (5.2.5) is invariant under the transformation

$$\mathcal{A}^{\varepsilon}_{N;n_1,n_2}: \quad \alpha \mapsto (-1)^{\sigma_1}(-1)^{\tau}\sigma\alpha\tau^{-1} \quad \sigma \in S_N \times S_q[S_2]\ ,\ \tau \in S_{n_1}[S_2] \times S_{n_2}[S_2] \tag{5.2.22}$$

where for $\sigma \in S_N \times S_q[S_2]$, σ_1 is the S_N component. This defines the sub-space $\mathcal{A}^{\varepsilon}_{N;n_1,n_2}$ of $\mathbb{C}(S_{2n})$. However, this is not the permutation state space responsible for operator construction. The definition (5.2.22) relies on N being finite, and therefore we need to restrict the group algebra $\mathbb{C}(S_{2n})$ to only incorporate those Young diagram components with $l(R) \leq N$, as explained in section 2.5. The intersection of this restriction with $\mathcal{A}^{\varepsilon}_{N;n_1,n_2}$ is called $\mathcal{A}^{\varepsilon;N}_{N;n_1,n_2}$, and it is this space which captures the true degrees of freedom of the system. We give a formal definition for the restricted space in terms of its generators in (5.6.80).

At this point it is worth clarifying the terminology 'finite N cut-off'. For the mesonic and symplectic sectors, this refers to the restriction of the group algebra to $l(R) \leq N$, as this is the only effect of N from the permutation point of view. For the baryonic sector, the terminology is potentially confusing since we require N to be finite in order for it to exist, yet we can also consider the space $\mathcal{A}^{\varepsilon}_{N;n_1,n_2}$ on which we have not implemented the 'finite N cut-off'. For ease of notation, we will refer to the restriction, even in the baryonic sector, as the 'finite N cut-off'. For similar reasons, we refer to $\mathcal{A}^{\varepsilon}_{N;n_1,n_2}$ as a permutation state space, even though it does not describe any physical state space of operators.

By similar logic to (5.2.9), $\mathcal{A}^{\varepsilon}_{N;n_1,n_2}$ and $\mathcal{A}^{\varepsilon;N}_{N;n_1,n_2}$ have a trivial product (unless $2n = N$). However, like the mesonic and symplectic cases, there are auxiliary algebras acting on the left and right that have interesting multiplication properties with $\mathcal{A}^{\varepsilon}_{N;n_1,n_2}$. On the left we have

$$\mathcal{B}^{\varepsilon}_{N,q} : \qquad \alpha \mapsto (-1)^{\sigma_1}(-1)^{\tau_1}\sigma\alpha\tau^{-1} \qquad \sigma,\tau \in S_N \times S_q[S_2] \tag{5.2.23}$$

where σ_1 is the S_N component of σ.

On the right we have $\mathcal{A}^{-}_{n_1,n_2}$ defined in (5.2.16). As with the permutation state space, we should consider the restricted versions $\mathcal{B}^{\varepsilon;N}_{N,q}$ and $\mathcal{A}^{-;N}_{n_1,n_2}$ as there is no large N limit for baryonic operators. These are defined in terms of their generators in section 5.4.

Since the actions of $\mathcal{B}^{\varepsilon;N}_{N,q}$ and $\mathcal{A}^{-;N}_{n_1,n_2}$ commute on $\mathcal{A}^{\varepsilon;N}_{N;n_1,n_2}$, it forms a representation of $\mathcal{B}^{\varepsilon;N}_{N,q} \times \mathcal{A}^{-;N}_{n_1,n_2}$. This representation is described in (5.6.83).

The half-BPS equivalent of (5.2.22) is

$$\mathcal{A}^{\varepsilon}_{n} : \qquad \alpha \mapsto (-1)^{\sigma_1}(-1)^{\tau}\sigma\alpha\tau^{-1} \qquad \sigma \in S_N \times S_q[S_2]\ ,\ \tau \in S_n[S_2] \tag{5.2.24}$$

which, along with its restriction $\mathcal{A}^{\varepsilon;N}_{N;n}$, is acted on the left and right by auxiliary algebras $\mathcal{B}^{\varepsilon;N}_{N,q}$ and $\mathcal{A}^{-;N}_{n}$ respectively.

5.3 Double cosets

In section 5.2 we defined various algebras and permutation state spaces by invariance (or anti-invariance) under multiplication on the left and right by subgroups of S_{2n}. These actions were (5.2.7), (5.2.8), (5.2.10), (5.2.11), (5.2.15), (5.2.16), (5.2.18), (5.2.19)

Permutation algebra/state space	G_L	G_R	$(-1)^{(\sigma,\tau)}$
Half-BPS state space $\mathcal{A}_n^\delta$	$S_n[S_2]$	$S_n[S_2]$	$(-1)^\tau$
Half-BPS state space $\mathcal{A}_n^\Omega$	$S_n[S_2]$	$S_n[S_2]$	$(-1)^\sigma$
Auxiliary algebra $\mathcal{A}_n^+$	$S_n[S_2]$	$S_n[S_2]$	1
Auxiliary algebra $\mathcal{A}_n^-$	$S_n[S_2]$	$S_n[S_2]$	$(-1)^\sigma(-1)^\tau$
Quarter-BPS state space $\mathcal{A}_{n_1,n_2}^\delta$	$S_n[S_2]$	$S_{n_1}[S_2]\times S_{n_2}[S_2]$	$(-1)^\tau$
Quarter-BPS state space $\mathcal{A}_{n_1,n_2}^\Omega$	$S_n[S_2]$	$S_{n_1}[S_2]\times S_{n_2}[S_2]$	$(-1)^\sigma$
Auxiliary algebra $\mathcal{A}_{n_1,n_2}^+$	$S_{n_1}[S_2]\times S_{n_2}[S_2]$	$S_{n_1}[S_2]\times S_{n_2}[S_2]$	1
Auxiliary algebra $\mathcal{A}_{n_1,n_2}^-$	$S_{n_1}[S_2]\times S_{n_2}[S_2]$	$S_{n_1}[S_2]\times S_{n_2}[S_2]$	$(-1)^\sigma(-1)^\tau$
Half-BPS state space $\mathcal{A}_{N;n}^\varepsilon$	$S_N\times S_q[S_2]$	$S_n[S_2]$	$(-1)^{\sigma_1}(-1)^\tau$
Quarter-BPS state space $\mathcal{A}_{N;n_1,n_2}^\varepsilon$	$S_N\times S_q[S_2]$	$S_{n_1}[S_2]\times S_{n_2}[S_2]$	$(-1)^{\sigma_1}(-1)^\tau$
Baryonic auxiliary algebra $\mathcal{B}_{N,q}^\varepsilon$	$S_N\times S_q[S_2]$	$S_N\times S_q[S_2]$	$(-1)^{\sigma_1}(-1)^{\tau_1}$

Table 5.1: The permutation algebras and state spaces we consider in this chapter. Each is defined by invariance under multiplication by G_L on the left and G_R on the right, up to a sign change of $(-1)^{(\sigma,\tau)}$ for $(\sigma,\tau)\in G_L\times G_R$.

(5.2.22), (5.2.23) and (5.2.24). If we ignore the minus signs in each of these, they give an action purely on S_{2n} rather than the wider algebra $\mathbb{C}(S_{2n})$. The orbits under these unsigned actions are called double cosets. Given the subgroup G_L on the left and G_R on the right, we denote the set of double cosets by

$$G_L\backslash S_{2n}/G_R \tag{5.3.1}$$

When the left and right groups are the same, $G_L = G_R$, these doubles cosets span algebras that are known in the mathematics literature as Hecke algebras. Further, when the Hecke algebra is commutative, the groups G and $G_L = G_R$ are known as a Gelfand pair. See [64] for more on these mathematical concepts.

When we include the minus signs in the actions, the double cosets are split into two categories, even and odd. Denote the sign associated to the action of $(\sigma,\tau)\in G_L\times G_R$ by $(-1)^{(\sigma,\tau)}$. Depending on the action being considered, this may or may not be equal to $(-1)^\sigma(-1)^\tau$. The choice of sign is restricted by the nature of a group action: it always satisfies

$$(-1)^{(\sigma\sigma',\tau\tau')} = (-1)^{(\sigma,\tau)}(-1)^{(\sigma',\tau')} \tag{5.3.2}$$

The permutations groups G_L and G_R and the associated sign $(-1)^{(\sigma,\tau)}$ for the different algebras are given in table 5.1.

Let π be a representative member of a double coset. We define this to be an odd double coset if there exists a $(\sigma,\tau)\in G_L\times G_R$ such that

$$(-1)^{(\sigma,\tau)}\sigma\pi\tau^{-1} = -\pi \tag{5.3.3}$$

If there is no such (σ, τ), the double coset is even. Note it does not matter which representative π we chose; if we take $\pi' = \alpha\pi\beta^{-1}$ to be an alternative representative then $\sigma' = \alpha\sigma\alpha^{-1}$ and $\tau' = \beta\tau\beta^{-1}$, so (5.3.3) is true either for all π in the double coset or none.

The distinction between even and odd double cosets is important when we consider summing over the action of $G_L \times G_R$. Consider $\pi \in S_{2n}$. By summing over the action on π, we obtain an element of the invariant algebra/space

$$\alpha_\pi = \sum_{\substack{\sigma \in G_L \\ \tau \in G_R}} (-1)^{(\sigma,\tau)} \sigma\pi\tau^{-1} \tag{5.3.4}$$

If π is in an odd double coset, then there is some $(\sigma_\pi, \tau_\pi) \in G_L \times G_R$ with $(-1)^{(\sigma_\pi,\tau_\pi)} = -1$ and $\sigma_\pi\pi\tau_\pi^{-1} = \pi$. The element (σ_π, τ_π) generates a subgroup $\{(\sigma_\pi, \tau_\pi)^k : 1 \le k \le m\}$ of $G_L \times G_R$, where m is the smallest integer such that $\sigma_\pi^m = \tau_\pi^m = 1$. The sign associated to a member of this subgroup is

$$(-1)^{(\sigma_\pi,\tau_\pi)^k} = \left[(-1)^{(\sigma_\pi,\tau_\pi)}\right]^k = (-1)^k \tag{5.3.5}$$

Therefore

$$(-1)^{(\sigma_\pi,\tau_\pi)^k} \sigma_\pi^k \pi \tau_\pi^{-k} = (-1)^k \pi \tag{5.3.6}$$

Since the sign associated to $(\sigma_\pi, \tau_\pi)^m = (1, 1)$ is 1, m must be even. Now take right coset representatives (β, γ) of $G_L \times G_R$ over this subgroup. Then

$$\begin{aligned}
\alpha_\pi &= \sum_{(\beta,\gamma)} (-1)^{(\beta,\gamma)} \beta \left(\sum_{k=1}^{m} (-1)^{(\sigma_\pi,\tau_\pi)^k} \sigma_\pi^k \pi \tau_\pi^{-k} \right) \gamma^{-1} \\
&= \sum_{(\beta,\gamma)} (-1)^{(\beta,\gamma)} \beta \left(\frac{m}{2}\pi - \frac{m}{2}\pi \right) \gamma^{-1} \\
&= 0
\end{aligned} \tag{5.3.7}$$

So the odd double cosets are those that vanish when summed over the (signed) group action.

The map $\pi \to \alpha_\pi$ in (5.3.4) projects an arbitrary element of $\mathbb{C}(S_{2n})$ into the invariant algebra/space, and therefore elements of the form α_π constitute a spanning set. Any two elements π, π' of the same double coset have the same α_π (up to a potential minus sign), and therefore if we choose π to run over set of double coset representatives, α_π generate the invariant algebra/space. (5.3.7) demonstrates that odd double cosets do not contribute, and therefore we restrict to π that are representatives of an even double coset. These α_π form a basis for the invariant algebras/spaces.

The existence of odd double cosets depend on the sign $(-1)^{(\sigma,\tau)}$ in the particular action. In the defining action of $\mathcal{A}_n^+$ and $\mathcal{A}_{n_1,n_2}^+$, the sign is $(-1)^{(\sigma,\tau)} = 1$, so clearly all double cosets are even. Similarly, if (σ,τ) are such that $\sigma\pi\tau = \pi$, then clearly σ and τ have the same sign and $(-1)^\sigma(-1)^\tau = 1$. Therefore $\mathcal{A}_n^-$ and $\mathcal{A}_{n_1,n_2}^-$ also do not have odd double cosets.

The other algebras/state spaces in table 5.1 all have odd double cosets that do not contribute. For $\mathcal{A}_n^{\delta/\Omega}$ and $\mathcal{A}_{n_1,n_2}^{\delta/\Omega}$, setting $\sigma = \tau = (1,2)$ implies the double coset containing the identity is odd. For $\mathcal{A}_n^{\varepsilon}$ and $\mathcal{A}_{n_1,n_2}^{\varepsilon}$, set $\sigma = \tau = (N+1, N+2)$, and again the double coset containing the identity is odd. Finally, the defining action for $\mathcal{B}_{N,q}$ has an odd double coset containing $\pi = (1, N+1, 2, N+2)$ since

$$(1,2)\pi(N+1, N+2) = \pi \tag{5.3.8}$$

For the actions defining the permutation state spaces $\mathcal{A}_{n_1,n_2}^{\delta/\Omega}$, $\mathcal{A}_n^{\delta/\Omega}$, $\mathcal{A}_{n_1,n_2}^{\varepsilon}$ and $\mathcal{A}_n^{\varepsilon}$, different elements in the same double coset produce the same operators when inserted in the appropriate contractions: (5.2.4), (5.2.5) or (5.2.6). The defining property (5.3.3) of odd double cosets means operators produced by odd double cosets are identically zero. Therefore the even double cosets are responsible for operator construction.

In the remainder of this section we give a more explicit understanding of the double cosets associated with the (unsigned) actions

$$\alpha \mapsto \sigma\alpha\tau^{-1} \qquad \sigma,\tau \in S_n[S_2] \tag{5.3.9}$$

$$\alpha \mapsto \sigma\alpha\tau^{-1} \qquad \sigma \in S_n[S_2]\ ,\ \tau \in S_{n_1}[S_2] \times S_{n_2}[S_2] \tag{5.3.10}$$

The first of these is associated to the left auxiliary algebras $\mathcal{A}_n^{\pm}$, as well as the half-BPS state spaces $\mathcal{A}_n^{\delta/\Omega}$. The second is associated to the quarter-BPS state spaces $\mathcal{A}_{n_1,n_2}^{\delta/\Omega}$.

As previously discussed, the actions (5.2.10) and (5.2.11) defining $\mathcal{A}_n^{\pm}$ do not have odd double cosets, so do not split the double cosets of (5.3.9). In principle the split into even and odd could be different for the different actions (5.2.18) and (5.2.19) corresponding to $\mathcal{A}_n^{\delta/\Omega}$. However, if $\sigma\pi\tau^{-1} = \pi$ it follows that σ and τ have the same sign, so splitting the double cosets by the sign $(-1)^\sigma$ is the same as splitting them by $(-1)^\tau$. Therefore there is only a single consistent way of splitting (5.3.9) into even and odd double cosets, and from now on we use this definition for even and odd, independent of the sign of the particular action being considered.

We have a similar situation for the defining actions (5.2.7) and (5.2.8) for $\mathcal{A}_{n_1,n_2}^{\delta/\Omega}$ splitting the double cosets of (5.3.10), and therefore there is no ambiguity in the definition of even and odd double cosets.

For each of (5.3.9) and (5.3.10) we will give a labelling set for the double cosets, provide descriptions of representative members, write down the size of a given double coset and identify which of these are even and odd. We then use sums over even double cosets to construct bases. For the permutation state spaces $\mathcal{A}^{\delta/\Omega}_{n_1,n_2}$ and $\mathcal{A}^{\delta/\Omega}_{n}$, these bases are responsible for the construction of multi-traces, and the labelling sets of the even double cosets are the same as those for the multi-traces.

For the auxiliary algebras $\mathcal{A}^{\pm}_{n_1,n_2}$ and $\mathcal{B}^{\varepsilon}_{N,q}$ and the baryonic state spaces $\mathcal{A}^{\varepsilon}_{N;n_1,n_2}$ and $\mathcal{A}^{\varepsilon}_{N;n}$ not covered by (5.3.9) and (5.3.10), explicit descriptions of the double cosets are more involved and we do not give them here. In the interests of completeness, the double coset bases are

$$\mathcal{A}^{+}_{n_1,n_2}: \qquad \alpha_\pi = \sum_{\sigma,\tau\in S_{n_1}[S_2]\times S_{n_2}[S_2]} \sigma\pi\tau^{-1} \tag{5.3.11}$$

$$\mathcal{A}^{-}_{n_1,n_2}: \qquad \alpha_\pi = \sum_{\sigma,\tau\in S_{n_1}[S_2]\times S_{n_2}[S_2]} (-1)^{\sigma}(-1)^{\tau}\sigma\pi\tau^{-1} \tag{5.3.12}$$

$$\mathcal{B}^{\varepsilon}_{N,q}: \qquad \alpha_\pi = \sum_{\sigma,\tau\in S_N\times S_q[S_2]} (-1)^{\sigma_1}(-1)^{\tau_1}\sigma\pi\tau^{-1} \tag{5.3.13}$$

$$\mathcal{A}^{\varepsilon}_{N;n_1,n_2}: \qquad \alpha_\pi = \sum_{\substack{\sigma\in S_N\times S_q[S_2]\\ \tau\in S_{n_1}[S_2]\times S_{n_2}[S_2]}} (-1)^{\sigma_1}\sigma\pi\tau^{-1} \tag{5.3.14}$$

$$\mathcal{A}^{\varepsilon}_{N;n}: \qquad \alpha_\pi = \sum_{\substack{\sigma\in S_N\times S_q[S_2]\\ \tau\in S_{n}[S_2]}} (-1)^{\sigma_1}\sigma\pi\tau^{-1} \tag{5.3.15}$$

where in each case π runs over the representatives of the even double cosets, and for $\sigma \in S_N \times S_q[S_2]$, σ_1 is the S_N component.

5.3.1 Action of $S_n[S_2] \times S_n[S_2]$: the half-BPS sector

Above (5.3.9), we explained that any two permutations in the same (even) double coset produce the same operator up to a possible minus sign. For mesonic operators at large N, the converse is true: if two permutations produce the same operator (up to a sign), they belong to the same double coset. Therefore the labels for multi-traces considered in section 4.1 are the same as the labels for even double cosets. In the half-BPS sector, this means the labels for even double cosets are partitions $q \vdash \frac{n}{2}$.

Consider the action (5.3.9). This is the algebra invariance produced if we replaced the anti-symmetric matrix X in the (half-bps) construction of operators (5.2.4) with a symmetric matrix. Since multi-traces of this matrix would be labelled by partitions $p \vdash n$, the double cosets under (5.3.9) are also labelled by p. From the previous paragraph, the odd double cosets would be those p which have an odd component while the even double cosets have $p = 2q$, where the partition $2q$ is defined to have components that are double the components of q.

To give representatives for the double cosets, consider a permutation $\sigma \in S_n$ of cycle type p. Embed S_n into S_{2n} by acting only on the odd numbers $\{1, 3, \ldots, 2n-1\}$, and let $\sigma^{(odd)} \in S_{2n}$ be the embedding of σ. Then $\sigma^{(odd)}$ is a representative member of the double coset labelled by p. As expected, the operator (5.2.4) constructed from $\sigma^{(odd)}$ is the standard trace operator of type p

$$
\begin{aligned}
\mathcal{O}^{\delta}_{\sigma^{(odd)}} &= C^{(\delta)}_I \left(\sigma^{(odd)}\right)^I_J \left(X^{\otimes n}\right)^J \\
&= \delta_{i_1 j_1}\delta_{i_2 j_2}\dots\delta_{i_n j_n}\left(\sigma^{(odd)}\right)^{i_1 j_1 i_2 j_2 \dots i_n j_n}_{k_1 l_1 k_2 l_2 \dots k_n l_n} X^{k_1 l_1} X^{k_2 l_2}\dots X^{k_n l_n} \\
&= \delta_{i_1 j_1}\dots\delta_{i_n j_n}\sigma^{i_1 i_2 \dots i_n}_{k_1 k_2 \dots k_n}\delta^{j_1}_{l_1}\delta^{j_2}_{l_2}\dots\delta^{j_n}_{l_n} X^{k_1 l_1} X^{k_2 l_2}\dots X^{k_n l_n} \\
&= \sigma^{i_1 i_2 \dots i_n}_{k_1 k_2 \dots k_n} X^{k_1 i_1} X^{k_2 i_2}\dots X^{k_n i_n} \\
&= X^{k_1 k_{\sigma(1)}} X^{k_2 k_{\sigma(2)}}\dots X^{k_n k_{\sigma(n)}} \\
&= \prod_i \left(\mathrm{Tr} X^i\right)^{p_i}
\end{aligned} \tag{5.3.16}
$$

where we have evaluated the last line by noting that this is a $U(N)$ type contraction as considered in (2.1.11) with the generic $U(N)$ matrix replace with the anti-symmetric $SO(N)$ matrix.

The calculation (5.3.16) (excluding the last line), is shown diagrammatically in 5.5, which makes the structure of the argument clearer. Intuitively, placing $\sigma^{(odd)}$ in the $SO(N)$ contraction formula (5.2.4) reduces it to the $U(N)$ contraction formula (2.1.11).

In (5.3.16), we have not used any symmetry or anti-symmetry properties of X, so the result applies for both. Taking X to be anti-symmetric, we deduce that if p has an odd component, the trace vanishes, and therefore we conclude again that it is these p which label the odd double cosets and $p = 2q$ label the even double cosets.

The same calculation can be performed for the symplectic contraction (5.2.6)

$$
\begin{aligned}
\mathcal{O}^{\Omega}_{\sigma} &= C^{(\Omega)}_I \left(\sigma^{(odd)}\right)^I_J \left[(\Omega X)^{\otimes n}\right]^J \\
&= \Omega_{i_1 j_1}\Omega_{i_2 j_2}\dots\Omega_{i_n j_n}\left(\sigma^{(odd)}\right)^{i_1 j_1 i_2 j_2 \dots i_n j_n}_{k_1 l_1 k_2 l_2 \dots k_n l_n} (\Omega X)^{k_1 l_1} (\Omega X)^{k_2 l_2}\dots(\Omega X)^{k_n l_n} \\
&= \Omega_{i_1 j_1}\Omega_{i_2 j_2}\dots\Omega_{i_n j_n}\sigma^{i_1 i_2 \dots i_n}_{k_1 k_2 \dots k_n}\delta^{j_1}_{l_1}\delta^{j_2}_{l_2}\dots\delta^{j_n}_{l_n} (\Omega X)^{k_1 l_1} (\Omega X)^{k_2 l_2}\dots(\Omega X)^{k_n l_n} \\
&= \Omega_{i_1 j_1} (\Omega X)^{j_1 k_1} \Omega_{i_2 j_2} (\Omega X)^{j_2 k_2}\dots\Omega_{i_n j_n} (\Omega X)^{j_n k_n} \sigma^{i_1 i_2 \dots i_n}_{k_1 k_2 \dots k_n} \\
&= \sigma^{i_1 i_2 \dots i_n}_{k_1 k_2 \dots k_n} \left(\Omega^2 X\right)^{i_1 k_1} \left(\Omega^2 X\right)^{i_2 k_2}\dots\left(\Omega^2 X\right)^{i_n k_n} \\
&= (-1)^n \sigma^{i_1 i_2 \dots i_n}_{k_1 k_2 \dots k_n} X^{i_1 k_1} X^{i_2 k_2}\dots X^{i_n k_n} \\
&= (-1)^n X^{i_1 i_{\sigma^{-1}(1)}} X^{i_2 i_{\sigma^{-1}(2)}}\dots X^{i_n i_{\sigma^{-1}(n)}} \\
&= \prod_i \left(\mathrm{Tr} X^i\right)^{p_i}
\end{aligned} \tag{5.3.17}
$$

This is shown more intuitively, excluding the last line, via a diagram in figure 5.6. The fact (5.3.17) matches (5.3.16) serves as another demonstration that the $SO(N)$ and $Sp(N)$ traces have the same form, as argued in section 4.1.2.

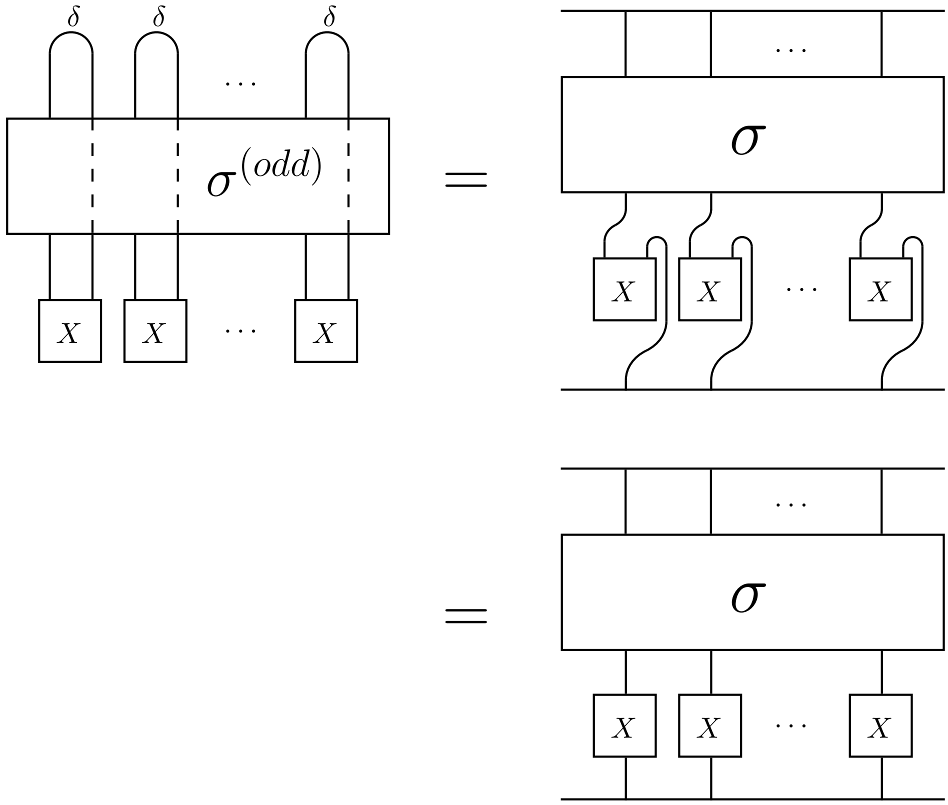


Figure 5.5: A diagrammatic version of (5.3.16). The dotted lines represent the fact that $\sigma^{(odd)}$ fixes all even numbers. The first row keeps the index positions in X constant, while the second breaks our index conventions and uses the index structure $X^i{}_j = X^{ij}$ to illustrate that using $\sigma^{(odd)} \in S_{2n}$ has changed the $SO(N)$ type contraction into the $U(N)$ type contraction (cf. figures 5.2 and 2.1)

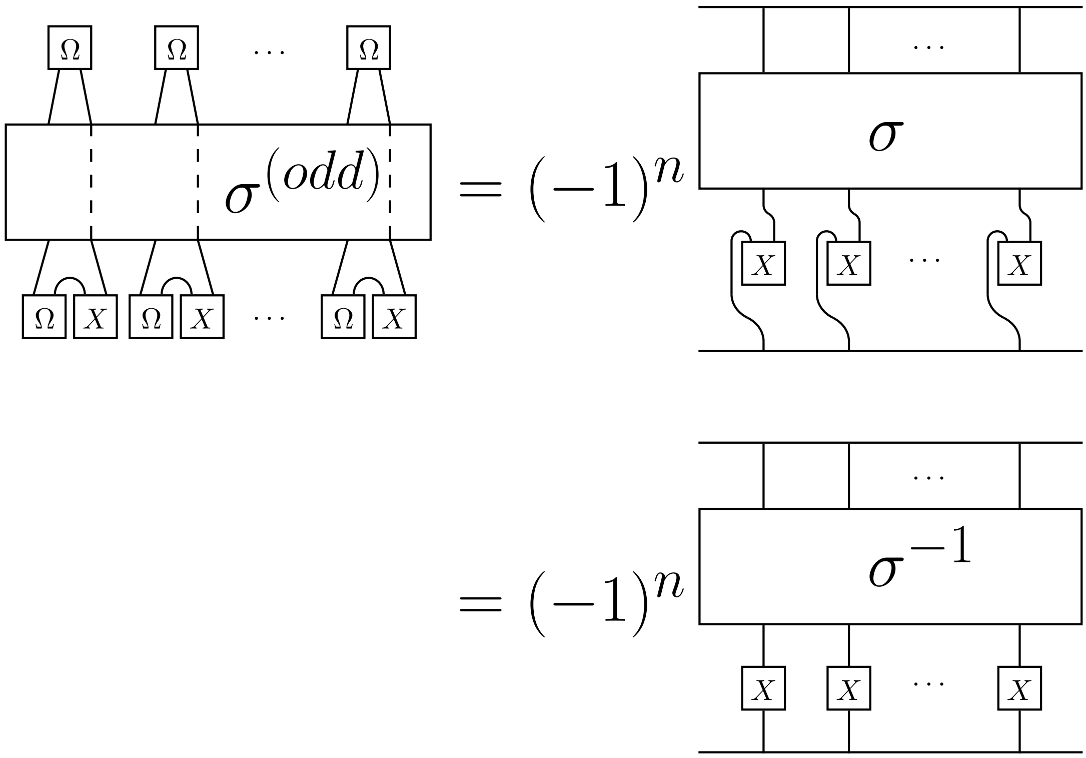


Figure 5.6: A diagrammatic version of (5.3.17). The dotted lines represent the fact that $\sigma^{(odd)}$ fixes all even numbers. By following the index lines on the left, we see that $\sigma^{(odd)}$ is contracted with n copies of the matrix $\Omega X \Omega^T$. Using the condition (4.0.1), this is just $-X^T$. We have pulled out the factors of -1 and the transpose means the X indices switch roles (compare with figure 5.5). In the second row, we convert this result into a $U(N)$ type contraction by breaking our index conventions and setting $X^i{}_j = X^{ij}$. The role switch of the X indices on the first line means σ is inverted on the second line.

Take a $\sigma \in S_n$ of cycle type p and consider the stabiliser of $\sigma^{(odd)}$ under (5.3.9). For each cycle of length i, it consists of a rotation group and a reflection element that together generate a copy of D_i, the dihedral group of order $2i$. Similarly to the $U(N)$ case in section 3.5.1, there is also a S_{p_i} factor that acts on the p_i cycles of length i. For a more detailed description of how these properties appear in the stabiliser of σ, see [1]. In total the stabiliser group is

$$Stab(\sigma) \cong \bigtimes_i \left(S_{p_i} \ltimes (D_i)^{p_i} \right) = \bigtimes_i S_{p_i} [D_i] \tag{5.3.18}$$

which has size

$$\prod_i (2i)^{p_i} (p_i)! = z_{2p} \tag{5.3.19}$$

Applying the orbit-stabiliser theorem, the size of a double coset is

$$\frac{|S_n[S_2] \times S_n[S_2]|}{|\text{stabiliser}|} = \frac{2^{2n}(n!)^2}{z_{2p}} \tag{5.3.20}$$

In terms of $q \vdash \frac{n}{2}$, the size of an even double coset is

$$\frac{2^{2n}(n!)^2}{z_{4q}} \tag{5.3.21}$$

where $4q$ is the partition of $2n$ with components quadruple those of q. For a formal mathematical proof of the fact that partitions label double cosets, as well as a derivation of their size, see [64, Chapter VII.2].

The double coset bases for $\mathcal{A}_n^{\pm}$ and $\mathcal{A}^{\delta/\Omega}$ are

$$\mathcal{A}_n^+ : \qquad \alpha_p = \sum_{\tau,\pi \in S_n[S_2]} \tau \sigma_p^{(odd)} \pi^{-1} \tag{5.3.22}$$

$$\mathcal{A}_n^- : \qquad \alpha_p = \sum_{\tau,\pi \in S_n[S_2]} (-1)^{\tau} (-1)^{\pi} \tau \sigma_p^{(odd)} \pi^{-1} \tag{5.3.23}$$

$$\mathcal{A}_n^{\delta} : \qquad \alpha_q = \frac{1}{2^{2n}\,(n!)^2} \sum_{\tau,\pi \in S_n[S_2]} (-1)^{\pi} \tau \sigma_{2q}^{(odd)} \pi^{-1} \tag{5.3.24}$$

$$\mathcal{A}_n^{\Omega} : \qquad \alpha_q = \frac{1}{2^{2n}\,(n!)^2} \sum_{\tau,\pi \in S_n[S_2]} (-1)^{\tau} \tau \sigma_{2q}^{(odd)} \pi^{-1} \tag{5.3.25}$$

where $\sigma_p \in S_n$ is a permutation of cycle type $p \vdash n$. The sign of α_q will in general depend on the choice of σ_{2q}, but this ambiguity is not an issue for our purposes, and we do not resolve it here.

5.3.2 Action of $S_n[S_2] \times (S_{n_1}[S_2] \times S_{n_2}[S_2])$: the quarter-BPS sector

This section closely follows the half-BPS discussion above, and much of the logic is repeated.

In the quarter-BPS sector, double cosets are defined to be orbits in S_{2n} under the action (5.3.10) where this is the unsigned version of (5.2.7) and (5.2.8). This is the invariance we would have obtained had we taken X and Y to be symmetric matrices in the construction of operators (5.2.4), and so the double cosets correspond to the traces of symmetric matrices, and are therefore labelled by $\bar{\mathcal{P}}$ as defined at the end of section 4.1.1.

Take $\sigma \in S_n$. Then by the same logic as in (5.3.16), we have

$$C_I^{(\delta)} \left(\sigma^{(odd)}\right)_J^I \left(X^{\otimes n_1} Y^{\otimes n_2}\right)^J = X^{k_1 k_{\sigma(1)}} \dots X^{k_{n_1} k_{\sigma(n_1)}} Y^{k_{n_1+1} k_{\sigma(n_1+1)}} \dots Y^{k_n k_{\sigma(n)}} \tag{5.3.26}$$

By comparing with the explanation of quarter-BPS $U(N)$ traces offered in section 3.5.3 we see that the trace is determined by the cycles of σ. Each cycle is a single trace, where a number in $\{1, 2, \dots, n_1\}$ corresponds to an X and a number in $\{n_1+1, n_1+2, \dots n\}$ corresponds to a Y. Arranging the Xs and Ys in the order specified by the cycle gives the trace. We say $\sigma \in S_n$ is of 'cycle type' $\bar{\mathcal{P}}$ if it produces (up to a sign) the multi-trace

$$T_{\bar{\mathcal{P}}} = \prod_{\bar{w}, i} \left(\mathrm{Tr}\bar{W}\right)^{(p_{\bar{w}})_i} \tag{5.3.27}$$

where $\bar{W}$ is the matrix word corresponding to the Lyndon word (up to reversal) $\bar{w}$ as described at the end of section 4.1.1. For any $\sigma \in S_n$ with 'cycle type' $\bar{\mathcal{P}}$, $\sigma^{(odd)}$ is a representative member of the double coset labelled by $\bar{\mathcal{P}}$.

The stabiliser under the action (5.3.10) for such a representative $\sigma^{(odd)}$ is more complex than the half-BPS statement (5.3.18). Each cycle, labelled by $\bar{w}, i$ has an associated rotation group $\mathbb{Z}_i$. If the word $\bar{w}$ is reversal invariant (i.e. of type 1), we also have a reflection symmetry, enhancing the rotation group to the dihedral group D_i. This is the same dihedral group that played a crucial role in determining the generating function for single traces in section 4.3. There is also the permutation factor $S_{p_{\bar{w},i}}$ permuting cycles with the same labels. For a more detailed description of this stabiliser group see [1]. This leads to the stabiliser

$$Stab(\sigma) \cong \left(\bigtimes_{\substack{\bar{w} \text{ of type 1} \\ i}} S_{p_{\bar{w},i}}[D_i] \right) \times \left(\bigtimes_{\substack{\bar{w} \text{ of type 2} \\ i}} S_{p_{\bar{w},i}}[\mathbb{Z}_i] \right) \tag{5.3.28}$$

which has size

$$\bar{Z}_{\bar{\mathcal{P}}} = \left(\prod_{\bar{w} \text{ of type } 1} z_{2p_{\bar{w}}} \right) \left(\prod_{\bar{w} \text{ of type } 2} z_{p_{\bar{w}}} \right) \tag{5.3.29}$$

Applying the orbit-stabiliser theorem, the size of the double coset is

$$\frac{2^{2n} n! n_1! n_2!}{\bar{Z}_{\bar{\mathcal{P}}}} \tag{5.3.30}$$

In the construction of the stabiliser group (5.3.28), all rotations of a cycle $\bar{w}, i$ are given by even permutations $\sigma \in S_n[S_2], \tau \in S_{n_1}[S_2] \times S_{n_2}[S_2]$ under the action (5.3.10), while the reflection action is given by σ, τ with signs $(-1)^\sigma = (-1)^\tau = (-1)^{il(\bar{w})}$. So the double coset is odd if there are one or more cycles labelled by $\bar{w}$ with odd length and odd i. This happens when one of the constituent partitions $p_{\bar{w}}$ in $\bar{\mathcal{P}}$ has an odd component, for $\bar{w}$ of type 1 and odd length. So for even double cosets, the partitions $p_{\bar{w}}$ for $\bar{w}$ of type 1 and odd length are of the form $p_{\bar{w}} = 2p_{\bar{w}\bar{w}}$ where $p_{\bar{w}\bar{w}}$ is a partition with half the sum, and $\bar{w}\bar{w}$ as is an orthogonal Lyndon word of type 1B. The remaining partitions $p_{\bar{w}}$, for $\bar{w}$ of type 1 and even length (an orthogonal Lyndon word of type 1A) or for $\bar{w}$ of type 2 (an orthogonal Lyndon word of type 2), can have any form for the even double cosets.

Recall that $\bar{\mathcal{P}}$ is defined as a set of partitions. Then replacing the partitions $p_{\bar{w}} = 2p_{\bar{w}\bar{w}}$ for an even double coset means this set of partitions is now of the form $\widetilde{\mathcal{P}}$ as defined in (4.1.5). So, as expected, the even double cosets have the same labels as the quarter-BPS multi-traces.

In terms of the label $\widetilde{\mathcal{P}}$, the stabiliser of an even double coset is

$$Stab(\sigma) \cong \left(\bigtimes_{\substack{\tilde{w} \text{ of type 1A} \\ i}} S_{p_{\tilde{w},i}}[D_i] \right) \times \left(\bigtimes_{\substack{\tilde{w} \text{ of type 1B} \\ i}} S_{p_{\tilde{w},i}}[D_{2i}] \right) \times \left(\bigtimes_{\substack{\tilde{w} \text{ of type 2} \\ i}} S_{p_{\tilde{w},i}}[\mathbb{Z}_i] \right) \tag{5.3.31}$$

with size

$$\widetilde{Z}_{\widetilde{\mathcal{P}}} = \left(\prod_{\tilde{w} \text{ of type 1A}} z_{2p_{\tilde{w}}} \right) \left(\prod_{\tilde{w} \text{ of type 1B}} z_{4p_{\tilde{w}}} \right) \left(\prod_{\tilde{w} \text{ of type } 2} z_{p_{\tilde{w}}} \right) \tag{5.3.32}$$

and the size of an even double coset is

$$\frac{|S_n[S_2] \times (Sn_1[S_2] \times S_{n_2}[S_2])|}{|\text{stabiliser}|} = \frac{2^{2n} n! n_1! n_2!}{\widetilde{Z}_{\widetilde{\mathcal{P}}}} \tag{5.3.33}$$

Similarly to the half-BPS case, even double cosets produce the traces for both orthogonal and symplectic gauge theory. Sums over the even double cosets form bases for the invariant algebras at large N. The bases for $\mathcal{A}^{\delta}_{n_1,n_2}$ and $\mathcal{A}^{\Omega}_{n_1,n_2}$ are

$$\mathcal{A}^{\delta}_{n_1,n_2}: \qquad \alpha_{\widetilde{\mathcal{P}}} = \frac{1}{2^{2n}n!n_1!n_2!} \sum_{\substack{\tau\in S_n[S_2] \\ \pi\in S_{n_1}[S_2]\times S_{n_2}[S_2]}} (-1)^{\pi} \tau \sigma^{(odd)}_{\widetilde{\mathcal{P}}} \pi^{-1} \tag{5.3.34}$$

$$\mathcal{A}^{\Omega}_{n_1,n_2}: \qquad \alpha_{\widetilde{\mathcal{P}}} = \frac{1}{2^{2n}n!n_1!n_2!} \sum_{\substack{\tau\in S_n[S_2] \\ \pi\in S_{n_1}[S_2]\times S_{n_2}[S_2]}} (-1)^{\tau} \tau \sigma^{(odd)}_{\widetilde{\mathcal{P}}} \pi^{-1} \tag{5.3.35}$$

where $\sigma_{\widetilde{\mathcal{P}}} \in S_n$ is of 'cycle type' $\widetilde{\mathcal{P}}$.

5.3.3 Equivalent S_n description

In the previous section we have described the equivalence classes in S_{2n} that lead, via the contractions (5.2.4) and (5.2.6), to the different $SO(N)$ and $Sp(N)$ multi-traces. These classes were orbits under the group action (5.3.10), and we separated the orbits into odd and even depending on whether they produced non-vanishing traces.

The $U(N)$-type contraction, (2.1.3), also produces $SO(N)$ traces if we treat X and Y as antisymmetric matrices (after performing the flavour projection to the 2-matrix system), and therefore we can give an equivalent description using equivalence classes in S_n. Explicitly, given $\sigma \in S_n$, we have

$$\sigma \sim \alpha\sigma\alpha^{-1} \qquad \alpha \in S_{n_1} \times S_{n_2} \tag{5.3.36}$$

and in addition, σ is related to any permutation that can be obtained by inverting some subset of the cycles of σ. If the cycle decomposition of σ is $\sigma = c_1c_2\dots c_r$ then

$$\sigma \sim c_1^{i_1} c_2^{i_2} \dots c_r^{i_r} \qquad i_j \in \{-1, 1\} \tag{5.3.37}$$

As before, we can split these equivalence classes into those that produce non-zero traces and those that don't. If σ contains a cycle c of odd length such that c is conjugate (under $S_{n_1} \times S_{n_2}$) to c^{-1}, then the contraction vanishes. If σ contains no such cycle, then it and the corresponding equivalence class produce a non-vanishing trace.

The combination of (5.3.36) and (5.3.37) in S_n is equivalent to (5.3.10) in S_{2n}. We see that the S_n version is more complicated, and explicitly depends on the cycle structure of σ. It therefore cannot be described as a group action on S_n and would be difficult to deal with as a result. We can clearly see the advantages of using S_{2n}.

5.4 Fourier bases for auxiliary algebras

In section 5.3 we schematically gave bases for the auxiliary algebras $\mathcal{A}^{\pm}_{n_1,n_2}$ in terms of double cosets, and more explicitly for $\mathcal{A}^{\pm}_{n}$. These bases were only valid at large N. In the first two parts of this section we give a different set of bases labelled by Young diagrams. These make explicit the Wedderburn-Artin decomposition of the algebras and allow identification of representations. They are also valid at finite N, and have definite eigenvalues when acting on the Schur and restricted Schur bases (introduced in section 5.6) of the appropriate permutation state spaces $\mathcal{A}^{\delta/\Omega}_{n}$, $\mathcal{A}^{\delta/\Omega}_{n_1,n_2}$, $\mathcal{A}^{\varepsilon}_{N;n}$ and $\mathcal{A}^{\varepsilon}_{N;n_1,n_2}$.

The baryonic auxiliary algebra $\mathcal{B}^{\varepsilon}_{N,q}$, defined in (5.2.23) was also considered in 5.3, and a schematic double coset basis was given. However, baryonic operators can only be defined at finite N, and therefore the correct auxiliary algebra to consider is the restricted version $\mathcal{B}^{\varepsilon;N}_{N,q}$. In the final part of this section we give a Young diagram basis for $\mathcal{B}^{\varepsilon}_{N,q}$ that allows us to define $\mathcal{B}^{\varepsilon;N}_{N,q}$ in terms of its generators. After this definition, we will in general study $\mathcal{B}^{\varepsilon;N}_{N,q}$ as the algebra of interest, and only mention the unrestricted version $\mathcal{B}^{\varepsilon}_{N,q}$ in passing when the distinction is important. This is in contrast to the $SO(N)$ mesonic and $Sp(N)$ cases, where we focus on the large N algebras and only mention the finite N versions in passing.

In each case, the Fourier basis can be obtained by transforming from the permutation basis of $\mathbb{C}(S_{2n})$ to the Fourier basis (2.5.1) and considering the effect of invariance under the defining action of each algebra. For a detailed account of a basis construction of this type see [1], or the construction of the mesonic covariant basis in appendix F.

There are two special elements of $\mathbb{C}(S_{2n})$ relevant for correlator calculations in $SO(N)$ and $Sp(N)$ theories. $\widetilde{\Omega}$ is important for the $SO(N)$ mesonic sector and the $Sp(N)$ theory, while Ω^{ε} is important for the baryonic sector of the $SO(N)$ theory. For the left auxiliary algebras, the corresponding Ω element has eigenvalues on the Fourier basis determined by the Young diagram labels. For a definition of the two elements and a description of their properties, see appendix A. For their role in correlators see section 5.5.

The equivalent Fourier bases for the permutation state spaces are given in section 5.6.

5.4.1 $\mathcal{A}^{\pm}_{n}$

The Fourier basis for $\mathcal{A}^{+}_{n}$ is labelled by $R \vdash 2n$ with even row lengths

$$\beta^{+}_{R} = \frac{d_R}{(2n)!} \sum_{\sigma \in S_{2n}} \langle R, [S]| D^{R}(\sigma) |R, [S]\rangle \sigma \tag{5.4.1}$$

where the vector $|R,[S]\rangle$ is invariant under $S_n[S_2]$. It was defined, along with the anti-symmetric version $|R,[A]\rangle$, in (5.1.4).

The basis for $\mathcal{A}_n^-$ is labelled by the conjugate set of Young diagrams to (5.4.1). This is given by $R \vdash 2n$ with even column lengths

$$\beta_R^- = \frac{d_R}{(2n)!} \sum_{\sigma \in S_{2n}} \langle R,[A]|D^R(\sigma)|R,[A]\rangle \sigma \tag{5.4.2}$$

It follows from the behaviour of $D^R(\sigma)$ and $|R,[S]\rangle$ under conjugation of R, given in (5.1.11) and (5.1.14), that these two bases are directly related to each other by anti-symmetrisation and conjugation of Young diagrams

$$\beta_{R^c}^+ = \text{Anti-Sym}(\beta_R^-) \tag{5.4.3}$$

where the operator Anti-Sym sends $\sigma \to (-1)^\sigma \sigma$ and extends linearly to $\mathbb{C}(S_{2n})$.

This is another example of the anti-symmetrisation relation (5.0.2) between the $SO(N)$ and $Sp(N)$ theories. $\mathcal{A}_n^+$, spanned by β_R^+, is the left auxiliary algebra for $SO(N)$ mesonic operators while $\mathcal{A}_n^-$, spanned by β_R^-, is the left auxiliary algebra for $Sp(N)$ operators, so (5.4.3) shows the exchange of the two algebras under anti-symmetrisation and $R \to R^c$.

When we restrict to $N < n$, the basis elements $\beta_R^\pm$ with $l(R) > N$ will annihilate the relevant permutation state spaces $\mathcal{A}_n^{\delta/\Omega;N}$ and $\mathcal{A}_{n_1,n_2}^{\delta/\Omega;N}$ under left multiplication. Therefore we restrict the auxiliary algebras to $\mathcal{A}_n^{\pm;N}$ by restricting the R labels to have $l(R) \leq N$.

The normalisations of (5.4.1) and (5.4.2) are chosen so that they have the multiplication property

$$\beta_R^+\beta_S^+ = \delta_{RS}\beta_R^+ \qquad\qquad \beta_R^-\beta_S^- = \delta_{RS}\beta_R^- \tag{5.4.4}$$

where we have evaluated the product using the orthogonality relation (2.3.4). This proves that $\mathcal{A}_n^\pm$ are commutative algebras. They can be realised as matrices with rows labelled by R (subject to the appropriate row and column length conditions), where $\beta_R^\pm$ has a 1 in the Rth diagonal entry and 0 everywhere else.

The Wedderburn-Artin theorem [84] states that any (semi-simple) algebra is isomorphic to a matrix algebra consisting of block diagonal components. The matrix interpretation above gives the Wedderburn-Artin decomposition of $\mathcal{A}_n^\pm$. As complex algebras, they are

$$\mathcal{A}_n^+ = \bigoplus_{\substack{R\vdash 2n \text{ with} \\ \text{even row lengths}}} \mathbb{C} \qquad\qquad \mathcal{A}_n^- = \bigoplus_{\substack{R\vdash 2n \text{ with} \\ \text{even column lengths}}} \mathbb{C} \tag{5.4.5}$$

Where $\mathbb{C}$ is understood as the space of 1×1 matrices. Using these decompositions, we can identify the distinct irreducible representations of $\mathcal{A}_n^{\pm}$. Since representations of $\mathbb{C}$ are labelled by a complex numbers, the irreducible representations of $\mathcal{A}^{\pm}$ are labelled by a Young diagram $R \vdash 2n$ (satisfying the appropriate conditions) and a complex number c.

When acted on by $\widetilde{\Omega}$, the $\beta_R^{\pm}$ form eigenvectors on both the left and the right. It follows from the action of $\widetilde{\Omega}$ on the vectors $|R,[S]\rangle$ and $|R,[A]\rangle$, given in (A.2.18) and (A.2.19), that

$$\widetilde{\Omega}\beta_R^{+} = \beta_R^{+}\widetilde{\Omega} = f_R^{\delta}\beta_R^{+} \qquad \widetilde{\Omega}\beta_R^{-} = \beta_R^{-}\widetilde{\Omega} = f_R^{\Omega}\beta_R^{-} \tag{5.4.6}$$

where

$$f_R^{\delta} = \prod_{\substack{b\in \text{ odd} \\ \text{columns of } R}} (N + c_b) \tag{5.4.7}$$

$$f_R^{\Omega} = \prod_{\substack{b\in \text{ odd} \\ \text{rows of } R}} (N + c_b) \tag{5.4.8}$$

Since the $\beta_R^{\pm}$ generate $\mathcal{A}_n^{\pm}$, it follows that $\widetilde{\Omega}$ commutes with these algebras. Note that $f_R^{\delta}, f_R^{\Omega} = 0$ if $l(R) > N$, so $\widetilde{\Omega}$ takes the unrestricted algebras $\mathcal{A}_n^{\pm}$ to the finite N versions $\mathcal{A}_n^{\pm;N}$.

The projectors $P_{[S]}$ and $P_{[A]}$, defined in (5.1.3), are invariant under the respective defining actions (5.2.10) and (5.2.11) of $\mathcal{A}_n^{+}$ and $\mathcal{A}_n^{-}$, and are therefore members of the algebras. They therefore commute with $\widetilde{\Omega}$

$$\widetilde{\Omega}P_{[S]} = P_{[S]}\widetilde{\Omega} \qquad \widetilde{\Omega}P_{[A]} = P_{[A]}\widetilde{\Omega} \tag{5.4.9}$$

5.4.2 $\mathcal{A}^{\pm}_{n_1,n_2}$

The Fourier basis for $\mathcal{A}^{+}_{n_1,n_2}$ is labelled by $R \vdash 2n$; $R_1, T_1 \vdash 2n_1$; $R_2, T_2 \vdash 2n_2$ and two Littelwood-Richardson multiplicity indices μ, ν for the triples $(R; R_1, R_2)$ and $(R; T_1, T_2)$ respectively. R_1, R_2, T_1 and T_2 are restricted to have even row lengths, while the only restriction on R is that $g_{R;R_1,R_2}, g_{R;T_1,T_2} > 0$. The basis is given by

$$\beta^{+}_{R,(R_1,R_2,\mu),(T_1,T_2,\nu)} = \frac{d_R}{(2n)!}\sum_{\sigma\in S_{2n}} \langle R_1, R_2, [S], \mu|\, D^R(\sigma)\, |T_1, T_2, [S], \nu\rangle\, \sigma \tag{5.4.10}$$

where the vector $|R_1, R_2, [S], \mu\rangle$ is defined in (5.1.16) and is invariant under $S_{n_1}[S_2] \times S_{n_2}[S_2]$. The anti-invariant version $|R_1, R_2, [A], \mu\rangle$ is defined in (5.1.17).

The basis for $\mathcal{A}^{-}_{n_1,n_2}$ is labelled by the conjugate set of Young diagrams to (5.4.10), given by $R \vdash 2n$; $R_1, T_1 \vdash 2n_1$; $R_2, T_2 \vdash 2n_2$ and two Littelwood-Richardson multiplicity indices μ, ν for the triples $(R; R_1, R_2)$ and $(R; T_1, T_2)$ respectively. R_1, R_2, T_1 and T_2 are restricted to have even columns, while the only restriction on R is that $g_{R;R_1,R_2}, g_{R;T_1,T_2} > 0$.

$$\beta^-_{R,(R_1,R_2,\mu),(T_1,T_2,\nu)} = \frac{d_R}{(2n)!} \sum_{\sigma \in S_{2n}} \langle R_1, R_2, [A], \mu| \, D^R(\sigma) \, |T_1, T_2, [A], \nu\rangle \, \sigma \tag{5.4.11}$$

These two bases are related to each other by conjugation of Young diagrams. It follows from the behaviour of $D^R(\sigma)$ and $|R_1, R_2, [S], \mu\rangle$ under $R \to R^c$, given in (5.1.11) and (5.1.22), that

$$\beta^+_{R^c,(R_1^c,R_2^c,\mu),(T_1^c,T_2^c,\nu)} = \text{Anti-Sym}\left(\beta^-_{R,(R_1,R_2,\mu),(T_1,T_2,\nu)}\right) \tag{5.4.12}$$

Similarly to the discussion under (5.4.3), this shows the right auxiliary algebras of the $SO(N)$ mesonic and $Sp(N)$ sector switch places under anti-symmetrisation.

When we restrict to $N < n$, the basis elements $\beta^\pm_{R,(R_1,R_2,\mu),(T_1,T_2,\nu)}$ with $l(R) > N$ will annihilate the relevant permutation state spaces $\mathcal{A}^{\delta/\Omega;N}_{n_1,n_2}$ and $\mathcal{A}^{\varepsilon;N}_{N;n_1,n_2}$ under right multiplication. Therefore we restrict the auxiliary algebras to $\mathcal{A}^{\pm;N}_{n_1,n_2}$ by restricting the R labels to have $l(R) \leq N$.

The normalisations of (5.4.10) and (5.4.11) are chosen so that they have the multiplication property

$$\beta^+_{R,(R_1,R_2,\mu),(T_1,T_2,\nu)} \beta^+_{S,(S_1,S_2,\lambda),(U_1,U_2,\rho)} = \delta_{RS} \; \delta_{(T_1,T_2,\nu)(S_1,S_2,\lambda)} \; \beta^+_{R,(R_1,R_2,\mu),(U_1,U_2,\rho)} \tag{5.4.13}$$

$$\beta^-_{R,(R_1,R_2,\mu),(T_1,T_2,\nu)} \beta^-_{S,(S_1,S_2,\lambda),(U_1,U_2,\rho)} = \delta_{RS} \; \delta_{(T_1,T_2,\nu)(S_1,S_2,\lambda)} \; \beta^-_{R,(R_1,R_2,\mu),(U_1,U_2,\rho)} \tag{5.4.14}$$

where we have evaluated the product using the orthogonality relation (2.3.4).

The multiplication relations (5.4.13) and (5.4.14) can be realised by matrices with rows labelled by R, R_1, R_2, μ, subject to the appropriate conditions. The matrices are block diagonal in the R label, and $\beta^\pm_{R,(R_1,R_2,\mu),(T_1,T_2,\nu)}$ has a 1 in the (R_1, R_2, μ)th row and (T_1, T_2, ν)th column of the Rth block, with zeroes everywhere else in that block and in all other blocks.

Define

$$m_R^+ = \sum_{\substack{R_1 \vdash 2n_1 \text{ with even row lengths} \\ R_2 \vdash 2n_2 \text{ with even row lengths}}} g_{R;R_1,R_2} \tag{5.4.15}$$

$$m_R^- = \sum_{\substack{R_1 \vdash 2n_1 \text{ with even column lengths} \\ R_2 \vdash 2n_2 \text{ with even column lengths}}} g_{R;R_1,R_2} \tag{5.4.16}$$

Then the Wedderburn-Artin decomposition of the algebras $\mathcal{A}^{\pm}_{n_1,n_2}$ is

$$\mathcal{A}^{+}_{n_1,n_2} = \bigoplus_{R \vdash 2n} \mathcal{M}(m_R^+) \tag{5.4.17}$$

$$\mathcal{A}^{-}_{n_1,n_2} = \bigoplus_{R \vdash 2n} \mathcal{M}(m_R^-) \tag{5.4.18}$$

where $\mathcal{M}(k)$ is the algebra of $k \times k$ matrices. Representations of $\mathcal{M}(k)$ are the same as representations of $GL(k)$, and therefore the irreducible representations of $\mathcal{A}^{\pm}_{n_1,n_2}$ are labelled by R (satisfying the appropriate conditions) and a $GL(m_R^{\pm})$ Young diagram.

5.4.3 $\mathcal{B}^{\varepsilon}_{N,q}$ and $\mathcal{B}^{\varepsilon;N}_{N,q}$

The Fourier basis for $\mathcal{B}^{\varepsilon}_{N,q}$ is

$$\beta^{\varepsilon}_R = \frac{d_R}{(2n)!} \sum_{\sigma \in S_{2n}} \Big(\big\langle [1^N] \big| \otimes \langle \bar{R}, [S] | \Big) D^R(\sigma) \Big(\big| [1^N] \big\rangle \otimes | \bar{R}, [S] \rangle \Big) \sigma \tag{5.4.19}$$

where $R \vdash 2n$ contains a copy of the $S_N \times S_{2q}$ representation $[1^N] \otimes \bar{R}$ for $\bar{R}$ a Young diagram with even length rows. In section D.2.1, a characterisation of these R is given, along with a proof that for these R, there is a unique associated $\bar{R}$ with Littewood-Richardson coefficient $g_{R;[1^N],\bar{R}} = 1$. The unit vector $\big|[1^N]\big\rangle \otimes |\bar{R}, [S]\rangle$, defined in (5.1.23), is the unique vector in R that is anti-invariant under S_N and invariant under $S_q[S_2]$.

Any basis element β^{ε}_R with $l(R) > N$ will annihilate the permutation state spaces $\mathcal{A}^{\varepsilon;N}_{N;n}$ and $\mathcal{A}^{\varepsilon;N}_{N;n_1,n_2}$. The appropriate sub-algebra is $\mathcal{B}^{\varepsilon;N}_{N,q}$, defined by

$$\mathcal{B}^{\varepsilon;N}_{N,q} = \text{Span}\left\{ \beta^{\varepsilon}_R : l(R) \leq N \right\} \tag{5.4.20}$$

Since R must include a copy of the $[1^N]$ representation of S_N, the restriction $l(R) \leq N$ implies $l(R) = N$, and the relation between R and $\bar{R}$ becomes $R = [1^N] + \bar{R}$. Therefore $\bar{R}$ could be used as a label for the restricted algebra.

The multiplication rule for (5.4.19) is

$$\beta^{\varepsilon}_R \beta^{\varepsilon}_S = \delta_{RS} \beta^{\varepsilon}_R \tag{5.4.21}$$

where we have evaluated the product using the orthogonality relation (2.3.4). This proves that both $\mathcal{B}^{\varepsilon;N}_{N,q}$ and $\mathcal{B}^{\varepsilon}_{N,q}$ are commutative algebras. They can be realised as matrices with rows labelled by R (subject to the appropriate conditions). The basis element β^{ε}_R has a 1 in the Rth diagonal entry and 0 everywhere else.

From this matrix interpretation of β^{ε}_R, the Wedderburn-Artin decomposition of $\mathcal{B}^{\varepsilon;N}_{N,q}$ is

$$\mathcal{B}^{\varepsilon;N}_{N,q} = \bigoplus_{\substack{R\vdash 2n \text{ with odd row lengths} \\ l(R)=N}} \mathbb{C} = \bigoplus_{\substack{\bar{R}\vdash 2q \text{ with even row lengths} \\ l(\bar{R})\leq N}} \mathbb{C} \tag{5.4.22}$$

Therefore the irreducible representations of $\mathcal{B}^{\varepsilon;N}_{N,q}$ are one dimensional, labelled by a Young diagram $R \vdash 2n$ (satisfying the appropriate conditions) and a complex number c. We could of course replace the R label with the equivalent $\bar{R} \vdash 2q$.

When acted on by Ω^{ε}, the β^{ε}_R form eigenvectors on both the left and right. It follows from the action of Ω^{ε} on $\left|[1^N]\right\rangle \otimes |\bar{R}, [S]\rangle$, given in (A.2.63) that

$$\Omega^{\varepsilon}\beta^{\varepsilon}_R = \beta^{\varepsilon}_R\Omega^{\varepsilon} = f^{\varepsilon}_R\beta^{\varepsilon}_R \tag{5.4.23}$$

where

$$f^{\varepsilon}_R = \prod_{\substack{b\in\text{odd} \\ \text{columns of } R}} (N + c_b) \tag{5.4.24}$$

This definition is identical to f^{δ}_R in (5.4.7), however the constraints on the allowable Young diagrams are different in the baryonic case, so we use the different notation to emphasise that these are a different class of R.

Since β^{ε}_R generate $\mathcal{B}^{\varepsilon;N}_{N,q}$, (5.4.23) implies that Ω^{ε} commutes with the entire algebra (both the unrestricted and restricted versions). Note that $f^{\varepsilon}_R = 0$ if $l(R) > N$, so Ω^{ε} maps the unrestricted algebra $\mathcal{B}^{\varepsilon}_{N,q}$ to $\mathcal{B}^{\varepsilon;N}_{N,q}$.

The projector $P_{[1^N]\otimes[S]}$, defined in (5.1.24), is invariant under the defining action (5.2.23) of $\mathcal{B}^{\varepsilon}_{N,q}$, and is therefore a member of the unrestricted algebra. It therefore commutes with Ω^{ε}

$$\Omega^{\varepsilon}P_{[1^N]\otimes[S]} = P_{[1^N]\otimes[S]}\Omega^{\varepsilon} \tag{5.4.25}$$

5.5 Correlators from permutations

In (2.6.7) and (2.6.12) we saw that in the $U(N)$ theory we can express correlators of operators purely in terms of permutations. In this section we develop analogous formulae for the mesonic and baryonic sectors of $SO(N)$ theory and the $Sp(N)$ theory. These formulae are given in (5.5.10), (5.5.27) and (5.5.22) respectively.

In each of these theories, including $U(N)$, the construction of operators and the formulae for correlators obey a common pattern.

Take $\alpha \in S_n$. Then there is a corresponding operator $\mathcal{O}_\alpha$. There are redundancies in the map $\alpha \to \mathcal{O}_\alpha$ described by the action of a permutation group G. Take $\sigma \in G$ and let the action of σ be denoted by $\sigma(\alpha)$. Then

$$\mathcal{O}_\alpha = \mathcal{O}_{\sigma(\alpha)} \qquad \alpha \in S_n \qquad \sigma \in G \tag{5.5.1}$$

In the $SO(N)$ and $Sp(N)$ theories, G is the direct product group $G_L \times G_R$ considered in section 5.3 and acts by both left and right multiplication. In the $U(N)$ theory, G is $S_{n_1} \times \cdots \times S_{n_M}$ and acts by conjugation as described in section 2.1.

Define the permutation state space $\mathcal{A}$ to be the sub-algebra of $\mathbb{C}(S_n)$ that is invariant under the action of G. Then the map from $\alpha \in \mathcal{A}$ to $\mathcal{O}_\alpha$ is redundancy-free.

There are two auxiliary algebras $\mathcal{A}^L$ and $\mathcal{A}^R$ that act naturally on $\mathcal{A}$ by multiplication on the left and right respectively. For the $SO(N)$ and $Sp(N)$ theories, $\mathcal{A}^L$ is defined to be sub-algebra of $\mathbb{C}(S_n)$ that is invariant (up to a sign) under left and right multiplication of G_L, while $\mathcal{A}^R$ is invariant (up to a sign) under left and right multiplication of G_R. In the $U(N)$ theory, $\mathcal{A}^L = \mathcal{A}^R = \mathcal{A}$.

In each theory there is a special N-dependent element $\Omega^{(G)} \in \mathbb{C}(S_n)$ that appears in correlator formulae. We have

$$\langle \mathcal{O}_\beta | \mathcal{O}_\alpha \rangle = \sum_{\sigma \in G} \delta\left(\Omega^{(G)} \sigma(\alpha) \beta^{-1}\right) \tag{5.5.2}$$

where for $\beta \in \mathbb{C}(S_{2n})$, the inverse is defined to invert each element of S_{2n} and then extend to $\mathbb{C}(S_{2n})$ linearly. We call this the linear inversion of β. Explicitly

$$\left(\sum_{\sigma \in S_{2n}} a_\sigma \sigma\right)^{-1} := \sum_{\sigma \in S_{2n}} a_\sigma \sigma^{-1} \tag{5.5.3}$$

For the $U(N)$ theory, the $\Omega^{(G)}$ element is Ω, defined in (2.3.17). For the $SO(N)$ mesonic sector and $Sp(N)$, $\Omega^{(G)} = \widetilde{\Omega}$ seen in the previous section, while for the $SO(N)$ baryonic sector $\Omega^{(G)} = \Omega^{\varepsilon}$.

Although the element $\Omega^{(G)}$ changes from theory to theory, it has similar properties in each

- The state space $\mathcal{A}$ has nice eigenvalues under left multiplication by $\Omega^{(G)}$.
- The left auxiliary algebra $\mathcal{A}^L$ commutes with $\Omega^{(G)}$ and has nice eigenvalue under left or right multiplication by $\Omega^{(G)}$.
- $\Omega^{(G)}$ enforces the finite N cut-off on Young diagrams in the algebras $\mathcal{A}$ and $\mathcal{A}^L$.
- In the leading N limit, $\Omega^{(G)}$ reduces to a multiple of the identity in S_n.
- $\Omega^{(G)}$ is constructed from Jucys-Murphy elements.

For the mathematical results behind these properties see appendix A.

In each case, expressing the two-point function in the form (5.5.2) allows us to define an alternative S_n inner product on operators by replacing $\Omega^{(G)}$ with the identity of S_n (and imposing a cut-off on Young diagrams by replacing δ with δ_N). In the $N \to \infty$ limit, the S_n inner product is the planar inner product.

In table 5.2, we give G, $\sigma(\alpha)$, Ω^G, $\mathcal{A}$, $\mathcal{A}^L$ and $\mathcal{A}^R$ for each of the $U(N)$, $SO(N)$ mesonic, $SO(N)$ baryonic and $Sp(N)$ theories in both the half and quarter-BPS sectors.

A summary of permutations and their role in $\mathcal{N}=4$ SYM with $U(N)$ gauge group was given in [62]. The structure described above generalises many of these structures to the $SO(N)$ and $Sp(N)$ gauge theories. With these techniques, it should be possible to simply extend many of the results obtained in the $U(N)$ theories to $SO(N)$ and $Sp(N)$. This applies not only to $\mathcal{N}=4$ SYM, but also to general quiver theories such as those considered in [68].

In [85] it was shown that the embedding properties of classical Lie algebras imply that the existence of a Schur basis for the half-BPS sector is a gauge group independent property. Since $\Omega^{(G)}$ acts nicely on these Schur operators, we expect there to be some relation. This is an interesting problem for future study.

5.5.1 Mesonic $SO(N)$ operators

The inner product of two $SO(N)$ matrix fields is

$$\left\langle X^{kl} | X^{ij} \right\rangle = \delta^i_k \delta^j_l - \delta^i_l \delta^j_k = \left\langle Y^{kl} | Y^{ij} \right\rangle \tag{5.5.4}$$

When extended to tensor products of X and Y using Wick contractions, this becomes

$$\left\langle \left(X^{\otimes n_1} Y^{\otimes n_2}\right)^J | \left(X^{\otimes n_1} Y^{\otimes n_2}\right)^I \right\rangle = \sum_{\sigma \in S_{n_1}[S_2] \times S_{n_2}[S_2]} (-1)^\sigma \sigma^I_J \ = 2^n n_1! n_2! \left(P_{[A]\otimes[A]}\right)^I_J \tag{5.5.5}$$

where the projector $P_{[A]\otimes[A]}$ was defined in (5.1.18).

We define

$$C^\delta(\beta) = C^{(\delta)}_I \beta^I_J C^{(\delta)\,J} \tag{5.5.6}$$

where $C^{(\delta)}_I$ is the mesonic contractor defined in (5.2.1). $C^\delta(\beta)$ is invariant under linear inversion (see (5.5.3)) of β and also under left and right multiplication of β by $S_n[S_2]$. This action of $S_n[S_2]$ on both sides was given in (5.3.9) and studied in section 5.3.1. It splits the permutations in S_{2n} into double cosets labelled by a partition $p \vdash n$. If we take σ_p to be any permutation in the p double coset, then

$$C^\delta(\sigma_p) = N^{l(p)} \tag{5.5.7}$$

	$U(N)$ half-BPS	$U(N)$ quarter-BPS	$SO(N)$ mesonic half-BPS	$SO(N)$ mesonic quarter-BPS
G	S_n	$S_{n_1} \times S_{n_2}$	$S_n[S_2] \times S_n[S_2]$	$S_n[S_2] \times (S_{n_1}[S_2] \times S_{n_2}[S_2])$
$\sigma(\alpha)$	$\sigma\alpha\sigma^{-1}$	$\sigma\alpha\sigma^{-1}$	$(-1)^{\tau}\sigma\alpha\tau^{-1}$	$(-1)^{\tau}\sigma\alpha\tau^{-1}$
$\Omega^{(G)}$	Ω	Ω	$\widetilde{\Omega}$	$\widetilde{\Omega}$
$\mathcal{A}$	centre of S_n	$\mathcal{A}_{n_1,n_2}$	$\mathcal{A}^{\delta}_{n}$	$\mathcal{A}^{\delta}_{n_1,n_2}$
$\mathcal{A}^L$	centre of S_n	$\mathcal{A}_{n_1,n_2}$	$\mathcal{A}^{+}_{n}$	$\mathcal{A}^{+}_{n}$
$\mathcal{A}^R$	centre of S_n	$\mathcal{A}_{n_1,n_2}$	$\mathcal{A}^{-}_{n}$	$\mathcal{A}^{-}_{n_1,n_2}$

	$Sp(N)$ half-BPS	$Sp(N)$ quarter-BPS
G	$S_n[S_2] \times S_n[S_2]$	$S_n[S_2] \times (S_{n_1}[S_2] \times S_{n_2}[S_2])$
$\sigma(\alpha)$	$(-1)^{\sigma}\sigma\alpha\tau^{-1}$	$(-1)^{\sigma}\sigma\alpha\tau^{-1}$
$\Omega^{(G)}$	$\widetilde{\Omega}$	$\widetilde{\Omega}$
$\mathcal{A}$	$\mathcal{A}^{\delta}_{n}$	$\mathcal{A}^{\delta}_{n_1,n_2}$
$\mathcal{A}^L$	$\mathcal{A}^{-}_{n}$	$\mathcal{A}^{-}_{n}$
$\mathcal{A}^R$	$\mathcal{A}^{+}_{n}$	$\mathcal{A}^{+}_{n_1,n_2}$

	$SO(N)$ baryonic half-BPS	$SO(N)$ baryonic quarter-BPS
G	$(S_N \times S_q[S_2]) \times S_n[S_2]$	$(S_N \times S_q[S_2]) \times (S_{n_1}[S_2] \times S_{n_2}[S_2])$
$\sigma(\alpha)$	$(-1)^{\sigma_1}(-1)^{\tau}\sigma\alpha\tau^{-1}$	$(-1)^{\sigma_1}(-1)^{\tau}\sigma\alpha\tau^{-1}$
$\Omega^{(G)}$	Ω^{ε}	Ω^{ε}
$\mathcal{A}$	$\mathcal{A}^{\varepsilon;N}_{n}$	$\mathcal{A}^{\varepsilon;N}_{n_1,n_2}$
$\mathcal{A}^L$	$\mathcal{B}^{\varepsilon;N}_{N,q}$	$\mathcal{B}^{\varepsilon;N}_{N,q}$
$\mathcal{A}^R$	$\mathcal{A}^{\varepsilon;N}_{N;n}$	$\mathcal{A}^{\varepsilon;N}_{N;n_1,n_2}$

Table 5.2: We give the theory dependent parts of the correlator formula (5.5.2) for each sector of interest in the $U(N)$, $SO(N)$ and $Sp(N)$ gauge theories. For $U(N)$, we use $\sigma \in G$, and when G is a direct product group $G = G_L \times G_R$ for the $SO(N)$ and $Sp(N)$ theories, we use $\sigma \in G_L$ and $\tau \in G_R$. For $\sigma \in S_N \times S_q[S_2]$, we define σ_1 to be the S_N component. The different $\Omega^{(G)}$ are defined in appendix A while the various permutation state spaces and algebras were introduced in section 5.2. For the non-baryonic sectors, all algebras are defined at large N. As the baryonic sector only exists at finite N, we give the finite N versions of the algebras defined in 5.2 here.

this is proved in (A.2.7).

The $\Omega^{(G)}$ relevant for $SO(N)$ mesonic operators is $\widetilde{\Omega}$, which is related to the sum of (5.5.7) over S_{2n}. Define p_σ to be the partition labelling the double coset of σ. Then

$$\sum_{\sigma \in S_{2n}} N^{l(p_\sigma)} \sigma = \sum_{\sigma \in S_{2n}} C^\delta(\sigma)\sigma = \widetilde{\Omega}\left(\sum_{\tau \in S_n[S_2]} \tau\right) = 2^n n! \widetilde{\Omega} P_{[S]} \tag{5.5.8}$$

where the projector $P_{[S]}$ is defined in (5.1.3). This result is proved in (A.2.5). Relative to the statement there, we have commuted $\widetilde{\Omega}$ past $P_{[S]}$ using (5.4.9).

Using (5.5.5), the two-point function of the operators (5.2.4) is

$$\left\langle \mathcal{O}^\delta_\beta | \mathcal{O}^\delta_\alpha \right\rangle = 2^n n_1! n_2! C^\delta\left(\alpha P_{[A]\otimes[A]} \beta^{-1}\right) \tag{5.5.9}$$

where β^{-1} is the linear inversion (5.5.3) of β.

We can re-express (5.5.9) by introducing a spurious sum over S_{2n}, using the invariance of C^δ under (linear) inversion, and substituting (5.5.8)

$$\begin{aligned}\left\langle \mathcal{O}^\delta_\beta | \mathcal{O}^\delta_\alpha \right\rangle &= \sum_{\tau \in S_{n_1}[S_2]\times S_{n_2}[S_2]} (-1)^\tau C^\delta\left(\alpha\tau^{-1}\beta^{-1}\right) \\ &= \sum_{\substack{\pi \in S_{2n} \\ \tau \in S_{n_1}[S_2]\times S_{n_2}[S_2]}} (-1)^\tau C^\delta(\pi)\, \delta\left(\pi^{-1}\alpha\tau^{-1}\beta^{-1}\right) \\ &= \sum_{\substack{\sigma \in S_n[S_2] \\ \tau \in S_{n_1}[S_2]\times S_{n_2}[S_2]}} (-1)^\tau \delta\left(\widetilde{\Omega}\sigma\alpha\tau^{-1}\beta^{-1}\right) \end{aligned} \tag{5.5.10}$$

This is the $SO(N)$ equivalent of the formula (2.6.12). In that formula, Ω imposed the finite N cut-off in the algebra $\mathbb{C}(S_n)$. Here, $\widetilde{\Omega}$ plays the same role, as explained below (A.2.19).

The leading large N behaviour of $\widetilde{\Omega}$ is

$$\widetilde{\Omega} = N^n\left[1 + O\left(\frac{1}{N}\right)\right] \tag{5.5.11}$$

So at large N, (5.5.10) reduces to N^n times the S_{2n} inner product, defined by

$$\left\langle \mathcal{O}^\delta_\beta | \mathcal{O}^\delta_\alpha \right\rangle_{S_{2n}} := \sum_{\substack{\sigma \in S_n[S_2] \\ \tau \in S_{n_1}[S_2]\times S_{n_2}[S_2]}} (-1)^\tau \delta\left(\sigma\alpha\tau^{-1}\beta^{-1}\right) \tag{5.5.12}$$

Similarly to the $U(N)$ S_n inner product (2.6.13), this can be viewed as the planar inner product for $SO(N)$ provided the coefficients of operators are N-independent. Using (5.5.12) and the properties of even double cosets as explained in section 5.3, we can evaluate the large N inner product of two $SO(N)$ multi-traces as defined in (4.1.1) and (4.1.6)

$$\langle T_p | T_q \rangle_{S_{2n}} = \delta_{pq} z_{4p} \tag{5.5.13}$$

$$\left\langle T_{\widetilde{\mathcal{P}}} | T_{\widetilde{\mathcal{Q}}} \right\rangle_{S_{2n}} = \delta_{\widetilde{\mathcal{P}}\widetilde{\mathcal{Q}}} \widetilde{Z}_{\widetilde{\mathcal{P}}} \tag{5.5.14}$$

This follows the general rule established for $U(N)$ multi-trace planar inner products in section 3.5. The size of the stabiliser of a double coset representative receives a physical interpretation as the planar inner product of two operators. From (5.5.10) we see that the finite N corrections to the planar result are controlled by $\widetilde{\Omega}$, just as the finite N corrections to the $U(N)$ correlator are controlled by Ω.

5.5.2 Symplectic operators

The two point function for symplectic matrices is

$$\left\langle X^{kl} | X^{ij} \right\rangle = \delta^i_k \delta^j_l - \Omega^i{}_l \Omega^j{}_k = \left\langle Y^{kl} | Y^{ij} \right\rangle \tag{5.5.15}$$

which is equivalent to

$$\left\langle (\Omega X)^{kl} \,|\, (\Omega X)^{ij} \right\rangle = \delta^i_k \delta^j_l + \delta^i_l \delta^j_k = \left\langle (\Omega Y)^{kl} \,|\, (\Omega Y)^{ij} \right\rangle \tag{5.5.16}$$

Using Wick contractions, we can apply this to tensor products of ΩX and ΩY

$$\begin{aligned}\left\langle \left[(\Omega X)^{\otimes n_1} (\Omega Y)^{\otimes n_2} \right]^J \,|\, \left[(\Omega X)^{\otimes n_1} (\Omega Y)^{\otimes n_2} \right]^I \right\rangle &= \sum_{\sigma \in S_{n_1}[S_2] \times S_{n_2}[S_2]} \sigma^I_J \\ &= 2^n n_1! n_2! \left(P_{[S]\otimes[S]} \right)^I_J \end{aligned} \tag{5.5.17}$$

Using the symplectic contractor $C^{(\Omega)}_I$ introduced in (5.2.3), we define

$$C^{\Omega}(\beta) = C^{(\Omega)}_I \beta^I_J C^{(\Omega)\,J} \tag{5.5.18}$$

This is invariant under (linear) inversion of β and anti-invariant under left and right multiplication of β by $S_n[S_2]$. The symplectic versions of (5.5.7) and (5.5.8) are

$$C^{\Omega}(\sigma_p) = (-1)^n (-1)^{\sigma_p} (-N)^{l(p)} \tag{5.5.19}$$

$$(-1)^n \sum_{\sigma \in S_{2n}} (-1)^{\sigma} (-N)^{l(p_\sigma)} \sigma = \sum_{\sigma \in S_{2n}} C^{\Omega}(\sigma) \sigma = \widetilde{\Omega} \left(\sum_{\tau \in S_n[S_2]} (-1)^{\tau} \tau \right) = 2^n n! \widetilde{\Omega} P_{[A]} \tag{5.5.20}$$

where the projector $P_{[A]}$ is defined in (5.1.3). These results are proved in (A.2.12) and (A.2.15) respectively. Relative to the statement in (A.2.15), we have commuted $\widetilde{\Omega}$ past $P_{[A]}$ using (5.4.9).

Comparing (5.5.19) and (5.5.20) with the $SO(N)$ equivalents (5.5.7) and (5.5.8) we see that they are related (up to a factor of $(-1)^n$) by anti-symmetrisation of σ and $N \to -N$. This is an example of the general connection (4.0.3) between gauge invariants of the $SO(N)$ and $Sp(N)$ theories.

Using (5.5.17), the two-point function of the operators (5.2.6) is

$$\left\langle \mathcal{O}^{\Omega}_{\beta} | \mathcal{O}^{\Omega}_{\alpha} \right\rangle = 2^n n_1! n_2! C^{\Omega}\left(\alpha P_{[S]\otimes[S]} \beta^{-1}\right) \tag{5.5.21}$$

where β^{-1} is the linear inversion (5.5.3) of β.

In analogy to (5.5.10), we can rearrange (5.5.21) by introducing a sum over S_{2n}, using the invariance of C^{Ω} under linear inversion, and substituting (5.5.20)

$$\left\langle \mathcal{O}^{\Omega}_{\beta} | \mathcal{O}^{\Omega}_{\alpha} \right\rangle = \sum_{\substack{\sigma \in S_n[S_2] \\ \tau \in S_{n_1}[S_2] \times S_{n_2}[S_2]}} (-1)^{\sigma} \delta\left(\widetilde{\Omega} \sigma \alpha \tau^{-1} \beta^{-1}\right) \tag{5.5.22}$$

This has the same properties as (5.5.10). $\widetilde{\Omega}$ imposes the finite N cut-off (see below (A.2.19)), and at large N, it reduces to N^n times the S_{2n} inner product, defined by

$$\left\langle \mathcal{O}^{\Omega}_{\beta} | \mathcal{O}^{\Omega}_{\alpha} \right\rangle_{S_{2n}} := \sum_{\substack{\sigma \in S_n[S_2] \\ \tau \in S_{n_1}[S_2] \times S_{n_2}[S_2]}} (-1)^{\sigma} \delta\left(\sigma \alpha \tau^{-1} \beta^{-1}\right) \tag{5.5.23}$$

As explained in section 4.1.2, the $Sp(N)$ multi-traces have exactly the same form as the $SO(N)$ traces, and their large N inner product is exactly as given in (5.5.13) and (5.5.14) for the half-BPS and quarter-BPS sectors respectively.

5.5.3 Baryonic $SO(N)$ operators

Define

$$C^{\varepsilon}(\beta) = C^{(\varepsilon)}_I \beta^I_J C^{(\varepsilon)\,J} \tag{5.5.24}$$

where $C^{(\varepsilon)}_I$ is the baryonic contractor introduced in (5.2.2). This is invariant under (linear) inversion of β and has nice transformation properties under left and right multiplication of β by $S_N \times S_q[S_2]$. Under the S_N factor $C^{\varepsilon}(\beta)$ is anti-invariant, while under the $S_q[S_2]$ factor it is invariant.

This action of $S_N \times S_q[S_2]$ on both sides was given in (5.2.23) and is the defining action for the baryonic auxiliary algebra $\mathcal{B}^{\varepsilon}_{N,q}$. It follows that if σ is in an odd double coset of this action we have $C^{\varepsilon}(\sigma) = 0$. From the definition of the baryonic contractor (5.5.24), it is simple to see that for σ in an even double coset, we have $C^{\varepsilon}(\sigma) = \pm N! N^k$

for some k. We expect that the even double cosets are characterised by a partition and k is exactly the length of this partition, giving a baryonic equivalent to (5.5.7) and (5.5.19).

For the baryonic sector, the element $\Omega^{(G)}$ is Ω^{ε}. This is related to sums of (5.5.24) over S_{2n} by

$$\sum_{\sigma\in S_{2n}} C^{\varepsilon}(\sigma)\sigma = \Omega^{\varepsilon}\left(\sum_{\sigma\in S_N\times S_q[S_2]} (-1)^{\sigma_1}\sigma\right) = N!2^q q!\Omega^{\varepsilon}P_{[1^N]\otimes[S]} \tag{5.5.25}$$

where for $\sigma \in S_N \times S_q[S_2]$, σ_1 is the S_N component and the projector $P_{[1^N]\otimes[S]}$ is defined in (5.1.24). This result is proved in (A.2.24). Relative to the statement there, we have commuted Ω^{ε} past $P_{[1^N]\otimes[S]}$ using (5.4.25).

We can use (5.5.5) to deduce the two-point function of two baryonic operators as defined in (5.2.5)

$$\left\langle \mathcal{O}^{\varepsilon}_{\beta}|\mathcal{O}^{\varepsilon}_{\alpha}\right\rangle = 2^n n_1!n_2!C^{\varepsilon}\left(\alpha P_{[A]\otimes[A]}\beta^{-1}\right) \tag{5.5.26}$$

where β^{-1} is the linear inversion (5.5.3) of β.

As in (5.5.10), we can rearrange (5.5.26) by introducing a sum over S_{2n}, using the invariance of C^{ε} under (linear) inversion and substituting (5.5.25)

$$\left\langle \mathcal{O}^{\varepsilon}_{\beta}|\mathcal{O}^{\varepsilon}_{\alpha}\right\rangle = \sum_{\substack{\sigma\in S_N\times S_q[S_2]\\ \tau\in S_{n_1}[S_2]\times S_{n_2}[S_2]}} (-1)^{\sigma_1}(-1)^{\tau}\delta\left(\Omega^{\varepsilon}\sigma\alpha\tau^{-1}\beta^{-1}\right) \tag{5.5.27}$$

This allows the definition of an S_{2n} inner product of baryonic operators

$$\left\langle \mathcal{O}^{\varepsilon}_{\beta}|\mathcal{O}^{\varepsilon}_{\alpha}\right\rangle_{S_{2n}} = \sum_{\substack{\sigma=(\sigma_1,\sigma_2)\in S_N\times S_q[S_2]\\ \tau\in S_{n_1}[S_2]\times S_{n_2}[S_2]}} (-1)^{\sigma_1}(-1)^{\tau}\delta\left(\sigma\alpha\tau^{-1}\beta^{-1}\right) \tag{5.5.28}$$

However baryonic operators are intrinsically finite N objects, so the S_{2n} inner product is not the large N limit of the physical inner product, unlike (5.5.12) and (5.5.23).

5.6 Fourier bases for permutation state spaces; the Schur and restricted Schur basis of operators; and correlators

In section 5.4 we gave Fourier bases for the various auxiliary algebras. In this section we do the same for the permutation state spaces $\mathcal{A}^{\delta/\Omega}_n$, $\mathcal{A}^{\varepsilon;N}_{N;n}$, $\mathcal{A}^{\delta/\Omega}_{n_1,n_2}$ and $\mathcal{A}^{\varepsilon;N}_{N;n_1,n_2}$. When inserted into the appropriate operator construction formulae (5.2.4), (5.2.5) and

(5.2.6), these Fourier bases lead to orthogonal Young diagram bases for the space of operators. In the half-BPS case, these are called the (mesonic, symplectic or baryonic) Schur basis, while the quarter-BPS versions are the (mesonic, symplectic or baryonic) restricted Schur basis.

The programme of finding Young diagram bases for the half and quarter-BPS sectors was first started, for the $U(N)$ theory, in [22]. This was applied to the $SO(N)$ theory in [56], which defined the Schur basis for mesonic operators and calculated their correlators. This was soon followed by [57], which introduced the symplectic and baryonic Schur bases, though did not calculate correlators for the baryonic basis. The extension to the restricted Schur basis for mesonic quarter-BPS operators first appeared in [59] including correlator results, and [60] did the equivalent for the symplectic Schur basis. The correlators for the baryonic Schur basis were found in [1], which also introduced the restricted Schur extension.

For each permutation state space, we split the discussion into four sections. We start by introducing the Fourier basis, the associated labelling set and describing the algebraic properties. This includes giving the action of the associated left and right auxiliary algebras and the corresponding decomposition as a representation of the product auxiliary algebra. We also give the eigenvalues of the basis under the left action of the appropriate one of $\widetilde{\Omega}$ and Ω^{ε}. After introducing the algebra basis we move on to the operators they construct. For the half-BPS sector we can interpret these in terms of symmetric functions of the $\frac{N}{2}$ distinct eigenvalues. The third section uses the labelling of the Fourier basis to give a combinatorial description of the size of the space. This connects to the large N generating functions found in section 4.3. For the half-BPS sector, we also have expressions for the finite N generating functions. Finally, we give the physical and S_{2n} correlators of the basis.

5.6.1 $\mathcal{A}_n^{\delta}$ and $\mathcal{A}^{\Omega}$: the mesonic and symplectic Schur bases

Algebra basis

The state spaces $\mathcal{A}_n^{\delta}$ and $\mathcal{A}_n^{\Omega}$ are defined by invariance under the group actions given in (5.2.18) and (5.2.19) respectively. From these actions, we can derive the Fourier bases

$$\alpha_R^{\delta} = \frac{1}{2^n n!}\sqrt{\frac{d_R}{(2n)!}} \sum_{\sigma\in S_{2n}} \langle R,[S]|D^R(\sigma)|R,[A]\rangle \sigma \tag{5.6.1}$$

$$\alpha_R^{\Omega} = \frac{1}{2^n n!}\sqrt{\frac{d_R}{(2n)!}} \sum_{\sigma\in S_{2n}} \langle R,[A]|D^R(\sigma)|R,[S]\rangle \sigma \tag{5.6.2}$$

where in both bases $R \vdash 2n$ is a Young digram with even length rows and columns. Note this implies n even; for n odd the spaces $\mathcal{A}_n^{\delta/\Omega}$ are zero-dimensional.

The normalisation of α_R^δ and α_R^Ω are chosen to give a nice form for correlators, given later in (5.6.23) and (5.6.24).

These two bases are closely related. Firstly, consider conjugating the Young diagram R. The behaviours of $D^R(\sigma)$, $|R,[S]\rangle$ and $|R,[A]\rangle$ under $R \to R^c$ were given in (5.1.11), (5.1.13) and (5.1.14) respectively. From these it follows that

$$\alpha_{R^c}^\delta = \text{Anti-Sym}\left(\alpha_R^\Omega\right) \tag{5.6.3}$$

So the mesonic and symplectic bases are related by anti-symmetrisation and conjugation of Young diagrams. In (5.4.3) we proved that under the same transformation the left auxiliary algebra for the mesonic operators switches with the left auxiliary algebra for symplectic operators. Since for half-BPS operators these are also the right auxiliary algebras, we see that this transformation exchanges all three spaces, the permutation state space and both auxiliary algebras, between the $SO(N)$ and $Sp(N)$ theories.

The bases are also related by interchange of the factors in the decomposition (2.5.2) of $\mathbb{C}(S_n)$, or equivalently they transform into each under linear inversion.

The left auxiliary algebra for $\mathcal{A}_n^\delta$ is $\mathcal{A}_n^+$, and the right auxiliary algebra is $\mathcal{A}_n^-$, while for $\mathcal{A}_n^\Omega$ it is the other way round. Using the Fourier bases (5.4.1) and (5.4.2) for $\mathcal{A}_n^\pm$ the actions are given by

$$\beta_S^+\alpha_R^\delta = \delta_{RS}\alpha_R^\delta \qquad\qquad \alpha_R^\delta\beta_S^- = \delta_{RS}\alpha_R^\delta \tag{5.6.4}$$

$$\beta_S^-\alpha_R^\Omega = \delta_{RS}\alpha_R^\Omega \qquad\qquad \alpha_R^\Omega\beta_S^+ = \delta_{RS}\alpha_R^\Omega \tag{5.6.5}$$

In section 5.4.1, we explained the algebra structure of $\mathcal{A}_n^\pm$ and showed that representations were labelled by a Young diagram $R \vdash 2n$ and a complex number c. For $\mathcal{A}_n^+$, R was restricted to have even row lengths, while for $\mathcal{A}_n^-$, R has even column lengths, so the R in (5.6.1) and (5.6.2) falls into both categories. Since $\mathcal{A}_n^+$ acts on the left of $\mathcal{A}_n^\delta$ and $\mathcal{A}_n^-$ acts on the right, the two actions commute, and $\mathcal{A}_n^\delta$ can be sorted into representations of the product algebra $\mathcal{A}_n^+ \times \mathcal{A}_n^-$. From (5.6.4), this decomposition is

$$V^\delta = \bigoplus_{\substack{R\vdash 2n \text{ with even} \\ \text{row and column lengths}}} V_{R,c=1}^+ \otimes V_{R,c=1}^- \tag{5.6.6}$$

Similarly, $\mathcal{A}_n^\Omega$ can be sorted into representations of $\mathcal{A}_n^- \times \mathcal{A}_n^+$

$$V^\Omega = \bigoplus_{\substack{R\vdash 2n \text{ with even} \\ \text{row and column lengths}}} V_{R,c=1}^- \otimes V_{R,c=1}^+ \tag{5.6.7}$$

In both representations there is no multiplicity space, and the R on the left matches the R on the right. As discussed in section 2.4.2 in the context of the double centraliser theorem, this shows that $\mathcal{A}_n^{\pm}$ are each others' centraliser within the endomorphism algebras of $\mathcal{A}_n^{\delta/\Omega}$.

To give a more concrete realisation of the algebraic structure of $\mathcal{A}_n^{\pm}$, in section 5.4.1 we gave a matrix interpretation of $\beta_R^{\pm}$ as matrices with rows labelled by R. We can repeat this for (5.6.6) and (5.6.7), to give a more intuitive understanding.

In this matrix picture α_R^{δ} is a the tensor product of column vector with a single 1 in the Rth row with respect to $\mathcal{A}_n^{+}$, and a row vector with a single 1 in the Rth column with respect to $\mathcal{A}_n^{-}$. For α_R^{Ω} just switch 'row' and 'column'.

The action of $\widetilde{\Omega}$ on $\alpha_R^{\delta/\Omega}$ follows from its action on the vectors $|R,[S]\rangle$ and $|R,[A]\rangle$ given in (A.2.18) and (A.2.19)

$$\begin{aligned} \widetilde{\Omega}\alpha_R^{\delta} &= f_R^{\delta}\alpha_R^{\delta} & \qquad \alpha_R^{\delta}\widetilde{\Omega} &= f_R^{\Omega}\alpha_R^{\delta} \\ \widetilde{\Omega}\alpha_R^{\Omega} &= f_R^{\Omega}\alpha_R^{\Omega} & \qquad \alpha_R^{\Omega}\widetilde{\Omega} &= f_R^{\delta}\alpha_R^{\Omega} \end{aligned} \tag{5.6.8}$$

So the eigenvalues of $\widetilde{\Omega}$ are given in terms of the Young diagram label R. The eigenvalues on the left are relevant for correlators. Since $f_R^{\delta} = f_R^{\Omega} = 0$ for $l(R) > N$, $\widetilde{\Omega}$ maps the large N state spaces $\mathcal{A}_n^{\delta/\Omega}$ to the restricted finite N versions $\mathcal{A}_n^{\delta/\Omega;N}$.

Operator basis

To construct the mesonic and symplectic Schur bases of operators we insert (5.6.1) and (5.6.2) into the contraction formulae (5.2.4) and (5.2.6) respectively

$$\mathcal{O}_R^{\delta} = \frac{1}{2^n n!}\sqrt{\frac{d_R}{(2n)!}}\sum_{\sigma\in S_{2n}} \langle R,[S]|D^R(\sigma)|R,[A]\rangle\; C_I^{(\delta)}\sigma_J^I\left(X^{\otimes n}\right)^J \tag{5.6.9}$$

$$\mathcal{O}_R^{\Omega} = \frac{1}{2^n n!}\sqrt{\frac{d_R}{(2n)!}}\sum_{\sigma\in S_{2n}} \langle R,[A]|D^R(\sigma)|R,[S]\rangle\; C_I^{(\Omega)}\sigma_J^I\left[(\Omega X)^{\otimes n}\right]^J \tag{5.6.10}$$

At finite N, those $\mathcal{O}_R^{\delta/\Omega}$ with $l(R) > N$ vanish, and the remaining $\mathcal{O}_R^{\delta/\Omega}$ form a basis for the reduced space.

When $n = O(N)$, these operators are dual to giant gravitons wrapped around 3-cycles within $AdS_5 \times \mathbb{RP}^5$.

In [57], the authors gave a different way of writing the Schur $SO(N)$ and $Sp(N)$ Schur operators in terms of the matrix X^2. Consider the $\frac{n}{2}$ times tensor product of X^2 as an operator on $V^{\otimes\frac{n}{2}}$, denoted by $\mathbb{X}^2$. Then in a completely analogous way to the X version (2.1.11), for a permutation $\sigma \in S_{\frac{n}{2}}$ of cycle type $p \vdash \frac{n}{2}$, we have

$$\mathrm{Tr}\left(\sigma\mathbb{X}^2\right) = \prod_i \left(\mathrm{Tr}X^{2i}\right)^{p_i} \tag{5.6.11}$$

Since $R \vdash 2n$ has even column lengths and row lengths, it can be 'quartered' into a partition $\frac{R}{4} \vdash \frac{n}{2}$ as described in (5.1.7). Then (5.6.9) and (5.6.10) can be rewritten as

$$\mathcal{O}_R^{\delta} = \frac{1}{\left(\frac{n}{2}\right)!} \sum_{\sigma \in S_{\frac{n}{2}}} 2^{-c(\sigma)} \chi_{\frac{R}{4}}(\sigma) \mathrm{Tr}\left(\sigma \mathbb{X}^2\right) \tag{5.6.12}$$

$$\mathcal{O}_R^{\Omega} = \frac{1}{\left(\frac{n}{2}\right)!} \sum_{\sigma \in S_{\frac{n}{2}}} 2^{-c(\sigma)} \chi_{\frac{R}{4}}(\sigma) \mathrm{Tr}\left(\sigma \mathbb{X}^2\right) \tag{5.6.13}$$

where $c(\sigma)$ is the number of cycles in σ. Note that this means that $SO(N)$ and $Sp(N)$ operators have the same expressions when written in terms of multi-traces.

In the $SO(N)$ gauge theory with N even, we can put X in the form

$$X = \begin{pmatrix} 0 & x_1 & 0 & 0 & \dots & 0 & 0 \\ -x_1 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & x_2 & \dots & 0 & 0 \\ 0 & 0 & -x_2 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 & x_{\frac{N}{2}} \\ 0 & 0 & 0 & 0 & \dots & -x_{\frac{N}{2}} & 0 \end{pmatrix} \tag{5.6.14}$$

which means

$$X^2 = \mathrm{Diag}\left(-x_1^2, -x_1^2, -x_2^2, -x_2^2, \dots, -x_{\frac{N}{2}}^2, -x_{\frac{N}{2}}^2\right) \tag{5.6.15}$$

If N is odd, then $x_{\frac{N}{2}}$ in (5.6.14) and (5.6.15) is replaced with $x_{\frac{N-1}{2}}$ and an extra 0 is added to the diagonal in both. In either case, it follows that

$$\mathrm{Tr}X^{2k} = (-1)^k 2 \sum_{i=1} x_i^{2k} = 2T_k\left(-x_1^2, -x_2^2, \dots, -x^2_{\lfloor \frac{N}{2} \rfloor}\right) \tag{5.6.16}$$

where T_k is the power-sum symmetric function defined in (2.7.7) and $\lfloor \frac{N}{2} \rfloor$ is $\frac{N}{2}$ rounded down to the closest integer. Applying this to each factor of (5.6.11)

$$\mathrm{Tr}\left(\sigma \mathbb{X}^2\right) = 2^{c(\sigma)} T_p\left(-x_1^2, -x_2^2, \dots, -x^2_{\lfloor \frac{N}{2} \rfloor}\right) \tag{5.6.17}$$

where $\sigma \in S_{\frac{n}{2}}$ is of cycle type $p \vdash \frac{n}{2}$ and T_p is the power-sum symmetric function defined in (2.7.8).

Re-expressing (5.6.12) in terms of $S_{\frac{n}{2}}$ conjugacy classes and comparing with the definition (2.7.10) of a Schur symmetric function, we have

$$\begin{aligned} \mathcal{O}_R^{\delta} &= \sum_{p \vdash \frac{n}{2}} \frac{1}{z_p} \chi_{\frac{R}{4}}(p) T_p\left(-x_1^2, -x_2^2, \dots, -x^2_{\lfloor \frac{N}{2} \rfloor}\right) \\ &= s_{\frac{R}{4}}\left(-x_1^2, -x_2^2, \dots, -x^2_{\lfloor \frac{N}{2} \rfloor}\right) \end{aligned} \tag{5.6.18}$$

So, in direct analogy to the $U(N)$ statement (2.7.11), the $SO(N)$ operators are Schur functions in the $\lfloor \frac{N}{2} \rfloor$ distinct eigenvalues of the matrix X^2.

For the $Sp(N)$ gauge theory, N is always even and we can put X in the form

$$X = \begin{pmatrix} x_1 & 0 & \dots & 0 & & & & \\ 0 & x_2 & \dots & 0 & & & & \\ \vdots & \vdots & \ddots & \vdots & & & & \\ 0 & 0 & \dots & x_{\frac{N}{2}} & & & & \\ & & & & -x_1 & 0 & \dots & 0 \\ & & & & 0 & -x_2 & \dots & 0 \\ & & & & \vdots & \vdots & \ddots & \vdots \\ & & & & 0 & 0 & \dots & -x_{\frac{N}{2}} \end{pmatrix} \tag{5.6.19}$$

which leads to a relative minus sign compared to (5.6.15) in the expression for X^2, and therefore in terms of eigenvalues

$$\mathcal{O}^{\Omega}_R = s_{\frac{R}{4}} \left(x_1^2, x_2^2, \dots, x_{\frac{N}{2}}^2 \right) \tag{5.6.20}$$

Combinatorics

The Young diagrams $\frac{R}{4} \vdash \frac{n}{2}$ are unrestricted at large N, and hence the number of Schur operators of degree n is $p\left(\frac{n}{2}\right)$ if n even and 0 if n odd. This matches the counting of multi-traces from (4.1.1), and the generating function is known [64]

$$\prod_{n=1}^{\infty} \frac{1}{1 - x^{2n}} \tag{5.6.21}$$

which matches (4.3.35) as expected.

At finite N, the cut-off is $l\left(\frac{R}{4}\right) \leq \frac{N}{2}$. The generating function for these cut-off partitions numbers is

$$\prod_{n=1}^{N} \frac{1}{1 - x^{2n}} \tag{5.6.22}$$

Correlators

The correlator of two $SO(N)$ mesonic Schur operators follows from the formula (5.5.10) for correlators of mesonic operators, orthogonality of matrix elements (2.3.4) and the action (5.6.8) of $\widetilde{\Omega}$ on $|R, [S]\rangle$

$$\left\langle \mathcal{O}^{\delta}_R | \mathcal{O}^{\delta}_S \right\rangle = \delta_{RS} f^{\delta}_R \tag{5.6.23}$$

Similarly the correlator for symplectic Schurs follows from (5.5.22), (2.3.4) and (5.6.8)

$$\langle \mathcal{O}_R^{\Omega} | \mathcal{O}_S^{\Omega} \rangle = \delta_{RS} f_R^{\Omega} \tag{5.6.24}$$

In the S_{2n} inner product both bases are orthonormal.

Consider the effect of the transformation (4.0.3) on (5.4.7). Conjugating a Young diagram R takes the odd rows of R to the odd columns of R^c, and for a box $b \in R$, the conjugated box $b^c \in R^c$ has contents $c_{b^c} = -c_b$ (see (2.3.19)). Therefore

$$\begin{aligned} f_{R^c}^{\delta}(-N) &= \prod_{\substack{b \in \text{ odd} \\ \text{columns of } R^c}} (-N + c_b) && (5.6.25) \\ &= (-1)^n \prod_{\substack{b \in \text{ odd} \\ \text{columns of } R^c}} (N - c_b) && (5.6.26) \\ &= (-1)^n \prod_{\substack{b \in \text{ odd} \\ \text{rows of } R}} (N + c_b) && (5.6.27) \\ &= (-1)^n f_R^{\Omega}(N) && (5.6.28) \end{aligned}$$

Since n is even for half-BPS operators, we have

$$\left\langle \mathcal{O}_{R^c}^{\delta} | \mathcal{O}_{S^c}^{\delta} \right\rangle \Big|_{N \to -N} = \langle \mathcal{O}_R^{\Omega} | \mathcal{O}_S^{\Omega} \rangle \tag{5.6.29}$$

This is an explicit example of the relation (4.0.3) between $SO(N)$ and $Sp(N)$ gauge invariant quantities.

As the Schur operators are Schur symmetric functions in the eigenvalues of X^2, product rules for these operators are expressed in terms of Littlewood-Richardson coefficients (D.1.4). Therefore three-point functions and higher are given in terms of these.

5.6.2 $\mathcal{A}_{N;n}^{\varepsilon;N}$: the baryonic Schur basis

Algebra basis

The permutation state space that constructs half-BPS baryonic operators is $\mathcal{A}_{N;n}^{\varepsilon;N}$. Such operators only exist when N is even, $2n \geq N$, and for the half-BPS case, when $q = n - \frac{N}{2}$ is even. $\mathcal{A}_{N;n}^{\varepsilon;N}$ is invariant under the action (5.2.24), but also requires a cut-off on Young diagrams. The Fourier basis for the unrestricted version $\mathcal{A}_{N;n}^{\varepsilon}$ is

$$\alpha_R^{\varepsilon} = \sqrt{\frac{d_R}{N!2^q q! 2^n n! (2n)!}} \sum_{\sigma \in S_{2n}} \Big(\langle [1^N] | \otimes \langle \bar{R}, [S] | \Big) D^R(\sigma) | R, [A] \rangle \sigma \tag{5.6.30}$$

where R has even column lengths and admits a representation $[1^N] \otimes \bar{R}$ of $S_N \times S_{2q}$ where $\bar{R}$ has even row lengths. The constraints this condition puts on R are given in section D.2.1. When we impose the cut-off $l(R) \leq N$, the constraints simplify, and R must be of the form $R = [1^N] + \bar{R}$. This implies $\bar{R}$ also has even column lengths, and is therefore of the same form as Young diagram used to label the mesonic Schur basis (5.6.9). This gives a simple description of the state space $\mathcal{A}^{\varepsilon;N}_{N;n}$

$$\mathcal{A}^{\varepsilon;N}_{N;n} = \text{Span}\left\{\alpha^{\varepsilon}_{R} : R = [1^N] + \bar{R} \text{ for } \bar{R} \text{ with even row and column lengths}\right\} \quad (5.6.31)$$

The auxiliary algebras $\mathcal{B}^{\varepsilon;N}_{N,q}$ and $\mathcal{A}^{-;N}_{n}$ act on the left and right of $\mathcal{A}^{\varepsilon;N}_{N;n}$ respectively. Using the Fourier bases (5.4.2) and (5.4.19), these actions are given by

$$\beta^{\varepsilon}_{S}\alpha^{\varepsilon}_{R} = \delta_{RS}\alpha^{\varepsilon}_{R} \qquad\qquad \alpha^{\varepsilon}_{R}\beta^{-}_{S} = \delta_{RS}\alpha^{\varepsilon}_{R} \quad (5.6.32)$$

The state space $\mathcal{A}^{\varepsilon;N}_{n}$ can be given as a representation of $\mathcal{B}^{\varepsilon;N}_{N,q} \times \mathcal{A}^{-;N}_{n}$. In sections 5.4.3 and 5.4.1 respectively, the representations of $\mathcal{B}^{\varepsilon;N}_{N,q}$ and $\mathcal{A}^{-;N}_{n}$ were classified. Both are labelled by a Young diagram R and a complex number c. For $\mathcal{B}^{\varepsilon;N}_{N,q}$, R is restricted to have form $R = [1^N] + \bar{R}$ for $\bar{R}$ with even row lengths and $l(\bar{R}) \leq N$, while for $\mathcal{A}^{-;N}_{n}$, R must have even column lengths and $l(R) \leq N$. Therefore the R in (5.6.30) falls into both categories. The representation is

$$V^{\varepsilon} = \bigoplus_{\substack{R \vdash 2n \text{ with odd} \\ \text{row and even column lengths} \\ l(R)=N}} V^{\varepsilon}_{R,c=1} \otimes V^{-}_{R,c=1} \quad (5.6.33)$$

The lack of a multiplicity space in this decomposition, and the matching of R between the two factors means $\mathcal{B}^{\varepsilon;N}_{N,q}$ and $\mathcal{A}^{-;N}_{n}$ are each others' centralisers within the endomorphism algebra of $\mathcal{A}^{\varepsilon}_{n}$.

In sections 5.4.3 and 5.4.1 we give a matrix interpretation of $\mathcal{B}^{\varepsilon;N}_{N,q}$ and $\mathcal{A}^{-;N}_{n}$, with rows labelled by the Young diagram R. In this picture, α^{ε}_{R} is a column vector with a 1 in the Rth row with respect to $\mathcal{B}^{\varepsilon;N}_{N,q}$ and a row vector with a 1 in the Rth column with respect to $\mathcal{A}^{-;N}_{n}$.

The action of Ω^{ε} on α^{ε}_{R} follows from its action (A.2.63) on the vector $\left|[1^N]\right\rangle \otimes |\bar{R}, [S]\rangle$

$$\Omega^{\varepsilon}\alpha^{\varepsilon}_{R} = f^{\varepsilon}_{R}\alpha^{\varepsilon}_{R} \quad (5.6.34)$$

There is also an action of $\widetilde{\Omega}$ on the right, though this is not relevant for correlators

$$\alpha^{\varepsilon}_{R}\widetilde{\Omega} = f^{\Omega}_{R}\alpha^{\varepsilon}_{R} \quad (5.6.35)$$

Since $f^{\varepsilon}_{R} = f^{\Omega}_{R} = 0$ for R with $l(R) > N$, Ω^{ε} imposes the cut-off in Young diagrams in the unrestricted space $\mathcal{A}^{\varepsilon}_{N;n}$.

Operator basis

To construct the baryonic Schur basis we insert the basis elements (5.6.30) into the baryonic contraction formula (5.2.5)

$$\mathcal{O}^{\varepsilon}_{R} = \sqrt{\frac{d_R}{N!2^q q! 2^n n!(2n)!}} \sum_{\sigma \in S_{2n}} \Big(\langle [1^N] | \otimes \langle \bar{R}, [S] | \Big) D^R(\sigma) |R, [A]\rangle \; C^{(\varepsilon)}_{I} \sigma^{I}_{J} \left(X^{\otimes n}\right)^J \tag{5.6.36}$$

If R is a single column of length N, this is dual to a giant graviton wrapped around a $\mathbb{RP}^3$ within the $\mathbb{RP}^5$ factor. More general R with $n = O(N)$ boxes is a multi-giant state, with one wrapped around a $\mathbb{RP}^3$ and the others wrapped on S^3.

We saw in (5.6.18) that the $SO(N)$ mesonic Schur basis are Schur functions in the eigenvalues of X^2. The baryonic Schur basis has an equivalent interpretation involving the Pfaffian of X. For a generic anti-symmetric matrix with N even, this is defined to be the square root of the determinant. For X of the form (5.6.14), it is simply

$$\mathrm{Pf}(X) = x_1 x_2 \dots x_{\frac{N}{2}} \tag{5.6.37}$$

This appears in baryonic operators through the identity

$$\varepsilon_{i_1 i_2 i_3 i_4 \dots i_{N-1} i_N} X^{i_1 i_2} X^{i_3 i_4} \dots X^{i_{N-1} i_N} = 2^{\frac{N}{2}} \left(\frac{N}{2}\right)! \, \mathrm{Pf}(X) \tag{5.6.38}$$

Consider a permutation $\tau \in S_q$ of cycle type p. This is embedded into S_{2q} by acting on the odd numbers, and embedded into S_{2n} by acting on the odd numbers greater than N. In both these embeddings we denote the equivalent to τ by $\tau^{(odd)}$. Then from (5.6.38) and (5.3.16)

$$\begin{aligned} C^{(\varepsilon)}_{I} \left(\tau^{(odd)}\right)^{I}_{J} \left(X^{\otimes n}\right)^J &= 2^{\frac{N}{2}} \left(\frac{N}{2}\right)! \, \mathrm{Pf}(X) C^{(\delta)}_{I} \left(\tau^{(odd)}\right)^{I}_{J} \left(X^{\otimes q}\right)^J \\ &= 2^{\frac{N}{2}} \left(\frac{N}{2}\right)! \, \mathrm{Pf}(X) \prod_i \left(\mathrm{Tr} X^i\right)^{p_i} \end{aligned} \tag{5.6.39}$$

To express this in terms of $\mathcal{O}^{R}_{\varepsilon}$, we use a similar approach to that taken in [57] to prove the equality of expressions (5.6.9) and (5.6.12). For $\alpha \in S_{2n}$, define

$$\mathcal{O}^{\varepsilon;\alpha}_{R} = \sum_{\sigma \in S_{2n}} \chi_R(\sigma) C^{(\varepsilon)}_{I} \left(\alpha\sigma\right)^{I}_{J} \left(X^{\otimes n}\right)^J \tag{5.6.40}$$

Using the resolution of the identity (2.3.6), it follows that

$$\frac{1}{(2n)!}\sum_{R\vdash 2n} d_R \mathcal{O}_R^{\varepsilon;\alpha} = C_I^{(\varepsilon)}\alpha_J^I \left(X^{\otimes n}\right)^J \tag{5.6.41}$$

We now rearrange (5.6.40) to compare it to (5.6.36)

$$\begin{aligned}
\mathcal{O}_R^{\varepsilon;\alpha} &= \sum_{\sigma\in S_{2n}} \chi_R(\alpha^{-1}\sigma) C_I^{(\varepsilon)}\sigma_J^I \left(X^{\otimes n}\right)^J \\
&= \sum_{\sigma\in S_{2n}} \chi_R(\alpha^{-1}\sigma) C_I^{(\varepsilon)} \left(P_{[1^N]\otimes[S]}\sigma P_{[A]}\right)_J^I \left(X^{\otimes n}\right)^J \\
&= \sum_{\sigma\in S_{2n}} \chi_R\left(\alpha^{-1}P_{[1^N]\otimes[S]}\sigma P_{[A]}\right) C_I^{(\varepsilon)}\sigma_J^I \left(X^{\otimes n}\right)^J
\end{aligned} \tag{5.6.42}$$

where the projectors $P_{[1^N]\otimes[S]}$ and $P_{[A]}$ are defined in (5.1.24) and (5.1.3) respectively. Using the expressions (5.1.25) and (5.1.6) for the projectors in a representation, we have

$$\begin{aligned}
\mathcal{O}_R^{\varepsilon;\alpha} = \langle R,[A]|\, D^R\left(\alpha^{-1}\right) \Big(\big|[1^N]\big\rangle \otimes |\bar{R},[S]\rangle \Big) \\
\sum_{\sigma\in S_{2n}} \Big(\big\langle[1^N]\big| \otimes \langle\bar{R},[S]| \Big) D^R(\sigma)\,|R,[A]\rangle\, C_I^{(\varepsilon)}\sigma_J^I \left(X^{\otimes n}\right)^J \\
= \Big(\big\langle[1^N]\big| \otimes \langle\bar{R},[S]| \Big) D^R(\alpha)\,|R,[A]\rangle \sqrt{\frac{N!2^q q!2^n n!(2n)!}{d_R}}\mathcal{O}_R^{\varepsilon}
\end{aligned} \tag{5.6.43}$$

The matrix element can be evaluated by decomposing R as a representation of $S_N\times S_{2q}$. We have

$$V_R^{S_{2n}} = \bigoplus_{\substack{r_N\vdash N\\ r_{2q}\vdash 2q}} V_{r_N}^{S_N}\otimes V_{r_{2q}}^{S_{2q}}\otimes V_{R;r_N,r_{2q}}^{mult} \tag{5.6.44}$$

where $V_{R;r_N,r_{2q}}^{mult}$ is the Littlewood-Richardson multiplicity space for the decomposition. The tensor product vector on the left of the matrix element means only the $r_N = [1^N]$, $r_{2q} = \bar{R}$ term will contribute. It is proved in appendix D that the multiplicity space is trivial for these values of r_N and r_{2q}.

For $\tau\in S_{2q}$ embedded into S_{2n} by acting on $\{N+1,\dots,2n\}$, the $r_N = [1^N]$, $r_{2q} = \bar{R}$ component of the decomposition of $D^R(\tau)$ is

$$D^R(\tau) \sim I_{[1^N]}\otimes D^{r_{2q}}(\tau) \tag{5.6.45}$$

where $I_{[1^N]}$ is the identity operator on $V_{[1^N]}^{S_N}$.

Similarly, the relevant component of the vector $|R,[A]\rangle$ is

$$|R,[A]\rangle \sim a_{[1^N],\bar{R}}^R \left(|[1^N]\rangle \otimes |\bar{R},[A]\rangle \right) \tag{5.6.46}$$

where $|[1^N]\rangle$ appears as this is the vector $|r_N, [A]\rangle$ anti-invariant under $S_n[S_2]$ for the representation $r_N = [1^N]$. The coefficient is

$$a^R_{[1^N],\bar{R}} = \left(\langle[1^N]| \otimes \langle\bar{R}, [A]|\right) |R, [A]\rangle = \sqrt{\chi_R \left(P_{[1^N]\otimes[A]} P_{[A]}\right)} \tag{5.6.47}$$

where for the second equality we have used the representative (5.1.6) of the projector $P_{[A]}$ defined in (5.1.3), and introduced a new projector $P_{[1^N]\otimes[A]}$ defined analogously to $P_{[1^N]\otimes[S]}$ in (5.1.24) with representative analogous to (5.1.25). We have not been able to find a simple formula for $a^R_{[1^N],\bar{R}}$.

Using the components (5.6.45) and (5.6.46), for $\tau \in S_q$ of cycle type $2p$ we can write the matrix element of (5.6.43) as

$$\begin{aligned}
&\left(\langle[1^N]| \otimes \langle\bar{R}, [S]|\right) D^R\left(\tau^{(odd)}\right) |R, [A]\rangle \\
&\quad = a^R_{[1^N],\bar{R}} \left(\langle[1^N]| \otimes \langle\bar{R}, [S]|\right) \left(I_{[1^N]} \otimes D^{\bar{R}}\left(\tau^{(odd)}\right)\right) \left(|[1^N]\rangle \otimes |\bar{R}, [A]\rangle\right) \\
&\quad = a^R_{[1^N],\bar{R}} \langle\bar{R}, [S]| D^{\bar{R}}\left(\tau^{(odd)}\right) |\bar{R}, [A]\rangle \\
&\quad = a^R_{[1^N],\bar{R}} \frac{2^{l(p)}}{2^q q!} \sqrt{\frac{(2q)!}{d_{\bar{R}}}} \chi_{\frac{\bar{R}}{4}}(p)
\end{aligned} \tag{5.6.48}$$

where we have used (5.1.10) to evaluate the final $\bar{R}$ matrix element.

Putting the pieces together, for $\sigma \in S_{\frac{n}{2}}$ of cycle type p, we have

$$\begin{aligned}
2^{\frac{N}{2}} \left(\frac{N}{2}\right)! \,\mathrm{Pf}(X) \mathrm{Tr}\left(\sigma \mathbb{X}^2\right) &= 2^{\frac{N}{2}} \left(\frac{N}{2}\right)! \,\mathrm{Pf}(X) \prod_i \left(\mathrm{Tr} X^{2i}\right)^{p_i} \\
&= \sum_{R \vdash 2n} a^R_{[1^N],\bar{R}} \sqrt{\frac{N! 2^n n! d_R (2q)!}{2^q q! (2n)! d_{\bar{R}}}} 2^{c(\sigma)} \chi_{\frac{\bar{R}}{4}}(\sigma) \mathcal{O}^{\varepsilon}_R
\end{aligned} \tag{5.6.49}$$

Inverting this relation using orthogonality relations (2.3.5) gives

$$2^{\frac{N}{2}} \left(\frac{N}{2}\right)! \,\mathrm{Pf}(X) \frac{1}{\left(\frac{q}{2}\right)!} \sum_{\sigma \in S_{\frac{q}{2}}} \chi_{\frac{\bar{R}}{4}}(\sigma) 2^{-c(\sigma)} \mathrm{Tr}(\sigma \mathbb{X}^2) = \sqrt{\frac{N! 2^n n!}{2^q q!}} \sqrt{\frac{d_R (2q)!}{(2n)! d_{\bar{R}}}} a^R_{[1^N],\bar{R}} \mathcal{O}^{\varepsilon}_R \tag{5.6.50}$$

Comparing with the mesonic Schur operators (5.6.12), we see the baryonic Schurs are, up to normalisation, the Pfaffian operator multiplied by a mesonic Schur

$$2^{\frac{N}{2}} \left(\frac{N}{2}\right)! \,\mathrm{Pf}(X) \mathcal{O}^{\delta}_{\bar{R}} = \sqrt{\frac{N! 2^n n!}{2^q q!}} \sqrt{\frac{d_R (2q)!}{(2n)! d_{\bar{R}}}} a^R_{[1^N],\bar{R}} \mathcal{O}^{\varepsilon}_R \tag{5.6.51}$$

In terms of eigenvalues, the baryonic Schur is, up to normalisation, the product of the Pfaffian (5.6.37) with the Schur symmetric function labelled by $\frac{\bar{R}}{4}$, as described in

(5.6.18).

Combinatorics

The combinatorics of the baryonic sector are determined in much the same way as the mesonic sector, with $\bar{R}$ playing the role of R from section 5.6.1. The only restriction on $\frac{\bar{R}}{4}$ is $l\left(\frac{\bar{R}}{4}\right) \leq \frac{N}{2}$, so the size of the degree n baryonic sector is the number of partitions of $\frac{q}{2}$ with length $\leq \frac{N}{2}$. The generating function for these is

$$x^{\frac{N}{2}} \prod_{n=1}^{\frac{N}{2}} \frac{1}{1-x^{2n}} \tag{5.6.52}$$

Comparing with (5.6.22), we see the generating function for the entire half-BPS sector of the $SO(N)$ gauge theory at finite even N is

$$\left(1+x^{\frac{N}{2}}\right) \prod_{n=1}^{\frac{N}{2}} \frac{1}{1-x^{2n}} \tag{5.6.53}$$

which matches the results of [57].

Correlators

The two-point function of two baryonic operators can be calculated using the formula (5.5.27) for the correlator of baryonic operators, orthogonality of matrix elements (2.3.4) and the action (5.6.34) of Ω^{ε} on $\left|\left[1^N\right]\right\rangle \otimes |\bar{R}, [S]\rangle$

$$\langle \mathcal{O}^{\varepsilon}_R | \mathcal{O}^{\varepsilon}_S \rangle = \delta_{RS} f^{\varepsilon}_R \tag{5.6.54}$$

Under the S_{2n} inner product, the $\mathcal{O}^{\varepsilon}_R$ are orthonormal.

This correlator should be reproducible by studying the stringy physics of branes wrapped around a non-trivial 3-cycle in the $\mathbb{RP}^5$ factor of $AdS_5 \times \mathbb{RP}^5$.

5.6.3 $\mathcal{A}^{\delta}_{n_1,n_2}$ and $\mathcal{A}^{\Omega}_{n_1,n_2}$: the mesonic and symplectic restricted Schur bases

Algebra basis

The state spaces $\mathcal{A}^{\delta}_{n_1,n_2}$ and $\mathcal{A}^{\Omega}_{n_1,n_2}$ are defined to be invariant under the group actions given in (5.2.7) and (5.2.8) respectively. From these actions, we can derive the Fourier bases

$$\alpha^{\delta}_{R,R_1,R_2,\lambda} = \frac{1}{2^n}\sqrt{\frac{d_R}{(2n)!n!n_1!n_2!}} \sum_{\sigma \in S_{2n}} \langle R, [S] | D^R(\sigma) | R_1, R_2, [A], \lambda \rangle \sigma \tag{5.6.55}$$

$$\alpha^{\Omega}_{R,R_1,R_2,\lambda} = \frac{1}{2^n}\sqrt{\frac{d_R}{(2n)!n!n_1!n_2!}} \sum_{\sigma\in S_{2n}} \langle R,[A]|D^R(\sigma)\,|R_1,R_2,[S],\lambda\rangle\,\sigma \qquad (5.6.56)$$

where the labelling sets and their restrictions are

$SO(N)$	$Sp(N)$	
$R\vdash 2n$ with even row lengths	$R\vdash 2n$ with even column lengths	
$R_1\vdash 2n_1$ with even column lengths	$R_1\vdash 2n_1$ with even row lengths	(5.6.57)
$R_2\vdash 2n_2$ with even column lengths	$R_2\vdash 2n_2$ with even row lengths	
$1\leq\lambda\leq g_{R;R_1R_2}$	$1\leq\lambda\leq g_{R;R_1R_2}$	

More formally, the μ,ν indices label basis vectors in the Littlewood-Richardson multiplicity space. In section D.3 we give a prescription for how to choose this basis. When $N<n$, we also impose the finite N condition $l(R)\leq N$.

Since Littlewood-Richardson coefficients are invariant under conjugation (D.0.6), at large N the labels for an $SO(N)$ operator are conjugate to the labels for an $Sp(N)$ operator. From the behaviour of $D^R(\sigma)$, $|R,[S]\rangle$ and $|R_1,R_2,[A],\lambda\rangle$ under conjugation, given in (5.1.11), (5.1.14) and (5.1.21) respectively, the two bases are related directly by conjugation of R and anti-symmetrisation

$$\alpha^{\delta}_{R^c,R_1^c,R_2^c,\lambda} = \text{Anti-Sym}\left(\alpha^{\Omega}_{R,R_1,R_2,\lambda}\right) \qquad (5.6.58)$$

This is a generalisation of the half-BPS version (5.6.3), and has the same interpretation as explained there.

The auxiliary algebras for $\mathcal{A}^{\delta}_{n_1,n_2}$ are $\mathcal{A}^{+}_{n}$ on the left and $\mathcal{A}^{-}_{n_1,n_2}$ on the right. For $\mathcal{A}^{\Omega}_{n_1,n_2}$, we have $\mathcal{A}^{-}_{n}$ on the left and $\mathcal{A}^{+}_{n_1,n_2}$ on the right. Using the Fourier bases defined in sections 5.4.1 and 5.4.2, these actions are

$$\beta^{+}_{S}\alpha^{\delta}_{R,R_1,R_2,\lambda} = \delta_{RS}\ \alpha^{\delta}_{R,R_1,R_2,\lambda} \qquad (5.6.59)$$

$$\alpha^{\delta}_{R,R_1,R_2,\lambda}\beta^{-}_{S,(S_1,S_2,\mu),(T_1,T_2,\nu)} = \delta_{RS}\ \delta_{(R_1,R_2,\lambda)(S_1,S_2,\mu)}\ \alpha^{\delta}_{R,T_1,T_2,\nu} \qquad (5.6.60)$$

$$\beta^{-}_{S}\alpha^{\Omega}_{R,R_1,R_2,\lambda} = \delta_{RS}\ \alpha^{\Omega}_{R,R_1,R_2,\lambda} \qquad (5.6.61)$$

$$\alpha^{\Omega}_{R,R_1,R_2,\lambda}\beta^{+}_{S,(S_1,S_2,\mu),(T_1,T_2,\nu)} = \delta_{RS}\ \delta_{(R_1,R_2,\lambda)(S_1,S_2,\mu)}\ \alpha^{\Omega}_{R,T_1,T_2,\nu} \qquad (5.6.62)$$

The state space $\mathcal{A}^{\delta}_{n_1,n_2}$ can be decomposed as representations of the direct product algebra $\mathcal{A}^{+}_{n}\times\mathcal{A}^{-}_{n_1,n_2}$. In sections 5.4.1 and 5.4.2, we classified the representations of $\mathcal{A}^{+}_{n}$ and $\mathcal{A}^{-}_{n_1,n_2}$. For $\mathcal{A}^{+}_{n}$, representations are labelled by a Young diagram R with even row lengths and a complex number c, while for $\mathcal{A}^{-}_{n_1,n_2}$, representations are labelled by a Young diagram R and a $GL(m^{-}_{R})$ Young digram r, where m^{-}_{R} is defined in (5.4.16) and R is restricted to admit $g_{R;R_1,R_2}>0$ for some $R_1\vdash 2n_1, R_2\vdash 2n_2$ with even column lengths. The R in (5.6.55) satisfies both conditions. From (5.6.59) and (5.6.60), the decomposition of $\mathcal{A}^{\delta}_{n_1,n_2}$ is

$$V^{\delta} = \bigoplus_{\substack{R\vdash 2n \text{ with} \\ \text{even row lengths}}} V^{+}_{R,c=1} \otimes V^{-}_{R,r=\square} \tag{5.6.63}$$

Similarly, $\mathcal{A}^{\Omega}_{n_1,n_2}$ has auxiliary algebras $\mathcal{A}^{-}_{n}$ on the left and $\mathcal{A}^{+}_{n_1,n_2}$ on the right. Representations of $\mathcal{A}^{-}_{n}$ are labelled by R with even column lengths and a complex number c, while representations of $\mathcal{A}^{+}_{n_1,n_2}$ are labelled by R (which admits $g_{R;R_1,R_2} > 0$ for some $R_1 \vdash 2n_2, R_2 \vdash n_2$ with even row lengths) and a $GL(m^{+}_{R})$ Young digram r, where m^{+}_{R} is defined in (5.4.15). Then as a $\mathcal{A}^{-}_{n} \times \mathcal{A}^{+}_{n_1,n_2}$ representation, $\mathcal{A}^{\Omega}_{n_1,n_2}$ is

$$V^{\Omega} = \bigoplus_{\substack{R\vdash 2n \text{ with} \\ \text{even column lengths}}} V^{-}_{R,c=1} \otimes V^{+}_{R,r=\square} \tag{5.6.64}$$

In both representations there is no multiplicity space, and the Rs on either side match, so $\mathcal{A}^{\pm}_{n}$ and $\mathcal{A}^{\mp}_{n_1,n_2}$ are each other's centraliser within the endomorphism algebras of $\mathcal{A}^{\delta/\Omega}_{n_1,n_2}$.

We gave a matrix interpretation of $\mathcal{A}^{\pm}_{n}$ and $\mathcal{A}^{\pm}_{n_1,n_2}$ in sections 5.4.1 and 5.4.2. In this picture $\alpha^{\delta}_{R,R_1,R_2,\lambda}$ is a column vector with a single 1 in the Rth row with respect to $\mathcal{A}^{+}_{n}$ and a row vector with a single 1 in the (R_1, R_2, λ)th column of the R block with respect to $\mathcal{A}^{-}_{n_1,n_2}$. Similarly, $\alpha^{\Omega}_{R,R_1,R_2,\lambda}$ is a column vector with a single 1 in the Rth row with respect to $\mathcal{A}^{-}_{n}$ and a row vector with a single 1 in the (R_1, R_2, λ)th column of the R block with respect to $\mathcal{A}^{+}_{n_1,n_2}$.

There is one more interesting multiplication property of $\mathcal{A}^{\delta}_{n_1,n_2}$ and $\mathcal{A}^{\Omega}_{n_1,n_2}$. When multiplied on the right by the basis (3.6.3) of the $U(N)$ algebra $\mathcal{A}_{2n_1,2n_2}$, we have

$$\alpha^{\delta}_{R,R_1,R_2,\lambda}\beta^{U(N)}_{S,S_1,S_2,\mu,\nu} = \delta_{RS}\delta_{R_1S_1}\delta_{R_2S_2}\delta_{\mu\lambda}\alpha^{\delta}_{R,R_1,R_2,\nu} \tag{5.6.65}$$
$$\alpha^{\Omega}_{R,R_1,R_2,\lambda}\beta^{U(N)}_{S,S_1,S_2,\mu,\nu} = \delta_{RS}\delta_{R_1S_1}\delta_{R_2S_2}\delta_{\mu\lambda}\alpha^{\Omega}_{R,R_1,R_2,\nu} \tag{5.6.66}$$

So $\mathcal{A}^{\delta}_{n_1,n_2}$ forms a representation over $\mathcal{A}_{2n_1,2n_2}$. In section 3.6.1 we give a description of the representations of $\mathcal{A}_{2n_1,2n_2}$. They are labelled by a triple R, R_1, R_2 with $g_{R;R_1,R_2} > 0$ and a $GL(g_{R;R_1,R_2})$ Young diagram r. Using this labelling, $\mathcal{A}^{\delta}_{n_1,n_2}$ and $\mathcal{A}^{\Omega}_{n_1,n_2}$ are

$$V^{\delta} = \bigoplus_{\substack{R\vdash 2n \text{ with even row lengths} \\ R_1\vdash 2n_1 \text{ with even column lengths} \\ R_2\vdash 2n_2 \text{ with even column lengths}}} V_{(R,R_1,R_2),r=\square} \tag{5.6.67}$$

$$V^{\Omega} = \bigoplus_{\substack{R\vdash 2n \text{ with even column lengths} \\ R_1\vdash 2n_1 \text{ with even row lengths} \\ R_2\vdash 2n_2 \text{ with even row lengths}}} V_{(R,R_1,R_2),r=\square} \tag{5.6.68}$$

Thinking of the $U(N)$ elements as block diagonal matrices as explained in section 3.6.1,

$\alpha^{\delta}_{R,R_1,R_2,\lambda}$ form row vectors with zero entries in all blocks except that corresponding to (R, R_1, R_2), in which it has a single 1 at the λth position.

This gives a nice interpretation of the form of the $SO(N)$ and $Sp(N)$ counting formulae (5.6.74) and (5.6.75) and their $U(N)$ equivalent (3.6.6). The $U(N)$ counting contains squares of Littlewood-Richardson coefficients because $\mathcal{A}_{n_1,n_2}$ lives in the adjoint representation of $\mathcal{A}_{n_1,n_2}$. The $SO(N)$ and $Sp(N)$ counting contains Littlewood-Richardson coefficients to the first power because they lie in the fundamental representation (of a subset of the blocks) of $\mathcal{A}_{n_1,n_2}$.

The action of $\widetilde{\Omega}$ on $\mathcal{A}^{\delta/\Omega}_{n_1,n_2}$ follows from its action on $|R,[S]\rangle$ and $|R,[A]\rangle$, given in (A.2.18) and (A.2.19)

$$\widetilde{\Omega}\alpha^{\delta}_{R,R_1,R_2,\lambda} = f^{\delta}_R \alpha^{\delta}_{R,R_1,R_2,\lambda} \qquad \widetilde{\Omega}\alpha^{\Omega}_{R,R_1,R_2,\lambda} = f^{\Omega}_R \alpha^{\Omega}_{R,R_1,R_2,\lambda} \tag{5.6.69}$$

Unlike the half-BPS sector equivalent (5.6.8), in the quarter-BPS sector $\widetilde{\Omega}$ only has definite eigenvalues when acting on $\mathcal{A}^{\delta}_{n_1,n_2}$ and $\mathcal{A}^{\Omega}_{n_1,n_2}$ on the left. Since $f^{\delta}_R = f^{\Omega}_R = 0$ if $l(R) > N$, it still enforces the finite N cut-off.

Operator basis

The restricted Schur basis operators for $SO(N)$ and $Sp(N)$ are constructed by inserting (5.6.55) and (5.6.56) into the contraction formulae (5.2.4) and (5.2.6) respectively

$$\mathcal{O}^{\delta}_{R,R_1,R_2,\lambda} = \frac{1}{2^n}\sqrt{\frac{d_R}{(2n)!n!n_1!n_2!}} \sum_{\sigma\in S_{2n}} \langle R,[S]|D^R(\sigma)|R_1,R_2,[A],\lambda\rangle\ C^{(\delta)}_I \sigma^I_J \left(X^{\otimes n_1} Y^{\otimes n_2}\right)^J \tag{5.6.70}$$

$$\mathcal{O}^{\Omega}_{R,R_1,R_2,\lambda} = \frac{1}{2^n}\sqrt{\frac{d_R}{(2n)!n!n_1!n_2!}} \sum_{\sigma\in S_{2n}} \langle R,[A]|D^R(\sigma)\,|R_1,R_2,[S],\lambda\rangle\, C^{(\Omega)}_I \sigma^I_J \left[(\Omega X)^{\otimes n_1} (\Omega Y)^{\otimes n_2}\right]^J \tag{5.6.71}$$

In the half-BPS sector, the Schur bases for $SO(N)$ and $Sp(N)$ have the same expressions in terms of multi-traces. This is not true in the quarter-BPS. Instead, it follows from (5.6.58) that restricted Schurs with conjugate labels are anti-symmetrisations of each other. Define the anti-symmetrisation operator on a single trace by

$$\text{Anti-Sym}\,(\text{Tr}W) = (-1)^{l(W)+1}\text{Tr}W \tag{5.6.72}$$

for a matrix word W. The definition is extended in the obvious way to multi-traces and linear combinations thereof. This is directly analogous to the operator Anti-Sym on permutations. Then

$$\mathcal{O}^{\delta}_{R^c,R_1^c,R_2^c,\lambda} = \text{Anti-Sym}\left(\mathcal{O}^{\Omega}_{R,R_1,R_2,\lambda}\right) \tag{5.6.73}$$

$SO(N)$ mesonic and $Sp(N)$ Schur operators in the half-BPS sector could be expressed as Schur symmetric functions in the distinct eigenvalues of X^2. In the quarter-BPS, the matrices X and Y cannot be simultaneously diagonalised, so there is no equivalent expression for the restricted Schur operators in terms of the two sets of eigenvalues.

Combinatorics

From the labelling sets for the restricted Schur operators, the number of field content (n_1, n_2) operators in the two theories is

$$N^{(\delta;N)}_{n_1,n_2} = \sum_{\substack{R\vdash 2n \text{ with even row lengths}\\ R_1\vdash 2n_1 \text{ with even column lengths}\\ R_2\vdash 2n_2 \text{ with even column lengths}\\ l(R)\leq N}} g_{R;R_1,R_2} \tag{5.6.74}$$

$$N^{(\Omega;N)}_{n_1,n_2} = \sum_{\substack{R\vdash 2n \text{ with even column lengths}\\ R_1\vdash 2n_1 \text{ with even row lengths}\\ R_2\vdash 2n_2 \text{ with even row lengths}\\ l(R)\leq N}} g_{R;R_1,R_2} \tag{5.6.75}$$

It follows from the invariance of Littlewood-Richardson coefficients under conjugation (D.0.6) that for large N the combinatorics for $SO(N)$ mesonic operators and $Sp(N)$ operators are the same. In fact, since baryonic operators do not exist at large N, this is the combinatorics of the entire quarter-BPS sector for both gauge groups. The large N generating function for both $N^{(\delta;\infty)}_{n_1,n_2}$ and $N^{(\Omega;\infty)}_{n_1,n_2}$ is given in (4.3.21). We prove this directly from the formula (5.6.74) in appendix E.

The combinatorics (5.6.74) and (5.6.75) have also been derived from group integrals in [59] for $SO(N)$ and [60] for $Sp(N)$.

Correlators

The correlator of two restricted Schur operators can be calculated from the formula (5.5.10) for the correlator of two mesonic operators, orthogonality of matrix elements (2.3.4) and the action (A.2.18) of $\widetilde{\Omega}$ on $|R,[S]\rangle$

$$\left\langle \mathcal{O}^{\delta}_{R,R_1,R_2,\lambda_1} | \mathcal{O}^{\delta}_{S,S_1,S_2,\lambda_2} \right\rangle = \delta_{RS}\delta_{R_1S_1}\delta_{R_2S_2}\delta_{\lambda_1\lambda_2} f^{\delta}_{R} \tag{5.6.76}$$

while the symplectic equivalent follows from (5.5.22), (2.3.4) and (A.2.19)

$$\langle \mathcal{O}^{\Omega}_{R,R_1,R_2,\lambda_1} | \mathcal{O}^{\Omega}_{S,S_1,S_2,\lambda_2} \rangle = \delta_{RS}\delta_{R_1S_1}\delta_{R_2S_2}\delta_{\lambda_1\lambda_2} f^{\Omega}_{R} \tag{5.6.77}$$

Under the S_{2n} inner product, both bases are orthonormal.

It follows from (5.6.28) that the $SO(N)$ and $Sp(N)$ operators with conjugate labels have norms related by

$$\left\langle \mathcal{O}^{\delta}_{R^c,R_1^c,R_2^c,\lambda_1} | \mathcal{O}^{\delta}_{S^c,S_1^c,S_2^c,\lambda_2} \right\rangle \Big|_{N\to -N} = (-1)^n \left\langle \mathcal{O}^{\Omega}_{R,R_1,R_2,\lambda_1} | \mathcal{O}^{\Omega}_{S,S_1,S_2,\lambda_2} \right\rangle \tag{5.6.78}$$

This is another example of the relation (4.0.3) between mesonic and symplectic gauge invariants.

More general correlation functions can be calculated using product rules for the restricted Schurs [60].

5.6.4 $\mathcal{A}^{\varepsilon;N}_{n_1,n_2}$: the baryonic restricted Schur basis

Algebra basis

The permutation state space that constructs quarter-BPS baryonic operators is $\mathcal{A}^{\varepsilon;N}_{n_1,n_2}$. Such operators only exist when N is even and $2n \geq N$. Unlike the half-BPS case, the quarter-BPS sector does admit operators when $q = n - \frac{N}{2}$ is odd. $\mathcal{A}^{\varepsilon;N}_{N;n_1,n_2}$ is invariant under the action (5.2.22), but also requires a cut-off on Young diagrams. The Fourier basis for the unrestricted version $\mathcal{A}^{\varepsilon}_{N;n_1,n_2}$ is

$$\alpha^{\varepsilon}_{R,R_1,R_2,\lambda} = \sqrt{\frac{d_R}{N!2^q q!2^n n_1!n_2!(2n)!}} \sum_{\sigma \in S_{2n}} \Big(\langle [1^N] \,| \otimes \langle \bar{R}, [S]| \Big) D^R(\sigma) |R_1, R_2, [A], \lambda\rangle \sigma \tag{5.6.79}$$

where $R \vdash 2n$ has two types of Littlewood-Richardson decompositions. Firstly, it admits a representation $R_1 \otimes R_2$ of $S_{2n_1} \times S_{2n_2}$ where R_1 and R_2 have even columns lengths. λ is a Littlewood-Richardson index for this decomposition. Secondly, it contains a representation $[1^N] \otimes \bar{R}$ of $S_N \times S_{2q}$ where $\bar{R}$ has even row lengths. The conditions for such an R are complex and are given in section D.2.1. When we impose the cut-off $l(R) \leq N$, they simplify considerably to $R = [1^N] + \bar{R}$. This allows a simple description of the state space $\mathcal{A}^{\varepsilon;N}_{N;n_1,n_2}$

$$\mathcal{A}^{\varepsilon;N}_{N;n_1,n_2} = \text{Span}\left\{ \alpha^{\varepsilon}_{R,R_1,R_2,\lambda} : R = [1^N] + \bar{R} \text{ for } \bar{R} \text{ with even row lengths}, 1 \leq \lambda \leq g_{R;R_1,R_2} \right\} \tag{5.6.80}$$

The auxiliary algebras for $\mathcal{A}^{\varepsilon;N}_{N;n_1,n_2}$ are $\mathcal{B}^{\varepsilon;N}_{N,q}$ on the left and $\mathcal{A}^{-}_{n_1,n_2}$ on the right. Using the Fourier bases defined in (5.4.19) and (5.4.11) respectively their actions are

$$\beta^{\varepsilon}_S \alpha^{\varepsilon}_{R,R_1,R_2,\lambda} = \delta_{RS} \alpha^{\varepsilon}_{R,R_1,R_2,\lambda} \tag{5.6.81}$$

$$\alpha^{\varepsilon}_{R,R_1,R_2,\lambda} \beta^{-}_{S,(S_1,S_2,\mu),(T_1,T_2,\nu)} = \delta_{RS}\ \delta_{(R_1,R_2,\lambda)(S_1,S_2,\mu)}\ \alpha^{\varepsilon}_{T,T_1,T_2,\nu} \tag{5.6.82}$$

Under these actions, $\mathcal{A}^{\varepsilon;N}_{N;n_1,n_2}$ can be decomposed as representations of the product algebra $\mathcal{B}^{\varepsilon;N}_{N,q} \times \mathcal{A}^{-;N}_{n_1,n_2}$. The irreducible representations of the two auxiliary algebras were classified in sections 5.4.3 and 5.4.2. For $\mathcal{B}^{\varepsilon;N}_{N,q}$, a representation is labelled by a complex number c and a Young diagram R of form $R = [1^N] + \bar{R}$ where $\bar{R}$ with even row lengths and $l(\bar{R}) \leq N$, while for $\mathcal{A}^{-;N}_{n_1,n_2}$, representations are labelled by a Young diagram $R \vdash 2n$ with $l(R) \leq N$ and a $GL(m^-_R)$ Young diagram r, where m^-_R is defined in (5.4.16) and R is restricted to admit $g_{R;R_1,R_2} > 0$ for some $R_1 \vdash 2n_1, R_2 \vdash n_2$ with even column lengths. The R in (5.6.79) satisfies both constraints. From (5.6.81) and (5.6.82), the decomposition of $\mathcal{A}^{\varepsilon;N}_{N;n_1,n_2}$ is

$$V^\varepsilon = \bigoplus_{\substack{R \vdash 2n \text{ with} \\ \text{odd row lengths} \\ l(R)=N}} V^\varepsilon_{R,c=1} \otimes V^-_{R,r=\square} \tag{5.6.83}$$

In both representations there is no multiplicity space and the R on either side match, so $\mathcal{B}^{\varepsilon;N}_{N,q}$ and $\mathcal{A}^{-;N}_{n_1,n_2}$ are each other's centraliser within the endomorphism algebra of $\mathcal{A}^{\varepsilon;N}_{N;n_1,n_2}$.

In sections 5.4.3 and 5.4.2 we gave a matrix interpretation of the Wedderburn-Artin decompositions of $\mathcal{B}^{\varepsilon;N}_{N,q}$ and $\mathcal{A}^{-;N}_{n-1,n_2}$. Using this language, $\alpha^\varepsilon_{R,R_1,R_2,\lambda}$ is a column vector with a single 1 in the Rth row with respect to $\mathcal{B}^{\varepsilon;N}_{N,q}$ and a row vector with a single 1 in the (R_1, R_2, λ)th column of the R block with respect to $\mathcal{A}^{-;N}_{n_1,n_2}$.

We can also give a $\mathcal{A}^{\varepsilon;N}_{n_1,n_2}$ as a representation of the $U(N)$ algebra $\mathcal{A}_{2n_1,2n_2}$ in a similar manner to (5.6.67) and (5.6.68). There is no mathematical difference between the baryonic case and the mesonic and symplectic version already considered, so we will not spell it out explicitly here.

The action of Ω^ε on $\alpha^\varepsilon_{R,R_1,R_2,\lambda}$ follows from its action (A.2.63) on the vector $\left|\left[1^N\right]\right\rangle \otimes |\bar{R},[S]\rangle$

$$\Omega^\varepsilon \alpha^\varepsilon_{R,R_1,R_2,\lambda} = f^\varepsilon_R \alpha^\varepsilon_{R,R_1,R_2,\lambda} \tag{5.6.84}$$

As $f^\varepsilon_R = 0$ for R with $l(R) > N$, Ω^ε enforces the cut-off in Young diagrams in the unrestricted space $\mathcal{A}^{\varepsilon;N}_{N;n_1,n_2}$.

Operator basis

To construct the baryonic Schur basis we insert the basis elements (5.6.79) into the baryonic contraction formula (5.2.5)

$$\mathcal{O}^\varepsilon_{R,R_1,R_2,\lambda} = \sqrt{\frac{d_R}{N!2^q q! 2^n n_1! n_2! (2n)!}} \sum_{\sigma \in S_{2n}} \left(\left\langle \left[1^N\right] \right| \otimes \langle \bar{R},[S]| \right) D^R(\sigma) |R_1, R_2, [A], \lambda\rangle C^{(\varepsilon)}_I \sigma^I_J \left(X^{\otimes n_1} Y^{\otimes n_2}\right)^J \tag{5.6.85}$$

Combinatorics

From the labelling of the operators (5.6.85), the dimension of the degree (n_1, n_2) space of baryonic operators is

$$N^{\varepsilon;N}_{n_1,n_2} = \sum_{\substack{R \vdash 2n \text{ with odd row lengths} \\ R_1 \vdash 2n_1 \text{ with even column lengths} \\ R_2 \vdash 2n_2 \text{ with even column lengths} \\ l(R)=N}} g_{R;R_1,R_2} \tag{5.6.86}$$

Correlators

The two-point function of two baryonic operators can be calculated using the baryonic correlator formula (5.5.27), the orthogonality of matrix elements (2.3.4) and the action (5.6.84) of Ω^ε on $\left|[1^N]\right\rangle \otimes |\bar{R}, [S]\rangle$

$$\left\langle \mathcal{O}^\varepsilon_{R,R_1,R_2,\lambda_1} | \mathcal{O}^\varepsilon_{S,S_1,S_2,\lambda_2} \right\rangle = \delta_{RS}\delta_{R_1S_1}\delta_{R_2S_2}\delta_{\lambda_1\lambda_2} f^\varepsilon_R \tag{5.6.87}$$

Under the S_{2n} inner product, the $\mathcal{O}^\varepsilon_{R,R_1,R_2,\lambda}$ are orthonormal.

5.7 Covariant bases

For each of the mesonic, symplectic and baryonic sectors, we can define $U(2)$ covariant bases in much the same way as we did for the $U(N)$ theory in section 3.6.2. The $U(N)$ covariant basis has been used, first in [63], and subsequently in chapter 7 of this thesis, to construct quarter-BPS operators at weak coupling. This is the first construction of the $SO(N)$ and $Sp(N)$ equivalents, and we expect that they could be used in a similar way.

In this section, we present the key concepts necessary for the construction of the bases, give a formula for the operators, and develop their combinatorics and correlators. The detailed mathematical work involved in the construction of the mesonic operators and the calculation of their correlators is given in appendix F. The symplectic and baryonic basis are mathematically very similar to the mesonic version, so we are more schematic for these two, and leave out some of the details both here and in appendix F.

When introducing the $U(N)$ covariant basis in section 3.6.2, we described some basic $U(2)$ representation theory. In this section we use the same notation.

Define $X_1 = X$ and $X_2 = Y$. Then the i index in X_i is in the fundamental representation of $U(2)$, as described in section 3.6.2. Similarly to that section, consider $V_2^{\otimes n}$, where V_2 is the fundamental of $U(2)$, and in particular the basis vector $a = e_{a_1} \otimes e_{a_2} \otimes \cdots \otimes e_{a_n}$ of $V_2^{\otimes n}$ where $a_j \in \{1, 2\}$ for each j. Then we define $\mathbb{X}_a = X_{a_1} \otimes X_{a_2} \otimes \cdots \otimes X_{a_n}$.

5.7.1 $SO(N)$ mesonic basis

Operator basis

Combining $\mathbb{X}_a$ with a permutation $\sigma \in S_{2n}$, we define

$$\mathcal{O}^{\delta}_{a,\sigma} = C^{(\delta)}_{I} \sigma^{I}_{J} (\mathbb{X}_a)^{J} \tag{5.7.1}$$

In appendix F we discuss this definition, determine the redundancies in labelling operators by a and σ and derive a different labelling set that removes these redundancies. These operators are

$$\mathcal{O}^{\delta}_{\Lambda,M_\Lambda,R,\mu} = \frac{1}{2^n n!}\sqrt{\frac{d_R}{d_\Lambda (2n)!}} \sum_{a,\sigma,J,k} C^{a}_{\Lambda,M_\Lambda,k} \left[\langle R,[S]|\, D^R(\sigma)\right]_J B^{S_{2n}\rightarrow S_n[S_2]\ ;J}_{R\ \ \rightarrow(\phi,\Lambda),\mu;k} \mathcal{O}^{\delta}_{a,\sigma} \tag{5.7.2}$$

where the labels are $R \vdash 2n$ with even column lengths satisfying $l(R) \leq N$, $\Lambda \vdash n$ with $l(\Lambda) \leq 2$, M_Λ a basis index for the Λ $U(2)$ representation and μ a multiplicity index.

The coefficients involved in the definition are:

- $C^{a}_{\Lambda,M_\Lambda,k}$ is a Clebsch-Gordon coefficient for the Schur-Weyl decomposition (2.4.3) of $V_2^{\otimes n}$. k is an index for the Λ representation of S_n.
- $\left[\langle R,[S]|\, D^R(\sigma)\right]_J$ is the representative of σ in the R representation of S_{2n} multiplied on the left by the $S_n[S_2]$-invariant vector $|R,[S]\rangle$. J is a basis index for R.
- $B^{S_{2n}\rightarrow S_n[S_2];J}_{R\rightarrow(\phi,\Lambda);k,\mu}$ is the branching coefficient taking the R representation of S_{2n} to the (ϕ,Λ) representation of $S_n[S_2]$ (see section F.1 for a description of $S_n[S_2]$ representation theory). μ is a multiplicity index for the $S_{2n} \rightarrow S_n[S_2]$ decomposition.

For more properties of these coefficients and an explanation of their appearance in (5.7.2), see appendix F.

Combinatorics

In [86] the authors give an expression for the multiplicity $\mathcal{M}^{\delta}_{R,\Lambda}$ of a representation (ϕ,Λ) of $S_n[S_2]$ when reduced from a representation R of S_{2n}. They give two formulae, the first in terms of terms of the *plethysm* of the symmetric functions s_Λ and the elementary symmetric function e_2. The plethsym of two Schur functions is described in the introduction to chapter 6, but we will not define the more general case for generic symmetric functions. The second formula is in terms of Littlewood-Richardson coefficients. For $\Lambda = [\Lambda_1, \Lambda_2]$, we have

$$\mathcal{M}^{\delta}_{R,\Lambda} = \sum_{\substack{R_1 \vdash 2\Lambda_1\,,\, R_2 \vdash 2\Lambda_2 \\ \text{with even column lengths}}} g_{R;R_1,R_2} - \sum_{\substack{R_1 \vdash 2(\Lambda_1+1)\,,\, R_2 \vdash 2(\Lambda_2-1) \\ \text{with even column lengths}}} g_{R;R_1,R_2} \tag{5.7.3}$$

These are exactly the Littlewood-Richardson coefficients found in the multiplicity (5.6.74) of the restricted Schur basis. Indeed, the first term in (5.7.3) is the number of linearly independent mesonic quarter-BPS operators with field content (Λ_1, Λ_2). The second term is the number of operators with field content $(\Lambda_1 + 1, \Lambda_2 - 1)$, so by subtracting those we remove the operators which are $U(2)$ descendants and are left only with those that are highest weight states of a $\Lambda = [\Lambda_1, \Lambda_2]$ representation.

Correlators

Consider the two point function of two $U(2)$ covariant tensor operators $\mathbb{X}_a$. We can use Wick contractions along with the two-point function (5.5.4) for the matrices X and Y to find the correlator of $\mathbb{X}_a$

$$\left\langle (\mathbb{X}_b)^J \,|\, (\mathbb{X}_a)^I \right\rangle = \sum_{\sigma \in S_n[S_2]} \delta_{a,\bar{\sigma}(b)} (-1)^{\sigma} \left(\sigma^{-1}\right)^I_J \tag{5.7.4}$$

where for $\sigma \in S_n[S_2]$, $\bar{\sigma} \in S_n$ is defined as the S_n component of σ from the semi-direct product $S_n[S_2] = S_n \ltimes (S_2)^n$.

In (5.5.10) we derived a formula for $SO(N)$ mesonic correlators involving the element $\widetilde{\Omega}$. There is an analogous formula for covariant mesonic operators. Using the expression (5.5.8) giving $\widetilde{\Omega}$ as a sum of the contraction C^{δ}, the correlator of the (5.7.1) operators is

$$\begin{aligned} \left\langle \mathcal{O}^{\delta}_{b,\tau} | \mathcal{O}^{\delta}_{a,\sigma} \right\rangle &= \sum_{\gamma \in S_n[S_2]} \delta_{a,\bar{\gamma}(b)} (-1)^{\gamma} C^{\delta}(\sigma\gamma^{-1}\tau^{-1}) \\ &= \sum_{\gamma,\pi \in S_n[S_2]} \delta_{a,\bar{\gamma}(b)} (-1)^{\gamma} \delta\left(\widetilde{\Omega}\pi\sigma\gamma^{-1}\tau^{-1}\right) \end{aligned} \tag{5.7.5}$$

In appendix F.3 we use (5.7.5) to evaluate the correlator of the covariant basis operators (5.7.2). We find

$$\left\langle \mathcal{O}^{\delta}_{\Gamma,M_{\Gamma},S,\nu} | \mathcal{O}^{\delta}_{\Lambda,M_{\Lambda},R,\mu} \right\rangle = \delta_{\Lambda\Gamma}\delta_{M_{\Lambda}M_{\Gamma}}\delta_{RS}\delta_{\mu\nu} f^{\delta}_R \tag{5.7.6}$$

In the S_{2n} inner product, the operators are orthonormal.

5.7.2 Symplectic basis

Operator basis

The matrix combination with definite symmetry in the symplectic theory is ΩX_i rather than X_i, so define

$$(\Omega\mathbb{X}_a)^I = (\Omega X_{a_1})^{i_1 i_2}(\Omega X_{a_2})^{i_3 i_4}\dots(\Omega X_{a_n})^{i_{2n-1}i_{2n}} \tag{5.7.7}$$

Then a generic quarter-BPS symplectic operator can be written

$$\mathcal{O}^{\Omega}_{a,\sigma} = C^{(\Omega)}_I \sigma^I_J (\Omega\mathbb{X}_a)^J \tag{5.7.8}$$

The covariant basis is

$$\mathcal{O}^{\Omega}_{\Lambda,M_\Lambda,R,\mu} = \frac{1}{2^n n!}\sqrt{\frac{d_R}{d_\Lambda(2n)!}}\sum_{a,\sigma,J,k} C^a_{\Lambda,M_\Lambda,k}\left[\langle R,[A]|\, D^R(\sigma)\right]_J B^{S_{2n}\to S_n[S_2]\ ;J}_{R\ \ \to(\Lambda,\phi),\mu;k}\,\mathcal{O}^{\Omega}_{a,\sigma} \tag{5.7.9}$$

where $R\vdash 2n$ has even row lengths and satisfies $l(R)\le N$, $\Lambda\vdash n$ with $l(\Lambda)\le 2$, M_Λ is a basis index for the $U(2)$ representation Λ and μ is a multiplicity index, this time for the reduction of R to the (Λ,ϕ) representation of $S_n[S_2]$ (compared with (ϕ,Λ) in (5.7.2)).

As with the Schur and restricted Schur bases, the mesonic and symplectic bases are related by conjugation and anti-symmetrisation. Define $E^{a,\sigma}_{\Lambda,M_\Lambda,R,\mu}$ to be the coefficient of $\mathcal{O}^{\delta}_{a,\sigma}$ in (5.7.2) and $F^{a,\sigma}_{\Lambda,M_\Lambda,R,\mu}$ to be the coefficient of $\mathcal{O}^{\Omega}_{a,\sigma}$ in (5.7.9). Then it follows from the behaviour under conjugation of the representative $D^R(\sigma)$, the vector $|R,[A]\rangle$ and the branching coefficient, given in (5.1.11), (5.1.13) and (F.3.5) respectively, that

$$E^{a,\sigma}_{\Lambda,M_\Lambda,R^c,\mu} = (-1)^{\sigma}F^{a,\sigma}_{\Lambda,M_\Lambda,R,\mu} \tag{5.7.10}$$

Combinatorics

The reduction of a representation R of S_{2n} to the representation (Λ,ϕ) of $S_n[S_2]$ was also investigated in [86]. The multiplicity $\mathcal{M}^{\Omega}_{R,\Lambda}$ can be given in terms of the plethysm of s_Λ with the power-sum symmetric polynomial p_2 or instead in terms of Littlewood-Richardson coefficients. For $\Lambda=[\Lambda_1,\Lambda_2]$ we have

$$\mathcal{M}^{\Omega}_{R,\Lambda} = \sum_{\substack{R_1\vdash 2\Lambda_1,\, R_2\vdash 2\Lambda_2\\ \text{with even row lengths}}} g_{R;R_1,R_2} - \sum_{\substack{R_1\vdash 2(\Lambda_1+1),\, R_2\vdash 2(\Lambda_2-1)\\ \text{with even row lengths}}} g_{R;R_1,R_2} \tag{5.7.11}$$

which matches the multiplicity from the restricted Schur basis as expected.

Correlators

By considering Wick contractions and using the formula (5.5.16) for the two-point function of ΩX and ΩY, the correlators of two copies of $\Omega\mathbb{X}$ is

$$\left\langle (\Omega\mathbb{X}_b)^J \,|\, (\Omega\mathbb{X}_a)^I \right\rangle = \sum_{\sigma\in S_n[S_2]} \delta_{a,\bar{\sigma}(b)} \left(\sigma^{-1}\right)^I_J \tag{5.7.12}$$

Using the relation (5.5.20) between sums of the contraction C^{Ω} and $\widetilde{\Omega}$, we can express the correlator of two generic covariant symplectic operators of the form (5.7.8) as

$$\begin{aligned}\left\langle \mathcal{O}^{\delta}_{b,\tau} | \mathcal{O}^{\delta}_{a,\sigma} \right\rangle &= \sum_{\gamma\in S_n[S_2]} \delta_{a,\bar{\gamma}(b)} C^{\Omega}(\sigma\gamma^{-1}\tau^{-1}) \\ &= \sum_{\gamma,\pi\in S_n[S_2]} \delta_{a,\bar{\gamma}(b)} (-1)^{\pi} \delta\left(\widetilde{\Omega}\pi\sigma\gamma^{-1}\tau^{-1}\right)\end{aligned} \tag{5.7.13}$$

Employing similar techniques to those used in appendix F to calculate the correlators of the mesonic basis, we can give the two-point function of the symplectic covariant basis operators (5.7.9)

$$\langle \mathcal{O}^{\Omega}_{\Gamma,M_\Gamma,S,\nu} | \mathcal{O}^{\Omega}_{\Lambda,M_\Lambda,R,\mu} \rangle = \delta_{\Lambda\Gamma}\delta_{M_\Lambda M_\Gamma}\delta_{RS}\delta_{\mu\nu} f^{\Omega}_R \tag{5.7.14}$$

They are orthonormal in the S_{2n} inner product.

5.7.3 Baryonic basis

Operator basis

For a permutation $\sigma \in S_{2n}$, define a generic covariant baryonic operator by

$$\mathcal{O}^{\varepsilon}_{a,\sigma} = C^{(\varepsilon)}_I \sigma^I_J (\mathbb{X}_a)^J \tag{5.7.15}$$

The covariant basis is

$$\mathcal{O}^{\Omega}_{\Lambda,M_\Lambda,R,\mu} = \sqrt{\frac{d_R}{d_\Lambda (2n)! N! 2^q q! 2^n n!}} \sum_{a,\sigma,J,k} C^{a}_{\Lambda,M_\Lambda,k} \left[\left(\langle [1^N]| \otimes \langle \bar{R}, [S]|\right) D^R(\sigma)\right]_J B^{S_{2n}\to S_n[S_2]\,;J}_{R\quad \to(\phi,\Lambda),\mu;k} \mathcal{O}^{\varepsilon}_{a,\sigma} \tag{5.7.16}$$

where $R = [1^N] + \bar{R}$ for $\bar{R} \vdash 2q$ with even row lengths satisfying $l(\bar{R}) \leq N$, $\Lambda \vdash n$ with $l(\Lambda) \leq 2$, M_Λ is a basis index for the $U(2)$ representation Λ and μ is a multiplicity index for the reduction from the R representation of S_{2n} to the (ϕ, Λ) representation of $S_n[S_2]$.

Combinatorics

The combinatorics of the baryonic basis are identical to the mesonic case, just with a different class of Young diagrams R, so we refer the reader to the description (5.7.3).

Correlators

Using the formula (5.7.4) for the correlator of two copies of $\mathbb{X}_a$ and the relation (5.5.25) between sums of the contraction C^ε and the element Ω^ε, we write the correlator of two generic baryonic operators of the form (5.7.15) as

$$\begin{aligned}\left\langle \mathcal{O}^\varepsilon_{b,\tau} | \mathcal{O}^\varepsilon_{a,\sigma} \right\rangle &= \sum_{\gamma \in S_n[S_2]} \delta_{a,\bar{\gamma}(b)} (-1)^\gamma C^\varepsilon(\sigma\gamma^{-1}\tau^{-1}) \\ &= \sum_{\gamma,\pi \in S_n[S_2]} \delta_{a,\bar{\gamma}(b)} (-1)^{\pi_1} (-1)^\gamma \delta\left(\Omega^\varepsilon \pi\sigma\gamma^{-1}\tau^{-1}\right) \end{aligned} \tag{5.7.17}$$

Then using the same methodology as that of appendix F to calculate the correlators of the mesonic basis, the correlators of baryonic covariant basis operators (5.7.16) are

$$\left\langle \mathcal{O}^\varepsilon_{\Gamma,M_\Gamma,S,\nu} | \mathcal{O}^\varepsilon_{\Lambda,M_\Lambda,R,\mu} \right\rangle = \delta_{\Lambda\Gamma}\delta_{M_\Lambda M_\Gamma}\delta_{RS}\delta_{\mu\nu} f^\varepsilon_R \tag{5.7.18}$$

In the S_{2n} inner product the operators are orthonormal.

Chapter 6

Orientifold quotient from $U(N)$ theory to $SO(N)$ or $Sp(N)$ theory

In [55], $\mathcal{N} = 4$ super Yang-Mills with $SO(N)$ and $Sp(N)$ gauge groups were demonstrated to be the dual of type IIB string theory on $AdS_5 \times \mathbb{RP}^5$. This string theory was obtained from the standard $AdS_5 \times S^5$ theory by identifying anti-podal points on the S^5 and reversing worldsheet orientation at the same time. Depending on topological considerations, this orientifold quotient can lead to either a $SO(N)$ or a $Sp(N)$ gauge group in the dual CFT. We now study this quotient in the BPS sector from the gauge theory point of view. The distinction between the orthogonal and symplectic quotient is much less subtle here, we either put the scalar fields X and Y in the adjoint of $\mathfrak{so}(N)$ or $\mathfrak{sp}(N)$.

The majority of this chapter examines the quotient in the half-BPS sector. A Young diagram basis of the $U(N)$ half-BPS sector was derived in [22], while in [56, 57] equivalents were found for the $SO(N)$ and $Sp(N)$ theories. When we perform the orientifold quotient on an arbitrary $U(N)$ state, it becomes a linear combination of the $SO(N)/Sp(N)$ basis. The coefficients in this expansion describe how a giant graviton in $AdS_5 \times S^5$ reduces to those in $AdS_5 \times \mathbb{RP}^5$. We investigate these coefficients using two different approaches.

Firstly, we find that these coefficients have a simple and elegeant expression in terms of a classic concept in the combinatorics of Young diagrams, called plethysms of Young diagrams.

Consider a Young diagram t with m boxes and a positive integer k. There is a representation V_t of $U(N)$ corresponding to t. We take N to be large here, more precisely $N \geq mk$. Now consider the tensor product $V_t^{\otimes k}$. This is a representation of $U(N)$ under the diagonal action where the group element $U \in U(N)$ acts as $U \otimes U \otimes \cdots U$. This diagonal action of $U(N)$ commutes with the S_k permutation group acting on $V_t^{\otimes k}$ by permuting the different factors of the tensor product. So we can decompose $V_t^{\otimes k}$ according to irreps of $U(N) \times S_k$ which correspond to pair (R, Λ) where R is a Young diagram with km boxes and Λ is a Young diagram with k boxes. The multiplicity of (R, Λ), denoted $\mathcal{P}(t, \Lambda, R)$ is known as a plethysm coefficient. They were defined by D. E. Littlewood [87] and remain the subject of important questions in combinatorics [88]. The sum over Λ of $\mathcal{P}(t, \Lambda, R)$ can be expressed in terms of Littlewood-Richardson coefficients. For the case where $k = 2$, the Young diagram Λ can be either the symmetric with a row of length 2, denoted as $\Lambda = [2]$, or it can be anti-symmetric, denoted as $\Lambda = [1, 1]$ for two rows of length 1. The sum $\mathcal{P}(t, [2], R) + \mathcal{P}(t, [1, 1], R)$ is a Littlewood-Richardson coefficient: the number of times R appears in $V_t^{\otimes 2}$ when this is decomposed into irreps of the diagonal $U(N)$. Thus $\mathcal{P}(t, [2], R)$ and $\mathcal{P}(t, [1, 1], R)$ are plethystic refinements of the Littlewood-Richardson coefficients. It turns out that the orientifold projection map can be expressed in terms of the plethysm coefficients $\mathcal{P}(t, [2], R)$ and $\mathcal{P}(t, [1, 1], R)$. A combinatorial rule for finding these coefficients was given in [89], refining the Littlewood-Richardson rule by replacing the standard Littlewood-Richardson tableaux with Yamanouchi domino tableaux.

The second approach uses another mathematical result described in [89]. Since both $U(N)$ and $SO(N)/Sp(N)$ Young diagram bases can be described as Schur symmetric functions, there is an operation on symmetric functions equivalent to the orientifold quotient, denoted by ϕ^2. [89] gives a formula for ϕ^2 on a $U(N)$ Young diagram R in terms of the *2-quotient*, a pair of Young diagrams (t_1, t_2) that gives an alternative parameterisation of R. Interestingly, this gives a $\mathbb{Z}_2$ symmetry on the $U(N)$ theory, interchanging t_1 and t_2, that does not affect the quotient operator. We conjecture this is dual to inversion of the S^5 and worldsheet orientation.

At the end of the chapter, we use the restricted Schur bases for $U(N)$ [44, 45] and $SO(N)/Sp(N)$ [59, 60] to examine the quotient in the free field quarter-BPS sectors.

This chapter concerns only the mesonic sector of the $SO(N)$ theory, since the $U(N)$ operators are all multi-traces, and replacing a generic X with an anti-symmetric X takes multi-traces to multi-traces. The baryonic operators, and associated branes wrapped on an $\mathbb{RP}^3$ within $\mathbb{RP}^5$ do not arise from the quotient in this way.

Some of the material in this chapter was originally presented in [1].

6.1 Projection coefficients in the half-BPS sector

Consider the quotient of a half-BPS $U(N)$ single trace operator. We have

$$\mathrm{Tr}X^k \xrightarrow{\mathbb{Z}_2} \begin{cases} \mathrm{Tr}X^k & k \text{ even} \\ 0 & k \text{ odd} \end{cases} \tag{6.1.1}$$

Note that on the left-hand side, X is an unconstrained complex matrix, while on the right it is either anti-symmetric or obeys the symplectic condition (4.0.1), depending on our choice of quotient.

Extending this quotient in the obvious way to multi-traces and linear combinations thereof, we can examine the behaviour of the Schur operators (2.3.14). Let $R \vdash n$ (with n even, otherwise the quotient always vanishes) index a $U(N)$ operator, then

$$\mathcal{O}_R^{U(N)} \xrightarrow{\mathbb{Z}_2} \sum_{\substack{T \vdash 2n \text{ with even} \\ \text{row and column lengths}}} \alpha_R^T \mathcal{O}_T^{\delta/\Omega} \tag{6.1.2}$$

where the $SO(N)$ and $Sp(N)$ operators are defined in (5.6.12) and (5.6.13). Note that since the expressions for $\mathcal{O}_T^{\delta}$ and $\mathcal{O}_T^{\Omega}$ in terms of multi-traces are identical, the projection coefficients α_R^T are independent of the gauge group. We will use δ/Ω in the upper label of operators throughout to indicate this property.

If we consider the definition (6.1.2) at large N, then since all coefficients of multi-traces in $\mathcal{O}^{U(N)}$ and $\mathcal{O}^{\delta/\Omega}$ are independent of N, so are the α_R^T. At finite N, the same quotient relation holds, though now some or all of the operators may vanish. Consistency of the finite N cut-offs on the two bases require that α_R^T is only non-zero when $l(T) \geq l(R)$.

As an example of the quotient we look at $n = 4$. Using the definition (2.3.14) the $U(N)$ operators are

$$\mathcal{O}^{U(N)}_{[4]} = \frac{1}{4}\mathrm{Tr}X^4 + \frac{1}{8}\left(\mathrm{Tr}X^2\right)^2 + \frac{1}{4}\left(\mathrm{Tr}X^2\right)\left(\mathrm{Tr}X\right)^2 + \frac{1}{3}\left(\mathrm{Tr}X^3\right)\left(\mathrm{Tr}X\right) + \frac{1}{24}\left(\mathrm{Tr}X\right)^4 \tag{6.1.3}$$

$$\mathcal{O}^{U(N)}_{[3,1]} = -\frac{1}{4}\mathrm{Tr}X^4 - \frac{1}{8}\left(\mathrm{Tr}X^2\right)^2 + \frac{1}{4}\left(\mathrm{Tr}X^2\right)\left(\mathrm{Tr}X\right)^2 + \frac{1}{8}\left(\mathrm{Tr}X\right)^4 \tag{6.1.4}$$

$$\mathcal{O}^{U(N)}_{[2,2]} = \frac{1}{4}\left(\mathrm{Tr}X^2\right)^2 - \frac{1}{3}\left(\mathrm{Tr}X^3\right)\left(\mathrm{Tr}X\right) + \frac{1}{12}\left(\mathrm{Tr}X\right)^4 \tag{6.1.5}$$

$$\mathcal{O}^{U(N)}_{[2,1,1]} = \frac{1}{4}\mathrm{Tr}X^4 - \frac{1}{8}\left(\mathrm{Tr}X^2\right)^2 - \frac{1}{4}\left(\mathrm{Tr}X^2\right)\left(\mathrm{Tr}X\right)^2 + \frac{1}{8}\left(\mathrm{Tr}X\right)^4 \tag{6.1.6}$$

$$\mathcal{O}^{U(N)}_{[1,1,1,1]} = \underbrace{-\frac{1}{4}\mathrm{Tr}X^4 + \frac{1}{8}\left(\mathrm{Tr}X^2\right)^2}_{\text{Survive the } \mathbb{Z}_2 \text{ quotient}} \underbrace{- \frac{1}{4}\left(\mathrm{Tr}X^2\right)\left(\mathrm{Tr}X\right)^2 + \frac{1}{3}\left(\mathrm{Tr}X^3\right)\left(\mathrm{Tr}X\right) + \frac{1}{24}\left(\mathrm{Tr}X\right)^4}_{\text{Annihilated by the } \mathbb{Z}_2 \text{ quotient}} \tag{6.1.7}$$

R	T		
	[6,6]	[4,4,2,2]	[2,2,2,2,2,2]
[6]	1	0	0
[5,1]	-1	0	0
[4,2]	1	1	0
[4,1,1]	0	-1	0
[3,3]	-1	-1	0
[3,2,1]	0	0	0
[3,1,1,1]	0	1	0
[2,2,2]	0	1	1
[2,2,1,1]	0	-1	-1
[2,1,1,1,1]	0	0	1
[1,1,1,1,1,1]	0	0	-1

Table 6.1: Projection coefficients α_R^T at $n = 6$.

while from (5.6.12) and (5.6.13), the $SO(N)/Sp(N)$ operators

$$\mathcal{O}^{\delta/\Omega}_{[4,4]} = \frac{1}{4}\mathrm{Tr}X^4 + \frac{1}{8}\left(\mathrm{Tr}X^2\right)^2 \tag{6.1.8}$$

$$\mathcal{O}^{\delta/\Omega}_{[2,2,2,2]} = -\frac{1}{4}\mathrm{Tr}X^4 + \frac{1}{8}\left(\mathrm{Tr}X^2\right)^2 \tag{6.1.9}$$

So the quotient is

$$\mathcal{O}^{U(N)}_{[4]} \xrightarrow{\mathbb{Z}_2} \mathcal{O}^{\delta/\Omega}_{[4,4]} \tag{6.1.10}$$

$$\mathcal{O}^{U(N)}_{[3,1]} \xrightarrow{\mathbb{Z}_2} -\mathcal{O}^{\delta/\Omega}_{[4,4]} \tag{6.1.11}$$

$$\mathcal{O}^{U(N)}_{[2,2]} \xrightarrow{\mathbb{Z}_2} \mathcal{O}^{\delta/\Omega}_{[4,4]} + \mathcal{O}^{\delta/\Omega}_{[2,2,2,2]} \tag{6.1.12}$$

$$\mathcal{O}^{U(N)}_{[2,1,1]} \xrightarrow{\mathbb{Z}_2} -\mathcal{O}^{\delta/\Omega}_{[2,2,2,2]} \tag{6.1.13}$$

$$\mathcal{O}^{U(N)}_{[1,1,1,1]} \xrightarrow{\mathbb{Z}_2} \mathcal{O}^{\delta/\Omega}_{[2,2,2,2]} \tag{6.1.14}$$

More examples of projection coefficients for $n = 6$ and $n = 8$ are shown in tables 6.1 and 6.2. These are calculated using the formula (6.1.18) derived presently.

To give an explicit expression for the projection coefficients, we recall that the size of the conjugacy class in S_n with cycle type $p \vdash n$ is $\frac{n!}{z_p}$. Using this we write the $U(N)$

R	T				
	[8,8]	[6,6,2,2]	[4,4,4,4]	[4,4,2,2,2,2]	[2,2,2,2,2,2,2,2]
[8]	1	0	0	0	0
[7,1]	-1	0	0	0	0
[6,2]	1	1	0	0	0
[6,1,1]	0	-1	0	0	0
[5,3]	-1	-1	0	0	0
[5,2,1]	0	0	0	0	0
[5,1,1,1]	0	1	0	0	0
[4,4]	1	1	1	0	0
[4,3,1]	0	0	-1	0	0
[4,2,2]	0	1	1	1	0
[4,2,1,1]	0	-1	0	-1	0
[4,1,1,1,1]	0	0	0	1	0
[3,3,2]	0	-1	0	-1	0
[3,3,1,1]	0	1	1	1	0
[3,2,2,1]	0	0	-1	0	0
[3,2,1,1,1]	0	0	0	0	0
[3,1,1,1,1,1]	0	0	0	-1	0
[2,2,2,2]	0	0	1	1	1
[2,2,2,1,1]	0	0	0	-1	-1
[2,2,1,1,1,1]	0	0	0	1	1
[2,1,1,1,1,1,1]	0	0	0	0	-1
[1,1,1,1,1,1,1,1]	0	0	0	0	1

Table 6.2: Projection coefficients α_R^T at $n = 8$.

operators (2.3.14) in terms of traces

$$\mathcal{O}_R^{U(N)} = \sum_{p \vdash n} \frac{\chi_R(p)}{z_p} \prod_i \left(\mathrm{Tr} X^i\right)^{p_i} \tag{6.1.15}$$

To give this as a sum of $SO(N)/Sp(N)$ Schur operators, invert (5.6.12)/(5.6.13) to give

$$\prod_i \left(\mathrm{Tr} X^{2i}\right)^{q_i} = 2^{l(q)} \sum_{t \vdash \frac{n}{2}} \chi_t(q) \mathcal{O}_T^{\delta/\Omega} \tag{6.1.16}$$

where $q \vdash \frac{n}{2}$ and T is obtained from t by replacing each box of t by the 2×2 block $\boxplus$. In the notation of section 5.6.1, $t = \frac{T}{4}$, and in the partition notation of section 2.2, $T = 2t \cup 2t$.

It follows that

$$\begin{aligned} \mathcal{O}_R^{U(N)} &\xrightarrow{\mathbb{Z}_2} \sum_{p \vdash \frac{n}{2}} \frac{\chi_R(2p)}{z_{2p}} 2^{l(p)} \sum_{t \vdash \frac{n}{2}} \chi_t(p) \mathcal{O}_T^{\delta/\Omega} \\ &= \sum_{t \vdash \frac{n}{2}} \left(\sum_{p \vdash \frac{n}{2}} \frac{1}{z_p} \chi_R(2p) \chi_t(p) \right) \mathcal{O}_T^{\delta/\Omega} \end{aligned} \tag{6.1.17}$$

where we have used $z_{2p} = 2^{l(p)} z_p$. Therefore

$$\alpha_R^T = \sum_{p \vdash \frac{n}{2}} \frac{1}{z_p} \chi_R(2p) \chi_t(p) \tag{6.1.18}$$

Introduce

$$\pi = \left(1, 1 + \frac{n}{2}\right)\left(2, 2 + \frac{n}{2}\right) \dots \left(\frac{n}{2}, n\right) \tag{6.1.19}$$

For a permutation $\sigma \in S_{\frac{n}{2}}$, embedded into S_n by acting on $\left\{1, 2, \dots, \frac{n}{2}\right\}$, the product $\sigma\pi$ has cycle type $2p$. So we have

$$\alpha_R^T = \frac{1}{\left(\frac{n}{2}\right)!} \sum_{\sigma \in S_{\frac{n}{2}}} \chi_R\left(\sigma\pi\right) \chi_t(\sigma) = \frac{1}{d_t} \chi_R\left(P_t \pi\right) \tag{6.1.20}$$

where P_t, defined in (2.3.13), is the projector onto the t representation of $S_{\frac{n}{2}}$.

Since π switches the sets $\left\{1, 2, \dots, \frac{n}{2}\right\}$ and $\left\{\frac{n}{2} + 1, \frac{n}{2} + 2, \dots, n\right\}$, the conjugate of P_t by π is the projector $\hat{P}_t$ onto the t representation of $S_{\frac{n}{2}}$ with a different embedding, acting on $\left\{\frac{n}{2} + 1, \frac{n}{2} + 2, \dots, n\right\}$ instead of $\left\{1, 2, \dots, \frac{n}{2}\right\}$. Then

$$\alpha_R^T = \frac{1}{d_t} \chi_R\left(P_t P_t \pi\right) = \frac{1}{d_t} \chi_R\left(P_t \pi \hat{P}_t\right) = \frac{1}{d_t} \chi_R\left(P_{t \otimes t} \pi\right) \tag{6.1.21}$$

where $P_{t \otimes t} = P_t \hat{P}_t = \hat{P}_t P_t$ is the projector onto the representation $t \otimes t$ of $S_{\frac{n}{2}} \times S_{\frac{n}{2}}$.

Decomposing R into representations $S_{\frac{n}{2}} \times S_{\frac{n}{2}}$, we have

$$V_R^{S_n} = \bigoplus_{r,s\vdash\frac{n}{2}} V_r^{S_{\frac{n}{2}}} \otimes V_s^{S_{\frac{n}{2}}} \otimes V_{R;r,s}^{mult} \tag{6.1.22}$$

where $V_{R;r,s}^{mult}$ is the multiplicity space of dimension $g_{R;r,s}$. Consider the action of π on a vector in the (r,s) subspace. Since π exchanges the two copies of $S_{\frac{n}{2}}$, it is mapped to a different vector in the (s,r) subspace. This induces a map between the multiplicity spaces $V_{R;r,s}^{mult}$ and $V_{R;s,r}^{mult}$. For the subspaces with $r=s$, this is a map from the multiplicity space to itself. Since $\pi^2=1$, we can split $V_{R;r,r}^{mult}$ into the $+1$ and -1 eigenspaces, which we denote by $V_{R;r,r}^{mult;\pm}$.

The projector $P_{t\otimes t}$ in (6.1.21) means only the $r=s=t$ term in the decomposition contributes. On this term, π splits into

$$\pi = \pi_t \otimes \pi_{mult} \tag{6.1.23}$$

where π_t acts on $t\otimes t$ by switching the factors, and π_{mult} acts on the multiplicity space. Only the diagonal terms in $t\otimes t$ contribute to the trace, and therefore

$$\chi_{t\otimes t}(\pi_t) = d_t \tag{6.1.24}$$

It follows that

$$\alpha_R^T = \frac{1}{d_t}\chi_{t\otimes t}(\pi_t)\mathrm{Tr}_{V_{R;t,t}^{mult}}(\pi_{mult}) = \mathrm{Dim}\left(V_{R;t,t}^{mult;+}\right) - \mathrm{Dim}\left(V_{R;t,t}^{mult;-}\right) \tag{6.1.25}$$

These dimensions are S_n plethysm coefficients. We now use Schur-Weyl duality, given in (2.4.3), to give the equivalent expression in terms of $U(N)$ plethysm coefficients that have been studied in [89]. Since the projection coefficients are N-independent, we work at large N to avoid issues with finite N cut-offs.

Let V be the fundamental of $U(N)$. Then $U(N)$ and S_n act on $V^{\otimes n}$, with the interaction between the two given by Schur-Weyl duality. Explicitly, for $\sigma\in S_n$ and $\mathcal{U}\in U(N)$, we have the decomposition

$$\mathrm{Tr}_{V^{\otimes n}}(\sigma\mathcal{U}) = \sum_{R\vdash n}\chi_R^{U(N)}(\mathcal{U})\,\chi_R^{S_n}(\sigma) \tag{6.1.26}$$

In direct analogy to the projector (2.3.13) we can define an operator that projects onto the R irrep of $U(N)$. Since $U(N)$ is a compact Lie group, the sum is replaced by an integral over the Haar measure (normalised so that the volume of the group is 1).

$$P_R^{U(N)} = \int d\mathcal{U}\;\chi_R^{U(N)}(\mathcal{U}^{-1})\,\mathcal{U} \tag{6.1.27}$$

which has representative 1 in the R representation of $U(N)$ and 0 in all others. We can use this to express α_R^T as a trace over the whole of $V^{\otimes n}$

$$\begin{aligned}\alpha_R^T &= \frac{1}{d_t}\chi_R^{S_n}\left(P_{t\otimes t}\,\pi\right)\\ &= \frac{1}{d_R^{U(N)}\,d_t}\chi_R^{U(N)}\left(P_R^{U(N)}\right)\chi_R^{S_n}\left(P_{t\otimes t}\,\pi\right)\\ &= \frac{1}{d_R^{U(N)}\,d_t}\mathrm{Tr}_{V^{\otimes n}}\left(P_R^{U(N)}P_{t\otimes t}\,\pi\right)\end{aligned} \tag{6.1.28}$$

Perform a Schur-Weyl decomposition on each factor in $V^{\otimes n} = V^{\otimes \frac{n}{2}} \otimes V^{\otimes \frac{n}{2}}$.

$$\begin{aligned}V^{\otimes n} &= \left(\bigoplus_{r\vdash \frac{n}{2}} V_r^{U(N)} \otimes V_r^{S_{\frac{n}{2}}}\right) \otimes \left(\bigoplus_{s\vdash \frac{n}{2}} V_s^{U(N)} \otimes V_s^{S_{\frac{n}{2}}}\right)\\ &= \bigoplus_{r,s\vdash \frac{n}{2}} V_r^{U(N)} \otimes V_s^{U(N)} \otimes V_r^{S_{\frac{n}{2}}} \otimes V_s^{S_{\frac{n}{2}}}\end{aligned} \tag{6.1.29}$$

Examine the action of π on this decomposition, just as we did for (6.1.22). Since π exchanges the two copies of $V^{\otimes \frac{n}{2}}$, it exchanges the spaces labelled by (r, s) and (s, r) for $r \neq s$. On the spaces with $r = s$, π splits into a tensor product operator

$$\pi = \pi^{U(N)} \otimes \pi^{S_n} \tag{6.1.30}$$

where $\pi^{U(N)}$ exchanges the factors of $V_r^{U(N)} \otimes V_r^{U(N)}$ and π^{S_n} is π_r as defined in (6.1.23). Therefore

$$\begin{aligned}\alpha_R^T &= \frac{1}{d_R^{U(N)}\,d_t}\mathrm{Tr}_{V^{\otimes n}}\left(P_{t\otimes t}P_R^{U(N)}\,\pi\right)\\ &= \frac{1}{d_R^{U(N)}\,d_t}\mathrm{Tr}_{V_t^{U(N)}\otimes V_t^{U(N)}\otimes V_t^{S_{\frac{n}{2}}}\otimes V_t^{S_{\frac{n}{2}}}}\left(P_R^{U(N)}\pi\right)\\ &= \frac{1}{d_R^{U(N)}\,d_t}\chi_{t\otimes t}^{U(N)}\left(P_R^{U(N)}\pi^{U(N)}\right)\chi_{t\otimes t}^{S_{\frac{n}{2}}}\left(\pi^{S_n}\right)\\ &= \frac{1}{d_R^{U(N)}}\chi_{t\otimes t}^{U(N)}\left(P_R^{U(N)}\pi^{U(N)}\right)\end{aligned} \tag{6.1.31}$$

Splitting the $U(N)$ representation $t \otimes t$ into its symmetric and anti-symmetric parts $S^2(t)$ and $\Lambda^2(t)$, we have

$$\alpha_R^T = \frac{1}{d_R^{U(N)}}\left[\chi_{S^2(t)}\left(P_R^{U(N)}\right) - \chi_{\Lambda^2(t)}\left(P_R^{U(N)}\right)\right] \tag{6.1.32}$$

Each of the two terms is just the multiplicity of the R irrep of $U(N)$ in $S^2(t)$ and $\Lambda^2(t)$

respectively. By definition, these are the $U(N)$ plethsym coefficients $\mathcal{P}(t,[2],R)$ and $\mathcal{P}(t,[1,1],R)$. Keeping track of the π eigenspaces (for example by using $\frac{1\pm\pi}{2}$ instead of π) in the above, we see that these plethsym coefficients agree with the S_n plethsym coefficients of (6.1.25)

$$\mathcal{P}(t,[2],R) := \mathrm{Mult}\left(R, S^2(t)\right) = \mathrm{Dim}\left(V^{mult;+}_{R;t,t}\right) \tag{6.1.33}$$

$$\mathcal{P}(t,[1,1],R) := \mathrm{Mult}\left(R, \Lambda^2(t)\right) = \mathrm{Dim}\left(V^{mult;-}_{R;t,t}\right) \tag{6.1.34}$$

where the partitions $[2]$ and $[1,1]$ denote the fact that we took the symmetric and anti-symmetric parts of $t\otimes t$.

So we have

$$\alpha^T_R = \mathcal{P}(t,[2],R) - \mathcal{P}(t,[1,1],R) \tag{6.1.35}$$

The Littlewood-Richardson coefficient is

$$g_{R;t,t} = \mathcal{P}(t,[2],R) + \mathcal{P}(t,[1,1],R) \tag{6.1.36}$$

so α^T_R is a refined version of $g_{R;t,t}$. We see that $g_{R;t,t}=0$ is a sufficient condition for $\alpha^T_R = 0$ and the parity of $g_{t,t;R}$ is the same as the parity of α^T_R.

The plethysm coefficients $\mathcal{P}(t,[2],R)$ and $\mathcal{P}(t,[1,1],R)$ were the subject of the paper [89]. They present two combinatorial rules, the first gives the difference $\mathcal{P}(t,[2],R)-\mathcal{P}(t,[1,1],R)=\alpha^T_R$ directly, while the second gives the two plethysm coefficients individually. Both rules involve Yamanouchi domino tableaux, which we now define.

6.2 Domino tableaux and combinatorics of plethysms

A domino tiling of shape $R\vdash n$ (n even) is a tiling of the shape R with 2×1 or 1×2 rectangles, which are called dominoes. A domino tableau is a tiling where each domino contains a positive integer, such that the numbers increase weakly along the rows and strictly down the columns. Note that each domino occupies 2 rows and 1 column (or 2 columns and 1 row), and the integers contained within the dominoes must be correctly ordered in both rows (columns). This is analogous to the semi-standard Young tableau introduced in section 3.6.2.

Each row in a domino tableau defines a word by reading the numbers in the row from right to left, where vertical dominoes, which span two row, only contribute to the upper row. The *row reading* of the tableau is then defined by concatenating these words starting with the top row and proceeding downwards.

A lattice word is a word on the alphabet of positive integers such that each prefix contains at least as many 1s as 2s, at least as many 2s as 3s, and, more generally, at

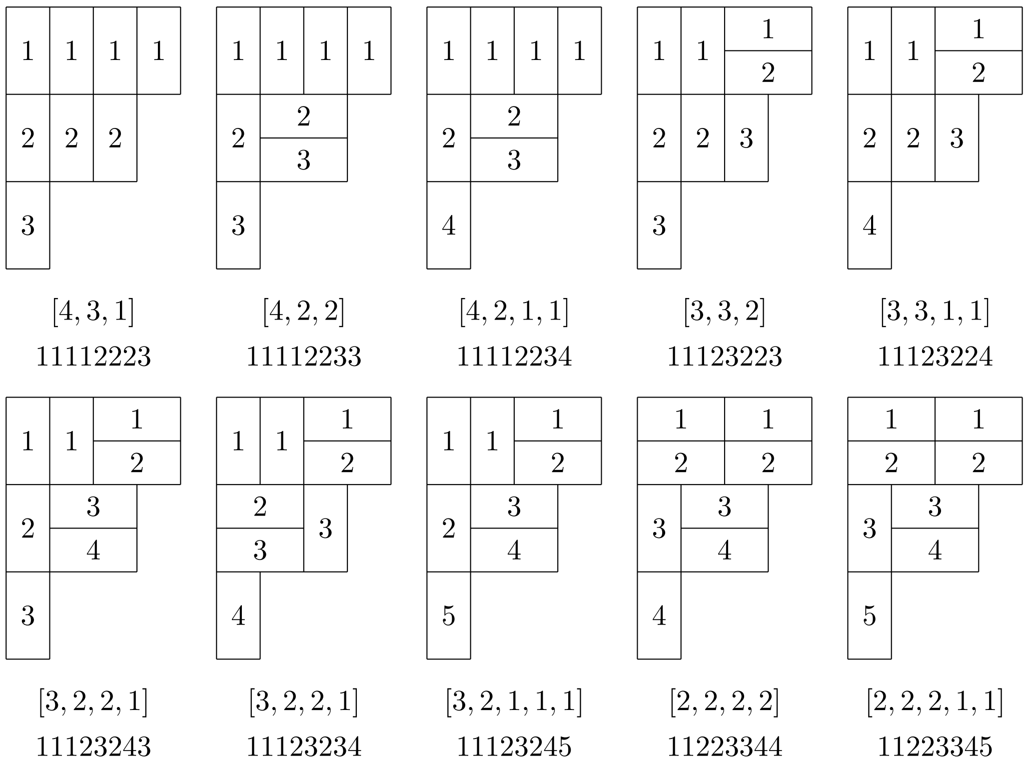


Figure 6.1: The possible Yamanouchi domino tableaux of shape [4,4,3,3,1,1]. The evaluation and row reading of each tableau is given beneath.

least as many is as $(i+1)$s for every i. A Yamanouchi domino tableau is a domino tableau for which the row reading is a lattice word.

In the original paper [89], they use the *column reading* instead of the row reading, which must be Yamanouchi words, defined to be the reversal of a lattice word. The two forms are equivalent. In this work we use lattice words to make clearer the analogy between Yamanouchi domino tableaux and the Littlewood-Richardson tableaux defined in appendix D.

For a given Yamanouchi domino tableau, let the number of integers i in the tableau be given by λ_i. We define the evaluation of the tableau to be $\lambda = [\lambda_1, \lambda_2, \ldots]$. Clearly $\sum_i \lambda_i = \frac{n}{2}$, and the lattice word condition on the row reading ensures that λ is a partition of $\frac{n}{2}$, i.e. the λ_i are weakly decreasing.

As an example of the above definitions, figure 6.1 gives the ten Yamanouchi domino tableaux of shape $[4, 4, 3, 3, 1, 1]$ along with their row readings and evaluations.

A key property of a domino tiling is the number of horizontal or vertical dominoes. Take $R \vdash n$, with components $R_1, R_2, \ldots, R_k$. Assume that R admits a domino tiling, and let r be such a tiling. Then define $h_i(r)$ to be the number of horizontal dominoes in row i of r, $v_i(r)$ be the number of vertical dominoes with their uppermost box in row

i, and $h(r)$ and $v(r)$ be the total number of horizontal and vertical dominoes. Then we have

$$\begin{aligned} R_1 &= 2h_1(r) + v_1(r) \\ R_2 &= 2h_2(r) + v_1(r) + v_2(r) \\ R_3 &= 2h_3(r) + v_2(r) + v_3(r) \\ &\vdots \\ R_{k-1} &= 2h_{k-1}(r) + v_{k-2}(r) + v_{k-1}(r) \\ R_k &= 2h_k(r) + v_{k-1}(r) \end{aligned} \tag{6.2.1}$$

Therefore

$$(-1)^{R_1+R_3+\ldots} = (-1)^{2(h_1(r)+h_3(r)+\ldots)+v_1(r)+v_2(r)+\ldots+v_{k-1}(r)} = (-1)^{v(r)} \tag{6.2.2}$$

Crucially, if a domino tiling of shape R exists, the parity of $v(r)$ (similarly the parity of $h(r)$) depends only on R, and not on how the dominoes are arranged. In light of this, we define $\varepsilon_2(R)$, the *2-sign* of R, to be $(-1)^{v(r)}$ if R admits a domino tiling, and 0 otherwise.

Under conjugation of R, horizontal dominoes turn into vertical ones and vice versa. Since $h(r) + v(R) = \frac{n}{2}$, it follows that

$$\varepsilon_2(R^c) = (-1)^{\frac{n}{2}} \varepsilon_2(R) \tag{6.2.3}$$

We can now give the first combinatorial rule, proved in [89], for finding α_R^T. Defining D_λ^R to be the number of Yamanouchi domino tableau of shape R and evaluation λ, we have

$$\alpha_R^T = \mathcal{P}(t, [2], R) - \mathcal{P}(t, [1,1], R) = \varepsilon_2(R) D_t^R \tag{6.2.4}$$

Note this means the sign of the non-zero α_R^T depends only on R and not T, since $D_t^R \geq 0$. This can be seen in tables 6.1 and 6.2, where each row consists only of zeroes and positive numbers, or zeroes and negative numbers.

For the second rule, consider $T \vdash 2n$, constructed from 2×2 blocks. Clearly we can tile T with dominoes by putting 2 horizontal dominoes in each 2×2 block. Therefore in any domino tableau of T, there must be an even number of horizontal (and vertical) dominoes. We split the domino tableau of shape T into two classes, based on the number of pairs of horizontal dominoes. If a tableau has an even number of pairs, we say it has spin 1, while if it has an odd number of pairs it has spin -1. For T of this type, we define $D_{+,R}^T$ and $D_{-,R}^T$ to be the number of Yamanouchi domino tableaux of evaluation R and positive and negative spin respectively. The second combinatorial rule, which gives the two plethysm coefficients individually, is

$$\mathcal{P}(t, [2], R) = D^T_{+,R} \qquad \mathcal{P}(t, [1, 1], R) = D^T_{-,R} \tag{6.2.5}$$

This leads to a second expression for (6.1.35)

$$\alpha^T_R = D^T_{+,R} - D^T_{-,R} \tag{6.2.6}$$

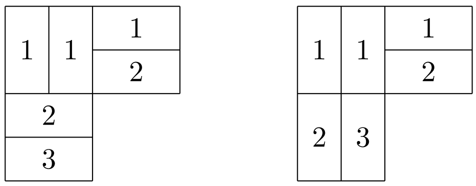
In tables 6.1 and 6.2 we gave some low n ($n = 6, 8$) examples of α^T_R, calculated using (6.1.18). In addition, the $n = 4$ coefficients can be read off from (6.1.10-6.1.14). These tables have been checked against both combinatorial rules (6.2.4) and (6.2.5). In all cases the results match.

From (6.1.36), the total number of domino tableau of any spin is the Littlewood-Richardson coefficient

$$g_{R;t,t} = D^T_{+,R} + D^T_{-,R} = D^T_R \tag{6.2.7}$$

So α^T_R may be viewed as a refinement of the Littlewood-Richardson coefficient $g_{R;t,t}$.

The two combinatoric methods of finding α^T_R are independent of each other. For example if we take $R = [3, 2, 1]$, then there are no domino tableau of shape R, so (6.2.4) gives 0 trivially. However if we look at Yamanouchi domino tableau of shape $T = [4, 4, 2, 2]$ (corresponding to $t = [2, 1]$) and evaluation R, we find two such tableaux, one contributing to each of the two plehtysm coefficients. These two tableaux are



(6.2.8)

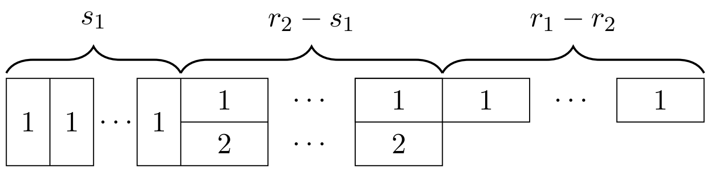
The first tableau has spin $+1$ while the second has spin -1. Using (6.2.5), $\mathcal{P}(t, [2], R) = \mathcal{P}(t, [1, 1], R) = 1$, and therefore $\alpha^T_R = 0$.

The two tableaux in (6.2.8) can also be interpreted with the roles of T and R switched. If we take $R = [4, 2, 2]$ and $t = [3, 2, 1]$ then these tableaux contribute to D^R_t, and by (6.2.4) we find $\alpha^T_R = 2$. This is the lowest n example of a projection coefficient taking a value with modulus greater than 1.

6.3 Brane interpretation of domino algorithm

We can also formulate a detailed brane interpretation of the domino algorithm. For a single column Young diagram R, a domino tiling exists only if the length of the column is even. Single giant gravitons with L units of angular momentum can be usefully thought of as composites of L quanta. Pairs of quanta are invariant under the orientifold action, consistent with the fact that only single column Young diagrams of even length survive the projection. The projection of these single column Young diagrams R are single column Young diagrams t, which should therefore also be interpreted as single giants in the orientifold theory. Similarly the quanta of angular momentum forming a single long row (AdS giant) are paired by the domino algorithm into $\mathbb{Z}_2$ invariant pairs, resulting in a single giant in the quotient.

Now consider a 2-row Young diagrams $R = [r_1, r_2]$, in the regime where r_1, r_2 are comparable to N and their difference is also comparable to N, e.g. $[r_1, r_2] = [2N, N]$. Consider a domino tiling with a number $s_1 < r_2$ of vertical dominoes, with the remaining boxes $[r_1 - s_1, r_2 - s_1]$ occupied by horizontal dominoes.



(6.3.1)

This has evaluation $t = \left[\frac{r_1+s_1}{2}, \frac{r_2-s_1}{2}\right]$. The vertical dominoes stretch across boxes in the first and second row, which can be viewed as quanta constituting the two branes described by R. The horizontal dominoes are constituents of the same brane. A horizontal domino in the first or second row of R contributes a box to the first or second row of t. The vertical dominoes, even though they span row one and two of R, contribute to the first row of t only. The domino combinatorics thus encodes, in a precise way, a recombination of angular momentum quanta between the two branes of angular momenta r_1, r_2 described by R, which accompanies the orientifold procedure. For multi-row Young diagrams, the domino algorithm pairs quanta of angular momentum in adjacent rows, equivalently adjacent giant gravitons in the LLM plane. An analogous discussion holds for multi-column states, where horizontally tiled dominoes pair quanta from distinct giants and vertically tiled dominoes pair quanta within a giant worldvolume.

It would be interesting to deduce connections between the brane interpretation of the orientifold projection coefficients discussed heuristically above, from more general frameworks for brane dynamics in the presence of orientifolds, as developed for example in [90, 91]. In the AdS/CFT context, a useful discussion of orientifolds is in [58].

6.4 The quotient operator as a product

Beyond the combinatorial rules (6.2.4) and (6.2.5), the paper [89] gives an expression for the quotient of a $U(N)$ operator $\mathcal{O}_R^{U(N)}$ as a product of two $SO(N)/Sp(N)$ operators.

Expressed in the language of symmetric functions, they define a map ϕ^2 which takes a Schur function s_R in the N variables $x_1, x_2, \ldots, x_N$ and returns a symmetric function $\phi^2(s_R)$ in the $\frac{N}{2}$ variables $y_1, y_2, \ldots, y_{\frac{N}{2}}$. This is defined by

$$\phi^2(s_R)\left(y_1, y_2, \ldots, y_{\frac{N}{2}}\right) = s_R\left(y_1, -y_1, y_2, -y_2, \ldots, y_{\frac{N}{2}}, -y_{\frac{N}{2}}\right) \tag{6.4.1}$$

The authors then explain that

$$\phi^2(s_R) = \varepsilon_2(R) s_{t_1}\left(y_1^2, y_2^2, \ldots, y_{\frac{N}{2}}^2\right) s_{t_2}\left(y_1^2, y_2^2, \ldots, y_{\frac{N}{2}}^2\right) \tag{6.4.2}$$

where the ordered pair of Young diagrams (t_1, t_2) are the *2-quotient* of R, which will be described shortly.

The $\mathbb{Z}_2$ quotient from the $U(N)$ theory replaces the generic matrix X with an anti-symmetric or symplectic matrix. In terms of the N eigenvalues of X, for $SO(N)$ (N even) the quotient acts as

$$x_1 \to ix_1\,,\ x_2 \to -ix_1\,,\ \ldots\,,\ x_{N-1} \to ix_{\frac{N}{2}}\,,\ x_N \to -ix_{\frac{N}{2}} \tag{6.4.3}$$

while for N odd, we have

$$x_1 \to ix_1\,,\ x_2 \to -ix_1\,,\ \ldots\,,\ x_{N-2} \to ix_{\frac{N-1}{2}}\,,\ x_{N-1} \to -ix_{\frac{N-1}{2}}\,,\ x_N \to 0 \tag{6.4.4}$$

and for $Sp(N)$

$$x_1 \to x_1\,,\ x_2 \to -x_1\,,\ \ldots\,,\ x_{N-1} \to x_{\frac{N}{2}}\,,\ x_N \to -x_{\frac{N}{2}} \tag{6.4.5}$$

Since the $U(N)$ Schur operators are Schur symmetric functions in the N eigenvalues of X, this means the $SO(N)$ quotient can be evaluated by setting $y_i = ix_i$ in (6.4.1), while the $Sp(N)$ quotient sets $y_i = x_i$. Using the formulae (5.6.18) and (5.6.20) for $SO(N)$ and $Sp(N)$ Schur operators in terms of Schur symmetric functions, we see that

$$\mathcal{O}_R^{U(N)} \xrightarrow{\mathbb{Z}_2} \varepsilon_2(R)\mathcal{O}_{T_1}^{\delta/\Omega}\mathcal{O}_{T_2}^{\delta/\Omega} \tag{6.4.6}$$

where $T_i = 2t_i \cup 2t_i$. Intuitively, T_i is obtained from t_i by replacing each box of t_i with the 2×2 block ⊞.

In the dual string theory, this suggests that the orientifold quotient of a giant graviton state in $AdS_5 \times S^5$ can be expressed as a composite system of two $SO(N)/Sp(N)$ giant graviton states.

The product of two Schur symmetric functions is expressed in terms of Littlewood-Richardson coefficients (D.1.4). Therefore

$$\mathcal{O}_R^{U(N)} \xrightarrow{\mathbb{Z}_2} \varepsilon_2(R)\mathcal{O}_{T_1}^{\delta/\Omega}\mathcal{O}_{T_2}^{\delta/\Omega}$$

$$
\begin{aligned}
&= \varepsilon_2(R) s_{t_1} s_{t_2} \\
&= \varepsilon_2(R) \sum_{\substack{t \vdash \frac{n}{2} \\ l(t) \leq \frac{N}{2}}} g_{t;t_1,t_2} s_t \\
&= \varepsilon_2(R) \sum_{\substack{t \vdash \frac{n}{2} \\ l(t) \leq \frac{N}{2}}} g_{t;t_1,t_2} \mathcal{O}_T^{\delta/\Omega}
\end{aligned} \tag{6.4.7}
$$

and consequently

$$
\alpha_R^T = \varepsilon_2(R) g_{t;t_1,t_2} \tag{6.4.8}
$$

We now explain how the 2-quotient Young diagrams (t_1, t_2) are derived from a partition $R \vdash n$.

6.4.1 2-core and 2-quotient of a partition

Take a partition $R \vdash n$. For the definitions of this section, we do not require n to be even. The 2-core and 2-quotient of R are discussed in [64, Chapter I].

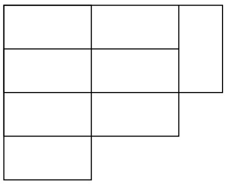
Consider removing dominoes from R until you obtain either the empty diagram, or a 'staircase' diagram of the form $\Delta_k = [k, k-1, \dots, 2, 1]$, from which no domino can be removed. Then the 2-core of R is the resulting Δ_k. This is independent of the order in which dominoes are removed from R.

A simple way to determine the 2-core is to colour the boxes of R. A box is white if the content, defined in (2.3.19), is even, and black if the content is odd. This forms a chessboard pattern on the boxes of R, with a white square in the top left. Let $n_w(R)$ be the number of white boxes in R and $n_b(R)$ the number of black boxes. Then $n_w - n_b$ is unchanged by adding a domino to the diagram. The staircase diagram Δ_k has $n_w - n_b = \frac{k+1}{2}$ if k is odd and $n_w - n_b = -\frac{k}{2}$ if k is even. Inverting these gives k in terms of $n_w - n_b$, and therefore $n_w - n_b$ determines the 2-core.

If the 2-core of R is Δ_k for $k > 0$, then R does not admit a domino tableau. Therefore the projection coefficients α_R^T will vanish unless R has 2-core Δ_0, the empty Young diagram.

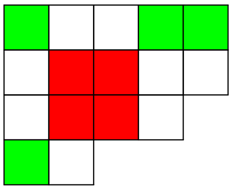
Split the rows of R into two types, *even* and *odd*, depending on whether the last box of the row has even or odd contents. Similarly, a column of R is even/odd if the last box in the column has even/odd contents. Then t_1 is defined to be the diagram composed of the boxes in the intersection between the even rows and the odd columns. Similarly t_2 is the intersection of the odd rows with the even columns.

As an example of this process, take $R = [5, 5, 4, 2]$. This can be tiled by dominoes

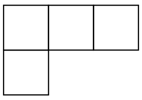
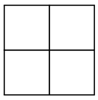

(6.4.9)

and therefore the 2-core of R is empty.

The 1st and 4th rows of R are even, while the 1st, 4th and 5th columns are odd. Correspondingly, the 2nd and 3rd rows are odd and the 2nd and 3rd columns are even. Colouring the intersection of even rows and odd columns green, and the intersection of odd rows and even columns red, we have

$R =$ 
(6.4.10)

so the 2-quotient (t_1, t_2) of R is

$t_1 =$  $= [3, 1]$ $t_2 =$  $= [2, 2]$ (6.4.11)

The algorithm above constructs the 2-core and 2-quotient for any partition R. Conversely, given a choice of Δ_k and a pair of partitions (t_1, t_2), there is a unique $R \vdash |\Delta_k| + 2(|t_1| + |t_2|)$ with the 2-core Δ_k and 2-quotient (t_1, t_2). We now give a construction of R with empty 2-core and 2-quotient (t_1, t_2), which we denote by $R(t_1, t_2)$.

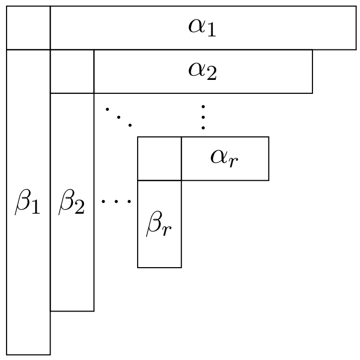
To describe $R(t_1, t_2)$ we introduce the *Frobenius* notation for a partition, as described in [64]. The *depth* of a partition p is the largest r such that there is a box in the (r, r)th position. So for example the depth of any hook diagram $[k, 1^l]$ is 1, while the depth of $[2, 2]$ and $[3, 2, 1]$ is 2.

For a partition $p = [\lambda_1, \lambda_2, \ldots, \lambda_k]$ of depth r, define $\alpha_i = \lambda_i - i$ for $1 \leq i \leq r$. This is the number of boxes to the right of (i, i) in the ith row of p. Given the conjugate of p is $p^c = [\mu_1, \mu_2, \ldots, \mu_l]$, define $\beta_i = \mu_i - i$. This is the number of boxes below (i, i) in the ith column of p. We have $\alpha_1 > \alpha_2 > \cdots > \alpha_r \geq 0$ and $\beta_1 > \beta_2 > \cdots > \beta_r \geq 0$, and we denote p by

$$p = (\alpha_1, \alpha_2, \ldots, \alpha_r | \beta_1, \beta_2, \ldots, \beta_r) \tag{6.4.12}$$

Intuitively, the pair α_i, β_i specify the hook of the box (i, i), and we can construct the entire partition from this information. Visually

$p =$



(6.4.13)

The size of p is

$$|p| = r + \sum_{i=1}^{r}(\alpha_i + \beta_i) \tag{6.4.14}$$

It is clear that conjugating p swaps the α_i and β_i.

We now move on to the construction of $R(t_1, t_2)$. In Frobenius notation, let

$$t_1 = (\alpha_1, \alpha_2, \ldots, \alpha_r | \beta_1, \beta_2, \ldots, \beta_r) \tag{6.4.15}$$

$$t_2 = (\gamma_1, \gamma_2, \ldots, \gamma_s | \delta_1, \delta_2, \ldots, \delta_s) \tag{6.4.16}$$

Then consider the sets

$$A = \{2\alpha_1, 2\alpha_2, \ldots, 2\alpha_r, 2\gamma_1 + 1, 2\gamma_2 + 1, \ldots 2\gamma_s + 1\} \tag{6.4.17}$$

$$B = \{2\beta_1 + 1, 2\beta_2 + 1, \ldots, 2\beta_r + 1, 2\delta_1, 2\delta_2, \ldots, 2\delta_s\} \tag{6.4.18}$$

The even members of A are just double the α_i, and it follows that they are all distinct. Similarly the odd members of A are all different, and therefore all members of A are distinct. Let a_i be the ith largest member of A for $1 \leq i \leq r + s$. Similarly, let b_i be the ith largest member of B. Then $R(t_1, t_2)$ is given in Frobenius notation by

$$R(t_1, t_2) = (a_1, a_2, \ldots, a_{r+s} | b_1, b_2, \ldots, b_{r+s}) \tag{6.4.19}$$

The length of $R(t_1, t_2)$ is

$$l\big(R(t_1, t_2)\big) = b_1 + 1 = \max(2\beta_1 + 2, 2\delta_1 + 1) = \max(2l(t_1), 2l(t_2) - 1) \tag{6.4.20}$$

Therefore the finite N cut-off on $R(t_1, t_2)$ is equivalent to

$$l(t_1), l(t_2) \leq \frac{N}{2} \tag{6.4.21}$$

As a special example of the 2-quotient, consider $t_1 = t_2 = r$. In this case $R = R(r,r)$ has even length rows and columns, and is built of 2×2 blocks $\boxplus$. As discussed earlier, this allows the definition of $\frac{R}{4}$ by replacing each 2×2 block with a single box. Then $\frac{R}{4} = r$. This gives an alternative expression for the relation between T and $t = \frac{T}{4}$ for the $SO(N)/Sp(N)$ theories:

$$T = R(t,t) \tag{6.4.22}$$

From the definition (6.4.19), under conjugation of R, we have

$$R(t_1,t_2)^c = R(t_2^c, t_1^c) \tag{6.4.23}$$

From this relation we can derive the conjugation behaviour of the projection coefficients. Using the expression (6.4.8) for α^T_R, the behaviour (6.2.3) of $\varepsilon_2(R)$ under conjugation, and properties of Littlewood-Richardson coefficients, it follows that

$$\alpha^{T^c}_{R^c} = \varepsilon_2\big(R^c\big) g_{t^c;t_2^c,t_1^c} = (-1)^{\frac{n}{2}} \varepsilon_2(R) g_{t;t_1,t_2} = (-1)^{\frac{n}{2}} \alpha^T_R \tag{6.4.24}$$

6.5 Simple families of projection coefficients

We can use the 2-core and 2-quotient to better understand the physics of the orientifold quotient. There are three families of R which have particularly nice properties.

Firstly, we have R with

$$\mathcal{O}_R^{U(N)} \xrightarrow{\mathbb{Z}_2} 0 \tag{6.5.1}$$

These are exactly the R with non-empty 2-core. As a special case, this includes all diagrams with $R \vdash n$ and n odd. More generally, these R are 'staircases + dominoes', i.e. they are constructed by taking a Δ_k with $k > 0$ and adding dominoes on to the diagram. One can check in examples (6.1.10-6.1.14) and tables 6.1 and 6.2 that all coefficients α^T_R with R of this form vanish.

The other two families of R have quotient

$$\mathcal{O}_R^{U(N)} \xrightarrow{\mathbb{Z}_2} \varepsilon_2(R)\mathcal{O}_T^{\delta/\Omega} \tag{6.5.2}$$

for a unique Young diagram T. This can occur in two distinct ways.

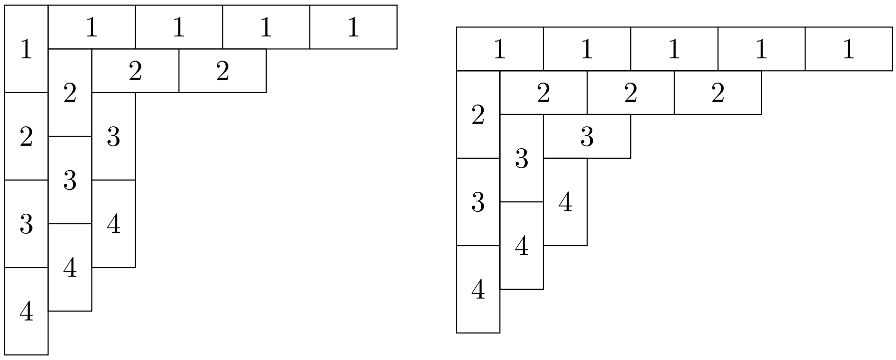
Firstly, consider $R = R(t,\phi)$ or $R = R(\phi,t)$, where ϕ is the empty partition. Then it follows from (6.4.6) that the orientifold quotient projects $\mathcal{O}_R^{U(N)}$ project to a single $SO(N)/Sp(N)$ operator. Let $t = (\alpha_1,\alpha_2,\dots,\alpha_r|\beta_1,\beta_2,\dots,\beta_r)$ in Frobenius notation. Then (6.4.19) reduces to

$$R(t,\phi) = (2\alpha_1, 2\alpha_2, \dots, 2\alpha_r | 2\beta_1+1, 2\beta_2+1, \dots, 2\beta_r+1) \tag{6.5.3}$$

$$R(\phi,t) = (2\alpha_1+1, 2\alpha_2+1, \dots, 2\alpha_r+1 | 2\beta_1, 2\beta_2, \dots, 2\beta_r) \tag{6.5.4}$$

For this t and these R, we have $\alpha_R^T = \varepsilon_2(R)$. To determine the sign, we need to construct a domino tableau of shape R. From the domino rule (6.2.4), there is a unique Yamanouchi domino tableau of shape R and evaluation t. We now describe this tableau, both to determine $\varepsilon_2(R)$ and to give an example of the domino rule at work.

Before doing the general case, consider an example with $t = (4,2,0|3,2,1) = [5,4,3,3]$. Then $R(t,\phi) = (8,4,0|7,5,3) = [9,6,3,3,3,3,2,1]$ and $R(\phi,t) = (9,5,1|6,4,2) = [10,7,4,3,3,2,1]$, shown below. The Yamanouchi domino tableaux that contribute to $\alpha^T_{R(t,\phi)}$ and $\alpha^T_{R(\phi,t)}$ respectively are



(6.5.5)

Counting the number of vertical dominoes, we conclude the 2-sign of $R(t,\phi)$ is -1, while the 2-sign of $R(\phi,t)$ is $+1$.

In more generality, the Yamanouchi domino tableau of shape $R(t,\phi)$ contains α_i horizontal dominoes in the ith row, each containing an i and $\beta_i + 1$ vertical dominoes in the ith column, numbered $i, i+1, \dots, i+\beta_i$. The Yamanouchi domino tableau of shape $R(\phi,t)$ contains $\alpha_i + 1$ horizontal dominoes in the ith row each containing an i and β_i vertical dominoes in the ith column, numbered $i+1, i+2, \dots, i+\beta_i$.

From these descriptions, it follows that

$$\varepsilon_2\Big(R(t,\phi)\Big) = (-1)^{r+\sum_{i=1}^r \beta_i} = (-1)^{\sum_{i=1}^r \alpha_i} \qquad \varepsilon_2\Big(R(\phi,t)\Big) = (-1)^{\sum_{i=1}^r \beta_i} \tag{6.5.6}$$

where we have used $r + \sum_i(\alpha_i + \beta_i) = n$.

The second way to obtain the quotient (6.5.2) is when the finite N cut-off on Young diagrams reduces the sum in (6.4.7) to a single Young diagram. From the LR rule described in appendix D.2, this happens when $t_1 = \left[k^{\frac{N}{2}}\right]$ for some $k > 0$ and t_2 is arbitrary, or vice versa. The corresponding t is obtained by placing k columns of length $\frac{N}{2}$ in front of the unrestricted t_i. Formally, this is $t = t_1 + t_2$.

To find the associated $R(t_1, t_2)$, one can use the definition (6.4.19), where the Frobenius notation for k columns of length $\frac{N}{2}$ is

$$\left[k^{\frac{N}{2}}\right] = (k-1, k-2, \dots, 0|N-1, N-2, \dots, N-k) \tag{6.5.7}$$

6.6 The orientifold $\mathbb{Z}_2$ action in the $U(N)$ gauge theory

In this section we propose a candidate ρ for the $\mathbb{Z}_2$ orientifold action in the $U(N)$ gauge theory. That is, the gauge theory equivalent of the map $x \to -x$ for $x \in S^5$ along with worldsheet reversal.

There is a natural candidate for ρ, namely the map induced by $X \to -X^T$. However, on a single matrix multi-trace operator of degree n, this would act merely as multiplication by $(-1)^n$. It was pointed out in [55] that the $AdS_5 \times \mathbb{RP}^5$ theory is composed only of states from the $AdS_5 \times S^5$ theory that are invariant under the $\mathbb{Z}_2$ orientifold action. Therefore this naive map would correctly predict that all operators with n odd would disappear in the $SO(N)$ theory, however it would also imply that the $U(N)$ and $SO(N)$ theories should have the same half-BPS sector for n even, which is incorrect.

Certainly a property ρ should posses is that an operator $\mathcal{O}_R^{U(N)}$ and its image $\rho\left(\mathcal{O}_R^{U(N)}\right)$ have the same orientifold quotient. The formula (6.4.8) then suggests an alternative candidate, since the Schur operators with Young diagrams $R(t_1, t_2)$ and $R(t_2, t_1)$ have the same quotient, up to a sign. We therefore conjecture that interchanging the 2-quotient of a $U(N)$ Young diagram should be interpreted in the dual AdS description as the geometric $\mathbb{Z}_2$ action of inverting the S^5 and reversing worldsheet orientation.

This action can be defined not just on operators of the form $R(t_1, t_2)$, but also on those with non-empty 2-core. Let $R(\Delta; t_1, t_2)$ be the Young diagram with 2-core Δ and 2-quotient (t_1, t_2). Then write

$$\rho\left[R(\Delta; t_1, t_2)\right] = R(\Delta; t_2, t_1) \tag{6.6.1}$$

On $U(N)$ operators, the $\mathbb{Z}_2$ action is

$$\rho\left(\mathcal{O}_R^{U(N)}\right) = \pm \mathcal{O}_{\rho(R)}^{U(N)} \tag{6.6.2}$$

where for R with empty 2-core, the sign is $\varepsilon_2(R)\varepsilon_2(\rho(R))$, while for R with non-empty 2-core, it is non-obvious which choice of sign we should take, and we leave it undetermined except for the consistency condition $\rho^2 = 1$.

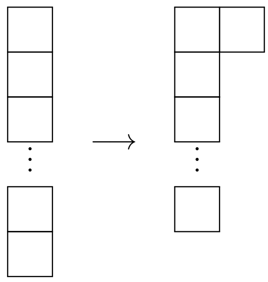
We check that ρ makes sense as the geometric action $x \to -x$ along with worldsheet inversion in two examples. One we expect to be invariant under the $\mathbb{Z}_2$ from geometric considerations in the dual AdS, and another we expect to be invariant from the gauge theory.

Firstly, take $R = [1^N]$, a single column of length N. This is dual to a single maximal giant graviton wrapped around an S^3 equator of the S^5. Under the action $x \to -x$, we expect this state to be invariant.

The 2-quotient (t_1, t_2) depends on whether N is even or odd. If N is even, then R has empty 2-core, and the 2-quotient is $\left(\left[1^{\frac{N}{2}}\right], \phi\right)$, whereas if N is odd, R has 2-core $[1]$ and the 2-quotient is $\left(\phi, \left[1^{\frac{N-1}{2}}\right]\right)$. Interchanging the 2-quotient, we have

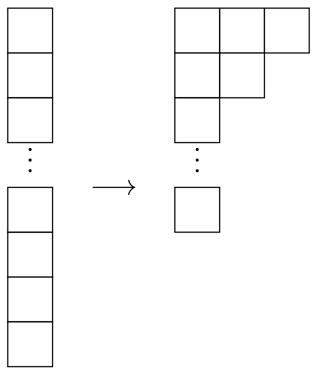
$$\left[1^N\right] \longrightarrow \begin{cases} \left[2, 1^{N-2}\right] & N \text{ even} \\ \left[3, 2, 1^{N-5}\right] & N \text{ odd} \end{cases} \tag{6.6.3}$$

Visually, if N even



(6.6.4)

and if N odd



(6.6.5)

So this state is not invariant under the 2-quotient interchange action. However, it does maintain its qualitative interpretation. For large N, the image Young diagram still has a single large column of length $\lesssim N$, interpreted as a single giant graviton. The extra boxes in latter columns are treated as small perturbations that do not change the qualitative behaviour.

It is worth noting that the maximal giant, wrapping an S^3 equator of S^5, is a classical state, and invariance under the geometric $\mathbb{Z}_2$ action does not necessarily transfer to the full quantum theory. The quantisation of sphere giants has been investigated in [27], and it would be interesting to see whether this approach can determine how this geometric $\mathbb{Z}_2$ behaves on the quantised Hilbert space.

For the second example, note that among those R with empty 2-core, the R with even length columns and rows are invariant under interchange of the 2-quotient. This was discussed above (6.4.22). In particular, one could consider $R = [2^k]$ for even k of order N. This corresponds to two sphere giants. The AdS dual state consists of two D3-branes wrapped on 3-sphere of the same size within the S^5. The two branes can be placed on anti-podal 3-spheres within S^5, related by the map $x \to -x$, thereby forming a classical state invariant under the geometric $\mathbb{Z}_2$.

In this argument we made a choice about where to place the branes in order to find a classically invariant state. Of course other classical states are possible where the branes are not anti-podal, and are therefore not invariant. In general, the branes wrap 3-spheres that rotate within the S^5, forming a $S^1 \times S^3$ worldvolume. Under a time-averaging procedure, the branes are spread out evenly along the S^1, and therefore the choice of position for the branes becomes irrelevant. We expect that the quantisation process will include such a step implicitly. As with the first example, it would be interesting to investigate whether this behaviour emerges from the quantisation process in [27].

It was observed below (6.2.7) that the projection coefficient α_R^T can be viewed as a refinement of the Littlewood-Richardson coefficient $g_{R;t,t}$. Interestingly, the same coefficient also appears as the extremal correlator $\left\langle \mathcal{O}_t^{U(N)}\mathcal{O}_t^{U(N)} \middle| \mathcal{O}_R^{U(N)} \right\rangle$ in the $U(N)$ theory [22]. Given the correspondence between Young diagrams and branes, this extremal correlator is naturally interpreted as the amplitude for the overlap between the composite system consisting of the pair of branes (t, t) and the brane R. The effect of the orientifold operation is to change the amplitude of interaction $t \otimes t \to R$ by introducing the sign in the projection coefficient (6.2.6).

An interesting direction of research is to investigate the action of ρ on the product $\mathcal{O}_t\mathcal{O}_t$ and in particular whether there is a relation to the sign change between (6.2.7) and (6.2.6). This may shed light on the orientifold quotient in the dual string theory, or even on the presence of Littlewood-Richardson coefficients in three point functions of giant gravitons. While there have been various tests on the agreement of brane physics in $AdS_5 \times S^5$ with the correlator formula in terms of LR coefficients [23–26, 92], a general understanding, directly from the spacetime perspective, of why the interaction of branes is given by the Littlewood-Richardson coeffients is not currently available.

6.7 $SO(N)/Sp(N)$ giant gravitons

Half-BPS operators labelled by Young diagrams R with a single column of length comparable to N are dual to single giant gravitons which are S^3 expanding in S^5. A single row with length of order N is dual to a single giant graviton wrapping an S^3 inside AdS_5. Multiple column or multiple row Young diagrams where the number of columns/rows is of order 1 and column/row lengths are comparable to N, are dual to multi-giants with the S^3 expanding in the appropriate factor. It is instructive to consider the domino algorithm for α_R^T in these regimes and develop a heuristic interpretation in terms of branes and orientifolds.

A natural first postulate is that the analogous picture for the connection between branes and rows or columns of the Young diagram works for t in the $SO(N)/Sp(N)$ theory. A single column t, with length comparable to N, is a single giant graviton with a large S^3 world-volume in the directions inside $\mathbb{RP}^5$ of $AdS^5 \times \mathbb{RP}^5$. Multiple long columns correspond to multi-giants of this type. A single long row with length of order N corresponds to a single giant, with large spatial world-volume in AdS^5. Multiple long rows correspond to multiple giants of this type. Note that among the giants which are large in the $\mathbb{RP}^5$ we also have those with worldvolume $\mathbb{R}P^3$ [58] corresponding to baryonic operators involving the ε-invariant. Since our focus here is on the projection to mesonic operators, these will not be part of the discussion that follows here.

It was demonstrated in [55] that the $AdS_5 \times \mathbb{RP}^5$ theory is composed only of states from the $AdS_5 \times S^5$ theory that are invariant under the $\mathbb{Z}_2$ orientifold action. Therefore to understand the brane interpretation of $SO(N)/Sp(N)$ operators, we can look at states in the $U(N)$ theory invariant under this $\mathbb{Z}_2$. The gauge theory version of this action was discussed in the previous section.

Take a Young diagram $t = [k_1, k_2, \ldots, k_r]$ with $r = O(1)$ and $k_i = O(N)$ for each i. Then a $U(N)$ operator, invariant under the $\mathbb{Z}_2$ action, that projects to $\mathcal{O}_T^{\delta/\Omega}$ is

$$\frac{1}{2}\left(\mathcal{O}_{R(t,\phi)} + (-1)^r \mathcal{O}_{R(\phi,t)}\right) \tag{6.7.1}$$

The Frobenius notation for t is $t = (k_1 - 1, k_2 - 2, \ldots, k_r - r | r - 1, r - 2, \ldots, 0)$, so using (6.5.4), the $U(N)$ Young diagrams are

$$R(t, \phi) = [2k_1 - 1, 2k_2 - 2, \ldots, 2k_r - r, r, r - 1, \ldots, 1] \tag{6.7.2}$$

$$R(\phi, t) = [2k_1, 2k_2 - 2, \ldots, 2k_r - r + 1, r - 1, r - 2, \ldots, 1] \tag{6.7.3}$$

These consist of of r long rows of length $O(N)$ with a staircase diagram of size $\frac{r(r\pm 1)}{2}$ attached beneath. Since $r = O(1)$, both diagrams correspond to small perturbations of r giant gravitons extended in AdS_5. Therefore t can naturally be interpreted as r AdS giants in the $AdS_5 \times \mathbb{RP}^5$ theory.

The conjugation property (6.4.24) of the projection coefficients mean t with r long columns also receives a similar interpretations to the $U(N)$ theory, as r giant gravitons wrapped around an S^3 within the $\mathbb{RP}^5$ factor.

6.8 Inverse projection coefficients and $U(N)$ correlators of $SO(N)/Sp(N)$ operators

The $U(N)$ half-BPS sector is spanned by multi-traces of the form

$$\prod_i \left(\mathrm{Tr} X^i\right)^{p_i} \tag{6.8.1}$$

where $p \vdash n$. The $SO(N)/Sp(N)$ half-BPS sectors are spanned by multi-traces of the form

$$\prod_i \left(\mathrm{Tr}X^{2i}\right)^{q_i} \tag{6.8.2}$$

where $q \vdash \frac{n}{2}$. Therefore one can consider the half-BPS sectors of the $SO(N)/Sp(N)$ theories as a subspace of the equivalent in the $U(N)$ theory.

This leads to the question, what does the $U(N)$ inner product look like on this subspace? The $SO(N)/Sp(N)$ theories have their own inner product, but this is a different pairing with a different structure. In this thesis, we have made extensive use of permutations as a way to describe bases of gauge-invariant operators in theories with different gauge groups. They give us a uniform way of talking about operators in different gauge theories, namely about how the indices of matrices X, Y are contracted without being specific about whether these are generic matrices in the Lie algebra $\mathfrak{u}(N)$, anti-symmetric matrices in $\mathfrak{so}(N)$, or matrices in $\mathfrak{sp}(N)$. These different theories, via AdS/CFT duality, correspond to different string theory backgrounds. In this sense, permutations are background independent structures, while the pairings we put on them are theory-dependent. Here we will see that exploring the $U(N)$ inner product which survive the projection to $SO(N)$ has interesting relations to an appropriately defined inverse of the plethysm coefficients α^T_R.

Consider the $SO(N)/Sp(N)$ operators (5.6.12), but where X is an arbitrary complex matrix rather than anti-symmetric. We can express this as a sum of $U(N)$ operators

$$\mathcal{O}^{\delta/\Omega}_T\Big|_{X\in\mathfrak{u}(N)} = \sum_{R\vdash n} \beta^R_T \mathcal{O}^{U(N)}_R \tag{6.8.3}$$

Then taking the $\mathbb{Z}_2$ orientifold quotient of this expression, the left hand side returns to the standard $SO(N)/Sp(N)$ Schur operator, while we can evaluate the right hand side using the coefficients α^T_R. We have

$$\mathcal{O}^{\delta/\Omega}_T = \sum_{R\vdash n}\sum_{T'} \beta^R_T \alpha^{T'}_R \mathcal{O}^{\delta/\Omega}_{T'} \tag{6.8.4}$$

Since this holds for all T, we have

$$\sum_{R\vdash n} \beta^R_T \alpha^{T'}_R = \delta^{T'}_T \tag{6.8.5}$$

so we call β^R_T inverse projection coefficients. They are not true inverses to α^T_R, since R has more degrees of freedom than T, and summing over T will not lead to $\delta^R_{R'}$.

To find β^R_T, we use the orthogonality relation (2.3.9) to invert the definition (2.3.14) and give traces in terms of $U(N)$ Schur operators

$$\prod_i \left(\mathrm{Tr}X^i\right)^{p_i} = \sum_{R\vdash n} \chi_R(p)\mathcal{O}_R^{U(N)} \tag{6.8.6}$$

Substituting into the definition (5.6.12) for $SO(N)/Sp(N)$ Schur operators, we have

$$\mathcal{O}_T^{\delta/\Omega}\Big|_{X\in\mathfrak{u}(N)} = \sum_{p\vdash\frac{n}{2}} \frac{1}{z_{2p}}\chi_t(p) \sum_{R\vdash n} \chi_R(2p)\mathcal{O}_R^{U(N)} \tag{6.8.7}$$

and therefore

$$\beta_T^R = \sum_{p\vdash\frac{n}{2}} \frac{1}{z_{2p}}\chi_R(2p)\chi_t(p) = \sum_{p\vdash\frac{n}{2}} \frac{1}{2^{l(p)}z_p}\chi_R(2p)\chi_t(p) \tag{6.8.8}$$

Note the similarities between this and the expression (6.1.18) for α_R^T. There is an extra factor of $2^{-l(p)}$ inside the sum, and in general β_T^R are non-integer. We have not been able to find a combinatoric interpretation of β_T^R.

Using the formula (2.6.11) for the $U(N)$ correlators, we have

$$\left\langle \mathcal{O}_T^{\delta/\Omega}\Big|_{X\in\mathfrak{u}(N)} \middle| \mathcal{O}_{T'}^{\delta/\Omega}\Big|_{X\in\mathfrak{u}(N)} \right\rangle = \sum_{R\vdash n} \beta_T^R\beta_{T'}^R f_R \tag{6.8.9}$$

So the $SO(N)/Sp(N)$ Schur operators are not orthogonal under the $U(N)$ inner product, even at large N.

6.9 The orientifold quotient in the free field quarter-BPS sector

Consider a 2-matrix $U(N)$ multi-trace $\mathrm{Tr}W^k$, where $W(X,Y)$ is a matrix Lyndon word on the letters X and Y. After replacing X and Y with $\mathfrak{so}(N)$ or $\mathfrak{sp}(N)$ matrices, this trace may vanish or not depending on the relation (4.1.3). In section 4.1 we investigated this formula in detail and gave a description of the non-vanishing traces in terms of orthogonal Lyndon words. These are not simple objects to work with, and so rather than working with multi-traces as we did in the half-BPS sector, we instead work with permutations which allow a simpler description of the quotient.

The key results that enabled the evaluation of the projection coefficients in the half-BPS sector were the expressions for $U(N)$, $SO(N)$ and $Sp(N)$ Schur operators in terms of Schur symmetric functions. The quarter-BPS sector does not have the same structure, and finding formulae for the coefficients is therefore more difficult. The domino combinatorics of section 6.2 allows us to give sufficient conditions for when the coefficients vanish, but we are not able to evaluate the non-zero coefficients.

In section 3.6 we gave two different orthogonal bases that generalise the Schur basis to the free field quarter-BPS sector, the restricted Schur basis and the covariant basis. In this section we study the quotient of the restricted Schur basis in detail, and briefly mention how one might approach the covariant basis.

In the restricted Schur basis, the $SO(N)$ and $Sp(N)$ mesonic operators do not have the same expression in terms of multi-traces, and therefore the exchange between the two theories is slightly more complex. Define the projection coefficients

$$\mathcal{O}^{U(N)}_{R,R_1,R_2,\mu,\nu} \xrightarrow{\mathbb{Z}_2} \begin{cases} \sum_{T,T_1,T_2,\lambda} a^{T,T_1,T_2,\lambda}_{R,R_1,R_2,\mu,\nu} \mathcal{O}^{\delta}_{T,T_1,T_2,\lambda} & \text{in the } SO(N) \text{ theory} \\ \sum_{T,T_1,T_2,\lambda} b^{T,T_1,T_2,\lambda}_{R,R_1,R_2,\mu,\nu} \mathcal{O}^{\Omega}_{T,T_1,T_2,\lambda} & \text{in the } Sp(N) \text{ theory} \end{cases} \tag{6.9.1}$$

Under anti-symmetrisation of traces and conjugation of the labels T, T_1, T_2, the $SO(N)$ and $Sp(N)$ transform into each other (5.6.73). For the $U(N)$ operators, we have the conjugation relation

$$\mathcal{O}^{U(N)}_{R^c,R_1^c,R_2^c,\mu,\nu} = \text{Anti-Sym}\left(\mathcal{O}^{U(N)}_{R,R_1,R_2,\mu,\nu}\right) \tag{6.9.2}$$

Therefore

$$a^{T,T_1,T_2,\lambda}_{R^c,R_1^c,R_2^c,\mu,\nu} = b^{T^c,T_1^c,T_2^c,\lambda}_{R,R_1,R_2,\mu,\nu} \tag{6.9.3}$$

So the projection coefficients of the $SO(N)$ theory determine the projection coefficients of the $Sp(N)$ theory. For the rest of this chapter we work with $SO(N)$ for simplicity. All results can be transferred across using (6.9.3).

Some of the free field operators considered in this section recombine into long multiplets when we turn on interactions. Therefore we cannot give an interpretation of these projection coefficients in terms of giant gravitons, which are dual to strong coupling quarter-BPS operators. In chapter 7, we give an approach to constructing weak coupling operators. It is believed that there is no further change in the quarter-BPS spectrum occurs as we travel from weak to strong coupling, and therefore it is these operators that should be used to study the orientifold physics of quarter-BPS giant gravitons. The work in this section is a first step along this path.

6.9.1 Quotient on $SO(N)$ restricted Schurs

It follows from (5.3.16) that for $\sigma \in S_n$

$$\text{Tr}\left(\sigma X^{\otimes n_1} Y^{\otimes n_2}\right) \xrightarrow{\mathbb{Z}_2} C^{(\delta)}_I \left(\sigma^{(odd)}\right)^I_{\ J} \left(X^{\otimes n_1} Y^{\otimes n_2}\right)^J \tag{6.9.4}$$

where $\sigma^{(odd)}$ is the permutation $\sigma \in S_n$ when S_n is embedded in S_{2n} by acting on only the odd numbers. This embedding is called $S_n^{(odd)}$.

From the definition (3.6.1) for a $U(N)$ restricted Schur operator

$$\mathcal{O}^{U(N)}_{R,R_1,R_2,\mu,\nu} \xrightarrow{\mathbb{Z}_2} \sqrt{\frac{d_R}{d_{R_1}d_{R_2}n!n_1!n_2!}} \sum_{\sigma\in S_n} \chi_{R,R_1,R_2,\mu,\nu}(\sigma) C^{(\delta)}_I \left(\sigma^{(odd)}\right)^I_J \left(X^{\otimes n_1} Y^{\otimes n_2}\right)^J \tag{6.9.5}$$

Inverting the definition (5.6.70) to give a $SO(N)$ multi-trace in terms of $SO(N)$ restricted Schurs, for any $\tau \in S_{2n}$

$$C^{(\delta)}_I \tau^I_J \left(X^{\otimes n_1} Y^{\otimes n_2}\right)^J = 2^n \sqrt{n!n_1!n_2!} \sum_{T,T_1,T_2,\lambda} \sqrt{\frac{d_T}{(2n)!}} \left\langle T,[S]\right| D^T(\tau) \left|T_1,T_2,[A],\lambda\right\rangle \mathcal{O}^{\delta}_{T,T_1,T_2,\lambda} \tag{6.9.6}$$

Therefore

$$\mathcal{O}^{U(N)}_{R,R_1,R_2,\mu,\nu} \xrightarrow{\mathbb{Z}_2} 2^n \sqrt{\frac{d_R}{d_{R_1}d_{R_2}}} \sum_{T,T_1,T_2,\lambda} \sqrt{\frac{d_T}{(2n)!}} \sum_{\sigma\in S_n} \chi_{R,R_1,R_2,\mu,\nu}(\sigma) \left\langle T,[S]\right| D^T(\sigma^{(odd)}) \left|T_1,T_2,[A],\lambda\right\rangle \mathcal{O}^{\delta}_{T,T_1,T_2,\lambda} \tag{6.9.7}$$

and the projection coefficient is

$$a^{T,T_1,T_2,\lambda}_{R,R_1,R_2,\mu,\nu} = 2^n \sqrt{\frac{d_R d_T}{d_{R_1}d_{R_2}(2n)!}} \sum_{\sigma\in S_n} \chi_{R,R_1,R_2,\mu,\nu}(\sigma) \left\langle T,[S]\right| D^T\left(\sigma^{(odd)}\right) \left|T_1,T_2,[A],\lambda\right\rangle \tag{6.9.8}$$

To investigate the properties of the projection coefficient, consider the intertwiner $P^R_{R_1,R_2;\mu\to\nu}$ used in the definition (3.6.2) of the restricted character. This can be constructed explicitly as sum of permutations using the algebra $\mathcal{A}^R_{R_1,R_2}$ discussed in appendix D.3. In particular, it is only non-zero in the representation R of S_n, and the representation $R_1 \otimes R_2$ when restricted to a representation of $S_{n_1} \times S_{n_2}$.

From the definition of $P^R_{R_1,R_2;\mu\to\nu}$ as the operator taking the μth copy of $R_1 \otimes R_2$ inside R to the νth copy, the transpose of the matrix representative of $P^R_{R_1,R_2;\mu\to\nu}$ switches the role of μ and ν

$$\left[D^R\left(P^R_{R_1,R_2;\mu\to\nu}\right)\right]^T = D^R\left(P^R_{R_1,R_2;\nu\to\mu}\right) \tag{6.9.9}$$

From the orthogonality of representations of S_n, this corresponds to inverting each permutation in the sum, referred to as linear inversion (5.5.3).

It follow from standard properties of characters that

$$\begin{aligned}\sum_{\sigma\in S_n}\chi_{R,R_1,R_2,\mu,\nu}(\sigma)D^T\left(\sigma^{(odd)}\right) &= \sum_{\sigma\in S_n}\chi_R\left(P^R_{R_1,R_2;\mu\to\nu}\sigma\right)D^T\left(\sigma^{(odd)}\right)\\ &= \sum_{\sigma\in S_n}\chi_R(\sigma)\,D^T\left(P^{R;(odd)}_{R_1,R_2;\nu\to\mu}\sigma^{(odd)}\right)\\ &= \frac{n!}{d_R}D^T\left(P^{R;(odd)}_{R_1,R_2;\nu\to\mu}P^{(odd)}_R\right)\\ &= \frac{n!}{d_R}D^T\left(P^{R;(odd)}_{R_1,R_2;\nu\to\mu}\right)\end{aligned} \tag{6.9.10}$$

So the projection coefficients are given by

$$a^{T,T_1,T_2,\lambda}_{R,R_1,R_2,\mu,\nu} = 2^n n!\sqrt{\frac{d_T}{d_R d_{R_1} d_{R_2}(2n)!}}\,\langle T,[S]|\,D^T\left(P^{R;(odd)}_{R_1,R_2;\nu\to\mu}\right)|T_1,T_2,[A],\lambda\rangle \tag{6.9.11}$$

In analogy to $S_n^{(odd)}$, there is another embedding $S_n^{(even)}$ of S_n into S_{2n} that acts only on the even numbers. Given $\sigma\in S_n$, the product $\sigma^{(diag)}=\sigma^{(odd)}\sigma^{(even)}$ is in the wreath product group $S_n[S_2]$. As $|T,[S]\rangle$ is invariant under $S_n[S_2]$, we have

$$\begin{aligned}\langle T,[S]|\,D^T\left(\sigma^{(odd)}\right)|T_1,T_2,[A],\lambda\rangle &= \langle T,[S]|\,D^T\left[\left(\sigma^{(diag)}\right)^{-1}\sigma^{(odd)}\right]|T_1,T_2,[A],\lambda\rangle\\ &= \langle T,[S]|\,D^T\left[\left(\sigma^{(even)}\right)^{-1}\right]|T_1,T_2,[A],\lambda\rangle\end{aligned} \tag{6.9.12}$$

From (6.9.9) and the discussion below it, we have

$$\begin{aligned}&\langle T,[S]|\,D^T\left(P^{R;(odd)}_{R_1,R_2;\nu\to\mu}\right)|T_1,T_2,[A],\lambda\rangle =\\ &\qquad\langle T,[S]|\,D^T\left(P^{R;(even)}_{R_1,R_2;\mu\to\nu}\right)|T_1,T_2,[A],\lambda\rangle\end{aligned} \tag{6.9.13}$$

There is another method to switch between the embeddings $S_n^{(even)}$ and $S_n^{(odd)}$ using the permutation

$$\pi = (1,2)(3,4)\dots(2n-1,2n) \tag{6.9.14}$$

For any $\sigma\in S_n$, we have

$$\pi\sigma^{(odd)}\pi = \sigma^{(even)} \tag{6.9.15}$$

We observe that π is in both $S_n[S_2]$ and $S_{n_1}[S_2]\times S_{n_2}[S_2]$ with sign $(-1)^n$, and therefore

$$\begin{aligned}\langle T,[S]|\,D^T\left(\sigma^{(odd)}\right)|T_1,T_2,[A],\lambda\rangle &= (-1)^n\,\langle T,[S]|\,D^T\left(\pi\sigma^{(odd)}\pi\right)|T_1,T_2,[A],\lambda\rangle\\ &= (-1)^n\,\langle T,[S]|\,D^T\left[\sigma^{(even)}\right]|T_1,T_2,[A],\lambda\rangle\end{aligned} \tag{6.9.16}$$

On the projectors, we have

$$\langle T, [S]| D^T \left(P^{R;(odd)}_{R_1,R_2;\nu\to\mu}\right) |T_1, T_2, [A], \lambda\rangle = \\ (-1)^n \langle T, [S]| D^T \left(P^{R;(even)}_{R_1,R_2;\nu\to\mu}\right) |T_1, T_2, [A], \lambda\rangle \qquad (6.9.17)$$

Comparing with (6.9.13), the projection coefficients are symmetric or anti-symmetric in the $U(N)$ multiplicity indices μ, ν depending on the parity of n

$$a^{T,T_1,T_2,\lambda}_{R,R_1,R_2,\mu,\nu} = (-1)^n a^{T,T_1,T_2,\lambda}_{R,R_1,R_2,\nu,\mu} \qquad (6.9.18)$$

We now investigate the structure of the projection coefficient in terms of Littlewood-Richardson multiplicity spaces, and prove that it can be expressed as a simple inner product on one of these spaces.

Perform a Littlewood-Richardson decomposition of the T representation of S_{2n} into representations of $S_n^{(odd)} \times S_n^{(even)}$

$$V_T^{S_{2n}} = \bigoplus_{S,S'\vdash n} V_S^{S_n^{(odd)}} \otimes V_{S'}^{S_n^{(even)}} \otimes V^{mult}_{T;S,S'} \qquad (6.9.19)$$

where $V^{mult}_{T;S,S'}$ is the multiplicity space of dimension $g_{T;S,S'}$. The intertwiner ensures that only the $S = R$ representation of $S_n^{(odd)}$ contributes to the projection coefficient.

Consider the embedding of S_n into S_{2n} defined by

$$\sigma \to \sigma^{(odd)}\sigma^{(even)} \qquad (6.9.20)$$

We call this embedding $S_n^{(diag)}$ as it is the diagonal subgroup of $S_n^{(odd)} \times S_n^{(even)}$. A representation $S \otimes S'$ of $S_n^{(odd)} \times S_n^{(even)}$ is a Clebsch-Gordon tensor product representation of $S_n^{(diag)}$. In particular, it contains the trivial representation of $S_n^{(diag)}$ if and only if $S = S'$. As $|T, [S]\rangle$ is invariant under permutations of the form $\sigma^{(odd)}\sigma^{(even)}$, it lies in the trivial representation of $S_n^{(diag)}$, and therefore only the term with $S = S' = R$ in (6.9.19) contribute to the projection coefficient.

Let $|i\rangle_R$ be a basis for the R representation of S_n. Then the unit S_n invariant vector in $R \otimes R$ is

$$|\text{triv}\rangle_{RR} = \sum_{i=1}^{d_R} \frac{1}{\sqrt{d_R}} |i\rangle_R \otimes |i\rangle_R \qquad (6.9.21)$$

and we can write $|T, [S]\rangle$

$$|T, [S]\rangle = \sum_{R\vdash n} |\text{triv}\rangle_{RR} \otimes |+\rangle^T_{RR} \qquad (6.9.22)$$

where $|+\rangle^T_{RR}$ is a vector in $V^{mult;+}_{T;R,R}$, the $+1$ eigenspace of $V^{mult}_{T;R,R}$ under π introduced below (6.1.22).

So provided T contains a symmetric copy of $R \otimes R$, we have

$$a^{T,T_1,T_2,\lambda}_{R,R_1,R_2,\mu,\nu} = 2^n n! \sqrt{\frac{d_T}{d_R d_{R_1} d_{R_2} (2n)!}}$$
$$\left(\langle \text{triv}|_{RR} \otimes \langle +|^T_{RR}\right) \left[D^{R;(odd)}\left(P^R_{R_1,R_2;\nu\to\mu}\right) \otimes D^{R;(even)}(1) \otimes I^T_{RR}\right] |T_1, T_2, [A], \lambda\rangle \tag{6.9.23}$$

where I^T_{RR} is the identity operator on the multiplicity space $V^{mult}_{T;R,R}$.

Now decompose the first (odd) copy of R further into representations $S_1 \otimes S_2$ of $S^{(odd)}_{n_1} \times S^{(odd)}_{n_2}$, and the second (even) copy into representations $S'_1 \otimes S'_2$ of $S^{(even)}_{n_1} \times S^{(even)}_{n_2}$. As before, the intertwiner ensures that only $S_1 = R_1$ and $S_2 = R_2$ contribute, while $|T_1, T_2, [A], \lambda\rangle$ enforces $S'_1 = S_1$ and $S'_2 = S_2$.

Each element of (6.9.23) has a corresponding decomposition. For $|T_1, T_2, [A], \lambda\rangle$, this is

$$\begin{aligned} |T_1, T_2, [A], \lambda\rangle &= |T_1, [A]\rangle \otimes |T_2, [A]\rangle \otimes |\lambda\rangle^T_{T_1T_2} \\ &= \sum_{\substack{S_1\vdash n_1 \\ S_2 \vdash n_2}} |\text{triv}\rangle_{S_1S_1} \otimes |-\rangle^{T_1}_{S_1S_1} \otimes |\text{triv}\rangle_{S_2S_2} \otimes |-\rangle^{T_2}_{S_2S_2} \otimes |\lambda\rangle^T_{T_1T_2} \end{aligned} \tag{6.9.24}$$

where $|\lambda\rangle^T_{T_1T_2}$ is a basis vector for $V^{mult}_{T;T_1,T_2}$ and $|-\rangle^{T_i}_{S_iS_i}$ is a vector in $V^{mult;(-1)^{n_i}}_{T_i;S_i,S_i}$.

The vector $|\text{triv}\rangle_{RR}$ decomposes as

$$\begin{aligned} |\text{triv}\rangle_{RR} &= \frac{1}{\sqrt{d_R}} \sum_{i=1}^{d_R} |i\rangle_R \otimes |i\rangle_R \\ &= \frac{1}{\sqrt{d_R}} \sum_{\substack{S_1\vdash n_1 \\ S_2 \vdash n_2}} \sum_{j=1}^{d_{S_1}} \sum_{k=1}^{d_{S_2}} \sum_{\rho=1}^{g_{R;S_1,S_2}} \left(|j\rangle_{S_1} \otimes |k\rangle_{S_2} \otimes |\rho\rangle^R_{S_1S_2}\right) \otimes \left(|j\rangle_{S_1} \otimes |k\rangle_{S_2} \otimes |\rho\rangle^R_{S_1S_2}\right) \\ &= \frac{1}{\sqrt{d_R}} \sum_{\substack{S_1\vdash n_1 \\ S_2 \vdash n_2}} \sqrt{d_{S_1} d_{S_2}} |\text{triv}\rangle_{S_1S_1} \otimes |\text{triv}\rangle_{S_2S_2} \otimes |\text{triv}\rangle^{RR}_{S_1S_1S_2S_2} \end{aligned} \tag{6.9.25}$$

where $|\text{triv}\rangle^{RR}_{S_1S_1S_2S_2}$ is in the tensor product multiplicity space $V^{mult}_{R;S_1,S_2} \otimes V^{mult}_{R;S_1,S_2}$ and is given by

$$|\text{triv}\rangle^{RR}_{S_1S_1S_2S_2} = \sum_{\rho=1}^{g_{R;S_1,S_2}} |\rho\rangle^R_{S_1S_2} \otimes |\rho\rangle^R_{S_1S_2} \tag{6.9.26}$$

Finally, the intertwiner is

$$D^R\left(P^R_{R_1,R_2;\mu\to\nu}\right) = D^{R_1\otimes R_2}(1) \otimes |\nu\rangle^R_{R_1R_2} \langle\mu|^{R;}_{R_1R_2} \tag{6.9.27}$$

where $|\mu\rangle^R_{R_1R_2}$ is a basis vector for $V^{mult}_{R;R_1,R_2}$.

Before plugging these into the projection coefficient, note that

$$\begin{aligned}
&\langle\text{triv}|^{RR}_{R_1R_1R_2R_2}\left[D^{R;(odd)}\left(P^R_{R_1,R_2;\nu\to\mu}\right)\otimes I^{R;(even)}_{R_1R_2}\right]\\
&\quad=\left(\sum_{\rho=1}^{g_{R;R_1,R_2}}\langle\rho|^{R;(odd)}_{R_1R_2}\otimes\langle\rho|^{R;(even)}_{R_1R_2}\right)\left[\left(|\mu\rangle^{R;(odd)}_{R_1R_2}\langle\nu|^{R;(odd)}_{R_1R_2}\right)\otimes I^{R;(even)}_{R_1R_2}\right]\\
&\quad=\sum_{\rho=1}^{g_{R;R_1,R_2}}\delta_{\rho\mu}\langle\nu|^{R;(odd)}_{R_1R_2}\otimes\langle\rho|^{R;(even)}_{R_1R_2}\\
&\quad=\langle\nu|^{R;(odd)}_{R_1R_2}\otimes\langle\mu|^{R;(even)}_{R_1R_2}
\end{aligned}\tag{6.9.28}$$

Putting these together, the expression (6.9.23) for the projection coefficients simplifies to

$$\begin{aligned}
a^{T,T_1,T_2,\lambda}_{R,R_1,R_2,\mu,\nu}&=\frac{2^n n!}{d_R}\sqrt{\frac{d_T}{(2n)!}}\left(\langle\text{triv}|_{R_1R_1}\otimes\langle\text{triv}|_{R_2R_2}\otimes\langle\text{triv}|^{RR}_{R_1R_1R_2R_2}\otimes\langle+|^T_{RR}\right)\\
&\qquad\left[D^{R_1\otimes R_1\otimes R_2\otimes R_2}(1)\otimes\left(|\mu\rangle^{R;(odd)}_{R_1R_2}\langle\nu|^{R;(odd)}_{R_1R_2}\right)\otimes I^{R;(even)}_{R_1R_2}\otimes I^T_{RR}\right]\\
&\qquad\left(|\text{triv}\rangle_{R_1R_1}\otimes|-\rangle^{T_1}_{R_1R_1}\otimes|\text{triv}\rangle_{R_2R_2}\otimes|-\rangle^{T_2}_{R_2R_2}\otimes|\lambda\rangle^T_{T_1T_2}\right)\\
&=\frac{2^n n!}{d_R}\sqrt{\frac{d_T}{(2n)!}}\langle\text{triv}|_{R_1R_1}|\text{triv}\rangle_{R_1R_1}\langle\text{triv}|_{R_2R_2}|\text{triv}\rangle_{R_2R_2}\\
&\qquad\left(\langle+|^T_{RR}\otimes\langle\nu|^{R;(odd)}_{R_1R_2}\otimes\langle\mu|^{R;(even)}_{R_1R_2}\right)\left(|-\rangle^{T_1}_{R_1R_1}\otimes|-\rangle^{T_2}_{R_2R_2}\otimes|\lambda\rangle^T_{T_1T_2}\right)\\
&=\frac{2^n n!}{d_R}\sqrt{\frac{d_T}{(2n)!}}\left(\langle+|^T_{RR}\otimes\langle\nu|^{R;(odd)}_{R_1R_2}\otimes\langle\mu|^{R;(even)}_{R_1R_2}\right)\left(|-\rangle^{T_1}_{R_1R_1}\otimes|-\rangle^{T_2}_{R_2R_2}\otimes|\lambda\rangle^T_{T_1T_2}\right)
\end{aligned}\tag{6.9.29}$$

This is an inner product in the vector space $V^{mult}_{T;R_1,R_1,R_2,R_2}$, the multiplicity space for the decomposition of T as a representation of S_{2n} into $R_1\otimes R_1\otimes R_2\otimes R_2$ as a representation of $S_{n_1}\otimes S_{n_1}\otimes S_{n_2}\otimes S_{n_2}$. The left hand side decomposes via the representation $R\otimes R$ of $S_n\otimes S_n$, while the right hand side decomposes via the representation $T_1\otimes T_2$ of $S_{2n_1}\times S_{2n_2}$. The difficulty in evaluating the projection coefficients is understanding how these two decompositions interact.

The combinatoric discussion of domino tableaux in section 6.2 does play a role, as it determines the dimensions of the various multiplicity spaces. Let $\widehat{R}=R(R,R)$ be the Young diagram with empty 2-core and 2-quotient (R,R), so that $R=\frac{\widehat{R}}{4}$, and similarly for $\widehat{R}_1$ and $\widehat{R_2}$. Then

$$\text{Dim}\left(V^{mult;+}_{T;R,R}\right) = D^{\widehat{R}}_{+,T} \tag{6.9.30}$$

$$\text{Dim}\left(V^{mult;(-1)^{n_i}}_{T_i;R_i,R_i}\right) = D^{\widehat{R}_i}_{(-1)^{n_i},T_i} \tag{6.9.31}$$

Since $|+\rangle^{T}_{RR}$, $|-\rangle^{T_i}_{R_iR_i}$ belong to these spaces, if $D^{\widehat{R}}_{+,T}$ or $D^{\widehat{R}_i}_{(-1)^{n_i},T_i}$ is zero then the corresponding projection coefficient vanishes.

6.9.2 Covariant projection coefficients

In sections 3.6.2 and 5.7 we defined $U(2)$ covariant bases for the $U(N)$ and $SO(N)/Sp(N)$ quarter-BPS sectors. One could consider the effect of the orientifold quotient in terms of these bases. This would most naturally be done using a covariant bases for $U(N)$ and $SO(N)/Sp(N)$ multi-traces and a covariant version of characters, equivalent to the restricted character used in restricted Schur operators.

Steps towards a covariant trace basis for the $U(N)$ theory, labelled by an integer partition which determines the single trace structure, are taken in section 7.3.2. We have not investigated such bases for the $SO(N)$ or $Sp(N)$ theories, and leave this as an interesting problem for the future.

Chapter 7

Quarter-BPS operators in the $U(N)$ theory at weak coupling

The construction of quarter BPS operators from the gauge theory side has been developed in [49, 93–95]. At zero field theory coupling, the quarter BPS states are general holomorphic operators built from two complex matrices X and Y. A subspace of these operators is annihilated by the one-loop dilatation operator and forms the weak coupling quarter BPS space. An important outcome of these papers is that the weak coupling quarter BPS operators form the orthogonal subspace, in the free field inner product, of the operators which contain commutators $[X, Y]$ within a trace. This a well-defined characterisation of the quarter-BPS operators at finite N.

In the free field $U(2)$ covariant constructions of quarter BPS operators [43, 46] having a total of n copies of X and Y, the labels consist of a Young diagram R with n boxes and columns no longer than N, a Young diagram Λ with n boxes and columns no longer than 2, along with a label τ which runs over the multiplicity of trivial S_n irreps in $R \otimes R \otimes \Lambda$. The Young diagram label Λ is also a representation of the global symmetry $U(2)$. The construction of weak-coupling quarter-BPS operators based on this new understanding of the finite N inner product was further developed in [51, 96]. The finite N construction of quarter BPS operators was given in terms of a projector $\mathcal{P}_N$ in $\mathbb{C}(S_n)$, which projects to the intersection of two subspaces of $\mathbb{C}(S_n)$ [51]. One subspace is associated with the symmetrised traces at large N, another with the finite N cut-off on the free field basis.

There has not been so far, a general construction of quarter-BPS operators at weak coupling and finite N, which includes a $U(2)$ Young diagram label alongside a $U(N)$ Young diagram label. In this chapter, we will address this open problem and give a basis of operators which are quarter-BPS at weak coupling, orthogonal with respect to the free field inner product, and labelled by a $U(2)$ Young diagram, a $U(N)$ Young diagram, alongside an associated multiplicity label depending on these two Young diagrams. The virtue of having a $U(N)$ Young diagram is that the disappearance of states upon a reduction of N to $N-1$ can be directly expressed in terms of this Young diagram - the disappearing states as N is reduced to $N-1$ are precisely the ones corresponding to Young diagrams with exactly N rows. We may therefore describe our basis as an SEP-compatible (SEP = stringy exclusion principle) basis which is also $U(2)$ covariant.

The key ingredient which allows us to find a manifestly SEP-compatible $U(2)$ covariant construction of quarter BPS states is the mathematics of multi-symmetric functions [97–99]. When gauge invariant functions of two matrices X, Y are evaluated on diagonal matrices $X = \mathrm{Diag}(x_1, x_2, \cdots, x_N)$ and $Y = \mathrm{Diag}(y_1, y_2, \cdots, y_N)$, we get polynomials which are invariant under the $\sigma \in S_N$ acting simultaneously as

$$\begin{aligned} x_i &\to x_{\sigma(i)} \\ y_i &\to y_{\sigma(i)} \end{aligned} \tag{7.0.1}$$

These polynomials are called *multi-symmetric functions*. More generally, we can have variables x_i^a with $a \in \{1, 2, \cdots, M\}$ and $i \in \{1, 2, \cdots, N\}$. Polynomials invariant under simultaneous S_N permutations of all the M vectors are more general multi-symmetric functions. There is a rich mathematics associated with changing between different bases of multi-symmetric functions for any M which is relevant in this chapter and is controlled by an underlying structure of *set partitions*.

The chapter is organised as follows. Section 7.1 is an introduction to the necessary background and key mathematical tools we will use to derive a basis for weak coupling quarter BPS operators. In section 7.2 we use the combinatorics of set partitions to derive results on the transformation between two bases for multi-symmetric functions. The first is the trace basis. Elements of this basis set are obtained by specifying a trace structure for matrices X, Y, or more generally $X^1, \cdots, X^M$ and specialising to diagonal matrices. Another basis is the multi-symmetric monomial basis, which allows a simple description of finite N cut-offs. In section 7.3, we start from the observation that every vector partition $\mathbf{p}$ defines an associated partition p, which is invariant under the action of the $U(2)$ transformations which interchange X, Y. We use results on plethysms of $SU(2)$ representations to obtain detailed expressions for refined multiplicities depending on a pair of Young diagram Λ, p, where Λ is a $U(2)$ Young diagram and p is a Young diagram constrained to have no more than N rows, which we refer to as a $U(N)$ Young diagram. In section 7.4 we describe an algorithm for producing a basis of operators labelled by the pair of Young diagrams (Λ, p) alongside the appropriate multiplicity label. The basis is orthogonal under the free field inner product. In section 7.5 we elucidate the vector space geometry within $\mathbb{C}(S_n)$, involving the interplay between a projector for the $U(2)$ flavour symmetry, a projector for the symmetrisation of traces $\mathcal{P}$ and an operator $\mathcal{F}_N$ whose kernel implements finite N constraints. This discussion allows us to show that the counting and two-point correlators of quarter-BPS operators at weak coupling can be expressed in terms of observables in two-dimensional topological field theory based on permutation group algebras with appropriate defects.

This chapter consists of work originally presented in [2]

7.1 Background on construction of quarter BPS operators

When we turn the coupling on in $\mathcal{N} = 4$ SYM, some of the short quarter-BPS multiplets at zero coupling recombine and form long non-BPS multiplets. We give two equivalent ways of characterising which 2-matrix multi-trace operators remain quarter-BPS, and which do not.

Firstly, non-BPS multi-traces of X, Y are SUSY descendants. It was explained in [49] that these are exactly commutator traces. That is, they are multi-traces (or linear combinations thereof) where at least one of the constituent single traces contains a commutator $[X, Y]$.

Secondly, consider the one-loop dilatation operator [50, 100]

$$\mathcal{H}_2 = -\mathrm{Tr}\left([X, Y]\left[\frac{\partial}{\partial X}, \frac{\partial}{\partial Y}\right]\right) \tag{7.1.1}$$

Quarter-BPS operators are annihilated by $\mathcal{H}_2$, and as $\mathcal{H}_2$ is hermitian in the free field inner product, they are orthogonal to the image. It is clear from the definition (7.1.1) that states in the image are commutator traces. While it is not immediately obvious that all commutator traces live in the image, our numerical calculations indicate that they are, and consistency with [49] implies they should be.

The dilatation operator (7.1.1) is Hermitian in the free field inner product (3.6.25), therefore the multi-traces that remain quarter-BPS as we move to weak coupling are those that are orthogonal to the commutator traces. This inner product is difficult to evaluate on multi-trace operators, so [51] took a different approach, instead using the S_n inner product (3.6.26) and relating this to the physical inner product using operators $\mathcal{F}_N$ and $\mathcal{G}_N$.

Comparing the two inner products (3.6.25) and (3.6.26), the only difference is a factor of Ω_N, defined in (2.3.17) (in this chapter, we add an N subscript to emphasise the dependence on N). $\mathcal{F}_N$ implements multiplication by Ω_N on multi-trace operators

$$\mathcal{F}_N \mathcal{O}_{a,\sigma} = \mathcal{O}_{a,\Omega_N \sigma} \tag{7.1.2}$$

where the covariant multi-trace operator $\mathcal{O}_{a,\sigma}$ was defined in (3.6.19). Comparing the definitions (3.6.25) and (3.6.26), it follows that for any quarter-BPS operators $\mathcal{O}_1, \mathcal{O}_2$

$$\langle \mathcal{O}_1 | \mathcal{O}_2 \rangle = \langle \mathcal{O}_1 | \mathcal{F}_N \mathcal{O}_2 \rangle_{S_n} = \langle \mathcal{F}_N \mathcal{O}_1 | \mathcal{O}_2 \rangle_{S_n} \tag{7.1.3}$$

Since the physical inner product includes a $\mathcal{F}_N$ factor, we will also call it the $\mathcal{F}$-weighted inner product, and will sometimes use a $\mathcal{F}$ subscript to emphasise the difference between it and the S_n inner product.

In general Ω_N is not invertible in the full algebra $\mathbb{C}(S_n)$, however it does have an inverse in those representations with $l(R) \leq N$. Define

$$\Omega_N^{-1} = \sum_{\substack{R \vdash n \\ l(R) \leq N}} \frac{1}{f_R} P_R \tag{7.1.4}$$

where the projector P_R was defined in (2.3.13) and f_R is given in (2.3.20). This is inverse to Ω_N in all representations R with $l(R) \leq N$. If $N \geq n$, it is inverse to Ω_N in the full group algebra $\mathbb{C}(S_n)$.

Define $\mathcal{G}_N$ to implement multiplication by Ω_N^{-1} on multi-trace operators

$$\mathcal{G}_N \mathcal{O}_{a,\sigma} = \mathcal{O}_{a,\Omega_N^{-1}\sigma} \tag{7.1.5}$$

Then since only Young diagrams with $l(R) \leq N$ contribute to operator construction, $\mathcal{F}_N$ and $\mathcal{G}_N$ are inverse to each other on the free field quarter-BPS operators. Therefore for two operators $\mathcal{O}_1, \mathcal{O}_2$, we have

$$\langle \mathcal{O}_1 | \mathcal{O}_2 \rangle_{S_n} = \langle \mathcal{O}_1 | \mathcal{G}_N \mathcal{O}_2 \rangle_{\mathcal{F}} = \langle \mathcal{G}_N \mathcal{O}_1 | \mathcal{O}_2 \rangle_{\mathcal{F}} \tag{7.1.6}$$

The $U(2)$ generators (3.6.10) act only on the label a in $\mathcal{O}_{a,\sigma}$, while $\mathcal{F}_N$ and $\mathcal{G}_N$ act only on σ, and therefore the two commute. This means the same hermiticity conditions (3.6.13) apply for the S_n inner product as for the physical inner product, and consequently operators with different $U(2)$ quantum numbers are orthogonal in both inner products.

The action of $\mathcal{F}_N$ and $\mathcal{G}_N$ is particularly simple on the covariant basis for free field operators (3.6.20)

$$\mathcal{F}_N \mathcal{O}_{\Lambda,M_\Lambda,R,\tau} = f_R \mathcal{O}_{\Lambda,M_\Lambda,R,\tau} \tag{7.1.7}$$

$$\mathcal{G}_N \mathcal{O}_{\Lambda,M_\Lambda,R,\tau} = \begin{cases} \frac{1}{f_R} \mathcal{O}_{\Lambda,M_\Lambda,R,\tau} & l(R) \leq N \\ 0 & l(R) > N \end{cases} \tag{7.1.8}$$

The relations (7.1.3) and (7.1.6) between the S_n and physical inner products can be used to construct BPS operators. Let $\mathcal{O}^c$ be a commutator trace, and $\mathcal{O}^s$ be a *pre-BPS* operator, defined to be orthogonal to commutator traces under the S_n inner product. Then

$$\langle \mathcal{O}^c | \mathcal{G}_N \mathcal{O}^s \rangle = \langle \mathcal{O}^c | \mathcal{O}^s \rangle_{S_n} = 0 \tag{7.1.9}$$

So the operator $\mathcal{G}_N \mathcal{O}^s$ is a quarter-BPS operator at weak coupling.

The natural next step is to determine the form of the operators $\mathcal{O}^s$. It was demonstrated in [51] that for $N \geq n$ these are symmetrised traces. For a single trace, the symmetrised version is

$$\mathrm{Tr}(X_{a_1} X_{a_2} \dots X_{a_n}) \to \frac{1}{n!} \sum_{\sigma \in S_n} \mathrm{Tr}(X_{a_{\sigma(1)}} X_{a_{\sigma(2)}} \dots X_{a_{\sigma(n)}}) =: \mathrm{Str}(X_{a_1} X_{a_2} \dots X_{a_n}) \tag{7.1.10}$$

where $a_i \in \{1, 2\}$ for each i and $X_1 = X$, $X_2 = Y$. For a multi-trace, this process is applied to each of the constituent single traces. A symmetrised trace is determined by the field content of each single trace factor. For a particular total field content (n_1, n_2), the possible symmetrised traces are labelled by *vector partitions* $\mathbf{p}$. A vector partition is a set of integer 2-vectors which sum to (n_1, n_2), which we denote by $\mathbf{p} \vdash (n_1, n_2)$. For a vector partition $\mathbf{p} = [(k_1, l_1), \dots (k_m, l_m)]$ the associated symmetrised trace is

$$T_{\mathbf{p}} = \mathrm{Str}\left(X^{k_1} Y^{l_1}\right) \mathrm{Str}\left(X^{k_2} Y^{l_2}\right) \dots \mathrm{Str}\left(X^{k_m} Y^{l_m}\right) \tag{7.1.11}$$

We conclude a generic quarter-BPS operator for $N \geq n$ can simply be written as

$$\mathcal{O}^{BPS} = \mathcal{G}_N T_{\mathbf{p}} \tag{7.1.12}$$

At finite N, non-trivial relations among different multi-traces reduce the dimensionality of the quarter-BPS sector, and correspondingly the pre-BPS operators as well. A finite N relation among traces could have three distinct behaviours with respect to the large N space of symmetrised traces

1. It is internal to the space of symmetrised traces. In this case, under an appropriate choice of basis, a symmetrised trace reduces to the zero operator. Correspondingly, the dimension of the pre-BPS and BPS sectors reduce by 1.

2. It is internal to the space of commutator traces. This does not affect the pre-BPS or BPS sectors.

3. It is a linear combination of symmetrised traces and commutator traces. In this case, under an appropriate choice of basis, a symmetrised trace reduces to a commutator trace. This means it is no longer pre-BPS, as it is not S_n orthogonal to descendants. Correspondingly, the dimension of the pre-BPS and BPS sectors reduce by 1.

Therefore, SEP-compatibility in the pre-BPS sector has a different interpretation to the BPS equivalent. A basis for pre-BPS operators is SEP-compatible if operators with labels longer than N reduce to *either* the zero operator *or* a commutator trace after applying finite N relations. After applying $\mathcal{G}_N$ to such a basis, we obtain an SEP-compatible basis for the quarter-BPS sector.

7.1.1 Steps in the construction of an SEP-compatible orthogonal BPS basis

The key ingredient that will allow the construction of an SEP-compatible basis for pre-BPS operators is an isomorphism proved by Vaccarino [97] and Domokos [98], summarised nicely by Procesi in [99].

From the definition (7.1.10), the non-commuting matrices X and Y commute within a symmetrised trace, and therefore we naively expect that that symmetrised traces of non-commuting matrices correspond to ordinary multi-traces of commuting matrices via

$$\mathrm{Str}(X^{k_1}Y^{l_1})\dots\mathrm{Str}(X^{k_m}Y^{l_m}) \longleftrightarrow \mathrm{Tr}\left(A^{k_1}B^{l_1}\right)\dots\mathrm{Tr}\left(A^{k_m}B^{l_m}\right) \tag{7.1.13}$$

where A and B are two commuting $N\times N$ matrices. The isomorphism of [97–99] makes this expectation rigorous.

Consider the ring $R(X,Y)$ generated by the matrix elements of two $N\times N$ matrices X,Y. This ring is acted on by $U(N)$ via simultaneous conjugation of the two matrices. Given a $\mathcal{U}\in U(N)$, we have

$$(X,Y)\to(\mathcal{U}X\mathcal{U}^\dagger,\mathcal{U}Y\mathcal{U}^\dagger) \tag{7.1.14}$$

Then invariants of $R(X,Y)$ under this action are multi-traces of X and Y, and correspond to the quarter-BPS sector at zero coupling. At weak coupling, we consider $R(X,Y)$ modulo the ideal I generated by the commutator $[X,Y]$. We call the quotient ring $R_s(X,Y)$. Each $U(N)$ invariant of the quotient ring corresponds to an equivalence class of multi-trace operators related by addition of a commutator trace. In each class there is a unique pre-BPS representative that is orthogonal to all commutator traces (under the S_n inner product). There is also a unique BPS operator orthogonal to commutator traces under the $\mathcal{F}_N$ inner product. Conversely, given a pre-BPS or BPS operator, there is a unique equivalence class to which it belongs. Therefore the invariants of $R_s(X,Y)$ give the combinatorics of the pre-BPS and BPS sectors, both at large N and finite N.

Finding the pre-BPS operator in a given equivalence class is simple when $N\geq n$; as discussed above (7.1.10), the representative is a symmetrised trace. When $N<n$, the multi-trace expansion of an operator is non-unique, and it is more difficult to identify the pre-BPS operator. In section 7.4, we describe how to find the pre-BPS operator by orthogonalisation.

On the other side of the isomorphism are *multi-symmetric functions*. Take two commuting $N \times N$ matrices $A = \text{Diag}(x_1, x_2, \cdots, x_N)$ and $B = \text{Diag}(y_1, y_2, \cdots, y_N)$. Then a multi-trace of A and B will be a polynomial in the $2N$ variables invariant under permutations

$$(x_i, y_i) \to \left(x_{\sigma(i)}, y_{\sigma(i)}\right) \tag{7.1.15}$$

for $\sigma \in S_N$. These are called multi-symmetric functions, and generalise the symmetric functions of section 2.7 to two families of variables. They are discussed in detail in section 7.2.

The theorem in [97–99] tells us that the ring of invariants of $R_s(X, Y)$ is isomorphic to the ring of multi-symmetric functions in $2N$ variables.

This isomorphism is simple to give explicitly. Take a multi-trace of X and Y and restrict the two matrices to be diagonal. This is now a multi-symmetric function in the $2N$ eigenvalues. Clearly the commutator $[X, Y]$ vanishes for the diagonal X and Y, and therefore any multi-traces related by a commutator trace lead to the same multi-symmetric function.

Conversely, given a multi-trace of two commuting matrices A and B, we use the map (7.1.13) to pick a representative of the isomorphic equivalence class. At large N, this correctly identifies the pre-BPS operator. However, for $N < n$, this map does not associate a unique symmetrised trace with a given multi-symmetric function. Finite N relations mean a multi-symmetric function can be written in multiple ways as the trace of commuting matrices. These different expressions give genuinely different operators in the gauge theory, related by commutator traces. For the multi-symmetric functions we use, we will give a defining representation as a multi-trace of commuting matrices, and then (7.1.13) defines the equivalent symmetrised trace operator in an unambiguous way.

We will generally use the same notation for either side of the isomorphism. For example we will use X and Y to refer to both the commuting matrices on the right of (7.1.13) and the non-commuting matrices of the super Yang-Mills theory on the left. Similarly, both a symmetrised trace and its isomorphic multi-symmetric function will be denoted $T_{\mathbf{p}}$. When the distinction is important, we will be clear which is under discussion.

For symmetric functions, we introduced two SEP-compatible bases, the monomial basis of section 2.7.1 and the Schur basis of section 2.7.3. There is no obvious analogue of the Schur basis for multi-symmetric functions, however the monomial basis does generalise, and provides a good finite N description for multi-symmetric functions. We denote these monomials by $M_{\mathbf{p}}$, where the label is a vector partition $\mathbf{p}$, as already seen for symmetrised traces in (7.1.11). The length of $\mathbf{p}$ determines the finite N behaviour. In section 7.2 we study the basis change between $T_{\mathbf{p}}$ and $M_{\mathbf{p}}$, both as multi-symmetric functions and their isomorphic image as symmetrised trace operators.

Under the map (7.1.13), the $M_{\mathbf{p}}$ give a basis for pre-BPS operators for $N \geq n$. When $N < n$, the operators with $l(\mathbf{p}) > N$ reduce to commutator traces. As discussed previously, this is a feature of an SEP-compatible basis of pre-BPS operators. However, the operators with $l(\mathbf{p}) \leq N$ are not S_n orthogonal to all commutator traces, and therefore are not pre-BPS. This is because the map (7.1.13) did not choose the correct pre-BPS operator from the equivalence class of operators related by addition of a commutator trace. We say $M_{\mathbf{p}}$ is SEP-compatible *modulo commutators* for the pre-BPS sector, and this is a key stepping stone to an SEP-compatible basis.

In section 7.3 we organise the $M_{\mathbf{p}}$ according to representations of the $U(2)$ symmetry, replacing the label $\mathbf{p}$ with $(\Lambda, M_\Lambda, p, \nu)$. Λ is a $U(2)$ Young diagram with n boxes, where n is the total number of X, Y matrices in the operator. M_Λ labels a basis state in the Λ representation of $U(2)$. p is an integer partition of n whose components are related to the vector partition $\mathbf{p}$ simply by summing each of the two-vector components of $\mathbf{p}$. We call p the associated partition of $\mathbf{p}$. Since $l(\mathbf{p}) = l(p)$, the SEP-compatibility (modulo commutators) of $M_{\mathbf{p}}$ is transferred to the covariant basis. This restricts $l(p) \leq N$, which is the usual constraint associated with a $U(N)$ Young diagram, and we will therefore refer to p as a $U(N)$ Young diagram label. The final label ν runs over a multiplicity space of dimension $\mathcal{M}_{\Lambda,p}$. Much of section 7.3 is devoted to calculating and understanding $\mathcal{M}_{\Lambda,p}$ as it describes the finite N combinatorics of the weak coupling quarter-BPS sector.

Section 7.4 takes the covariant monomials $M_{\Lambda,M_\Lambda,p,\nu}$ and uses orthogonalisation algorithms to produce an SEP-compatible basis of pre-BPS operators. There are three separate steps in producing an orthogonal SEP-compatible basis of BPS operators.

1. For $l(p) \leq N < n$, the covariant monomial $M_{\Lambda,M_\Lambda,p,\nu}$ differs from a pre-BPS operator by a commutator trace. Orthogonalising $M_{\Lambda,M_\Lambda,p,\nu}$ against $M_{\Lambda,M_\Lambda,q,\eta}$ with $l(q) > N$ using the S_n inner product identifies pre-BPS operators denoted $\bar{M}_{\Lambda,M_\Lambda,q,\eta}$. If $N \geq n$, this step is trivial.

2. Applying the operator $\mathcal{G}_N$ to the pre-BPS operators produces BPS operators. We orthogonalize these BPS operators using the physical $\mathcal{F}_N$ inner product.

3. We normalize these orthogonal operators using the S_n inner product. This ensures that the basis is SEP-compatible: if we apply the construction using $\mathcal{G}_N$ and matrices of size N, and subsequently substitute in our expressions matrices of size $\widehat{N}$ while making substitutions $N \to \widehat{N}$, then all operators with $l(p) > \widehat{N}$ vanish and the non-zero operators with $l(p) \leq \widehat{N}$ are those produced by applying the 3-step construction directly at $\widehat{N}$.

The first step is explained in detail in section 7.5 in a more general context where the

	SEP-compatible	$U(2)$ covariant	BPS at $N < n$	Orthogonal
$T^{BPS}_{\mathbf{p}}$	×	×	×	×
$M^{BPS}_{\mathbf{p}}$	modulo commutators	×	×	×
$M^{BPS}_{\Lambda,M_\Lambda,p,\nu}$	modulo commutators	✓	×	×
$\bar{M}^{BPS}_{\Lambda,M_\Lambda,p,\nu}$	modulo commutators	✓	✓	×
$S^{BPS}_{\Lambda,M_\Lambda,p,\nu}$	✓	✓	✓	✓

Table 7.1: Properties of the different BPS bases constructed. All are BPS at $N \geq n$.

2-matrix problem is generalized to allow any number of matrices. Section 7.4 puts together all the steps and proves that the outcome $S^{BPS}_{\Lambda,M_\Lambda,p,\nu}$ indeed form an orthogonal SEP-compatible basis for BPS operators.

The orthogonalisation and $\mathcal{G}_N$ application processes involved in the construction are linear, so there is some flexibility in the order of the application of the different steps. Figure 7.1 shows the algorithm we have implemented in SAGE to obtain the basis $S^{BPS}_{\Lambda,M_\Lambda,p,\nu}$ starting from symmetrised traces $T_{\mathbf{p}}$. The red arrows indicate the route taken here, while the other arrows indicate different routes where $\mathcal{G}_N$ is applied at a different stage.

The operators $T^{BPS}_{\mathbf{p}}$, $M^{BPS}_{\mathbf{p}}$ and $M^{BPS}_{\Lambda,M_\Lambda,p,\nu}$, obtained by applying $\mathcal{G}_N$ (and normalising) to $T_{\mathbf{p}}$, $M_{\mathbf{p}}$ or $M_{\Lambda,M_\Lambda,p,\nu}$ respectively, are BPS bases at $N \geq n$, but in general for $N < n$ are no longer BPS, although the latter two do capture some of the finite N behaviour. Table 7.1 shows the properties of the different BPS bases.

In the case where Λ is taken to be $[n]$, the $S^{BPS}_{\Lambda,M_\Lambda,p,\nu}$ are half-BPS operators built purely from X. In section 7.4.9 we show that the construction for Young diagram p reproduces the Schur operator (2.3.14) labelled by Young diagram $R = p$. Further, for $\Lambda = [n-1, 1]$, numerical calculations suggest that $S^{BPS}_{\Lambda,M_\Lambda,p,\nu}$ match the free field quarter-BPS operators (3.6.20) with $R = p$. This justifies the view that p is a $U(2)$ invariant, quarter-BPS generalization of the R-label of the half-BPS sector.

An important perspective on the half-BPS R label comes from the analysis of the asymptotics of LLM geometries. Specifically $U(N)$ Casimirs of R are measurable from the asymptotics of the supergravity fields [101]. We propose that that the $U(2)$ quadratic Casimir for Λ as well as the sequence of Casimirs identifying p should be measurable from the multipole moments that can be read off from the long-distance expansions of the sugra fields of LLM geometries corresponding to quarter-BPS operators at $n \sim N^2$. Precision holography of LLM geometries is also developed using correlators of small operators in the LLM background [38] which should provide complementary insights into the labels Λ, p.

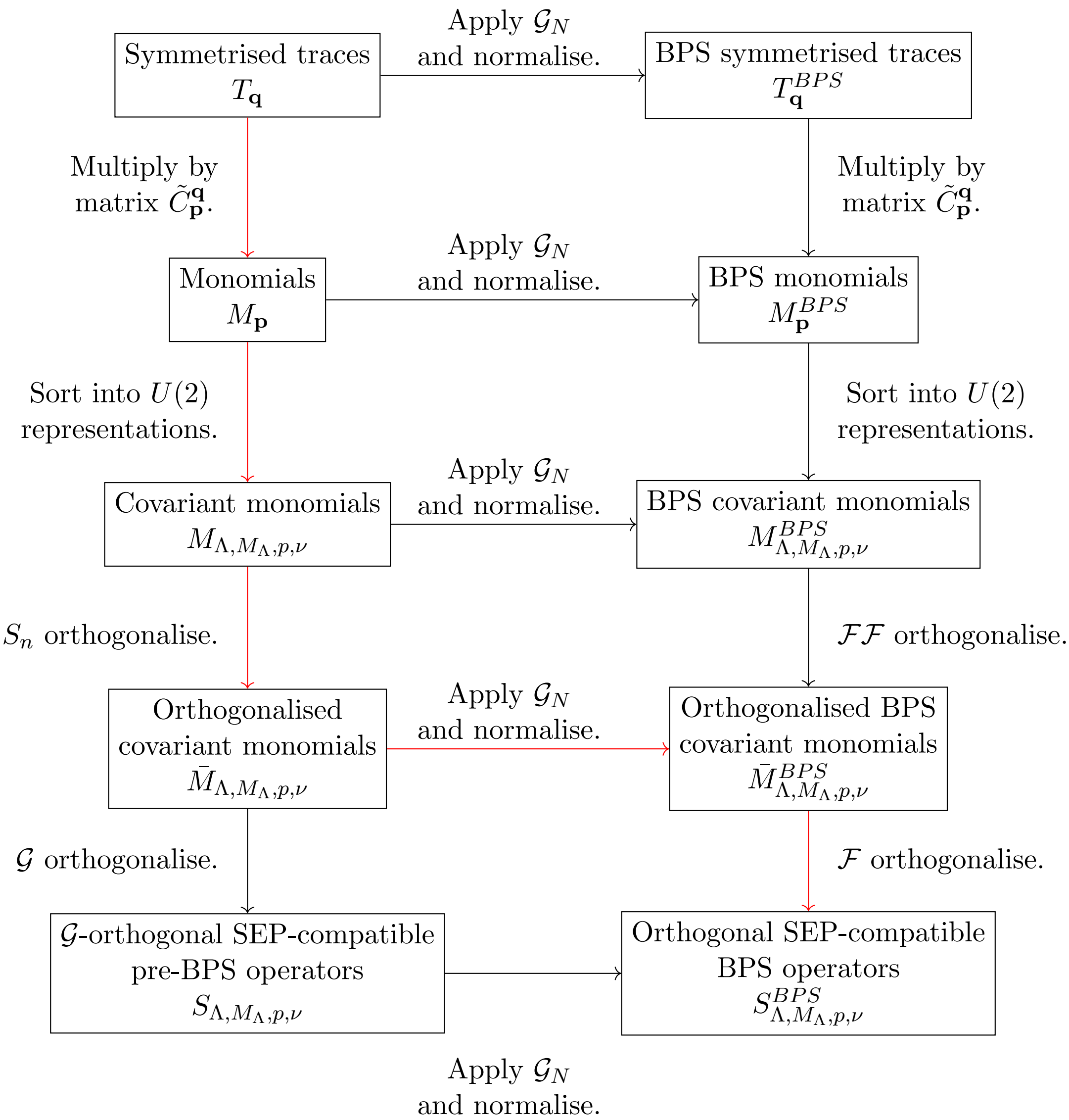


Figure 7.1: An outline of the algorithm starting with symmetrised trace operators $T_{\mathbf{p}}$ and deriving a $U(2)$ covariant, orthogonal, SEP-compatible basis $S^{BPS}_{\Lambda,M_\Lambda,p,\nu}$ for BPS operators. The route taken here is down the left side and across the bottom, shown in red. The first step is studied in detail in section 7.2, the second step in section 7.3 and the last three steps in section 7.4.

The R Young diagram label of the half BPS sector is related to the free fermions underlying the holomorphic sector of the complex matrix model which describes half-BPS combinatorics [22, 36, 102]. Remarkably, this free fermion description also shows up in the droplet description of half-BPS LLM geometries [37]. The droplet description generalizes to the quarter-BPS LLM geometries, with some significant differences [41, 103, 104] . We have colourings of regions in a four-dimensional space instead of a two-dimensional plane. The two colours are now associated with the collapse of an $S^1 \subset S^5$ or an $S^3 \subset S^5$. A natural conjecture is that the p-label of quarter-BPS operators is analogously associated to colourings of regions in $\mathbb{R}^4$ as the R-label of the half-BPS operators is associated to colourings of the plane.

7.2 Finite N combinatorics from many-boson states: multi-symmetric functions and set partitions

As explained in section 7.1, the key result that will enable us to give an SEP-compatible construction of quarter-BPS operators is an isomorphism of Vaccarino and Domokos [97, 98] between multi-symmetric functions and the ring of gauge invariants of two matrices modulo commutator traces. This section focuses on the multi-symmetric function side of this isomorphism.

An important aspect of multi-symmetric functions, which plays a central role in finding BPS operators, is the transformation between two bases for these multi-symmetric functions. The first basis will be referred to as the "monomial multi-symmetric basis" and the second as the "multi-trace basis". The physical importance of these two bases, and their transformations, can be understood using perspectives from many-body quantum mechanics [105]. This draws on an important insight from the AdS/CFT correspondence: the strong coupling limit of the quarter BPS sector in $\mathcal{N} = 4$ SYM corresponds to a Hilbert space of N bosons in a two-dimensional harmonic oscillator [27, 40, 106].

We begin this section by developing the link between multi-symmetric functions and the Hilbert space of N identical bosons in a two-dimensional harmonic oscillator. We then introduce the monomial and multi-trace bases and investigate the combinatorics of the matrix that transforms between the two. This leads to the interesting mathematics of the poset of set partitions and the associated Möbius function.

7.2.1 Multi-symmetric functions as wavefunctions of a harmonic oscillator

The Lagrangian for one particle in a two-dimensional harmonic oscillator is

$$L = \frac{1}{2}\left(\dot{x}^2 + \dot{y}^2\right) - \frac{1}{2}\left(x^2 + y^2\right) \tag{7.2.1}$$

In terms of creation and annihilation operators, the one-particle Hamiltonian is

$$H = a_x^\dagger a_x + a_y^\dagger a_y \tag{7.2.2}$$

Define the coherent state

$$\langle x, y| = \langle 0|e^{xa_x + ya_y} \tag{7.2.3}$$

We have

$$\langle x, y|(a_x^\dagger)^\lambda (a_y^\dagger)^\mu |0\rangle = x^\lambda y^\mu \tag{7.2.4}$$

In this coherent state representation, the Hamiltonian acts as the degree operator for the 2-variable polynomial

$$\begin{aligned}\langle x, y|H(a_x^\dagger)^\lambda (a_y^\dagger)^\mu |0\rangle &= \left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)\langle x, y|(a_x^\dagger)^\lambda (a_y^\dagger)^\mu |0\rangle \\ &= \left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)(x^\lambda y^\mu) \\ &= (\lambda + \mu)(x^\lambda y^\mu)\end{aligned} \tag{7.2.5}$$

For the system of N-particles in the two-dimensional harmonic oscillator, we have the coherent state

$$\langle x_1, y_1; x_2, y_2; \ldots; x_N, y_N| = \langle 0|e^{\sum_{i=1}^N x_i a_{i;x} + y_i a_{i;y}} \tag{7.2.6}$$

The energy eigenstates of the Hamiltonian correspond to the product of one-particle wavefunctions

$$\begin{aligned}&\langle x_1, y_1; x_2, y_2; \cdots; x_N, y_N|(a_{1;x}^\dagger)^{\lambda_1}(a_{1;y}^\dagger)^{\mu_1}(a_{2;x}^\dagger)^{\lambda_2}(a_{2;y}^\dagger)^{\mu_2}\ldots(a_{k;x}^\dagger)^{\lambda_k}(a_{k;y}^\dagger)^{\mu_k}|0\rangle \\ &\qquad = x_1^{\lambda_1} y_1^{\mu_1}\cdots x_k^{\lambda_k} y_k^{\mu_k}\end{aligned} \tag{7.2.7}$$

It is useful to write

$$\begin{aligned}x_1^{\lambda_1} y_1^{\mu_1}\ldots x_k^{\lambda_k} y_k^{\mu_k} &= \psi_{\lambda_1,\mu_1}(x_1, y_1)\psi_{\lambda_2,\mu_2}(x_2, y_2)\cdots\psi_{\lambda_k,\mu_k}(x_k, y_k) \\ &= \psi_{\lambda_1,\mu_1}(x_1, y_1)\ldots\psi_{\lambda_k,\mu_k}(x_k, y_k)\psi_{0,0}(x_{k+1}, y_{k+1})\ldots\psi_{0,0}(x_N, y_N)\end{aligned} \tag{7.2.8}$$

In a system of N identical bosons, we must symmetrise the product of annihilation operators using S_N permutations. The product wavefunctions and their symmetrisations are a standard tool in many-body quantum mechanics (see. e.g. [105]). The permutations $\sigma \in S_N$ act as

$$(x_i, y_i) \to (x_{\sigma(i)}, y_{\sigma(i)}) \tag{7.2.9}$$

These states are polynomials, symmetric under these simultaneous permutations of x, y pairs, which are exactly multi-symmetric functions. In fact these form the monomial multi-symmetric functions that we will study presently. They have the nice property that finite N effects are nicely encoded in the fact that, by definition, $k \leq N$.

A quantum state where a single particle is excited, after symmetrisation, has a coherent state representation

$$\phi_{\lambda_1,\mu_1}(x_i, y_i) = \sum_{i=1}^{N} x_i^{\lambda_1} y_i^{\mu_1} \tag{7.2.10}$$

When we have two particles excited, the symmetrisation of the product wavefunction is proportional to

$$\sum_{i_1 \neq i_2}^{N} x_{i_1}^{\lambda_1} y_{i_1}^{\mu_1} x_{i_2}^{\lambda_2} y_{i_2}^{\mu_2} \tag{7.2.11}$$

The restriction $i_1 \neq i_2$, when extended to $i_1 \neq i_2 \cdots \neq i_k$, is closely related to the finite N property, but also has the consequence that the 2-particle wavefunction 7.2.11 is not equal to the product of 1-particle wavefunctions. It is rather a linear combination of the product wavefunction $\phi_{\lambda_1,\mu_1}\phi_{\lambda_2,\mu_2}$ along with a 1-particle wavefunction $\phi_{\lambda_1+\lambda_2,\mu_1+\mu_2}$.

Defining diagonal matrices $X = \mathrm{Diag}(x_1, \cdots, x_N)$ and $Y = \mathrm{Diag}(y_1, y_2, \cdots, y_N)$, we observe that the 1-particle wavefunction is a trace

$$\phi_{\lambda_1,\mu_1}(x_i, y_i) = \mathrm{Tr}(X^{\lambda_1} Y^{\mu_1}) \tag{7.2.12}$$

We now draw on an idea from collective field theory, where one associates creation operators to invariant traces [107–109] to define a map from traces and products of traces to Fock space states

$$\mathrm{Tr}(X^{\lambda} Y^{\mu}) \to B^{\dagger}_{\lambda,\mu}|0\rangle \tag{7.2.13}$$

$$\prod_{a=1}^{k} \mathrm{Tr}(X^{\lambda_a} Y^{\mu_a}) \to \prod_{a=1}^{k} B^{\dagger}_{\lambda_a,\mu_a}|0\rangle \tag{7.2.14}$$

This map is a homomorphism between the product structure on the polynomials and the product structure on oscillators. It can also be obtained from a coherent state construction

$$\langle X,Y| = \langle 0| e^{\sum_{\lambda,\mu} B_{\lambda,\mu} \mathrm{Tr}(X^\lambda Y^\mu)}$$
$$\langle X,Y| \prod_{a=1}^{k} B^\dagger_{\lambda_a,\mu_a} |0\rangle = \prod_{a=1}^{k} \mathrm{Tr}(X^{\lambda_a} Y^{\mu_a}) \tag{7.2.15}$$

In section 7.2.3 we will be studying in detail the transformation between the monomial multi-symmetric functions and the trace wavefunctions. As a result of a triangular property of this transformation, the finite N cutoff on multi-symmetric functions can also be described by restricting the number of factors in the product of traces to be less than or equal to N.

7.2.2 Monomial and trace bases for multi-symmetric functions

We now give a formal definition of multi-symmetric functions in a completely analogous way to the symmetric functions of section 2.7, just with two families of variables $x_1, \ldots, x_N$ and $y_1, \ldots, y_N$ instead of one. They are polynomials in these $2N$ variables that are invariant under all S_N permutations on the pairs (x_i, y_i). Given a polynomial $f(x_1, x_2, \ldots, x_N; y_1, y_2, \ldots, y_N)$, f is a multi-symmetric function if

$$f(x_1, x_2, \ldots, x_N; y_1, y_2, \ldots, y_N) = f\left(x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(N)}; y_{\sigma(1)}, y_{\sigma(2)}, \ldots, y_{\sigma(N)}\right) \tag{7.2.16}$$

for all $\sigma \in S_N$.

We can take a large N limit and work with formal power series in an infinite number of variables rather than polynomials. To return to the finite N case, we set $x_{N+1} = y_{N+1} = x_{N+2} = y_{N+2} = \cdots = 0$.

We can also define multi-symmetric functions with M families of variables $x_i^{(k)}$, for $1 \le k \le M$, $1 \le i \le N$, invariant under S_N permutations of the i index. These would be relevant for systems of M commuting matrices.

For a mathematical overview of multi-symmetric functions and their properties see [110].

The monomial and multi-trace (power-sum) bases for symmetric polynomials defined in (2.7.6) and (2.7.8) have direct analogues in the multi-symmetric case. As before, they are graded bases, this time graded by both the x degree n_1 and the y degree n_2.

A vector partition $\mathbf{p}$ of (n_1, n_2) is defined to be a sequence of pairs of non-negative integers (at least one of each pair must be non-zero) summing to (n_1, n_2). We use a bold $\mathbf{p}$ to distinguish between vector and integer partitions, and write $\mathbf{p} \vdash (n_1, n_2)$ to denote that $\mathbf{p}$ sums to (n_1, n_2). The basis elements at degree (n_1, n_2) are labelled by

$\mathbf{p} \vdash (n_1, n_2)$ with length $l(\mathbf{p}) \leq N$.

To construct the monomial basis, take a vector partition $\mathbf{p} = [(\lambda_1, \mu_1), (\lambda_2, \mu_2), \ldots, (\lambda_k, \mu_k)]$ of (n_1, n_2) with $l(\mathbf{p}) = k \leq N$ and consider the un-symmetrised monomial

$$x_1^{\lambda_1} y_1^{\mu_1} x_2^{\lambda_2} y_2^{\mu_2} \ldots x_k^{\lambda_k} y_k^{\mu_k} \tag{7.2.17}$$

After adding all distinct permutations of the lower indices, one arrives at the monomial basis element. Explicitly

$$m_{\mathbf{p}} = \frac{1}{Z_{\mathbf{p}}} \sum_{\sigma \in S_N} x_{\sigma(1)}^{\lambda_1} y_{\sigma(1)}^{\mu_1} x_{\sigma(2)}^{\lambda_2} y_{\sigma(2)}^{\mu_2} \ldots x_{\sigma(k)}^{\lambda_k} y_{\sigma(k)}^{\mu_k} \tag{7.2.18}$$

where the factor in front removes the normalisation introduced by redundancies in the elements of $\mathbf{p}$, so that the coefficient in front of each individual monomial is 1. Using multiplicity notation for vector partitions, let $\mathbf{p} = [(0,1)^{\mathbf{p}_{(0,1)}}, (1,0)^{\mathbf{p}_{(1,0)}}, \ldots]$. Then the normalisation is given by

$$Z_{\mathbf{p}} = \prod_{i,j} \mathbf{p}_{(i,j)}! \tag{7.2.19}$$

As in the symmetric case, we will use a modified version of the monomial basis, obtained by leaving out this normalisation factor

$$M_{\mathbf{p}} = Z_{\mathbf{p}} m_{\mathbf{p}} = \sum_{\sigma \in S_N} x_{\sigma(1)}^{\lambda_1} y_{\sigma(1)}^{\mu_1} x_{\sigma(2)}^{\lambda_2} y_{\sigma(2)}^{\mu_2} \ldots x_{\sigma(k)}^{\lambda_k} y_{\sigma(k)}^{\mu_k} \tag{7.2.20}$$

As discussed below (7.2.16), we can lower N to $N-1$ by setting $x_N = y_N = 0$, causing a reduction in the size of the space. Starting from $N > n_1 + n_2$ and reducing stepwise, this implies those monomial functions with $l(\mathbf{p}) > N$ vanish identically, while the remaining $M_{\mathbf{p}}$ with $l(\mathbf{p}) \leq N$ form a basis for the smaller space. So the monomial basis is SEP-compatible for multi-symmetric functions.

Note that the the isomorphism, as described around (7.1.14), states that multi-symmetric functions are isomorphic to invariants of matrices X, Y modulo commutator traces. Therefore the isomorphic image of $M_{\mathbf{p}}$, also referred to as $M_{\mathbf{p}}$, is not necessarily zero if $l(\mathbf{p}) > N$, but could instead be a commutator trace. This is the version of SEP-compatibility for pre-BPS operators as discussed in section 7.1.

As multi-symmetric functions, the monomial functions $M_{\mathbf{p}}$ are SEP-compatible. Using the map (7.1.13) to give the equivalent symmetrised trace operators, they form a basis for pre-BPS operators at $N \geq n$. As we decrease $N < n$, the SEP-compatibility implies any operator with $l(\mathbf{p}) > N$ reduces to a commutator trace. However, the operators with $l(\mathbf{p}) \leq N$ are not in general S_n orthogonal to commutator traces when $N < n$, and therefore they do not form a basis for pre-BPS operators. This is due to (7.1.13) not selecting the right representative of the equivalence class of operators isomorphic to the multi-symmetric function, as discussed below (7.1.14). We say the Yang-Mills operators $M_{\mathbf{p}}$ are SEP-compatible modulo commutators, and in section 7.4, describe how to transform this into a genuinely SEP-compatible basis for pre-BPS operators.

The multi-trace basis for multi-symmetric functions, also called the power-sum basis in the mathematics literature, is built out of

$$T_{(n_1,n_2)} = \sum_{i=1}^{N} x_i^{n_1} y_i^{n_2} \tag{7.2.21}$$

Given a vector partition $\mathbf{p} = [(\lambda_1, \mu_1), (\lambda_2, \mu_2), \ldots (\lambda_k, \mu_k)]$, the associated multi-symmetric function is

$$T_{\mathbf{p}} = \prod_{i=1}^{k} T_{(\lambda_i,\mu_i)} \tag{7.2.22}$$

Introduce two $N \times N$ diagonal matrices X and Y with diagonal elements $x_1, x_2, \ldots x_N$ and $y_1, y_2, \ldots y_N$ respectively. Then $T_{(n_1,n_2)} = \mathrm{Tr} X^{n_1} Y^{n_2}$, and the multi-trace multi-symmetric functions are exactly given by the multi-traces of these two matrices, justifying the name.

The isomorphism of [97, 98] identifies the multi-symmetric functions (7.2.22) with the symmetrised trace operators (7.1.11), establishing the connection between multi-symmetric functions and the quarter-BPS sector of $\mathcal{N} = 4$ super Yang-Mills at weak coupling.

Note that while (7.2.22) and (7.1.11) are conceptually different, the isomorphism between the two means we abuse notation slightly and use the same symbol $T_{\mathbf{p}}$ for both.

At finite N, non-trivial relationships appear between the different multi-traces leading to a reduction in the dimensionality of the space of multi-symmetric functions. Those multi-traces labelled by $\mathbf{p}$ with $l(\mathbf{p}) \leq N$ form a basis for the reduced space. However, unlike the monomial functions, the remaining multi-traces (labelled by $\mathbf{p}$ with $l(\mathbf{p}) > N$) do not vanish, but become complicated linear combinations of the reduced basis.

Define a matrix $C^{\mathbf{p}}_{\mathbf{q}}$, indexed by vector partitions $\mathbf{p}$ and $\mathbf{q}$ to be the change of basis matrix from $M_{\mathbf{p}}$ to $T_{\mathbf{q}}$, with inverse $\widetilde{C}$

$$T_{\mathbf{q}} = \sum_{\mathbf{p}} C^{\mathbf{p}}_{\mathbf{q}} M_{\mathbf{p}} \qquad\qquad M_{\mathbf{p}} = \sum_{\mathbf{q}} \widetilde{C}^{\mathbf{q}}_{\mathbf{p}} T_{\mathbf{q}} \tag{7.2.23}$$

At finite N, the $\mathbf{p}$ label for monomials is SEP-compatible (modulo commutators). Therefore the second of the equations above gives the finite N relations imposed on commuting matrices. On the other side of the isomorphism, this gives the linear combinations of symmetrised traces that reduce to commutator traces at finite N.

The C and $\widetilde{C}$ matrices have very interesting combinatorial properties, which we will now investigate in depth.

7.2.3 Basis change for multi-symmetric functions

In this section, we show that the properties of the linear transformations C and $\widetilde{C}$ are illuminated by considering set partitions. Set partitions form a partially ordered set (poset), and the Möbius inversion formula for posets plays an important role.

To find an expression for $C^{\mathbf{p}}_{\mathbf{q}}$, first expand the product in the definition (7.2.22) for multi-trace functions. For $\mathbf{p} = [(\lambda_1, \mu_1), \ldots, (\lambda_k, \mu_k)]$,

$$T_{\mathbf{p}} = \prod_{j=1}^{k} \left(\sum_{i=1}^{N} x_i^{\lambda_j} y_i^{\mu_j} \right) \tag{7.2.24}$$

$$= \sum_{j_1,\ldots,j_k=1}^{N} x_{j_1}^{\lambda_1} y_{j_1}^{\mu_1} x_{j_2}^{\lambda_2} y_{j_2}^{\mu_2} \ldots x_{j_k}^{\lambda_k} y_{j_k}^{\mu_k} \tag{7.2.25}$$

To further sort this sum, note the different ways the js could coincide. If, for example, $k = 3$, we could have

$$j_1 = j_2 = j_3 \qquad\qquad j_1 = j_2 \neq j_3 \tag{7.2.26}$$

$$j_1 = j_3 \neq j_2 \qquad\qquad j_1 \neq j_2 = j_3 \tag{7.2.27}$$

$$j_1, j_2, j_3 \text{ all distinct} \tag{7.2.28}$$

These correspond to the 5 different ways of partitioning the set $\{1, 2, 3\}$ into subsets

$$\pi_1 = \{\{1, 2, 3\}\} \qquad\qquad \pi_2 = \{\{1, 2\}, \{3\}\} \tag{7.2.29}$$

$$\pi_3 = \{\{1, 3\}, \{2\}\} \qquad\qquad \pi_4 = \{\{1\}, \{2, 3\}\} \tag{7.2.30}$$

$$\pi_5 = \{\{1\}, \{2\}, \{3\}\} \tag{7.2.31}$$

Continuing with the example, we can sort the sum (7.2.25) into the different partitions

$$\begin{aligned} T_{\mathbf{p}} = &\sum_{j} x_j^{\lambda_1+\lambda_2+\lambda_3} y_j^{\mu_1+\mu_2+\mu_3} + \sum_{j_1, j_3 \text{ distinct}} x_{j_1}^{\lambda_1+\lambda_2} y_{j_1}^{\mu_1+\mu_2} x_{j_3}^{\lambda_3} y_{j_3}^{\mu_3} \\ &+ \sum_{j_1, j_2 \text{ distinct}} x_{j_1}^{\lambda_1+\lambda_3} y_{j_1}^{\mu_1+\mu_3} x_{j_2}^{\lambda_2} y_{j_2}^{\mu_2} + \sum_{j_1, j_2 \text{ distinct}} x_{j_1}^{\lambda_1} y_{j_1}^{\mu_1} x_{j_2}^{\lambda_2+\lambda_3} y_{j_2}^{\mu_2+\mu_3} \\ &+ \sum_{j_1, j_2, j_3 \text{ distinct}} x_{j_1}^{\lambda_1} y_{j_1}^{\mu_1} x_{j_2}^{\lambda_2} y_{j_2}^{\mu_2} x_{j_3}^{\lambda_3} y_{j_3}^{\mu_3} \end{aligned} \tag{7.2.32}$$

The first term is just the monomial function associated to the vector partition $\pi_1(\mathbf{p}) = [(\lambda_1 + \lambda_2 + \lambda_3, \mu_1 + \mu_2 + \mu_3)]$. Similarly, the second term is related to the monomial function with vector partition $\pi_2(\mathbf{p}) = [(\lambda_1+\lambda_2, \mu_1+\mu_2), (\lambda_3, \mu_3)]$ via

$$\sum_{j_1, j_2 \text{ distinct}} x_{j_1}^{\lambda_1+\lambda_2} y_{j_1}^{\mu_1+\mu_2} x_{j_2}^{\lambda_3} y_{j_2}^{\mu_3} = \begin{cases} 2m_{\pi_2(p)} & (\lambda_1+\lambda_2, \mu_1+\mu_2) = (\lambda_3, \mu_3) \\ m_{\pi_2(p)} & \text{otherwise} \end{cases} \tag{7.2.33}$$

We can simplify this expression by noting that if $(\lambda_1+\lambda_2, \mu_1+\mu_2) = (\lambda_3, \mu_3)$ then $Z_{\pi_2(\mathbf{p})} = 2$, and otherwise $Z_{\pi_2(\mathbf{p})} = 1$. Therefore

$$\sum_{j_1, j_2 \text{ distinct}} x_{j_1}^{\lambda_1+\lambda_2} y_{j_1}^{\mu_1+\mu_2} x_{j_2}^{\lambda_3} y_{j_2}^{\mu_3} = Z_{\pi_2(\mathbf{p})} m_{\pi_2(\mathbf{p})} = M_{\pi_2(\mathbf{p})} \tag{7.2.34}$$

Similarly the third, fourth and fifth terms of (7.2.32) are just $M_{\pi_3(\mathbf{p})}, M_{\pi_4(\mathbf{p})}$ and $M_{\pi_5(\mathbf{p})}$ respectively, where

$$\pi_3(\mathbf{p}) = [(\lambda_1+\lambda_3, \mu_1+\mu_3), (\lambda_2, \mu_2)] \tag{7.2.35}$$

$$\pi_4(\mathbf{p}) = [(\lambda_1, \mu_1), (\lambda_2+\lambda_3, \mu_2+\mu_3)] \tag{7.2.36}$$

$$\pi_5(\mathbf{p}) = \mathbf{p} \tag{7.2.37}$$

Putting this together, we have

$$T_{\mathbf{p}} = \sum_{i=1}^{5} M_{\pi_i(\mathbf{p})} \tag{7.2.38}$$

Repeating this analysis more generally, let the set of set partitions of $\{1, 2, 3, \ldots k\}$ be denoted by $\Pi(k)$. Then given a set partition $\pi \in \Pi(k)$ and a vector partition $\mathbf{p} = [(\lambda_1, \mu_1), \ldots, (\lambda_k, \mu_k)]$ of length $l(\mathbf{p}) = k$, we define the vector partition $\pi(k)$ to be that with components

$$\left(\sum_{i \in b} \lambda_i \quad , \quad \sum_{i \in b} \mu_i \right) \tag{7.2.39}$$

where the blocks $b \in \pi$ run over the subsets into which $\{1, 2, 3, \ldots k\}$ have been partitioned. Conceptually, this should be thought of as summing up $\mathbf{p}$ into a new, shorter vector partition, where the summation structure is given by π.

Given this notation, we can now write the generalisation of (7.2.38) to any k

$$T_{\mathbf{p}} = \sum_{\pi \in \Pi(l(\mathbf{p}))} M_{\pi(\mathbf{p})} \tag{7.2.40}$$

Proving this result in the general case is just an exercise in repeating the logic that led from (7.2.25) to (7.2.38).

So the coefficient of $M_{\mathbf{p}}$ in $T_{\mathbf{q}}$ is just the number of set partitions $\pi \in \Pi(l(\mathbf{q}))$ that have $\pi(\mathbf{q}) = \mathbf{p}$.

$$\boxed{C^{\mathbf{p}}_{\mathbf{q}} = \sum_{\pi \in \Pi(l(\mathbf{q}))} \delta_{\mathbf{p}\ \pi(\mathbf{q})}} \tag{7.2.41}$$

We can see that for vector partitions of a particular length $k = l(\mathbf{q})$, it is the set partitions of $\{1, 2, 3, \ldots, k\}$ that control the behaviour.

The poset (partially ordered set) structure of set of set partitions is well studied [111], and will help further explain the structure of the matrix C and its inverse $\tilde{C}$. The partial ordering is defined by saying that one set partition, π, is less than another, π', if every block $b \in \pi$ is contained within some block $b' \in \pi'$. We call π a refinement of π' or π' a coarsening of π.

Intuitively, if $\pi < \pi'$, then the blocks of π are smaller in size than those in π'. However, this means that there are more blocks in π than in π', so confusingly $\pi < \pi'$ implies that $|\pi| > |\pi'|$.

Now instead of looking at $T_{\mathbf{p}}$, we look at $T_{\pi(\mathbf{p})}$, for some $\pi \in \Pi(l(\mathbf{p}))$. Clearly we can still use the formula (7.2.40) just by replacing $\mathbf{p}$ with $\pi(\mathbf{p})$. Then summing over $\pi' \in \Pi(l(\pi(\mathbf{p})))$ with summand $M_{\pi'(\pi(\mathbf{p}))}$ is equivalent to summing over all coarsenings $\pi'' \geq \pi$ with summand $M_{\pi''(\mathbf{p})}$, so we can write

$$T_{\pi(\mathbf{p})} = \sum_{\pi' \geq \pi} M_{\pi'(\mathbf{p})} \tag{7.2.42}$$

Considering T_π and M_π as functions from vector partitions to multi-symmetric functions, we have

$$T_\pi = \sum_{\pi' \geq \pi} M_{\pi'} \tag{7.2.43}$$

Equations like (7.2.43) are standard the theory of posets [111], and can be inverted using the Möbius inversion formula (7.2.49). We explain this formula in more detail in section 7.2.4, including a combinatoric interpretation that allows a simple explanation of the inversion property.

In this case, the Möbius inversion formula implies

$$M_\pi = \sum_{\pi' \geq \pi} \mu(\pi, \pi') T_{\pi'} \tag{7.2.44}$$

where the Möbius function $\mu(\pi, \pi')$ is defined in (7.2.53).

Choosing a vector partition $\mathbf{p}$ on which to act, we have

$$M_{\pi(\mathbf{p})} = \sum_{\pi' \geq \pi} \mu(\pi, \pi') T_{\pi'(\mathbf{p})} \tag{7.2.45}$$

We can now use this to find an explicit expression for $\tilde{C}^{\mathbf{q}}_{\mathbf{p}}$. Let $k = l(\mathbf{p})$ and $\pi = \pi_k$ to be the minimal set partition in $\Pi(k)$, in which every element has its own block so that $\pi_k(\mathbf{p}) = \mathbf{p}$. Applying (7.2.45) gives

$$M_{\mathbf{p}} = \sum_{\pi \in \Pi(k)} \mu(\pi_k, \pi) T_{\pi(\mathbf{p})} \tag{7.2.46}$$

and therefore

$$\boxed{\widetilde{C}^{\mathbf{p}}_{\mathbf{q}} = \sum_{\pi \in \Pi(k)} \mu(\pi_k, \pi)\ \delta_{\mathbf{p}\ \pi(\mathbf{q})}} \tag{7.2.47}$$

7.2.4 Möbius function for the poset of set partitions and combinatoric interpretation

In this section we introduce the Möbius function for a general poset, and give its value on the poset of set partitions. There is a combinatoric interpretation for the Möbius function in terms of permutations on the blocks of the set partitions, and this interpretation allows us to simply see why the Möbius inversion formula works in this case.

The Möbius function is defined recursively for a generic poset by

$$\mu(\pi, \pi') = \begin{cases} 1 & \pi = \pi' \\ -\sum\limits_{\pi \le \pi'' < \pi'} \mu(\pi, \pi'') & \pi < \pi' \\ 0 & \text{otherwise} \end{cases} \tag{7.2.48}$$

The key utility of this definition is in the Möbius inversion formula, which states that given two functions f, g from a poset into a vector space, the following two relations are equivalent

$$\begin{aligned} f(\pi) &= \sum_{\pi' \ge \pi} g(\pi') \\ g(\pi) &= \sum_{\pi' \ge \pi} \mu(\pi, \pi') f(\pi') \end{aligned} \tag{7.2.49}$$

In order to give an explicit expression for the Möbius function on set partitions, consider $\pi = \{b_1, b_2, \ldots, b_k\}$ for $k = |\pi|$. We then look at the set partitions of π itself. For example if $k = 3$ the five possible set partitions of π are

$$\begin{aligned} \rho_1 &= \{\{b_1, b_2, b_3\}\} & \rho_2 &= \{\{b_1, b_2\}, \{b_3\}\} \\ \rho_3 &= \{\{b_1, b_3\}, \{b_2\}\} & \rho_4 &= \{\{b_1\}, \{b_2, b_3\}\} \\ & & \rho_5 = \{\{b_1\}, \{b_2\}, \{b_3\}\} \end{aligned} \tag{7.2.50}$$

The set of set partitions of π is denoted by $\Pi(\pi)$, and there is an obvious correspondence between this and $\Pi(|\pi|)$. For any particular $\pi \in \Pi(n)$ and $\rho \in \Pi(\pi)$, we define $\rho(\pi)$ to be the following set partition in $\Pi(n)$.

$$\left\{ \bigcup_{b \in B} b : B \in \rho \right\} \tag{7.2.51}$$

So for the examples in (7.2.50), we have

$$\begin{aligned} \rho_1(\pi) &= \{b_1 \cup b_2 \cup b_3\} & \rho_2(\pi) &= \{b_1 \cup b_2, b_3\} \\ \rho_3(\pi) &= \{b_1 \cup b_3, b_2\} & \rho_4(\pi) &= \{b_1, b_2 \cup b_3\} \\ & \rho_5(\pi) = \{b_1, b_2, b_3\} = \pi \end{aligned} \tag{7.2.52}$$

Given $\pi \leq \pi'$, by definition each block $b \in \pi$ is a subset of a block $b' \in \pi'$. Therefore there is a set partition $\rho \in \Pi(\pi)$ such that $\rho(\pi) = \pi'$, we call this set partition π'/π.

Using the definition of π'/π for $\pi' \geq \pi$, we can now give an expression for $\mu(\pi, \pi')$, which is a standard result in the field of posets [111]. Firstly, by definition $\mu(\pi, \pi')$ vanishes unless $\pi' < \pi$, so we assume $\pi' \geq \pi$. This means π'/π exists, and we can write

$$\boxed{\mu(\pi, \pi') = (-1)^{|\pi| - |\pi'|} \prod_{b \in \pi'/\pi} (|b| - 1)!} \tag{7.2.53}$$

There is a combinatoric interpretation for the magnitude of $\mu(\pi, \pi')$ in terms of permutations, where the sign of μ is given by the sign of these permutations. In order to describe this, consider a permutation $\sigma \in S_n$ and take an arbitrary subset $A \subseteq \{1, 2, \ldots, n\}$. Then σ acts on A by permuting the numbers 1 to n, leading to a distinct subset $\sigma(A)$. We can then define the subgroup $G(\pi) \leq S_n$ by

$$G(\pi) = \{\sigma : \sigma(b) = b \text{ for all blocks } b \in \pi\} \tag{7.2.54}$$

For π with block sizes of $[\lambda_1, \lambda_2, \ldots, \lambda_k] \vdash n$, we have

$$G(\pi) \cong S_{\lambda_1} \times S_{\lambda_2} \times \cdots \times S_{\lambda_k} \tag{7.2.55}$$

Intuitively, the S_{λ_i} factor permutes the elements of the corresponding block with size λ_i. The exact embedding of $S_{\lambda_1} \times \cdots \times S_{\lambda_k}$ into S_n depends on the set partition.

Take a $\sigma \in S_n$. The cycle structure of σ defines a partition $\pi(\sigma) \in \Pi(n)$. Formally, the set partition $\pi(\sigma)$ is simply the set of orbits of $\{1, 2, \ldots, n\}$ under the action of σ. We also define a set of permutations associated with each $\pi \in \Pi(n)$

$$\text{Perms}(\pi) = \{\sigma : \pi(\sigma) = \pi\} \tag{7.2.56}$$

For any $\sigma \in S_n$ with $\pi(\sigma) = \pi$, $\text{Perms}(\pi)$ is just the conjugacy class of σ under conjugation by $G(\pi)$.

Clearly $\text{Perms}(\pi)$ are disjoint for different π, and between them they cover S_n.

$$\bigsqcup_{\pi \in \Pi(n)} \text{Perms}(\pi) = S_n \tag{7.2.57}$$

We have a similar result for $G(\pi)$, obtained by taking the decomposition (7.2.55) and applying (7.2.57) to each factor individually.

$$\bigsqcup_{\pi' \leq \pi} \text{Perms}(\pi') = G(\pi) \tag{7.2.58}$$

To illustrate the above, we now give some examples. If we fix $\pi = \{\{1,2,3\},\{4,5\},\{6\}\}$ then

$$G(\pi) = S_{\{1,2,3\}} \times S_{\{4,5\}} \times S_{\{6\}} \cong S_3 \times S_2 \times S_1 \tag{7.2.59}$$

$$\text{Perms}(\pi) = \{\ (1,2,3)(4,5)\ ,\ (1,3,2)(4,5)\ \} \tag{7.2.60}$$

Enumerating the elements of $G(\pi)$, we can see that it splits as specified in (7.2.58).

$$\begin{aligned}
G(\pi) &= \{e, (1,2), (1,3), (2,3), (1,2,3), (1,3,2),\\
&\qquad (4,5), (1,2)(4,5), (1,3)(4,5), (2,3)(4,5), (1,2,3)(4,5), (1,3,2)(4,5)\}\\
&= \{e\} \sqcup \{(1,2)\} \sqcup \{(1,3)\} \sqcup \{(2,3)\} \sqcup \{(1,2,3), (1,3,2)\}\\
&\quad \sqcup \{(4,5)\} \sqcup \{(1,2)(4,5)\} \sqcup \{(1,3)(4,5)\} \sqcup \{(2,3)(4,5)\}\\
&\quad \sqcup \{(1,2,3)(4,5), (1,3,2)(4,5)\}\\
&= \text{Perms}\big(\{\{1\},\{2\},\{3\},\{4\},\{5\},\{6\}\}\big) \sqcup \text{Perms}\big(\{\{1,2\},\{3\},\{4\},\{5\},\{6\}\}\big)\\
&\quad \sqcup \text{Perms}\big(\{\{1,3\},\{2\},\{4\},\{5\},\{6\}\}\big) \sqcup \text{Perms}\big(\{\{1\},\{2,3\},\{4\},\{5\},\{6\}\}\big)\\
&\quad \sqcup \text{Perms}\big(\{\{1,2,3\},\{4\},\{5\},\{6\}\}\big) \sqcup \text{Perms}\big(\{\{1\},\{2\},\{3\},\{4,5\},\{6\}\}\big)\\
&\quad \sqcup \text{Perms}\big(\{\{1,2\},\{3\},\{4,5\},\{6\}\}\big) \sqcup \text{Perms}\big(\{\{1,3\},\{2\},\{4,5\},\{6\}\}\big)\\
&\quad \sqcup \text{Perms}\big(\{\{1\},\{2,3\},\{4,5\},\{6\}\}\big) \sqcup \text{Perms}\big(\{\{1,2,3\},\{4,5\},\{6\}\}\big) && (7.2.61)\\
&= \bigsqcup_{\pi' \leq \pi} \text{Perms}(\pi') && (7.2.62)
\end{aligned}$$

Equations (7.2.54-7.2.62) are based on using permutations $\sigma \in S_n$ and set partitions $\pi \in \Pi(n)$. However, if we pick $\pi \in \Pi(n)$, we can use the exact same constructions for permutations of π itself - we call this group S_π - and set partitions $\rho \in \Pi(\pi)$. Then

$\mathrm{Perms}(\pi'/\pi)$ provides our combinatoric interpretation for μ

$$\boxed{\mu(\pi,\pi') = \sum_{\sigma\in\mathrm{Perms}(\pi'/\pi)} \mathrm{sgn}(\sigma) = (-1)^{|\pi|-|\pi'|}\left|\mathrm{Perms}(\pi'/\pi)\right|} \tag{7.2.63}$$

So the magnitude of μ is just the number of permutations in a certain conjugacy class, and its sign is just the sign of these permutations.

This permutation interpretation of μ allows us to easily prove the Möbius inversion formula for set partitions. Fix π and π'' with $\pi'' \geq \pi$ and consider the sum

$$\sum_{\pi''\geq\pi'\geq\pi} \mu(\pi,\pi') \tag{7.2.64}$$

The simplest way to parameterise the sum over π' is to look at the possible $\pi'/\pi \in \Pi(\pi)$. The condition $\pi'' \geq \pi'$ becomes $(\pi''/\pi) \geq (\pi'/\pi)$, so instead of summing over $\pi' \in \Pi(n)$, we sum over $\pi'/\pi = \rho \in \Pi(\pi)$.

$$\begin{aligned}\sum_{\pi''\geq\pi'\geq\pi} \mu(\pi,\pi') &= \sum_{\rho\leq(\pi''/\pi)} \sum_{\sigma\in\mathrm{Perms}(\rho)} \mathrm{sgn}(\sigma)\\ &= \sum_{\sigma\in G(\pi''/\pi)} \mathrm{sgn}(\sigma)\\ &= \begin{cases} 1 & G(\pi''/\pi) \cong S_1\times S_1\times\cdots\times S_1\\ 0 & \text{otherwise}\end{cases}\end{aligned} \tag{7.2.65}$$

Where we have used (7.2.58) to change the sum into one over $G(\pi''/\pi)$, and the final line is a simple fact from permutation group theory. Now the only case for which $G(\pi''/\pi) \cong S_1\times\cdots\times S_1$ is when $\pi'' = \pi$, so we conclude that

$$\sum_{\pi''\geq\pi'\geq\pi} \mu(\pi,\pi') = \delta_{\pi\pi''} \tag{7.2.66}$$

Using this result and substituting (7.2.43), we have

$$\begin{aligned}\sum_{\pi'\geq\pi} \mu(\pi,\pi')T_{\pi'} &= \sum_{\pi''\geq\pi'\geq\pi} \mu(\pi,\pi')M_{\pi''}\\ &= M_\pi\end{aligned} \tag{7.2.67}$$

This proves the Möbius inversion formula for set partitions.

For a more thorough overview of the Möbius function on general posets and for the poset of set partitions see [111].

7.2.5 More general C and $\tilde{C}$ matrices for M-matrix systems

The structures explained in sections 7.2.3 and 7.2.4, involving the poset of set partitions and the associated Möbius function, can be used not just for the two matrix system, but for an M-matrix system. In (7.2.43) we introduced T_π and M_π as functions from 2-vector partitions of length $|\pi|$ into the space of multi-symmetric functions defined on two families of variables, x_i and y_i. However, we could equally consider them as functions from M-vector partitions of length $|\pi|$ into the space of multi-symmetric functions defined on M families of variables, $x_i^{(1)}, x_i^{(2)}, \ldots, x_i^{(M)}$. These multi-symmetric functions have monomial and multi-trace bases defined in direct analogy to the 2-vector versions in section (7.2.2). For each M there are corresponding C and $\widetilde{C}$ matrices, defined in a completely analogous way to (7.2.41) and (7.2.47).

To think about these possibilities in a unified way, we define a more general C and $\widetilde{C}$ that transform between T_π and M_π.

$$T_\pi = \sum_{\pi'} C_\pi^{\pi'} M_{\pi'} \qquad\qquad M_\pi = \sum_{\pi'} \widetilde{C}_\pi^{\pi'} T_{\pi'} \tag{7.2.68}$$

We already have expressions for these from (7.2.43) and (7.2.44), given by

$$C_\pi^{\pi'} = \zeta(\pi, \pi') = \begin{cases} 1 & \pi' \geq \pi \\ 0 & \text{otherwise} \end{cases} \qquad\qquad \widetilde{C}_\pi^{\pi'} = \mu(\pi, \pi') \tag{7.2.69}$$

where the first equation defines $\zeta(\pi, \pi')$. The above also serves as the definition for the ζ function of a general poset, just with π, π' arbitrary elements of the poset rather than set partitions. We have already seen the Möbius function for a general poset in (7.2.48). An equivalent way of stating the Möbius inversion formula seen in (7.2.49) is that the ζ and μ are inverses of each other when multiplied as matrices

$$\sum_{\pi'} \zeta(\pi, \pi')\mu(\pi', \pi'') = \delta_{\pi\pi''} \tag{7.2.70}$$

The C and $\widetilde{C}$ for vector partitions (both 2-vectors and M-vectors) can easily be obtained from these more general objects. For $\mathbf{p}, \mathbf{q} \vdash (n_1, n_2)$ and some $\pi \in \Pi(n)$ such that $\pi([(1,0)^{n_1}, (0,1)^{n_2}]) = \mathbf{p}$, we have

$$C_{\mathbf{p}}^{\mathbf{q}} = \sum_{\pi' \in \Pi(n)} C_\pi^{\pi'} \qquad\qquad \widetilde{C}_{\mathbf{p}}^{\mathbf{q}} = \sum_{\pi' \in \Pi(n)} \widetilde{C}_\pi^{\pi'} \tag{7.2.71}$$

where the sums run over π' with $\pi'\left([(1,0)^{n_1}, (0,1)^{n_2}]\right) = \mathbf{q}$. Analogous formulae hold for M-vectors.

We can think of the (7.2.71) as a flavour projection from the general system of

M_π, T_π to the M-flavour system consisting of $M_{\mathbf{p}}$ and $T_{\mathbf{p}}$. Physically, one flavour corresponds to the half-BPS sector, two flavours to the quarter-BPS sector, and three to the eighth-BPS sector. We give an alternative viewpoint on the flavour projection using permutations in section 7.5.1.

As an example, consider $n = 4$. There are 15 different set partitions in $\Pi(4)$, so to simplify things we only give the transformations for the five different orbits under S_4, corresponding to the integer partitions of 4. The C matrix can be read off from the relationships showing T_π in terms of M_π.

$$T_{\{\{1,2,3,4\}\}} = M_{\{\{1,2,3,4\}\}} \tag{7.2.72}$$
$$T_{\{\{1\},\{2,3,4\}\}} = M_{\{\{1\},\{2,3,4\}\}} + M_{\{\{1,2,3,4\}\}} \tag{7.2.73}$$
$$T_{\{\{1,2\},\{3,4\}\}} = M_{\{\{1,2\},\{3,4\}\}} + M_{\{\{1,2,3,4\}\}} \tag{7.2.74}$$
$$\begin{aligned} T_{\{\{1,2\},\{3\},\{4\}\}} = {} & M_{\{\{1,2\},\{3\},\{4\}\}} + M_{\{\{1,2,3\},\{4\}\}} + M_{\{\{1,2,4\},\{3\}\}} \\ & + M_{\{\{1,2\},\{3,4\}\}} + M_{\{\{1,2,3,4\}\}} \end{aligned} \tag{7.2.75}$$
$$\begin{aligned} T_{\{\{1\},\{2\},\{3\},\{4\}\}} = {} & M_{\{\{1\},\{2\},\{3\},\{4\}\}} + M_{\{\{1,2\},\{3\},\{4\}\}} + M_{\{\{1,3\},\{2\},\{4\}\}} \\ & + M_{\{\{1,4\},\{2\},\{3\}\}} + M_{\{\{1\},\{2,3\},\{4\}\}} + M_{\{\{1\},\{2,4\},\{3\}\}} \\ & + M_{\{\{1\},\{2\},\{3,4\}\}} + M_{\{\{1,2\},\{3,4\}\}} + M_{\{\{1,3\},\{2,4\}\}} \\ & + M_{\{\{1,4\},\{2,3\}\}} + M_{\{\{1,2,3\},\{4\}\}} + M_{\{\{1,2,4\},\{3\}\}} \\ & + M_{\{\{1,3,4\},\{2\}\}} + M_{\{\{1\},\{2,3,4\}\}} + M_{\{\{1,2,3,4\}\}} \end{aligned} \tag{7.2.76}$$

The $\widetilde{C}$ matrix can be shown in an analogous way by writing M_π in terms of T_π.

$$M_{\{\{1,2,3,4\}\}} = T_{\{\{1,2,3,4\}\}} \tag{7.2.77}$$
$$M_{\{\{1\},\{2,3,4\}\}} = T_{\{\{1\},\{2,3,4\}\}} - T_{\{\{1,2,3,4\}\}} \tag{7.2.78}$$
$$M_{\{\{1,2\},\{3,4\}\}} = T_{\{\{1,2\},\{3,4\}\}} - T_{\{\{1,2,3,4\}\}} \tag{7.2.79}$$
$$\begin{aligned} M_{\{\{1,2\},\{3\},\{4\}\}} = {} & T_{\{\{1,2\},\{3\},\{4\}\}} - T_{\{\{1,2,3\},\{4\}\}} - T_{\{\{1,2,4\},\{3\}\}} \\ & - T_{\{\{1,2\},\{3,4\}\}} + 2T_{\{\{1,2,3,4\}\}} \end{aligned} \tag{7.2.80}$$
$$\begin{aligned} M_{\{\{1\},\{2\},\{3\},\{4\}\}} = {} & T_{\{\{1\},\{2\},\{3\},\{4\}\}} - T_{\{\{1,2\},\{3\},\{4\}\}} - T_{\{\{1,3\},\{2\},\{4\}\}} \\ & - T_{\{\{1,4\},\{2\},\{3\}\}} - T_{\{\{1\},\{2,3\},\{4\}\}} - T_{\{\{1\},\{2,4\},\{3\}\}} \\ & - T_{\{\{1\},\{2\},\{3,4\}\}} + T_{\{\{1,2\},\{3,4\}\}} + T_{\{\{1,3\},\{2,4\}\}} \\ & + T_{\{\{1,4\},\{2,3\}\}} + 2T_{\{\{1,2,3\},\{4\}\}} + 2T_{\{\{1,2,4\},\{3\}\}} \\ & + 2T_{\{\{1,3,4\},\{2\}\}} + 2T_{\{\{1\},\{2,3,4\}\}} - 6T_{\{\{1,2,3,4\}\}} \end{aligned} \tag{7.2.81}$$

We can then apply these to the vector partition $[(1,0)^2,(0,1)^2]$ to get C and $\widetilde{C}$ for field content $(2,2)$. Again we choose to display them by writing out the relations between

$T_{\mathbf{p}}$ and $M_{\mathbf{p}}$, but this time all possibilities are included. The C matrix is

$$\mathrm{Str}X^2Y^2 = M_{[(2,2)]} \tag{7.2.82}$$
$$\mathrm{Tr}X^2Y\mathrm{Tr}Y = M_{[(2,1),(0,1)]} + M_{[(2,2)]} \tag{7.2.83}$$
$$\mathrm{Tr}X^2\mathrm{Tr}Y^2 = M_{[(2,0),(0,2)]} + M_{[(2,2)]} \tag{7.2.84}$$
$$\mathrm{Tr}XY^2\mathrm{Tr}X = M_{[(1,2),(1,0)]} + M_{[(2,2)]} \tag{7.2.85}$$
$$(\mathrm{Tr}XY)^2 = M_{[(1,1),(1,1)]} + M_{[(2,2)]} \tag{7.2.86}$$
$$\mathrm{Tr}X^2\left(\mathrm{Tr}Y\right)^2 = M_{[(2,0),(0,1),(0,1)]} + M_{[(2,0),(0,2)]} + 2M_{[(2,1),(0,1)]} + M_{[(2,2)]} \tag{7.2.87}$$
$$\begin{aligned}\mathrm{Tr}XY\mathrm{Tr}X\mathrm{Tr}Y = {} & M_{[(1,1),(1,0),(0,1)]} + M_{[(1,1),(1,1)]} + M_{[(1,2),(1,0)]} \\ & + M_{[(2,1),(0,1)]} + M_{[(2,2)]}\end{aligned} \tag{7.2.88}$$
$$\left(\mathrm{Tr}X\right)^2\mathrm{Tr}Y^2 = M_{[(1,0),(1,0),(0,2)]} + 2M_{[(1,2),(1,0)]} + M_{[(2,0),(0,2)]} + M_{[(2,2)]} \tag{7.2.89}$$
$$\begin{aligned}\left(\mathrm{Tr}X\right)^2\left(\mathrm{Tr}Y\right)^2 = {} & M_{[(1,0),(1,0),(0,1),(0,1)]} + M_{[(1,0),(1,0),(0,2)]} + 4M_{[(1,1),(1,0),(0,1)]} \\ & + M_{[(2,0),(0,1),(0,1)]} + 2M_{[(1,1),(1,1)]} + 2M_{[(1,2),(1,0)]} \\ & + M_{[(2,0),(0,2)]} + 2M_{[(2,1),(0,1)]} + M_{[(2,2)]}\end{aligned} \tag{7.2.90}$$

The $\widetilde{C}$ matrix for $(2,2)$ is

$$M_{[(2,2)]} = \mathrm{Str}X^2Y^2 \tag{7.2.91}$$
$$M_{[(2,1),(0,1)]} = \mathrm{Tr}X^2Y\mathrm{Tr}Y - \mathrm{Str}X^2Y^2 \tag{7.2.92}$$
$$M_{[(2,0),(0,2)]} = \mathrm{Tr}X^2\mathrm{Tr}Y^2 - \mathrm{Str}X^2Y^2 \tag{7.2.93}$$
$$M_{[(1,2),(1,0)]} = \mathrm{Tr}XY^2\mathrm{Tr}X - \mathrm{Str}X^2Y^2 \tag{7.2.94}$$
$$M_{[(1,1),(1,1)]} = (\mathrm{Tr}XY)^2 - \mathrm{Str}X^2Y^2 \tag{7.2.95}$$
$$M_{[(2,0),(0,1),(0,1)]} = \mathrm{Tr}X^2\left(\mathrm{Tr}Y\right)^2 - \mathrm{Tr}X^2\mathrm{Tr}Y^2 - 2\mathrm{Tr}X^2Y\mathrm{Tr}Y + 2\mathrm{Str}X^2Y^2 \tag{7.2.96}$$
$$\begin{aligned}M_{[(1,1),(1,0),(0,1)]} = {} & \mathrm{Tr}XY\mathrm{Tr}X\mathrm{Tr}Y - (\mathrm{Tr}XY)^2 - \mathrm{Tr}XY^2\mathrm{Tr}X \\ & - \mathrm{Tr}X^2Y\mathrm{Tr}Y + 2\mathrm{Str}X^2Y^2\end{aligned} \tag{7.2.97}$$
$$M_{[(1,0),(1,0),(0,2)]} = \left(\mathrm{Tr}X\right)^2\mathrm{Tr}Y^2 - 2\mathrm{Tr}XY^2\mathrm{Tr}X - \mathrm{Tr}X^2\mathrm{Tr}Y^2 + 2\mathrm{Str}X^2Y^2 \tag{7.2.98}$$
$$\begin{aligned}M_{[(1,0),(1,0),(0,1),(0,1)]} = {} & \left(\mathrm{Tr}X\right)^2\left(\mathrm{Tr}Y\right)^2 - \left(\mathrm{Tr}X\right)^2\mathrm{Tr}Y^2 - 4\mathrm{Tr}XY\mathrm{Tr}X\mathrm{Tr}Y \\ & - \mathrm{Tr}X^2\left(\mathrm{Tr}Y\right)^2 + 2\left(\mathrm{Tr}XY\right)^2 + 4\mathrm{Tr}XY^2\mathrm{Tr}X \\ & + \mathrm{Tr}X^2\mathrm{Tr}Y^2 + 4\mathrm{Tr}X^2Y\mathrm{Tr}Y - 6\mathrm{Str}X^2Y^2\end{aligned} \tag{7.2.99}$$

Note that we have used

$$\mathrm{Str}X^2Y^2 = \frac{2}{3}\mathrm{Tr}X^2Y^2 + \frac{1}{3}\mathrm{Tr}(XY)^2 \tag{7.2.100}$$

rather than just $\mathrm{Tr}X^2Y^2$. This means the expressions (7.2.82-7.2.99) give the relations between $T_{\mathbf{p}}$ and $M_{\mathbf{p}}$ both as multi-symmetric functions and symmetrised trace operators in $\mathcal{N}=4$ super Yang-Mills.

7.2.6 Relation to other combinatorial quantities

Stirling numbers of the second kind, $S(n,k)$, are defined to be the number of ways of partitioning a set of n objects into k non-empty subsets. Combinatorically, these are a coarsened version of the 2-vector and set partition C matrices. Starting with the 2-vector version, $S(n,k)$ is given by

$$S(n,k) = \sum_{\substack{\mathbf{p}\vdash(n_1,n_2)\\ l(\mathbf{p})=k}} C^{\mathbf{p}}_{[(1,0)^{n_1},(0,1)^{n_2}]} \tag{7.2.101}$$

where $n_1+n_2=n$.

Alternatively, consider an arbitrary $m>n$ and $\mathbf{q}\vdash(m_1,m_2)$ with $m_1+m_2=m$ to be a vector partition with $l(\mathbf{q})=n$, then

$$S(n,k) = \sum_{\substack{\mathbf{p}\vdash(m_1,m_2)\\ l(\mathbf{p})=k}} C^{\mathbf{p}}_{\mathbf{q}} \tag{7.2.102}$$

Define $\pi_n\in\Pi(n)$ to be the unique set partition of length n, meaning each number has its own block. Then in terms of the more general set partition C

$$S(n,k) = \sum_{\substack{\pi\in\Pi(n)\\ |\pi|=k}} C^{\pi}_{\pi_n} \tag{7.2.103}$$

Or alternatively, taking $\pi\in\Pi(m)$ to be any set partition with $|\pi|=n$, then

$$S(n,k) = \sum_{\substack{\pi'\in\Pi(m)\\ |\pi'|=k}} C^{\pi'}_{\pi} \tag{7.2.104}$$

Unsigned Stirling numbers of the first kind, $|s(n,k)|$, are defined to be the number of permutations in S_n with k cycles. The signed Stirling numbers $s(n,k)$ have the same magnitude, but are multiplied by the sign of the permutations $(-1)^{n-k}$. This is related to the 2-vector and set partition $\widetilde{C}$ matrices in the same way as $S(n,k)$ was related to C. Using the same notation as (7.2.101-7.2.104), we have

$$s(n,k) = \sum_{\substack{\mathbf{p}\vdash(n_1,n_2)\\ l(\mathbf{p})=k}} \widetilde{C}^{\mathbf{p}}_{[(1,0)^{n_1},(0,1)^{n_2}]} \qquad s(n,k) = \sum_{\substack{\mathbf{p}\vdash(m_1,m_2)\\ l(\mathbf{p})=k}} \widetilde{C}^{\mathbf{p}}_{\mathbf{q}} \tag{7.2.105}$$

$$s(n,k) = \sum_{\substack{\pi \in \Pi(n) \\ |\pi|=k}} \widetilde{C}^{\pi}_{\pi_n} \qquad s(n,k) = \sum_{\substack{\pi' \in \Pi(m) \\ |\pi'|=k}} \widetilde{C}^{\pi'}_{\pi} \tag{7.2.106}$$

Bell numbers, B_n, count the number of set partitions of n objects. In terms of C, these are

$$B_n = \sum_k S(n,k) = \sum_{\mathbf{p} \vdash (n_1,n_2)} C^{\mathbf{p}}_{[(1,0)^{n_1},(0,1)^{n_2}]} = \sum_{\mathbf{p} \vdash (m_1,m_2)} C^{\mathbf{p}}_{\mathbf{q}} = \sum_{\pi \in \Pi(n)} C^{\pi}_{\pi_n} = \sum_{\pi' \in \Pi(m)} C^{\pi'}_{\pi} \tag{7.2.107}$$

7.3 Counting: $U(2) \times U(N)$ Young diagram labels and multiplicities at weak coupling

The space of states spanned by symmetrised traces $T_{\mathbf{p}}$ of general matrices X, Y admits a $U(2)$ action on the pair X, Y as in section 3.6.2. These symmetrised traces are representatives of the elements of the ring of gauge invariants modulo commutators. Specialising to diagonal matrices $X = \text{Diag}(x_1, x_2, \cdots, x_N), Y = \text{Diag}(y_1, y_2, \cdots, y_N)$ gives the isomorphism [97, 99] to multi-symmetric polynomials in x_i, y_i discussed in section 7.1.1. For economy of notation, we are generally using $T_{\mathbf{p}}$ also for the image $\iota(T_{\mathbf{p}})$ of the isomorphism. There is an analogous $U(2)$ action on multi-symmetric functions which transforms the pairs x_i, y_i. Applying the isomorphism and then doing a $U(2)$ transformation is equivalent to doing a $U(2)$ transformation on symmetrised traces and then applying the isomorphism. In other words the isomorphism between gauge invariants modulo commutators and multi-symmetric polynomials is a $U(2)$ equivariant isomorphism. The $U(2)$ transformations (3.6.10) on the monomial multi-symmetric functions, $M_{\mathbf{p}}$, are obtained either by expressing them in terms of $T_{\mathbf{p}}$ using the $\tilde{C}$ transformation or equivalently using the $U(2)$ on the pairs (x_i, y_i). In this latter picture, the $U(2)$ generators are

$$\begin{aligned} \mathcal{J}_0 &= \sum_{i=1}^{N} \left(x_i \frac{\partial}{\partial x_i} + y_i \frac{\partial}{\partial y_i} \right) & \mathcal{J}_3 &= \sum_{i=1}^{N} \left(x_i \frac{\partial}{\partial x_i} - y_i \frac{\partial}{\partial y_i} \right) \\ \mathcal{J}_+ &= \sum_{i=1}^{N} x_i \frac{\partial}{\partial y_i} & \mathcal{J}_- &= \sum_{i=1}^{N} y_i \frac{\partial}{\partial x_i} \end{aligned} \tag{7.3.1}$$

A $U(2)$ covariant basis will be sorted by $U(2)$ representations Λ and an index M_Λ labelling the basis states. As in section 3.6.2, M_Λ runs over the semi-standard tableaux of shape Λ and determines the field content. In order to parameterise the space for a specific Λ, we observe that for each vector partition $\mathbf{p} \vdash (n_1, n_2)$, there is an associated integer partition $p(\mathbf{p}) \vdash n_1 + n_2$ obtained by summing the pairs

$$\mathbf{p} = [(\lambda_1, \mu_1), (\lambda_2, \mu_2), \dots, (\lambda_k, \mu_k)] \to p(\mathbf{p}) = [\lambda_1 + \mu_1, \lambda_2 + \mu_2, \dots, \lambda_k + \mu_k] \quad (7.3.2)$$

Consider the action of $U(2)$ on a simple monomial $x^\lambda y^\mu$. We have

$$\begin{aligned} \mathcal{J}_0\, x^\lambda y^\mu &= (\lambda + \mu) x^\lambda y^\mu & \mathcal{J}_3\, x^\lambda y^\mu &= (\lambda - \mu) x^\lambda y^\mu \\ \mathcal{J}_+\, x^\lambda y^\mu &= \mu\, x^{\lambda+1} y^{\mu-1} & \mathcal{J}_-\, x^\lambda y^\mu &= \lambda\, x^{\lambda-1} y^{\mu+1} \end{aligned} \quad (7.3.3)$$

The operators $\mathcal{J}_\pm$ send $\lambda \to \lambda \pm 1$, $\mu \to \mu \mp 1$ while $\mathcal{J}_0$ and $\mathcal{J}_3$ leave λ, μ invariant. For all $U(2)$ generators, the sum $\lambda + \mu$ is unchanged. More generally, for a monomial $x_1^{\lambda_1} y_1^{\mu_1} \dots x_k^{\lambda_k} y_k^{\mu_k}$, the sums $\lambda_i + \mu_i$ are unchanged in each monomial term arising from the action of the $U(2)$ generators.

Applying this analysis to each of the monomials in $M_{\mathbf{p}}$, we see that $U(2)$ preserves the associated partition $p(\mathbf{p})$, and therefore p serves as another label in the $U(2)$ covariant basis. We denote the multiplicity of a given pair Λ, p in the covariant monomial basis by $\mathcal{M}_{\Lambda,p}$.

For a given associated partition $p = [1^{p_1}, 2^{p_2}, \dots]$ we have monomial multi-symmetric functions $M_{\mathbf{p}}$ with $p(\mathbf{p}) = p$. The constituent monomials in $M_{\mathbf{p}}$ (recall the defining equation (2.7.6)) contain products of p_i factors each with i variables that can be x or y and are transformed between the two using $\mathcal{J}_\pm$. We will show that these fit into the representation

$$\mathcal{R}_p^{U(2)} = \bigotimes_i \mathrm{Sym}^{p_i}\left(\mathrm{Sym}^i\left(V_2\right)\right) \quad (7.3.4)$$

where V_2 is the 2-dimensional fundamental representation of $U(2)$. We can decompose $\mathcal{R}_p^{U(2)}$ in terms of irreducible representations $R_\Lambda^{U(2)}$

$$\mathcal{R}_p^{U(2)} = \bigoplus_{\substack{\Lambda \vdash n \\ l(\Lambda) \leq 2}} R_\Lambda^{U(2)} \otimes V_{\Lambda,p}^{mult} \quad (7.3.5)$$

for some multiplicity space $V_{\Lambda,p}^{mult}$. The direct sum is restricted to run only over $\Lambda \vdash n$ since $\mathcal{R}_p^{U(2)}$ is a subspace of $(V_2)^{\otimes n}$, and therefore the $U(1)$ weight of all sub-representations is n. The analogous representation of the global symmetry $U(3)$ in the case of eighth-BPS states is discussed in [28, 51]. The multiplicity of $R_\Lambda^{U(2)}$ in $\mathcal{R}_p^{U(2)}$ is just the dimension of the multiplicity space $V_{\Lambda,p}^{mult}$, and is also the multiplicity of the pair Λ, p in the covariant monomial basis

$$\mathcal{M}_{\Lambda,p} = \mathrm{Mult}\left(\Lambda, \mathcal{R}_p^{U(2)}\right) = \mathrm{Dim}\left(V_{\Lambda,p}^{mult}\right) \quad (7.3.6)$$

To find this multiplicity we split $U(2)$ into its $U(1)$ and $SU(2)$ components as discussed in section 3.6.2. As already mentioned, $\mathcal{R}_p^{U(2)}$ is in the weight n representation of $U(1)$, so

$$\mathcal{R}_p^{U(2)} = R_n^{U(1)} \otimes \mathcal{R}_p^{SU(2)} \tag{7.3.7}$$

where $\mathcal{R}_p^{SU(2)}$ is

$$\mathcal{R}_p^{SU(2)} = \bigotimes_i \mathrm{Sym}^{p_i}\left(\mathrm{Sym}^i\left(R_{\frac{1}{2}}\right)\right) = \bigotimes_i \mathrm{Sym}^{p_i}\left(R_{\frac{i}{2}}\right) \tag{7.3.8}$$

for R_j the spin j representation of $SU(2)$. Then the $U(2)$ decomposition (7.3.5) of $\mathcal{R}_p^{U(2)}$ is equivalent to the $SU(2)$ decomposition

$$\mathcal{R}_p^{SU(2)} = \bigoplus_j R_j \otimes V^{mult}_{[\frac{n}{2}+j,\frac{n}{2}-j],p} \tag{7.3.9}$$

where we have used the correspondence, discussed in section 3.6.2, between a $U(2)$ representation $\Lambda = \left[\frac{n}{2}+j, \frac{n}{2}-j\right]$ of $U(1)$ weight n and an $SU(2)$ representation of spin j. The question of calculating the dimension of the multiplicity space in (7.3.9) is called an $SU(2)$ plethysm problem and is addressed in [112]. We will use a formula derived there shortly.

The monomials $M_{\mathbf{p}}$ with $p(\mathbf{p}) = p$ define states $|\mathbf{p}\rangle$ in $\mathcal{R}_p^{U(2)}$, whose normalisation is given by the S_n inner product on $M_{\mathbf{p}}$

$$\langle \mathbf{p}|\mathbf{q}\rangle = \langle M_{\mathbf{p}}|M_{\mathbf{q}}\rangle \tag{7.3.10}$$

There is a change of basis to $U(2)$ orthonormal covariant states of the form

$$|\Lambda, M_\Lambda, p, \nu\rangle \tag{7.3.11}$$

where ν is a multiplicity index with $1 \le \nu \le \mathcal{M}_{\Lambda,p}$. This change of basis is implemented using Clebsch-Gordan coefficients

$$|\Lambda, M_\Lambda, p, \nu\rangle = \sum_{\mathbf{p}\,:\,p(\mathbf{p})=p} B^{\mathbf{p}}_{\Lambda,M_\Lambda,p,\nu}|\mathbf{p}\rangle \tag{7.3.12}$$

We define the covariant monomial operators by

$$M_{\Lambda,M_\Lambda,p,\nu} = \sum_{\mathbf{p}\,:\,p(\mathbf{p})=p} B^{\mathbf{p}}_{\Lambda,M_\Lambda,p,\nu} M_{\mathbf{p}} \tag{7.3.13}$$

As an example, consider the multi-symmetric monomials for field content $(2,2)$, given explicitly in (7.2.91-7.2.99). We only give the M_Λ and p labels, as the shape of the Young tableau specifies Λ, and the multiplicity for these operators is trivial. The covariant monomials are

$$M_{\begin{smallmatrix}1&1&2&2\end{smallmatrix},\,[4]} = \sqrt{\frac{3}{2}} M_{[(2,2)]} \tag{7.3.14}$$

$$M_{\begin{smallmatrix}1&1&2&2\end{smallmatrix},\,[3,1]} = \sqrt{\frac{3}{14}} \left(M_{[(2,1),(0,1)]} + M_{[(1,2),(1,0)]}\right) \tag{7.3.15}$$

$$M_{\begin{smallmatrix}1&1&2&2\end{smallmatrix},\,[2,2]} = \frac{1}{3\sqrt{2}} \left(M_{[(2,0),(0,2)]} + 2M_{[(1,1),(1,1)]}\right) \tag{7.3.16}$$

$$M_{\begin{smallmatrix}1&1&2&2\end{smallmatrix},\,[2,1,1]} = \frac{1}{4\sqrt{15}} \left(M_{[(2,0),(0,1),(0,1)]} + 4M_{[(1,1),(1,0),(0,1)]} + M_{[(1,0),(1,0),(0,2)]}\right) \tag{7.3.17}$$

$$M_{\begin{smallmatrix}1&1&2&2\end{smallmatrix},\,[1,1,1,1]} = \frac{1}{4\sqrt{6}} M_{[(1,0),(1,0),(0,1),(0,1)]} \tag{7.3.18}$$

$$M_{\begin{smallmatrix}1&1&2\\2\end{smallmatrix},\,[3,1]} = \frac{1}{\sqrt{2}} \left(M_{[(2,1),(0,1)]} - M_{[(1,2),(1,0)]}\right) \tag{7.3.19}$$

$$M_{\begin{smallmatrix}1&1&2\\2\end{smallmatrix},\,[2,1,1]} = \frac{1}{4} \left(M_{[(2,0),(0,1),(0,1)]} - M_{[(1,0),(1,0),(0,2)]}\right) \tag{7.3.20}$$

$$M_{\begin{smallmatrix}1&1\\2&2\end{smallmatrix},\,[2,2]} = \frac{1}{\sqrt{6}} \left(M_{[(2,0),(0,2)]} - M_{[(1,1),(1,1)]}\right) \tag{7.3.21}$$

$$M_{\begin{smallmatrix}1&1\\2&2\end{smallmatrix},\,[2,1,1]} = \frac{1}{6} \left(M_{[(2,0),(0,1),(0,1)]} - 2M_{[(1,1),(1,0),(0,1)]} + M_{[(1,0),(1,0),(0,2)]}\right) \tag{7.3.22}$$

The associated partition has length $l(p(\mathbf{p})) = l(\mathbf{p})$, and therefore the SEP compatibility (modulo commutators) of the $M_{\mathbf{p}}$ basis is transferred to the new basis.

If p has length $l(p) > N$ then the multi-symmetric function $M_{\Lambda,M_\Lambda,p,\nu}$ vanishes identically, while on the other side of the isomorphism, the operator $M_{\Lambda,M_\Lambda,p,\nu}$ reduces to a commutator trace and therefore is no longer pre-BPS. Operators with $l(p) \leq N$ are in general not pre-BPS, but differ from such an operator by a commutator trace. In section 7.4 we show how to remove this commutator trace component to derive a pre-BPS basis. For now, we note that the multiplicity $\mathcal{M}_{\Lambda,p}$ determines the finite N combinatorics of the quarter-BPS sector.

The half-BPS operators $\mathcal{O}_R$ defined in (2.3.14) are dual to giant gravitons. There are two types of giant gravitons: those that have an extended $S^3 \subset S^5$ as part of the world-volume, and those that have an extended $S^3 \subset AdS_5$. We will refer to these as sphere giants and AdS giants respectively: they are also sometimes distinguished in the AdS/CFT literature as giants versus dual-giants respectively. $U(2)$ rotations of these half-BPS giants produces giant graviton states in the $\Lambda = [n]$ representation, where n is the number of boxes in the Young diagram R. The nature of the Young diagram R is related to the type of giant graviton system. As we deform Λ to $\Lambda = [n-m, m]$ we move away from the half-BPS sector. The deformations of sphere giant states are described in terms of moduli spaces of polynomials in three complex variables [39] while deformations of AdS giant states are described in terms of a family of solutions with

$S^3 \subset AdS_5$ world-volumes orbiting great circles on the S^5 [40].

In section 7.4 we will produce a basis $S^{BPS}_{\Lambda,M_\Lambda,p,\nu}$ for the quarter-BPS sector with the same labels as (7.3.13). For $\Lambda = [n]$, this basis agrees with the half-BPS Schur basis (2.3.14) by identifying p with R. This matching between the Young diagrams p labelling the quarter-BPS sector and R labelling the half-BPS states suggests that for a particular diagram p, we can follow the half-BPS sector states into the quarter-BPS sector by considering $\Lambda = [n-m, m]$ and slowly increasing the length of the second row, m. We expect that if we keep p fixed in this half to quarter transition, we qualitatively preserve the physical nature of the giant graviton: Young diagrams with a few long rows of lengths order N correspond to AdS-giants while diagrams with a few long columns of lengths order N correspond to sphere giants. It is reasonable to think of Young diagrams p (for more general Λ) with k rows of length comparable to N as an AdS-giant system formed as some form of composite of k giants. Likewise, in the following discussion, we will think of a Young diagram p with k rows of length order N as some composite involving k sphere giants. There will be interesting differences between sphere giants and AdS giants, so the precise meaning of "composite system of k giants" is something which should be explored through future comparisons between bulk physics and CFT correlators.

The multiplicities (7.3.6) interpolate from half-BPS in the case $\Lambda = [n]$ to more general quarter-BPS for $\Lambda = [n - \Lambda_2, \Lambda_2]$, with small Λ_2 being close to half-BPS. These multiplicities should be reproducible from the stringy physics of $D3$-branes in $AdS_5 \times S^5$. In each part of this section, we discuss the giant graviton interpretation of the multiplicity results.

7.3.1 Λ, p multiplicities and plethysms of $SU(2)$ characters

We consider the space of multi-symmetric functions $M_{\mathbf{p}}$ with a given associated partition p, and how this can be split into $U(2)$ representations.

As discussed in section 3.6.2, $U(2)$ can be split into a product of $U(1)$ and $SU(2)$. The $U(1)$ weight of a given p is just $n = |p|$, so to derive the $U(2)$ representation we first study the $SU(2)$ part, then recombine with the $U(1)$ piece at the end.

For the sake of simplicity, we will primarily work with non-symmetrised monomials, since such a choice determines the associated multi-symmetric function by adding all permuted monomials. The construction of the multi-symmetric function from the non-symmetrised monomial can affect the $SU(2)$ structure, and we will describe this in more detail as it occurs.

Start by considering $p = [n]$. This allows $\mathbf{p} = [(\lambda, \mu)]$ for $\lambda + \mu = n$. The non-symmetrised monomials corresponding to these are just $x_1^\lambda y_1^\mu$, whose action under $U(2)$ we gave in (7.3.3). From the action of $\mathcal{J}_\pm, \mathcal{J}_3$, they lie in the spin $\frac{n}{2}$ representation of

$SU(2)$. Symmetrising the monomials does not change the $SU(2)$ structure, so $p = [n]$ produces the $R_{\frac{n}{2}}$ representation of $SU(2)$.

Next consider $p = [k_1, k_2]$. This allows $\mathbf{p} = [(\lambda_1, \mu_1), (\lambda_2, \mu_2)]$, with corresponding non-symmetrised monomials $x_1^{\lambda_1} y_1^{\mu_1} x_2^{\lambda_2} y_2^{\mu_2}$ subject to $\lambda_i + \mu_i = k_i$ for $i = 1, 2$. There are $(k+1)(l+1)$ different states, living in the tensor product representation $R_{\frac{k_1}{2}} \otimes R_{\frac{k_2}{2}}$. When $k_1 \neq k_2$, this is the correct $SU(2)$ representation for the symmetrised version as well. However, if $k_1 = k_2$, then the states $x_1^{\lambda_1} y_1^{\mu_1} x_2^{\lambda_2} y_2^{\mu_2}$ and $x_1^{\lambda_2} y_1^{\mu_2} x_2^{\lambda_1} y_2^{\mu_1}$ both lead to the same multi-symmetric function and should be identified with each other. The correct representation here is the symmetric part of the tensor product, written as $\mathrm{Sym}^2 \left(R_{\frac{k_1}{2}} \right)$.

As our final example, take $p = [k_1, k_2, k_3]$, allowing $\mathbf{p} = [(\lambda_1, \mu_1), (\lambda_2, \mu_2), (\lambda_3, \mu_3)]$. By the same considerations as the previous two examples, the non-symmetrised monomials $x_1^{\lambda_1} y_1^{\mu_1} x_2^{\lambda_2} y_2^{\mu_2} x_3^{\lambda_3} y_3^{\lambda_3}$ fit into the $R_{\frac{k_1}{2}} \otimes R_{\frac{k_2}{2}} \otimes R_{\frac{k_3}{2}}$ representation of $SU(2)$. If all three of the ks are distinct, this is the correct representation for the symmetrised monomials. If two of the ks coincide and the third is distinct, e.g. $k_1 = k_2 \neq k_3$, then the $M_{\mathbf{p}}$ live in $\mathrm{Sym}^2 \left(R_{\frac{k_1}{2}} \right) \otimes R_{\frac{k_3}{2}}$. Finally, if $k_1 = k_2 = k_3$, then there are 6 permutations of the basic monomial that lead to the same multi-symmetric function and should be identified. These are

$$\begin{array}{lll} x_1^{\lambda_1} y_1^{\mu_1} x_2^{\lambda_2} y_2^{\mu_2} x_3^{\lambda_3} y_3^{\lambda_3} & x_1^{\lambda_2} y_1^{\mu_2} x_2^{\lambda_3} y_2^{\mu_3} x_3^{\lambda_1} y_3^{\lambda_1} & x_1^{\lambda_3} y_1^{\mu_3} x_2^{\lambda_1} y_2^{\mu_1} x_3^{\lambda_2} y_3^{\lambda_2} \\ x_1^{\lambda_1} y_1^{\mu_1} x_2^{\lambda_3} y_2^{\mu_3} x_3^{\lambda_2} y_3^{\lambda_2} & x_1^{\lambda_3} y_1^{\mu_3} x_2^{\lambda_2} y_2^{\mu_2} x_3^{\lambda_1} y_3^{\lambda_1} & x_1^{\lambda_2} y_1^{\mu_2} x_2^{\lambda_1} y_2^{\mu_1} x_3^{\lambda_3} y_3^{\lambda_3} \end{array} \tag{7.3.23}$$

In an analogous way to the single coincidence, this leads to us using the completely symmetric part of the triple tensor product, written $\mathrm{Sym}^3 \left(R_{\frac{k_1}{2}} \right)$. This is the part of $R_{\frac{k_1}{2}}^{\otimes 3}$ that is invariant under all S_3 permutations.

From the principles established in these three examples, we can generalise to a generic integer partition p. The multi-symmetric functions with associated partition $p = [1^{p_1}, 2^{p_2}, \dots]$ fit into the representation of $SU(2)$ given by

$$\mathcal{R}_p^{SU(2)} = R_p^{SU(2)} = \bigotimes_i \mathrm{Sym}^{p_i} \left(R_{\frac{i}{2}} \right) \tag{7.3.24}$$

Restoring the $U(1)$ weight, as a $U(2)$ representation this is

$$\begin{aligned} \mathcal{R}_p^{U(2)} &= R_n^{U(1)} \otimes \mathcal{R}_p^{SU(2)} \\ &= \bigotimes_i R_{ip_i}^{U(1)} \otimes \mathrm{Sym}^{p_i} \left(\mathrm{Sym}^i \left(R_{\frac{1}{2}} \right) \right) \\ &= \bigotimes_i \mathrm{Sym}^{p_i} \left(R_i^{U(1)} \otimes \mathrm{Sym}^i \left(R_{\frac{1}{2}} \right) \right) \end{aligned}$$

$$
\begin{aligned}
&= \bigotimes_i \mathrm{Sym}^{p_i}\left(\mathrm{Sym}^i\left(R_1^{U(1)} \otimes R_{\frac{1}{2}}\right)\right) \\
&= \bigotimes_i \mathrm{Sym}^{p_i}\left(\mathrm{Sym}^i\left(V_2\right)\right) \qquad (7.3.25)
\end{aligned}
$$

where we have used $R_{\frac{i}{2}} = \mathrm{Sym}^i\left(R_{\frac{1}{2}}\right)$ for $SU(2)$ representations and the fundamental representation of $U(2)$ is $V_2 = R_1^{U(1)} \otimes R_{\frac{1}{2}}$.

So the problem of finding $\mathcal{M}_{\Lambda,p}$ reduces to a $U(2)$ representation theory problem of finding the multiplicity of $R_\Lambda^{U(2)}$ within the representation $\mathcal{R}_p^{U(2)}$, or equivalently the $SU(2)$ representation theory problem of finding the multiplicity of R_j within $\mathcal{R}_p^{SU(2)}$ and using the correspondence $j \leftrightarrow \Lambda = \left[\frac{n}{2}+j, \frac{n}{2}-j\right]$ for $U(2)$ representations of $U(1)$ weight n.

We will solve the $SU(2)$ problem. In order to do this, we calculate the character of the representation (7.3.24) and compare it to the known characters of the spin representations. From standard $SU(2)$ representation theory we know that

$$
\chi_{R_j}\left(q^{J_3}\right) = q^j + q^{j-1} + \cdots + q^{-j} = q^{-j}\frac{(1-q^{2j+1})}{(1-q)} \qquad (7.3.26)
$$

So the multiplicity of R_j inside a direct sum representation R is given by

$$
\mathrm{Mult}\left(R_j, R\right) = \mathrm{Coeff}\left[q^{-j}, (1-q)\chi_R\left(q^{J_3}\right)\right] \qquad (7.3.27)
$$

Taking a single factor of (7.3.24), the character of $\mathrm{Sym}^{p_i}\left(R_{\frac{i}{2}}\right)$ was calculated in [112] and is given by

$$
\begin{aligned}
\chi_{\mathrm{Sym}^{p_i}\left(R_{\frac{i}{2}}\right)}\left(q^{J_3}\right) &= q^{-\frac{ip_i}{2}}\frac{(1-q^{p_i+1})}{(1-q)}\frac{(1-q^{p_i+2})}{(1-q^2)}\cdots\frac{(1-q^{p_i+i})}{(1-q^i)} \\
&= q^{-\frac{ip_i}{2}}F_{i,p_i}(q) \qquad (7.3.28)
\end{aligned}
$$

where

$$
F_{i,p_i} = \frac{(1-q^{p_i+1})}{(1-q)}\frac{(1-q^{p_i+2})}{(1-q^2)}\cdots\frac{(1-q^{p_i+i})}{(1-q^i)} \qquad (7.3.29)
$$

So the multiplicity of R_j inside $\mathcal{R}_p^{SU(2)}$ is

$$
\begin{aligned}
\mathrm{Mult}\left(R_j, \mathcal{R}_p^{SU(2)}\right) &= \mathrm{Coeff}\left(q^{-j}, (1-q)\prod_i q^{-\frac{ip_i}{2}}F_{i,p_i}(q)\right) \\
&= \mathrm{Coeff}\left(q^{-j}, (1-q)q^{-\frac{n}{2}}\prod_i F_{i,p_i}(q)\right) \qquad (7.3.30)
\end{aligned}
$$

Since the $\Lambda = [n-m, m]$ representation of $U(2)$ corresponds to spin $j = \frac{n}{2} - m$, this means

$$\begin{aligned} \mathrm{Mult}\Big([n-m,m],\mathcal{R}_p^{U(2)}\Big) &= \mathrm{Coeff}\left(q^{m-\frac{n}{2}},(1-q)q^{-\frac{n}{2}}\prod_i F_{i,p_i}(q)\right) \\ &= \mathrm{Coeff}\left(q^{m},(1-q)\prod_i F_{i,p_i}(q)\right) \end{aligned} \tag{7.3.31}$$

Writing

$$F_p(q) = \prod_i F_{i,p_i}(q) \tag{7.3.32}$$

we can give a simple formula for the multiplicity in terms of the coefficients of F_p

$$\boxed{\mathcal{M}_{[n-m,m],p} = \mathrm{Coeff}\left(q^m,F_p\right) - \mathrm{Coeff}\left(q^{m-1},F_p\right)} \tag{7.3.33}$$

We now take two distinct approaches to studying F_p. Firstly we derive a generic formula for $\mathcal{M}_{\Lambda,p}$ that allows simple computational calculations of the multiplicity for any Λ, p of reasonable size. Secondly, we study sets of p which have identical multiplicities for all Λ and give explicit results of $\mathcal{M}_{\Lambda,p}$ for the simplest such sets.

7.3.2 Covariant trace bases

In the previous section we argued from the vector partition structure of the monomial multi-symmetric functions that the $M_{\mathbf{p}}$ fit in to the representation $\mathcal{R}_p^{U(2)}$ of $U(2)$, where p is the integer partition associated to $\mathbf{p}$. Performing a similar process on the multi-trace multi-symmetric functions $T_{\mathbf{p}}$ (or equivalently symmetrised trace operators), the $U(2)$ action not only preserves $p(\mathbf{p})$, it has exactly the same form as the action on monomials $M_{\mathbf{p}}$. That is, given $\mathcal{U} \in U(2)$ with action

$$\mathcal{U} M_{\mathbf{p}} = \sum_{\mathbf{q}} a_{\mathbf{p}}^{\mathbf{q}} M_{\mathbf{q}} \tag{7.3.34}$$

for some coefficients $a_{\mathbf{p}}^{\mathbf{q}}$, then the action of $\mathcal{U}$ on symmetrised traces is

$$\mathcal{U} T_{\mathbf{p}} = \sum_{\mathbf{q}} a_{\mathbf{p}}^{\mathbf{q}} T_{\mathbf{q}} \tag{7.3.35}$$

Therefore sorting $M_{\mathbf{p}}$ into a $U(2)$ covariant basis is mathematically identical to sorting $T_{\mathbf{p}}$ into a $U(2)$ covariant basis. It follows that the linear maps $C, \tilde{C}$ relating $M_{\mathbf{p}}$ and $T_{\mathbf{p}}$ are $U(2)$ equivariant, and we can define a $U(2)$ covariant symmetrised trace basis

$$T_{\Lambda,M_\Lambda,p,\nu} = \sum_{\mathbf{p}\,:\,p(\mathbf{p})=p} B^{\mathbf{p}}_{\Lambda,M_\Lambda,p,\nu} T_{\mathbf{p}} \tag{7.3.36}$$

In [51], the authors proved that the multiplicity of Λ, p in the symmetrised trace covariant basis is

$$\mathcal{M}_{\Lambda,p} = \chi_\Lambda(\mathbb{P}_p) \tag{7.3.37}$$

where χ_Λ is the S_n character of Λ and $\mathbb{P}_p$ is an element of $\mathbb{C}(S_n)$ that projects onto symmetrised traces with cycle type p. We discuss this projector in section 7.3.7.

The formulae (7.3.33) and (7.3.37) give $\mathcal{M}_{\Lambda,p}$ from $U(2)$ and S_n representation theory respectively. The former is more amenable to explicit calculations.

As discussed above (7.2.23), the symmetrised trace operators $T_{\mathbf{p}}$ with $l(\mathbf{p}) \leq N$ form a basis for symmetrised traces (but not pre-BPS operators) at finite N. Since the p label in (7.3.36) has the same length as $\mathbf{p}$, this property also holds for $T_{\Lambda,M_\Lambda,p,\nu}$.

In addition to the symmetrised trace covariant basis, there is a corresponding $U(2)$ covariant basis for commutator traces. It follows from the definitions (3.6.10) that the $U(2)$ generators act on a simple commutator as

$$R^i_j[X,Y] = \delta^i_j[X,Y] \tag{7.3.38}$$

Any commutator trace, generically containing a more complicated commutator than $[X,Y]$, can be written as a linear combination of traces containing $[X,Y]$. So (7.3.38) shows that the space of commutator traces forms a $U(2)$ representation. By similar considerations to $M_{\mathbf{p}}$ and $T_{\mathbf{p}}$, these can be further sorted by an integer partition $p \vdash n$ that describes the factorisation of a commutator multi-trace into single traces.

In [49], the authors used superspace techniques in the $SU(N)$ gauge theory to develop candidate quarter-BPS operators and SUSY descendent operators. These are exactly the covariant symmetrised trace and commutator trace bases respectively, though they did not include partitions with components of size 1, since in the $SU(N)$ theory, traces of individual matrices vanish.

The covariant bases for symmetrised and commutator traces are used in appendices G.1, G.2 and G.3 to describe the final BPS operators at $n = 5, 6$. In this section we will focus on the covariant monomials and not comment further on the covariant symmetrised or commutator traces.

7.3.3 General multiplicity formula

We now find an expression for

$$\operatorname{Coeff}(q^m, F_p(q)) \tag{7.3.39}$$

This is done explicitly for $m = 0, 1, 2, 3$, from which we extrapolate the general result.

The relevant parts of p to describe the coefficients in (7.3.32) are

$$c_{j,k} = |\{i : i > j, p_i \geq k\}| \qquad j \geq 0, k \geq 1 \tag{7.3.40}$$

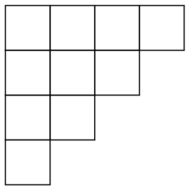
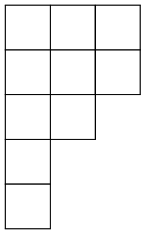
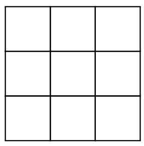
p	Young diagram	Table of $c_{j,k}$	Table of s_l
$[4,3,2,1]$		$\begin{array}{c\|c} c_{j,k} & k=1 \\ \hline j=0 & 4 \\ j=1 & 3 \\ j=2 & 2 \\ j=3 & 1 \end{array}$	$s_1=4$, $s_2=3$, $s_3=2$, $s_4=1$
$[3,3,2,1,1]$		$\begin{array}{c\|cc} c_{j,k} & k=1 & k=2 \\ \hline j=0 & 3 & 2 \\ j=1 & 2 & 1 \\ j=2 & 1 & 1 \end{array}$	$s_1=3$, $s_2=4$, $s_3=2$, $s_4=1$
$[3,3,3]$		$\begin{array}{c\|ccc} c_{j,k} & k=1 & k=2 & k=3 \\ \hline j=0 & 1 & 1 & 1 \\ j=1 & 1 & 1 & 1 \\ j=2 & 1 & 1 & 1 \end{array}$	$s_1=1$, $s_2=2$, $s_3=3$, $s_4=2$, $s_5=1$

Figure 7.2: Examples of the non-zero $c_{j,k}$ and s_l for various integer partitions p.

Let $Y_j(p)$, $j \geq 0$ be the Young diagram of p with the first j columns removed. Then intuitively, $c_{j,k}$ is the number of vertical edges of length k or greater in $Y_j(p)$. It follows that $c_{j,1}$ is the number of corners in $Y_j(p)$, and $c_{0,1}$ is the number of corners in the full Young diagram $Y(p)$. Figure 7.2 shows some examples to illustrate this. The full set of $c_{j,k}$ completely determines the partition p.

It will also be useful to define

$$s_l = \sum_{j+k=l} c_{j,k} \tag{7.3.41}$$

We have included examples of the s_l in figure 7.2. In contrast to the $c_{j,k}$, the s_l do not define the partition p. For example, $p=[2]$ and $p=[1,1]$ both have $s_1=1$, $s_2=1$ and all others zero. The sets of partitions which have identical s_l for all l are studied in section 7.3.4.

To find the coefficients of $q^{0,1,2,3}$ in $F_p(q)$, we look at the low order terms from the definitions (7.3.29) and (7.3.32). For all $i > 0$, F_{i,p_i} contains the factor

$$\frac{(1-q^{p_i+1})}{(1-q)} = 1 + q + q^2 + \cdots + q^{p_i} \tag{7.3.42}$$

For all $i > 1$, F_{i,p_i} contains, in addition to the above, the factor

$$\frac{(1-q^{p_i+2})}{(1-q^2)} = (1 + q^2 + q^4 + \cdots)(1-q^{p_i+2}) \tag{7.3.43}$$

$$= 1 + q^2 - q^{p_i+2} + O(q^4) \tag{7.3.44}$$

For all $i > 2$, the factor F_{i,p_i} contains, in addition to the above,

$$\frac{(1-q^{p_i+3})}{(1-q^3)} = (1 + q^3 + \cdots)(1-q^{p_i+3}) \tag{7.3.45}$$

$$= 1 + q^3 - q^{p_i+3} + O(q^4) \tag{7.3.46}$$

All other factors in the definition (7.3.29) of F_{i,p_i} are of the form $1 + O(q^4)$ so we can ignore them for our purposes, giving

$$F_p = f_1 f_2 f_3 + O(q^4) \tag{7.3.47}$$

where

$$f_1 = \prod_{i>0}(1 + q + q^2 + \cdots + q^{p_i}) \tag{7.3.48}$$

$$f_2 = \prod_{i>1}(1 + q^2 - q^{p_i+2} + \cdots) \tag{7.3.49}$$

$$f_3 = \prod_{i>2}(1 + q^3 - q^{p_i+3} + \cdots) \tag{7.3.50}$$

From this we can read off

$$\text{Coeff}\left(q^0, F_p\right) = 1 \tag{7.3.51}$$

All the qs in the expansion of F_p come from f_1, with the coefficient given by the number of $p_i \geq 1$. From the definitions (7.3.40) and (7.3.41), we can express this as

$$\text{Coeff}\left(q, F_p\right) = c_{0,1} = s_1 \tag{7.3.52}$$

There are three ways to arrive at a q^2 from the product (7.3.47).

1. We can take a q^2 from a factor of f_2 and 1 from every other factor. Within f_2, this happens whenever $p_i \geq 1$ for $i > 1$, so there are $c_{1,1}$ different ways of doing this. Therefore this route contributes $c_{1,1}$ to the coefficient of q^2.

2. We can take a q^2 from a factor of f_1. Each factor contains a q^2 term only if $p_i \geq 2$, so the number of different ways of doing this is $c_{0,2}$.

3. We can take a q from a pair of the f_1 factors. There are $\binom{c_{0,1}}{2}$ different ways of doing this.

So we arrive at the expression

$$\text{Coeff}\left(q^2, F_p\right) = c_{0,2} + c_{1,1} + \binom{c_{0,1}}{2} = s_2 + \frac{s_1(s_1-1)}{2} \tag{7.3.53}$$

Looking at q^3, there are six distinct ways to arrive at a q^3 from the product (7.3.47).

1. We can take a q^3 from a factor of f_3. There are $c_{2,1}$ different ways of doing this.

2. We can take a q^3 from a factor of f_2. This can only be done if $p_i = 1$, as it comes from the term q^{p_i+1}. The number of factors with $p_i = 1$ is given by $c_{1,1} - c_{1,2}$. Noting that any q^3 obtained in this manner comes with a minus sign, this contributes $c_{1,2} - c_{1,1}$ to the coefficient.

3. We can take a q^3 from a factor of f_1. There are $c_{0,3}$ ways of doing this.

4. We can take a q^2 from a factor of f_2 and a q from a factor of f_1. There are $c_{1,1}c_{0,1}$ ways of doing this.

5. We can take a q^2 from a factor of f_1 and a q from a different factor of f_1. There are $c_{0,2}(c_{0,1}-1)$ different ways of doing this.

6. We can take a q from three different factors of f_1. There are $\binom{c_{0,1}}{3}$ different ways of doing this.

Collecting everything, we have

$$\begin{aligned}\text{Coeff}\left(q^3, F_p\right) &= c_{2,1} + c_{1,2} - c_{1,1} + c_{0,3} + c_{1,1}c_{0,1} + c_{0,2}(c_{0,1}-1) + \binom{c_{0,1}}{3} \\ &= c_{2,1} + c_{1,2} + c_{0,3} + (c_{1,1}+c_{0,2})(c_{0,1}-1) + \binom{c_{0,1}}{3} \\ &= s_3 + s_2(s_1-1) + \frac{s_1(s_1-1)(s_1-2)}{6}\end{aligned} \tag{7.3.54}$$

A similar process for the coefficient of q^4 leads to

$$\begin{aligned}\text{Coeff}\left(q^4, F_p\right) &= c_{0,4} + c_{1,3} + c_{2,2} + c_{3,1} + (c_{0,3} + c_{1,2} + c_{2,1})(c_{0,1}-1) \\ &\quad + \binom{c_{0,2}+c_{1,1}}{2} + (c_{0,2}+c_{1,1})\binom{c_{0,1}-1}{2} + \binom{c_{0,1}}{4} \\ &= s_4 + s_3(s_1-1) + \frac{s_2(s_2-1)}{2} + \frac{s_2(s_1-1)(s_1-2)}{2}\end{aligned}$$

$$+\frac{s_1(s_1-1)(s_1-2)(s_1-3)}{24} \tag{7.3.55}$$

In (7.3.51), (7.3.52), (7.3.53), (7.3.54) and (7.3.55) we have expressed the first 5 coefficients in the expansion of F_p in terms of the s_l. The terms in these sums correspond to the partitions of the exponent of q. For example in (7.3.55), the terms correspond respectively to the partitions $[4],[3,1],[2,2],[2,1,1]$ and $[1,1,1,1]$. This leads us to suggest the general formula

$$\boxed{\mathrm{Coeff}\left(q^m,F_p\right)=\sum_{\lambda\vdash m}\left[\prod_k\left(s_{\lambda_k}-k+1\right)\right]\left[\prod_i\frac{1}{\mu_i!}\right]} \tag{7.3.56}$$

where we have used both the component notation $\lambda=[\lambda_1,\lambda_2,...]$ and the multiplicity notation $\lambda=<\mu_1,\mu_2,\cdots>$ for λ.

In work for this thesis, we have algebraically proved this formula for $m\leq 6$, and have numerically checked it up to $m=20$. A proof for general m and p is a problem for future work.

It is interesting to note that since F_p is a palindromic polynomial (arising from the $q\to q^{-1}$ invariance of $SU(2)$ characters), these coefficients form a palindromic sequence. Explicitly,

$$\mathrm{Coeff}\left(q^m,F_p\right)=\mathrm{Coeff}\left(q^{n-m},F_p\right) \tag{7.3.57}$$

As the sums over λ in (7.3.56) get extremely complicated for large m, this is quite surprising, and leads us to suspect there is more hidden structure in the sum (7.3.56).

Combining (7.3.33) with (7.3.56) gives us an explicit formula for $\mathcal{M}_{\Lambda,p}$

$$\boxed{\begin{aligned}\mathcal{M}_{[n-m,m],p}=&\sum_{\lambda\vdash m}\left[\prod_k\left(s_{\lambda_k}-k+1\right)\right]\left[\prod_i\frac{1}{\mu_i(\lambda)!}\right]\\&-\sum_{\lambda\vdash m-1}\left[\prod_k\left(s_{\lambda_k}-k+1\right)\right]\left[\prod_i\frac{1}{\mu_i(\lambda)!}\right]\end{aligned}} \tag{7.3.58}$$

Applying this to $m=0$ to 4, the formulae are

$$\mathcal{M}_{[n],p}=1 \tag{7.3.59}$$

$$\mathcal{M}_{[n-1,1],p}=s_1-1 \tag{7.3.60}$$

$$\mathcal{M}_{[n-2,2],p}=s_2+\frac{s_1(s_1-3)}{2} \tag{7.3.61}$$

$$\mathcal{M}_{[n-3,3],p}=s_3+s_2(s_1-2)+\frac{s_1(s_1-1)(s_1-5)}{6} \tag{7.3.62}$$

$$\mathcal{M}_{[n-4,4],p} = s_4 + s_3(s_1-2) + s_2(s_2-1) + \frac{s_2(s_2-1)}{2} + \frac{s_2(s_1-1)(s_1-4)}{2} + \frac{s_1(s_1-1)(s_1-2)(s_1-7)}{24} \tag{7.3.63}$$

These formulae are independent of N, so to get finite N multiplicities we impose the finite N cut-off on p. Including this, the general multiplicity formula is

$$\mathcal{M}_{[n-m,m],p} = \begin{cases} (7.3.58) & l(p) \leq N \\ 0 & l(p) > N \end{cases} \tag{7.3.64}$$

We can also look at the total Λ multiplicity $\mathcal{M}_\Lambda$ by summing over all $p \vdash n$

$$\begin{aligned} \mathcal{M}_{[n-m,m]} &= \sum_{p\vdash n, l(p)\leq N} \mathcal{M}_{[n-m,m],p} \\ &= \sum_{p\vdash n, l(p)\leq N} \left(\sum_{\lambda\vdash m} \left[\prod_k (s_{\lambda_k} - k + 1)\right] \left[\prod_i \frac{1}{\mu_i(\lambda)!}\right] \right. \\ &\qquad \left. - \sum_{\lambda\vdash m-1} \left[\prod_k (s_{\lambda_k} - k + 1)\right] \left[\prod_i \frac{1}{\mu_i(\lambda)!}\right] \right) \end{aligned} \tag{7.3.65}$$

From representation theory considerations [10], the sectors with $\Lambda = [n]$ and $[n-1,1]$ do not undergo a step-change as we turn on the coupling constant. Therefore the weak coupling combinatorics of these sectors should match the free field combinatorics of section 3.6.2. A priori, the combinatorics should agree when considering the entire $\Lambda = [n]$ or $[n-1,1]$ sector. We find a stronger result: the combinatorics of the Young diagram label R in (3.6.20) matches the partition p of this section.

From (7.3.63), for $\Lambda = [n]$ the multiplicity of any given p is 1, while for $\Lambda = [n-1,1]$ recall that $s_1 = c_{0,1}$ is the number of corners of p, so the multiplicity of p is simply the number of corners subtract 1. As expected, these match (3.6.22) and (3.6.23) respectively and therefore

$$C(R,R,\Lambda) = \mathcal{M}_{\Lambda,R} \tag{7.3.66}$$

for $\Lambda = [n]$ and $[n-1,1]$.

7.3.4 Hermite reciprocity and p-orbits of fixed $\mathcal{M}_{\Lambda,p}$

There are collections of p which lead to the same multiplicities for all Λ. To understand these, we look at the definition (7.3.29) of F_{i,p_i}. If $i > p_i$, the numerator and denominator start cancelling, and we end up with

$$F_{i,p_i} = \frac{\left(1-q^{i+1}\right)}{(1-q)}\frac{\left(1-q^{i+2}\right)}{(1-q^2)}\cdots\frac{\left(1-q^{i+p_i}\right)}{(1-q^{p_i})} = F_{p_i,i} \tag{7.3.67}$$

We can rewrite this to be explicitly symmetric in $i \leftrightarrow p_i$

$$F_{i,p_i} = F_{p_i,i} = \frac{\left(1-q^{\max(i,p_i)+1}\right)}{(1-q)}\frac{\left(1-q^{\max(i,p_i)+2}\right)}{(1-q^2)}\cdots\frac{\left(1-q^{i+p_i}\right)}{\left(1-q^{\min(i,p_i)}\right)} \tag{7.3.68}$$

This symmetry is known as Hermite reciprocity [113] and can be viewed as a property of $SU(2)$ characters.

We can use this $i \leftrightarrow p_i$ symmetry to do transformations on partitions that keep the product F_p the same, and by extension all the associated Λ multiplicities.

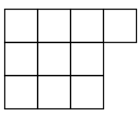
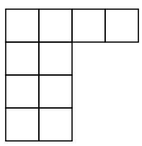
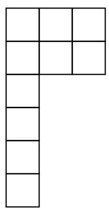
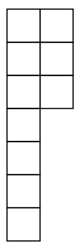
As our first example, take p to be rectangular, so $p = [i^{p_i}]$ for some particular choice of i. Then the conjugate partition $p^c = [(p_i)^i]$ has the same F, leading to the same multiplicities for all Λ. Note that $p^c = p$ if $i = p_i$.

Now suppose $p = [i^{p_i}, j^{p_j}]$ for $i < j$. Then there are three candidates for partitions with the same F, namely

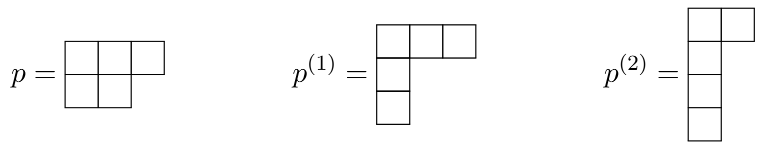
$$p^{(1)} = \left[(p_i)^i, j^{p_j}\right] \tag{7.3.69}$$
$$p^{(2)} = \left[i^{p_i}, (p_j)^j\right] \tag{7.3.70}$$
$$p^{(3)} = \left[(p_i)^i, (p_j)^j\right] \tag{7.3.71}$$

The partition given by $p^{(1)}$ will only produce the same F if $j \neq p_i$. If $j = p_i$, then $p^{(1)}$ should be written as $[j^{i+p_j}]$ and the Fs no longer match. Similarly $p^{(2)}$ will only match if $i \neq p_j$ and $p^{(3)}$ if $p_i \neq p_j$.

To visualise the transformations taking p to $p^{(1,2,3)}$, split p into two rectangles stacked on top of each other. Then $p^{(1)}$ is obtained by rotating the i rectangle through 90 degrees, reordering the two rectangles if appropriate, and re-stacking them. In the same manner, $p^{(2)}$ is obtained by rotating the j rectangle, and $p^{(3)}$ by rotating both. We take $p = [4, 3, 3]$ as an example

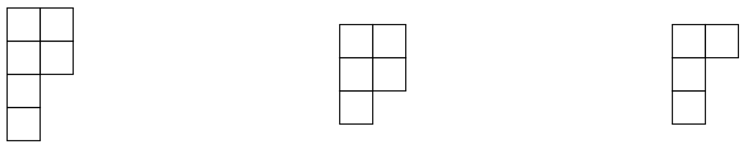
$p =$  $p^{(1)} =$  $p^{(2)} =$  $p^{(3)} =$  (7.3.72)

When one of the dimensions of the first rectangle coincides with one of the dimensions of the second rectangle, one or more of these four options will reduce from two distinct rectangles into one larger rectangle, and hence to a different F. If there is one coincidence, for example $p = [3, 2]$ where we have $p_2 = p_3 = 1$, we only have three partitions with the same F



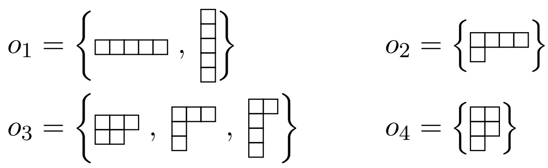
(7.3.73)

If there are two coincidences, then the partition is not related to any other via these transformations. There are three distinct ways for these two coincidences to occur. Firstly, three of the dimensions could be the same, while the fourth is different, for example $p = [2, 2, 1, 1]$. Secondly, both rectangles are squares, with distinct sizes, for example $p = [2, 2, 1]$. Finally, the two rectangles are identical, but are non-square, for example $p = [2, 1, 1]$. These three partitions are shown below



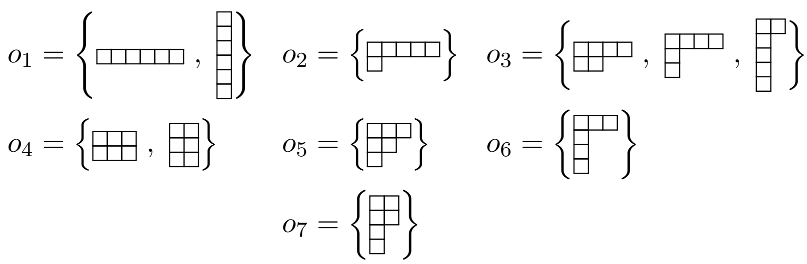
(7.3.74)

The generalisation to more rectangles is straightforward. A partition made from k rectangles can be related to as many as 2^k others by rotating a subset of the k rectangles. These rotations are only valid if the widths of all the rotated rectangles are distinct. As an example, consider all partitions of 5. These fall into 4 orbits under these rotations



(7.3.75)

The equivalent classification for partitions of 6 is

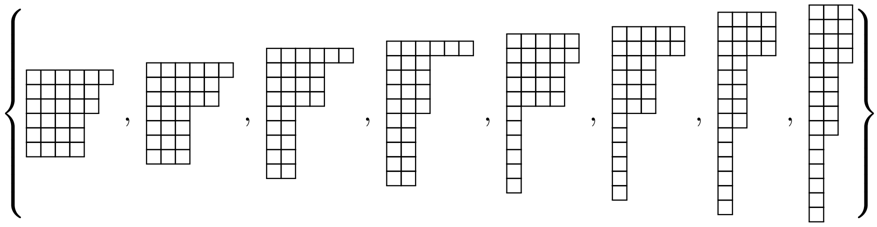


(7.3.76)

In appendices G.1, G.2 and G.3, we give explicit formulae for the $n = 5, 6$ basis elements. As expected, the families of partitions above have the same multiplicities for all Λ.

The first orbit constructed from 2 rectangles to have all 4 dimensions distinct, and therefore achieve the maximum size of $4 = 2^2$ is found at $n = 10$ and is shown in

(7.3.72). The first orbit constructed from 3 rectangles to have all 6 dimensions distinct and have maximum size $8 = 2^3$ is found at $n = 28$, and consists of



(7.3.77)

In the half-BPS sector, the Young diagram label of the Schur basis 2.3.14 gives us the properties of the corresponding giant graviton in the AdS/CFT correspondence. A partition with $k \sim 1$ rows of length $L \sim N$ describes k giant gravitons (or a single giant graviton wrapped k times) extended in AdS_5 of angular momentum L. Similarly, a partition with k columns of length of length L describes k giant gravitons extended in S^5 of angular momentum L.

These orbits of partitions with identical multiplicities at all Λ allow us to identify families of different giant graviton states that behave the same under quarter-BPS deformations.

Taking the simplest example of a single rectangle with $p = [k^L]$, this rectangle rotation symmetry means k coincident giant gravitons rotating with angular momentum L in AdS_5 behave the same way as k coincident giant gravitons rotating with angular momentum L in S^5.

A system of two AdS giant gravitons with different angular momenta has $p = [k_1, k_2]$ with $k_1, k_2 \sim N$ and $k_1 \neq k_2$. Then we can rotate each of the rows individually to get $p = [k_1, 1^{k_2}]$ or $p = [k_2, 1^{k_1}]$, however we cannot rotate both at the same time. Therefore the behaviour of two non-coincident sphere giants under quarter-BPS deformations is different to that of two non-coincident AdS giants. This is studied further in the next section.

7.3.5 Calculation of multiplicities for simplest orbits

For some of the simplest orbits of partitions under rectangle rotation, we can describe the Λ multiplicities explicitly. These are dual to one or two giant gravitons wrapped around the AdS_5 or S^5 factors of $AdS_5 \times S^5$.

Recall from (7.3.33) that $(1-q)F_p$ is the generating function for the Λ multiplicities. More specifically, the coefficient of q^m is the multiplicity of $\Lambda = [n-m, m]$ for $m \leq \left\lfloor \frac{n}{2} \right\rfloor$.

We start with $p = [1^n]$ or equivalently $p = [n]$.

$$(1-q)F_{[n]} = (1-q)F_{1,n} = 1 - q^{n+1} \tag{7.3.78}$$

So $\Lambda = [n]$ appears with multiplicity 1, and all other Λ have multiplicity 0. In the dual string theory, this means a single half-BPS giant graviton cannot deform into the quarter-BPS sector.

We next consider rectangles with side lengths 2 and $k \geq 2$, so $p = [2^k]$ or $[k, k]$.

$$\begin{aligned}(1-q)F_{[k,k]} = (1-q)F_{2,k} &= \frac{(1-q^{k+1})(1-q^{k+2})}{1-q^2} \\ &= (1+q^2+q^4+\dots)(1-q^{k+1}-q^{k+2}+q^{2k+3}) \\ &= 1+q^2+q^4+\dots+q^{\lfloor k \rfloor_2}+O(q^{k+1})\end{aligned} \tag{7.3.79}$$

where $\lfloor k \rfloor_2 = 2\left\lfloor \frac{k}{2} \right\rfloor$ is k rounded down to the nearest multiple of 2. Since we are only interested in the terms with exponent $\leq k = \frac{n}{2}$, we can ignore the $O(q^{k+1})$ parts of the expression. Then $\Lambda = [n-m, m]$ appears with multiplicity 1 if m is even, and 0 otherwise.

The dual interpretation of $p = [k, k]$ is two coincident AdS giants, while $p = [2^k]$ is two coincident sphere giants. Then (7.3.79) states that these states can be deformed deep into the quarter-BPS sector. In some sense, the quarter-BPS state 'furthest' from half-BPS is $\Lambda = [\frac{n}{2}, \frac{n}{2}]$, and this arrangement of giants can be deformed right up to that limit if $\frac{n}{2}$ is even (and only one away if $\frac{n}{2}$ odd). However, not all quarter-BPS deformations are available. In particular the 'smallest' deformation $\Lambda = [n-1, 1]$ does not exist, and we must deform by 'twice' as much for each step into the quarter-BPS.

Now look at a combination of two rectangles, both with one dimension of length 1. Let the other dimensions be $k \geq l$. If $k = l$, then the orbit has size 1, namely $p = [k, 1^k]$. Otherwise, the orbit consists of three partitions, $p = [k, l]$, $p = [k, 1^l]$, $p = [l, 1^k]$. The considerations of the orbit size do not affect the calculation of multiplicities. This calculation is

$$\begin{aligned}(1-q)F_{[k,l]} = (1-q)F_{1,k}F_{1,l} &= \frac{(1-q^{k+1})(1-q^{l+1})}{1-q} \\ &= (1+q+q^2+\dots)(1-q^{l+1}-q^{k+1}+q^{k+l+2}) \\ &= 1+q+q^2+\dots+q^l+O(q^{k+1})\end{aligned} \tag{7.3.80}$$

So $\Lambda = [n-m, m]$ appears with multiplicity 1 if $m \leq l$ and 0 otherwise.

For $k > l$, based on the argument that keeping $Y(p)$ fixed and deforming $\Lambda = [n]$ to $\Lambda = [n-n_2, n_2]$ preserves the qualitative physics of the giant states, we expect the partition $p = [k, l]$ corresponds to two non-coincident AdS giants when k, l are of order N. The multiplicity of Λ we are getting above is precisely the multiplicity of $U(2)$ reps in $(\Lambda = [k]) \otimes (\Lambda = [l])$. Indeed the $U(2)$ representation for the quantum states constructed from multi-symmetric functions $M_{\mathbf{p}}$ with associated partition $p = [k, l]$ is $Sym^k(V_2) \otimes Sym^l(V_2) = \mathcal{R}^{U(2)}_{p=[k]} \otimes \mathcal{R}^{U(2)}_{p=[l]}$. The case $p = [k, k]$ corresponds, by the same argument, to bound state of two AdS giants of angular momentum k. In this case the multi-symmetric construction gives the $U(2)$ representation $Sym^2(Sym^k(V_2)) = Sym^2(\Lambda = [k])$, which is the symmetric subspace of $\mathcal{R}^{U(2)}_{p=[k]} \otimes \mathcal{R}^{U(2)}_{p=[k]}$. The projection to the symmetric part accounts for the missing powers of q in (7.3.79) compared to the $p = [k, l]$ case (7.3.80).

To look at two non-coincident sphere giants, we consider $p = [l, l, k-l]$ for $k > l$. After rotating both rectangles, this is equivalent to $p = [2^l, 1^{k-l}]$, corresponding to two sphere giants of momenta k and l respectively. This has

$$\begin{aligned}(1-q)F_{[l,l,k-l]} = (1-q)F_{2,l}F_{1,k-l} &= \frac{(1-q^{l+1})(1-q^{l+2})(1-q^{k-l+1})}{(1-q)(1-q^2)} \\ &= \left(1 - q^{k-l+1} - q^{l+1} - q^{l+2} + q^{2l+3} + O(q^k)\right) \\ &\qquad (1 + q + 2q^2 + 2q^3 + 3q^4 + 3q^5 + \dots)\end{aligned} \tag{7.3.81}$$

Where we can ignore terms of order k and higher as these exponents are greater than $\frac{n}{2} = \frac{k+l}{2}$.

Let $a_m = 0$ for $m < 0$ and $a_m = \lfloor \frac{m}{2} \rfloor + 1$ for $m \geq 0$, the coefficient of q^m in the second factor of (7.3.81). Then the coefficient of q^m in (7.3.81) is

$$\mathcal{M}_{[n-m,m],[l,l,k-l]} = a_m - a_{m-k+l-1} - a_{m-l-1} - a_{m-l-2} + a_{m-2l-3} \tag{7.3.82}$$

The exact formulae for the multiplicities depend on the relative sizes of k and l. If $k \leq 2l$, then

$$\mathcal{M}_{[n-m,m],[l,l,k-l]} = \begin{cases} \lfloor \frac{m}{2} \rfloor + 1 & 0 \leq m \leq k-l \\ \lfloor \frac{m}{2} \rfloor - \lfloor \frac{m-k+l-1}{2} \rfloor & k-l+1 \leq m \leq l \\ \lfloor \frac{m}{2} \rfloor - \lfloor \frac{m-k+l-1}{2} \rfloor - m + l & l+1 \leq m \leq \frac{k+l}{2} \end{cases} \tag{7.3.83}$$

where we have used

$$\left\lfloor \frac{c-1}{2} \right\rfloor + \left\lfloor \frac{c-2}{2} \right\rfloor = \frac{c-1}{2} + \frac{c-2}{2} - \frac{1}{2} = c-2 \tag{7.3.84}$$

for $c = m - l$.

If $2l \leq k \leq 3l$, then

$$\mathcal{M}_{[n-m,m],[l,l,k-l]} = \begin{cases} \lfloor \frac{m}{2} \rfloor + 1 & 0 \leq m \leq l \\ \lfloor \frac{m}{2} \rfloor - m + l + 1 & l+1 \leq m \leq k-l \\ \lfloor \frac{m}{2} \rfloor - \lfloor \frac{m-k+l-1}{2} \rfloor - m + l & k-l+1 \leq m \leq \frac{k+l}{2} \end{cases} \tag{7.3.85}$$

Finally, if $k \geq 3l$

$$\mathcal{M}_{[n-m,m],[l,l,k-l]} = \begin{cases} \left\lfloor \frac{m}{2} \right\rfloor + 1 & 0 \leq m \leq l \\ \left\lfloor \frac{m}{2} \right\rfloor - m + l + 1 & l+1 \leq m \leq 2l \\ 0 & 2l+1 \leq m \leq \frac{k+l}{2} \end{cases} \tag{7.3.86}$$

For two sphere giants of momenta $k, l \lessapprox N$, the multiplicities fall into category (7.3.83). Roughly speaking, the multiplicity of $\Lambda = [n-m, m]$ increases as $\frac{m}{2}$ until reaching $\frac{k-l}{2}$. It then stays constant until m reaches l before turning around and decreasing for $m \geq l$, reaching 0 at $m = \frac{n}{2}$.

From the construction based on multi-symmetric functions, the states for $p = [2^l, 1^{k-l}]$ form, in all the cases, the $U(2)$ representation

$$\begin{aligned} \mathrm{Sym}^l(\mathrm{Sym}^2(V_2)) \otimes \mathrm{Sym}^{k-l}(V_2) &= \mathrm{Sym}^2(\mathrm{Sym}^l(V_2)) \otimes \mathrm{Sym}^{k-l}(V_2) \\ &= \mathcal{R}^{U(2)}_{p=[2^l]} \otimes \mathcal{R}^{U(2)}_{p=[1^{k-l}]} \\ &= \mathcal{R}^{U(2)}_{p=[l,l]} \otimes \mathcal{R}^{U(2)}_{p=[k-l]} \end{aligned} \tag{7.3.87}$$

So the construction implies that the 2-sphere-giant system for $p = [2^k, 1^{k-l}]$ have the same multiplicities as a composite consisting of the 2-sphere-giant bound state $p = [2^l]$ along with a 1-sphere giant system $[1^{(k-l)}]$, while Hermite reciprocity further implies that these multiplicities are also the same as those of an AdS 2-giant bound state of angular momentum l composed with a single AdS giant of angular momentum $k - l$.

We can see a marked difference between the behaviour of two non-coincident sphere giants compared to two non-coincident AdS giants. In (7.3.80) the multiplicity of each Λ was at most 1, so there was a unique way of deforming the arrangement of AdS giants at each stage on their way into the quarter-BPS sector. Furthermore, the furthest possible deformation was $m = l$, the lesser of the two momenta of the gravitons. With (7.3.83) the multiplicities can be larger than 1, and are non-zero right up to $m = \frac{n}{2}$. So there are a multitude of ways of deforming sphere giants, and they can be deformed all the way into the quarter-BPS. Interestingly, when the two momenta are more uneven, and m can get as high as $m = 2l$, there is a cut-off on the possible deformations. This is twice the equivalent cut-off for non-coincident AdS giants.

We have interpreted $p = [2^l, 1^{k-l}]$ as corresponding to two non-coincident sphere giants in order to compare with the equivalent system of AdS giants. However, when $l, k-l \sim N$, the rotation $p = [l, l, k-l]$ is exactly the system of two coincident AdS giants of momenta l and a third giant of momenta $k-l$. So two separated sphere giants have the same behaviour as a system of three AdS giants.

It is worth remarking that there are important differences in how the same Hilbert spaces of N free bosons in a harmonic oscillator are arrived at in the two problems of quantizing moduli spaces of sphere giants [27] and the moduli space of AdS giants [40]. In [27] quarter-BPS multi-giant systems are described by Fock space oscillators associated with higher order polynomials in x, y. In [40] there is a relatively simpler phase space of classical AdS solutions which is $\mathbb{C}^2$ (and $\mathbb{C}^3$ in the more general eighth-BPS case) and the full Hilbert space is obtained by considering an N particle boson system based on this 1-particle system. This serves to explain why the gauge theory construction of BPS operators we are giving here, which is intimately tied to a weak-coupling gauge theory realization of the multi-free boson Hilbert space, leads to simpler compositeness structures for the AdS giants as discussed above.

7.3.6 Partitions with one dominant row or column

There is another family of partitions that have nice properties. Consider $p \vdash n$ in which the first row dominates the partition, i.e. $p = [\lambda_1, \hat{p}]$, where $\lambda_1 \geq \frac{n}{2}$ and $\hat{p} \vdash \hat{n} = n - \lambda_1$.

With one exception, when $\lambda_1 = \frac{n}{2}$ and $\hat{p} = [\lambda_1]$ (this case has already been considered in (7.3.79)), this leads to

$$F_p = F_{\lambda_1, 1} F_{\hat{p}} = \frac{1 - q^{\lambda_1 + 1}}{1 - q} F_{\hat{p}} \tag{7.3.88}$$

and therefore

$$(1 - q) F_p = (1 - q^{\lambda_1 + 1}) F_{\hat{p}} \tag{7.3.89}$$

Using the second equation in (7.3.33), we have

$$\begin{aligned} \mathcal{M}_{[n-m,m],[\lambda_1,\hat{p}]} &= \text{Coeff}\left[q^m, (1-q) \prod_i F_{i,p_i}(q)\right] \\ &= \text{Coeff}\left[q^m, (1 - q^{\lambda_1+1}) F_{\hat{p}}\right] \end{aligned} \tag{7.3.90}$$

Since $m \leq \frac{n}{2}$ for Λ to be valid Young diagram and $\lambda_1 + 1 > \frac{n}{2}$ by the dominant first row property, it follows that

$$\text{Coeff}\left(q^m, q^{\lambda_1+1} F_{\hat{p}}\right) = 0 \tag{7.3.91}$$

and we can simplify (7.3.90) to

$$\mathcal{M}_{[n-m,m],[\lambda_1,\hat{p}]} = \text{Coeff}\left(q^m, F_{\hat{p}}\right) \tag{7.3.92}$$

Thus, the generating function for the Λ multiplicities is just $F_{\hat{p}}$ and does not depend on λ_1. We can now use our study of the coefficients of F from section 7.3.3 to give the Λ multiplicities. Note that the dominant first row condition has allowed us to obtain a simple formula for $(1-q)F_p$ in terms of $F_{\hat{p}}$. As a result the multiplicities are being obtained simply from the coefficients of a known generating function ($F_{\hat{p}}$). This is simpler than the procedure of section 7.3.3 where the Λ multiplicities were obtained from the difference of two consecutive coefficients in F_p.

Using the formulae (7.3.59)-(7.3.63) we can write

$$F_{\hat{p}}(q) = \sum_m q^m \mathcal{M}_{[n-m,m],[\lambda_1,\hat{p}]} = 1 + s_1 q + \left(s_2 + \frac{s_1(s_1-1)}{2}\right) q^2 + \left(s_3 + s_2(s_1-1) + \frac{s_1(s_1-1)(s_1-2)}{6}\right) q^3 + \dots \tag{7.3.93}$$

Where the s_i refer to $s_i(\hat{p})$, not $s_i(p)$.

As previously observed in (7.3.57), the coefficients of $F_{\hat{p}}$ form a palindromic sequence, starting and ending with 1 at q^0 and $q^{\hat{n}}$. Adding this to (7.3.93), we have

$$\begin{aligned}\sum_m q^m \mathcal{M}_{[n-m,m],[\lambda_1,\hat{p}]} = 1 + s_1 q &+ \left(s_2 + \frac{s_1(s_1-1)}{2}\right) q^2 \\ &+ \left(s_3 + s_2(s_1-1) + \frac{s_1(s_1-1)(s_1-2)}{6}\right) q^3 \\ &+ \dots \\ &+ \left(s_3 + s_2(s_1-1) + \frac{s_1(s_1-1)(s_1-2)}{6}\right) q^{\hat{n}-3} \\ &+ \left(s_2 + \frac{s_1(s_1-1)}{2}\right) q^{\hat{n}-2} + s_1 q^{\hat{n}-1} + q^{\hat{n}}\end{aligned} \tag{7.3.94}$$

In summary, for p of the form $p = [\lambda_1, \hat{p}]$, where $\hat{p} \vdash \hat{n}$ and $\lambda_1 \geq \frac{n}{2}$, the multiplicities of $\Lambda = [n]$ and $\Lambda = [n-\hat{n}, \hat{n}]$ are exactly 1, the multiplicities of $\Lambda = [n-1,1]$ and $\Lambda = [n-\hat{n}+1, \hat{n}-1]$ are the number of corners in $\hat{p}$, and the general multiplicity of $\Lambda = [n-m,m]$ can be read from (7.3.56) with $p = \hat{p}$ if $m \leq \hat{n}$ or is 0 if $m > \hat{n}$. Furthermore, when $m \leq \hat{n}$, sending $m \to \hat{n} - m$ does not affect the multiplicity :

$$\mathcal{M}_{[n-m,m],[\lambda_1,\hat{p}]} = \mathcal{M}_{[n-\hat{n}+m,\hat{n}-m],[\lambda_1,\hat{p}]} \tag{7.3.95}$$

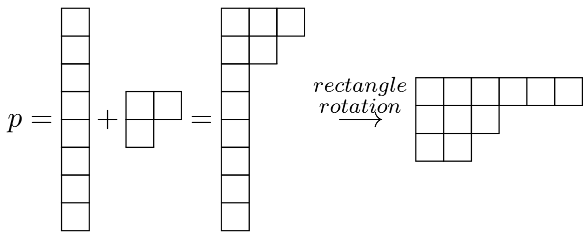
These properties have interesting implications. For a given n, $\hat{n} \leq \frac{n}{2}$, there are a large class of partitions with a dominant single row of length $n-\hat{n}$ for which the combinatorics of the deep quarter-BPS sector are determined by the combinatorics of the near half-BPS sector. For $\Lambda = [n-\hat{n}, \hat{n}]$, there is a multiplicity of exactly 1 for any of the partitions in this class, which is the same combinatorics as the half-BPS $\Lambda = [n]$. For $\Lambda = [n-\hat{n}+1, \hat{n}-1]$, the multiplicity is the same as the next to half-BPS $\Lambda = [n-1,1]$. For $\Lambda = [n]$ and $[n-1,1]$, there is no change in spectrum as we turn on the coupling constant. Therefore the combinatorics of the $\Lambda = [n-\hat{n}, \hat{n}], [n-\hat{n}+1, \hat{n}-1]$ sectors (for this class of partitions) at weak coupling are determined by free field considerations.

It would be interesting to find out whether this unreasonable effectiveness of the free theory has any connections to arguments in [47, 114] that important features of black hole physics in $AdS_5 \times S^5$ are captured by the free theory.

More generally, if $m \ll \hat{n} \lesssim \frac{n}{2}$, then $\Lambda = [n-m, m]$ is a small deviation from half-BPS while $\Lambda = [n-\hat{n}+m, \hat{n}-m]$ is a long way into the quarter-BPS sector, and yet their combinatorics are identical for this class of partitions. An interesting question is whether for these states with dominant first row in p (or dominant first column) have a well-defined semi-classical brane or space-time interpretation which can explain the coincidence of multiplicities between near-half-BPS and far-into-quarter-BPS regimes. Near half-BPS states have been studied in the context of the BMN limit of AdS/CFT [16]. In the context of giant gravitons, the physics of perturbations, in some sense small, of well-separated multi-giants has been understood [31, 35, 52].

Using the rectangle rotation described in section 7.3.4, similar properties hold for a single large column. Consider p with a first column of length μ_1 and a partition $\bar{p}$ attached to the right. This is denoted by $p = [1^{\mu_1}] + \bar{p}$. In terms of rectangles we can use for the rotation symmetry, this is a partition $[1^{l(\bar{p})}] + \bar{p}$ with a single column below it of length $\mu_1 - l(\bar{p})$. So setting $\lambda_1 = \mu_1 - l(\bar{p})$ and $\hat{p} = [1^{l(\bar{p})}] + \bar{p}$, p is in the same rotation orbit as $[\lambda_1, \hat{p}]$, and we can apply the logic of this section directly to p.

As an example, consider $\mu_1 = 8$ and $\bar{p} = [2, 1]$, with corresponding $\lambda_1 = 6$, $\hat{p} = [3, 2]$. This is easiest to see visually



(7.3.96)

It is clear that the conditions on a single dominant column are more difficult to work with than those for a single dominant row. Let μ_1 and μ_2 be the length of the first and second columns respectively, then to use the analysis of this section, we require $\mu_1 - \mu_2 \geq \frac{n}{2}$. This is a far smaller class of diagrams than given by the analogous condition $\lambda_1 \geq \frac{n}{2}$ for a diagram with a single dominant row.

7.3.7 Identifying a multiplicity space basis

In discussing the decomposition (7.3.9) we have not specified a choice of basis for $V^{mult}_{\Lambda,p}$, instead introducing a multiplicity index ν in the state (7.3.11). In this section we outline an algebraic approach to choosing a basis and characterising ν.

As seen in (7.3.4), $R_p^{U(2)}$ is a subspace of $(V_2)^{\otimes n}$. There is an S_n action on $(V_2)^{\otimes n}$ by permutation of the tensor factors, and $R_p^{U(2)}$ is the subspace invariant under the subgroup G_p of S_n. For $p = [1^{p_1}, 2^{p_2}, \dots] \vdash n$, this subgroup, which is discussed in [51], is

$$G_p = \bigtimes_i S_{p_i} [S_i] \tag{7.3.97}$$

where $S_{p_i} [S_i]$ is the wreath product of S_{p_i} with S_i. This is defined as the semi-direct product of S_{p_i} with $(S_i)^{p_i}$, where S_{p_i} acts on $(S_i)^{p_i}$ by permutation of factors.

G_p contains as a subgroup the group $G(\pi)$ given in (7.2.55), where $\pi \in \Pi(n)$ is a set partition with block size sizes given by p. G_p consists of $G(\pi)$ with the addition of the S_{p_i} factors.

The projector onto the G_p-invariant space is

$$\mathbb{P}_p = \frac{1}{|G_p|} \sum_{\sigma \in G_p} \sigma \tag{7.3.98}$$

which acts on a permutation $\tau \in S_n$ via the adjoint action

$$\mathbb{P}_p(\tau) = \frac{1}{|G_p|} \sum_{\sigma \in G_p} \sigma \tau \sigma^{-1} \tag{7.3.99}$$

This projector was used in [51] to derive the formula (7.3.37) for $\mathcal{M}_{\Lambda,p}$.

On the full space $(V_2)^{\otimes n}$, the $U(2)$ and S_n actions commute, and therefore they still commute on the $R_p^{U(2)}$ subspace. Since $R_p^{U(2)}$ is the G_p-invariant subspace, we should consider the action of the permutation subalgebra invariant under G_p-conjugation, rather than the full group algebra $\mathbb{C}(S_n)$. This algebra is

$$\mathcal{A}_p = \mathbb{P}_p [\mathbb{C}(S_n)] = \left\{ \alpha \in \mathbb{C}(S_n) \,|\, \sigma \alpha \sigma^{-1} = \alpha \,,\, \forall \sigma \in G_p \right\} \tag{7.3.100}$$

Now $\mathcal{A}_p$ acts on $R_p^{U(2)}$, but commutes with $U(2)$, which means in the decomposition (7.3.9) it acts only on the multiplicity space components. So to choose a basis for $V_{\Lambda,p}^{mult}$, we can choose a maximally commuting set of operators in $\mathcal{A}_p$ and label the multiplicity space basis by the eigenvalues of these operators.

The algebras $\mathcal{A}_p$ are in general quite complicated, and finding a maximally commuting set of operators within them is an involved computational problem that we do not attempt to find a general solution for. They are a generalisation of the algebras studied in [63, 115].

7.4 Construction of orthogonal $U(2)\times U(N)$ Young-diagram-labelled basis

In Section 7.5 we show that for any n, N, we can construct BPS operators by applying $\mathcal{G}_N$ to the subspace $\bar{\mathcal{M}}_N^{\leq} \subset \mathbb{C}(S_n)$, and using the map (3.6.19) from permutations to gauge invariant operators built from $N \times N$ matrices. The physical inner product on such operators obtained from 2-point functions uses the element $\mathcal{F}_N$: $\delta((\mathcal{G}_N\sigma_1)\mathcal{F}_N(\mathcal{G}_N\sigma_2)) = \delta(\sigma_1\mathcal{G}_N\sigma_2)$. An orthogonal basis is obtained by choosing an ordering (section 7.4.1) on the labels of the basis elements of $\bar{\mathcal{M}}_N^{\leq}$ and Gram-Schmidt orthogonalising. The orthogonal basis elements $S^{BPS}_{\Lambda,M_\Lambda,p,\nu}$ are normalised in the S_n inner product given in (3.6.26) (in section 7.5 this is the $g_{n,N}$ inner product). This construction algorithm gives a basis of BPS operators which is not only orthogonal but also SEP-compatible.

In this section we explain the construction of this orthogonal SEP-compatible basis of BPS operators from the covariant monomials $M_{\Lambda,M_\Lambda,p,\nu}$. We work with $N \times N$ matrices X and Y, of which there are n in total, where we can consider $N \geq n$ or $N < n$.

The final output will be a basis of BPS operators of the form

$$S^{BPS}_{\Lambda,M_\Lambda,p,\nu} = \sum_{\substack{R,\tau \\ l(R)\leq N}} s^{R,\tau}_{p,\nu}(\Lambda;N)\mathcal{O}_{\Lambda,M_\Lambda,R,\tau} \tag{7.4.1}$$

where $\mathcal{O}_{\Lambda,M_\Lambda,R,\tau}$ are the free field operators defined in (3.6.20) and the expansion coefficients $s^{R,\tau}_{p,\nu}(\Lambda;N)$ are functions of N. These will in general consist of a polynomial numerator and a denominator that is the square root of a polynomial.

Let us give a precise statement of SEP-compatibility for these operators. Take some $\widehat{N} \leq N$, and evaluate these operators on matrices X and Y of size $\widehat{N}\times\widehat{N}$ instead of size $N \times N$. This means the free field operators with $l(R) > \widehat{N}$ vanish, and the coefficients are evaluated at $\widehat{N}$ rather than N. Then the operators with $l(p) > \widehat{N}$ will vanish and the operators with $l(p) \leq \widehat{N}$ will form a basis for the reduced BPS sector. Moreover, these are exactly the operators that would be produced by applying the construction algorithm directly with matrices of size $\widehat{N} \times \widehat{N}$.

Sections 7.4.1 through 7.4.4 describe how to construct $S^{BPS}_{\Lambda,M_\Lambda,p,\nu}$ and prove that this basis is indeed BPS and SEP-compatible. In section 7.4.5 we give an equivalent, shorter construction that is represented in figure 7.1. The remaining sections investigate various properties of the bases.

7.4.1 Orthogonalisation and SEP compatibility

In order to construct the basis (7.4.1), we use results from section 7.5. Define

$$\mathcal{M}^{>}_{\widehat{N}}(N) = \text{Span}\left\{M_{\Lambda,M_\Lambda,p,\nu}(N) : l(p) > \widehat{N}\right\} \tag{7.4.2}$$

$$\mathcal{M}^{\leq}_{\widehat{N}}(N) = \text{Span}\left\{M_{\Lambda,M_\Lambda,p,\nu}(N) : l(p) \leq \widehat{N}\right\} \tag{7.4.3}$$

for some $\widehat{N} \leq N$. The operators $M_{\Lambda,M_\Lambda,p,\nu}(N)$ are constructed by employing the permutation to operator map in (3.6.19) with matrices X, Y of size N. Orthogonalise $\mathcal{M}^{\leq}_{\widehat{N}}$ against $\mathcal{M}^{>}_{\widehat{N}}$ using the S_n inner product (the $g_{n,\widehat{N}}$ inner product in section 7.5), and denote the orthogonalised space by $\bar{\mathcal{M}}^{\leq}_{\widehat{N}}$. Note here the distinction between N and $\widehat{N}$. The operators are defined using matrices of size $N \times N$, while $\widehat{N}$ is used to separate the operators into two classes depending on the length of p, the partition label for operators.

The result (7.5.48) and the discussion below it prove several useful facts.

1. Setting $\widehat{N} = N$, $\bar{\mathcal{M}}^{\leq}_{N}$ is the entire pre-BPS sector.

2. The subspace $\bar{\mathcal{M}}^{\leq}_{\widehat{N}}(N)$ is within the span of free field operators with $l(R) \leq \widehat{N}$. In particular, operators within $\bar{\mathcal{M}}^{\leq}_{\widehat{N}}$ do not receive any contribution from free field operators with $l(R) > \widehat{N}$. To see this, note that $\bar{\mathcal{M}}^{\leq}_{\widehat{N}} = Im(\mathcal{P}) \cap Im(\mathcal{F}_{\widehat{N}})$. The general gauge invariant operators for matrices of size N are constructed using permutation group algebra elements cut-off by $l(R) \leq N$. The definition of $\bar{\mathcal{M}}^{\leq}_{\widehat{N}}(N)$ involves the stronger restriction $l(R) \leq \widehat{N}$.

3. $\bar{\mathcal{M}}^{\leq}_{\widehat{N}}(N)$ gives a subspace of pre-BPS operators for matrices of size N. This subspace is such that, when we reduce N to $\widehat{N}$ by lowering the size of the matrices X and Y to $\widehat{N} \times \widehat{N}$, these operators remain pre-BPS, and in fact form the entire pre-BPS sector.

The first of these results tells us the minimum work necessary to create BPS operators. Take an operator $M_{\Lambda,M_\Lambda,p,\nu}(N)$ with $l(p) \leq N$, and orthogonalise it against $\mathcal{M}^{>}_{N}$ to give a new operator $\bar{M}_{\Lambda,M_\Lambda,p,\nu}(N)$. These form a basis for the quarter-BPS sector. If $N \geq n$, then $\mathcal{M}^{>}_{N}$ is empty, and no orthogonalisation is necessary.

However, for $\widehat{N} < N$, operators $\bar{M}_{\Lambda,M_\Lambda,p,\nu}(N)$ with $l(p) \leq \widehat{N}$ are not necessarily orthogonal to $\mathcal{M}^{>}_{\widehat{N}}$, and therefore upon lowering N to $\widehat{N} < N$, these are no longer pre-BPS. In other words, this is not an SEP-compatible basis, and more work is needed to find one. From the second and third points above, we have a sequence of pre-BPS spaces

$$\bar{\mathcal{M}}^{\leq}_{1}(N) \subset \cdots \subset \bar{\mathcal{M}}^{\leq}_{N-1}(N) \subset \bar{\mathcal{M}}^{\leq}_{N}(N) \tag{7.4.4}$$

such that for any $\widehat{N} \leq N$, the corresponding subspace $\bar{\mathcal{M}}^{\leq}_{\widehat{N}}$ is the entire pre-BPS sector when we lower N to $\widehat{N}$.

It is now clear how we construct an SEP-compatible basis. Each operator $M_{\Lambda,M_\Lambda,p,\nu}(N)$ must be orthogonalised in the S_n inner product against all operators $M_{\Lambda,M_\Lambda,q,\eta}(N)$ with $l(q) > l(p)$. Then for any $\widehat{N}$, the orthogonalised operators with $l(p) \leq \widehat{N}$ form a basis for $\bar{\mathcal{M}}^{\leq}_{\widehat{N}}$. Note that operators in different Λ sectors are already orthogonal from the hermiticity properties of $U(2)$, so we do not need to consider these. In the subsequent discussion we will describe in more detail the steps involved in the construction of SEP-compatible orthogonal BPS operators starting from $M_{\Lambda,M_\Lambda,p,\nu}(N)$. We will henceforth drop the label N and simply write $M_{\Lambda,M_\Lambda,p,\nu}$, with the fact that we are describing the construction for matrices of size N being understood. The parameter $\widehat{N} < N$ will come up in discussion of SEP compatibility of the construction.

In section 7.1.1, we explained that if $N < n$, then $M_{\Lambda,M_\Lambda,p,\nu}$ picked an operator that differed from a pre-BPS operator by addition of a commutator trace. Intuitively, the orthogonalisation is removing the commutator trace part to leave only the pre-BPS operator.

Before implementing the orthogonalisation, recall from section 7.1 that for $N \geq n$, applying $\mathcal{G}_N$ to any basis of symmetrised traces gives BPS operators, without any complicated orthogonalisation procedure. From the above, we can give a weaker bound on N within a specific Λ sector.

Let p^*_Λ be the longest (largest in the ordering (7.4.6)) partition with $\mathcal{M}_{\Lambda,p^*_\Lambda} > 0$. In section 7.4.8 we prove that for $\Lambda = [\Lambda_1, \Lambda_2]$, we have

$$l(p^*_\Lambda) = n - \left\lceil \frac{\Lambda_2}{2} \right\rceil \tag{7.4.5}$$

Then if $N \geq l(p^*_\Lambda)$, then $\mathcal{M}^{>}_{N}$ has no operators transforming on the Λ representation of $U(2)$ and the operators $M_{\Lambda,M_\Lambda,p,\nu}$ do not need to be orthogonalised before applying $\mathcal{G}_N$ to get BPS operators.

Returning to the construction, to carry out the procedure we will use Gram-Schmidt orthogonalisation, which requires choosing an ordering on partitions. To ensure the correct properties when we lower N to some $\widehat{N} \leq N$, we must begin with the longest partition, and proceed to the shortest. For those with the same length, any ordering would suffice to create an SEP-compatible basis. A natural choice is to compare the length of the second column and start with the longer. If this is also the same length, the comparisons proceed along the columns until one is longer. If the partitions are the same and the operators occupy a multiplicity space, we do not specify an ordering. Any will suffice.

More formally, this ordering on partitions is the conjugate of the standard lexicographic ordering of partitions, which compares partitions based on the length of the first row, followed by the second row, etc. Given $p^c = [\lambda_1, \lambda_2, \dots], q^c = [\mu_1, \mu_2, \dots]$, then

$$p > q \iff \text{there is a } k \geq 1 \text{ such that } \begin{cases} \lambda_i = \mu_i & i < k \\ \lambda_k > \mu_k \end{cases} \tag{7.4.6}$$

The operators obtained by performing the orthogonalisation are denoted by $\bar{Q}_{\Lambda, M_\Lambda, p, \mu}$.

In section 7.4.8 we prove that $\mathcal{M}_{\Lambda, p^*_\Lambda} = 1$, so we drop the multiplicity. For this partition we have

$$\bar{Q}_{\Lambda, M_\Lambda, p^*_\Lambda} = M_{\Lambda, M_\Lambda, p^*_\Lambda} \tag{7.4.7}$$

If we are performing the algorithm with $N < l(p^*_\Lambda)$, then the associated operator $M_{\Lambda, M_\Lambda, p^*_\Lambda}$ will reduce to a commutator trace or vanish. In the former case, there is no difference to the algorithm, while in the latter, we instead start the orthogonalisation with the largest p such that $\mathcal{M}_{\Lambda, p} > 0$ and the associated operator does not vanish. This partition will not necessarily have multiplicity 1.

For the remaining p, ν the orthogonalised operators are defined inductively

$$\bar{Q}_{\Lambda, M_\Lambda, p, \nu} = M_{\Lambda, M_\Lambda, p, \nu} - \sum_{(q,\eta) > (p,\nu)} \frac{\left\langle M_{\Lambda, M_\Lambda, p, \nu}, \bar{Q}_{\Lambda, M_\Lambda, q, \eta} \right\rangle_{S_n}}{\left\langle \bar{Q}_{\Lambda, M_\Lambda, q, \eta}, \bar{Q}_{\Lambda, M_\Lambda, q, \eta} \right\rangle_{S_n}} \bar{Q}_{\Lambda, M_\Lambda, q, \eta} \tag{7.4.8}$$

where by $(q, \eta) > (p, \nu)$ we mean either $q > p$ in the ordering (7.4.6) or $q = p$ and $\eta > \nu$.

Similarly to the first step, it may occur that $\bar{Q}_{\Lambda, M_\Lambda, p, \nu} = 0$ for some operators with $l(p) > N$. Such operators are excluded from the rest of the orthogonalisation algorithm. It is implicit that the sum in (7.4.8) does not run over these values of q, η.

In order to compare the different $\bar{Q}_{\Lambda, M_\Lambda, p, \nu}$, we normalise to have S_n norm 1

$$Q_{\Lambda, M_\Lambda, p, \nu} = \frac{\bar{Q}_{\Lambda, M_\Lambda, p, \nu}}{\sqrt{\left\langle \bar{Q}_{\Lambda, M_\Lambda, p, \nu} | \bar{Q}_{\Lambda, M_\Lambda, p, \nu} \right\rangle_{S_n}}} \tag{7.4.9}$$

The new operators $Q_{\Lambda, M_\Lambda, p\nu}$ are an SEP-compatible basis for pre-BPS operators. They are orthonormal under the S_n inner product, and form a stepping stone on the way to producing (7.4.1).

From the arguments above, we know that when expanding $Q_{\Lambda, M_\Lambda, p, \nu}$ in terms of the free field BPS operators (3.6.20), only those operators with $l(R) \leq l(p)$ contribute.

7.4.2 An example: field content $(2, 2)$

We now give an explicit example of this construction for the field content $(2, 2)$ sector at $N \geq n = 4$. While doing so, we will observe various features that generalise to

M_Λ	1 1 2 2	1 1 2 2	1 1 2 2	1 1 2 2	1 1 2 2	1 1 2 / 2	1 1 2 / 2	1 1 / 2 2	1 1 / 2 2	1 1 / 2 2
R	[4]	[3,1]	[2,2]	[2,1,1]	[1,1,1,1]	[3,1]	[2,1,1]	[3,1]	[2,2]	[2,1,1]
Normalisation Coefficient	$\frac{\sqrt{6}}{24}$	$\frac{\sqrt{6}}{24}$	$\frac{\sqrt{6}}{12}$	$\frac{\sqrt{6}}{24}$	$\frac{\sqrt{6}}{24}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{\sqrt{3}}{12}$	$\frac{\sqrt{6}}{12}$	$\frac{\sqrt{3}}{12}$
$(\mathrm{Tr}X)^2(\mathrm{Tr}Y)^2$	1	3	1	3	1	0	0	0	0	0
$\mathrm{Tr}X^2(\mathrm{Tr}Y)^2$	1	1	0	-1	-1	1	-1	1	-1	-1
$\mathrm{Tr}XY\mathrm{Tr}X\mathrm{Tr}Y$	4	4	0	-4	-4	0	0	-2	2	2
$(\mathrm{Tr}X)^2\mathrm{Tr}Y^2$	1	1	0	-1	-1	-1	1	1	-1	-1
$\mathrm{Tr}X^2Y\mathrm{Tr}Y$	4	0	-2	0	4	2	2	0	0	0
$\mathrm{Tr}XY^2\mathrm{Tr}X$	4	0	-2	0	4	-2	-2	0	0	0
$\mathrm{Tr}X^2\mathrm{Tr}Y^2$	1	-1	1	-1	1	0	0	2	0	2
$(\mathrm{Tr}XY)^2$	2	-2	2	-2	2	0	0	-2	0	-2
$\mathrm{Tr}X^2Y^2$	4	-4	0	4	-4	0	0	2	2	-2
$\mathrm{Tr}(XY)^2$	2	-2	0	2	-2	0	0	-2	-2	2

Table 7.2: The covariant basis for field content $(2,2)$ in terms of multi-traces. Each element is identified by its M_Λ and R labels. We give the overall normalisation and the coefficient of each multi-trace within the operator.

higher orders. These are discussed in later subsections.

We begin with the operators $M_{\Lambda,M_\Lambda,p,\nu}$ given in (7.3.14-7.3.22). The ordering (7.4.6) for partitions of $n=4$ is

$$[1,1,1,1] > [2,1,1] > [2,2] > [3,1,1] > [4] \tag{7.4.10}$$

Throughout the rest of this section we will continue using field content $(2,2)$ operators as an example. In order to do this will need to express operators in terms of the free field covariant basis defined in (3.6.20). The full set of covariant operators is given in terms of multi-traces in table 7.2.

$\Lambda = [4]$ and $[3,1]$ sectors

Orthogonalising in the $\Lambda = [4]$ sector, and normalising with respect to the S_n inner product, we obtain

$$\begin{aligned} Q_{\boxed{1}\boxed{1}\boxed{2}\boxed{2},\,[1,1,1,1]} = \frac{1}{4\sqrt{6}}\Big[& -4\mathrm{Tr}X^2Y^2 - 2\mathrm{Tr}\,(XY)^2 + 4\mathrm{Tr}X^2Y\mathrm{Tr}Y + 4\mathrm{Tr}X\mathrm{Tr}XY^2 \\ & + 2\,(\mathrm{Tr}XY)^2 + \mathrm{Tr}X^2\mathrm{Tr}Y^2 - \mathrm{Tr}X^2\,(\mathrm{Tr}Y)^2 \\ & -4\mathrm{Tr}X\mathrm{Tr}XY\mathrm{Tr}Y - (\mathrm{Tr}X)^2\,\mathrm{Tr}Y^2 + (\mathrm{Tr}X)^2\,(\mathrm{Tr}Y)^2\Big] \end{aligned} \tag{7.4.11}$$

$$Q_{\young(1122),\,[2,1,1]} = \frac{1}{4\sqrt{6}}\Big[4\mathrm{Tr}X^2Y^2 + 2\mathrm{Tr}\left(XY\right)^2 - 2\left(\mathrm{Tr}XY\right)^2 - \mathrm{Tr}X^2\mathrm{Tr}Y^2 - \mathrm{Tr}X^2\left(\mathrm{Tr}Y\right)^2 - 4\mathrm{Tr}X\mathrm{Tr}XY\mathrm{Tr}Y - \left(\mathrm{Tr}X\right)^2\mathrm{Tr}Y^2 + 3\left(\mathrm{Tr}X\right)^2\left(\mathrm{Tr}Y\right)^2\Big] \tag{7.4.12}$$

$$Q_{\young(1122),\,[2,2]} = \frac{1}{2\sqrt{6}}\Big[2\left(\mathrm{Tr}XY\right)^2 + \mathrm{Tr}X^2\mathrm{Tr}Y^2 - 2\mathrm{Tr}X^2Y\mathrm{Tr}Y - 2\mathrm{Tr}X\mathrm{Tr}XY^2 + \left(\mathrm{Tr}X\right)^2\left(\mathrm{Tr}Y\right)^2\Big] \tag{7.4.13}$$

$$Q_{\young(1122),\,[3,1]} = \frac{1}{4\sqrt{6}}\Big[-4\mathrm{Tr}X^2Y^2 - 2\mathrm{Tr}\left(XY\right)^2 - 2\left(\mathrm{Tr}XY\right)^2 - \mathrm{Tr}X^2\mathrm{Tr}Y^2 + \mathrm{Tr}X^2\left(\mathrm{Tr}Y\right)^2 + 4\mathrm{Tr}X\mathrm{Tr}XY\mathrm{Tr}Y + \left(\mathrm{Tr}X\right)^2\mathrm{Tr}Y^2 + 3\left(\mathrm{Tr}X\right)^2\left(\mathrm{Tr}Y\right)^2\Big] \tag{7.4.14}$$

$$Q_{\young(1122),\,[4]} = \frac{1}{4\sqrt{6}}\Big[4\mathrm{Tr}X^2Y^2 + 2\mathrm{Tr}\left(XY\right)^2 + 4\mathrm{Tr}X^2Y\mathrm{Tr}Y + 4\mathrm{Tr}X\mathrm{Tr}XY^2 + 2\left(\mathrm{Tr}XY\right)^2 + \mathrm{Tr}X^2\mathrm{Tr}Y^2 + \left(\mathrm{Tr}X\right)^2\left(\mathrm{Tr}Y\right)^2 + \mathrm{Tr}X^2\left(\mathrm{Tr}Y\right)^2 + 4\mathrm{Tr}X\mathrm{Tr}XY\mathrm{Tr}Y + \left(\mathrm{Tr}X\right)^2\mathrm{Tr}Y^2\Big] \tag{7.4.15}$$

where we have suppressed the trivial multiplicity indices, and omitted Λ as this is determined by the shape of the semi-standard tableau M_Λ.

For $\Lambda = [3,1]$, the orthogonalisation process produces

$$Q_{\young(112,2),\,[2,1,1]} = \frac{1}{4}\Big[-2\mathrm{Tr}X^2Y\mathrm{Tr}Y + 2\mathrm{Tr}X\mathrm{Tr}XY^2 + \mathrm{Tr}X^2\left(\mathrm{Tr}Y\right)^2 - \left(\mathrm{Tr}X\right)^2\mathrm{Tr}Y^2\Big] \tag{7.4.16}$$

$$Q_{\young(112,2),\,[3,1]} = \frac{1}{4}\Big[2\mathrm{Tr}X^2Y\mathrm{Tr}Y - 2\mathrm{Tr}X\mathrm{Tr}XY^2 + \mathrm{Tr}X^2\left(\mathrm{Tr}Y\right)^2 - \left(\mathrm{Tr}X\right)^2\mathrm{Tr}Y^2\Big] \tag{7.4.17}$$

For both these sectors, the operators obtained are identical to the free field BPS operators, given for field content $(2,2)$ in table 7.2, where the weak coupling label p matches the zero coupling label R. At $n = 4$, this is largely pre-determined by SEP-compatibility of the two bases.

Since the $\Lambda = [4]$ and $[3,1]$ sectors remain unchanged as we turn on interactions, the free field BPS operators are also the weak coupling BPS operators. As they are also eigen-operators of $\mathcal{F}_N$, the space of free field BPS operators is the same as the space of pre-BPS operators. Moreover, in these sectors there are no commutator traces, and therefore SEP-compatibility for the pre-BPS operators is the same as that for BPS operators, an operator with $l(p) > N$ vanishes identically.

Therefore for $\Lambda = [4]$ and $[3,1]$, $Q_{\Lambda,M_\Lambda,p,\nu}$ and $\mathcal{O}_{\Lambda,M_\Lambda,R,\tau}$ are both SEP-compatible bases for the same space. This means the spaces spanned by partitions of a given length are the same. These spaces are all one-dimensional with the exception of $\Lambda = [4]$, $p = [2,2]$ and $p = [3,1]$. Therefore the matching of all but these two is trivial. A priori, the matching between p and R for $p = [3,1]$ and $[2,2]$ is surprising.

More generally, we find the same behaviour for all $n \leq 7$, which are within reach of numerical calculations. The p label in $Q_{\Lambda,M_\Lambda,p,\nu}$ matches the R label in $\mathcal{O}_{\Lambda,M_\Lambda,R,\tau}$ for $\Lambda = [n]$ and $[n-1,1]$. Since neither basis specifies exactly how the multiplicities ν and τ are chosen, these will not necessarily match, but they do span the same space. We go into this in more detail in section 7.4.9. For $\Lambda = [n]$, the matching follows from the fact the Kostka numbers converting the monomial basis to the Schur basis (2.7.12) are upper diagonal in partition indices. For $\Lambda = [n-1,1]$ we leave the matching at general n as a conjecture.

$\Lambda = [2,2]$ sector

The orthogonalised basis of pre-BPS operators for $\Lambda = [2,2]$ is

$$Q_{\young(11,22),\,[2,1,1]} = \frac{1}{6}\left(-2t_{[2,2]} + t_{[2,1,1]}\right) \tag{7.4.18}$$

$$Q_{\young(11,22),\,[2,2]} = \frac{1}{3\sqrt{2}}\left(t_{[2,2]} + t_{[2,1,1]}\right) \tag{7.4.19}$$

where the trace combinations are

$$t_{[2,1,1]} = \mathrm{Tr}X^2\left(\mathrm{Tr}Y\right)^2 - 2\mathrm{Tr}X\mathrm{Tr}XY\mathrm{Tr}Y + \left(\mathrm{Tr}X\right)^2\mathrm{Tr}Y^2 \tag{7.4.20}$$

$$t_{[2,2]} = \mathrm{Tr}X^2\mathrm{Tr}Y^2 - \left(\mathrm{Tr}XY\right)^2 \tag{7.4.21}$$

These trace combinations are the $\Lambda = [2,2]$, $M_\Lambda = \young(11,22)$ parts of the symmetrised trace covariant basis defined in (7.3.36).

In order to check the SEP-compatibility of (7.4.18) and (7.4.19), we express them as a linear combination of the zero coupling basis given in table 7.2. We have

$$Q_{\young(11,22),\,[2,1,1]} = -\frac{1}{2\sqrt{3}}\left(\mathcal{O}_{\young(11,22),\yng(3,1)} + \sqrt{2}\mathcal{O}_{\young(11,22),\yng(2,2)} + 3\mathcal{O}_{\young(11,22),\yng(2,1,1)}\right) \tag{7.4.22}$$

$$Q_{\young(11,22),\,[2,2]} = \frac{1}{\sqrt{3}}\left(\sqrt{2}\mathcal{O}_{\young(11,22),\yng(3,1)} - \mathcal{O}_{\young(11,22),\yng(2,2)}\right) \tag{7.4.23}$$

The only commutator trace at field content $(2,2)$ is

$$\begin{aligned}\mathrm{Tr}\,X[X,Y]Y &= \mathrm{Tr}X^2Y^2 - \mathrm{Tr}(XY)^2 \\ &= \frac{\sqrt{3}}{2}\left(\mathcal{O}_{\young(11,22),\yng(3,1)} + \sqrt{2}\mathcal{O}_{\young(11,22),\yng(2,2)} - \mathcal{O}_{\young(11,22),\yng(2,1,1)}\right)\end{aligned} \tag{7.4.24}$$

It is simple to check that for $N \geq 3$

$$\left\langle Q_{\ytableaushort{11,22},\,[2,1,1]} \middle| \operatorname{Tr} X[X,Y]Y \right\rangle_{S_n} = \left\langle Q_{\ytableaushort{11,22},\,[2,2]} \middle| \operatorname{Tr} X[X,Y]Y \right\rangle_{S_n} = 0 \tag{7.4.25}$$

At $N = 2$, comparing (7.4.22) with (7.4.24) and recalling that $\mathcal{O}_{\ytableaushort{11,22},\,\ydiagram{2,1,1}}$ vanishes, we have

$$Q_{\ytableaushort{11,22},\,[2,1,1]} = -\frac{1}{3}\operatorname{Tr} X[X,Y]Y \tag{7.4.26}$$

Therefore (7.4.22) is no longer pre-BPS. The other operator (7.4.23) is still orthogonal to the commutator.

At $N = 1$, both (7.4.22) and (7.4.23) vanish identically. Combined with the behaviour at $N = 2$, this demonstrates that these two operators form an SEP-compatible basis for pre-BPS operators in the $\Lambda = [2,2]$ sector.

At $N = 2$, the finite N relations mean that (7.4.22) can be written as both a symmetrised trace and a commutator trace. This is discussed in more generality in section 7.5, where we develop the finite N vector space geometry responsible for transforming between the two types of traces.

7.4.3 Normalisation conventions for BPS operators

The final step to obtain an SEP-compatible basis for weakly coupled BPS operators is to apply $\mathcal{G}_N$ to $Q_{\Lambda,M_\Lambda,p,\nu}$, where $l(p) \leq N$. However, the operators $\mathcal{G}_N Q_{\Lambda,M_\Lambda,p,\nu}$ contain denominators of the form $(N-i)$ for $i \leq l(p)$ which make it difficult to see how they should behave when we lower N to $\widehat{N} = i$.

In this section we prove that by normalising $\mathcal{G}_N Q_{\Lambda,M_\Lambda,p,\nu}$ under the S_n inner product, we remove these denominators and obtain an SEP-compatible basis of BPS operators. We start by continuing the example of $\Lambda = [2,2]$ from the previous subsection to show some of the behaviour that occurs. Working at $N \geq 3$, we have

$$\begin{aligned}\mathcal{G}_N\, Q_{\ytableaushort{11,22},\,[2,1,1]} = -\frac{1}{2\sqrt{3}N(N-1)(N+1)} \Bigg(&\frac{1}{N+2}\mathcal{O}_{\ytableaushort{11,22},\,\ydiagram{3,1}} + \frac{\sqrt{2}}{N}\mathcal{O}_{\ytableaushort{11,22},\,\ydiagram{2,2}} \\ &+ \frac{3}{N-2}\mathcal{O}_{\ytableaushort{11,22},\,\ydiagram{2,1,1}} \Bigg)\end{aligned} \tag{7.4.27}$$

$$\mathcal{G}_N\, Q_{\ytableaushort{11,22},\,[2,2]} = \frac{1}{\sqrt{3}N(N-1)(N+1)} \left(\frac{\sqrt{2}}{N+2}\mathcal{O}_{\ytableaushort{11,22},\,\ydiagram{3,1}} - \frac{1}{N}\mathcal{O}_{\ytableaushort{11,22},\,\ydiagram{2,2}} \right) \tag{7.4.28}$$

Now consider lower N to $\widehat{N} = 2$ and imposing the finite $\widehat{N}$ cut-off. It is unclear how we should treat (7.4.27), since the operator $\mathcal{O}_{\ytableaushort{11,22},\,\ydiagram{2,1,1}}$ vanishes, yet we also have a division by zero. Let us resolve the ambiguity by declaring that this term should indeed vanish. Then at $\widehat{N} = 2$ we have

$$\mathcal{G}_N \, Q_{\ytableaushort{11,22},\,[2,1,1]} = -\frac{1}{12\sqrt{3}} \left(\frac{1}{4} \mathcal{O}_{\ytableaushort{11,22},\,\ydiagram{3,1}} + \frac{1}{\sqrt{2}} \mathcal{O}_{\ytableaushort{11,22},\,\ydiagram{2,2}} \right) \tag{7.4.29}$$

This is a perfectly well defined operator, yet $\mathcal{G}_N Q_{\Lambda,M_\Lambda,p,\nu}$ was meant to be an SEP-compatible basis. For BPS operators, this means (7.4.27) should vanish after reducing N to $\widehat{N} = 2$.

The resolution of this problem is to normalise $\mathcal{G}_N Q_{\Lambda,M_\Lambda,p,\nu}$ in the S_n inner product. Define

$$Q^{BPS}_{\Lambda,M_\Lambda,p,\nu} = \frac{\mathcal{G}_N \, Q_{\Lambda,M_\Lambda,p,\nu}}{\sqrt{\langle \mathcal{G}_N \, Q_{\Lambda,M_\Lambda,p,\nu} | \mathcal{G}_N \, Q_{\Lambda,M_\Lambda,p,\nu} \rangle_{S_n}}} \tag{7.4.30}$$

For $\Lambda = [2,2]$ we have

$$\begin{aligned} Q^{BPS}_{\ytableaushort{11,22},\,[2,1,1]} = -\frac{1}{2\sqrt{P_1(N)}} \Bigg(& N(N-2)\mathcal{O}_{\ytableaushort{11,22},\,\ydiagram{3,1}} + \sqrt{2}(N+2)(N-2)\mathcal{O}_{\ytableaushort{11,22},\,\ydiagram{2,2}} \\ & + 3N(N+2)\mathcal{O}_{\ytableaushort{11,22},\,\ydiagram{2,1,1}} \Bigg) \end{aligned} \tag{7.4.31}$$

$$Q^{BPS}_{\ytableaushort{11,22},\,[2,2]} = \frac{1}{\sqrt{P_2(N)}} \left(\sqrt{2} N \mathcal{O}_{\ytableaushort{11,22},\,\ydiagram{3,1}} - (N+2)\mathcal{O}_{\ytableaushort{11,22},\,\ydiagram{2,2}} \right) \tag{7.4.32}$$

where the normalisation polynomials are

$$P_1(N) = 3N^4 + 8N^3 + 6N^2 + 8 \tag{7.4.33}$$

$$P_2(N) = 3N^2 + 4N + 4 \tag{7.4.34}$$

There is now no ambiguity in the definitions of the operators after lowering N to $\widehat{N} = 1, 2$, and they vanish identically for $\widehat{N} < l(p)$, thereby forming an SEP-compatible basis for BPS operators.

We now generalise to arbitrary Λ. Take some operator $Q_{\Lambda,M_\Lambda,p,\nu}$ with $l(p) \leq N$ and expand it in terms of free field operators

$$Q_{\Lambda,M_\Lambda,p,\nu} = \sum_{\substack{R,\tau \\ l(R) \leq l(p)}} q^{R,\tau}_{p,\nu} \mathcal{O}_{\Lambda,M_\Lambda,R,\tau} \tag{7.4.35}$$

for some coefficients $q^{R,\tau}_{p,\nu}$. The limit $l(R) \leq l(p)$ on the sum was discussed below (7.4.9). Define the set

$$Y_{p,\nu} = \left\{ R : q^{R,\tau}_{p,\nu} \neq 0 \text{ for some } \tau \right\} \tag{7.4.36}$$

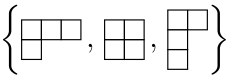
Intuitively, $Y_{p,\nu}$ is the set of Young diagrams R that contribute to $Q_{\Lambda,M_\lambda,p,\nu}$. Define

$R^{max}_{p,\nu}$ to be the minimal Young diagram that contains every R in $Y_{p,\nu}$ and $R^{min}_{p,\nu}$ to be the maximal Young diagram that is contained within every R in $Y_{p,\nu}$. Define

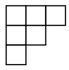
$$f^{max}_{p,\nu}(N) = f_{R^{max}_{p,\nu}}(N) \qquad\qquad f^{min}_{p,\nu}(N) = f_{R^{min}_{p,\nu}}(N) \tag{7.4.37}$$

where $f_R(N)$ is a polynomial in N depending on the shape of R. It is defined in (2.3.20) as a product of linear factors. Intuitively, $f^{max}_{p,\nu}(N)$ is the lowest common multiple of the different f_R for $R \in Y_{p,\nu}$, while $f^{min}_{p,\nu}(N)$ is the highest common factor.

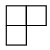
As an example, consider the previous example with $\Lambda = [2, 2]$ and $p = [2, 1, 1]$. We have

$$Y_{[2,1,1]} = \tag{7.4.38}$$


and

$$R^{max}_{[2,1,1]} =$$


$$f^{max}_{[2,1,1]}(N) = (N + 2)(N + 1)N^2(N - 1)(N - 2) \tag{7.4.39}$$

$$R^{min}_{[2,1,1]} =$$


$$f^{min}_{[2,1,1]}(N) = (N + 1)N(N - 1) \tag{7.4.40}$$

Using $f^{max}_{p,\nu}(N)$, we can factor out the denominators in $\mathcal{G}_N Q_{\Lambda,M_\Lambda,p,\nu}$. We have

$$\mathcal{G}_N Q_{\Lambda,M_\Lambda,p,\nu} = \sum_{\substack{R,\tau \\ l(R)\leq l(p)}} \frac{q^{R,\tau}_{p,\nu}}{f_R(N)} \mathcal{O}_{\Lambda,M_\Lambda,R,\tau} = \frac{1}{f^{max}_{p,\nu}(N)} \sum_{\substack{R,\tau \\ l(R)\leq l(p)}} q^{R,\tau}_{p,\nu} \frac{f^{max}_{p,\nu}(N)}{f_R(N)} \mathcal{O}_{\Lambda,M_\Lambda,R,\tau} \tag{7.4.41}$$

where the coefficients $\frac{f^{max}_{p,\nu}(N)}{f_R(N)}$ are simple polynomials in N made up of products of linear factors. Therefore the S_n norm of $\mathcal{G}_N Q_{\Lambda,M_\Lambda,p,\nu}$ is

$$|\mathcal{G}_N\, Q_{\Lambda,M_\Lambda,p,\nu}|^2_{S_n} = \langle \mathcal{G}_N\, Q_{\Lambda,M_\Lambda,p,\nu} | \mathcal{G}_N\, Q_{\Lambda,M_\Lambda,p,\nu} \rangle_{S_n} = \frac{P_{p,\nu}(N)}{\left[f^{max}_{p,\nu}(N)\right]^2} \tag{7.4.42}$$

where the numerator polynomial is

$$P_{p,\nu}(N) = \sum_{\substack{R,\tau \\ l(R)\leq l(p)}} \left(q^{R,\tau}_{p,\nu} \frac{f^{max}_{p,\nu}(N)}{f_R(N)} \right)^2 \tag{7.4.43}$$

This polynomial is a sum of squares, and therefore can only vanish if all terms are zero. Since $\frac{f^{max}_{p,\nu}(N)}{f_R(N)}$ is a product of simple linear factors in N, this in turn can only occur if there is linear factor common to every term. From the definition of $f^{max}_{p,\nu}$ as the lowest common multiple of the f_R that appear in the sum, this does not happen. Therefore $P_{p,\nu}(N)$ is positive for any N. In particular, it remains positive when we evaluate it on

$\widehat{N} \leq N$. We have

$$Q^{BPS}_{\Lambda,M_\Lambda,p,\nu} = \frac{1}{\sqrt{P_{p,\nu}(N)}} \sum_{\substack{R,\tau \\ l(R)\leq l(p)}} q^{R,\tau}_{p,\nu} \frac{f^{max}_{p,\nu}(N)}{f_R(N)} \mathcal{O}_{\Lambda,M_\Lambda,R,\tau} \tag{7.4.44}$$

We now prove this vanishes when we lower N to some $\widehat{N} < l(p)$. The properties of $P_{p,\nu}(N)$ mean there are no divisions by zero to concern us, and we focus on the coefficients $\frac{f^{max}_{p,\nu}(N)}{f_R(N)}$.

By construction, $Q_{\Lambda,M_\Lambda,p,\nu}$ reduces to a commutator trace when we lower N to $\widehat{N} < l(p)$. Therefore it must contain at least one $\mathcal{O}_{\Lambda,M_\Lambda,R,\tau}$ with $l(R) = l(p)$, otherwise it would remain S_n orthogonal to commutator traces if we lowered N to $\widehat{N} = l(p) - 1$. Therefore

$$f^{max}_{p,\nu}(N) \supset \prod_{i=0}^{l(p)-1} (N - i) \tag{7.4.45}$$

where $\supset$ means $f^{max}_{p,\nu}$ contains these as factors. By definition

$$f_R(N) \supset \prod_{i=0}^{l(R)-1} (N - i) \tag{7.4.46}$$

it follows that if $R \in Y_{p,\nu}$, the coefficient in front of $\mathcal{O}_{\Lambda,M_\Lambda,R,\tau}$ contains

$$\frac{f^{max}_{p,\nu}(N)}{f_R(N)} \supset \prod_{i=l(R)}^{l(p)-1} (N - i) \tag{7.4.47}$$

This ensures that if $l(p) > \widehat{N}$, all terms in the expansion (7.4.44) vanish when we lower N to $\widehat{N}$. If $l(R) > \widehat{N}$, $\mathcal{O}_{\Lambda,M_\Lambda,R,\tau}$ vanishes by definition, while if $l(R) \leq \widehat{N}$, the factors in (7.4.47) set it to zero.

Therefore $Q^{BPS}_{\Lambda,M_\Lambda,p,\nu}$ vanishes identically when we lower N to $\widehat{N} < l(p)$, and hence this is an SEP-compatible basis for weakly coupled BPS operators. This justifies the statement made in section 7.1 that applying $\mathcal{G}_N$ to an SEP-compatible basis of pre-BPS operators gives an SEP-compatible basis of BPS operators.

An alternative viewpoint is to look at the physical $\mathcal{F}$-weighted inner product. We have

$$|Q^{BPS}_{\Lambda,M_\Lambda,p,\nu}|^2_{\mathcal{F}} = \langle Q_{\Lambda,M_\Lambda,p,\nu} | Q_{\Lambda,M_\Lambda,p,\nu} \rangle_{\mathcal{F}} = \frac{f^{max}_{p,\nu}(N) P^{(F)}_{p,\nu}(N)}{P_{p,\nu}(N)} \tag{7.4.48}$$

where the new polynomial in the numerator is

$$P^{(F)}_{p,\nu}(N) = \sum_{\substack{R,\tau \\ l(R)\leq l(p)}} \left(q^{R,\tau}_{p,\nu}\right)^2 \frac{f^{max}_{p,\nu}(N)}{f_R(N)} \tag{7.4.49}$$

The overall factor of $f^{max}_{p,\nu}(N)$ in (7.4.48) ensures that it vanishes when we lower N to some $\widehat{N} < l(p)$. As a consistency check, we also prove that it is non-vanishing when $\widehat{N} \geq l(p)$.

This relies on noticing that the linear factors in $\frac{f^{max}_{p,\nu}(N)}{f_R(N)}$ are all of the form $(N-i)$ for $-n \leq i \leq l(p)-1$. Therefore $\frac{f^{max}_{p,\nu}(\widehat{N})}{f_R(\widehat{N})} > 0$ when $\widehat{N} \geq l(p)$. It follows that $P^{(F)}_{p,\nu}(\widehat{N}) > 0$, and the result follows.

In the planar ($N \to \infty$) limit, applying $\mathcal{G}_N$ reduces to division by N^n, and therefore after S_n-normalising we have

$$Q^{BPS}_{\Lambda,M_\Lambda,p,\nu}\big|_{N\to\infty} = Q_{\Lambda,M_\Lambda,p,\nu} \tag{7.4.50}$$

Since $Q_{\Lambda,M_\Lambda,p,\nu}$ is a symmetrised trace at large N, this means commutator traces are sub-leading in the large N multi-trace expansion of $Q^{BPS}_{\Lambda,M_\Lambda,p,\nu}$.

Having constructed an SEP-compatible basis of BPS operators, the natural next question to ask is whether we can find a formula for their correlators. This uses the physical $\mathcal{F}$-weighted inner product. From the hermiticity of $U(2)$, the $Q^{BPS}_{\Lambda,M_\Lambda,p,\nu}$ are $\mathcal{F}$-orthogonal in the Λ, M_Λ labels, but in general are not in p,ν. Therefore studying correlators involves calculating a matrix of inner products. In the next section we $\mathcal{F}$-orthogonalise the $Q^{BPS}_{\Lambda,M_\Lambda,p,\nu}$ in order to produce an $\mathcal{F}$-orthogonal SEP-compatible basis in which it is easier to study properties of correlators.

In this section we have normalised operators using the S_n inner product. This is in some sense un-natural, as the $Q^{BPS}_{\Lambda,M_\Lambda,p,\nu}$ are not orthogonal in this inner product. There is another, alternative normalisation we could consider. In (7.4.44) we have a complicated normalisation factor of $(P_{p,\nu}(N))^{-1/2}$, which if removed, would mean the coefficients of the free field operators (and multi-traces) would be expressible purely as polynomials in N. This is a natural normalisation to consider, and is given simply by

$$f^{max}_{p,\nu}\mathcal{G}_N\, Q_{\Lambda,M_\Lambda,p,\nu} \tag{7.4.51}$$

However, this is more difficult than the S_n normalisation, since it involves knowing which free field operators appear in the expansion of $Q_{\Lambda,M_\Lambda,p,\nu}$, and the free field operators are computationally expensive to construct. In contrast, the S_n normalisation can be deduced purely from an expression in terms of multi-traces, which is obtainable explicitly from the construction of $Q_{\Lambda,M_\Lambda,p,\nu}$.

In section 7.4.9 we prove that for $\Lambda = [n]$, the $Q_{\Lambda,M_\Lambda,p,\nu}$ exactly reproduce the free field basis $\mathcal{O}_{\Lambda,M_\Lambda,R,\tau}$ up to a choice of multiplicity basis, and conjecture that the same happens for $\Lambda = [n-1,1]$. If this is true, then for these Λ, applying $\mathcal{G}$ and S_n normalising leaves the operators unchanged and we have

$$Q^{BPS}_{\Lambda,M_\Lambda,p,\nu} = Q_{\Lambda,M_\Lambda,p,\nu} = \mathcal{O}_{\Lambda,M_\Lambda,R=p,\tau=\nu} \tag{7.4.52}$$

7.4.4 $\mathcal{F}$-orthogonalisation

To define an orthogonal SEP-compatible basis, we Gram-Schmidt orthogonalise the $Q^{BPS}_{\Lambda,M_\Lambda,p,\nu}$ operators using the physical $\mathcal{F}$-weighted inner product and the same ordering defined in (7.4.6). We denote the orthogonalised operators by $\bar{S}^{BPS}_{\Lambda,M_\Lambda,p,\nu}$, and after normalising them to have S_n norm 1, $S^{BPS}_{\Lambda,M_\Lambda,p,\nu}$.

Let $p^*_{\Lambda;N}$ be the largest partition with $\mathcal{M}_{\Lambda,p^*_{\Lambda;N}} > 0$ and $l(p^*_{\Lambda;N}) \leq N$. Then in analogy to (7.4.7), (7.4.8) and (7.4.9), we have

$$\bar{S}^{BPS}_{\Lambda,M_\Lambda,p^*_{\Lambda;N}} = Q^{BPS}_{\Lambda,M_\Lambda,p^*_{\Lambda;N}} \tag{7.4.53}$$

$$\bar{S}^{BPS}_{\Lambda,M_\Lambda,p,\nu} = Q^{BPS}_{\Lambda,M_\Lambda,p,\nu} - \sum_{\substack{(q,\eta)>(p,\nu)\\ l(q)\leq N}} \frac{\left\langle Q^{BPS}_{\Lambda,M_\Lambda,p,\nu}, \bar{S}^{BPS}_{\Lambda,M_\Lambda,q,\eta}\right\rangle_{\mathcal{F}}}{\left\langle \bar{S}^{BPS}_{\Lambda,M_\Lambda,q,\eta}, \bar{S}^{BPS}_{\Lambda,M_\Lambda,q,\eta}\right\rangle_{\mathcal{F}}} \bar{S}^{BPS}_{\Lambda,M_\Lambda,q,\eta} \tag{7.4.54}$$

$$S^{BPS}_{\Lambda,M_\Lambda,p,\nu} = \frac{\bar{S}^{BPS}_{\Lambda,M_\Lambda,p,\nu}}{\sqrt{\left\langle \bar{S}^{BPS}_{\Lambda,M_\Lambda,p,\nu} | \bar{S}^{BPS}_{\Lambda,M_\Lambda,p,\nu}\right\rangle_{S_n}}} \tag{7.4.55}$$

where $l(p) \leq N$.

Note the difference in the starting point of the orthogonalisation compared to (7.4.7). When S_n orthogonalising the pre-BPS operators, we began with $p^*_\Lambda = p^*_{\Lambda;\infty}$ even if $l(p^*_\Lambda) > N$, whereas this time we apply the finite N cut-off to the partitions being orthogonalised.

From the construction it follows that $Q^{BPS}_{\Lambda,M_\Lambda,q,\eta}$ only contributes to $S^{BPS}_{\Lambda,M_\Lambda,p,\nu}$ if $q \geq p$. Upon lowering N to $\widehat{N}$, since the Q^{BPS} operators with $l(q) > \widehat{N}$ vanish identically, the S^{BPS} will also vanish for $l(p) > \widehat{N}$, and therefore this is an SEP-compatible basis.

Note this relies on $Q^{BPS}_{\Lambda,M_\Lambda,q,\eta}$ not appearing with a coefficient of $\frac{1}{N-i}$ for $i \leq l(q)$, as this would upset the SEP-compatibility in a way similar to that described in (7.4.29). As in the previous section, the S_n normalisation ensures this does not occur.

In the planar ($N \to \infty$) limit, the physical $\mathcal{F}$-weighted inner product reduces to N^n times the S_n inner product, therefore $\mathcal{F}$-orthogonalising is equivalent to S_n-orthogonalising. From (7.4.50), the $Q^{BPS}_{\Lambda,M_\Lambda,p,\nu}$ operators reduce to the pre-BPS operators $Q_{\Lambda,M_\Lambda,p,\nu}$ in the planar limit, which are already S_n orthonormal. Therefore

$$S^{BPS}_{\Lambda,M_\Lambda,p,\nu}\Big|_{N\to\infty} = Q_{\Lambda,M_\Lambda,p,\nu} \tag{7.4.56}$$

Below (7.4.9) we explained that only those free field operators with $l(R) \leq l(p)$ contribute to $Q_{\Lambda,M_\Lambda,p,\nu}$. After orthogonalisation, the $S^{BPS}_{\Lambda,M_\Lambda,p,\nu}$ can admit operators with $l(R) > l(p)$, but (7.4.56) proves that these are sub-leading at large N.

Furthermore, just as discussed for $Q^{BPS}_{\Lambda,M_\Lambda,p,\nu}$ below (7.4.50), commutator traces are sub-leading in the large N multi-trace expansion of $S^{BPS}_{\Lambda,M_\Lambda,p,\nu}$.

The $\Lambda = [2,2]$ example

We begin with the operators (7.4.31) and (7.4.32). After $\mathcal{F}$-orthogonalising and normalising with respect to the S_n inner product, we obtain

$$S^{BPS}_{\ytableaushort{11,22},[2,1,1]} = -\frac{1}{2\sqrt{P_1(N)}}\Big(N(N-2)\mathcal{O}_{\ytableaushort{11,22},\ydiagram{3,1}} + \sqrt{2}(N-2)(N+2)\mathcal{O}_{\ytableaushort{11,22},\ydiagram{2,2}} + 3N(N+2)\mathcal{O}_{\ytableaushort{11,22},\ydiagram{2,1,1}}\Big) \tag{7.4.57}$$

$$S^{BPS}_{\ytableaushort{11,22},[2,2]} = \frac{1}{\sqrt{2P_2(N)}}\Big((2N-1)\mathcal{O}_{\ytableaushort{11,22},\ydiagram{3,1}} - \sqrt{2}(N+1)\mathcal{O}_{\ytableaushort{11,22},\ydiagram{2,2}} + \mathcal{O}_{\ytableaushort{11,22},\ydiagram{2,1,1}}\Big) \tag{7.4.58}$$

where the normalisation polynomials P_1 and P_2 are given by

$$P_1(N) = 3\,N^4 + 8\,N^3 + 6\,N^2 + 8 \tag{7.4.59}$$

$$P_2(N) = 3N^2 + 2 \tag{7.4.60}$$

Written in terms of traces, these operators are

$$S^{BPS}_{\ytableaushort{11,22},[2,1,1]} = \frac{1}{2\sqrt{3P_1(N)}}\Big[\left(N^2 + 2\,N - 2\right)t_{[2,1,1]} - 2N\,(N+1)\,t_{[2,2]} + 4\,(N+1)\,\mathrm{Tr}X[X,Y]Y\Big] \tag{7.4.61}$$

$$S^{BPS}_{\ytableaushort{11,22},[2,2]} = \frac{1}{\sqrt{6P_2(N)}}\Big[N\left(t_{[2,1,1]} + t_{[2,2]}\right) - 2\mathrm{Tr}X[X,Y]Y\Big] \tag{7.4.62}$$

7.4.5 A shorter algorithm

In the previous sections we have given a method to derive an orthogonal SEP-compatible basis $S^{BPS}_{\Lambda,M_\Lambda,p,\nu}$. This method goes through two orthogonalisation procedures, applying $\mathcal{G}_N$ inbetween. The first step used the S_n inner product and involved all the partitions, the second used the $\mathcal{F}$-weighted inner product and only included those partitions with $l(p) \leq N$. This means the partitions with $l(p) \leq N$ are orthogonalised among each other twice. We now prove that one may simplify the first orthogonalisation procedure to only orthogonalise against those p with $l(p) > N$ and still obtain the same final output. This simplifies the computational requirements, and is also conceptually simpler, for reasons that will be outlined below.

To prove this streamlined procedure produces the same BPS operators, we first recall some useful facts.

General properties of Gram-Schmidt orthogonalisation

Consider two bases $\{v_i\}$ and $\{e_i\}$ of a vector space V, where the second is orthonormal. Then orthogonalising v_i results in the basis e_i (up to normalisation constants) if and only if the matrix connecting the two is lower diagonal.

$$\{v_i\} \xrightarrow{\text{GS}} \{e_i\} \qquad \Longleftrightarrow \qquad e_i = \sum_{j \geq i} A_i^j v_j \tag{7.4.63}$$

Note that the orthogonalisation process here is the opposite way round to the standard Gram-Schmidt orthogonalisation one would find in a textbook. Ordinarily, one starts with minimal i and proceeds to larger i, and therefore obtains an upper diagonal A matrix. In (7.4.63), we start with maximal i and decrease, since this is the approach taken in (7.4.8).

Now introduce another basis $\{u_i\}$ for V, not necessarily orthogonal. This is related to $\{v_i\}$ by a a matrix B_i^j

$$u_i = \sum_i B_i^j v_j \tag{7.4.64}$$

Then it follows from (7.4.63) that if B_i^j is lower diagonal, $\{u_i\}$ and $\{v_i\}$ orthogonalise to the same orthonormal basis $\{e_i\}$ (up to normalisation constants).

Back to SEP-compatible bases

As explained below (7.4.3), the S_n orthogonalisation of $\mathcal{M}_N^{\leq}$ against $\mathcal{M}_N^{>}$ gives us pre-BPS operators. The continued S_n orthogonalisation among partitions with $l(p) \leq N$ gives us SEP-compatibility, but is not required for the operators to be pre-BPS.

To split these two steps us, define the pre-BPS basis $\bar{M}_{\Lambda,M_\Lambda,p,\nu}$ by

$$\bar{M}_{\Lambda,M_\Lambda,p^*_\Lambda} = M_{\Lambda,M_\Lambda,p^*_\Lambda} \tag{7.4.65}$$

$$\bar{M}_{\Lambda,M_\Lambda,p,\nu} = M_{\Lambda,M_\Lambda,p,\nu} - \sum_{\substack{(q,\eta)>(p,\nu) \\ l(q)>N}} \frac{\left\langle M_{\Lambda,M_\Lambda,p,\nu}, \bar{M}_{\Lambda,M_\Lambda,q,\eta} \right\rangle_{S_n}}{\left\langle \bar{M}_{\Lambda,M_\Lambda,q,\eta}, \bar{M}_{\Lambda,M_\Lambda,q,\eta} \right\rangle_{S_n}} \bar{M}_{\Lambda,M_\Lambda,q,\eta} \tag{7.4.66}$$

where for p with $l(p) > N$, we have the same caveats as mentioned around (7.4.8) regarding vanishing of operators. The $\bar{M}_{\Lambda,M_\Lambda,p,\nu}$ operators were briefly mentioned above (7.4.4). The corresponding BPS operators, normalised in the S_n inner product, are

$$\bar{M}^{BPS}_{\Lambda,M_\Lambda,p,\nu} = \frac{\mathcal{G}_N \bar{M}_{\Lambda,M_\Lambda,p,\nu}}{\left|\mathcal{G}_N \bar{M}_{\Lambda,M_\Lambda,p,\nu}\right|_{S_n}} \tag{7.4.67}$$

From their construction, orthogonalising $\bar{M}_{\Lambda,M_\Lambda,p,\nu}$ all the way down the partitions using the S_n inner product would result in the S_n orthogonal basis $Q_{\Lambda,M_\Lambda,p,\nu}$ for pre-

BPS operators. From (7.4.63), these two bases are related by a lower diagonal matrix. Applying $\mathcal{G}_N$ and S_n normalising both bases, the equivalent BPS operators are also related by a (rescaled) lower diagonal matrix. Then from the discussion below (7.4.64), it follows that orthogonalising the two bases in the physical $\mathcal{F}$-weighted inner product will result in the same final basis $S^{BPS}_{\Lambda,M_\Lambda,p,\nu}$.

Therefore one may obtain the $S^{BPS}_{\Lambda,M_\Lambda,p,\nu}$ basis in a simpler manner by $\mathcal{F}$-orthogonalising the $\bar{M}_{\Lambda,M_\Lambda,p,\nu}$ basis, much as we did in (7.4.53-7.4.55)

$$\bar{S}^{BPS}_{\Lambda,M_\Lambda,p^*_{\Lambda;N}} = \bar{M}^{BPS}_{\Lambda,M_\Lambda,p^*_{\Lambda;N}} \tag{7.4.68}$$

$$\bar{S}^{BPS}_{\Lambda,M_\Lambda,p,\nu} = \bar{M}^{BPS}_{\Lambda,M_\Lambda,p,\nu} - \sum_{\substack{(q,\eta)>(p,\nu)\\ l(q)\leq N}} \frac{\left\langle \bar{M}^{BPS}_{\Lambda,M_\Lambda,p,\nu}, \bar{S}^{BPS}_{\Lambda,M_\Lambda,q,\eta}\right\rangle_{\mathcal{F}}}{\left\langle \bar{S}^{BPS}_{\Lambda,M_\Lambda,q,\eta}, \bar{S}^{BPS}_{\Lambda,M_\Lambda,q,\eta}\right\rangle_{\mathcal{F}}} \bar{S}^{BPS}_{\Lambda,M_\Lambda,q,\eta} \tag{7.4.69}$$

$$S^{BPS}_{\Lambda,M_\Lambda,p,\nu} = \frac{\bar{S}^{BPS}_{\Lambda,M_\Lambda,p,\nu}}{\sqrt{\left\langle \bar{S}^{BPS}_{\Lambda,M_\Lambda,p,\nu} | \bar{S}^{BPS}_{\Lambda,M_\Lambda,p,\nu}\right\rangle_{S_n}}} \tag{7.4.70}$$

where $l(p) \leq N$.

This approach to producing the $S^{BPS}_{\Lambda,M_\Lambda,p,\nu}$ operators still involves two orthogonalisation steps, but the first is now computationally less demanding, and can be completely skipped if $N \geq l(p^*_\Lambda)$. Moreover, there is a clearer conceptual separation between the two steps. The first one obtains pre-BPS operators, while the second one finds an SEP-compatible basis.

This process skips the $Q_{\Lambda,M_\Lambda,p,\nu}$ operators, but they still have physical relevance as the planar limit of $S^{BPS}_{\Lambda,M_\Lambda,p,\nu}$, and were mathematically useful in proving the SEP-compatibility of these operators.

7.4.6 Choice of SEP-compatible basis

In the section 7.4.4 we derived an orthogonal SEP-compatible basis of operators for the $\Lambda = [2,2]$ sector. This sector was also investigated in appendix C of [51], and a different orthogonal SEP-compatible basis was found. One can check that the two bases span the same two dimensional space for any $N \geq 3$, and when $N = 2$ we find exact agreement of operators.

The SEP-compatibility determines the behaviour of such a basis for $N = 1, 2$, but for higher N there is a large degree of freedom. Define the $\mathcal{F}$-normalised operators $\widehat{S}^{BPS}_{\Lambda,M_\Lambda,p,\nu}$ for a generic Λ by

$$\widehat{S}^{BPS}_{\Lambda,M_\Lambda,p,\nu} = \frac{S^{BPS}_{\Lambda,M_\Lambda,p,\nu}}{\left\langle S^{BPS}_{\Lambda,M_\Lambda,p,\nu} | S^{BPS}_{\Lambda,M_\Lambda,p,\nu}\right\rangle_{\mathcal{F}}} \tag{7.4.71}$$

Let $c(N)$ and $s(N)$ be two functions of N satisfying

$$c(2) = 1 \qquad s(2) = 0 \qquad c(N)^2 + s(N)^2 = 1 \tag{7.4.72}$$

We can use c and s to rotate the $\Lambda = [2,2]$ $\mathcal{F}$-normalised basis operators to a new configuration

$$\mathcal{O}_1 = c(N)\widehat{S}^{BPS}_{\begin{array}{|c|c|}\hline 1&1\\\hline 2&2\\\hline\end{array},[2,1,1]} + s(N)\widehat{S}_{\begin{array}{|c|c|}\hline 1&1\\\hline 2&2\\\hline\end{array},[2,2]} \tag{7.4.73}$$

$$\mathcal{O}_2 = -s(N)\widehat{S}^{BPS}_{\begin{array}{|c|c|}\hline 1&1\\\hline 2&2\\\hline\end{array},[2,1,1]} + c(N)\widehat{S}_{\begin{array}{|c|c|}\hline 1&1\\\hline 2&2\\\hline\end{array},[2,2]} \tag{7.4.74}$$

To avoid problems with vanishing denominators at $N = 2$, we normalise $\mathcal{O}_1, \mathcal{O}_2$ to have norm 1 in the S_n inner product. These then define an alternative orthogonal, SEP-compatible basis for weak coupling quarter-BPS operators in the $\Lambda = [2,2]$ sector.

As $s(N)$ is determined by $c(N)$, there is effectively a function's worth of freedom in defining an orthogonal SEP-compatible basis. Clearly the vast majority of these will have definitions with far more complicated coefficients than those in (7.4.57) and (7.4.58) (or equivalently (7.4.61) and (7.4.62)). An interesting question is whether we can uniquely characterise a basis by having the 'nicest' coefficients. For example, the coefficients in the basis of [51] are of the form

$$N + 1 \pm \sqrt{2N^2 + 1} \tag{7.4.75}$$

These involve a sum of polynomial and surd terms, whereas the basis (7.4.57) and (7.4.58) has coefficients that are polynomial in N up to an overall normalisation. One possible criterion would be to demand a basis with polynomial coefficients (up to overall normalisation) whose polynomials have minimal degree. If unique, these operators would in some sense be the 'simplest' orthogonal, SEP-compatible basis. It is reasonable to conjecture that $S^{BPS}_{\Lambda,M_\Lambda,p,\nu}$ form this basis.

For more general Λ, take an N-dependent orthogonal rotation matrix $R(\Lambda; N)$ of size $\mathcal{M}_\Lambda \times \mathcal{M}_\Lambda$, where

$$\mathcal{M}_\Lambda = \sum_{\substack{p \vdash n \\ l(p) \leq N}} \mathcal{M}_{\Lambda,p} \tag{7.4.76}$$

When evaluated at $\widehat{N} \leq N$, the matrix should split into diagonal blocks

$$R(\Lambda; \widehat{N}) = \begin{pmatrix} R^{\leq}(\Lambda; \widehat{N}) & 0 \\ 0 & R^{>}(\Lambda; \widehat{N}) \end{pmatrix} \tag{7.4.77}$$

where $R^{>}(\Lambda; \widehat{N})$ rotates those partitions with length $l(p) > \widehat{N}$ among themselves and $R^{\leq}(\Lambda; \widehat{N})$ rotates those partitions with length $l(p) \leq \widehat{N}$ among themselves.

Then consider the S_n normalised versions of the operators

$$\sum_{\substack{q,\eta \\ l(q)\leq N}} R(\Lambda;N)^{p,\nu}_{q,\eta}\, \widehat{S}^{BPS}_{\Lambda,M_\Lambda,p,\nu} \tag{7.4.78}$$

These form an alternative orthogonal, SEP-compatible basis for the weakly coupled quarter-BPS sector.

7.4.7 Physical norms of BPS operators

For the free field operators (3.6.20), the physical norm f_R is a polynomial in N that is closely related to the corresponding Young diagram R. It has mathematical significance as the numerator of the Weyl dimension formula for $U(N)$ representations. We now investigate the physical norms of the weak coupling BPS operators $S^{BPS}_{\Lambda,M_\Lambda,p,\nu}$, beginning with an example at $\Lambda=[2,2]$. These operators are given in (7.4.57) and (7.4.58), and have physical $\mathcal{F}$-weighted norms

$$\left| S^{BPS}_{\begin{smallmatrix}1&1\\2&2\end{smallmatrix},\,[2,1,1]} \right|^2 = \frac{\left(3\,N^2+4\,N-2\right)(N+2)(N+1)N^2(N-1)(N-2)}{P_1} \tag{7.4.79}$$

$$\left| S^{BPS}_{\begin{smallmatrix}1&1\\2&2\end{smallmatrix},\,[2,2]} \right|^2 = \frac{\left(3\,N^2+4\,N-2\right)(N+1)N^2(N-1)}{P_2} \tag{7.4.80}$$

This has two key features that will generalise. Firstly, the liner factors in the norms reflect the SEP-compatibility, enforcing that the first operator vanish for when we lower N to $\widehat{N}=1,2$ and the second operator vanish when we lower N to $\widehat{N}=1$. However, both norms have more linear factors than just those required by SEP-compatibility. For these two p, the numerators contain f_p as a factor. This does not generalise to all p; in (G.2.175) and (G.2.179) we see that the numerators in the norms of $p=[2,2,2]$ and $[3,3]$ operators are one linear factor short of containing f_p. It is unclear whether these are exceptions, or whether at large n, very few operators contain f_p in the numerator of the norm. It would be interesting to understand the linear factors that appear in the numerator and whether these have a physical interpretation.

Secondly, the numerators share a factor of $(3N^2+2N-2)$. In appendices G.1, G.2 and G.3 we see that consecutive partitions (in the ordering (7.4.6)) share a complicated polynomial factor in the numerators. We believe this generalises to larger n, though it may be an artefact of the orthogonalisation process. This is discussed further in appendix G.2.4.

While the numerators of BPS norms have interesting properties, we have not found any structure in the denominators. They arise by dividing through by the square root of the S_n norm, and from our numerical calculations do not seem to factorise into smaller units.

Norms of operators with multiplicity

The norms (7.4.79) and (7.4.80) can be considered as characteristic functions of Λ along with the partitions $[2,1,1]$ and $[2,2]$ respectively, just as the norms of the free field basis are characteristic polynomials of the Young diagrams.

For the free field covariant basis (3.6.20), the physical norms depended only on R and not the $U(2)$ Young diagram Λ or the multiplicity τ. For the weak coupling basis, the norms can now depend on Λ, p and ν. The dependence on Λ and p is completely determined by the construction, while the dependence on ν is dictated by the choice of multiplicity space basis. We now outline a way of extracting functions of N that do not depend on this choice, and are therefore associated to the pair p, Λ.

In (7.4.71), a rescaled BPS basis $\widehat{S}^{BPS}_{\Lambda,M_\Lambda,p,\nu}$ was defined, orthonormal in the physical $\mathcal{F}$-weighted inner product. A different choice of multiplicity basis would result in an orthogonal rotation of these operators, and any trace over the multiplicity index is therefore independent of the this choice.

In particular, consider the matrix of S_n inner products. For a trivial multiplicity space, this would be a 1×1 matrix containing the reciprocal of the norm $|S^{BPS}_{\Lambda,M_\Lambda,p,\nu}|^2$. The appropriate generalisation to non-trivial multiplicity should therefore be to take the reciprocal of the trace, and we should also divide by the dimension of the multiplicity space. So the invariant function is

$$f_{\Lambda,p} = \left(\frac{1}{\mathcal{M}_{\Lambda,p}} \sum_{\nu=1}^{\mathcal{M}_{\Lambda,p}} \left\langle \widehat{S}^{BPS}_{\Lambda,M_\Lambda,p,\nu} | \widehat{S}^{BPS}_{\Lambda,M_\Lambda,p,\nu} \right\rangle \right)^{-1} \tag{7.4.81}$$

Note that the the hermiticity properties of $U(2)$ imply that we can choose any semi-standard tableau M_Λ and it will not affect the calculation.

We can also use the square/cube/... of the S_n inner product matrix to extract further basis-invariant functions of N. Let A be the S_n inner product matrix. Then we have

$$f^{(k)}_{\Lambda,p} = \left(\frac{1}{\mathcal{M}_{\Lambda,p}} \mathrm{Tr} A^k \right)^{-\frac{1}{k}} \tag{7.4.82}$$

Where we have taken the kth root in order to have functions of the same degree in N. This stack of powers only goes up to $k \leq \mathcal{M}_{\Lambda,p}$ before the invariants are no longer independent.

In appendix G.2, we see two examples of non-trivial multiplicity spaces in the $\Lambda = [4,2]$ sector, both of dimension two. In section G.2.3 we show the calculation for $p = [2,2,1,1]$ in some detail, while we are more schematic for $p = [3,2,1]$.

In both examples, the numerator of $f^{(2)}_{\Lambda,p}$ is the same as $f^{(1)}_{\Lambda,p}$, though the denominator is not. As discussed below (7.4.80), this is further evidence that we should only look at the numerators of the BPS norms, giving a single characteristic function for a given Λ, p.

7.4.8 Longest p for a given Λ and explicit quarter-BPS operators

Due to the computational nature of the orthogonalisation process to derive SEP-compatible BPS operators, it is difficult to give explicit formulae for many of the $S^{BPS}_{\Lambda,M_\Lambda,p,\nu}$ operators. The exception is for p^*_Λ, the longest partition with $\mathcal{M}_{\Lambda,p} > 0$. Shortly, we will prove that this has $\mathcal{M}_{\Lambda,p^*_\Lambda} = 1$, so we drop the multiplicity index. Provided $N \geq l\left(p^*_\Lambda\right)$, the operators with partition p^*_Λ do not get orthogonalised, so we have the formula

$$\boxed{S^{BPS}_{\Lambda,M_\Lambda,p^*_\Lambda} = \frac{\mathcal{G}_N M_{\Lambda,M_\Lambda,p^*_\Lambda}}{\left|\mathcal{G}_N M_{\Lambda,M_\Lambda,p^*_\Lambda}\right|_{S_n}}} \tag{7.4.83}$$

where

$$M_{\Lambda,M_\Lambda,p^*_\Lambda} = \sum_{\mathbf{p}\vdash(n_1,n_2)\,:\,p(\mathbf{p})=p^*_\Lambda} B^{\mathbf{p}}_{\Lambda,M_\Lambda,p^*_\Lambda} M_{\mathbf{p}} \tag{7.4.84}$$

$$= \sum_{\substack{\mathbf{p}\vdash(n_1,n_2)\,:\,p(\mathbf{p})=p^*_\Lambda \\ \mathbf{q}\vdash(n_1,n_2)}} B^{\mathbf{p}}_{\Lambda,M_\Lambda,p^*_\Lambda} \tilde{C}^{\mathbf{q}}_{\mathbf{p}} T_{\mathbf{q}} \tag{7.4.85}$$

and we have used (7.3.13) to express $M_{\Lambda,M_\Lambda,p^*_\Lambda}$ in terms of $M_{\mathbf{p}}$, (7.2.23) to write $M_{\mathbf{p}}$ in terms of $T_{\mathbf{q}}$, and $T_{\mathbf{q}}$ is the symmetrised trace operator (7.1.11).

Explicit formulae for other $S^{BPS}_{\Lambda,M_\Lambda,p,\nu}$ operators in each Λ sector are much more difficult to write down as they involve first orthogonalising down the partitions. Of course one may find non-orthogonal BPS operators by applying $\mathcal{G}_N$ to the covariant monomials $M_{\Lambda,M_\Lambda,p,\nu}$ (provided $N \geq l\left(p^*_\Lambda\right)$).

We can use the results of sections 7.3.5 and 7.3.6 to find p^*_Λ explicitly. As discussed at the end of section 7.3.6, we consider a partition $p = [1^{\mu_1}] + \bar{p}$ with a single dominant column of length μ_1 attached to a smaller partition $\bar{p} \vdash \bar{n} = n - \mu_1$. By rectangle rotations, this has the same multiplicities as a single dominant row partition with first row of length $\lambda_1 = \mu_1 - l(\bar{p})$ above a smaller partition $\hat{p} = [1^{l(\bar{p})}] + \bar{p}$, where $\hat{p} \vdash \hat{n} = l(\bar{p}) + \bar{n}$ and we are using the notation of section 7.3.6. We give an example of these relations between single dominant column and single dominant row partitions in (7.3.96).

Applying the general formula (7.3.94) for a partition with a single dominant row, we see that $\Lambda = [\Lambda_1, \Lambda_2]$ only has non-trivial multiplicity with p if

$$\Lambda_2 \leq n - \mu_1 + l(\bar{p}) \tag{7.4.86}$$

Rearranging to constrain μ_1 in terms of Λ and $\bar{p}$

$$\mu_1 \leq \Lambda_1 + l(\bar{p}) \tag{7.4.87}$$

Not only does (7.3.94) give this constraint on μ_1, it also tells us that the multiplicity is 1 if the inequality is saturated. If it is not quite saturated, and instead $\mu_1 = \Lambda_1 + l(\bar{p}) - 1$, then the multiplicity is the number of corners in p minus 1. Since p has a dominant first column, this is just the number of corners in $\bar{p}$.

Therefore the maximum possible μ_1 is obtained when $l(\bar{p})$ is at its largest. This occurs when $\bar{p} = [1^{\bar{n}}]$ and $l(\bar{p}) = \bar{n} = n - \mu_1$. Plugging this in, we have

$$\mu_1 \leq n - \frac{\Lambda_2}{2} \tag{7.4.88}$$

Therefore the maximal μ_1 is $n - \left\lceil \frac{\Lambda_2}{2} \right\rceil$, with associated $\bar{p} = \left[1^{\left\lceil \frac{\Lambda_2}{2} \right\rceil}\right]$. If Λ_2 is even then (7.4.87) is saturated and the multiplicity is 1. If Λ_2 is odd, then the multiplicity is the number of corners in $\bar{p}$, which is also 1. These multiplicities agree with the explicit calculation for two column partitions in (7.3.82).

Stated fully, for $\Lambda = [\Lambda_1, \Lambda_2] \vdash n$, the longest p with non-trivial multiplicity is

$$p^*_\Lambda = \left[n - \left\lceil \frac{\Lambda_2}{2} \right\rceil, \left\lceil \frac{\Lambda_2}{2} \right\rceil\right]^c = \left[2^{\left\lceil \frac{\Lambda_2}{2} \right\rceil}, 1^{n-2\left\lceil \frac{\Lambda_2}{2} \right\rceil}\right] \tag{7.4.89}$$

and this multiplicity is 1.

7.4.9 Orthogonalisation at $\Lambda = [n]$ and $[n-1, 1]$

In section 7.4.2, we observed that Gram-Schmidt orthogonalising the $M_{\Lambda, M_\Lambda, p, \nu}$ with $\Lambda = [4]$ or $[3, 1]$ in the S_n inner product led to the free field operators $\mathcal{O}_{\Lambda, M_\Lambda, R, \tau}$. We now prove that this behaviour is general for $\Lambda = [n]$, and motivate a conjecture that this also happens for $\Lambda = [n-1, 1]$.

For this subsection, when we use Λ, we will be referring specifically to $\Lambda = [n]$ or $[n-1, 1]$, and stated results will apply only to those Λ.

Recall that for these Λ, a free field BPS operator is also a weak coupling pre-BPS operators. Therefore we have expansions of the form

$$\mathcal{O}_{\Lambda, M_\Lambda, R, \tau} = \sum_{p,\nu} b^{p,\nu}_{R,\tau} M_{\Lambda, M_\Lambda, p, \nu} \qquad M_{\Lambda, M_\Lambda, p, \nu} = \sum_{R,\tau} \left(b^{-1}\right)^{R,\tau}_{p,\nu} \mathcal{O}_{\Lambda, M_\Lambda, R, \tau} \tag{7.4.90}$$

Next, recall from (3.6.28) that the free field operators are orthogonal in the S_n inner product. Therefore from the property (7.4.63) of Gram-Schmidt orthogonalisation, the $M_{\Lambda,M_\Lambda,p,\nu}$ will orthogonalise to $\mathcal{O}_{\Lambda,M_\Lambda,R,\tau}$ if and only if $b^{p,\nu}_{R,\tau}$ is lower diagonal. That is, if the coefficients satisfy

$$b^{p,\nu}_{R,\tau} = 0 \quad \text{if} \quad (p,\nu) < (R,\tau) \tag{7.4.91}$$

where the comparison between multiplicities makes sense since the size of the multiplicity space for a given $p = R$ is the same for the two bases. This is proved in (7.3.66).

More generally, since neither basis specifies a choice of multiplicity space basis, it is sufficient to prove that

$$b^{p,\nu}_{R,\tau} = 0 \quad \text{if} \quad p < R \tag{7.4.92}$$

then after choosing the multiplicity space bases appropriately, (7.4.91) and the result will follow.

$\Lambda = [n]$

In the $\Lambda = [n]$ sector at field content $(n,0)$, the covariant monomials $M_{\Lambda,M_\Lambda,p,\nu}$ reduce to the monomial symmetric functions M_p defined in (2.7.6), while the free field operators $\mathcal{O}_{\Lambda,M_\Lambda,R,\tau}$ reduce to the Schur operators s_R defined in terms of monomials in (2.7.12). From these definitions, the two bases are related by the (rescaled) Kostka numbers

$$s_R = \sum_{p \vdash n} \frac{K_{Rp}}{\prod_i p_i!} M_p \tag{7.4.93}$$

The Kostka numbers are the number of semi-standard Young tableaux of shape R and evaluation p, where these terms are defined in section 3.6.2. To prove (7.4.92) in this case we need

$$K_{Rp} = 0 \qquad \text{for} \qquad R > p \tag{7.4.94}$$

Consider $R > p$ with column lengths $R^c = [\lambda_1, \lambda_2, \ldots]$ and $p^c = [\rho_1, \rho_2, \ldots]$. By definition there is some l for which $\lambda_i = \rho_i$ for all $i < l$ and $\lambda_l > \rho_l$. Now take a semi-standard Young tableaux of shape R and evaluation p. The entries in the first column must strictly increase, so the entry at the bottom of the first column is $\geq \lambda_1$. Since the evaluation is p, the available numbers to use are $1, 2, \ldots, \rho_1 = \lambda_1$, so we must fill this column with exactly the numbers 1 to λ_1. Similarly the second column must be filled with the numbers 1 to λ_2 and so on until we reach the lth column. At this point, the entry at the bottom of the lth column must be $\geq \lambda_l$, while the maximum available number to use is $\rho_l < \lambda_l$. So the Young tableaux cannot have evaluation p, and therefore $K_{Rp} = 0$.

This proves (7.4.92) for $\Lambda = [n]$ and the highest weight state M_Λ. Applying the $U(2)$ lowering operator $\mathcal{J}_-$, the same will happen for any M_Λ within the $\Lambda = [n]$ sector, and this gives the result.

$\Lambda = [n-1, 1]$

For $\Lambda = [n-1, 1]$, there are two principal reasons we might expect $Q_{\Lambda,M_\Lambda,p,\nu}$ to agree with $\mathcal{O}_{\Lambda,M_\Lambda,R,\tau}$. Firstly, we know both bases are SEP-compatible. Since the space of operators that vanish when $N \to N-1$ is well-defined, it follows that for a fixed length $k = l(p) = l(R)$, the two bases must have the same span

$$\text{Span}\left\{S_{\Lambda,M_\Lambda,p,\nu} : l(p) = k\right\} = \text{Span}\left\{\mathcal{O}_{\Lambda,M_\Lambda,R,\tau} : l(R) = k\right\} \tag{7.4.95}$$

Secondly, from (7.4.91) the multiplicity for a given p matches the multiplicity for $R = p$. Mathematically

$$\text{Dim}\Big(\text{Span}\left\{S_{\Lambda,M_\Lambda,p,\nu} : 1 \leq \nu \leq \mathcal{M}_{\Lambda,p}\right\}\Big) = \text{Dim}\Big(\text{Span}\left\{\mathcal{O}_{\Lambda,M_\Lambda,R,\tau} : 1 \leq \tau \leq C(R,R,\Lambda)\right\}\Big) \tag{7.4.96}$$

A rigorous proof that the $Q_{\Lambda,M_\Lambda,p,\nu}$ and $\mathcal{O}_{\Lambda,M_\Lambda,R,\tau}$ operators match is more difficult. Numerical calculations indicate it holds true up to at least $n = 7$, and we leave the general case as a conjecture.

7.4.10 Alternative algorithm

There is an alternative approach to capturing the finite N behaviour of the pre-BPS sector starting from the free field operators. Following a similar process to that given in sections 7.4.1-7.4.5, one may use this to derive an orthogonal SEP-compatible basis of BPS operators. This alternative algorithm is outlined in figure 7.3. At first glance, there is no reason to expect agreement between this and the $S^{BPS}_{\Lambda,M_\Lambda,p,\nu}$ basis defined in (7.4.68). However our numerical calculations show that they do agree up to $n = 6$, and we conjecture that this is a general result.

Start by considering the free field basis $\mathcal{O}_{\Lambda,M_\Lambda,R,\tau}$. This has a symmetrised trace component and a commutator trace component. As discussed around (7.3.38), there is a $U(2)$-covariant basis for commutator traces that we will denote by $c_{\Lambda,M_\Lambda,p,\xi}$, where $p \vdash n$ is a partition that describes the trace structure of the commutator trace and ξ is a multiplicity index. For an example of these operators see (G.2.26-G.2.30), where we give the highest weight states in the $\Lambda = [4, 2]$ covariant commutator trace basis.

Then $\mathcal{O}_{\Lambda,M_\Lambda,R,\tau}$ can be written

$$\mathcal{O}_{\Lambda,M_\Lambda,R,\tau} = \sum_{p,\nu} b^{p,\nu}_{R,\tau} M_{\Lambda,M_\Lambda,p,\nu} + \sum_{p,\xi} d^{p,\xi}_{R,\tau} c_{\Lambda,M_\Lambda,p,\xi} \tag{7.4.97}$$

The coefficients $b^{p,\nu}_{R,\tau}$ are a generalisation of those seen in (7.4.90) to generic Λ.

The expansion coefficients in (7.4.97) are only defined uniquely at $N \geq n$. For $N < n$, finite N relations make the choice non-unique. We will choose to use the large N coefficient, even when working at $N < n$. These coefficients are valid for all N and are independent of N.

After removing the commutator trace component, we are left with

$$\mathcal{O}^{symm}_{\Lambda,M_\Lambda,R,\tau} = \mathcal{O}_{\Lambda,M_\Lambda,R,\tau} - \sum_{p,\xi} d^{p,\xi}_{R,\tau} c_{\Lambda,M_\Lambda,p,\xi} = \sum_{p,\nu} b^{p,\nu}_{R,\tau} M_{\Lambda,M_\Lambda,p,\nu} \tag{7.4.98}$$

which is a redundant spanning set for symmetrised traces. These operators were considered in [51], where they were referred to as $\mathcal{O}^{S}_{\Lambda,M_\Lambda,R,\tau}$.

Since $\mathcal{O}^{symm}_{\Lambda,M_\Lambda,R,\tau}$ are symmetrised traces, they form pre-BPS operators for $N \geq n$. We also know from the construction that if $N < l(R)$, the operator $\mathcal{O}^{symm}_{\Lambda,M_\Lambda,R,\tau}$ reduces to a commutator trace. However, if $l(R) \leq N < n$, it is not necessarily true that $\mathcal{O}^{symm}$ is orthogonal to all commutator traces. This is the same situation as the monomial basis, discussed in section 7.1.1.

We may therefore use the same processes described in sections 7.4.1-7.4.5 in order to find an orthogonal SEP-compatible basis for BPS operators. In this section, we sill use the route given in sections 7.4.1-7.4.4 rather than the shorter one from section 7.4.5.

In particular, we produce a basis $\mathcal{O}^{orth}_{\Lambda,M_\Lambda,R,\rho}$ by following the S_n-orthogonalisation procedure (7.4.7-7.4.9). We then apply $\mathcal{G}_N$, $\mathcal{F}$-orthogonalise the resulting operators and S_n-normalise. The final basis is then denoted by $\mathcal{O}^{BPS}_{\Lambda,M_\Lambda,R,\rho}$. This algorithm is outlined in figure 7.3.

As the operators $\mathcal{O}^{symm}_{\Lambda,M_\Lambda,R,\tau}$ are linearly dependent, some of the them will vanish during the orthogonalisation process. Unlike the orthogonalisation of monomials, this can occur for both $l(R) > N$ and $l(R) \leq N$. At such a point, remove that operator and continue with the orthogonalisation. This means the multiplicities for each pair Λ, R with $l(R) \leq N$ could reduce, and in some cases will reduce to zero. Denote the reduced multiplicity for a pair by $\mathcal{M}^{orth}_{\Lambda,R}$. To indicate this reduction, we use a multiplicity index ρ for the orthogonalised operators rather than τ.

The $\mathcal{O}^{orth}_{\Lambda,M_\Lambda,R,\rho}$ with $l(R) \leq N$ form a S_n-orthogonal SEP-compatible basis for pre-BPS operators. The Λ, M_Λ labels match the equivalents in $Q_{\Lambda,M_\Lambda,p,\nu}$, and by similar reasoning to (7.4.95), the length of R must match the length of p. However, a priori, there is no reason to suspect that the R label should match the p label, or even that the multiplicities should be the same for any given partition.

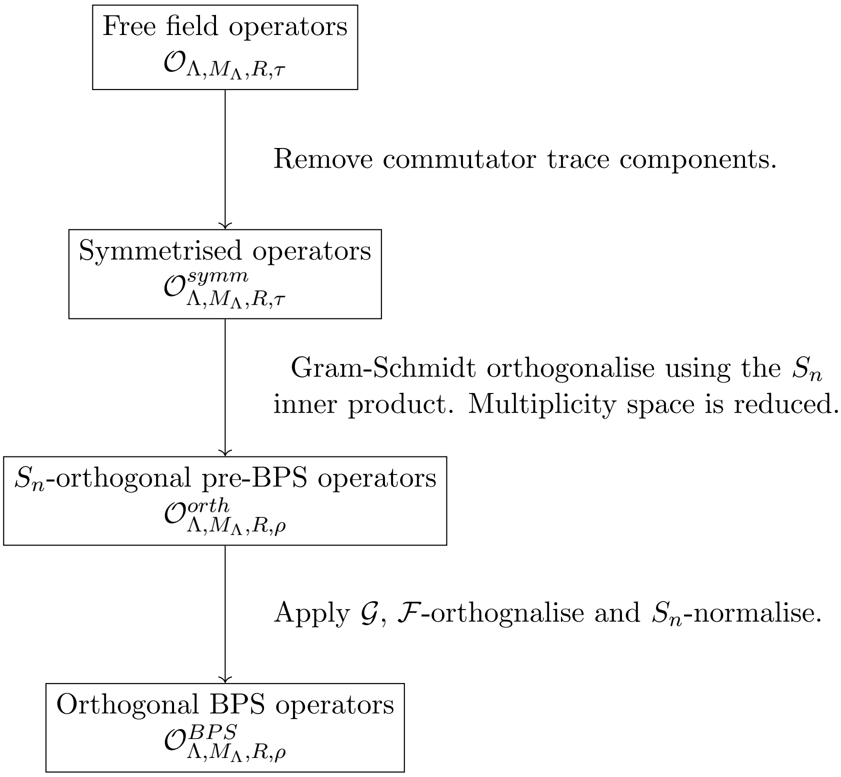


Figure 7.3: Outline of the alternative algorithm of this section. Our numerical calculations suggest that $\mathcal{O}^{BPS}_{\Lambda,M_\Lambda,R,\rho}$ agrees (up to a choice of multiplicity basis) with the operators $S^{BPS}_{\Lambda,M_\Lambda,p,\nu}$ derived from the algorithm in figure 7.1.

From our numerical calculations up to $n = 6$, we find that for each pair Λ, R the multiplicities match, and the span of the operators with those labels is the same. Mathematically,

$$\mathcal{M}^{orth}_{\Lambda,R} = \mathcal{M}_{\Lambda,p=R} \tag{7.4.99}$$

$$\text{Span}\left\{\mathcal{O}^{orth}_{\Lambda,M_\Lambda,R,\rho} : 1 \leq \rho \leq \mathcal{M}^{orth}_{\Lambda,R}\right\} = \text{Span}\left\{Q_{\Lambda,M_\Lambda,p=R,\nu} : 1 \leq \nu \leq \mathcal{M}_{\Lambda,p}\right\} \tag{7.4.100}$$

We conjecture that this is a general result for all Λ, R.

From this, it clearly follows that the BPS bases $\mathcal{O}^{BPS}_{\Lambda,M_\Lambda,R,\rho}$ and $S^{BPS}_{\Lambda,M_\Lambda,p,\nu}$ also match.

We can consider (7.4.99) and (7.4.100) as a generalisation to all Λ of the orthogonalisation results discussed in section 7.4.9 for $\Lambda = [n]$ and $[n-1,1]$. In that case, we showed that to prove the results, it was sufficient for the coefficients in (7.4.90) to be lower diagonal in partition indices. We now prove a proposition that generalises this lower diagonality to arbitrary Λ.

The proposition involves the coefficients $b^{p,\nu}_{R,\tau}$ in (7.4.98). Consider the sub-matrix of $b^{p,\nu}_{R,\tau}$ with $p, R \geq q$ for some partition q and denote this by $(b_q)^{p,\nu}_{R,\tau}$. Then b_q is an

$m_{q;1} \times m_{q;2}$ matrix where

$$m_{q;1} = \sum_{p \geq q} \mathcal{M}_{\Lambda,p} \qquad m_{q;2} = \sum_{R \geq q} C(R,R,\Lambda) \tag{7.4.101}$$

Denote the rank of b_q by r_q. This satisfies $r_q \leq m_{q;1}$.

Proposition

Suppose the coefficients $b^{p,\nu}_{R,\tau}$ are lower diagonal in the partition indices, so that

$$\mathcal{O}^{symm}_{\Lambda,M_\Lambda,R,\tau} = \sum_{\substack{p,\nu \\ p \geq R}} b^{p,\nu}_{R,\tau} M_{\Lambda,M_\Lambda,p,\nu} \tag{7.4.102}$$

In addition, suppose that b_q has maximal rank $r_q = m_{q;1}$ for each q. Then (7.4.99) and (7.4.100) hold.

Proof

Define

$$V_R = \text{Span}\{\mathcal{O}^{symm}_{\Lambda,M_\Lambda,S,\tau} : S \geq R, 1 \leq \tau \leq C(S,S,\Lambda)\} \tag{7.4.103}$$
$$\widetilde{V}_R = \text{Span}\{M_{\Lambda,M_\Lambda,p,\nu} : p \geq R, 1 \leq \nu \leq \mathcal{M}_{\Lambda,p}\} \tag{7.4.104}$$

It follows from (7.4.102) and the maximal rank condition for b_R that $V_R = \widetilde{V}_R$.

Denote the partition immediately higher than R by $R+1$. By construction, for R with $l(R) \leq N$, the additional monomials included in $\widetilde{V}_R$ by lowering $R+1$ to R are linearly independent, and therefore the dimension increases by $\mathcal{M}_{\Lambda,p=R}$.

For V_R, we have $C(R,R,\Lambda)$ new operators included by lowering $R+1$ to R, but the dimension only increases by $\mathcal{M}^{orth}_{\Lambda,R}$. Then since $V_R = \widetilde{V}_R$, we have

$$\mathcal{M}^{orth}_{\Lambda,R} = \mathcal{M}_{\Lambda,p=R} \tag{7.4.105}$$

Therefore for R with $l(R) \leq N$ we may choose the multiplicity space basis such that $\mathcal{O}^{symm}_{\Lambda,M_\Lambda,R,\tau}$ is linearly independent of V_{R+1} if $1 \leq \tau \leq \mathcal{M}_{\Lambda,p=R}$, and is linearly dependent on V_{R+1} if $\tau > \mathcal{M}_{\Lambda,p=R}$. Under the orthogonalisation procedure, those operators with $\tau > \mathcal{M}_{\Lambda,p=R}$ will vanish. We can therefore equivalently start the procedure with a reduced set of operators $\mathcal{O}^{symm}_{\Lambda,M_\Lambda,R,\rho}$ where we only consider $1 \leq \rho \leq \mathcal{M}_{\Lambda,p=R}$. The coefficients relating these with $M_{\Lambda,M_\Lambda,p,\nu}$ are still lower diagonal in partition indices.

Now split the orthogonalisation into two steps, first orthogonalising against the p or R with $l(p), l(R) > N$ using the S_n inner product. For the monomials, this results in the $\bar{M}\Lambda, M_\Lambda, p, \nu$ operators given in (7.4.66). Let the equivalent operators for the $\mathcal{O}^{symm}$ orthogonalisation be $\bar{O}^{symm}_{\Lambda,M_\Lambda,R,\rho}$. These are both basis for the set of pre-BPS operators, and are still related by coefficients that are lower diagonal in partition indices. Therefore, as discussed below (7.4.64), they must orthogonalise to the same basis, in particular

$$\mathcal{O}^{orth}_{\Lambda,M_\Lambda,R,\rho} = Q_{\Lambda,M_\Lambda,p=R,\nu=\rho} \tag{7.4.106}$$

□

We have proved that (7.4.102), along with the maximal rank condition, is sufficient for (7.4.99) and (7.4.100). By a similar argument, it is also a necessary condition, though we will not prove this here.

The maximal rank condition $r_q = m_{q;1}$ is equivalent to saying that we can choose bases for the free field and covariant monomial multiplicity spaces such that

$$b^{p;\nu}_{R=p,\tau} = 0 \qquad \text{if } \tau > \nu \tag{7.4.107}$$

$$b^{p;\nu}_{R=p,\tau} \neq 0 \qquad \text{if } \tau = \nu \tag{7.4.108}$$

where $1 \leq \nu \leq \mathcal{M}_{\Lambda,p}$ and $1 \leq \tau \leq C(p,p,\Lambda)$. Intuitively, this says that b is lower diagonal in the multiplicity block with non-zero elements on the diagonal.

The coefficients $b^{p;\nu}_{R,\tau}$ are in some sense a covariant generalisation of the Kostka numbers, which have a nice combinatoric interpretation. It would be interesting to investigate whether there is a choice of normalisation for $\mathcal{O}^{symm}_{\Lambda,M_\Lambda,p,\tau}$ and $M_{\Lambda,M_\Lambda,p,\nu}$ such that these coefficients are integers, and whether they have any combinatoric interpretation.

7.5 Vector space Geometry in $\mathbb{C}(S_n)$: BPS states from Projectors for the intersection of finite N and symmetrisation constraints in symmetric group algebras

The construction algorithm for quarter BPS states in section 7.4 involves a $U(2)$ global symmetry which provides labels for the states constructed. Alongside the $U(2)$ state labels, there is a $U(N)$ Young diagram $Y(p)$ which emerges from the combinatorics of multi-symmetric functions and their relation to the space of gauge invariant 2-matrix operators modulo commutators $[X,Y]$. We have observed in Section 7.2.5 that the combinatorics of multi-symmetric functions admits a generalization to the multi-matrix case where we have M different matrices $X^1, X^2, \cdots, X^M$. In this section we take a different viewpoint on the M-matrix system, using permutations to describe these operators as explained in chapter 2.

This is used to investigate the vector space geometry in $\mathbb{C}(S_n)$ that lies behind the constructions of BPS bases in the previous sections, involving the interplay between a projector $\mathcal{P}_H$ for the $U(M)$ flavour symmetry, a projector for the symmetrisation of traces $\mathcal{P}$ and an operator $\mathcal{F}_N$ whose kernel implements finite N constraints. Restricting to the image of $\mathcal{F}_N$, there is a well-defined inverse $\mathcal{G}_N$. These operators are M-matrix analogues of $\mathcal{F}_N$ and $\mathcal{G}_N$ discussed in section 7.1. It was proved in [51] that BPS states are in

$$\mathrm{Im}(\mathcal{G}_N\mathcal{P}_N) \tag{7.5.1}$$

where $\mathcal{P}_N$ is an orthogonal (with respect to the S_n inner product) projector acting on $\mathbb{C}(S_n)$ with

$$\mathrm{Im}\,\mathcal{P}_N \equiv \mathrm{Im}\,\mathcal{P} \cap \mathrm{Im}\,\mathcal{F}_N \tag{7.5.2}$$

The isomorphism between multi-symmetric functions and the ring of gauge invariants modulo commutators and the associated combinatorics of set partitions explained in section 7.2 allows us to give a general explicit construction of $\mathcal{P}_N$. This general discussion also serves to explain why the construction algorithm in section 7.4 is able to handle the finite N constraints on BPS operators systematically. The flavour projection $\mathcal{P}_H$, for any chosen flavour group H, commutes with $\mathcal{P}$ and $\mathcal{F}_N$ and can be done at the end.

7.5.1 Finite N relations and flavour projection in $\mathbb{C}(S_n)$

In section 2.1 we explained the vector space isomorphism between permutations in $\mathbb{C}(S_n)$ and multi-traces of n matrices of size $N \times N$ for $N \geq n$. Explicitly

$$\sigma \leftrightarrow \mathcal{O}_\sigma = \mathrm{Tr}(\sigma\mathbb{Z}) \tag{7.5.3}$$

Further, in section 2.5 we described how to deal with $N < n$ by removing the Fourier basis elements β^R_{ij} with $l(R) > N$. In this section we use a slightly different notation. Define $\mathcal{F}_N$ and $\mathcal{G}_N$ on $\mathbb{C}(S_n)$ by

$$\mathcal{F}_N\sigma = \Omega_N\sigma \qquad\qquad \mathcal{G}_N\sigma = \Omega_N^{-1}\sigma \tag{7.5.4}$$

These are the n-matrix analogues of (7.1.2) and (7.1.5). Then the image and kernel of $\mathcal{F}_N$ exactly splits $\mathbb{C}(S_n)$ into those permutations that survive the finite N cut-off and those that don't

$$\mathrm{Im}\,\mathcal{F}_N = \mathrm{Span}\{\beta^R_{IJ} : l(R) \leq N\} \tag{7.5.5}$$

$$\operatorname{Ker}\mathcal{F}_N = \operatorname{Span}\{\beta^R_{IJ} : l(R) > N\} \tag{7.5.6}$$

So (7.5.3) gives a vector space isomorphism between $\operatorname{Im}\mathcal{F}_N$ (or the quotient space $\mathbb{C}(S_n)/\operatorname{Ker}\mathcal{F}_N$) and multi-traces of n matrices of size $N \times N$, where N is now unrestricted.

There is a natural inner product on $\mathbb{C}(S_n)$. On permutations $\sigma, \tau \in S_n$ it is defined by

$$g_n(\sigma, \tau) = \delta\left(\sigma\tau^{-1}\right) \tag{7.5.7}$$

If $N \geq n$, this corresponds to the S_n inner product (3.6.26) when mapped to operators using (7.5.3). For $N < n$ we introduce a different inner product on $\mathbb{C}(S_n)$ given by

$$g_{n,N}(\sigma, \tau) = \delta_N(\sigma\tau^{-1}) \tag{7.5.8}$$

This corresponds to the S_n inner product for any N, including $N < n$.

Note that if $N < n$, $g_{n,N}$ is a degenerate inner product on $\mathbb{C}(S_n)$. It vanishes on $\operatorname{Ker}\mathcal{F}_N$, and is identical to g_n on $\operatorname{Im}\mathcal{F}_N$. In particular, for an element $\alpha \in \operatorname{Im}\mathcal{F}_N$, we have

$$g_{n,N}(\alpha, \tau) = g_n(\alpha, \tau) \tag{7.5.9}$$

for any $\tau \in S_n$.

It will be useful late to note that the Fourier basis elements are orthogonal in the g_n inner product (from standard character orthogonality relations), and therefore

$$(\operatorname{Im}\mathcal{F}_N)^\perp = \operatorname{Ker}\mathcal{F}_N \tag{7.5.10}$$

where $(.)^\perp$ denotes the orthogonal complement in the g_n inner product.

In (2.1.9) we introduced the sub-algebra $\mathcal{A}_H$ of $\mathbb{C}(S_n)$ that describes the degree $(n_1, n_2, \ldots, n_M)$ subspace of an M-matrix system. Define the flavour projector $\mathcal{P}_H$ onto $\mathcal{A}_H$ by

$$\mathcal{P}_H(\sigma) = \frac{1}{|H|}\sum_{\tau \in H} \tau\sigma\tau^{-1} \tag{7.5.11}$$

To check this is indeed a projector, we prove that it has the two properties

$$(\mathcal{P}_H)^2 = \mathcal{P}_H \qquad\qquad (\mathcal{P}_H)^\dagger = \mathcal{P}_H \tag{7.5.12}$$

where the Hermitian conjugate is with respect to the g_n inner product. These are both simple consequences of the definition. We have

$$(\mathcal{P}_H)^2(\alpha) = \mathcal{P}_H(\mathcal{P}_H(\alpha))$$

$$
\begin{aligned}
&= \frac{1}{|H|^2} \sum_{\sigma,\tau \in H} \sigma\tau\alpha\tau^{-1}\sigma^{-1} \\
&= \frac{1}{|H|^2} \sum_{\sigma,\tau \in H} \sigma\alpha\sigma^{-1} \\
&= \frac{1}{|H|} \sum_{\mu \in H} \sigma\alpha\sigma^{-1} \\
&= \mathcal{P}_H(\alpha)
\end{aligned} \tag{7.5.13}
$$

and

$$
\begin{aligned}
g(\alpha, \mathcal{P}_H(\beta)) &= \frac{1}{|H|} \sum_{\sigma \in H} \delta(\alpha\sigma\beta^{-1}\sigma^{-1}) \\
&= \frac{1}{|H|} \sum_{\sigma \in H} \delta(\sigma^{-1}\alpha\sigma\beta^{-1}) \\
&= \frac{1}{|H|} \sum_{\sigma \in H} \delta(\sigma\alpha\sigma^{-1}\beta^{-1}) \\
&= g(\mathcal{P}_H(\alpha), \beta)
\end{aligned} \tag{7.5.14}
$$

Finally, we check that $\mathcal{A}_H$ is the image of $\mathcal{P}_H$. Any $\alpha \in \mathcal{A}_H$ is invariant under conjugation by $\tau \in H$, and therefore $\mathcal{P}_H(\alpha) = \alpha$. Conversely, for any $\alpha \in \mathbb{C}(S_n)$, we have

$$
\tau\mathcal{P}_H(\alpha)\tau^{-1} = \tau \sum_{\sigma \in H} \sigma\alpha\sigma^{-1}\tau^{-1} = \sum_{\sigma \in H} \sigma\alpha\sigma^{-1} = \mathcal{P}_H(\alpha) \tag{7.5.15}
$$

and therefore $\mathcal{P}_H(\alpha) \in \mathcal{A}_H$.

The map (7.5.3) gives an isomorphism between $\mathcal{A}_H$ (or the quotient space $\mathbb{C}(S_n)/\operatorname{Ker}\mathcal{P}_H$) and the space of M-matrix multi-traces of degree $(n_1, \ldots, n_M)$.

7.5.2 Symmetrised traces from $\mathbb{C}(S_n)$

A symmetrised trace of $Z_1, Z_2, \ldots, Z_n$ is defined in a completely analogous manner to the 2-matrix version in (7.1.10), allowing $a_i \in \{1, 2, \ldots, n\}$ instead of $\{1, 2\}$. Degree $(1, \ldots, 1)$ symmetrised traces are labelled by set partitions $\pi \in \Pi(n)$. These naturally correspond to n-vector partitions of weight $(1, \ldots, 1)$.

Take $b \subseteq \{1, 2, \ldots, n\}$. Then there is an associated symmetrised single trace

$$
T_b = \operatorname{Str}\left(\prod_{i \in b} Z_i\right) \tag{7.5.16}
$$

where the symmetrisation implicit in Str means the ordering of the product is irrelevant.

For a set partition $\pi \in \Pi(n)$ we have

$$T_\pi = \prod_{b\in\pi} T_b \tag{7.5.17}$$

where b runs over the blocks of π.

The equivalent permutation picture comes from the set of permutations $\mathrm{Perms}(\pi)$, defined in (7.2.56). We have

$$T_\pi = \frac{1}{|\mathrm{Perms}(\pi)|} \sum_{\sigma\in\mathrm{Perms}(\pi)} \sigma \tag{7.5.18}$$

where we use the same notation T_π for both the sum over permutations and the associated symmetrised trace operator. For the remainder of the section we only work with the permutation sum, so this ambiguity will not be an issue.

More generally, one can define a symmetrisation projector $\mathcal{P}$ which projects a permutation onto the space isomorphic to symmetrised traces. This is

$$\mathcal{P}(\sigma) = \frac{1}{|G(\pi(\sigma))|} \sum_{\tau\in G(\pi(\sigma))} \tau\sigma\tau^{-1} \tag{7.5.19}$$

where the set partition $\pi(\sigma)$ is defined naturally from the cycle structure of σ and is discussed above (7.2.56). The subgroup $G(\pi)$ for a given set partition is defined in (7.2.54) and permutes each block of π within itself but does not mix the different blocks. As mentioned below (7.2.56), the set $\mathrm{Perms}(\pi(\sigma))$ is just the conjugacy class of σ under $G(\pi(\sigma))$, and therefore

$$\mathcal{P}(\sigma) = T_{\pi(\sigma)} \tag{7.5.20}$$

As with the flavour projection, we now prove

$$\mathcal{P}^2 = \mathcal{P} \qquad\qquad \mathcal{P}^\dagger = \mathcal{P} \tag{7.5.21}$$

These follow immediately from the definition. We have

$$\begin{aligned}
\mathcal{P}(\mathcal{P}(\sigma)) &= \frac{1}{|G(\pi(\sigma))|^2} \sum_{\tau\in G(\pi(\sigma))} \sum_{\mu\in G(\pi(\tau\sigma\tau^{-1}))} \mu\tau\sigma\tau^{-1}\mu^{-1} \\
&= \frac{1}{|G(\pi(\sigma))|^2} \sum_{\tau\in G(\pi(\sigma))} \sum_{\mu\in G(\pi(\tau\sigma\tau^{-1}))} \tau\tau^{-1}\mu\tau\sigma\tau^{-1}\mu^{-1}\tau\tau^{-1} \\
&= \frac{1}{|G(\pi(\sigma))|^2} \sum_{\tau,\tilde{\mu}\in G(\pi(\sigma))} \tau\tilde{\mu}\sigma\tilde{\mu}^{-1}\tau^{-1}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{|G(\pi(\sigma))|^2} \sum_{\tau \in G(\pi(\sigma))} |G(\pi(\sigma))| \tau \alpha \tau^{-1} \\
&= \mathcal{P}(\alpha)
\end{aligned} \tag{7.5.22}
$$

where in the third line, we have defined $\tilde{\mu} = \tau^{-1}\mu\tau$. The conjugation by τ takes $\mu \in G(\pi(\tau\sigma\tau^{-1}))$ to $\tilde{\mu} \in G(\pi(\sigma))$. To prove $\mathcal{P}$ is Hermitian, we note that

$$
g(\sigma, \mathcal{P}(\tau)) = \frac{1}{|G(\pi(\tau))|} \sum_{\mu \in G(\pi(\tau))} \delta(\sigma\mu\tau^{-1}\mu^{-1}) \tag{7.5.23}
$$

is only non-zero if σ, τ belong to the same $\text{Perms}(\pi)$. In particular, they have $\pi(\sigma) = \pi(\tau)$. Therefore

$$
\begin{aligned}
g(\sigma, \mathcal{P}(\tau)) &= \frac{1}{|G(\pi(\sigma))|} \sum_{\mu \in G(\pi(\sigma))} \delta(\mu^{-1}\sigma\mu\tau^{-1}) \\
&= \frac{1}{|G(\pi(\sigma))|} \sum_{\mu \in G(\pi(\sigma))} \delta(\mu\sigma\mu^{-1}\tau^{-1}) \\
&= g(\mathcal{P}(\sigma), \tau)
\end{aligned} \tag{7.5.24}
$$

Therefore the map (7.5.3) gives an isomorphism between the symmetrised traces of n matrices with degree $(1, \ldots, 1)$ and $\operatorname{Im}\mathcal{P}$ (or the quotient $\mathbb{C}(S_n)/\operatorname{Ker}\mathcal{P}$). This is true when $N \geq n$. To deal with $N < n$ we have to include the finite N relations as well, which is discussed later.

As $\mathcal{P}$ is satisfies (7.5.21), it is expressible in the standard projector form

$$
\mathcal{P} = \sum_i |i\rangle\langle i| \tag{7.5.25}
$$

for orthonormal basis states $|i\rangle$ for $\text{Im}(\mathcal{P})$. These states $|i\rangle = \alpha_i$ belong to $\mathbb{C}(S_n)$, so to avoid doubling of notation, we will write this as

$$
\mathcal{P} = \sum_i \alpha_i \otimes \alpha_i \tag{7.5.26}
$$

which acts on $\sigma \in S_n$ via

$$
\mathcal{P}(\sigma) = \sum_i g(\alpha_i, \sigma)\alpha_i \tag{7.5.27}
$$

It is clear from (7.5.20) that T_π spans $\operatorname{Im}\mathcal{P}$. For $\pi \neq \pi'$, $\text{Perms}(\pi)$ is disjoint from $\text{Perms}(\pi')$, and therefore

$$
g(T_\pi, T_{\pi'}) = \frac{1}{|\text{Perms}(\pi)||\text{Perms}(\pi')|} \sum_{\substack{\sigma \in \text{Perms}(\pi) \\ \tau \in \text{Perms}(\pi')}} \delta(\sigma\tau^{-1})
$$

$$= \frac{1}{|\text{Perms}(\pi)|}\delta_{\pi,\pi'} \tag{7.5.28}$$

So an orthonormal basis for $\text{Im}\,\mathcal{P}$ is given by

$$\alpha_\pi = \sqrt{|\text{Perms}(\pi)|}\,T_\pi \tag{7.5.29}$$

and the corresponding expression for $\mathcal{P}$ is

$$\mathcal{P} = \sum_{\pi\in\Pi(n)} |\text{Perms}(\pi)|\; T_\pi \otimes T_\pi \tag{7.5.30}$$

In section 7.2.3 we defined another T_π as a map from 2-vector partitions to multi-symmetric functions. Composing these with the map (7.1.13) between multi-symmetric functions and symmetrised trace operators, we can identify $T_\pi \in \mathbb{C}(S_n)$ with these using the flavour projector $\mathcal{P}_H$ with $H = S_{n_1} \times S_{n_2}$.

Let $\mathbf{p} \vdash (n_1, n_2)$ be a vector partition, and π a set partition such that $\pi([(1,0)^{n_1}, (0,1)^{n_2}]) = \mathbf{p}$, where the action of a set partition on a vector partition was given in (7.2.39). Then

$$T_{\mathbf{p}} = \text{Tr}\left[\mathcal{P}_H\left(T_\pi\right)\mathbb{X}\right] \tag{7.5.31}$$

where $T_{\mathbf{p}}$ is the 2-matrix symmetrised trace operator given in (7.1.11).

Intuitively, the flavour projection and symmetrisation projectors should commute, since symmetrising a trace and renaming matrices from Z_i to X_j are commuting operations. Indeed

$$\begin{aligned}
\mathcal{P}P_H(\sigma) &= \frac{1}{|G(\pi(\tau\sigma\tau^{-1}))|}\frac{1}{|H|}\sum_{\substack{\tau\in H\\ \mu\in G(\pi(\tau\sigma\tau^{-1}))}} \mu\tau\sigma\tau^{-1}\mu^{-1}\\
&= \frac{1}{|G(\pi(\sigma))||H|}\sum_{\substack{\tau\in H\\ \mu\in G(\pi(\tau\sigma\tau^{-1}))}} \tau\left(\tau^{-1}\mu\tau\right)\sigma(\tau^{-1}\mu\tau)^{-1}\tau^{-1}\\
&= \frac{1}{|G(\pi(\sigma))||H|}\sum_{\substack{\tau\in H\\ \tilde{\mu}\in G(\pi(\sigma))}} \tau\tilde{\mu}\sigma\tilde{\mu}^{-1}\tau^{-1}\\
&= P_H\mathcal{P}(\sigma)
\end{aligned} \tag{7.5.32}$$

$\mathcal{P}$ was first considered in [51], though a slightly different group $G(\pi)$ was used in the definition. This involves wreath products and is given in section 7.3.7 for an integer partition. The difference in the defining group does not affect the action of the projector.

7.5.3 Multi-symmetric function isomorphism for n matrices

In section 7.1.1 we described the isomorphism of [97,98] between $U(N)$ gauge invariant of 2 complex matrices X_1 and X_2, modulo the ideal generated by commutators, and the ring of multi-symmetric functions in 2 families of variables. This was then used to construct a basis for 2-matrix symmetrised traces that respected the finite N behaviour. We now generalise this to the n-matrix case, and use the previous section to identify the space of multi-symmetric functions with sub-algebras of $\mathbb{C}(S_n)$. This will in turn allow a construction of the projector $\mathcal{P}_N$ that describes the interaction of the finite N cut-off with the symmetrisation projector $\mathcal{P}$.

Consider the M matrix variables $X_1, X_2, \cdots, X_M$. For each $a \in \{1, \cdots, M\}$, we have N^2 variables

$$(X_a)_{ij} \tag{7.5.33}$$

where $i, j \in \{1, \cdots, N\}$. Consider the ring of polynomials in these MN^2 variables. In this ring, there is an ideal generated by the elements of the commutators

$$[X_a, X_b]_{ik} = \sum_j (X_a)_{ij}(X_b)_{jk} - (X_b)_{ij}(X_a)_{jk} \tag{7.5.34}$$

where $a \neq b \in \{1, 2, \cdots, M\}$. We can form a quotient ring from this ideal. The ring of polynomial functions in the M matrix variables admits an action by $\mathcal{U} \in U(N)$ (or $GL(N, \mathbb{C})$):

$$X_a \to \mathcal{U} X_a \mathcal{U}^{-1} \tag{7.5.35}$$

The ideal generated by the commutators is invariant under the action of $U(N)$, so there is a quotient ring of $U(N)$ invariant polynomials. This is the ring of gauge invariants modulo commutator traces. This quotient ring of gauge invariants consists of multi-traces where any two traces differing by commutator traces define the same element of the ring. This is denoted by A_P^G in Theorem 3 of [97].

There is a polynomial ring D generated by x_i^a for $a \in \{1, \cdots, M\}$ and $i \in \{1, \cdots, N\}$. These polynomials have an S_N action given by

$$x_i^a \to x_{\sigma(i)}^a \tag{7.5.36}$$

The S_N invariant polynomials form multi-symmetric functions in M families of variables, and the ring of these functions is denoted D^{S_N}. Theorem 3 of [97] states that these two rings D^{S_N} and A_P^G are isomorphic.

To summarise, we have an isomorphism between gauge invariant polynomial functions of M matrices, modulo commutator traces, and permutation invariant polynomial functions of the M diagonal matrices. The map from the ring of $U(N)$ gauge invariant polynomial functions of matrices, modulo the commutator trace, to the space of S_N invariant polynomials is obtained by evaluating the gauge invariant functions on diagonal matrices. This map, denoted by ι, is proved to be an isomorphism in [97,98].

In the following, we will use a special case of this isomorphism where we have $M = n$ matrices and we consider gauge invariants containing exactly one of each matrix.

The space of matrix invariants appearing in this special case is important for the construction of BPS states. As discussed in section 7.1, the construction of quarter-BPS states is based on finding the orthogonal complement to the operators which are expressible as commutator traces at finite N. This orthogonalisation admits a generalization to the present case of $M = n$ matrices and gauge invariants containing one matrix of each type. Using the permutation description of n-matrix traces given in the previous section, it can be expressed as a problem in $\mathbb{C}(S_n)$ or constructing the orthogonal complement of $\mathrm{Ker}\,\mathcal{P} + \mathrm{Ker}\,\mathcal{F}_N$.

To see this, recall that permutations in $\mathrm{Ker}\,\mathcal{P}$ correspond to commutator traces via (7.5.3), while those in $\mathrm{Ker}\,\mathcal{F}_N$ correspond to the zero operator. Therefore any permutation in $\mathrm{Ker}\,\mathcal{P} + \mathrm{Ker}\,\mathcal{F}_N$ is a commutator trace. It follows that

Lemma 1

$$\Big(\mathbb{C}[X_1, X_2, \cdots, X_n] \;/\; \langle\{[X_a, X_b] : 1 \le a < b \le n\}\rangle\Big)^{U(N)}\Bigg|_{(1,1,\cdots,1)}$$

$$= \mathbb{C}(S_n)/\big(\mathrm{Ker}\,\mathcal{P} + \mathrm{Ker}\,\mathcal{F}_N\big) \tag{7.5.37}$$

Composing this with the isomorphism of [97,98] gives an identification between multi-symmetric functions and the quotient space of permutations.

The ring of multi-symmetric functions in n families of variables is spanned by multi-traces of n commuting matrices or monomials functions, denoted by T_π and M_π respectively. As discussed in section 7.5.2, these are labelled by set partitions $\pi \in \Pi(n)$ when the degree of each family of variables is 1, since this is equivalent to a n-vector partition of $(1, \dots, 1)$. The size N of each family of variables limit the number of subsets in the π to be less than N. This is denoted by $|\pi| \le N$ and follows immediately from the definition of the M_π, given for 2 families of variables in (7.2.20).

We use the same notation T_π and M_π for the multi-symmetric functions and the equivalent permutations. For T_π, this is given in (7.5.18). The M_π and T_π are related by

$$M_\pi = \sum_{\pi'} \tilde{C}_\pi^{\pi'} T_{\pi'} \qquad\qquad T_\pi = \sum_{\pi'} C_\pi^{\pi'} M_{\pi'} \tag{7.5.38}$$

where the C and $\tilde{C}$ matrices are described in section 7.2.5.

We now investigate the decomposition of $\mathbb{C}(S_n)$ in terms of the images and kernels of $\mathcal{P}$ and $\mathcal{F}_N$.

Lemma 2

Consider two subspaces S_1, S_2 of a vector space, equipped with an inner product. Let $S_1^\perp, S_2^\perp$ be the orthogonal complements to S_1, S_2 respectively. Let $S_1 + S_2$ be the set of vectors of the form $v_1 + v_2$, where $v_1 \in S_1, v_2 \in S_2$. It is a standard result that

$$(S_1 + S_2)^\perp = S_1^\perp \cap S_2^\perp \tag{7.5.39}$$

which is stated as "The orthogonal complement of a sum of vector spaces is the intersection of orthogonal complements".

Proof

Suppose $w \in S_1^\perp \cap S_2^\perp$, then

$$v_1 \cdot w = v_2 \cdot w = 0 \tag{7.5.40}$$

for all $v_1 \in S_1, v_2 \in S_2$. It follows that $w \cdot (v_1 + v_2) = w \cdot v_1 + w \cdot v_2 = 0$. So we conclude that $w \in (S_1 + S_2)^\perp$.

Conversely, suppose $w \notin S_1^\perp \cap S_2^\perp$, then $w \notin S_i^\perp$ for $i = 1$ or 2. This means there is some $v \in S_i$, such that $w \cdot v \neq 0$. But $v \in S_1 + S_2$, so $w \notin (S_1 + S_2)^\perp$. $\square$

Taking $S_1 = \operatorname{Im} \mathcal{P}$ and $S_2 = \operatorname{Im} \mathcal{F}_N$, we have an orthogonal decomposition for $\mathbb{C}(S_n)$ with respect to g_n

Lemma 2

$$\mathbb{C}(S_n) = (\operatorname{Im} \mathcal{P} \cap \operatorname{Im} \mathcal{F}_N) \oplus_{g_n} (\operatorname{Ker} \mathcal{P} + \operatorname{Ker} \mathcal{F}_N) \tag{7.5.41}$$

Using the fact that the monomial multi-symmetric functions form a basis, we have

$$\operatorname{Im} \mathcal{P} = \mathcal{M} = \operatorname{Span}\{M_\pi : \pi \in \Pi(n)\} \tag{7.5.42}$$

We will also define

$$\begin{aligned} \mathcal{M}_N^{\leq} &= \operatorname{Span}\{M_\pi : \pi \in \Pi(n), |\pi| \leq N\} \\ \mathcal{M}_N^{>} &= \operatorname{Span}\{M_\pi : \pi \in \Pi(n), |\pi| > N\} \end{aligned} \tag{7.5.43}$$

For $n < N$, we have to consider both operators $\mathcal{F}_N$ and $\mathcal{P}$. They are both hermitian operators wrt the g_n inner product, but they do not commute. The space $\text{Ker}\,\mathcal{P} + \text{Ker}\,\mathcal{F}_N$, spanned by sums of vectors in $\text{Ker}\,\mathcal{P}$ and $\text{Ker}\,\mathcal{F}_N$ is in general bigger than $\text{Ker}\,\mathcal{P}$. There is non-trivial intersection

$$\text{Im}\,\mathcal{P} \cap (\text{Ker}\,\mathcal{P} + \text{Ker}\,\mathcal{F}_N) \tag{7.5.44}$$

The non-triviality of this intersection is reflected in the fact that some symmetrised traces can also be written as as a symmetrised trace at finite N. An example of this is given in (7.4.26).

Since $\mathcal{P}$ is a Hermitian projector, we have the orthogonal decomposition

$$\mathbb{C}(S_n) = \text{Im}\,\mathcal{P} \oplus_{g_n} \text{Ker}\,\mathcal{P} \tag{7.5.45}$$

It follows that we have an orthogonal decomposition of $\text{Ker}\,\mathcal{P} + \text{Ker}\,\mathcal{F}_N$

Lemma 3

$$\text{Ker}\,\mathcal{P} + \text{Ker}\,\mathcal{F}_N = \big((\text{Ker}\,\mathcal{P} + \text{Ker}\,\mathcal{F}_N) \cap \text{Im}\,\mathcal{P}\big) \oplus_{g_n} \text{Ker}\,\mathcal{P} \tag{7.5.46}$$

Lemma 4

$$\boxed{(\text{Ker}\,\mathcal{P} + \text{Ker}\,\mathcal{F}_N) \cap \text{Im}\,\mathcal{P} = \mathcal{M}_N^{>} = \text{Span}\{M_\pi, |\pi| > N\}}$$

Proof

$\mathcal{M}_N^{>}$ is exactly the subspace of $\mathcal{M}$ which is not in the image of the isomorphism ι. Therefore

$$\mathcal{M}_N^{>} \subset \text{Ker}\,\mathcal{P} + \text{Ker}\,\mathcal{F}_N \subset \mathbb{C}(S_n) \tag{7.5.47}$$

Additionally $\mathcal{M}_N^{>} \subset \text{Im}\,\mathcal{P} = (\text{Ker}\,\mathcal{P})^{\perp}$. Then using Lemma 4, the result follows. □

Using Lemmas 2, 3 and 4 we have the result

Theorem

$$\boxed{\mathbb{C}(S_n) = (\text{Im}\,P \cap \text{Im}\,\mathcal{F}_N) \oplus_{g_n} \mathcal{M}_N^{>} \oplus_{g_n} \text{Ker}\,\mathcal{P}} \tag{7.5.48}$$

Using the definition of $\mathcal{P}_N$ in (7.5.2) we can also write this as

$$\mathbb{C}(S_n) = \mathrm{Im}\,\mathcal{P}_N \oplus_{g_n} \mathcal{M}_N^{>} \oplus_{g_n} \mathrm{Ker}\,\mathcal{P} \tag{7.5.49}$$

This gives a procedure, based on the combinatorics of multi-symmetric functions, for constructing the projector $\mathcal{P}_N$. This projector is built by constructing the projector for the susbspace of $\mathcal{M}$ orthogonal, with respect to the inner product g_n on $\mathbb{C}(S_n)$, to $\mathcal{M}_N^{>}$. Since $\mathcal{M}_N^{\leq} + \mathcal{M}_N^{>} = \mathrm{Im}\,\mathcal{P}$, this can be built by taking the vectors in $\mathcal{M}_N^{\leq}$ and subtracting vectors in $\mathcal{M}_N^{>}$ to ensure the orthogonality. The construction of $\mathrm{Im}\,\mathcal{P} \cap \mathrm{Im}\,\mathcal{F}_N$ uses the following elements:

- Vectors M_π in $\mathbb{C}(S_n)$ labelled by set partitions, spanning $\mathrm{Im}\,\mathcal{P}$.
- Finite N cut-off implemented using the set partition labels: the condition $|\pi| \leq N$ which defines $\mathcal{M}_N^{\leq}$.
- Orthogonalization of $\mathcal{M}_N^{\leq}$ to $\mathcal{M}_N^{>}$ with respect to the inner product g_n.

This procedure is used in section 7.4 to construct quarter-BPS bases.

The result of this procedure is a vector subspace of $\mathrm{Im}\,\mathcal{F}_N$, and therefore the orthogonalisation can equivalently be done using the inner product $g_{n,\widehat{N}}$ for any $\widehat{N} \geq N$, as on these permutations the g_n and $g_{n,\widehat{N}}$ inner products are the same.

We now give a construction of the projector $\mathcal{P}_N$ that captures this process. The formula for this is given in (7.5.62).

7.5.4 Finite N symmetrisation operator on $\mathbb{C}(S_n)$

We now construct the projector $\mathcal{P}_N$ onto $\mathrm{Im}\,\mathcal{P} \cap \mathrm{Im}\,\mathcal{F}_N$ and prove it has the projector properties

$$(\mathcal{P}_N)^2 = \mathcal{P}_N \qquad\qquad (\mathcal{P}_N)^\dagger = \mathcal{P}_N \tag{7.5.50}$$

and commutes with the flavour projector

$$\mathcal{P}_N \mathcal{P}_H = \mathcal{P}_H \mathcal{P}_N \tag{7.5.51}$$

Before producing $\mathcal{P}_N$, we give an alternative formula for $\mathcal{P}$, the large N symmetrisation projector. Using (7.5.30) and substituting using (7.5.38), we have

$$\mathcal{P} = \sum_{\pi,\pi',\pi'' \in \Pi(n)} |\mathrm{Perms}(\pi)| C_\pi^{\pi'} C_\pi^{\pi''} M_{\pi'} \otimes M_{\pi''} = \sum_{\pi,\pi' \in \Pi(n)} \left(CDC^T\right)^{\pi}_{\pi'} M_\pi \otimes M_{\pi'} \tag{7.5.52}$$

where D is the diagonal matrix

$$D^{\pi}_{\pi'} = |\mathrm{Perms}(\pi)| \delta_{\pi,\pi'} \tag{7.5.53}$$

To understand the appearance of CDC^T, note that this is the inverse metric on the subspace $\text{Im}(\mathcal{P})$ of $\mathbb{C}(S_n)$.

We know g is a positive definite inner product on the entirety of $\mathbb{C}(S_n)$. It is therefore also a positive definite inner product on the subspace $\text{Im}\,\mathcal{P}$. Hence there is an inverse metric on this subspace, which we call G. Using (7.5.38) we have

$$\begin{aligned} g(M_\pi, M_{\pi'}) &= \sum_{\pi_1,\pi_2\in\Pi(n)} \widetilde{C}_\pi^{\pi_1}\widetilde{C}_{\pi'}^{\pi_2} g(T_{\pi_1}, T_{\pi_2}) \\ &= \sum_{\pi_1\in\Pi(n)} \frac{\widetilde{C}_\pi^{\pi_1}\widetilde{C}_{\pi'}^{\pi_1}}{|\text{Perms}(\pi_1)|} \\ &= \left(\widetilde{C}^T D^{-1}\widetilde{C}\right)^\pi_{\pi'} \end{aligned} \tag{7.5.54}$$

Since C and $\tilde{C}$ are inverses of each other, this implies

$$G(M_\pi, M_{\pi'}) = (CDC^T)^\pi_{\pi'} \tag{7.5.55}$$

We can therefore write

$$\mathcal{P} = \sum_{\pi,\pi'\in\Pi(n)} (CDC^T)^\pi_{\pi'}\, M_\pi\otimes M_{\pi'} = \sum_{\pi,\pi'\in\Pi(n)} G(M_\pi, M_{\pi'}) M_\pi\otimes M_{\pi'} \tag{7.5.56}$$

This form for a projector is a generalisation of (7.5.25) to a basis of the image that is not orthonormal. We now find a basis for $\text{Im}\,\mathcal{P}_N$, and can use the form above to write down $\mathcal{P}_N$.

At finite N, we want to project onto the orthogonal complement of $\mathcal{M}_N^>$ within $\text{Im}\,\mathcal{P}$. The M_π with $|\pi|\leq N$ do not suffice for this as they are not orthogonal to M_π with $|\pi|>N$; we need to orthogonalise them first.

As already noted, g is an inner product on any subspace of $\mathbb{C}(S_n)$. This time the relevant subspace is $\mathcal{M}_N^>$. This means that the matrix of inner products $g(M_\pi, M_{\pi'})$ for $|\pi|,|\pi'|>N$ is invertible and has an inverse metric that we call $G^>$. Note that $G^>$ is distinct to G, which is the inverse inner product on $\text{Im}\,\mathcal{P}=\mathcal{M}$. Practically, the difference is

$$\sum_{\pi'\in\Pi(n)} g(M_\pi, M_{\pi'}) G(M_{\pi'}, M_{\pi''}) = \delta_{\pi\pi''} \qquad \pi,\pi'' \text{ unrestricted} \tag{7.5.57}$$

$$\sum_{\substack{\pi'\in\Pi(n)\\|\pi'|>N}} g(M_\pi, M_{\pi'}) G^>(M_{\pi'}, M_{\pi''}) = \delta_{\pi\pi''} \qquad |\pi|,|\pi''|>N \tag{7.5.58}$$

We can use $G^>$ to construct a basis for $\mathcal{M}_N^\leq$, labelled by those set partitions with

$|\pi| \leq N$.

$$\bar{M}_\pi = M_\pi - \sum_{\substack{\pi_1,\pi_2\in\Pi(n)\\|\pi_1|,|\pi_2|>N}} G^>(M_{\pi_1}, M_{\pi_2}) g(M_\pi, M_{\pi_1}) M_{\pi_2} \tag{7.5.59}$$

The simplest way of looking at this is to notice that the second term is using a projector of the form (7.5.56) applied to M_π. This is the projector

$$\mathcal{P}^> = \sum_{\substack{\pi_1,\pi_2\in\Pi(n)\\|\pi_1|,|\pi_2|>N}} G^>(M_{\pi_1}, M_{\pi_2})(M_{\pi_1} \otimes M_{\pi_2}) \tag{7.5.60}$$

that orthogonally projects onto $\mathcal{M}_N^>$. The construction of $\bar{M}_\pi$ just applies $1 - \mathcal{P}^>$ to M_π to produce something orthogonal to $\mathcal{M}_N^>$ while remaining in $\mathrm{Im}\,\mathcal{P}$. We can now use the $\bar{M}_\pi$ to define the finite N symmetrisation projector. Again, we need to produce a new inverse metric $G^\leq$ on the space spanned by $\bar{M}_\pi$. This satisfies

$$\sum_{\substack{\pi'\in\Pi(n)\\|\pi'|\leq N}} g(\bar{M}_\pi, \bar{M}_{\pi'}) G^\leq(\bar{M}_{\pi'}, \bar{M}_{\pi''}) = \delta_{\pi\pi''} \qquad |\pi|, |\pi''| \leq N \tag{7.5.61}$$

Using this, we construct the finite N symmetrisation projector

$$\boxed{\mathcal{P}_N = \sum_{\substack{\pi,\pi'\in\Pi(n)\\|\pi|,|\pi'|\leq N}} G^\leq(\bar{M}_\pi, \bar{M}_{\pi'})(\bar{M}_\pi \otimes \bar{M}_{\pi'})} \tag{7.5.62}$$

We now prove the properties (7.5.50) and (7.5.51) for $\mathcal{P}_N$.

$\mathcal{P}_N$ is a projector

To prove this, we act with the square of the projector

$$\begin{aligned}
\mathcal{P}_N\mathcal{P}_N(\alpha) &= \mathcal{P}_N \sum_{\substack{\pi_1,\pi_2\in\Pi(n)\\|\pi_1|,|\pi_2|\leq N}} G^\leq(\bar{M}_{\pi_1}, \bar{M}_{\pi_2})\bar{M}_{\pi_1}\; g(\bar{M}_{\pi_2}, \alpha) \\
&= \sum_{\substack{\pi_1,\pi_2,\pi_3,\pi_4\in\Pi(n)\\|\pi_1|,|\pi_2|,|\pi_3|,|\pi_4|\leq N}} G^\leq(\bar{M}_{\pi_3}, \bar{M}_{\pi_4}) G^\leq(\bar{M}_{\pi_1}, \bar{M}_{\pi_2}) g(\bar{M}_{\pi_4}, \bar{M}_{\pi_1}) g(\bar{M}_{\pi_2}, \alpha)\bar{M}_{\pi_3} \\
&= \sum_{\substack{\pi_1,\pi_2,\pi_3\in\Pi(n)\\|\pi_1|,|\pi_2|,|\pi_3|\leq N}} \delta(\pi_1, \pi_3) G^\leq(\bar{M}_{\pi_1}, \bar{M}_{\pi_2}) g(\bar{M}_{\pi_2}, \alpha)\bar{M}_{\pi_3} \\
&= \sum_{\substack{\pi_1,\pi_2\in\Pi(n)\\|\pi_1|,|\pi_2|\leq N}} G^\leq(\bar{M}_{\pi_1}, \bar{M}_{\pi_2})\bar{M}_{\pi_1} g(\bar{M}_{\pi_2}, \alpha) \\
&= \mathcal{P}_N(\alpha)
\end{aligned} \tag{7.5.63}$$

where we have used (7.5.61) to get from the second to third line.

$\mathcal{P}_N$ is hermitian

This follows from the symmetry between π and π' in (7.5.62)

$$\begin{aligned} g(\alpha, \mathcal{P}_N(\beta)) &= \sum_{\substack{\pi_1,\pi_2\in\Pi(n) \\ |\pi_1|,|\pi_2|\leq N}} G^{\leq}(\bar{M}_{\pi_1}, \bar{M}_{\pi_2}) g(\bar{M}_{\pi_2}, \beta) g(\alpha, \bar{M}_{\pi_1}) \\ &= g(\mathcal{P}_N(\alpha), \beta) \end{aligned} \tag{7.5.64}$$

$\mathcal{P}_N$ commutes with $\mathcal{P}_H$

This relies on some smaller results. We start with

$$\sigma^{-1} T_\pi \sigma = T_{\sigma(\pi)} \tag{7.5.65}$$

where we define $\sigma(\pi)$ as the set partition obtained by substituting $i \to \sigma(i)$ in the set partition π. It is useful to recall the fact that $\sigma^{-1}\mu\sigma$ is the permutation obtained by the substitution $i \to \sigma(i)$ in the cycle decomposition of μ, and therefore

$$\text{Perms}(\sigma(\pi)) = \sigma^{-1}\text{Perms}(\pi)\sigma = \left\{\sigma^{-1}\mu\sigma : \mu \in \text{Perms}(\pi)\right\} \tag{7.5.66}$$

It follows that

$$\begin{aligned} \sigma^{-1} T_\pi \sigma &= \frac{1}{|\text{Perms}(\pi)|} \sum_{\mu\in\text{Perms}(\pi)} \sigma^{-1}\mu\sigma \\ &= \frac{1}{|\text{Perms}(\sigma(\pi))|} \sum_{\tilde{\mu}\in\text{Perms}(\sigma(\pi))} \tilde{\mu} \\ &= T_{\sigma(\pi)} \end{aligned} \tag{7.5.67}$$

We also observe that

$$C^{\pi_1}_{\pi_2} = C^{\sigma(\pi_1)}_{\sigma(\pi_2)} \qquad\qquad \widetilde{C}^{\pi_1}_{\pi_2} = \widetilde{C}^{\sigma(\pi_1)}_{\sigma(\pi_2)} \tag{7.5.68}$$

This is because the incidence relations of the poset of set partitions are unchanged when we go from set partitions of $\{1, 2, \cdots, n\}$ to set partitions of $\{\sigma(1), \cdots, \sigma(n)\}$.

It follows from (7.5.65) and (7.5.68) that

$$\begin{aligned} \sigma^{-1} M_\pi \sigma &= \sum_{\pi'\in\Pi(n)} \widetilde{C}^{\pi'}_{\pi} \sigma^{-1} T_{\pi'} \sigma \\ &= \sum_{\pi'\in\Pi(n)} \widetilde{C}^{\sigma(\pi')}_{\sigma(\pi)} T_{\sigma(\pi')} \end{aligned}$$

$$\begin{aligned} &= \sum_{\pi' \in \Pi(n)} \widetilde{C}^{\pi'}_{\sigma(\pi)} T_{\pi'} \\ &= M_{\sigma(\pi)} \end{aligned} \tag{7.5.69}$$

where in going from the 2nd to 3rd line we have reparameterised the sum by $\pi' \to \sigma(\pi')$, which clearly just permutes the set partitions in $\Pi(n)$ among each other.

It is immediate from (7.5.7) that

$$g(\sigma\alpha\sigma^{-1}, \sigma\beta\sigma^{-1}) = g(\alpha, \beta) \tag{7.5.70}$$

Applying this to $\alpha = M_\pi, \beta = M_{\pi'}$ and using (7.5.69), we have

$$g(M_{\sigma(\pi)}, M_{\sigma(\pi')}) = g(M_\pi, M_{\pi'}) \tag{7.5.71}$$

We would like to show that G also has this property. To see this, note that G is defined by the property (7.5.57), so we need to show that the matrix $G(M_{\sigma(\pi)}, M_{\sigma(\pi')})$ satisfies the same relation.

$$\begin{aligned} \sum_{\pi' \in \Pi(n)} g(M_\pi, M_{\pi'}) G(M_{\sigma(\pi')}, M_{\sigma(\pi'')}) &= \sum_{\pi' \in \Pi(n)} g(M_\pi, M_{\sigma^{-1}(\pi')}) G(M_{\pi'}, M_{\sigma(\pi'')}) \\ &= \sum_{\pi' \in \Pi(n)} g(M_{\sigma(\pi)}, M_{\pi'}) G(M_{\pi'}, M_{\sigma(\pi'')}) \\ &= \delta_{\sigma(\pi)\sigma(\pi'')} \\ &= \delta_{\pi\pi''} \end{aligned} \tag{7.5.72}$$

Therefore

$$G(M_{\sigma(\pi)}, M_{\sigma(\pi')}) = G(M_\pi, M_{\pi'}) \tag{7.5.73}$$

Next note that $|\pi| = |\sigma(\pi)|$, so when changing variables from π to $\sigma(\pi)$, the restrictions $|\pi| > N$ or $|\pi| \leq N$ are maintained. This means we can repeat the steps in (7.5.72) but using $G^>$ or $G^\leq$ instead. Hence

$$G^>(M_{\sigma(\pi)}, M_{\sigma(\pi')}) = G^>(M_\pi, M_{\pi'}) \qquad G^\leq(M_{\sigma(\pi)}, M_{\sigma(\pi')}) = G^\leq(M_\pi, M_{\pi'}) \tag{7.5.74}$$

where π, π' satisfy the appropriate constraints on their length for the two operations.

Using the definition (7.5.59), as well as (7.5.69), (7.5.70) and (7.5.74)

$$\sigma^{-1}\bar{M}_\pi\sigma = \sigma^{-1}M_\pi\sigma - \sum_{\substack{\pi_1,\pi_2 \in \Pi(n) \\ |\pi_1|,|\pi_2| > N}} G^>(M_{\pi_1}, M_{\pi_2}) g(M_\pi, M_{\pi_1}) \sigma^{-1} M_{\pi_2} \sigma$$

$$
\begin{aligned}
&= M_{\sigma(\pi)} - \sum_{\substack{\pi_1,\pi_2\in\Pi(n)\\|\pi_1|,|\pi_2|>N}} G^{>}(M_{\pi_1}, M_{\pi_2}) g(M_\pi, M_{\pi_1}) M_{\sigma(\pi_2)} \\
&= M_{\sigma(\pi)} - \sum_{\substack{\pi_1,\pi_2\in\Pi(n)\\|\pi_1|,|\pi_2|>N}} G^{>}(M_{\pi_1}, M_{\sigma^{-1}(\pi_2)}) g(M_\pi, M_{\pi_1}) M_{\pi_2} \\
&= M_{\sigma(\pi)} - \sum_{\substack{\pi_1,\pi_2\in\Pi(n)\\|\pi_1|,|\pi_2|>N}} G^{>}(M_{\sigma(\pi_1)}, M_{\pi_2}) g(M_\pi, M_{\pi_1}) M_{\pi_2} \\
&= M_{\sigma(\pi)} - \sum_{\substack{\pi_1,\pi_2\in\Pi(n)\\|\pi_1|,|\pi_2|>N}} G^{>}(M_{\pi_1}, M_{\pi_2}) g(M_\pi, M_{\sigma^{-1}(\pi_1)}) M_{\pi_2} \\
&= M_{\sigma(\pi)} - \sum_{\substack{\pi_1,\pi_2\in\Pi(n)\\|\pi_1|,|\pi_2|>N}} G^{>}(M_{\pi_1}, M_{\pi_2}) g(M_{\sigma(\pi)}, M_{\pi_1}) M_{\pi_2} \\
&= \bar{M}_{\sigma(\pi)}
\end{aligned}
\tag{7.5.75}
$$

We can now prove that $\mathcal{P}_N$ and $\mathcal{P}_H$ commute

$$
\begin{aligned}
\mathcal{P}_N P_H(\alpha) &= \frac{1}{|H|}\sum_{\sigma\in H}\sum_{\substack{\pi_1,\pi_2\in\Pi(n)\\|\pi_1|,|\pi_2|\le N}} G^{\le}(\bar{M}_{\pi_2}, \bar{M}_{\pi_1}) \bar{M}_{\pi_2} g(\bar{M}_{\pi_1}, \sigma\alpha\sigma^{-1}) \\
&= \frac{1}{|H|}\sum_{\sigma\in H}\sum_{\substack{\pi_1,\pi_2\in\Pi(n)\\|\pi_1|,|\pi_2|\le N}} G^{\le}(\bar{M}_{\pi_2}, \bar{M}_{\pi_1}) \bar{M}_{\pi_2} g(\sigma^{-1}\bar{M}_{\pi_1}\sigma, \alpha) \\
&= \frac{1}{|H|}\sum_{\sigma\in H}\sum_{\substack{\pi_1,\pi_2\in\Pi(n)\\|\pi_1|,|\pi_2|\le N}} G^{\le}(\bar{M}_{\pi_2}, \bar{M}_{\pi_1}) \bar{M}_{\pi_2} g(\bar{M}_{\sigma(\pi_1)}, \alpha) \\
&= \frac{1}{|H|}\sum_{\sigma\in H}\sum_{\substack{\pi_1,\pi_2\in\Pi(n)\\|\pi_1|,|\pi_2|\le N}} G^{\le}(\bar{M}_{\pi_2}, \bar{M}_{\sigma^{-1}(\pi_1)}) \bar{M}_{\pi_2} g(\bar{M}_{\pi_1}, \alpha) \\
&= \frac{1}{|H|}\sum_{\sigma\in H}\sum_{\substack{\pi_1,\pi_2\in\Pi(n)\\|\pi_1|,|\pi_2|\le N}} G^{\le}(\bar{M}_{\sigma(\pi_2)}, \bar{M}_{\pi_1}) \bar{M}_{\pi_2} g(\bar{M}_{\pi_1}, \alpha) \qquad (7.5.76) \\
&= \frac{1}{|H|}\sum_{\sigma\in H}\sum_{\substack{\pi_1,\pi_2\in\Pi(n)\\|\pi_1|,|\pi_2|\le N}} G^{\le}(\bar{M}_{\pi_2}, \bar{M}_{\pi_1}) \bar{M}_{\sigma^{-1}(\pi_2)} g(\bar{M}_{\pi_1}, \alpha) \\
&= \frac{1}{|H|}\sum_{\sigma\in H}\sum_{\substack{\pi_1,\pi_2\in\Pi(n)\\|\pi_1|,|\pi_2|\le N}} G^{\le}(\bar{M}_{\pi_2}, \bar{M}_{\pi_1}) \sigma\bar{M}_{\pi_2}\sigma^{-1} g(\bar{M}_{\pi_1}, \alpha) \\
&= P_H \mathcal{P}_N(\alpha)
\end{aligned}
\tag{7.5.77}
$$

We can interpret this in words as follows. Recall that permutations σ generate gauge invariant operators via (7.5.3). Imagine we start with the n-flavour gauge invariant operator generated by σ, and then symmetrise the traces, and map that to symmetrised permutations. This means applying $\mathcal{P}$ then setting

$$\begin{aligned} Z_1, Z_2, \ldots, Z_{n_1} &\to X \\ Z_{n_1+1}, Z_{n_1+2}, \ldots, Z_n &\to Y \end{aligned} \tag{7.5.78}$$

On the other hand, we could specialise the n-flavour gauge invariants to 2-flavour gauge invariants before projecting to symmetrised traces. Intuitively, thinking about traces, we don't see any reason for a difference between the two orders of arriving at symmetrised traces of two matrices. So we expect the two projectors to commute. Indeed they do as shown above.

7.6 Hidden 2D topology: Permutation TFT2 for the counting and correlators at weak coupling

The connection between delta functions on symmetric group algebras and two-dimensional topological field theories (TFT2) is explained in [51]. We will give the delta function formulae and explain the TFT2 defects.

Lemma 1

In the problem of gauge invariants of n matrices, each occuring once, the counting of symmetrised traces at large N is given by

$$\sum_{\alpha \in S_n} \delta\big(\mathcal{P}(\alpha^{-1})\mathcal{P}(\alpha)\big) \tag{7.6.1}$$

Proof

The symmetrised traces form the image of the hermitian projector $\mathcal{P}$. So the dimension of the space of symmetrised traces is calculated as

$$\begin{aligned} \text{Dim}\,(\text{Im}\,\mathcal{P}) &= \sum_{\alpha \in S_n} g(\alpha, \mathcal{P}(\alpha)) \\ &= \sum_{\alpha \in S_n} g(\alpha, \mathcal{P}(\mathcal{P}(\alpha))) \\ &= \sum_{\alpha \in S_n} g(\mathcal{P}(\alpha), \mathcal{P}(\alpha)) \\ &= \sum_{\alpha \in S_n} \delta(\mathcal{P}(\alpha^{-1})\mathcal{P}(\alpha)) \end{aligned} \tag{7.6.2}$$

□

Proposition 2

The counting of quarter-BPS operators in the large N limit in the free theory is given by

$$\sum_{\alpha \in S_n} \delta\big(\mathcal{P}_H(\alpha)\alpha^{-1}\big) \tag{7.6.3}$$

where $\mathcal{P}_H$ is the flavour projector onto two flavours with $H = S_{n_1} \times S_{n_2}$ as described in (7.5.11).

Proof

We know that permutations can be used to construct 2-matrix gauge invariants and there is an equivalence up to conjugation by $H = S_{n_1} \times S_{n_2}$. Using Burnside's Lemma to count the free field operators, we have

$$\frac{1}{|H|} \sum_{\gamma \in H} \sum_{\alpha \in S_n} \delta(\gamma \alpha \gamma^{-1} \alpha^{-1}) = \sum_{\alpha \in S_n} \delta\big(\mathcal{P}_H(\alpha)\alpha^{-1}\big) \tag{7.6.4}$$

This is the free field counting of 2-matrix operators [51]. □

Proposition 3

The counting of 2-matrix symmetrised operators in the (n_1, n_2) sector is

$$\sum_{\alpha \in S_n} \delta(\mathcal{P}_H \mathcal{P}(\alpha) \mathcal{P}(\alpha^{-1})) \tag{7.6.5}$$

Proof

Both $\mathcal{P}, \mathcal{P}_H$ are hermitian with respect to the standard inner product on $\mathbb{C}S_n$ and they commute, so they can be simultaneously diagonalised. The dimension of the intersection of their images is equal to the trace of their product

$$\begin{aligned}
\sum_{\alpha \in S_n} g(\alpha, \mathcal{P}\mathcal{P}_H(\alpha)) &= \sum_{\alpha \in S_n} g(\alpha, \mathcal{P}^2 \mathcal{P}_H(\alpha)) \\
&= \sum_{\alpha \in S_n} g(\mathcal{P}(\alpha), \mathcal{P}\mathcal{P}_H(\alpha)) \\
&= \sum_{\alpha \in S_n} \delta(\mathcal{P}(\alpha^{-1}) \mathcal{P} P_H(\alpha)) \\
&= \sum_{\alpha \in S_n} \delta(\mathcal{P}_H \mathcal{P}(\alpha) \mathcal{P}(\alpha^{-1}))
\end{aligned} \tag{7.6.6}$$

□

Proposition 4

The counting formula for the finite N quarter-BPS operators is

$$\sum_{\alpha \in S_n} \delta(\mathcal{P}_H \mathcal{P}_N(\alpha) \mathcal{P}_N(\alpha^{-1})) \tag{7.6.7}$$

Proof

Given that we have proved the projector, hermiticity, and commutativity properties of $\mathcal{P}_N$ and $\mathcal{P}_H$, we can calculate the dimension of the image of $\mathcal{P}_N \mathcal{P}_H$ by repeating the steps we had for $\mathcal{P}$ and $\mathcal{P}_H$

$$\begin{aligned} \text{Dim}\left(\text{Im}(\mathcal{P}_N \mathcal{P}_H)\right) &= \sum_{\alpha \in S_n} g(\alpha, \mathcal{P}_N \mathcal{P}_H(\alpha)) \\ &= \sum_{\alpha \in S_n} g(\alpha, \mathcal{P}_N^2 \mathcal{P}_H(\alpha)) \\ &= \sum_{\alpha \in S_n} g(\mathcal{P}_N(\alpha), \mathcal{P}_N \mathcal{P}_H(\alpha)) \\ &= \sum_{\alpha \in S_n} \delta(\mathcal{P}_N(\alpha^{-1}) \mathcal{P}_N \mathcal{P}_H(\alpha)) \\ &= \sum_{\alpha \in S_n} \delta(\mathcal{P}_H \mathcal{P}_N(\alpha) \mathcal{P}_N(\alpha^{-1})) \end{aligned} \tag{7.6.8}$$

□

Proposition 5

The finite N two-point function for BPS states can be written as

$$\left\langle \text{Tr}\left(\mathcal{G}_N \mathcal{P}_N \alpha_1 X^{\otimes n_1} Y^{\otimes n_2}\right) , \text{Tr}\left(\mathcal{G}_N \mathcal{P}_N \alpha_2 X^{\otimes n_1} Y^{\otimes n_2}\right) \right\rangle = \delta\Big(\mathcal{P}_H \mathcal{P}_N(\alpha_1) \mathcal{P}_N(\alpha_2^{-1}) \Omega_N^{-1}\Big) \tag{7.6.9}$$

This follows as in [51]. $\Omega_N^{-1} \mathcal{P}_N(\alpha)$ span the BPS states as α runs over $\mathbb{C}(S_n)$. The free field inner product is $g_{FF}(\alpha, \beta) = g(\alpha, \Omega_N \beta)$. The step forward in this thesis is that we have an explicit construction of $\mathcal{P}_N$ using set partitions.

Now we will draw the TFT2 pictures corresponding to these delta function formulae. Figure 7.4 gives us the counting of weak coupling BPS operators. Figure 7.5 gives the TFT2 formulation for the 2-point function of quarter BPS operators at weak coupling. The one new ingredient in these TFT2 constructions is the $\mathcal{P}_N$-defect which can be associated to a circle. The defect is defined by declaring that it modifies the permutation α associated to that circle in the TFT2 to $\mathcal{P}_N(\alpha)$.

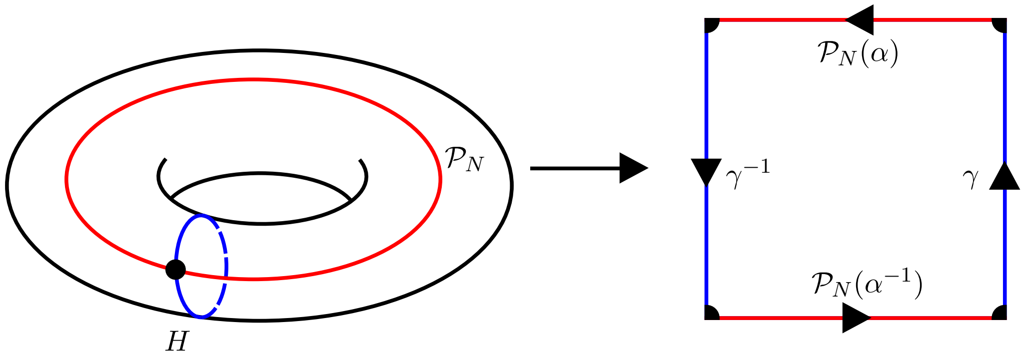


Figure 7.4: TFT2 partition function for finite N weak coupling BPS counting

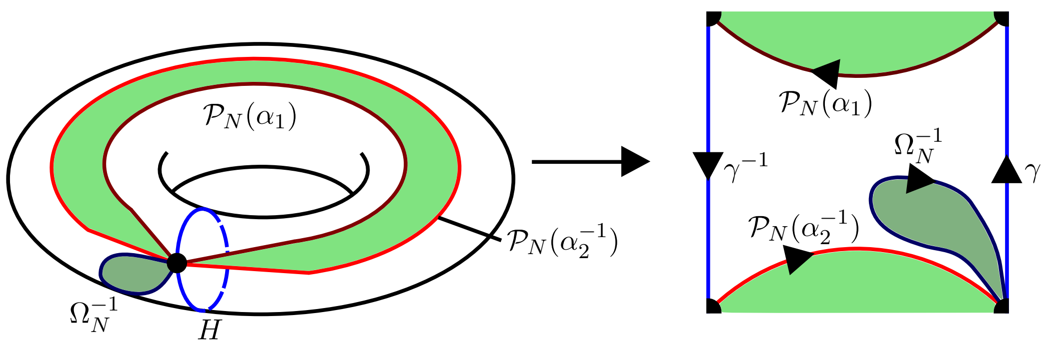


Figure 7.5: TFT2 partition function for finite N BPS 2-point function

Chapter 8

Conclusions

In this thesis we have explored aspects of the half and quarter-BPS sectors of $\mathcal{N} = 4$ super Yang-Mills theory with gauge groups $U(N)$, $SO(N)$ and $Sp(N)$. These results, interpreted through the AdS/CFT correspondence, have implications for our understanding of non-perturbative effects in string theory.

8.1 Word combinatorics

Motivated by the matching of generating functions between the planar quarter-BPS sector in the $U(N)$ gauge theory and an integrally graded word monoid, in chapter 3 we found a bijection between aperiodic single traces and Lyndon words, the factorisation units of the monoid. This bijection led to a decomposition in both the vector space structure of the quarter-BPS sector and the corresponding generating function.

In chapter 4 we derived the same structure for the $SO(N)$ and $Sp(N)$ gauge theories, where the Lyndon words of the $U(N)$ theory are replaced by orthogonal Lyndon words that satisfy a minimal periodicity condition, and gave two independent derivations of the planar generating function.

The generating function (3.0.1) has been generalised to arbitrary $U(N)$ quivers [68, 69]. The structure of the function, with its infinite product of a root function, was found to be very general, and the root function had an interpretation in terms of counting words made from loops in the quiver. A natural question to ask is whether there is an analogue of Lyndon words in this setting. In a different direction, it would be interesting to investigate the extension of the $SO(N)/Sp(N)$ counting function (4.3.21) to a general $SO(N)/Sp(N)$ quiver.

8.2 Permutation structures in gauge theory

Since the derivation of a Young diagram basis for the half-BPS sector [22], permutations have proved a powerful tool in studying $U(N)$ gauge theory and matrix models. See [62] for a summary of the various interesting applications that have been found. In particular, Schur-Weyl duality has been a crucial element in describing finite N effects. In chapter 5 we develop a permutation description of equivalent power for $SO(N)$ and $Sp(N)$ gauge groups. This mathematical formalism can be used to study phenomenon in the dual unoriented string theory as well as general $SO(N)/Sp(N)$ matrix models. Some subjects of particular interest that should be approachable with these techniques are

- Investigating the spectrum of the one-loop dilatation operator in the quarter-BPS sector of the $SO(N)/Sp(N)$ theory, in a manner similar to that of chapter 7.
- Counting and correlators in general $SO(N)/Sp(N)$ quiver theories.

More generally, the appearance of permutation structures in theories with different gauge groups offers an interesting interpretation of permutations as a background independent structure in string theory. The $U(N)$, $SO(N)$ and $Sp(N)$ inner products on permutations can be viewed as different (background-dependent) pairings on permutations which are background independent characterisations of gauge invariants. An interesting exercise is to revisit previous applications of permutations to stringy physics and disentangle the aspects of permutations and associated representation theory which contain information about specific backgrounds, and those that are common to different backgrounds, or relate different backgrounds.

8.3 Orientifold quotient

In chapter 6 we developed a detailed gauge theory description of the orientifold map that takes type IIB string theory on $AdS_5 \times S^5$, dual to a $U(N)$ gauge group, to strings on $AdS_5 \times \mathbb{RP}^5$, dual to $SO(N)$ or $Sp(N)$ gauge group. This quotient was expressed in terms of coefficients α_R^T for which we gave two distinct formulae, both related to Littlewood-Richardson coefficients.

The first expression was in terms of domino tableaux that have a strong physical interpretation as pairing up quanta of angular momenta in a precise way given by the combinatorics of Young diagrams. The second was as a product of two $SO(N)/Sp(N)$ Schur operators, involving the mathematical concept of a 2-quotient of a partition. This led to insight into the $U(N)$ theory, and in particular a $\mathbb{Z}_2$ action that is a candidate for field theory dual of the orientifold action that takes $x \to -x$ for $x \in S^5$ while also reversing worldsheet orientation.

Both interpretations of the coefficients have implications for the study of giant gravitons in the theories dual to $U(N)$, $SO(N)$ and $Sp(N)$ SYM. It will be intriguing to see string theory derivations of this structure from D-brane physics.

8.4 Quarter-BPS sector of $U(N)$ theory at weak coupling

In chapter 7 we gave a construction of quarter BPS operators in $\mathcal{N} = 4$ super Yang-Mills with $U(N)$ gauge group, built from two matrices X, Y and annihilated by the 1-loop dilatation operator of the $SU(2)$ sector. The construction depends on parameters n, N which are arbitrary, with n being the number of X, Y matrices in the operator. The construction produces an orthogonal basis of operators obeying an SEP-compatibility condition. The labels for the basis operators include a $U(2)$ Young diagram Λ and a $U(N)$ Young diagram p, alongside multiplicity labels. The SEP-compatibility means that finite N effects are captured simply by restricting the length of p to be less than N. We have detailed formulae for the dimensions of the multiplicity spaces as a function of Λ, p.

The understanding of holographic map between the quarter-BPS sector between $\mathcal{N} = 4$ SYM and $AdS_5 \times S^5$ is far less well-developed than the half-BPS sector. The Young diagrams labels for half-BPS states have provided valuable tools for precision mapping of states between SYM and the dual space-time. In the quarter BPS sector, there is a rich combinatoric structure involving Λ, p and the plethysm problem underlying the multiplicities $\mathcal{M}_{\Lambda,p}$, which control the structure of states. If will be fascinating to uncover the role of these structures in the dual space-time. Concretely, reproducing the refined multiplicity formulae for specified Λ, p from the the weakly coupled gravitational dual, is an interesting problem.

Another interesting extension would be to investigate the quarter-BPS sector at weak coupling in $SO(N)/Sp(N)$ theories. Since the half-BPS sector can be expressed in terms of symmetric functions in the squares of the eigenvalues, one might expect that multi-symmetric functions could be used to capture the finite N behaviour of the weakly coupled quarter-BPS operators. Beyond this, one could then use the orientifold quotient to explore the relation between the $U(N)$ and $SO(N)/Sp(N)$ quarter-BPS sectors at weak coupling.

Appendix A

The Young basis and Jucys-Murphy elements

A.1 Young basis for S_n

For a Young diagram $R \vdash n$, a Young tableau of shape R is produced by placing a positive integer into each box of R. The tableau is called semi-standard if the numbers increase weakly along the rows and strictly down the columns. It is called standard if in addition the n integers are the numbers 1 to n. As an example, the possible standard Young tableaux of shape $R = [3, 2]$ are

$$\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & 5 \\ \cline{1-2}\end{array} \quad \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 & 5 \\ \cline{1-2}\end{array} \quad \begin{array}{|c|c|c|}\hline 1 & 2 & 5 \\ \hline 3 & 4 \\ \cline{1-2}\end{array} \quad \begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 & 5 \\ \cline{1-2}\end{array} \quad \begin{array}{|c|c|c|}\hline 1 & 3 & 5 \\ \hline 2 & 4 \\ \cline{1-2}\end{array} \tag{A.1.1}$$

We can construct the irreducible representation $R \vdash n$ of S_n by setting the basis vectors to be the standard Young tableaux of shape R. The permutations $(i, i+1)$ generate S_n, so we only need to define the action for these.

Consider a standard Young tableau r of shape R. Let $s_i(r)$ be the tableau formed by swapping the numbers i and $i+1$ in r. This tableau could be standard or non-standard.

Let b_i be the box labelled by i in r. Then the distance $\rho_{i,i+1}(r)$ between the boxes b_i and b_{i+1} is simply the difference in their contents, as defined in (2.3.19).

$$\rho_{i,i+1}(r) = c_{b_{i+1}} - c_{b_i} \tag{A.1.2}$$

Intuitively, this measures how many boxes it takes to travel from i to $i+1$ in r, where the distance increases by 1 for each step upwards or to the right, and decreases by 1 for each step downwards or to the left.

We can now write the representatives of the permutation $(i, i+1)$ on the Young basis

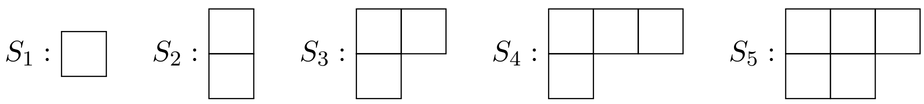
$$D^R\left[(i,i+1)\right]|r\rangle = \begin{cases} \frac{1}{\rho_{i,i+1}(r)}|r\rangle - \sqrt{1-\frac{1}{\rho_{i,i+1}^2}}|s_i(r)\rangle & s_i(r) \text{ a standard Young tableau} \\ \frac{1}{\rho_{i,i+1}(r)}|r\rangle & \text{otherwise} \end{cases} \tag{A.1.3}$$

The simplest consequence of this is if a contiguous block of number $i, i+1, \ldots, i+j$ are in ascending order in a single row, then that Young tableau is symmetric under all permutations of $i, i+1, \ldots, i+j$. Conversely, if they lie in order in a column, that Young tableau is anti-symmetric under all such permutations.

For example the first Young tableau in (A.1.1) is symmetric under permutations of $\{1,2,3\}$ and $\{4,5\}$, while the last one is anti-symmetric under permutations of $\{1,2\}$ and $\{3,4\}$.

The Young basis has another crucial property. The position of the number n in a tableau r tells us which representation of S_{n-1} the vector $|r\rangle$ lives in, when S_{n-1} is embedded into S_n by acting on $\{1,2,\ldots n-1\}$. By removing n from r, we obtain a tableau $\hat{r}$ of shape $\hat{R} \vdash n-1$. Then $|r\rangle$ lives in the $\hat{R}$ representation of S_{n-1}, and is the $|\hat{r}\rangle$ vector in this representation. So for example, the first, second and fourth tableaux in (A.1.1) live in the $\hat{R} = [3,1]$ representation of S_4, and form the Young basis for this representation.

By iterating this process, the positions of the numbers in r determine the representation $|r\rangle$ lives in for each of $S_1 \subseteq S_2 \subseteq \cdots \subseteq S_n$. For example the fourth tableaux in (A.1.1) belongs to the following representations of each



(A.1.4)

Conversely, given the sequence of representations that a vector lives in determines the corresponding Young tableau.

For a more thorough discussion of the Young basis of a representation of S_n see [116]. A different construction involving the Jucys-Murphy elements defined in the next section is used in [117].

An important subtlety to note is that the representation (A.1.3) uses the opposite convention for permutation multiplication as this thesis, given below (2.1.1). The effect on the representation theory is to transpose all representation matrices. We will primarily be concerned with commuting products of linear combinations of swaps (i,j). The fact the products commute means the order of multiplication is irrelevant, and since swaps are self-inverse, the representation matrix of any linear combination is symmetric. Therefore this difference in convention does not affect any of our calculations.

The Young basis played an important role in understanding the behaviour of perturbations around several giant gravitons that are well separated spatially [31, 35, 52]. From a Young diagram perspective, this corresponds to thinking about several rows, each of $O(N)$ length, where the difference in lengths is also $O(N)$. This situation is called the 'distant corners' approximation, and in this limit the representation (A.1.3) simplifies substantially.

A.2 Jucys-Murphy elements

The Jucys-Murphy elements are special elements of $\mathbb{C}(S_n)$, defined by

$$J_k = \sum_{i=1}^{k-1} (i, k) \tag{A.2.1}$$

for $k = 1, \ldots, n$. The Young basis vectors of R defined in A.1 are eigenvectors for the Jucys-Murphy elements, with eigenvalues given by the contents of the Young tableau. On a given standard Young tableau r of shape R, the eigenvalue of J_k is the contents of the box labelled by k in r.

For example if we have $R = [3, 2, 1]$ the contents of the cells are

$$\begin{array}{|c|c|c|} \hline 0 & 1 & 2 \\ \hline -1 & 0 & \\ \cline{1-2} -2 & & \\ \cline{1-1} \end{array}$$

so the eigenvalues of the Jucys-Murphy elements on 4 of the 16 different standard Young tableaux are

	$\begin{array}{ccc} 1 & 2 & 3 \\ 4 & 5 & \\ 6 & & \end{array}$	$\begin{array}{ccc} 1 & 3 & 5 \\ 2 & 6 & \\ 4 & & \end{array}$	$\begin{array}{ccc} 1 & 2 & 6 \\ 3 & 4 & \\ 5 & & \end{array}$	$\begin{array}{ccc} 1 & 4 & 6 \\ 2 & 5 & \\ 3 & & \end{array}$
J_2	1	-1	1	-1
J_3	2	1	-1	-2
J_4	-1	-2	0	1
J_5	0	2	-2	0
J_6	-2	0	2	2

The Jucys-Murphy elements span a maximal commuting sub-algebra of $\mathbb{C}(S_n)$, and therefore one can choose a basis of any irreducible representation to be simultaneous eigenvectors of all the J_k. The Young basis is exactly this choice [117]. For a more thorough treatment of the Jucys-Murphy elements and their properties, see [118, 119].

Any symmetric polynomial of the J_k lies in the centre of S_n. In particular, it is a standard result, see for example [118], that Ω as defined in (2.3.17) can also be written

$$\Omega = \prod_{i=1}^{n} (N + J_i) \tag{A.2.2}$$

The expression (2.3.20) for the representative of Ω in a representation R of S_n follows immediately from (A.2.2). Consider the action of Ω on a standard Young tableau r of shape R. Each of the J_i will pick up the contents of the box containing i. Since i runs over all entries in a tableau, this covers all boxes in R and the eigenvalue is just the product of $N + c_b$ for $b \in R$, independent of the tableau r.

The importance of Ω for this work stems from its involvement in $U(N)$ correlators, seen in (2.6.2), (2.6.7) and (2.6.12). There are two other elements of $\mathbb{C}(S_n)$, defined in terms of Jucys-Murphy elements, that are relevant for correlators of $SO(N)/Sp(N)$ mesonic operators and $SO(N)$ baryonic operators respectively. The first of these, $\widetilde{\Omega}$, had been used before in [56,57,59,60] to calculate correlators in the Schur and restricted Schur bases for $SO(N)$ and $Sp(N)$, but its importance for the $O(\frac{1}{N})$ expansion of the multi-trace basis had not been understood. The second, Ω^{ε}, has not previously been studied.

For each of $\widetilde{\Omega}$ and Ω^{ε}, we give a definition in terms of Jucys-Murphy elements, analogous to (A.2.2), the key result linking it to $SO(N)/Sp(N)$ mesonic or baryonic correlators, analogous to (2.3.17), and the action on the appropriate invariant vectors from section 5.1.3.

A.2.1 $\widetilde{\Omega}$

Consider $S_n[S_2]$, defined in section 5.1.2, as a subgroup of S_{2n}. Then one can choose the set $\mathcal{B}$ of right coset representatives of $S_n[S_2]$ such that

$$\sum_{\beta \in \mathcal{B}} C^{\delta}(\beta)\, \beta = \widetilde{\Omega} := \prod_{i=1}^{n} (N + J_{2i-1}) \tag{A.2.3}$$

where

$$C^{\delta}(\beta) = C_I^{(\delta)} \beta^I_J C^{(\delta)\, J} \tag{A.2.4}$$

and $C_I^{(\delta)}$ is n-fold tensor product of δ_{ij}, defined formally in (5.2.1). Since J_{2i-1} commute with each other, we do not need to give an ordering for the product. (A.2.3) was the key result that enabled the evaluation of the mesonic correlator in [56, 59], and it is proved inductively in [118].

Consider the sum over all elements of $S_n[S_2]$, and multiply by (A.2.3) on the right. Since the β in (A.2.3) are representatives of the right cosets of $S_n[S_2]$ and C^{δ} is invariant under multiplication by elements of $S_n[S_2]$, the left-hand side is a sum over all of S_{2n}

$$\sum_{\tau \in S_{2n}} C^{\delta}(\tau)\,\tau = \left(\sum_{\sigma \in S_n[S_2]} \sigma \right) \widetilde{\Omega} = 2^n n! P_{[S]} \widetilde{\Omega} \tag{A.2.5}$$

where $P_{[S]}$ is the projector onto the symmetric representation of $S_n[S_2]$ defined in (5.1.3).

This result is analogous to (2.3.17) for Ω, with $N^{c(\sigma)}$ replaced by $C^{\delta}(\sigma)$. We can make this analogy closer by expressing $C^{\delta}(\sigma)$ as a power of N.

Recall from section 5.3 that we can choose representatives of the double cosets (these are not the right cosets of (A.2.3)) of $S_n[S_2]$ to be a permutation $\sigma_p \in S_n$ of cycle type $p \vdash n$ embedded into S_{2n} as $\sigma_p^{(odd)}$ by acting on the odd number $\{1, 3, \dots, 2n-1\}$. Then we have

$$\begin{aligned} C^{\delta}\left(\sigma_p^{(odd)}\right) &= C_I^{(\delta)} \left(\sigma_p^{(odd)}\right)^I_J C^{(\delta)\,J} \\ &= \delta_{i_1 j_1} \delta_{i_2 j_2} \dots \delta_{i_n j_n} \left(\sigma_p^{(odd)}\right)^{i_1 j_1 i_2 j_2 \dots i_n j_n}_{k_1 l_1 k_2 l_2 \dots k_n l_n} \delta^{k_1 l_1} \delta^{k_2 l_2} \dots \delta^{k_n l_n} \\ &= \delta_{i_1 j_1} \dots \delta_{i_n j_n} (\sigma_p)^{i_1 i_2 \dots i_n}_{k_1 k_2 \dots k_n} \delta^{j_1}_{l_1} \delta^{j_2}_{l_2} \dots \delta^{j_n}_{l_n} \delta^{k_1 l_1} \delta^{k_2 l_2} \dots \delta^{k_n l_n} \\ &= (\sigma_p)^{i_1 i_2 \dots i_n}_{i_1 i_2 \dots i_n} = N^{c(\sigma_p)} = N^{l(p)} \end{aligned} \tag{A.2.6}$$

where $c(\sigma_p)$ is the number of cycles in σ_p, and we have used the standard result (2.1.7) for the trace of a permutation on a tensor space. This calculation is very similar to (5.3.16), though here X is taken to be the identity matrix. An intuitive understanding of how $\sigma^{(odd)}$ turns an $SO(N)$ type contraction pattern into a $U(N)$ type contraction of $\sigma \in S_n$ is given in figure 5.5.

Since σ_p is a representative of the double coset, a generic π in the p double coset can be written $\pi = \tau_1 \sigma_p \tau_2$ for $\tau_1, \tau_2 \in S_n[S_2]$. Since $C^{\delta}(\sigma)$ is invariant under $S_n[S_2]$ multiplication on the left or right

$$C^{\delta}(\pi) = N^{l(p)} \tag{A.2.7}$$

Defining p_τ to be the partition labelling the double coset of $\tau \in S_{2n}$, we can rewrite (A.2.5) as

$$\sum_{\tau \in S_{2n}} N^{l(p_\tau)} \tau = \left(\sum_{\sigma \in S_n[S_2]} \sigma \right) \widetilde{\Omega} = 2^n n! P_{[S]} \widetilde{\Omega} \tag{A.2.8}$$

We can now compare directly with the $U(N)$ version (2.3.17). In both cases, there is a partition p_τ associated to a permutation τ. Then Ω and $\widetilde{\Omega}$ are related to the sum over τ (in the relevant permutation group) of $N^{l(p_\tau)}\tau$.

Above (A.2.2) we noted that any symmetric polynomial in the Jucys-Murphy elements is in the centre of S_n. For odd Jucys-Murphy elements J_{2i-1} there is a similar result [120]. Any symmetric polynomial in the odd Jucys-Murphy elements, when multiplied on the left or right by $\left(\sum_{\sigma \in S_n[S_2]} \sigma\right)$, can be written as a sum over double cosets. As described in section 5.3, this means it belongs to the Hecke algebra $S_n[S_2] \backslash S_{2n} / S_n[S_2]$.

There is a symplectic equivalent to $C^\delta(\sigma)$, given by

$$C^\Omega(\sigma) = C_I^{(\Omega)} \sigma_J^I C^{(\Omega)\,J} \tag{A.2.9}$$

where $C_I^{(\Omega)}$ is a n-fold tensor product of Ω_{ij}, defined formally in (5.2.3).

Take $\sigma_p \in S_n$ with the corresponding $\sigma_p^{(odd)} \in S_{2n}$ a representative of the p double coset. Then

$$\begin{aligned}
C^\Omega\left(\sigma_p^{(odd)}\right) &= C_I^{(\Omega)} \left(\sigma_p^{(odd)}\right)_J^I C^{(\Omega)\,J} \\
&= \Omega_{i_1 j_1} \Omega_{i_2 j_2} \ldots \Omega_{i_n j_n} (\sigma_p)_{k_1 l_1 k_2 l_2 \ldots k_n l_n}^{i_1 j_1 i_2 j_2 \ldots i_n j_n} \Omega^{k_1 l_1} \Omega^{k_2 l_2} \ldots \Omega^{k_n l_n} \\
&= \Omega_{i_1 j_1} \Omega_{i_2 j_2} \ldots \Omega_{i_n j_n} (\sigma_p)_{k_1 k_2 \ldots k_n}^{i_1 i_2 \ldots i_n} \delta_{l_1}^{j_1} \delta_{l_2}^{j_2} \ldots \delta_{l_n}^{j_n} \Omega^{k_1 l_1} \Omega^{k_2 l_2} \ldots \Omega^{k_n l_n} \\
&= (\sigma_p)_{k_1 k_2 \ldots k_n}^{i_1 i_2 \ldots i_n} (\Omega \Omega^T)_{i_1}^{k_1} (\Omega \Omega^T)_{i_2}^{k_2} \ldots (\Omega \Omega^T)_{i_n}^{k_n} \\
&= (\sigma_p)_{i_1 i_2 \ldots i_n}^{i_1 i_2 \ldots i_n} = N^{c(\sigma_p)} = N^{l(p)}
\end{aligned} \tag{A.2.10}$$

where we have used $\Omega\Omega^T = 1$.

In S_{2n} the cycle type of σ_p is $p + [1^n]$, and therefore the sign of σ_p is $(-1)^{n+l(p)}$. We deduce

$$C^\Omega(\sigma_p) = (-1)^n (-1)^{\sigma_p} (-N)^{l(p)} \tag{A.2.11}$$

Since $C^\Omega(\sigma)$ is anti-invariant under multiplication by $S_n[S_2]$ on either side, for a generic $\pi = \tau_1 \sigma_p \tau_2$ in the p double coset we have

$$C^\Omega(\pi) = (-1)^n (-1)^\pi (-N)^{l(p)} \tag{A.2.12}$$

Comparing with (A.2.7), we see that C^Ω and C^δ are related by anti-symmetrisation of permutations and $N \to -N$, up to a factor of $(-1)^n$. This is an example of a very general relation between the mesonic sector of the $SO(N)$ theory and the $Sp(N)$ theory, explained in (4.0.3) and (5.0.2).

More practically, this relation allows us to derive a symplectic result equivalent to (A.2.8). Define the anti-symmetrisation of a permutation τ to be

$$\text{Anti-Sym}(\tau) = (-1)^\tau \tau \tag{A.2.13}$$

and extend linearly to the whole of $\mathbb{C}(S_n)$. Then $C^\Omega(\tau)$ and $C^\delta(\tau)$ are related by

$$C^\Omega(\tau) = (-1)^n \text{ Anti-Sym}(\tau)|_{N\to -N} \tag{A.2.14}$$

Recall from the definition (A.2.1) that Jucys-Murphy elements are composed purely of transpositions, and hence they will pick up a minus sign under anti-symmetrisation. Therefore

$$\begin{aligned}
\sum_{\tau\in S_{2n}} C^\Omega(\tau)\tau &= (-1)^n \text{Anti-Sym}\left[\sum_{\tau\in S_{2n}} C^\delta(\tau)\tau\right]_{N\to -N} \\
&= (-1)^n \text{Anti-Sym}\left[\left(\sum_{\sigma\in S_n[S_2]} \sigma\right)\widetilde{\Omega}\right]_{N\to -N} \\
&= (-1)^n \left[\left(\sum_{\sigma\in S_n[S_2]} (-1)^\sigma \sigma\right)\prod_{i=1}^{n}(N - J_{2i-1})\right]_{N\to -N} \\
&= (-1)^n \left(\sum_{\sigma\in S_n[S_2]} (-1)^\sigma \sigma\right)\prod_{i=1}^{n}(-N - J_{2i-1}) \\
&= \left(\sum_{\sigma\in S_n[S_2]} (-1)^\sigma \sigma\right)\widetilde{\Omega} = 2^n n! P_{[A]}\widetilde{\Omega}
\end{aligned} \tag{A.2.15}$$

where $P_{[A]}$ is the projector onto the anti-symmetric representation of $S_n[S_2]$, defined in (5.1.3).

Action of $\widetilde{\Omega}$

Consider the state $|R,[S]\rangle$ (defined in (5.1.4)) in a representation $R \vdash 2n$ with even row lengths. This state can be written as a sum over standard Young tableaux r

$$|R,[S]\rangle = \sum_{\substack{r \text{ of} \\ \text{shape } R}} a_r |r\rangle \tag{A.2.16}$$

It is proved in [118] that the r in this sum (those with non-zero a_r) must have a certain form. The numbers $2i-1$ and $2i$ must appear in a pair, with the even number directly to the right of the odd. For example, given $R = [4,2,2]$, the r that contribute in the sum (A.2.16) are

$$r_1 = \begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 4 \\ \hline 5 & 6 \\ \cline{1-2} 7 & 8 \\ \cline{1-2}\end{array} \qquad r_2 = \begin{array}{|c|c|c|c|}\hline 1 & 2 & 5 & 6 \\ \hline 3 & 4 \\ \cline{1-2} 7 & 8 \\ \cline{1-2}\end{array} \qquad r_3 = \begin{array}{|c|c|c|c|}\hline 1 & 2 & 7 & 8 \\ \hline 3 & 4 \\ \cline{1-2} 5 & 6 \\ \cline{1-2}\end{array} \tag{A.2.17}$$

As a result, the odd numbers in r must lie in the odd-numbered columns of R. When we act with $\widetilde{\Omega}$ on these r, the J_{2i-1} pick up the contents of all the boxes in the odd-numbered columns. Therefore

$$D^R\left(\widetilde{\Omega}\right)|R,[S]\rangle = \prod_{\substack{b\,\in\,\text{odd}\\ \text{columns of } R}} (N+c_b)\,|R,[S]\rangle = f_R^{\delta}|R,[S]\rangle \tag{A.2.18}$$

where this defines f_R^{δ}.

Similarly, for $R \vdash 2n$ with even column lengths, $|R,[A]\rangle$ (defined in (5.1.4)) is an eigenvector for $\widetilde{\Omega}$

$$D^R\left(\widetilde{\Omega}\right)|R,[A]\rangle = \prod_{\substack{b\,\in\,\text{odd}\\ \text{rows of } R}} (N+c_b)\,|R,[A]\rangle = f_R^{\Omega}|R,[A]\rangle \tag{A.2.19}$$

where this defines f_R^{Ω}.

If $l(R) > N$, then the Young diagram contains a box b in the $(N+1)$th row and the first column. The contents of b is $c_b = -N$, and therefore the factor associated to b is $N + c_b = 0$. This box is in an odd column, and therefore $f_R^{\delta} = 0$ for $l(R) > N$. For the $Sp(N)$ theory, N must be even, and hence this box is also in an odd row, meaning $f_R^{\Omega} = 0$ for $l(R) > N$. So $\widetilde{\Omega}$ enforces the finite N cut-off in R on the invariant vectors $|R,[S]\rangle$ and $|R,[A]\rangle$.

A.2.2 Ω^{ε}

The last element we consider is relevant for correlators of baryonic operators, and therefore the group of interest is now $S_N \times S_q[S_2]$ where $q = n - \frac{N}{2}$ (recall N must be even for baryonic operators to exist) and we embed S_N in S_{2n} by acting on $\{1, 2, \ldots, N\}$ while $S_q[S_2]$ acts on the pairs $\{N+2i-1, N+2i\}$ for $1 \leq i \leq q$. We define

$$\Omega^{\varepsilon} = N! \prod_{i=1}^{q} \left[N + \sum_{j=1}^{N} (j, N+2i-1) + \sum_{j=1}^{N} (j, N+2i) + \sum_{j=1}^{2i-2} (N+j, N+2i-1) \right] \tag{A.2.20}$$

where the product is ordered $[i=q][i=q-1]\ldots[i=1]$. This can be written in terms of Jucys-Murphy elements.

$$\Omega^{\varepsilon} = N! \prod_{i=1}^{q} \left[N + J_{N+2i-1} + J_{N+2i} - \bar{J}_{2i} \right] \tag{A.2.21}$$

where $\bar{J}_i$ is the Jucys-Murphy element for the subgroup S_{2q}, embedded into S_{2n} by acting on $\{N+1, N+2, \ldots, N+2q = n\}$.

$$\bar{J}_k = \sum_{i=1}^{k-1} (N+i, N+k) \tag{A.2.22}$$

In analogy to (A.2.3), we prove the proposition

Proposition
The set $\mathcal{B}$ of right coset representatives of $S_N \times S_q[S_2]$ inside S_{2n} can be chosen such that

$$\sum_{\beta \in \mathcal{B}} C^{\varepsilon}(\beta)\beta = \Omega^{\varepsilon} \tag{A.2.23}$$

where $C^{\varepsilon}(\beta)$ is defined in (5.5.24). For $\sigma \in S_N \times S_q[S_2]$, let σ_1 be the S_N component. It then immediately follows from (A.2.23) that

$$\sum_{\tau \in S_{2n}} C^{\varepsilon}(\tau)\tau = \left(\sum_{\sigma \in S_N \times S_q[S_2]} (-1)^{\sigma_1} \sigma \right) \Omega^{\varepsilon} \tag{A.2.24}$$

where we have multiplied on the left by $N! 2^q q! P_{[1^N] \otimes [S]}$ as defined in (5.1.24) and used the definition of $\mathcal{B}$ and the invariance of $C^{\varepsilon}(\beta)$ to change the sums over $\mathcal{B}$ and $S_N \times S_q[S_2]$ into a sum over S_{2n}.

Proof
We begin by characterising the right cosets of $S_N \times S_q[S_2]$. Define the set of pairs

$$w_0 = \Big\{ \{N+1, N+2\}, \{N+3, N+4\}, \ldots, \{N+2q-1, N+2q\} \Big\} \tag{A.2.25}$$

Then $\sigma \in S_{2n}$ acts on w_0 (and other possible pairings) by

$$\sigma(w_0) = \Big\{ \{\sigma(N+1), \sigma(N+2)\}, \{\sigma(N+3), \sigma(N+4)\}, \ldots, \{\sigma(N+2q-1), \sigma(N+2q)\} \Big\} \tag{A.2.26}$$

It follows from the definition of $S_q[S_2]$ that

$$\sigma \in S_N \times S_q[S_2] \iff \sigma(w_0) = w_0 \tag{A.2.27}$$

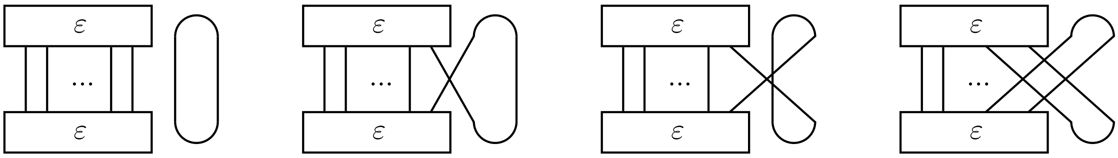


Figure A.1: Diagrammatic calculation of $C^{\varepsilon}(\beta)$ for $\beta = 1, (N, N+1), (N, N+2)$ and $(N-1, N+1)(N, N+2)$ respectively. Two εs fully contracted contribute $\varepsilon_{i_1 \dots i_N}\varepsilon^{i_1 \dots i_N} = N!$ while a loop gives $\delta_{ij}\delta^{ij} = N$. Since ε is anti-symmetric and δ is symmetric, a contraction between the two gives 0.

Therefore the left cosets of $S_N \times S_q[S_2]$ are labelled by a choice of q pairs in $\{1, 2, \dots, 2n\}$. For a given pairing

$$w = \Big\{\{i_{1,1}, i_{1,2}\}, \{i_{2,1}, i_{2,2}\}, \dots, \{i_{q,1}, i_{q,2}\}\Big\} \tag{A.2.28}$$

any $\beta \in S_{2n}$ in the corresponding right coset satisfies

$$\beta(w_0) = w \tag{A.2.29}$$

Let W_q denote the set of possible pairings, and β_w the coset representative for $w \in W_q$.

We prove (A.2.23) by induction on q at fixed N. First we consider the base case with $q = 1$. The possible $w \in W_1$, along with the associated β_w and $C^{\varepsilon}(\beta_w)$ are

w	$\{\{N+1, N+2\}\}$	$\{\{k, N+1\}\}$	$\{\{k, N+2\}\}$	$\{\{l_1, l_2\}\}$
β_w	1	$(k, N+2)$	$(k, N+1)$	$(l_1, N+1)(l_2, N+2)$
$C^{\varepsilon}(\beta_w)$	$N!N$	$N!$	$N!$	0

where $1 \le k, l_1, l_2 \le N$ and l_1, l_2 are distinct. It is simple to check that these β_w satisfy the conditions in (A.2.29) and therefore serve as coset representatives. The calculations for $C^{\varepsilon}(\beta_w)$ are shown diagrammatically in figure A.1. For simplicity the figure shows $k = N$ in the pairings $p = \{\{k, N+1\}\}, \{\{k, N+2\}\}$ and $l_1 = N-1, l_2 = N$ in the pairing $p = \{\{l_1, l_2\}\}$, but the results hold for all k, l_1, l_2.

Summing the contributions from each of the $w \in W_1$, we have

$$\sum_{w \in W_1} C^{\varepsilon}(\beta_w)\beta_w = N! \left[N + \sum_{j=1}^{N}(j, N+1) + \sum_{j=1}^{N}(j, N+2) \right] \tag{A.2.30}$$

as claimed in (A.2.23). We could also consider the case $q = 0$, where the product on the right of (A.2.23) is empty, and we take the representative of the only coset to be the identity, so the result is trivial.

Assume the claim is true for $q - 1$. In particular this means that there is a map from $W_{q-1} \to S_{N+2q-2}$, namely $\bar{w} \to \beta_{\bar{w}}$, such that for each $\bar{w}$, $\beta_{\bar{w}}$ satisfies (A.2.29), and the $\beta_{\bar{w}}$ combine so as to satisfy (A.2.23) for $q-1$.

Consider the case at q. The pairings $w \in W_q$ fall into 5 categories depending on how $N+2q-1$ and $N+2q$ pair (or don't pair) up with the first $N+2q-2$ numbers.

1. $\{N+2q-1, N+2q\}$ is a pair
2. $\{k_1, N+2q-1\}$ and $\{k_2, N+2q\}$ are pairs, for some $k_1, k_2 < N+2q-1, k_1 \neq k_2$
3. $N+2q$ is unpaired and $\{k, N+2q-1\}$ is a pair, for some $k < N+2q-1$
4. $N+2q-1$ is unpaired and $\{k, N+2q\}$ is a pair, for some $k < N+2q-1$
5. $N+2q-1$ and $N+2q$ are both unpaired

We split up the sum over W_q into five sums, one for each type of pairing.

Type 1

Let $W_{q;1}$ be the set of pairings that are of type 1. Given $w \in W_{q;1}$, first note that w reduces uniquely to a $\bar{w} \in W_{q-1}$ given by $\bar{w} = w \backslash \{N+2q-1, N+2q\}$. Using this $\bar{w}$, we choose the coset representative of w to be

$$\beta_w = \beta_{\bar{w}} \tag{A.2.31}$$

By which we mean that β_w acts as $\beta_{\bar{w}}$ on $\{1, 2, \dots, N+2q-2\}$ and as the identity on $\{N+2q-1, N+2q\}$. It is simple to check that this satisfies the conditions (A.2.29).

To calculate $C^{\varepsilon}(\beta_w)$, add an extra label q onto the contractor $C_I^{(\varepsilon)}$ to record how many indices it has. So $C^{(\varepsilon;q)}_{i_1 \dots i_{N+2q}} = C^{(\varepsilon;q-1)}_{i_1 \dots i_{N+2q-2}} \delta_{i_{N+2q-1} i_{N+2q}}$. This allows us to relate $C^{\varepsilon}(\beta_w)$ and $C^{\varepsilon}(\beta_{\bar{w}})$. The calculation is shown diagrammatically at the top left of figure A.2. We find

$$C^{\varepsilon}(\beta_w) = N C^{\varepsilon}(\beta_{\bar{w}}) \tag{A.2.32}$$

Given a $\bar{w} \in W_{q-1}$, there is a unique $w \in W_{q;1}$ which reduces to $\bar{w}$, namely $w = \bar{w} \cup \{N+2q-1, N+2q\}$. Therefore

$$\sum_{w \in W_{q;1}} C^{\varepsilon}(\beta_w)\beta_w = N \sum_{\bar{w} \in W_{q-1}} C^{\varepsilon}(\beta_{\bar{w}})\beta_{\bar{w}} \tag{A.2.33}$$

Type 2

We follow the same route as for type 1. Let $W_{q;2}$ be the set of pairings that are of type 2. Given $w \in W_{q,2}$, we define $\bar{w} \in W_{q-1}$ by $\bar{w} = (w \cup \{k_1, k_2\}) \backslash \{\{k_1, N+2q-$

$1\}, \{k_2, N+2q\}\}$. We then choose the coset representative of w to be

$$\beta_w = \left(\beta_{\bar{w}}^{-1}(k_2), N+2q-1\right)\beta_{\bar{w}} = \beta_{\bar{w}}\left(k_2, N+2q-1\right) \tag{A.2.34}$$

Again, one can check that this satisfies the conditions (A.2.29).

The calculation for $C^{\varepsilon}(\beta_w)$ is shown diagrammatically in figure A.2 in the middle of the top row. For simplicity, the calculation shown has $k_2 = N+2q-2$, but it is clear that for any k_2 we arrive at the relation

$$C^{\varepsilon}(\beta_w) = C^{\varepsilon}(\beta_{\bar{w}}) \tag{A.2.35}$$

Consider an arbitrary $\bar{w} \in W_{q-1}$. We can explicitly write this out as

$$\bar{w} = \Big\{\{l_{1,1}, l_{1,2}\}, \{l_{2,1}, l_{2,2}\}, \ldots, \{l_{q-1,1}, l_{q-1,2}\}\Big\} \tag{A.2.36}$$

There are $2(q-1)$ different $w \in W_{q;2}$ which reduce to $\bar{w}$, two for each pair of $\bar{w}$. For $1 \leq \iota \leq q-1$, these are

$$\begin{aligned} w_{i,1} &= \Big(\bar{w} \cup \Big\{\{l_{i,2}, N+2q-1\}, \{l_{i,1}, N+2q\}\Big\}\Big) \setminus \Big\{\{l_{i,1}, l_{i,2}\}\Big\} \\ w_{i,2} &= \Big(\bar{w} \cup \Big\{\{l_{i,1}, N+2q-1\}, \{l_{i,2}, N+2q\}\Big\}\Big) \setminus \Big\{\{l_{i,1}, l_{i,2}\}\Big\} \end{aligned} \tag{A.2.37}$$

We split the sum over $W_{q;2}$ into a sum over W_{q-1}, $i = 1, 2, \ldots, q-1$ and $j = 1, 2$. For $w_{i,1}$, we have $k_2 = l_{i,1}$ and for $w_{i,2}$ we have $k_2 = l_{i,2}$, so using (A.2.34) for the coset representatives

$$\sum_{w \in W_{q;2}} C^{\varepsilon}(\beta_w)\beta_w = \sum_{i=1}^{q-1}\sum_{j=1}^{2}\left(\beta_{\bar{w}}^{-1}(l_{i,j}), N+2q-1\right)\sum_{\bar{w} \in W_{q-1}} C^{\varepsilon}(\beta_{\bar{w}})\beta_{\bar{w}} \tag{A.2.38}$$

From (A.2.29) we know that $\left\{\beta_{\bar{w}}^{-1}(\{l_{i,j}\})\right\} = \{N+1, N+2, \ldots, N+2q-2\}$, so we can simplify this to

$$\sum_{w \in W_{q;2}} C^{\varepsilon}(\beta_w)\beta_w = \sum_{j=1}^{2q-2}(N+j, N+2q-1)\sum_{\bar{w} \in W_{q-1}} C^{\varepsilon}(\beta_{\bar{w}})\beta_{\bar{w}} \tag{A.2.39}$$

Types 3 and 4

Let $W_{q;3}$ be the set of pairings that are of type 3. Given $w \in W_{q;3}$, we define $\bar{w} \in W_{q-1}$ by $\bar{w} = w \setminus \{k, N+2q-1\}$. We then choose the coset representative of w to be

$$\beta_w = \left(\beta_{\bar{w}}^{-1}(k), N+2q\right)\beta_{\bar{w}} = \beta_{\bar{w}}\left(k, N+2q\right) \tag{A.2.40}$$

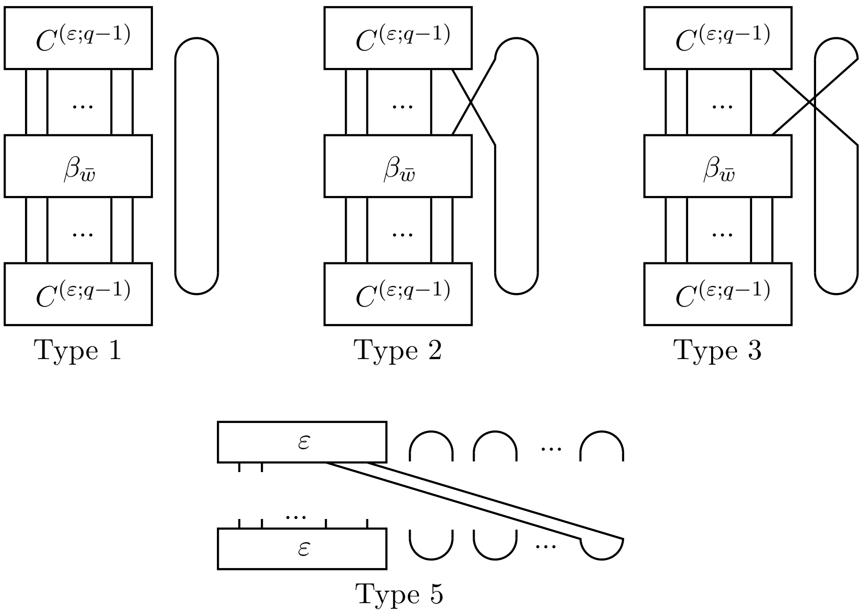


Figure A.2: Diagrammatic calculation of $C^{\varepsilon}(\beta)$ for various $\beta \in S_{N+2q}$ corresponding to type 1, 2, 3 and 5 pairings of $\{N+2q-1, N+2q\}$ with $\{1, 2, \ldots N+2q-2\}$. The top row shows $\beta = \beta_{\bar{w}}, \beta_{\bar{w}}(N+2q-2, N+2q-1)$ and $\beta_{\bar{w}}(N+2q-2, N+2q)$ respectively, where $\beta_{\bar{w}} \in S_{N+2q-2}$. The bottom row shows a β with $\beta(N-1) = N+2q-1$ and $\beta(N) = N+2q$. These two values of β are enough to ensure $C^{\varepsilon}(\beta) = 0$, so the remaining parts of β are not included in the diagram.

The calculation for $C^{\varepsilon}(\beta_w)$ is shown diagrammatically at the top right of figure A.2, and demonstrates that

$$C^{\varepsilon}(\beta_w) = C^{\varepsilon}(\beta_{\bar{w}}) \tag{A.2.41}$$

For simplicity, the calculation shown has $k = N + 2q - 2$, but clearly k can be arbitrary and we still arrive at the same result.

Take $\bar{w} \in W_{q-1}$. This contains $q - 1$ pairs from the set $\{1, 2, \dots N + 2q - 2\}$, so there are N numbers that are omitted. Let these be $\{l_1, \dots, l_N\}$. The different w which reduce to $\bar{w}$ are then given by $\bar{w} \cup \{l_i, N + 2q - 1\}$ for $i = 1, 2, \dots, N$ with the corresponding representative given by (A.2.40) with $k = l_i$. Splitting the sum over $w \in W_{q;3}$ into a sum over $\bar{w} \in W_{q-1}$ and $i = 1, 2, \dots, N$, we have

$$\sum_{w \in W_{q;3}} C^{\varepsilon}(\beta_w)\beta_w = \sum_{i=1}^{N} (\beta_{\bar{w}}(l_i), N + 2q) \sum_{\bar{w} \in W_{q-1}} C^{\varepsilon}(\beta_{\bar{w}})\beta_{\bar{w}} \tag{A.2.42}$$

From (A.2.29) we know that $\left\{\beta_{\bar{w}}^{-1}(\{l_i\})\right\} = \{1, 2, \dots, N\}$, therefore this simplifies to

$$\sum_{w \in W_{q;3}} C^{\varepsilon}(\beta_w)\beta_w = \sum_{j=1}^{N} (j, N + 2q) \sum_{\bar{w} \in W_{q-1}} C^{\varepsilon}(\beta_{\bar{w}})\beta_{\bar{w}} \tag{A.2.43}$$

We can repeat the above process with $N + 2q - 1$ and $N + 2q$ swapped to give the sum over type 4 pairings

$$\sum_{w \in W_{q;4}} C^{\varepsilon}(\beta_w)\beta_w = \sum_{j=1}^{N} (j, N + 2q - 1) \sum_{\bar{w} \in W_{q-1}} C^{\varepsilon}(\beta_{\bar{w}})\beta_{\bar{w}} \tag{A.2.44}$$

Type 5

Let $W_{q;5}$ be the set of pairings that are of type 5. Given $w \in W_{q,5}$, we can choose the coset representative β_w such that

$$\beta_w(N-1) = N + 2q - 1 \qquad \beta_w(N) = N + 2q \tag{A.2.45}$$

We do not need to specify the remaining values of β_w^{-1} as this is enough to show that $C^{\varepsilon}(\beta_w)$ vanishes. The calculation is shown diagrammatically on the bottom row of figure A.2. This means

$$\sum_{w \in W_{q;5}} C^{\varepsilon}(\beta_w)\beta_w = 0 \tag{A.2.46}$$

Adding together (A.2.33), (A.2.39), (A.2.43), (A.2.44) and (A.2.46), we get

$$\sum_{w\in W_q} C^\varepsilon(\beta_w)\beta_w = \left[N + \sum_{j=1}^{N}(j, N+2q-1) + \sum_{j=1}^{N}(j, N+2q) \right.$$
$$\left. + \sum_{j=1}^{2q-2}(N+j, N+2q-1)\right] \sum_{\bar{w}\in W_{q-1}} C^\varepsilon(\beta_{\bar{w}})\beta_{\bar{w}} \qquad \text{(A.2.47)}$$

The factor on the left is just the $i = q$ factor in (A.2.23), so plugging in the inductive assumption proves the proposition.

□

Action of Ω^ε

Consider the vector $\left|[1^N]\right\rangle \otimes |\bar{R}, [S]\rangle$, defined in (5.1.23), inside the representation R of S_{2n}. The restrictions on R and $\bar{R}$ and how the two diagrams are related is given at the end of section D.2.1.

There are two ways of expressing $\left|[1^N]\right\rangle \otimes |\bar{R}, [S]\rangle$ as a sum over Young tableaux. Firstly, since it is a vector in the R representation of S_{2n}, we can write it as a sum over standard Young tableaux r of shape R

$$\left|[1^N]\right\rangle \otimes |\bar{R}, [S]\rangle = \sum_{\substack{r \text{ of} \\ \text{shape } R}} a_r \,|r\rangle \qquad \text{(A.2.48)}$$

and secondly, since it is in the $[1^N] \otimes \bar{R}$ representation of $S_N \times S_{2q}$, we can write is as a sum over tensor products of two Young tableaux of shapes $[1^N]$ and $\bar{R}$ respectively. There is only one standard Young tableau of shape $[1^N]$, so we will suppress this tensor factor and just write $|\bar{r}\rangle$, where $\bar{r}$ is the tableau of shape $\bar{R}$

$$\left|[1^N]\right\rangle \otimes |\bar{R}, [S]\rangle = \sum_{\substack{\bar{r} \text{ of} \\ \text{shape } \bar{R}}} b_r \,|\bar{r}\rangle \qquad \text{(A.2.49)}$$

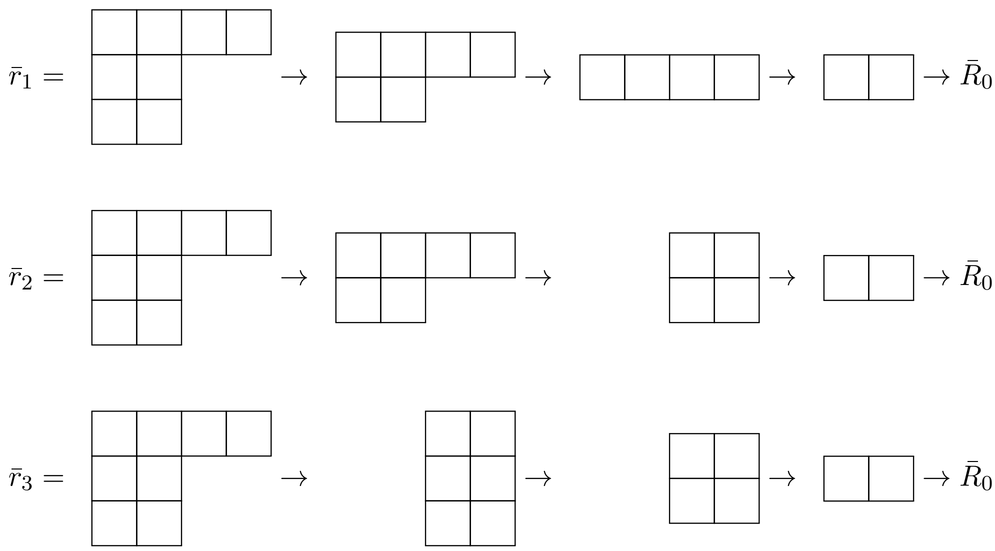
We now investigate how these two expansions of $\left|[1^N]\right\rangle \otimes |\bar{R}, [S]\rangle$ relate to each other.

Start by considering the tableaux in the second expansion (A.2.49). As explained below (A.2.16), these are restricted so that the numbers $2i-1$ and $2i$ appear in pairs, with the even number immediately to the right of the odd. As explained above (A.1.4), the positions of the numbers in a Young tableau describe the behaviour under embedded subgroups, and therefore the positions of each pair $\{2i-1, 2i\}$ in the distinct tableaux describe how $\bar{r}$ fits into representations of $S_{2(q-1)}$, $S_{2(q-2)}$ etc. In this case, if we remove the numbers $2q-1$ and $2q$ from a tableau $\bar{r}$, the fact the two are paired mean the new reduced tableau also has a shape with even row lengths, and therefore admits a $S_{q-1}[S_2]$ invariant vector. Further, the reduced tableau will contribute to this invariant vector.

As discussed below (A.1.4), the decomposition of a vector in terms of its representations in embedded subgroups is equivalent to giving its tableau. For $\bar{r}$ contributing (A.2.49), this decomposition does not need to include the diagrams with an odd number of boxes, as those are determined as intermediate stages between the ones with an even number of boxes. For a given tableau $\bar{r}$, we denote the equivalent even decomposition by

$$\bar{r} = \bar{R}_q \to \bar{R}_{q-1} \to \cdots \to \bar{R}_2 \to \bar{R}_1 \to \bar{R}_0 \tag{A.2.50}$$

where $\bar{R}_q = \bar{R}$, $\bar{R}_1 = \square\square$ and $\bar{R}_0$ is the empty Young diagram. For example, the three tableaux in (A.2.17) that contribute to the invariant vector for $\bar{R} = [4, 2, 2]$ have even decompositions



(A.2.51)

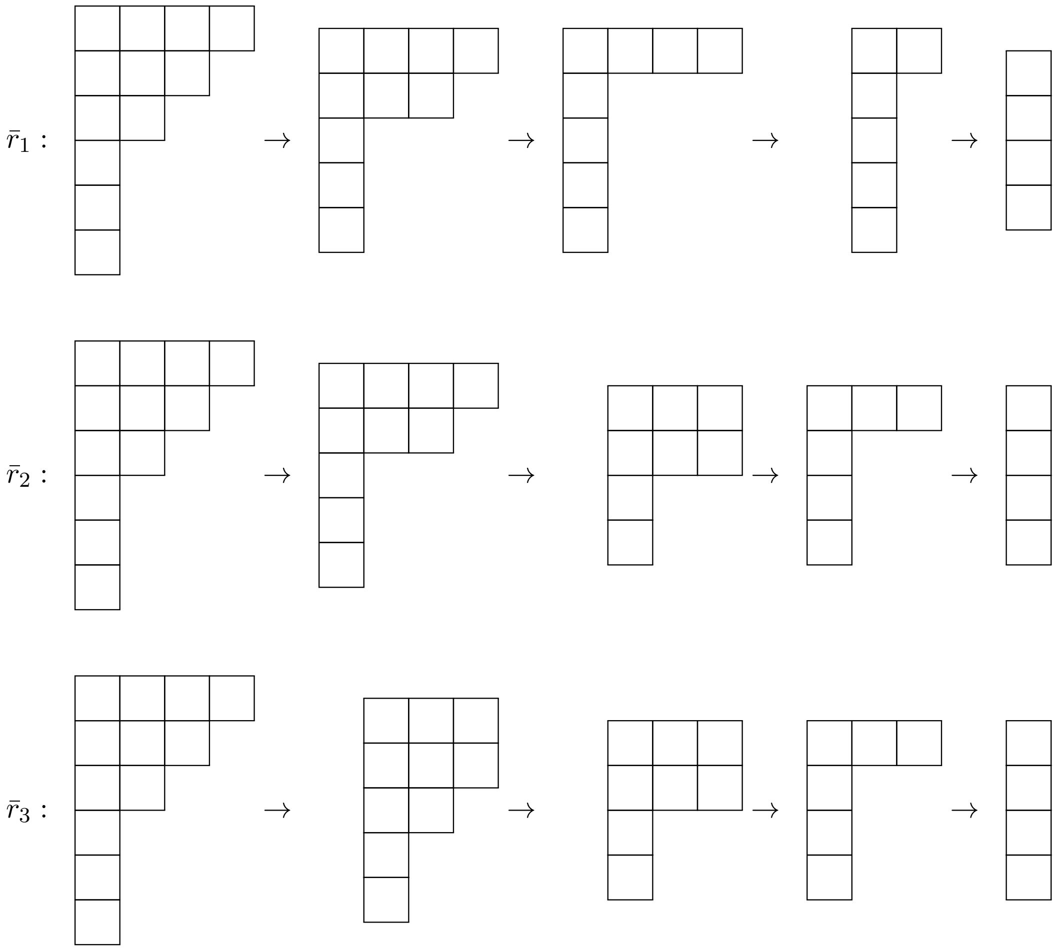
Under this new labelling, (A.2.49) reads as

$$\big|[1^N]\big\rangle \otimes |\bar{R}, [S]\rangle = \sum_{\bar{R}_q \to \cdots \to \bar{R}_1} b\big(\bar{R}_q \to \cdots \to \bar{R}_1\big)\, |\bar{R}_q \to \cdots \to \bar{R}_1\rangle \tag{A.2.52}$$

This gives us control of the $S_i[S_2]$ behaviour of $\big|[1^N]\big\rangle \otimes |\bar{R}, [S]\rangle$ for each $1 \le i \le q$.

Now consider how each term of (A.2.52) behaves under S_{2n} permutations, and in particular the decomposition as we reduce to $S_{2(n-1)}$, $S_{2(n-2)}$ and so on. Using the correspondence between R and $\bar{R}$ (this time with $N+2q-2$ and $q-2$ boxes respectively) established at the end of section D.2.1, the diagram $\bar{R}_{q-1}$ determines a R_{q-1}, the diagram $\bar{R}_{q-2}$ determines a R_{q-2}, etc. The full even decomposition $\bar{R}_q \to \cdots \to \bar{R}_0$ of $\bar{R}$ gives a corresponding even decomposition $R_q \to \cdots \to R_0$ where $R_q = R$ and

$R_0 = [1^N]$. Taking $N = 4$ and $R = [4, 3, 2, 1, 1, 1]$, the $\bar{R}$ even decompositions (A.2.51) have R equivalents



(A.2.53)

To describe the R decomposition explicitly from the $\bar{R}$ version, use the notation of section D.2.1. We have $R = [1^{N+k}] + \bar{S}$, where $\bar{S}$ has k odd rows, such that if we add a single box to each of these rows we get $\bar{R}$. Let the set of odd rows be $v = \{v_1, v, \ldots, v_k\}$, and denote the relation between $\bar{S}$ and $\bar{R}$ as $\bar{R} = \bar{S} +_v [1^k]$.

We can obtain $\bar{R}_{q-1}$ from $\bar{R} = \bar{R}_q$ by removing two boxes from a single row u. Then if $u \in v$, the corresponding $\bar{S}_{q-1}$ is obtained from $\bar{S}_q = \bar{S}$ by removing a single box from row u of $\bar{S}_q$, and the corresponding R_{q-1} is

$$R_{q-1} = [1^{N+k-1}] + \bar{S}_{q-1} \tag{A.2.54}$$

i.e. we have also removed a box from the first column of R. So $k \to k_{q-1} = k - 1$ and $v \to v_{q-1} = v/\{u\}$.

If $u \notin v$, $\bar{S}_{q-1}$ is obtained from $\bar{S}_q$ by removing two boxes from row u, and R_{q-1} is

$$R_{q-1} = [1^{N+k}] + \bar{S}_{q-1} \tag{A.2.55}$$

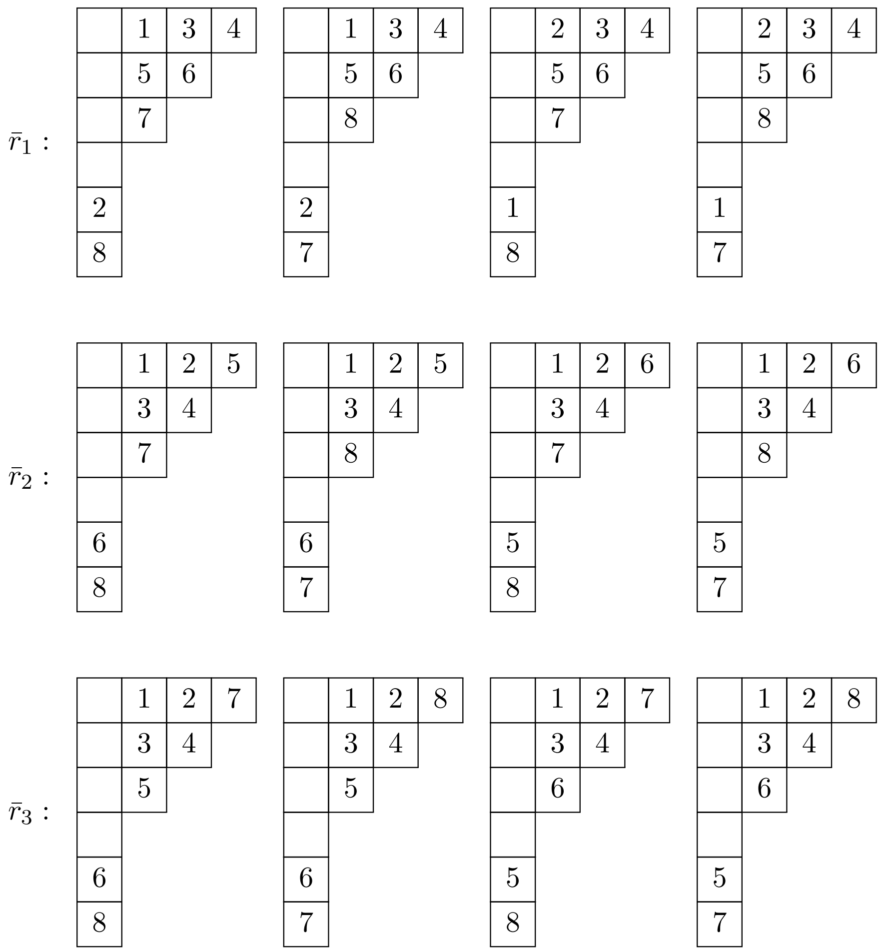
so the first column is unchanged, as are $k_{q-1} = k$ and $v_{q-1} = v$. Iterating this process describes the R even decomposition associated to $\bar{R}_q \to \cdots \to \bar{R}_0$.

This even decomposition of R gives some, but not all, of the information necessary to construct (A.2.48). When $R_i \to R_{i-1}$ has the form (A.2.54), there are two associated Young tableaux. One has $N + 2i - 1$ at the bottom of the 1st column and $N + 2i$ in the uth row, while they can also be the other way round. For the action of Ω^{ε}, these two tableaux are equivalent since they have the same eigenvalue under $J_{N+2i-1} + J_{N+2i}$ from the product (A.2.21). Even decompositions of form (A.2.55) completely fix the position of $N+2i-1$ and $N+2i$, so there are no more ambiguities in the Young tableaux corresponding to a particular even decomposition of R and $\bar{R}$. We can therefore write

$$|\bar{r}\rangle = |\bar{R}_q \to \cdots \to \bar{R}_1\rangle = \sum_r c_r \, |r\rangle \tag{A.2.56}$$

where r runs over the standard Young tableaux of shape R whose even decompositions agree with the decomposition $R_q \to \cdots \to R_0$ corresponding to $\bar{R}_q \to \cdots \to \bar{R}_0$. This allows an identification between the two sums (A.2.48) and (A.2.49).

As an example of the tableaux r that contribute to a sum of the form (A.2.56), consider the three decompositions in (A.2.53). Each has four Young tableaux r that contribute



(A.2.57)

where to fit the numbers in the tableaux we have used i to represent $N+i$.

Using the identification (A.2.56), each term in the product (A.2.21) for Ω^{ε} has a definite eigenvalue on the separate $\bar{r}$ in (A.2.49). Consider just the factor $N+J_{N+2i-1}+J_{N+2i}-\bar{J}_{2i}$. Write $c(j,\bar{r})$ for the contents of the box labelled by j in $\bar{r}$. Then $\bar{J}_{2i}$ has eigenvalue $c(2i,\bar{r})$ on $\bar{r}$. If the ith stage of the reduction corresponding to $\bar{r}$ has the form (A.2.54), then $J_{N+2i-1}+J_{N+2i}$ has eigenvalue $(-N-k_i+1)+(c(2i-1,\bar{r})+1)$, while if it has form (A.2.54), then $J_{N+2i-1}+J_{N+2i}$ has eigenvalue $c(2i-1,\bar{r})+c(2i,\bar{r})+2$. Noting that $c(2i-1,\bar{r})+1=c(2i,\bar{r})$, the total eigenvalue is

$$D^R(N+J_{N+2i-1}+J_{N+2i}-\bar{J}_{2i})\,|\bar{r}\rangle = \begin{cases} (N+c(2i-1,\bar{r})+2)\,|\bar{r}\rangle & R_i \to R_{i-1} \text{ has form (A.2.55)} \end{cases} \tag{A.2.58}$$

So the factors in (A.2.21) commute on $|\bar{r}\rangle$ and hence it is an eigenvector of Ω^{ε}, where the eigenvalue is the product of (A.2.58) over $i=1,2,\ldots,q$ (then multiplied by $N!$).

Suppose R has $l(R)>N$. Then $k\geq 1$, and for every decomposition $R_q\to\cdots\to R_0$, there will be some i where $1 = k_i \to k_{i-1} = 0$. At this point, the eigenvalue (A.2.58) will be 0. Therefore the eigenvalue of Ω^ε on every $\bar{r}$ will be 0. Re-summing to form $\left|[1^N]\right\rangle \otimes |\bar{R}, [S]\rangle$, Ω^ε will annihilate the invariant vector. Therefore Ω^ε enforces the finite N cut-off in R.

For $l(R) = N$, we have $R = [1^N] + \bar{R}$, and therefore every stage of the decomposition $R_q \to \cdots \to R_0$ is of the form (A.2.55). Taking the product to get the eigenvalue of Ω^ε

$$\begin{aligned} D^R(\Omega^\varepsilon)|\bar{r}\rangle &= N! \prod_{i=1}^{q} (N + c(2i-1, \bar{r}) + 2) |\bar{r}\rangle \\ &= N! \prod_{\substack{b \in \text{odd} \\ \text{columns of } \bar{R}}} (N + c_b + 2) |\bar{r}\rangle \end{aligned} \tag{A.2.59}$$

Since this eigenvalue is independent of $\bar{r}$, we can sum over $\bar{r}$ to form $\left|[1^N]\right\rangle \otimes |\bar{R}, [S]\rangle$

$$D^R(\Omega^\varepsilon)\left|[1^N]\right\rangle \otimes |\bar{R}, [S]\rangle = N! \prod_{\substack{b \in \text{odd} \\ \text{columns of } \bar{R}}} (N + c_b + 2) \left|[1^N]\right\rangle \otimes |\bar{R}, [S]\rangle \tag{A.2.60}$$

Interpret this eigenvalue in terms of the boxes of R rather than $\bar{R}$. Firstly, we only consider $\bar{R}$ with even length rows. Therefore we have

$$\prod_{\substack{b \in \text{odd} \\ \text{columns of } \bar{R}}} (N + c_b + 2) = \prod_{\substack{b \in \text{even} \\ \text{columns of } \bar{R}}} (N + c_b + 1) \tag{A.2.61}$$

In R, $\bar{R}$ has been moved one place to the right, so the contents of each cell increases by 1. In this context, the even columns of $\bar{R}$ become the odd columns of R, excluding the first column. Since the first column of R has length N, the product of $(N + c_b)$ on this column is $N!$, so we have

$$N! \prod_{\substack{b \in \text{odd} \\ \text{columns of } \bar{R}}} (N + c_b + 2) = \prod_{\substack{b \in \text{odd} \\ \text{columns of } R}} (N + c_b) \tag{A.2.62}$$

Substituting into (A.2.60), we have

$$D^R(\Omega^\varepsilon)\left|[1^N]\right\rangle \otimes |\bar{R}, [S]\rangle = \prod_{\substack{b \in \text{odd} \\ \text{columns of } R}} (N + c_b) \left|[1^N]\right\rangle \otimes |\bar{R}, [S]\rangle = f_R^\varepsilon \left|[1^N]\right\rangle \otimes |\bar{R}, [S]\rangle \tag{A.2.63}$$

where this defines f_R^ε. Note that from this definition, $f_R^\varepsilon = 0$ if $l(R) > N$, so (A.2.63) holds true for all R, not just those with $l(R) = N$.

Appendix B

Möbius inversion formula for positive integers

Proposition: The Möbius Inversion Formula
Let $\{a_n\}$ and $\{b_n\}$ be two sequences indexed by the positive integers. If a_n can be expressed as

$$a_n = \sum_{d|n} b_d = \sum_{d|n} b_{\frac{n}{d}} \tag{B.0.1}$$

where d runs over all divisors of n, denoted by $d|n$, then

$$b_n = \sum_{d|n} \mu\left(\frac{n}{d}\right) a_d = \sum_{d|n} \mu(d) a_{\frac{n}{d}} \tag{B.0.2}$$

where μ is the Möbius function defined by

$$\mu(d) = \begin{cases} 1 & d = 1 \\ (-1)^n & d \text{ a product of } n \text{ distinct prime factors} \\ 0 & d \text{ has a repeated prime in its prime factorisation} \end{cases} \tag{B.0.3}$$

The proof of this proposition relies on
Lemma

$$\sum_{d|n} \mu(d) = \begin{cases} 1 & n = 1 \\ 0 & n > 1 \end{cases} \tag{B.0.4}$$

Proof of Lemma
This is obvious for $n = 1$, so we will only prove the case $n > 1$. Writing n in terms of its prime factors, we have

$$n = p_1^{r_1} p_2^{r_2} \dots p_k^{r_k}$$

where $r_i \geq 1$ for each i. The divisors of n which contribute to the sum (B.0.4) are those which are square free. Explicitly, they can be written

$$d = p_1^{s_1} p_2^{s_2} \dots p_k^{s_k}$$

where $s_i \in \{0, 1\}$ for each i.

We define S to be the set of distinct prime factors of n: $S = \{p_1, p_2, \dots p_k\}$. Then subsets of S correspond exactly to the divisors d defined above

$$d = p_1^{s_1} p_2^{s_2} \dots p_k^{s_k} \qquad \longleftrightarrow \qquad \{p_i : s_i = 1\} \subseteq S \tag{B.0.5}$$

From the definition (B.0.3), we see that

$$\mu(d) = (-1)^{|\text{subset of } S \text{ corresponding to } d|}$$

So

$$\sum_{d|n} \mu(d) = \# \text{ of subsets of } S \text{ with even size} - \# \text{ of subsets of } S \text{ with odd size}$$

But we have a bijective map between even subsets and odd subsets given by

$$A \longrightarrow \begin{cases} A \cup \{p_1\} & p_1 \notin A \\ A/\{p_1\} & p_1 \in A \end{cases}$$

and therefore

$$\sum_{d|n} \mu(d) = 0$$

□

Proof of Proposition

The first step in the proof is to note that the a_n determine the b_n uniquely via the relation (B.0.1). Indeed, we have $b_1 = a_1$, $b_2 = a_2 - a_1$, $b_3 = a_3 - a_1$. To prove it in general, we use strong induction with these three as the base cases. Assuming b_n is determined by the sequences of as for all $n \leq k$, we can rearrange (B.0.1) to get

$$b_{k+1} = a_{k+1} - \sum_{\substack{d|(k+1) \\ d \neq k+1}} b_d$$

Then since the sum over d only includes $d \leq k$, we know inductively that b_d is determined by the as, and hence b_{k+1} is also determined by the as.

We now notice that the b_n, as defined in (B.0.2), satisfy (B.0.1):

$$\begin{aligned}\sum_{d|n} b_d &= \sum_{d|n}\sum_{e|d}\mu\left(\frac{d}{e}\right)a_e \\ &= \sum_{e|n} a_e \sum_{f|\frac{n}{e}}\mu(f) \\ &= a_n\end{aligned}$$

In going from the 1st to the 2nd line we have reordered the sums and reparameterised by $f = \frac{d}{e}$, and in going from the 2nd to the 3rd we have used the lemma (B.0.4).

Since the b_n have a unique solution, (B.0.2) must therefore be the correct formula for the b_n, as claimed. □

Note that in this proposition, there was nothing special about addition, the result and proof follow exactly the same way if we replace the addition by multiplication. Explicitly, given

$$b_n = \prod_{d|n} a_d = \prod_{d|n} a_{\frac{n}{d}}$$

we can invert uniquely to get

$$a_n = \prod_{d|n} b_d^{\mu\left(\frac{n}{d}\right)} = \prod_{d|n} b_{\frac{n}{d}}^{\mu(d)} \tag{B.0.6}$$

In chapters 3, we come across relations of the form

$$a_{n_1,n_2} = \sum_{d|n_1,n_2} b_{\frac{n_1}{d},\frac{n_2}{d}} \tag{B.0.7}$$

we now prove a generalisation of the Möbius inversion formula for two variables that will apply to the above. This generalisation is

Lemma

The b_{n_1,n_2} are determined uniquely by (B.0.7), with

$$b_{n_1,n_2} = \sum_{d|n_1,n_2} \mu(d) a_{\frac{n_1}{d},\frac{n_2}{d}} \tag{B.0.8}$$

Proof

To prove this, consider fixing $\bar{n}_1, \bar{n}_2$ to be coprime. We then define

$$\bar{a}_k = a_{k\bar{n}_1,k\bar{n}_2} \qquad \bar{b}_k = b_{k\bar{n}_1,k\bar{n}_2} \tag{B.0.9}$$

In terms of these sequences (B.0.7) reads

$$\bar{a}_k = \sum_{d|k\bar{n}_1,k\bar{n}_2} b_{\frac{k}{d}\bar{n}_1,\frac{k}{d}\bar{n}_2} \tag{B.0.10}$$

$$= \sum_{d|k} \bar{b}_{\frac{k}{d}} \tag{B.0.11}$$

where we have used the fact that $\bar{n}_1, \bar{n}_2$ are coprime to conclude that $d|k\bar{n}_1, k\bar{n}_2$ is equivalent to $d|k$. Then by the standard Möbius inversion formula, we have

$$\bar{b}_k = \sum_{d|k} \mu(d)\bar{a}_{\frac{k}{d}} \tag{B.0.12}$$

or in terms of as and bs

$$b_{k\bar{n}_1,k\bar{n}_2} = \sum_{d|k\bar{n}_1,k\bar{n}_2} \mu(d) a_{\frac{k\bar{n}_1}{d},\frac{k\bar{n}_2}{d}} \tag{B.0.13}$$

This is true for all k, and coprime $\bar{n}_1$, $\bar{n}_2$. So to prove (B.0.8) for an arbitrary n_1, n_2 we pick $k = \gcd(n_1, n_2)$, $\bar{n}_1 = \frac{n_1}{k}$, $\bar{n}_2 = \frac{n_2}{k}$.
□

The Möbius inversion formula can be used to prove some useful identities. We start with the well known identity

$$\sum_{d|n} \phi(d) = n \tag{B.0.14}$$

where $\phi(n)$ is the Euler totient function that counts the number of numbers less than n that are coprime to n. Applying the Möbius inversion formula gives

$$\frac{\phi(n)}{n} = \sum_{d|n} \frac{\mu(d)}{d} \tag{B.0.15}$$

and applying it again gives

$$\mu(n) = \sum_{d|n} d\mu(d)\phi\left(\frac{n}{d}\right) = \sum_{d|n} \frac{n}{d}\mu\left(\frac{n}{d}\right)\phi(d) \tag{B.0.16}$$

The Möbius inversion formula can be suitably generalised to any poset. In chapter 7 we use the Möbius inversion formula for the poset of set partitions.

Appendix C

List of sequences and generating functions

We introduce a lot of different single and multi-trace counting sequences in chapters 3 and 4. For simplicity we present all of them in one place. For each sequence we give the definition of the (n_1, n_2)th term, the first few terms, the generating function and (for the single trace sequences) the plethystic exponential of the generating function. We also give the vector spaces which have these functions as Hilbert series.

Many of the results here can be found together with their derivations in sections 3.2 and only considered at infinite N, while the multi-trace sequences are defined for finite N, but we have only found their generating functions at infinite N.

After listing the sequences, we give the relations between them and their generating functions.

C.1 Single trace sequences

All of the following definitions are valid provided we have one of $n_1, n_2 \neq 0$. For all single-trace sequences, we implicitly set the $n_1 = n_2 = 0$ term to 0.

C.1.1 A_{n_1,n_2}

The A_{n_1,n_2} count single traces of generic matrices ($U(N)$ single traces). They are defined by

$$A_{n_1,n_2} = \frac{1}{n} \sum_{d|n_1,n_2} \phi(d) \binom{\frac{n}{d}}{\frac{n_1}{d}} \tag{C.1.1}$$

Their generating function is

$$f_{U(N)}(x, y) = -\sum_{d=1}^{\infty} \frac{\phi(d)}{d} \log(1 - x^d - y^d) \tag{C.1.2}$$

which is the Hilbert series for the vector space T_{ST}. The plethystic exponential is

$$F_{U(N)}(x,y) = \prod_{n_1,n_2} \frac{1}{(1-x^{n_1}y^{n_2})^{A_{n_1,n_2}}} = \prod_{k=1}^{\infty} \frac{1}{1-x^k-y^k} \tag{C.1.3}$$

which is the Hilbert series for the vector space $T = \mathrm{Sym}\,(T_{ST})$.

The values of A_{n_1,n_2} for $n_1, n_2 \leq 10$ are shown below. These numbers form sequence A047996 in the OEIS [82].

	0	1	2	3	4	5	6	7	8	9	10
0	0	1	1	1	1	1	1	1	1	1	1
1	1	1	1	1	1	1	1	1	1	1	1
2	1	1	2	2	3	3	4	4	5	5	6
3	1	1	2	4	5	7	10	12	15	19	22
4	1	1	3	5	10	14	22	30	43	55	73
5	1	1	3	7	14	26	42	66	99	143	201
6	1	1	4	10	22	42	80	132	217	335	504
7	1	1	4	12	30	66	132	246	429	715	1144
8	1	1	5	15	43	99	217	429	810	1430	2438
9	1	1	5	19	55	143	335	715	1430	2704	4862
10	1	1	6	22	73	201	504	1144	2438	4862	9252

C.1.2 a_{n_1,n_2}

The a_{n_1,n_2} count aperiodic single traces of generic matrices ($U(N)$ aperiodic single traces), or equivalently Lyndon words. They are defined by

$$a_{n_1,n_2} = \frac{1}{n} \sum_{d|n_1,n_2} \mu(d) \binom{\frac{n}{d}}{\frac{n}{d}} \tag{C.1.4}$$

Their generating function is

$$\bar{f}_{U(N)}(x,y) = -\sum_{d=1}^{\infty} \frac{\mu(d)}{d} \log(1-x^d-y^d) \tag{C.1.5}$$

which is the Hilbert series for the vector space $T_{ST}^{(1)}$. The plethystic exponential is

$$\bar{F}_{U(N)}(x,y) = \prod_{n_1,n_2} \frac{1}{(1-x^{n_1}y^{n_2})^{a_{n_1,n_2}}} = \frac{1}{1-x-y} \tag{C.1.6}$$

which is the Hilbert series for the vector space $T^{(1)} = \mathrm{Sym}\left(T_{ST}^{(1)}\right)$.

The values of a_{n_1,n_2} for $n_1, n_2 \leq 10$ are shown below. Omitting the first row and column, these numbers form sequence A245558 in the OEIS [82]. The properties of A_{n_1,n_2} and a_{n_1,n_2}, the relationship between them, and a generalisation to other sequences were investigated in [73].

	0	1	2	3	4	5	6	7	8	9	10
0	0	1	0	0	0	0	0	0	0	0	0
1	1	1	1	1	1	1	1	1	1	1	1
2	0	1	1	2	2	3	3	4	4	5	5
3	0	1	2	3	5	7	9	12	15	18	22
4	0	1	2	5	8	14	20	30	40	55	70
5	0	1	3	7	14	25	42	66	99	143	200
6	0	1	3	9	20	42	75	132	212	333	497
7	0	1	4	12	30	66	132	245	429	715	1144
8	0	1	4	15	40	99	212	429	800	1430	2424
9	0	1	5	18	55	143	333	715	1430	2700	4862
10	0	1	5	22	70	200	497	1144	2424	4862	9225

C.1.3 $A^{inv}_{n_1,n_2}$

The $A^{inv}_{n_1,n_2}$ count matrix words (up to cyclic rotations) which don't change when reversed (up to cyclic rotations). They are defined by

$$A^{inv}_{n_1,n_2} = \binom{\lfloor \frac{n_1}{2} \rfloor + \lfloor \frac{n_2}{2} \rfloor}{\lfloor \frac{n_1}{2} \rfloor} \tag{C.1.7}$$

Their generating function is

$$f_{inv}(x,y) = \frac{x^2 + xy + y^2 + x + y}{1 - x^2 - y^2} \tag{C.1.8}$$

which is the Hilbert series for the vector space $T_{ST;inv}$. The plethystic exponential is

$$F_{inv}(x,y) = \prod_{n_1,n_2} \frac{1}{(1 - x^{n_1} y^{n_2})^{A^{inv}_{n_1,n_2}}} = \prod_{k=1}^{\infty} \exp\left[\frac{x^{2k} + x^k y^k + y^{2k} + x^k + y^k}{k(1 - x^{2k} - y^{2k})}\right] \tag{C.1.9}$$

which is the Hilbert series for the vector space $T_{inv} = \mathrm{Sym}\,(T_{ST;inv})$

The values of $A^{inv}_{n_1,n_2}$ for $n_1, n_2 \leq 10$ are shown below. These numbers form sequence A119963 in the OEIS [82].

	0	1	2	3	4	5	6	7	8	9	10
0	1	1	1	1	1	1	1	1	1	1	1
1	1	1	1	1	1	1	1	1	1	1	1
2	1	1	2	2	3	3	4	4	5	5	6
3	1	1	2	2	3	3	4	4	5	5	6
4	1	1	3	3	6	6	10	10	15	15	21
5	1	1	3	3	6	6	10	10	15	15	21
6	1	1	4	4	10	10	20	20	35	35	56
7	1	1	4	4	10	10	20	20	35	35	56
8	1	1	5	5	15	15	35	35	70	70	126
9	1	1	5	5	15	15	35	35	70	70	126
10	1	1	6	6	21	21	56	56	126	126	252

C.1.4 $a^{inv}_{n_1,n_2}$

The $a^{inv}_{n_1,n_2}$ count aperiodic matrix words (up to cyclic rotations) which don't change (up to cyclic rotations) when reversed. They are defined by

$$a^{inv}_{n_1,n_2} = \sum_{d|n_1,n_2} \mu(d) \binom{\lfloor \frac{n_1}{2d} \rfloor + \lfloor \frac{n_2}{2d} \rfloor}{\lfloor \frac{n_1}{2d} \rfloor} \tag{C.1.10}$$

Their generating function is

$$\bar{f}_{inv}(x,y) = \sum_{d=1}^{\infty} \mu(d) \frac{x^{2d} + x^d y^d + y^{2d} + x^d + y^d}{1 - x^{2d} - y^{2d}} \tag{C.1.11}$$

which is the Hilbert series for the vector space $T^{(1)}_{ST;inv}$. The plethystic exponential is

$$\bar{F}_{inv}(x,y) = \prod_{n_1,n_2} \frac{1}{(1 - x^{n_1} y^{n_2})^{a^{inv}_{n_1,n_2}}} = \prod_{k=1}^{\infty} \exp\left[\frac{x^{2k} + x^k y^k + y^{2k} + x^k + y^k}{k(1 - x^{2k} - y^{2k})} \sum_{d|k} d\mu(d) \right] \tag{C.1.12}$$

which is the Hilbert series for the vector space $T^{(1)}_{inv} = \mathrm{Sym}\left(T^{(1)}_{ST;inv}\right)$

The values of $a^{inv}_{n_1,n_2}$ for $n_1, n_2 \leq 10$ are shown below. The diagonal entries $a^{inv}_{m,m}$ form sequence A045680 in the OEIS [82].

	0	1	2	3	4	5	6	7	8	9	10
0	0	1	0	0	0	0	0	0	0	0	0
1	1	1	1	1	1	1	1	1	1	1	1
2	0	1	1	2	2	3	3	4	4	5	5
3	0	1	2	1	3	3	3	4	5	4	6
4	0	1	2	3	4	6	8	10	12	15	18
5	0	1	3	3	6	5	10	10	15	15	20
6	0	1	3	3	8	10	17	20	32	33	53
7	0	1	4	4	10	10	20	19	35	35	56
8	0	1	4	5	12	15	32	35	64	70	120
9	0	1	5	4	15	15	33	35	70	68	126
10	0	1	5	6	18	20	53	56	120	126	245

C.1.5 B_{n_1,n_2}

The B_{n_1,n_2} count single traces of anti-symmetric matrices ($SO(N)$ single traces). They are defined by

$$B_{n_1,n_2} = \frac{1}{2n+2m} \sum_{d|n_1,n_2} \phi(d) \binom{\frac{n}{d}}{\frac{n_1}{d}} + \frac{(-1)^n}{2} \binom{\lfloor \frac{n_1}{2} \rfloor + \lfloor \frac{n_2}{2} \rfloor}{\lfloor \frac{n_1}{2} \rfloor} \tag{C.1.13}$$

Their generating function is

$$f_{SO(N)}(x,y) = \frac{1}{2} \left[-\sum_{d=1}^{\infty} \frac{\phi(d)}{d} \log(1 - x^d - y^d) + \frac{x^2 + xy + y^2 - x - y}{1 - x^2 - y^2} \right] \tag{C.1.14}$$

which is the Hilbert series for the vector space $\widetilde{T}_{ST}$. The plethystic exponential is

$$\begin{aligned} F_{SO(N)}(x,y) &= \prod_{n_1,n_2} \frac{1}{(1 - x^{n_1} y^{n_2})^{B_{n_1,n_2}}} \\ &= \prod_{k=1}^{\infty} \frac{1}{\sqrt{1 - x^k - y^k}} \exp\left[\frac{x^{2k} + x^k y^k + y^{2k} - x^k - y^k}{2k(1 - x^{2k} - y^{2k})} \right] \end{aligned} \tag{C.1.15}$$

which is the Hilbert series for the vector space $\widetilde{T} = \mathrm{Sym}\left(\widetilde{T}_{ST}\right)$.

The values of B_{n_1,n_2} for $n_1, n_2 \leq 10$ are shown below. For $n = n_1 + n_2$ even, these numbers match sequence A052307 in the OEIS [82].

	0	1	2	3	4	5	6	7	8	9	10
0	0	0	1	0	1	0	1	0	1	0	1
1	0	1	0	1	0	1	0	1	0	1	0
2	1	0	2	0	3	0	4	0	5	0	6
3	0	1	0	3	1	5	3	8	5	12	8
4	1	0	3	1	8	4	16	10	29	20	47
5	0	1	0	5	4	16	16	38	42	79	90
6	1	0	4	3	16	16	50	56	126	150	280
7	0	1	0	8	10	38	56	133	197	375	544
8	1	0	5	5	29	42	126	197	440	680	1282
9	0	1	0	12	20	79	150	375	680	1387	2368
10	1	0	6	8	47	90	280	544	1282	2368	4752

C.1.6 b_{n_1,n_2}

The b_{n_1,n_2} count minimally periodic single traces of anti-symmetric matrices, or equivalently orthogonal Lyndon words. They are defined by

$$b_{n_1,n_2} = \frac{1}{2} \sum_{d|n_1,n_2} \mu(d) \left[\frac{1}{n} \binom{\frac{n}{d}}{\frac{n}{d}} + (-1)^{\frac{n}{d}} \binom{\lfloor \frac{n}{2d} \rfloor + \lfloor \frac{m}{2d} \rfloor}{\lfloor \frac{n}{2d} \rfloor} \right] \tag{C.1.16}$$

Their generating function is

$$\bar{f}_{SO(N)}(x,y) = \frac{1}{2} \sum_{d=1}^{\infty} \mu(d) \left[-\frac{1}{d} \log(1 - x^d - y^d) + \frac{x^{2d} + x^d y^d + y^{2d} - x^d - y^d}{1 - x^{2d} - y^{2d}} \right] \tag{C.1.17}$$

which is the Hilbert series for the vector space $\widetilde{T}_{ST}^{(min)}$. The plethystic exponential is

$$\begin{aligned} \bar{F}_{SO(N)}(x,y) &= \prod_{n_1,n_2} \frac{1}{(1 - x^{n_1} y^{n_2})^{b_{n_1,n_2}}} \\ &= \frac{1}{\sqrt{1-x-y}} \prod_{k=1}^{\infty} \exp\left[\frac{1}{2k} \frac{x^{2k} + x^k y^k + y^{2k} - x^k - y^k}{1 - x^{2k} - y^{2k}} \sum_{d|k} d\mu(d) \right] \end{aligned} \tag{C.1.18}$$

which is the Hilbert series for the vector space $\widetilde{T}^{(min)} = \mathrm{Sym}\left(\widetilde{T}_{ST}^{(min)}\right)$.

The values of b_{n_1,n_2} for $n_1, n_2 \leq 10$ are shown below. The diagonal entries $b_{m,m}^{inv}$ form sequence A045628 in the OEIS [82].

	0	1	2	3	4	5	6	7	8	9	10
0	0	0	1	0	0	0	0	0	0	0	0
1	0	1	0	1	0	1	0	1	0	1	0
2	1	0	1	0	3	0	3	0	5	0	5
3	0	1	0	2	1	5	3	8	5	11	8
4	0	0	3	1	6	4	16	10	26	20	47
5	0	1	0	5	4	15	16	38	42	79	90
6	0	0	3	3	16	16	46	56	125	150	275
7	0	1	0	8	10	38	56	132	197	375	544
8	0	0	5	5	26	42	125	197	432	680	1278
9	0	1	0	11	20	79	150	375	680	1384	2368
10	0	0	5	8	47	90	275	544	1278	2368	4735

C.1.7 $b^{(odd)}_{n_1,n_2}$

The $b^{(odd)}_{n_1,n_2}$ count single traces of anti-symmetric matrices with a specified odd number of periods. Note that n_1, n_2 refer to the number of Xs and Ys contained in the aperiodic root of the trace, rather than in the whole trace. They are defined by

$$b^{(odd)}_{n_1,n_2} = \frac{1}{2}\sum_{d|n_1,n_2}\mu(d)\left[\frac{1}{n}\binom{\frac{n}{d}}{\frac{n_1}{d}} + (-1)^n\binom{\lfloor\frac{n_1}{2d}\rfloor+\lfloor\frac{n_2}{2d}\rfloor}{\lfloor\frac{n_1}{2d}\rfloor}\right] \tag{C.1.19}$$

Their generating function is

$$\bar{f}^{(odd)}_{SO(N)}(x,y) = \frac{1}{2}\sum_{d=1}^{\infty}\mu(d)\left[-\frac{1}{d}\log(1-x^d-y^d) + \frac{x^{2d}+x^dy^d+y^{2d}+(-x)^d+(-y)^d}{1-x^{2d}-y^{2d}}\right] \tag{C.1.20}$$

which is the Hilbert series for the vector space $\widetilde{T}^{(odd)}_{ST}$. The plethystic exponential is

$$\bar{F}^{(odd)}_{SO(N)}(x,y) = \prod_{n_1,n_2}\frac{1}{(1-x^{n_1}y^{n_2})^{b^{(odd)}_{n_1,n_2}}} \tag{C.1.21}$$

$$= \frac{1}{\sqrt{1-x-y}}\prod_{k=1}^{\infty}\exp\left[\sum_{d|k}\frac{d\mu(d)}{2k}\frac{x^{2k}+x^ky^k+y^{2k}+(-1)^d(x^k+y^k)}{1-x^{2k}-y^{2k}}\right] \tag{C.1.22}$$

which is the Hilbert series for the vector space $\widetilde{T}^{(odd)} = \mathrm{Sym}\left(\widetilde{T}^{(odd)}_{ST}\right)$.

The values of $b^{(odd)}_{n_1,n_2}$ for $n_1, n_2 \le 10$ are shown below

	0	1	2	3	4	5	6	7	8	9	10
0	0	0	0	0	0	0	0	0	0	0	0
1	0	1	0	1	0	1	0	1	0	1	0
2	0	0	1	0	2	0	3	0	4	0	5
3	0	1	0	2	1	5	3	8	5	11	8
4	0	0	2	1	6	4	14	10	26	20	44
5	0	1	0	5	4	15	16	38	42	79	90
6	0	0	3	3	14	16	46	56	122	150	275
7	0	1	0	8	10	38	56	132	197	375	544
8	0	0	4	5	26	42	122	197	432	680	1272
9	0	1	0	11	20	79	150	375	680	1384	2368
10	0	0	5	8	44	90	275	544	1272	2368	4735

C.1.8 $b^{(even)}_{n_1,n_2}$

The $b^{(even)}_{n_1,n_2}$ count single traces of anti-symmetric matrices with a specified even number of periods. Note that n_1, n_2 refer to the number of Xs and Ys contained in the aperiodic root of the trace, rather than in the whole trace. They are defined by

$$b^{(even)}_{n_1,n_2} = \frac{1}{2}\sum_{d|n_1,n_2} \mu(d)\left[\frac{1}{n}\binom{\frac{n}{d}}{\frac{n_1}{d}} + \binom{\lfloor\frac{n_1}{2d}\rfloor + \lfloor\frac{n_2}{2d}\rfloor}{\lfloor\frac{n_1}{2d}\rfloor}\right] \tag{C.1.23}$$

Their generating function is

$$\bar{f}^{(even)}_{SO(N)}(x,y) = \frac{1}{2}\sum_{d=1}^{\infty}\mu(d)\left[-\frac{1}{d}\log(1-x^d-y^d) + \frac{x^{2d}+x^dy^d+y^{2d}+x^d+y^d}{1-x^{2d}-y^{2d}}\right] \tag{C.1.24}$$

which is the Hilbert series for the vector space $\widetilde{T}^{(even)}_{ST}$. The plethystic exponential is

$$\begin{aligned}\bar{F}^{(even)}_{SO(N)}(x,y) &= \prod_{n_1,n_2}\frac{1}{(1-x^{n_1}y^{n_2})^{b^{(even)}_{n_1,n_2}}}\\ &= \frac{1}{\sqrt{1-x-y}}\prod_{k=1}^{\infty}\exp\left[\frac{1}{2k}\frac{x^{2k}+x^ky^k+y^{2n}+x^k+y^k}{1-x^{2k}-y^{2k}}\sum_{d|k}d\mu(d)\right]\end{aligned} \tag{C.1.25}$$

which is the Hilbert series for the vector space $\widetilde{T}^{(even)} = \mathrm{Sym}\left(\widetilde{T}^{(even)}_{ST}\right)$.

The values of $b^{(even)}_{n_1,n_2}$ for $n_1, n_2 \leq 10$ are shown below

	0	1	2	3	4	5	6	7	8	9	10
0	0	1	0	0	0	0	0	0	0	0	0
1	1	1	1	1	1	1	1	1	1	1	1
2	0	1	1	2	2	3	3	4	4	5	5
3	0	1	2	2	4	5	6	8	10	11	14
4	0	1	2	4	6	10	14	20	26	35	44
5	0	1	3	5	10	15	26	38	57	79	110
6	0	1	3	6	14	26	46	76	122	183	275
7	0	1	4	8	20	38	76	132	232	375	600
8	0	1	4	10	26	57	122	232	432	750	1272
9	0	1	5	11	35	79	183	375	750	1384	2494
10	0	1	5	14	44	110	275	600	1272	2494	4735

C.2 Multi-trace sequences

$N^{U(N)}_{n_1,n_2}$

The $N^{U(N)}_{n_1,n_2}$ count the multi-traces of generic $N \times N$ matrices, where N can be finite or infinite. They are defined by

$$N^{U(N)}_{n_1,n_2} = \sum_{\substack{R \vdash n \\ R_1 \vdash n_1 \\ R_2 \vdash n_2 \\ l(R) \leq N}} g^2_{R;R_1,R_2} \tag{C.2.1}$$

At infinite N, A_{n_1,n_2} and $N^{U(N)}_{n_1,n_2}$ are related by the plethystic exponential, so the generating function is given by (C.1.3), which is the Hilbert series for T.

The values of $N^{U(\infty)}_{n_1,n_2}$ for $n_1, n_2 \leq 10$ are shown below. These numbers form sequence A322210 in the OEIS [82].

	0	1	2	3	4	5	6	7	8	9	10
0	1	1	2	3	5	7	11	15	22	30	42
1	1	2	4	7	12	19	30	45	67	97	139
2	2	4	10	18	34	56	94	146	228	340	506
3	3	7	18	38	74	133	233	385	623	977	1501
4	5	12	34	74	158	297	550	951	1614	2627	4202
5	7	19	56	133	297	602	1166	2133	3775	6437	10692
6	11	30	94	233	550	1166	2382	4551	8424	14953	25835
7	15	45	146	385	951	2133	4551	9142	17639	32680	58659
8	22	67	228	623	1614	3775	8424	17639	35492	68356	127443
9	30	97	340	977	2627	6437	14953	32680	68356	136936	264747
10	42	139	506	1501	4202	10692	25835	58659	127443	264747	530404

C.2.1 $N^{SO(N);\delta}_{n_1,n_2}$

The $N^{SO(N);\delta}_{n_1,n_2}$ count the multi-traces of anti-symmetric $N \times N$ matrices, where N can be finite or infinite. They are defined by

$$N^{SO(N);\delta}_{n_1,n_2} = \sum_{\substack{R \vdash 2n \text{ with even row lengths} \\ R_1 \vdash 2n_1 \text{ with even column lengths} \\ R_2 \vdash 2n_2 \text{ with even column lengths} \\ l(R) \leq N}} g_{R;R_1,R_2} \tag{C.2.2}$$

At infinite N, B_{n_1,n_2} and $N^{SO(N);\delta}_{n_1,n_2}$ are related by the plethystic exponential, so the generating function is given by (C.1.15), which is the Hilbert series for $\widetilde{T}$.

The values of $N^{SO(\infty);\delta}_{n_1,n_2}$ for $n_1, n_2 \leq 10$ are shown below. These numbers form sequence A045680 in the OEIS [82].

	0	1	2	3	4	5	6	7	8	9	10
0	1	0	1	0	2	0	3	0	5	0	7
1	0	1	0	2	0	4	0	7	0	12	0
2	1	0	4	0	9	0	19	0	35	0	62
3	0	2	0	9	1	23	4	52	10	105	22
4	2	0	9	1	33	6	85	21	198	56	410
5	0	4	0	23	6	86	33	243	114	600	313
6	3	0	19	4	85	33	297	152	845	512	2137
7	0	7	0	52	21	243	152	879	664	2646	2227
8	5	0	35	10	198	114	845	664	3003	2742	9168
9	0	12	0	105	56	600	512	2646	2742	9702	11033
10	7	0	62	22	410	313	2137	2227	9168	11033	33704

C.3 Relations between different sequences

The a_{n_1,n_2} are the Möbius transform of the A_{n_1,n_2}.

$$A_{n_1,n_2} = \sum_{d|n_1,n_2} a_{\frac{n}{d},\frac{m}{d}} \qquad a_{n_1,n_2} = \sum_{d|n_1,n_2} \mu(d) A_{\frac{n}{d},\frac{m}{d}} \tag{C.3.1}$$

$$f_{U(N)}(x,y) = \sum_{k=1}^{\infty} \bar{f}_{U(N)}(x^k, y^k) \qquad \bar{f}_{U(N)}(x,y) = \sum_{k=1}^{\infty} \mu(k) f_{U(N)}(x^k, y^k) \tag{C.3.2}$$

$$F_{U(N)}(x,y) = \prod_{k=1}^{\infty} \bar{F}_{U(N)}(x^k, y^k) \qquad \bar{F}_{U(N)}(x,y) = \prod_{k=1}^{\infty} F_{U(N)}(x^k, y^k)^{\mu(k)} \tag{C.3.3}$$

The $a^{inv}_{n_1,n_2}$ are the Möbius transform of the $A^{inv}_{n_1,n_2}$.

$$A^{inv}_{n_1,n_2} = \sum_{d|n_1,n_2} a^{inv}_{\frac{n}{d},\frac{m}{d}} \qquad a^{inv}_{n_1,n_2} = \sum_{d|n_1,n_2} \mu(d) A^{inv}_{\frac{n}{d},\frac{m}{d}} \tag{C.3.4}$$

$$f_{inv}(x,y) = \sum_{k=1}^{\infty} \bar{f}_{inv}(x^k, y^k) \qquad \bar{f}_{inv}(x,y) = \sum_{k=1}^{\infty} \mu(k) f_{inv}(x^k, y^k) \tag{C.3.5}$$

$$F_{inv}(x,y) = \prod_{k=1}^{\infty} \bar{F}_{inv}(x^k, y^k) \qquad \bar{F}_{inv}(x,y) = \prod_{k=1}^{\infty} F_{inv}(x^k, y^k)^{\mu(k)} \tag{C.3.6}$$

The B_{n_1,n_2} can be expressed in terms of the A_{n_1,n_2} and the $A^{inv}_{n_1,n_2}$.

$$B_{n_1,n_2} = \frac{1}{2}\left[A_{n_1,n_2} + (-1)^n A^{inv}_{n_1,n_2}\right] \tag{C.3.7}$$

$$f_{SO(N)}(x,y) = \frac{1}{2}\left[f_{U(N)}(x,y) + f_{inv}(-x,-y)\right] \tag{C.3.8}$$

The b_{n_1,n_2} are the Möbius transform of the B_{n_1,n_2}.

$$B_{n_1,n_2} = \sum_{d|n_1,n_2} b_{\frac{n}{d},\frac{m}{d}} \qquad b_{n_1,n_2} = \sum_{d|n_1,n_2} \mu(d) B_{\frac{n}{d},\frac{m}{d}} \tag{C.3.9}$$

$$f_{SO(N)}(x,y) = \sum_{k=1}^{\infty} \bar{f}_{SO(N)}(x^k,y^k) \quad \bar{f}_{SO(N)}(x,y) = \sum_{k=1}^{\infty} \mu(k) f_{SO(N)}(x^k,y^k) \tag{C.3.10}$$

$$F_{SO(N)}(x,y) = \prod_{k=1}^{\infty} \bar{F}_{SO(N)}(x^k,y^k) \quad \bar{F}_{SO(N)}(x,y) = \prod_{k=1}^{\infty} F_{SO(N)}(x^k,y^k)^{\mu(k)} \tag{C.3.11}$$

The $b^{(odd)}_{n_1,n_2}$ and $b^{(even)}_{n_1,n_2}$ can be expressed in terms of the a_{n_1,n_2} and the $a^{inv}_{n_1,n_2}$.

$$b^{(odd)}_{n_1,n_2} = \frac{1}{2}\left[a_{n_1,n_2} + (-1)^n a^{inv}_{n_1,n_2}\right] \qquad b^{(even)}_{n_1,n_2} = \frac{1}{2}\left[a_{n_1,n_2} + a^{inv}_{n_1,n_2}\right] \tag{C.3.12}$$

$$\bar{f}^{(odd)}_{SO(N)}(x,y) = \frac{1}{2}\left[\bar{f}_{U(N)}(x,y) + \bar{f}_{inv}(-x,-y)\right] \quad \bar{f}^{(even)}_{SO(N)}(x,y) = \frac{1}{2}\left[\bar{f}_{U(N)}(x,y) + \bar{f}_{inv}(x,y)\right] \tag{C.3.13}$$

Appendix D

Littlewood-Richardson coefficients

Given a representation $R \vdash n$ of S_n, we can act on R with the subgroup $S_{n_1} \times S_{n_2}$, where $n_1 + n_2 = n$, and therefore it is a representation of this subgroup. The irreducible representations of $S_{n_1} \times S_{n_2}$ are of the form $R_1 \otimes R_2$ for $R_1 \vdash n_1, R_2 \vdash n_2$, so we have a decomposition

$$V_R^{S_n} = \bigoplus_{\substack{R_1 \vdash n_1 \\ R_2 \vdash n_2}} V_{R_1}^{S_{n_1}} \otimes V_{R_2}^{S_{n_2}} \otimes V_{R;R_1,R_2}^{mult} \tag{D.0.1}$$

where $V_{R;R_1,R_2}^{mult}$ is the multiplicity space for this decomposition. The Littlewood-Richardson (abbreviated to LR) coefficient is defined to be

$$g_{R;R_1,R_2} = \text{Dim}\, V_{R;R_1,R_2}^{mult} \tag{D.0.2}$$

Since the subgroup $S_{n_2} \times S_{n_1}$ is conjugate to $S_{n_1} \times S_{n_2}$ within S_n, it follows that $g_{R;R_1,R_2}$ is symmetric in R_1 and R_2.

Take permutations $\sigma \in S_{n_1}$ and $\tau \in S_{n_2}$, and define $\sigma \circ \tau$ to the permutation in S_n that acts as σ on $\{1, 2, \dots, n_1\}$ and τ on $\{n_1 + 1, \dots, n\}$. Then from the decomposition (D.0.1), we have

$$\chi_R(\sigma \circ \tau) = \sum_{\substack{R_1 \vdash n_1 \\ R_2 \vdash n_2}} g_{R;R_1,R_2} \chi_{R_1}(\sigma) \chi_{R_2}(\tau) \tag{D.0.3}$$

Using the orthogonality relation (2.3.5)

$$g_{R;R_1,R_2} = \frac{1}{n_1! n_2!} \sum_{\substack{\sigma \in S_{n_1} \\ \tau \in S_{n_2}}} \chi_{R_1}(\sigma) \chi_{R_2}(\tau) \chi_R(\sigma \circ \tau) \tag{D.0.4}$$

LR coefficients have nice behaviour under conjugation of Young diagrams. From the relation (2.3.10) between representations with conjugate Young diagrams, we can transpose each of the Young diagram labels in (D.0.1) by taking a tensor product with the sign representation of S_n. This gives

$$V_{R^c}^{S_n} = \bigoplus_{\substack{R_1 \vdash n_1 \\ R_2 \vdash n_2}} V_{R_1^c}^{S_{n_1}} \otimes V_{R_2^c}^{S_{n_2}} \otimes V_{R;R_1,R_2}^{mult} \tag{D.0.5}$$

and therefore

$$g_{R;R_1,R_2} = g_{R^c;R_1^c,R_2^c} \tag{D.0.6}$$

D.1 Schur function multiplication

Consider the tensor product $V^{\otimes n}$, for the V the fundamental representation of $U(N)$. This is acted on by permutations in S_n by permuting the tensor factors, so as in (D.0.1), we can break the space down into representations of $S_{n_1} \times S_{n_2}$. There are two ways to approach this. Firstly, we use Schur-Weyl duality (2.4.3) and then apply (D.0.1)

$$V^{\otimes n} = \bigoplus_{\substack{R_1 \vdash n_1 \\ R_2 \vdash n_2}} V_{R_1}^{S_{n_1}} \otimes V_{R_2}^{S_{n_2}} \otimes \left(\bigoplus_{R \vdash n} V_R^{U(N)} \otimes V_{R;R_1,R_2}^{mult} \right) \tag{D.1.1}$$

Secondly, we can split $V^{\otimes n} = V^{\otimes n_1} \otimes V^{\otimes n_2}$ and use Schur-Weyl duality on each of the two factors

$$V^{\otimes n} = \bigoplus_{\substack{R_1 \vdash n_1 \\ R_2 \vdash n_2}} V_{R_1}^{S_{n_1}} \otimes V_{R_2}^{S_{n_2}} \otimes \left(V_{R_1}^{U(N)} \otimes V_{R_2}^{U(N)} \right) \tag{D.1.2}$$

Comparing the two expansions, we see

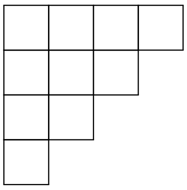
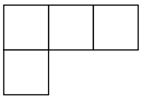
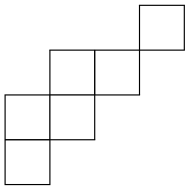
$$V_{R_1}^{U(N)} \otimes V_{R_2}^{U(N)} = \bigoplus_{R \vdash n} V_R^{U(N)} \otimes V_{R;R_1,R_2}^{mult} \tag{D.1.3}$$

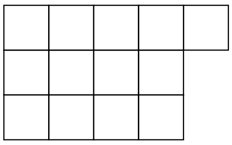
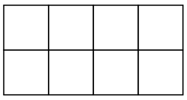
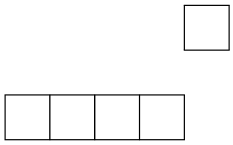
So the LR coefficients are Clebsch-Gordon coefficients for $U(N)$ representations. It follows from the identification (2.7.15) of the $U(N)$ characters as Schur symmetric functions that

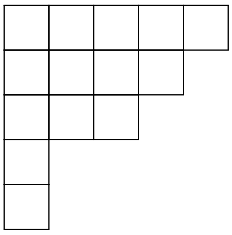
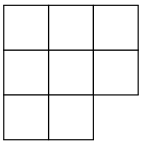
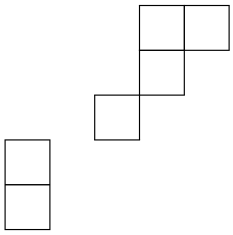
$$s_{R_1} s_{R_2} = \sum_{R \vdash n} g_{R;R_1,R_2} s_R \tag{D.1.4}$$

D.2 Littlewood-Richardson rule and tableaux

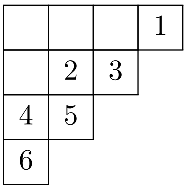
The Littlewood-Richardson rule is a combinatoric description of the LR coefficient $g_{R;R_1,R_2}$. Given $R \vdash n$ and $R_1 \vdash n_1$ such that R_1 fits within R, we define the skew Young diagram R/R_1 to be the diagram obtained by removing R_1 from R. The boxes remaining in R/R_1 do not need to be connected. We give some examples

$R =$  $R_1 =$  $R/R_1 =$  (D.2.1)

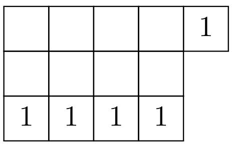
$R =$  $R_1 =$  $R/R_1 =$  (D.2.2)

$R =$  $R_1 =$  $R/R_1 =$  (D.2.3)

A skew tableau r of shape R/R_1 is the skew diagram R/R_1 with positive integers placed in each box. The skew tableau is called semi-standard if the numbers increase weakly along the rows and strictly down the columns. The *evaluation* of r is the sequence $w(r) = [k_1, k_2, \ldots]$ where k_i is the number of is inside r. We are concerned with the case where the k_i weakly decrease, meaning $w(r)$ is a partition of n_2. We give some examples of semi-standard skew tableaux of shapes (D.2.1-D.2.3) with their respective evaluations. We leave empty boxes in the tableaux to show where R_1 has been removed

$r =$ 

$w(r) = [1^6]$ (D.2.4)

$r =$ 

$w(r) = [5]$ (D.2.5)

$$r = \begin{ytableau} \; & \; & \; & 1 & 1 \\ \; & \; & \; & 2 \\ \; & \; & 2 \\ 3 \\ 4 \end{ytableau} \qquad\qquad w(r) = [2,2,1,1] \tag{D.2.6}$$

The *row reading* of a tableau is the word obtained by concatenating the reversed rows. The row reading of a tableau is a *lattice word* if every prefix of the word contains at least as many is as $(i+1)$s for every $i > 0$. A Littlewood-Richardson tableau is a semi-standard skew tableau where the row reading is a lattice word.

In the example (D.2.4), the row reading of r is 132546. If we take the prefix 13, then 2 does not appear at all, while 3 appears once. Therefore this is not a lattice word, and hence r is not a LR tableau. For (D.2.5) and (D.2.6), the row readings are 11111 and 112234, which are both lattice words, and thus the two r are LR tableaux.

The LR rule [121,122] states that the coefficient $g_{R;R_1,R_2}$ is given by the number of LR tableaux r of shape R/R_1 with evaluation $w(r) = R_2$.

We give two examples of calculations of particular coefficients, before moving on to the calculation of coefficients for a general class of diagrams. Firstly, consider $R = [5,3,2,1,1]$, and $R_1 = R_2 = [3,2,1]$. Then $g_{R;R_1,R_2} = 4$, with tableaux

$$\begin{gathered} r_1 = \begin{ytableau} \; & \; & \; & 1 & 1 \\ \; & \; & 1 \\ \; & 2 \\ 2 \\ 3 \end{ytableau} \qquad r_2 = \begin{ytableau} \; & \; & \; & 1 & 1 \\ \; & \; & 2 \\ \; & 1 \\ 2 \\ 3 \end{ytableau} \\ r_3 = \begin{ytableau} \; & \; & \; & 1 & 1 \\ \; & \; & 2 \\ \; & 2 \\ 1 \\ 3 \end{ytableau} \qquad r_4 = \begin{ytableau} \; & \; & \; & 1 & 1 \\ \; & \; & 2 \\ \; & 3 \\ 1 \\ 2 \end{ytableau} \end{gathered} \tag{D.2.7}$$

Our second example is relevant for the $SO(N)$ restricted Schur basis (5.6.70). The triple $R = [6,4,4,2,2,2,2]$, $R_1 = [4,4,2,2,1,1]$ and $R_2 = [3,3,1,1]$ are the lowest order example of a non-trivial multiplicity in this basis with $g_{R;R_1,R_2} = 2$. The tableaux are

$$r_1 = \begin{ytableau} ~ & ~ & ~ & ~ & 1 & 1 \\ ~ & ~ & ~ & ~ \\ ~ & ~ & 1 & 2 \\ ~ & ~ \\ ~ & 2 \\ ~ & 3 \\ 2 & 4 \end{ytableau} \qquad r_2 = \begin{ytableau} ~ & ~ & ~ & ~ & 1 & 1 \\ ~ & ~ & ~ & ~ \\ ~ & ~ & 2 & 2 \\ ~ & ~ \\ ~ & 1 \\ ~ & 3 \\ 2 & 4 \end{ytableau} \tag{D.2.8}$$

D.2.1 Baryonic tableaux

The baryonic state spaces and auxiliary algebras in chapter 4 are anti-invariant under S_N, meaning they lie in the single column $[1^N]$ representation, and invariant under $S_q[S_2]$, meaning they live in a representation $\bar{R} \vdash 2q$ of S_{2q}, where $\bar{R}$ has even row lengths. We are interested in which $R \vdash n = N + 2q$ can admit such representations of $S_N \times S_q[S_2]$.

As an example, take $R_1 = [1^4]$ (i.e. $N = 4$) and $R_2 = [4, 4, 2]$. Then the possible R with non-zero $g_{R;R_1,R_2}$ are $R = [5, 5, 3, 1]$, $[5, 5, 2, 1^2]$, $[5, 4, 3, 1^2]$, $[5, 4, 2, 1^3]$, $[4, 4, 3, 1^3]$ and $[4, 4, 2, 1^4]$. The corresponding tableaux are

$$r_1 = \begin{ytableau} ~ & 1 & 1 & 1 & 1 \\ ~ & 2 & 2 & 2 & 2 \\ ~ & 3 & 3 \\ ~ \end{ytableau} \qquad r_2 = \begin{ytableau} ~ & 1 & 1 & 1 & 1 \\ ~ & 2 & 2 & 2 & 2 \\ ~ & 3 \\ ~ \\ 3 \end{ytableau} \qquad r_3 = \begin{ytableau} ~ & 1 & 1 & 1 & 1 \\ ~ & 2 & 2 & 2 \\ ~ & 3 & 3 \\ ~ \\ 2 \end{ytableau}$$

$$r_4 = \begin{ytableau} ~ & 1 & 1 & 1 & 1 \\ ~ & 2 & 2 & 2 \\ ~ & 3 \\ ~ \\ 2 \\ 3 \end{ytableau} \qquad r_5 = \begin{ytableau} ~ & 1 & 1 & 1 \\ ~ & 2 & 2 & 2 \\ ~ & 3 & 3 \\ ~ \\ 1 \\ 2 \end{ytableau} \qquad r_6 = \begin{ytableau} ~ & 1 & 1 & 1 \\ ~ & 2 & 2 & 2 \\ ~ & 3 \\ ~ \\ 1 \\ 2 \\ 3 \end{ytableau} \tag{D.2.9}$$

Let $\hat{r}$ be r with the first column removed. Then in each of the above tableaux, we see that the jth row of $\hat{r}_i$ consists only of js. We now prove this fact for all LR tableaux r of shape $R/[1^N]$ by inducting down the rows.

By the lattice word property of the row reading of r, the first row of $\hat{r}$ must end with a 1. By the semi-standard property, this means the entire row must be composed of 1s. This establishes the base case of the induction.

Assume that the first $j-1$ rows consist purely of the row number. Then since the entries increase strictly down the columns, the jth row must consist only of numbers $\geq j$. By the lattice word property of the row reading of r, the row must end with a j, then since the row is weakly increasing, the jth row of $\hat{r}$ must consist entirely of js.

Now consider the first column of r. Since the numbers strictly increase down the columns, it must contain at most 1 of each number. Therefore the numbers in this column pick out a subset of the rows of $\bar{R}$, and the shape of $\hat{r}$ is given by $\bar{R}$ with a single box removed from each of the selected rows. This subset is not without restriction, since $\bar{R}$ with the boxes removed must still remain a Young diagram. In (D.2.9), the subsets used are respectively

$$\phi, \{3\}, \{2\}, \{2,3\}, \{1,2\}, \{1,2,3\} \tag{D.2.10}$$

where ϕ is the empty subset. The subsets $\{1\}$ and $\{1,3\}$ are not included as removing these boxes from $\bar{R}$ would not result in a valid Young diagram.

For each choice of subset, we obtain a different R, and therefore the coefficient is $g_{R;[1^N],\bar{R}} = 1$.

Since $\bar{R}$ has even row lengths, those with boxes removed have odd row lengths. This means for a given R we can easily identify if a $S_N \times S_q[S_2]$ invariant is possible, and to which $\bar{R}$ it belongs.

Consider $R \vdash 2n$ with a first column of length $\geq N$. It can be written as a single column $[1^{N+k}]$ combined with a Young diagram $\bar{S}$ of length $l(\bar{S}) \leq N+k$. Formally, $R = [1^{N+k}] + \bar{S}$, where $+$ is the addition operator for components of partitions defined in (2.2.7).

From $\bar{S}$, we can uniquely define an associated $\bar{R}$ by adding a single box to the end of each odd length row. Then R admits the representation $[1^N] \otimes \bar{R}$ of $S_N \times S_{2q}$ if (and only if) the number of odd length rows in $\bar{S}$ is k. If $l(\bar{R}) > N$, there is the additional condition that $N + k \geq l(\bar{R})$. This characterises the allowable R in the baryonic state spaces and auxiliary algebras, and proves that the $\bar{R}$ associated to a valid R is unique and has LR coefficient 1.

The relation is even simpler when we restrict R to have at most N rows. In this case we have $k = 0$, $\bar{S} = \bar{R}$ and $R = [1^N] + \bar{R}$.

D.3 Basis for Littlewood-Richardson multiplicity space

Consider the action of the algebra $\mathcal{A}_{n_1,n_2}$, defined in section 2.1, on the decomposition (D.0.1). By construction, $\mathcal{A}_{n_1,n_2}$ commutes with $S_{n_1} \times S_{n_2}$, and therefore by Schur's lemma it must act proportional to the identity on the space $V_{R_1}^{S_{n_1}} \otimes V_{R_2}^{S_{n_2}}$. Hence $\mathcal{A}_{n_1,n_2}$ acts only on the multiplicity space $V^{mult}_{R;R_1,R_2}$ in each term of the decomposition (D.0.1).

Define a sub-algebra $\mathcal{A}^R_{R_1,R_2}$ of $\mathcal{A}_{n_1,n_2}$ by projecting onto the R representation of S_n and the $R_1 \otimes R_2$ representation of $S_{n_1} \times S_{n_2}$. Explicitly,

$$\mathcal{A}^R_{R_1,R_2} = P_R P_{R_1 \otimes R_2} \mathcal{A}_{n_1,n_2} = \mathrm{Span}\left\{\beta_{R,R_1,R_2,\mu,\nu} : 1 \leq \mu, \nu \leq g_{R;R_1,R_2}\right\} \tag{D.3.1}$$

where the projector $P_{R_1 \otimes R_2}$ is the tensor product of the projectors defined in (2.3.13) and the $\beta_{R,R_1,R_2,\mu,\nu}$ are defined in (3.6.3).

The projection onto $R_1 \otimes R_2$ means $\mathcal{A}^R_{R_1,R_2}$ acts only on the $R_1 \otimes R_2$ term of (D.0.1) and annihilates all others. Therefore $\mathcal{A}^R_{R_1,R_2}$ acts purely on the multiplicity space $V^{mult}_{R;R_1,R_2}$.

One can then use the behaviour of vectors in $V^{mult}_{R;R_1,R_2}$ under $\mathcal{A}^R_{R_1,R_2}$ to choose an orthogonal basis. Simply choose a maximal commuting set of operators, and use the eigenbasis.

For a more complete description of how one chooses these operators, or the maximal commuting sub-algebra they span, see [63].

Appendix E

Alternative derivation of free field large N generating function for $SO(N)$ and $Sp(N)$

We now derive the generating function (4.3.21) for the quarter-BPS sector of the free field theory with $SO(N)$ or $Sp(N)$ gauge group directly from the expressions (5.6.74) and (5.6.75) respectively for the number of operators with n_1 Xs and n_2 Ys in the two theories. This derivation works at infinite N, ignoring the finite N constraint $l(R) \leq N$. In this regime, the combinatorics of the $SO(N)$ and $Sp(N)$ theories are identical, so we do not distinguish between them in this appendix.

The first step is to find an alternative formula for (5.6.74) that lends itself more easily to explicit calculation of the generating function. This is done using results from the theory of symmetric functions, and gives an expression involving the coefficients of the cycle index polynomial of $S_n[S_2]$.

Using this alternative formula we can express the generating function as a product of integrals, each of which can be explicitly evaluated.

E.1 An alternative counting formula

Start with the expression (5.6.74) at large N and re-express $g_{R;R_1,R_2}$ in terms of characters using (D.0.4)

$$N^{\delta}_{n_1,n_2} = \sum_{\substack{R \vdash 2n \text{ with even row lengths} \\ R_1 \vdash 2n_1 \text{ with even column lengths} \\ R_2 \vdash 2n_2 \text{ with even column lengths}}} \frac{1}{(2n_1)!(2n_2)!} \sum_{\substack{\sigma \in S_{2n_1} \\ \tau \in S_{2n_2}}} \chi_{R_1}(\sigma)\chi_{R_2}(\tau)\chi_R(\sigma \circ \tau) \tag{E.1.1}$$

where $\sigma \circ \tau$ means the permutation in S_{2n} that acts as σ on the first $2n_1$ objects and τ on the last $2n_2$. Since the characters only depend on the cycle type of σ and τ, we can rewrite this as

$$N^{\delta}_{n_1,n_2} = \sum_{\substack{R\vdash 2n \text{ with even row lengths} \\ R_1\vdash 2n_1 \text{ with even column lengths} \\ R_2\vdash 2n_2 \text{ with even column lengths}}} \sum_{\substack{p\vdash 2n_1 \\ q\vdash 2n_2}} \frac{\chi_{R_1}(p)\chi_{R_2}(q)\chi_R(p\cup q)}{z_p z_q} \tag{E.1.2}$$

where $p \cup q$ was defined in (2.2.9), and z_p and z_q arise because the number of permutations in S_{2n} with cycle type p is given by $\frac{(2n)!}{z_p}$. This is explained in section 2.3.1.

We now evaluate

$$\sum_{\substack{R\vdash 2n \text{ with even} \\ \text{row/column lengths}}} \chi_R(q) \tag{E.1.3}$$

From (2.7.10) we can rewrite this as

$$z_q \sum_{\substack{R \text{ with even} \\ \text{row/column lengths}}} \text{Coeff}\,(T_q; s_R) = z_q \,\text{Coeff} \left(T_q; \sum_{\substack{R \text{ with even} \\ \text{row/column lengths}}} s_R \right) \tag{E.1.4}$$

where T_q is the power-sum symmetric function defined in (2.7.8) and s_R is the Schur symmetric function defined in (2.7.10). Coeff $(T_q; s_R)$ is the coefficient of T_q when s_R is written as a sum over power-sum symmetric functions.

The sum in (E.1.4), can range over all partitions with even row lengths, rather than just those with $|R| = 2n$, since the coefficient of T_q is 0 in any s_R with $|R| \neq 2n$.

In MacDonald's book [64, Chapter I.5] he shows that

$$s(t_1, t_2, \ldots) = \sum_{\substack{R \text{ with even} \\ \text{row lengths}}} s_R = \prod_i \frac{1}{1-t_i^2} \prod_{i<j} \frac{1}{1-t_i t_j} \tag{E.1.5}$$

To find the coefficient of T_q inside s we first look at $\log s$

$$\log s = -\sum_i \log(1-t_i^2) - \sum_{i<j} \log(1-t_i t_j) \tag{E.1.6}$$

$$= \sum_{r=1}^{\infty} \frac{1}{2r} \left(\sum_{i,j} t_i^r t_j^r + \sum_i t_i^{2r} \right) \tag{E.1.7}$$

$$= \sum_{r=1}^{\infty} \frac{1}{2r} \left(T_r^2 + T_{2r} \right) \tag{E.1.8}$$

$$= \sum_{r=1}^{\infty} \frac{1}{r} Z^{S_2}(T_r, T_{2r}) \tag{E.1.9}$$

where Z^{S_2} is the cycle index polynomial of the group S_2 as defined in (4.3.7) and T_r is a component of a power-sum symmetric function as defined in (2.7.7). Therefore

$$s = \exp\left[\sum_{r=1}^{\infty} \frac{1}{r} Z^{S_2}(T_r, T_{2r})\right] \tag{E.1.10}$$

We recall two useful facts. Firstly, the generating function for the cycle index polynomials of S_n is [65, Chapter 5.13]

$$\sum_{n=0}^{\infty} x^n Z^{S_n}(t_1, t_2, \ldots) = \exp\left[\sum_{m=1}^{\infty} \frac{1}{m} x^m t_m\right] \tag{E.1.11}$$

Secondly, the cycle index polynomial of a wreath product group is [123, Chapter 15.5]

$$Z^{G[H]}(t_1, t_2, \ldots) = Z^G(r_1, r_2, \ldots) \tag{E.1.12}$$

where

$$r_i = Z^H(t_i, t_{2i}, t_{3i}, \ldots) \tag{E.1.13}$$

Combining (E.1.11) and (E.1.12) tells us that the generating function for the cycle index polynomials of $S_n[S_2]$ is

$$\sum_{n=0}^{\infty} x^n Z^{S_n[S_2]}(t_1, t_2, \ldots) = \sum_{n=0}^{\infty} x^n \sum_{q \vdash 2n} Z_q^{S_n[S_2]} \prod_i t_i^{q_i} = \exp\left[\sum_{r=1}^{\infty} \frac{1}{r} x^r Z^{S_2}(t_r, t_{2r})\right] \tag{E.1.14}$$

Comparing (E.1.14) with (E.1.10), we have

$$s = \sum_{n=0}^{\infty} Z^{S_n[S_2]}(T_1, T_2, \ldots) \qquad\qquad = \sum_{n=0}^{\infty} \sum_{q \vdash 2n} Z_q^{S_n[S_2]} T_q \tag{E.1.15}$$

Therefore the sum over even length rows in (E.1.3) is

$$\sum_{\substack{R \vdash 2n \text{ with even} \\ \text{row lengths}}} \chi_R(q) = z_q \operatorname{Coeff}(T_q; s) = z_q Z_q^{S_n[S_2]} \tag{E.1.16}$$

A Young diagram has even row lengths if and only if its conjugate has even column lengths, so to evaluate the column version of (E.1.3), we just conjugate the summation variable R. Since R^c =sgn$\otimes R$, the characters are related by

$$\chi_{R^c}(q) = (-1)^q \chi_R(q) \tag{E.1.17}$$

Therefore

$$\sum_{\substack{R\vdash 2n \text{ with even} \\ \text{column lengths}}} \chi_R(q) = (-1)^q Z_q^{S_n[S_2]} z_q \tag{E.1.18}$$

Plugging (E.1.16) and (E.1.18) into (E.1.2) gives

$$N^{\delta}_{n_1,n_2} = \sum_{\substack{p\vdash 2n_1 \\ q\vdash 2n_2}} (-1)^{p\cup q} z_{p\cup q} Z_p^{S_{n_1}[S_2]} Z_q^{S_{n_2}[S_2]} Z_{p\cup q}^{S_n[S_2]} \tag{E.1.19}$$

E.2 The generating function

The generating function for $N^{\delta}_{n_1,n_2}$ is

$$F_{SO(N)}(x,y) = \sum_{n_1,n_2} x^{n_1} y^{n_2} N^{\delta}_{n_1,n_2} = \sum_{n_1,n_2} x^{n_1} y^{n_2} \sum_{\substack{p\vdash 2n_1 \\ q\vdash 2n_2}} (-1)^{p\cup q} Z_p^{S_{n_1}[S_2]} Z_q^{S_{n_2}[S_2]} Z_{p\cup q}^{S_n[S_2]} z_{p\cup q} \tag{E.2.1}$$

Our approach is to build candidate generating functions by introducing the terms on the right hand side one by one. We begin by using (E.1.14) twice

$$\exp\left[\sum_{k=1}^{\infty} \frac{1}{2k}(x^k + y^k)(t_k^2 + t_{2k})\right] = \sum_{n_1,n_2} x^{n_1} y^{n_2} \sum_{\substack{p\vdash 2n \\ q\vdash 2m}} Z_p^{S_n[S_2]} Z_q^{S_n[S_2]} \prod_i t_i^{p_i+q_i} \tag{E.2.2}$$

The third cycle index in (E.1.19) comes with a factor of $(-1)^{p\cup q}$. To introduce this into (E.1.14), we just replace t_k with $-t_k$ for n even. Multiplying through by this modified version with a new set of variables s_k and no overall level (no equivalent to x, y) gives

$$\exp\left[\sum_{k=1}^{\infty} \frac{1}{2k}(s_k^2 - s_{2k}) + \sum_{k=1}^{\infty} \frac{1}{2k}(x^k + y^k)(t_k^2 + t_{2k})\right] \\ = \sum_{n_1,n_2,m} x^{n_1} y^{n_2} \sum_{\substack{p\vdash 2n_1 \\ q\vdash 2n_2 \\ r\vdash 2m}} (-1)^r Z_p^{S_{n_1}[S_2]} Z_q^{S_{n_2}[S_2]} Z_r^{S_m[S_2]} \prod_i t_i^{p_i+q_i} s_i^{r_i} \tag{E.2.3}$$

We introduce a factor of $z_{p\cup q}$ and enforce $r = p \cup q$ in two steps, corresponding to the two parts of

$$z_{p\cup q} = \prod_i i^{p_i+q_i}(p_i+q_i)! \tag{E.2.4}$$

To obtain the powers of i, we replace t_k and s_k with $\sqrt{k}\, s_k$ and $\sqrt{k}\, t_k$.

$$\exp\left[\sum_{k=1}^{\infty}\left(\frac{1}{2}s_k^2 - \frac{1}{\sqrt{2k}} s_{2k}\right)\right] \exp\left[\sum_{k=1}^{\infty}(x^k + y^k)\left(\frac{1}{2}t_k^2 + \frac{1}{\sqrt{2k}} t_{2k}\right)\right] \tag{E.2.5}$$

$$= \sum_{n_1,n_2,m} x^{n_1} y^{n_2} \sum_{\substack{p\vdash 2n_1\\ q\vdash 2n_2\\ r\vdash 2m}} (-1)^p (-1)^q Z_p^{S_{n_1}[S_2]} Z_q^{S_{n_2}[S_2]} Z_r^{S_m[S_2]} \prod_i i^{\frac{1}{2}(p_i+q_i+r_i)} t_i^{p_i+q_i} s_i^{r_i} \tag{E.2.6}$$

To replace $\prod_i t_i^{p_i+q_i} s_i^{r_i}$ with $\delta_{r_i,p_i+q_i}(p_i+q_i)!$, we use the integral

$$\int_{\mathbb{C}} \frac{\mathrm{d}z\mathrm{d}\bar{z}}{2\pi} e^{-z\bar{z}} z^p \bar{z}^r = \delta_{p,r} p! \tag{E.2.7}$$

Replacing t_k with z_k, s_k with $\bar{z}_k$, multiplying by $e^{-\sum_k z_k \bar{z}_k}$, and integrating over a copy of $\mathbb{C}$ for each k gives us

$$\begin{aligned}
F_{SO(N)}(x,y) &= \sum_{n_1,n_2} x^{n_1} y^{n_2} \sum_{\substack{p\vdash 2n\\ q\vdash 2m}} (-1)^{p\cup q} Z_p^{S_n[S_2]} Z_q^{S_m[S_2]} Z_{p\cup q}^{S_{n+m}[S_2]} z_{p\cup q} \\
&= \int \left(\prod_{k=1}^{\infty} \frac{\mathrm{d}z_k \mathrm{d}\bar{z}_k}{2\pi}\right) \exp\left[\sum_{k=1}^{\infty}\left(\frac{1}{2}\bar{z}_k^2 - \frac{1}{\sqrt{2k}}\bar{z}_{2k}\right)\right] \\
&\qquad \exp\left[\sum_{k=1}^{\infty}(x^k+y^k)\left(\frac{1}{2}z_k^2 + \frac{1}{\sqrt{2k}}z_{2k}\right)\right] \exp\left[-\sum_{k=1}^{\infty} z_k \bar{z}_k\right] \\
&= \prod_{k\text{ odd}} \int \frac{\mathrm{d}z\mathrm{d}\bar{z}}{2\pi} \exp\left[\frac{1}{2}(\bar{z}^2 - 2z\bar{z} + (x^k+y^k)z^2)\right] \\
&\prod_{k\text{ even}} \int \frac{\mathrm{d}z\mathrm{d}\bar{z}}{2\pi} \exp\left[\frac{1}{2}\left(\bar{z}^2 - 2z\bar{z} + (x^k+y^k)z^2 - \frac{2}{\sqrt{k}}\left(\bar{z} - \left(x^{\frac{k}{2}} + y^{\frac{k}{2}}\right)z\right)\right)\right]
\end{aligned} \tag{E.2.8}$$

To compute these two integrals, we split z into its real and imaginary parts. Using $z = u + iv$, $\bar{z} = u - iv$, and for simplicity writing $\lambda = x^k + y^k, \mu = x^{\frac{k}{2}} + y^{\frac{k}{2}}$, we have

$$\bar{z}^2 - 2z\bar{z} + \lambda z^2 = -(1-\lambda)(u+iv)^2 - 4v^2 \tag{E.2.9}$$

$$\begin{aligned}
\bar{z}^2 - 2z\bar{z} + \lambda z^2 - \frac{2}{\sqrt{k}}(\bar{z} - \mu z) &= -(1-\lambda)\left(u + iv + \frac{1-\mu}{\sqrt{k}(1-\lambda)}\right)^2 \\
&\quad - 4\left(v - \frac{i}{2\sqrt{k}}\right)^2 + \frac{\lambda - 2\mu + \mu^2}{k(1-\lambda)}
\end{aligned} \tag{E.2.10}$$

Changing variables from $(z,\bar{z})$ to (u,v) (and remembering that $\mathrm{d}z\mathrm{d}\bar{z} = 2\mathrm{d}u\mathrm{d}v$), both odd and even integrals can be evaluated using the standard Gaussian integral

$$\int_{-\infty}^{\infty} \mathrm{d}u\, e^{-a(u+b)^2} = \sqrt{\frac{\pi}{a}} \tag{E.2.11}$$

where a, b are complex numbers with $\mathrm{Re}(a) > 0$. The integrals are

$$\int \frac{\mathrm{d}z\mathrm{d}\bar{z}}{2\pi} \exp\left[\frac{1}{2}(\bar{z}^2 - 2z\bar{z} + \lambda z^2)\right] = \frac{1}{\sqrt{1-\lambda}} \tag{E.2.12}$$

and

$$\int \frac{\mathrm{d}z\mathrm{d}\bar{z}}{2\pi} \exp\left[\frac{1}{2}\left(\bar{z}^2 - 2z\bar{z} + \lambda z^2 - \sqrt{\frac{2}{k}}\left(\bar{z} - \mu z\right)\right)\right] = \frac{1}{\sqrt{1-\lambda}} \exp\left[\frac{\lambda - 2\mu + \mu^2}{2k(1-\lambda)}\right] \tag{E.2.13}$$

Plugging these into (E.2.8) gives

$$F_{SO(N)}(x,y) = \left(\prod_{k\ \mathrm{odd}} \frac{1}{\sqrt{1 - x^k - y^k}}\right) \left(\prod_{k\ \mathrm{even}} \frac{1}{\sqrt{1-x^k-y^k}} \exp\left[\frac{x^k + x^{\frac{k}{2}} y^{\frac{k}{2}} + y^k - x^{\frac{k}{2}} - y^{\frac{k}{2}}}{k(1-x^k-y^k)}\right]\right) \tag{E.2.14}$$

$$= \prod_{k=1}^{\infty} \frac{1}{\sqrt{1-x^k-y^k}} \exp\left[\frac{x^{2k} + x^k y^k + y^{2k} - x^k - y^k}{2k(1 - x^{2k} - y^{2k})}\right] \tag{E.2.15}$$

which matches the result (4.3.21).

Appendix F

Construction and correlators of $SO(N)$ covariant basis

F.1 Generic representations of $S_n[S_2]$

Recall from section 5.1.2 that $S_n[S_2]$ is defined abstractly as a semi-direct product of S_n with $(S_2)^n$. For $n_1 + n_2 = n$, we can split S_n into $S_{n_1} \times S_{n_2}$ and perform the corresponding split into $(S_2)^{n_1} \times (S_2)^{n_2}$, so that the S_{m_i} component acts by permuting the factors of $(S_2)^{m_i}$. Define the subgroup G_{n_1,n_2} of $S_n[S_2]$ to be the semi-direct product of these two splits

$$G_{n_1,n_2} = (S_{n_1} \times S_{n_2}) \ltimes [(S_2)^{n_1} \times (S_2)^{n_2}] = [S_{n_1} \ltimes (S_2)^{n_1}] \times [S_{n_2} \ltimes (S_2)^{n_2}] \tag{F.1.1}$$

We can then consider representations of G_{n_1,n_2}. In particular, let $R_1 \vdash n_1$ and $R_2 \vdash n_2$ be representations of S_{n_1} and S_{n_2} respectively. Denoting the the trivial representation of $(S_2)^{n_1}$ by $triv_{n_1}$ and the anti-symmetric representation of $(S_2)^{n_2}$ by $sign_{n_2}$, we define the representation (R_1, R_2) of G_{n_1,n_2} to be R_1 on the S_{n_1} factor, $triv_{n_1}$ on the $(S_2)^{n_1}$ factor, R_2 on the S_{n_2} factor and $sign_{n_2}$ on the $(S_2)^{n_2}$ factor.

$$V^{G_{n_1,n_2}}_{(R_1,R_2)} = V^{S_{n_1}}_{R_1} \otimes V^{(S_2)^{n_1}}_{triv_{n_1}} \otimes V^{S_{n_2}}_{R_2} \otimes V^{(S_2)^{n_2}}_{sign_{n_2}} \tag{F.1.2}$$

We can use the (R_1, R_2) representation of G_{n_1,n_2} to induce a representation of the full group $S_n[S_2]$. It is proved in [124] that this induced representation is irreducible. To understand the (R_1, R_2) representation of the full group $S_n[S_2]$ requires an understanding of the cosets of G_{n_1,n_2} within $S_n[S_2]$, however for our purposes, it will be enough to understand the behaviour of just G_{n_1,n_2} on the irreducible representations of $S_n[S_2]$, for which (F.1.2) is a complete description.

From this description, the representation $[S]$ (defined in section 5.1.2) of $S_n[S_2]$ is $([n], \phi)$, where ϕ is the empty partition of 0, and the representation $[A]$ is $(\phi, [n])$.

Using this understanding, when we embed $S_n[S_2]$ into S_{2n}, we can express an irreducible representation of $S_n[S_2]$ by only looking at the subgroup G_{n_1,n_2}. Explicitly, we have the decomposition

$$\begin{aligned} V_R^{S_{2n}} &= \bigoplus_{\substack{R_1\vdash n_1\,,\,R_2\vdash n_2\\ n_1+n_2=n}} V_{(R_1,R_2)}^{S_n[S_2]} \otimes V_{R;(R_1,R_2)}^{mult} \\ &= \bigoplus_{\substack{R_1\vdash n_1\,,\,R_2\vdash n_2\\ n_1+n_2=n}} V_{(R_1,R_2)}^{G_{n_1,n_2}} \otimes V_{R;(R_1,R_2)}^{mult} \\ &= \bigoplus_{\substack{R_1\vdash n_1\,,\,R_2\vdash n_2\\ n_1+n_2=n}} \left[V_{R_1}^{S_{n_1}} \otimes V_{triv_{n_1}}^{(S_2)^{n_1}} \otimes V_{R_2}^{S_{n_2}} \otimes V_{sign_{n_2}}^{(S_2)^{n_2}}\right] \otimes V_{R;(R_1,R_2)}^{mult} \end{aligned} \tag{F.1.3}$$

where $V_{R;(R_1,R_2)}^{mult}$ is a multiplicity space.

In the decomposition (F.1.3), when we act with S_{n_1} on the left hand side, we embed S_{n_1} into S_{2n} by acting on the set of n_1 pairs $\{1,2\},\{3,4\},\dots,\{2n_1-1,2n_1\}$, and similarly S_{n_2} is embedded by acting on the pairs $\{2n_1+1,2n_1+2\},\dots,\{2n-1,2n\}$.

Under conjugation of R the sign of all representatives is switched. The representations of S_{n_1} and S_{n_2} are unchanged, since all permutations in these embeddings are even. However, $(S_2)^{n_1}$ now has the representation $sign_{n_1}$ and $(S_2)^{n_2}$ has $triv_{n_2}$. Therefore

$$V_{R^c;(R_1,R_2)}^{mult} = V_{R;(R_2,R_1)}^{mult} \tag{F.1.4}$$

F.2 Operator construction

In (5.7.1) we defined a generic $U(2)$ covariant operator, labelled by $\sigma \in S_{2n}$ and $a \in V_2^{\otimes n}$, where V_2 is the fundamental representation of $U(2)$.

$$\mathcal{O}_{a,\sigma}^{\delta} = C_I^{(\delta)} \sigma_J^I \left(\mathbb{X}_a\right)^J \tag{F.2.1}$$

This definition has several invariant properties. Firstly, it follows from the structure of $C_I^{(\delta)}$, defined in (5.2.1), that

$$\mathcal{O}_{a,\tau\sigma}^{\delta} = \mathcal{O}_{a,\sigma}^{\delta} \tag{F.2.2}$$

for any $\tau \in S_n[S_2]$. Secondly, the anti-symmetry of X and Y imply that

$$\mathcal{O}_{a,\sigma\tau}^{\delta} = (-1)^{\tau} \mathcal{O}_{a,\sigma}^{\delta} \tag{F.2.3}$$

for any $\tau \in (S_2)^n$.

Finally, consider $\tau \in S_n$ embedded into S_{2n}. The definition of $S_n[S_2]$ consists of a semi-direct product between S_n and $(S_2)^n$ (see section 5.1.2). When $S_n[S_2]$ is embedded into S_{2n}, this S_n subgroup from the definition permutes the n pairs

$\{1,2\},\{3,4\},\ldots,\{2n-1,2n\}$ by taking odd numbers to odd numbers and evens to evens. In chapter 6 this embedding is referred to as $S_n^{(diag)}$. Denote this embedding of $\tau\in S_n$ into S_{2n} by $\tau^{(diag)}$. Then using a non-standard labelling for the $2n$ indices of $\tau^{(diag)}$, we have

$$\left(\tau^{(diag)}\right)^I_J=\left(\tau^{(diag)}\right)^{i_{1,1}i_{1,2}i_{2,1}i_{2,2}\cdots i_{n,1}i_{n,2}}_{j_{1,1}j_{1,2}j_{2,1}j_{2,2}\cdots j_{n,1}j_{n,2}}=\delta^{i_{1,1}}_{j_{\tau(1),1}}\delta^{i_{1,2}}_{j_{\tau(1),2}}\ldots\delta^{i_{n,1}}_{j_{\tau(n),1}}\delta^{i_{n,2}}_{j_{\tau(n),2}} \tag{F.2.4}$$

It follows that the action of $\tau^{(diag)}$ on $\mathbb{X}_a$ is

$$\begin{aligned}\left(\tau^{(diag)}\right)^I_J(\mathbb{X}_a)^J&=\left(\tau^{(diag)}\right)^{i_{1,1}i_{1,2}i_{2,1}i_{2,2}\cdots i_{n,1}i_{n,2}}_{j_{1,1}j_{1,2}j_{2,1}j_{2,2}\cdots j_{n,1}j_{n,2}}(X_{a_1})^{j_{1,1}j_{1,2}}\ldots(X_{a_n})^{j_{n,1}j_{n,2}}\\&=(X_{a_1})^{i_{\tau^{-1}(1),1}i_{\tau^{-1}(1),2}}\ldots(X_{a_n})^{i_{\tau^{-1}(n),1}i_{\tau^{-1}(n),2}}\\&=\left(X_{a_{\tau(1)}}\otimes X_{a_{\tau(2)}}\otimes\cdots\otimes X_{a_{\tau(n)}}\right)^J\\&=\left(\mathbb{X}_{\tau(a)}\right)^J\end{aligned} \tag{F.2.5}$$

where $\tau\in S_n$ acts on $a\in V_2^{\otimes n}$ by permutation of factors. Then

$$\begin{aligned}\mathcal{O}^\delta_{a,\sigma\tau^{(diag)}}&=C^{(\delta)}_I\sigma^I_J\left(\tau^{(diag)}\right)^J_K(\mathbb{X}_a)^K\\&=C^{(\delta)}_I\sigma^I_J\left(\mathbb{X}_{\tau(a)}\right)^J\\&=\mathcal{O}^\delta_{\tau(a),\sigma}\end{aligned} \tag{F.2.6}$$

We can view (F.2.1) as a map into the space of $SO(N)$ operators from

$$V_2^{\otimes n}\otimes\mathbb{C}(S_{2n}) \tag{F.2.7}$$

where there is a finite N cut-off on $\mathbb{C}(S_{2n})$ as described in section 2.5. It follows from (F.2.2), eqF.2.3 and (F.2.6) that this is a redundant description of the space of operators. The redundancies on (F.2.7) are

$$(a,\sigma)\to(a,\tau\sigma)\qquad\tau\in S_n[S_2] \tag{F.2.8}$$

$$(a,\sigma)\to(a,(-1)^\tau\sigma\tau)\qquad\tau\in(S_2)^n \tag{F.2.9}$$

$$(a,\sigma)\to(\tau(a),\sigma(\tau^{(diag)})^{-1})\qquad\tau\in S_n \tag{F.2.10}$$

To remove these redundancies, we re-express (F.2.7) using Schur-Weyl duality, defined in (2.4.3), and the decomposition of $\mathbb{C}(S_{2n})$ into representations of S_n, given in (2.5.2). Taking account of the finite N constraints on $\mathbb{C}(S_{2n})$, we have

$$V_2^{\otimes n}\otimes\mathbb{C}(S_{2n})=\bigoplus_{\substack{\Lambda\vdash n\,,\,l(\Lambda)\le 2\\ R\vdash 2n\,,\,l(R)\le N}}V_\Lambda^{U(2)}\otimes V_\Lambda^{S_n}\otimes V_R^{S_{2n};left}\otimes V_R^{S_{2n};right} \tag{F.2.11}$$

In terms of the $\mathcal{O}^\delta$ operators, this corresponds to setting

$$\mathcal{O}^\delta_{\Lambda,M_\Lambda,m_\Lambda,R,I,J} = \sum_{a,\sigma} C^a_{\Lambda,M_\Lambda,m_\Lambda} D^R_{IJ}(\sigma)\mathcal{O}_{a,\sigma} \tag{F.2.12}$$

where $C^a_{\Lambda,M_\Lambda,m_\Lambda}$ is the Clebsch-Gordon coefficient for the Schur-Weyl decomposition and I, J are basis indices for R.

On the new labelling space (F.2.11), the redundancies (F.2.8-F.2.10) are

$$(M_\Lambda, v_\Lambda, v^l_R, v^r_R) \to (M_\Lambda, v_\Lambda, D^R(\tau)v^l_R, v^r_R) \qquad \tau \in S_n[S_2] \tag{F.2.13}$$

$$(M_\Lambda, v_\Lambda, v^l_R, v^r_R) \to (M_\Lambda, v_\Lambda, v^l_R, (-1)^\tau D^R(\tau)v^r_R) \qquad \tau \in (S_2)^n \tag{F.2.14}$$

$$(M_\Lambda, v_\Lambda, v^l_R, v^r_R) \to (M_\Lambda, D^\Lambda(\tau)v_\Lambda, v^l_R, D^R(\tau^{(diag)})v^r_R) \qquad \tau \in S_n \tag{F.2.15}$$

where v^l_R and v^r_R are vectors in $V^{S_{2n};left}_R$ and $V^{S_{2n};right}_R$ respectively, v_Λ is a vector in $V^{S_n}_\Lambda$ and M_Λ is a vector in $V^{U(N)}_\Lambda$.

Studying (F.2.13), we see that to remove the redundancy we need to choose an $S_n[S_2]$-invariant vector in $V^{S_{2n}}_{R;left}$. Such a (non-zero) vector only exists if R has even row lengths, and in that case there is a unique choice for the vector, $|R,[S]\rangle$ (explained in 5.1.2). This means the index I in (F.2.12) should be contracted with the components $|R,[S]\rangle_I$.

To understand (F.2.14) and (F.2.15), we decompose $V^{S_{2n};right}_R$ into representations of $S_n[S_2]$ as given in (F.1.3). To ensure anti-symmetry under the whole of $(S_2)^n$, as given by (F.2.14), we only need to consider $n_1 = 0$, $n_2 = n$. In terms of the operators (F.2.12), this means introducing branching coefficients $B^{S_{2n}\to S_n[S_2]\;;J}_{R\;\;\to(\phi,\Lambda),\mu;l}$ from the R representation of S_{2n} to the (ϕ, R_2) representations of $S_n[S_2]$, where l is a basis index for R_2 and μ is a multiplicity index for the decomposition.

Dropping the one-dimensional vectors space, we have reduced (F.2.11) to

$$\bigoplus_{\substack{\Lambda\vdash n\,,\,l(\Lambda)\leq 2\\ R\vdash 2n\,,\,l(R)\leq N\\ R \text{ with even row lengths}\\ R_2\vdash n}} V^{U(2)}_\Lambda \otimes V^{S_n}_\Lambda \otimes V^{S_n}_{R_2} \otimes V^{mult}_{R;(\phi,R_2)} \tag{F.2.16}$$

where the only remaining redundancy (F.2.15) acts as

$$(M_\Lambda, v_\Lambda, v_{R_2}, \mu) \to (M_\Lambda, D^\Lambda(\tau)v_\Lambda, D^{R_2}(\tau)v_{R_2}, \mu) \qquad \tau \in S_n \tag{F.2.17}$$

This implies we are looking for an S_n-invariant vector in $V^{S_n}_\Lambda \otimes V^{S_n}_{R_2}$. By standard representation theory of S_n, this exists if and only if $\Lambda = R_2$, and then with multiplicity 1. Therefore after removing all the redundancies from the description, we are left with the labelling set

$$\bigoplus_{\substack{\Lambda\vdash n\,,\,l(\Lambda)\leq 2\\ R\vdash 2n\,,\,l(R)\leq N\\ R\text{ with even row lengths}}} V_{\Lambda}^{U(2)}\otimes V_{R;(\phi,\Lambda)}^{mult} \tag{F.2.18}$$

and in terms of operators

$$\mathcal{O}^{\delta}_{\Lambda,M_\Lambda,R,\mu}=\frac{1}{2^n n!}\sqrt{\frac{d_R}{d_\Lambda(2n)!}}\sum_{a,\sigma,J,k}C^a_{\Lambda,M_\Lambda,k}\Big[\langle R,[S]|\,D^R(\sigma)\Big]_J B^{S_{2n}\to S_n[S_2]\ ;J}_{R\quad\to(\phi,\Lambda),\mu;k}\mathcal{O}^{\delta}_{a,\sigma} \tag{F.2.19}$$

where the sum over k comes from projecting to the S_n invariant vector inside $V_\Lambda^{S_n}\otimes V_\Lambda^{S_n}$. The normalisation is chosen to give nice correlators in the next section.

F.3 Correlators

To calculate the two-point functions for (F.2.19), we start with two generic $U(2)$ covariant operators from (F.2.1). This correlator is given in (5.7.5) and reproduced below

$$\left\langle \mathcal{O}^{\delta}_{b,\tau}|\mathcal{O}^{\delta}_{a,\sigma}\right\rangle=\sum_{\gamma,\pi\in S_n[S_2]}\delta_{a,\bar\gamma(b)}(-1)^{\gamma}\delta\left(\widetilde{\Omega}\pi\sigma\gamma^{-1}\tau^{-1}\right) \tag{F.3.1}$$

where for $\gamma\in S_n[S_2]$, $\bar\gamma\in S_n$ is defined as the S_n component of γ from the semi-direct product $S_n[S_2]=S_n\ltimes(S_2)^n$. We also define $\hat\gamma$ to be the $(S_2)^n$ component.

The correlator of two operators is anti-linear in the first argument and linear in the second argument. All coefficients in (F.2.19) are real except for $C^a_{\Lambda,M_\Lambda,k}$, which picks up a complex conjugate as we remove it from the correlator.

We will need several properties of the coefficients in (F.2.19). Firstly, since $C^a_{\Lambda,M_\Lambda,k}$ are the coefficients for the decomposition (2.4.3), it follows by applying $\sigma\in S_n$ to both sides that

$$C^a_{\Lambda,M_\Lambda,k}=D^\Lambda_{kl}(\sigma)C^{\sigma(a)}_{\Lambda,M_\Lambda,l} \tag{F.3.2}$$

The coefficients $C^a_{\Lambda,M_\Lambda,k}$ are a unitary change of basis for $V_2^{\otimes n}$, and therefore

$$\sum_a C^a_{\Lambda,M_\Lambda,k}\left(C^a_{\Gamma,M_\Gamma,l}\right)^*=\delta_{\Lambda\Gamma}\delta_{M_\Lambda M_\Gamma}\delta_{kl} \tag{F.3.3}$$

Finally, the branching coefficients $B^{S_{2n}\to S_n[S_2]\ ;J}_{R\quad\to(\phi,\Lambda),\mu;k}$ take a basis index J in the R representation of S_{2n} to a basis index k in the μth copy of the (ϕ,Λ) representation of $S_n[S_2]$, and therefore for $\gamma\in S_n[S_2]$

$$\sum_{IJ}B^{S_{2n}\to S_n[S_2]\ ;I}_{R\quad\to(\phi,\Lambda),\mu;k}\,B^{S_{2n}\to S_n[S_2]\ ;J}_{R\quad\to(\phi,\Lambda),\nu;l}\,D^R_{IJ}(\gamma)=\delta_{\mu\nu}(-1)^{\hat\gamma}D^\Lambda_{kl}\left(\bar\gamma\right) \tag{F.3.4}$$

where $\bar{\gamma}$ is the S_n component of γ and $\hat{\gamma}$ is the $(S_2)^n$ component.

There is one more property of the branching coefficients that will be useful not for the calculation of correlators but for the comparison between symplectic and mesonic covariant operators. It follows from the identification (F.1.4) of conjugate multiplicity spaces that

$$B^{S_{2n}\to S_n[S_2]\ ;I}_{R^c\ \to(\phi,\Lambda),\mu;k} = B^{S_{2n}\to S_n[S_2]\ ;I}_{R\ \to(\Lambda,\phi),\mu;k} \tag{F.3.5}$$

Using (F.3.2-F.3.4), we build up to the calculation of correlators for (F.2.19). Start with

$$\begin{aligned}\left\langle \sum_{\tau\in S_{2n}} D^S_{KL}(\tau)\mathcal{O}_{b,\tau}| \sum_{\sigma\in S_{2n}} D^R_{IJ}(\sigma)\mathcal{O}_{a,\sigma}\right\rangle &= \sum_{\substack{\sigma,\tau\in S_{2n}\\ \gamma,\pi\in S_n[S_2]}} \delta_{a,\bar{\gamma}(b)}(-1)^\gamma D^R_{IJ}(\sigma)D^S_{KL}(\tau)\delta\left(\widetilde{\Omega}\pi\sigma\gamma^{-1}\tau^{-1}\right)\\ &= \sum_{\substack{\sigma,\tau\in S_{2n}\\ \gamma,\pi\in S_n[S_2]}} \delta_{a,\bar{\gamma}(b)}(-1)^\gamma D^R_{IJ}(\widetilde{\Omega}\pi\sigma\gamma)D^S_{KL}(\tau)\delta\left(\sigma\tau^{-1}\right)\\ &= \sum_{\substack{\sigma\in S_{2n}\\ \gamma,\pi\in S_n[S_2]}} \delta_{a,\bar{\gamma}(b)}(-1)^\gamma D^R_{IJ}(\widetilde{\Omega}\pi\sigma\gamma)D^S_{KL}(\sigma)\\ &= \delta_{RS}\frac{(2n)!}{d_R} \sum_{\gamma,\pi\in S_n[S_2]} \delta_{a,\bar{\gamma}(b)}(-1)^\gamma D^R_{IK}(\widetilde{\Omega}\pi)D^R_{LJ}(\gamma)\end{aligned} \tag{F.3.6}$$

where we have, respectively for each line, changed summation variables, summed over τ using the δ function, and summed over σ using the orthogonality of matrix elements (2.3.4).

Introducing the vector $|R,[S]\rangle$

$$\begin{aligned}&\left\langle \sum_{\tau\in S_{2n}} \Big[\langle S,[S]|\, D^S(\tau)\Big]_J \mathcal{O}_{b,\tau}| \sum_{\sigma\in S_{2n}} \Big[\langle R,[S]|\, D^R(\sigma)\Big]_I \mathcal{O}_{a,\sigma}\right\rangle\\ &\qquad= \delta_{RS}\frac{(2n)!}{d_R} \sum_{\gamma,\pi\in S_n[S_2]} \delta_{a,\bar{\gamma}(b)}(-1)^\gamma \langle R,[S]|\, D^R(\widetilde{\Omega}\pi)\,|R,[S]\rangle\, D^R_{JI}(\gamma)\\ &\qquad= \delta_{RS}\frac{(2n)!2^n n! f^\delta_R}{d_R} \sum_{\gamma\in S_n[S_2]} \delta_{a,\bar{\gamma}(b)}(-1)^\gamma D^R_{JI}(\gamma)\end{aligned} \tag{F.3.7}$$

where we have used the action of $\widetilde{\Omega}$ (A.2.18) and the invariance of $|R,[S]\rangle$ under $S_n[S_2]$.

Introducing the coefficients $C^a_{\Lambda,M_\Lambda,k}$

$$\left\langle \sum_{\substack{\tau\in S_{2n}\\ b}} C^b_{\Gamma,M_\Gamma,l}\Big[\langle S,[S]|\, D^S(\tau)\Big]_J \mathcal{O}_{b,\tau}| \sum_{\substack{\sigma\in S_{2n}\\ a}} C^a_{\Lambda,M_\Lambda,k}\Big[\langle R,[S]|\, D^R(\sigma)\Big]_I \mathcal{O}_{a,\sigma}\right\rangle$$

$$
\begin{aligned}
&= \delta_{RS} \frac{(2n)!2^n n! f_R^\delta}{d_R} \sum_{\substack{\gamma \in S_n[S_2] \\ a,b}} \delta_{a,\bar{\gamma}(b)} (-1)^\gamma D^R_{JI}(\gamma) C^a_{\Lambda, M_\Lambda, k} \left(C^b_{\Gamma, M_\Gamma, l} \right)^* \\
&= \delta_{RS} \frac{(2n)!2^n n! f_R^\delta}{d_R} \sum_{\substack{\gamma \in S_n[S_2] \\ a}} (-1)^\gamma D^R_{JL}(\gamma) C^a_{\Lambda, M_\Lambda, k} \left(C^{\bar{\gamma}^{-1}(a)}_{\Gamma, M_\Gamma, l} \right)^* \\
&= \delta_{RS} \frac{(2n)!2^n n! f_R^\delta}{d_R} \sum_{\substack{\gamma \in S_n[S_2] \\ a,m}} (-1)^\gamma D^R_{JI}(\gamma) C^a_{\Lambda, M_\Lambda, k} D^\Lambda_{lm}(\bar{\gamma}) \left(C^a_{\Gamma, M_\Gamma, m} \right)^* \\
&= \delta_{RS} \delta_{\Lambda\Gamma} \delta_{M_\Lambda M_\Gamma} \frac{(2n)!2^n n! f_R^\delta}{d_R} \sum_{\gamma \in S_n[S_2]} (-1)^\gamma D^R_{JI}(\gamma) D^\Lambda_{lk}(\bar{\gamma}) \qquad \text{(F.3.8)}
\end{aligned}
$$

where we summed over b using the δ function, then used (F.3.2) and (F.3.3) to remove the Schur-Weyl coefficients.

Finally introducing the branching coefficients

$$
\begin{aligned}
&\left\langle \mathcal{O}^\delta_{\Gamma, M_\Gamma, S, \nu} | \mathcal{O}^\delta_{\Lambda, M_\Lambda, R, \mu} \right\rangle \\
&\quad = \delta_{RS} \delta_{\Lambda\Gamma} \delta_{M_\Lambda M_\Gamma} \frac{f_R^\delta}{d_\Lambda 2^n n!} \sum_{\substack{\gamma \in S_n[S_2] \\ I,J}} (-1)^\gamma B_{R \;\;\to (\phi,\Lambda),\mu;k}^{S_{2n} \to S_n[S_2]\;;I} \; B_{R \;\;\to (\phi,\Lambda),\nu;l}^{S_{2n} \to S_n[S_2]\;;J} D^R_{JI}(\gamma) D^\Lambda_{lk}(\bar{\gamma}) \\
&\quad = \delta_{RS} \delta_{\Lambda\Gamma} \delta_{M_\Lambda M_\Gamma} \delta_{\mu\nu} \frac{f_R^\delta}{d_\Lambda 2^n n!} \sum_{\hat{\gamma} \in (S_2)^n} (-1)^{\hat{\gamma}} (-1)^{\hat{\gamma}} \sum_{\bar{\gamma} \in S_n} D^\Lambda_{lk}(\bar{\gamma}) D^\Lambda_{lk}(\bar{\gamma}) \\
&\quad = \delta_{RS} \delta_{\Lambda\Gamma} \delta_{M_\Lambda M_\Gamma} \delta_{\mu\nu} \frac{f_R^\delta}{(d_\Lambda)^2} \delta_{kk} \delta_{ll} \\
&\quad = \delta_{RS} \delta_{\Lambda\Gamma} \delta_{M_\Lambda M_\Gamma} \delta_{\mu\nu} f_R^\delta \qquad \text{(F.3.9)}
\end{aligned}
$$

where we have used (F.3.4), split the sum over $S_n[S_2]$ into two over S_n and $(S_2)^n$ respectively, noticed $(-1)^\gamma = (-1)^{\hat{\gamma}}$, applied the orthogonality of matrix coefficients (2.3.4) and then used $\delta_{kk} = d_\Lambda$ for k a basis index of the Λ representation of S_n.

F.4 Basis of multiplicity space

In a similar manner to section D.3, we can give a basis for the multiplicity space $V^{mult}_{R;(R_1,R_2)}$ defined by (F.1.3), giving a systematic way of choosing the multiplicity label μ in (F.2.19).

Let $\mathcal{B}_n$ be the sub-algebra of $\mathbb{C}(S_{2n})$ that commutes with $S_n[S_2]$, or equivalently, the sub-algebra that is invariant under conjugation by $S_n[S_2]$. Then since $\mathcal{B}_n$ commutes with $S_n[S_2]$, by Schur's lemma it must act purely on the multiplicity spaces in the decomposition (F.1.3).

We can define a more refined version of $\mathcal{B}_n$ by projecting to the (R_1, R_2) representation of G_{n_1,n_2}. Define the projector

$$P_{(R_1,R_2)} = \frac{d_{R_1} d_{R_2}}{2^n n_1! n_2!} \sum_{\substack{\sigma_1 \in S_{n_1}, \sigma_2 \in S_{n_2} \\ \tau_1 \in (S_2)^{n_1}, \tau_2 \in (S_2)^{n_2}}} \chi_{R_1}(\sigma_1) \chi_{R_2}(\sigma_2) (-1)^{\tau_2} \sigma_1 \tau_1 \sigma_2 \tau_2 \tag{F.4.1}$$

where S_{n_1}, S_{n_2} are embedded into S_{2n} as described in section F.1. Then the more refined algebra is

$$\mathcal{B}_{(R_1,R_2)} = P_{(R_1,R_2)} \mathcal{B}_n \tag{F.4.2}$$

Then $\mathcal{B}_{(R_1,R_2)}$ acts only on the space $V^{mult}_{R;(R_1,R_2)}$ in the decomposition (F.1.3). A basis for $V^{mult}_{R;(R_1,R_2)}$ can then be chosen by taking the eigenbasis of a maximal commuting sub-algebra of $\mathcal{B}_{(R_1,R_2)}$.

Appendix G

Examples of quarter BPS operators in specific Λ sectors

In this appendix we give explicit examples of quarter-BPS operators constructed using the algorithm presented in chapter 7.

G.1 $\Lambda = [3,2]$ sector

In this section we give the operators in the $\Lambda = [3,2]$ sector with M_Λ the highest weight state, corresponding to field content $(3,2)$. Other states in the $U(2)$ representation can be reached by applying the lowering operator $\mathcal{J}_-$.

Throughout this section we will work with $\Lambda = [3,2]$ and $M_\Lambda = \begin{array}{|c|c|c|}\hline 1&1&1\\\hline 2&2\\\cline{1-2}\end{array}$, so we will suppress these indices in operator labels.

For each BPS operator, we will first present it as a sum over the free field basis (3.6.20) and then as a sum over symmetrised traces and commutator traces, for which we use the covariant bases discussed in section 7.3.2. The covariant symmetrised trace basis is

$$t_{[3,2]} = \mathrm{Tr}X^3\mathrm{Tr}Y^2 - 2\mathrm{Tr}X^2Y\mathrm{Tr}XY + \mathrm{Tr}X^2\mathrm{Tr}XY^2 \tag{G.1.1}$$

$$t_{[3,1,1]} = \mathrm{Tr}X^3\left(\mathrm{Tr}Y\right)^2 - 2\mathrm{Tr}X\mathrm{Tr}X^2Y\mathrm{Tr}Y + \left(\mathrm{Tr}X\right)^2\mathrm{Tr}XY^2 \tag{G.1.2}$$

$$t_{[2,2,1]} = \mathrm{Tr}X\mathrm{Tr}X^2\mathrm{Tr}Y^2 - \mathrm{Tr}X\left(\mathrm{Tr}XY\right)^2 \tag{G.1.3}$$

$$t_{[2,1,1,1]} = \mathrm{Tr}X\mathrm{Tr}X^2\left(\mathrm{Tr}Y\right)^2 - 2\left(\mathrm{Tr}X\right)^2\mathrm{Tr}XY\mathrm{Tr}Y + \left(\mathrm{Tr}X\right)^3\mathrm{Tr}Y^2 \tag{G.1.4}$$

and the covariant commutator trace basis is

$$c_{[5]} = \mathrm{Tr}X^3Y^2 - \mathrm{Tr}X^2YXY = \mathrm{Tr}X^2[X,Y]Y \tag{G.1.5}$$

$$c_{[4,1]} = \mathrm{Tr}X\mathrm{Tr}X^2Y^2 - \mathrm{Tr}X\mathrm{Tr}\left(XY\right)^2 = \mathrm{Tr}X\mathrm{Tr}X[X,Y]Y \tag{G.1.6}$$

For these two bases, the partition label describes the cycle structure of the multi-traces.

The free field operators can be written in terms of symmetrised and commutator traces

$$\mathcal{O}_{[4,1]} = \frac{1}{6\sqrt{10}} \left(3t_{[3,2]} + t_{[3,1,1]} + 4t_{[2,2,1]} + t_{[2,1,1,1]} + 6c_{[5]} + 4c_{[4,1]}\right) \tag{G.1.7}$$

$$\mathcal{O}_{[3,2]} = \frac{1}{6\sqrt{2}} \left(t_{[3,1,1]} + t_{[2,2,1]} + t_{[2,1,1,1]} - 3c_{[5]} - 2c_{[4,1]}\right) \tag{G.1.8}$$

$$\mathcal{O}_{[3,1,1],\,\mathrm{odd}} = \frac{1}{2\sqrt{30}} \left(2t_{[3,2]} - t_{[2,1,1,1]} - 4c_{[4,1]}\right) \tag{G.1.9}$$

$$\mathcal{O}_{[3,1,1],\,\mathrm{even}} = \frac{1}{2\sqrt{5}} \left(t_{[3,1,1]} - t_{[2,2,1]} + c_{[5]}\right) \tag{G.1.10}$$

$$\mathcal{O}_{[2,2,1]} = \frac{1}{6\sqrt{2}} \left(t_{[3,1,1]} + t_{[2,2,1]} - t_{[2,1,1,1]} - 3c_{[5]} + 2c_{[4,1]}\right) \tag{G.1.11}$$

$$\mathcal{O}_{[2,1,1,1]} = \frac{1}{6\sqrt{10}} \left(-3t_{[3,2]} + t_{[3,1,1]} + 4t_{[2,2,1]} - t_{[2,1,1,1]} + 6c_{[5]} - 4c_{[4,1]}\right) \tag{G.1.12}$$

The odd/even labels for the $R = [3,1,1]$ multiplicity come from the odd/even permutations used to produce the respective traces. All other zero coupling operators are defined uniquely by Λ and R.

G.1.1 BPS operators

Following the algorithm, the BPS operators in the $\Lambda = [3,2]$ sector are

$$\begin{aligned} S^{BPS}_{[2,1,1,1]} = \frac{1}{2\sqrt{15P_1}} \Big(&(N-2)(N-3)\left[2N\mathcal{O}_{[4,1]} - \sqrt{5}(N+3)\mathcal{O}_{[3,2]}\right] \\ &+ N(N+3)(N-3)\left[4\sqrt{3}\mathcal{O}_{[3,1,1],\,\mathrm{odd}} + 3\sqrt{2}\mathcal{O}_{[3,1,1],\,\mathrm{even}}\right] \\ &-5(N+3)(N+2)\left[\sqrt{5}(N-3)\mathcal{O}_{[2,2,1]} + 4N\mathcal{O}_{[2,1,1,1]}\right]\Big) \end{aligned} \tag{G.1.13}$$

$$\begin{aligned} = \frac{1}{2\sqrt{6P_1}} \Big[&\left(N^3 + 5\,N^2 + 2\,N - 18\right)t_{[2,1,1,1]} - 4\left(N^2 + 3\,N - 3\right)(N+1)t_{[2,2,1]} \\ &- \left(N^2 + 3\,N - 6\right)(N+2)t_{[3,1,1]} + 3N(N+2)(N+1)t_{[3,2]} \\ &+ 4(2\,N + 9)(N+1)c_{[4,1]} - 18(N+2)(N+1)c_{[5]}\Big] \end{aligned} \tag{G.1.14}$$

$$\begin{aligned} S^{BPS}_{[2,2,1]} = \frac{1}{2\sqrt{3P_2}} \Big(&-\sqrt{5}(N-1)(N-2)\left[2N\mathcal{O}_{[4,1]} - \sqrt{5}(N+3)\mathcal{O}_{[3,2]}\right] \\ &- N(N+3)(N-1)\sqrt{5}\left[4\sqrt{5}\mathcal{O}_{[3,1,1],\,\mathrm{odd}} + 3\sqrt{2}\mathcal{O}_{[3,1,1],\,\mathrm{even}}\right] \\ &-\left(15\,N^3 + 48\,N^2 + 19\,N + 6\right)\mathcal{O}_{[2,2,1]} + 2\sqrt{5}N(3\,N+8)\mathcal{O}_{[2,1,1,1]}\Big) \end{aligned} \tag{G.1.15}$$

$$= \frac{1}{2\sqrt{6P_2}} \Big[\left(5\,N^3 + 12\,N^2 - 12\,N + 6\right)t_{[2,1,1,1]} + 2(3\,N-2)(N-1)t_{[2,2,1]}$$

$$- \left(5\,N^3 + 12\,N^2 - 2\,N - 4\right) t_{[3,1,1]} - N\left(5\,N^2 + 8\,N - 2\right) t_{[3,2]}$$
$$- 4(8\,N + 3)c_{[4,1]} + 6\left(5\,N^2 + 8\,N - 2\right)c_{[5]}\Big] \tag{G.1.16}$$

$$S^{BPS}_{[3,1,1]} = \frac{1}{\sqrt{15P_3}} \Big(-(2\,N-1)(N-2)\left[2N\mathcal{O}_{\yng(4,1)} - \sqrt{5}(N+3)\mathcal{O}_{\yng(3,2)}\right]$$
$$- \sqrt{3}\left(3\,N^3 + 9\,N^2 - 5\,N - 2\right)\mathcal{O}_{\yng(3,1,1),\,\mathrm{odd}}$$
$$+ 3\sqrt{2}\left(3\,N^3 + 6\,N^2 - 4\,N + 2\right)\mathcal{O}_{\yng(3,1,1),\,\mathrm{even}}$$
$$-(N+2)(N+1)\left[\sqrt{5}\mathcal{O}_{\yng(2,2,1)} - 2\mathcal{O}_{\yng(2,1,1,1)}\right]\Big) \tag{G.1.17}$$
$$= \frac{1}{2\sqrt{6P_3}}\Big[\left(N^3 + 3\,N^2 - 5\,N + 2\right)t_{[2,1,1,1]} - 4N^2(N+1)t_{[2,2,1]}$$
$$+ 2\left(2\,N^3 + 4\,N^2 - 5\,N + 2\right)t_{[3,1,1]} - 2N^2(N+1)t_{[3,2]}$$
$$+ 8(N+1)(N-1)c_{[4,1]} + 12N(N+1)c_{[5]}\Big] \tag{G.1.18}$$

$$S^{BPS}_{[3,2]} = \frac{1}{2\sqrt{15P_4}}\Big(2\left(5\,N^2 - 5\,N + 2\right)\mathcal{O}_{\yng(4,1)} + \sqrt{5}\left(4\,N^2 + 5\,N - 2\right)\mathcal{O}_{\yng(3,2)}$$
$$+ 4\sqrt{3}(N-1)\mathcal{O}_{\yng(3,1,1),\,\mathrm{odd}} - 3\sqrt{2}N\mathcal{O}_{\yng(3,1,1),\,\mathrm{even}}$$
$$+(N+2)\left[\sqrt{5}\mathcal{O}_{\yng(2,2,1)} - 2\mathcal{O}_{\yng(2,1,1,1)}\right]\Big) \tag{G.1.19}$$
$$= \frac{\sqrt{6}}{12\sqrt{P_4}}\left[N^2(t_{[2,1,1,1]} + 2t_{[2,2,1]} + t_{[3,1,1]} + t_{[3,2]}) - 4(N-1)c_{[4,1]} - 6Nc_{[5]}\right] \tag{G.1.20}$$

where the normalisation polynomials are

$$P_1 = 10\,N^6 + 74\,N^5 + 199\,N^4 + 252\,N^3 + 351\,N^2 + 648\,N + 702 \tag{G.1.21}$$
$$P_2 = 50\,N^6 + 220\,N^5 + 192\,N^4 - 78\,N^3 + 541\,N^2 - 156\,N + 78 \tag{G.1.22}$$
$$P_3 = 15\,N^6 + 50\,N^5 + 17\,N^4 - 66\,N^3 + 115\,N^2 - 60\,N + 20 \tag{G.1.23}$$
$$P_4 = 3\,N^4 + 5\,N^2 - 4\,N + 2 \tag{G.1.24}$$

In [49], these operators were studied, though in the $SU(N)$ gauge theory rather than the $U(N)$ theory. This means all traces whose cycle structure $p \vdash n$ contained one or more 1s do not contribute. In the $\Lambda = [3,2]$ sector, they found the single operator

$$\mathcal{O} = Nt_{[3,2]} - 6c_{[5]} \tag{G.1.25}$$

One can check that in each of the expansions above, $t_{[3,2]}$ and $c_{[5]}$ only appear in this ratio. We have found that by expanding the gauge group to $U(N)$ and allowing traces of a single matrix, there are three additional quarter-BPS operators.

G.1.2 Norms of BPS operators

The physical $\mathcal{F}$-weighted norms of the BPS operators are

$$\left|S^{BPS}_{[2,1,1,1]}\right|^2 = \frac{(N+3)(N+2)(N+1)N^2(N-1)(N-2)(N-3)Q_1}{P_1} \tag{G.1.26}$$

$$\left|S^{BPS}_{[2,2,1]}\right|^2 = \frac{(N+1)N^2(N-1)(N-2)Q_1Q_2}{P_2} \tag{G.1.27}$$

$$\left|S^{BPS}_{[3,1,1]}\right|^2 = \frac{(N+2)(N+1)N(N-1)(N-2)Q_2Q_3}{P_3} \tag{G.1.28}$$

$$\left|S^{BPS}_{[3,2]}\right|^2 = \frac{(N+2)(N+1)N^3(N-1)Q_3}{P_4} \tag{G.1.29}$$

Where the polynomials in the numerators are

$$Q_1 = 10\,N^3 + 37\,N^2 + 11\,N - 36 \tag{G.1.30}$$

$$Q_2 = 5\,N^3 + 11\,N^2 - 7\,N + 2 \tag{G.1.31}$$

$$Q_3 = 3\,N^3 + 5\,N^2 - 5\,N + 2 \tag{G.1.32}$$

We discuss the combination of linear factors and Q polynomials in the numerators in section G.2.4.

G.2 $\Lambda = [4,2]$ sector

We give the BPS basis for the $\Lambda = [4,2]$ sector with M_Λ the highest weight state corresponding to field content $(4,2)$.

Throughout this section we will work with $\Lambda = [4,2]$ and $M_\Lambda = \begin{smallmatrix} 1 & 1 & 1 & 1 \\ 2 & 2 & & \end{smallmatrix}$, so we will suppress these in operator labels.

G.2.1 Free field covariant basis from traces

When writing our operators as sums over the free field covariant basis (3.6.20), we have made a choice about how to span the free field multiplicity space for $R = [4,2], [4,1,1], [3,2,1], [3,1,1,1], [2,2,1,1]$. These choices are:

$$\mathcal{O}_{[4,2],1} = \frac{\sqrt{10}}{120}\big(-t_{[4,2]} - t_{[4,1,1]} + 6t_{[3,3]} + 6t_{[3,2,1],1} + 6t_{[3,2,1],2} - 3t_{[2,2,2]} - 3t_{[2,2,1,1],1} + 6c_{[6],2} + 6c_{[5,1]} - 4c_{[4,2]} + 2c_{[4,1,1]}\big) \tag{G.2.1}$$

$$\mathcal{O}_{[4,2],2} = \frac{\sqrt{10}}{240}\big(-2t_{[4,1,1]} - 6t_{[3,2,1],1} - 6t_{[3,1,1,1]} - 3t_{[2,2,1,1],1} + 9t_{[2,2,1,1],2} - 3t_{[2,1,1,1,1]}$$

$$+\,24c_{[6],1}+12c_{[5,1]}+12c_{[4,2]}+4c_{[4,1,1]}\big) \tag{G.2.2}$$

$$\mathcal{O}_{\tiny\yng(4,1,1),1}=\frac{\sqrt{3}}{108}\big(-t_{[4,2]}+3\,t_{[4,1,1]}+6\,t_{[3,3]}+6\,t_{[3,2,1],2}+6\,t_{[3,1,1,1]}-3\,t_{[2,2,2]}+3\,t_{[2,2,1,1],1}$$
$$+\,6\,t_{[2,2,1,1],2}+12\,c_{[6],1}+6\,c_{[6],2}+6\,c_{[5,1]}+2\,c_{[4,2]}\big) \tag{G.2.3}$$

$$\mathcal{O}_{\tiny\yng(4,1,1),2}=\frac{\sqrt{15}}{1080}\big(-10\,t_{[4,2]}-48\,t_{[3,3]}-18\,t_{[3,2,1],1}-12\,t_{[3,2,1],2}+6\,t_{[3,1,1,1]}-12\,t_{[2,2,2]}$$
$$+\,3\,t_{[2,2,1,1],1}-21\,t_{[2,2,1,1],2}+9\,t_{[2,1,1,1,1]}$$
$$-\,24\,c_{[6],1}-48\,c_{[6],2}+60\,c_{[5,1]}+20\,c_{[4,2]}+60\,c_{[4,1,1]}\big) \tag{G.2.4}$$

$$\mathcal{O}_{\tiny\yng(3,2,1),1}=\frac{\sqrt{10}}{90}\big(-t_{[4,2]}+6\,t_{[3,3]}-3\,t_{[3,1,1,1]}+3\,t_{[2,2,1,1],1}-3\,t_{[2,2,1,1],2}-3\,c_{[5,1]}+2\,c_{[4,2]}\big) \tag{G.2.5}$$

$$\mathcal{O}_{\tiny\yng(3,2,1),2}=\frac{\sqrt{5}}{15}\big(t_{[3,2,1],2}+t_{[2,2,2]}-c_{[6],1}+c_{[6],2}+c_{[4,1,1]}\big) \tag{G.2.6}$$

$$\mathcal{O}_{\tiny\yng(3,2,1),3}=\frac{\sqrt{5}}{90}\big(-t_{[4,1,1]}-6\,t_{[3,2,1],1}-3\,t_{[3,2,1],2}+3\,t_{[2,1,1,1,1]}+12\,c_{[6],1}+12\,c_{[6],2}-4\,c_{[4,1,1]}\big) \tag{G.2.7}$$

$$\mathcal{O}_{\tiny\yng(3,1,1,1),1}=\frac{\sqrt{3}}{108}\big(-t_{[4,2]}-3\,t_{[4,1,1]}+6\,t_{[3,3]}-6\,t_{[3,2,1],2}+6\,t_{[3,1,1,1]}+3\,t_{[2,2,2]}+3\,t_{[2,2,1,1],1}$$
$$+\,6\,t_{[2,2,1,1],2}-12\,c_{[6],1}-6\,c_{[6],2}+6\,c_{[5,1]}+2\,c_{[4,2]}\big) \tag{G.2.8}$$

$$\mathcal{O}_{\tiny\yng(3,1,1,1),2}=\frac{\sqrt{15}}{1080}\big(-10\,t_{[4,2]}-48\,t_{[3,3]}+18\,t_{[3,2,1],1}+12\,t_{[3,2,1],2}+6\,t_{[3,1,1,1]}+12\,t_{[2,2,2]}$$
$$+\,3\,t_{[2,2,1,1],1}-21\,t_{[2,2,1,1],2}-9\,t_{[2,1,1,1,1]}$$
$$+\,24\,c_{[6],1}+48\,c_{[6],2}+60\,c_{[5,1]}+20\,c_{[4,2]}-60\,c_{[4,1,1]}\big) \tag{G.2.9}$$

$$\mathcal{O}_{\tiny\yng(2,2,1,1),1}=\frac{\sqrt{10}}{120}\big(-t_{[4,2]}+t_{[4,1,1]}+6\,t_{[3,3]}-6\,t_{[3,2,1],1}-6\,t_{[3,2,1],2}+3\,t_{[2,2,2]}-3\,t_{[2,2,1,1],1}$$
$$-\,6\,c_{[6],2}+6\,c_{[5,1]}-4\,c_{[4,2]}-2\,c_{[4,1,1]}\big) \tag{G.2.10}$$

$$\mathcal{O}_{\tiny\yng(2,2,1,1),2}=\frac{\sqrt{10}}{240}\big(2\,t_{[4,1,1]}+6\,t_{[3,2,1],1}-6\,t_{[3,1,1,1]}-3\,t_{[2,2,1,1],1}+9\,t_{[2,2,1,1],2}+3\,t_{[2,1,1,1,1]}$$
$$-\,24\,c_{[6],1}+12\,c_{[5,1]}+12\,c_{[4,2]}-4\,c_{[4,1,1]}\big) \tag{G.2.11}$$

The zero coupling operators with $R=[5,1],[3,3],[2,2,2],[2,1^4]$ are defined uniquely (up to a minus sign) by Λ and R. We use

$$\mathcal{O}_{\tiny\yng(5,1)}=-\frac{\sqrt{10}}{720}\big(8t_{[4,2]}+2t_{[4,1,1]}+24t_{[3,3]}+30t_{[3,2,1],1}+6t_{[3,1,1,1]}+12t_{[2,2,2]}+3t_{[2,2,1,1],1}-21t_{[2,2,1,1],2}$$
$$+3t_{[2,1,1,1,1]}+72c_{[6],1}+24c_{[6],2}+60c_{[5,1]}+20c_{[4,2]}+20c_{[4,1,1]}\big) \tag{G.2.12}$$

$$\mathcal{O}_{\tiny\yng(3,3)}=\frac{\sqrt{10}}{360}\big(t_{[4,2]}+t_{[4,1,1]}-6t_{[3,3]}-12t_{[3,2,1],1}-18t_{[3,2,1],2}-6t_{[3,1,1,1]}-3t_{[2,2,2]}+6t_{[2,2,1,1],1}+3t_{[2,2,1,1],2}$$
$$-3t_{[2,1,1,1,1]}+30c_{[6],2}+30c_{[5,1]}-20c_{[4,2]}+10c_{[4,1,1]}\big) \tag{G.2.13}$$

$$\mathcal{O}_{\tiny\yng(2,2,2)}=\frac{\sqrt{10}}{360}\big(t_{[4,2]}-t_{[4,1,1]}-6t_{[3,3]}+12t_{[3,2,1],1}+18t_{[3,2,1],2}-6t_{[3,1,1,1]}+3t_{[2,2,2]}+6t_{[2,2,1,1],1}+3t_{[2,2,1,1],2}$$
$$+3t_{[2,1,1,1,1]}-30c_{[6],2}+30c_{[5,1]}-20c_{[4,2]}-10c_{[4,1,1]}\big) \tag{G.2.14}$$

$$\mathcal{O}_{\tiny\yng(2,1,1,1,1)}=\frac{\sqrt{10}}{720}\big(-8t_{[4,2]}+2t_{[4,1,1]}-24t_{[3,3]}+30t_{[3,2,1],1}-6t_{[3,1,1,1]}+12t_{[2,2,2]}-3t_{[2,2,1,1],1}+21t_{[2,2,1,1],2}$$
$$+3t_{[2,1,1,1,1]}+72c_{[6],1}+24c_{[6],2}-60c_{[5,1]}-20c_{[4,2]}+20c_{[4,1,1]}\big) \tag{G.2.15}$$

where the symmetrised trace combinations we use are defined by

$$t_{[4,2]} = 3\mathrm{Tr}X^4\mathrm{Tr}Y^2 - 6\mathrm{Tr}X^3Y\mathrm{Tr}XY + 2\mathrm{Tr}X^2\mathrm{Tr}X^2Y^2 + \mathrm{Tr}X^2\mathrm{Tr}\,(XY)^2 \tag{G.2.16}$$

$$t_{[4,1,1]} = 3\mathrm{Tr}X^4\,(\mathrm{Tr}Y)^2 - 6\mathrm{Tr}X\mathrm{Tr}X^3Y\mathrm{Tr}Y + 2\,(\mathrm{Tr}X)^2\,\mathrm{Tr}X^2Y^2 + (\mathrm{Tr}X)^2\,\mathrm{Tr}\,(XY)^2 \tag{G.2.17}$$

$$t_{[3,3]} = \mathrm{Tr}X^3\mathrm{Tr}XY^2 - \left(\mathrm{Tr}X^2Y\right)^2 \tag{G.2.18}$$

$$t_{[3,2,1],1} = \mathrm{Tr}X\mathrm{Tr}X^3\mathrm{Tr}Y^2 - 2\mathrm{Tr}X\mathrm{Tr}X^2Y\mathrm{Tr}XY + \mathrm{Tr}X\mathrm{Tr}X^2\mathrm{Tr}XY^2 \tag{G.2.19}$$

$$t_{[3,2,1],2} = \mathrm{Tr}X^3\mathrm{Tr}XY\mathrm{Tr}Y - \mathrm{Tr}X\mathrm{Tr}X^3\mathrm{Tr}Y^2 - \mathrm{Tr}X^2\mathrm{Tr}X^2Y\mathrm{Tr}Y + \mathrm{Tr}X\mathrm{Tr}X^2Y\mathrm{Tr}XY \tag{G.2.20}$$

$$t_{[3,1,1,1]} = \mathrm{Tr}X\mathrm{Tr}X^3\,(\mathrm{Tr}Y)^2 - 2\,(\mathrm{Tr}X)^2\,\mathrm{Tr}X^2Y\mathrm{Tr}Y + (\mathrm{Tr}X)^3\,\mathrm{Tr}XY^2 \tag{G.2.21}$$

$$t_{[2,2,2]} = \left(\mathrm{Tr}X^2\right)^2\mathrm{Tr}Y^2 - \mathrm{Tr}X^2\,(\mathrm{Tr}XY)^2 \tag{G.2.22}$$

$$t_{[2,2,1,1],1} = \left(\mathrm{Tr}X^2\mathrm{Tr}Y\right)^2 - 2\mathrm{Tr}X\mathrm{Tr}X^2\mathrm{Tr}XY\mathrm{Tr}Y + (\mathrm{Tr}X\mathrm{Tr}XY)^2 \tag{G.2.23}$$

$$t_{[2,2,1,1],2} = (\mathrm{Tr}X\mathrm{Tr}XY)^2 - (\mathrm{Tr}X)^2\,\mathrm{Tr}X^2\mathrm{Tr}Y^2 \tag{G.2.24}$$

$$t_{[2,1,1,1,1]} = (\mathrm{Tr}X)^2\,\mathrm{Tr}X^2\,(\mathrm{Tr}Y)^2 - 2\,(\mathrm{Tr}X)^3\,\mathrm{Tr}XY\mathrm{Tr}Y + (\mathrm{Tr}X)^4\,\mathrm{Tr}Y^2 \tag{G.2.25}$$

along with the commutators

$$c_{[6],1} = \mathrm{Tr}X^4Y^2 - \mathrm{Tr}X^3YXY = \mathrm{Tr}X^3[X,Y]Y \tag{G.2.26}$$

$$c_{[6],2} = \mathrm{Tr}X^3YXY - \mathrm{Tr}\left(X^2Y\right)^2 = \mathrm{Tr}X^2[X,Y]Y^2 \tag{G.2.27}$$

$$c_{[5,1]} = \mathrm{Tr}X\mathrm{Tr}X^3Y^2 - \mathrm{Tr}X\mathrm{Tr}X^2YXY = \mathrm{Tr}X\mathrm{Tr}X^2[X,Y]Y \tag{G.2.28}$$

$$c_{[4,2]} = \mathrm{Tr}X^2\mathrm{Tr}X^2Y^2 - \mathrm{Tr}X^2\mathrm{Tr}\,(XY)^2 = \mathrm{Tr}X^2\mathrm{Tr}X[X,Y]Y \tag{G.2.29}$$

$$c_{[4,1,1]} = (\mathrm{Tr}X)^2\,\mathrm{Tr}X^2Y^2 - (\mathrm{Tr}X)^2\,\mathrm{Tr}\,(XY)^2 = (\mathrm{Tr}X)^2\,\mathrm{Tr}X[X,Y]Y \tag{G.2.30}$$

These are respectively the covariant symmetrised trace and commutator trace bases for the $\Lambda = [4,2]$ sector with M_Λ the highest weight state, as discussed in section 7.3.2.

G.2.2 Quarter-BPS basis

We now give the end result of the construction algorithm for quarter-BPS operators in the $\Lambda = [4,2]$ sector. The operators in this section are very lengthy to write out, so in the interests of brevity we only express them as a sum of free field operators. An expression in terms of trace can be found by substituting (G.2.1-G.2.15).

For $p = [3,2,1]$ and $[2,2,1,1]$ there are two BPS operators. For these, we have chosen the multiplicity space basis using the alternative orthogonalisation algorithm of section 7.4.10, beginning with the choice of free field multiplicities in (G.2.5-G.2.7) and (G.2.10-G.2.11) respectively.

We present the operators starting from the longest partition $p = [2,1,1,1,1]$ and progressing to the shortest, $p = [4,2]$.

$$S^{BPS}_{[2,1,1,1,1]} = \frac{1}{6\sqrt{3P_0}}\Big((N-1)(N-3)(N-4)\Big[3\sqrt{3}(N-2)\left\{N\mathcal{O}_{[5,1]} - (N+4)\mathcal{O}_{[4,2],2}\right\}$$

$$
\begin{aligned}
&-\sqrt{2}N(N+4)\left\{2\sqrt{5}\mathcal{O}_{[4,1,1],1} - 11\mathcal{O}_{[4,1,1],2}\right\} - 2\sqrt{3}\,(N+4)(N+3)\left\{\mathcal{O}_{[3,2,1],1} + 4\,\sqrt{2}\mathcal{O}_{[3,2,1],3}\right\}\Big] \\
&+\sqrt{2}N(N+4)(N+3)(N-1)(N-4)\left[10\,\sqrt{5}\mathcal{O}_{[3,1,1,1],1} + 29\mathcal{O}_{[3,1,1,1],2}\right] \\
&+(N+4)(N+3)(N+2)\Big[10\sqrt{3}(N-3)(N-4)\mathcal{O}_{[2,2,2]} \\
&+3\sqrt{3}\,(N-1)(N-4)\left\{2\mathcal{O}_{[2,2,1,1],1} + 13\mathcal{O}_{[2,2,1,1],2}\right\} + 65\sqrt{3}N(N-1)\mathcal{O}_{[2,1,1,1,1]}\Big]\Big)
\end{aligned} \tag{G.2.31}
$$

where the normalisation polynomial is

$$
\begin{aligned}
P_0 = {} & 195\,N^{10} + 2298\,N^9 + 9767\,N^8 + 17008\,N^7 + 21041\,N^6 + 74974\,N^5 + 135005\,N^4 - 144704\,N^3 \\
& - 399936\,N^2 - 62976\,N + 707328
\end{aligned} \tag{G.2.32}
$$

For $p = [2,2,1,1]$ there is a two-dimensional multiplicity space. The first operator is

$$
\begin{aligned}
S^{BPS}_{[2,2,1,1],1} = \frac{1}{6\sqrt{30P_1}} \Big(& -20\sqrt{3}N(N+1)(N-2)(N-3)P_{1,1}\mathcal{O}_{[5,1]} \\
& + \sqrt{3}(N-3)P_{1,2}\left[3(N+1)(N-2)\mathcal{O}_{[4,2],1} + 5(N+3)(N-2)\mathcal{O}_{[3,3]} - 12\sqrt{2}(N+3)(N+1)\mathcal{O}_{[3,2,1],2}\right] \\
& + 2\sqrt{3}(N+1)(N-3)P_{1,3}\left[3(N-2)\mathcal{O}_{[4,2],2} + 8\,\sqrt{2}(N+3)\mathcal{O}_{[3,2,1],3}\right] \\
& + \sqrt{10}N(N+1)P_{1,4}\left[(N-3)\mathcal{O}_{[4,1,1],1} - 5(N+3)\mathcal{O}_{[3,1,1,1],1}\right] \\
& - 2\sqrt{2}(N+1)\left[5N(N-3)P_{1,5}\mathcal{O}_{[4,1,1],2} + \sqrt{6}(N+3)(N-3)P_{1,6}\mathcal{O}_{[3,2,1],1} + 5N(N+3)P_{1,7}\mathcal{O}_{[3,1,1,1],2}\right] \\
& + 5\sqrt{3}(N+3)(N+2)(N+1)P_{1,8}\left[5(N-3)\mathcal{O}_{[2,2,2]} + 3(N-1)\mathcal{O}_{[2,2,1,1],1}\right] \\
& +10\sqrt{3}(N+3)(N+2)(N+1)\left[3P_{1,9}\mathcal{O}_{[2,2,1,1],2} - 26NP_{1,10}\mathcal{O}_{[2,1,1,1,1]}\right]\Big)
\end{aligned} \tag{G.2.33}
$$

where the normalisation and coefficient polynomials are

$$
\begin{aligned}
P_1 = {} & 1254825\,N^{16} + 25236900\,N^{15} + 212913135\,N^{14} + 949347864\,N^{13} + 2265287922\,N^{12} + 2296326096\,N^{11} \\
& - 483268806\,N^{10} - 64991400\,N^9 + 7717590681\,N^8 + 4250132076\,N^7 - 14563157385\,N^6 \\
& - 5596987632\,N^5 + 20300164460\,N^4 + 5660498272\,N^3 - 5514459136\,N^2 + 14594125824\,N \\
& + 12396386304
\end{aligned} \tag{G.2.34}
$$

$$P_{1,1} = 78\,N^4 + 180\,N^3 - 411\,N^2 - 510\,N + 788 \tag{G.2.35}$$

$$P_{1,2} = 195\,N^5 + 1149\,N^4 + 687\,N^3 - 3927\,N^2 - 1552\,N + 4448 \tag{G.2.36}$$

$$P_{1,3} = 195\,N^5 + 1257\,N^4 + 801\,N^3 - 5871\,N^2 - 3656\,N + 9024 \tag{G.2.37}$$

$$P_{1,4} = 975\,N^5 + 6177\,N^4 + 3891\,N^3 - 27411\,N^2 - 16176\,N + 40544 \tag{G.2.38}$$

$$P_{1,5} = 507\,N^5 + 3225\,N^4 + 2037\,N^3 - 14487\,N^2 - 8664\,N + 21632 \tag{G.2.39}$$

$$P_{1,6} = 195\,N^5 + 1041\,N^4 + 573\,N^3 - 1983\,N^2 + 552\,N - 128 \tag{G.2.40}$$

$$P_{1,7} = 1443\,N^5 + 9129\,N^4 + 5745\,N^3 - 40335\,N^2 - 23688\,N + 59456 \tag{G.2.41}$$

$$P_{1,8} = 117\,N^4 + 720\,N^3 + 1041\,N^2 + 240\,N + 992 \tag{G.2.42}$$

$$P_{1,9} = 429\,N^5 + 2247\,N^4 + 1215\,N^3 - 3585\,N^2 + 2056\,N - 2112 \tag{G.2.43}$$

$$P_{1,10} = 54\,N^3 + 273\,N^2 + 120\,N - 572 \tag{G.2.44}$$

The second operator is

$$\begin{aligned}
S^{BPS}_{[2,2,1,1],2} = \frac{1}{3\sqrt{6P_2}} \Big(& \sqrt{3}(N-2)(N-3)NP_{2,1}\mathcal{O}_{[5,1]} \\
& - \sqrt{3}(N-3)P_{2,2}\left[3(N+1)(N-2)\mathcal{O}_{[4,2],1} + 5(N+3)(N-2)\mathcal{O}_{[3,3]} - 12\sqrt{2}(N+3)(N+1)\mathcal{O}_{[3,2,1],2}\right] \\
& + \sqrt{3}(N-3)P_{2,3}\left[3(N-2)\mathcal{O}_{[4,2],2} + 8\sqrt{2}(N+3)\mathcal{O}_{[3,2,1],3}\right] \\
& + \sqrt{10}NP_{2,4}\left[(N-3)\mathcal{O}_{[4,1,1],1} - 5(N+3)\mathcal{O}_{[3,1,1,1],1}\right] - \sqrt{2}(N-3)NP_{2,5}\mathcal{O}_{[4,1,1],2} \\
& + 2\sqrt{3}(N+3)(N-3)P_{2,6}\mathcal{O}_{[3,2,1],1} + \sqrt{2}N(N+3)P_{2,7}\mathcal{O}_{[3,1,1,1],2} \\
& -5\sqrt{3}(N+3)(N+2)(N-3)P_{2,8}\mathcal{O}_{[2,2,2]} + 3\sqrt{3}P_{2,9}\mathcal{O}_{[2,2,1,1],1} - 6\sqrt{3}P_{2,10}\mathcal{O}_{[2,2,1,1],2} - 2\sqrt{3}NP_{2,11}\mathcal{O}_{[2,1,1,1,1]}\Big)
\end{aligned} \tag{G.2.45}$$

where the normalisation and coefficient polynomials are

$$\begin{aligned}
P_2 = {} & 64575225\,N^{16} + 1221543180\,N^{15} + 9292923450\,N^{14} + 34312809600\,N^{13} + 49747071546\,N^{12} \\
& - 49520811024\,N^{11} - 212528733480\,N^{10} + 81502221096\,N^9 + 872883407025\,N^8 + 609873915684\,N^7 \\
& - 949480261506\,N^6 - 778095650280\,N^5 + 986491220724\,N^4 + 591265527264\,N^3 - 532623199736\,N^2 \\
& - 150593123520\,N + 181872634752
\end{aligned} \tag{G.2.46}$$

$$P_{2,1} = 135\,N^5 + 423\,N^4 + 999\,N^3 + 1653\,N^2 + 1716\,N + 74 \tag{G.2.47}$$

$$P_{2,2} = 351\,N^5 + 1485\,N^4 - 783\,N^3 - 3669\,N^2 + 5448\,N - 1832 \tag{G.2.48}$$

$$P_{2,3} = 189\,N^6 + 903\,N^5 - 429\,N^4 - 4851\,N^3 - 1590\,N^2 + 98\,N - 1320 \tag{G.2.49}$$

$$P_{2,4} = 27\,N^6 - 30\,N^5 - 1560\,N^4 - 5250\,N^3 - 4959\,N^2 - 3420\,N - 808 \tag{G.2.50}$$

$$P_{2,5} = 675\,N^6 + 2589\,N^5 - 7527\,N^4 - 35553\,N^3 - 24606\,N^2 - 13386\,N - 7192 \tag{G.2.51}$$

$$P_{2,6} = 1593\,N^6 + 8247\,N^5 + 2379\,N^4 - 22659\,N^3 + 5526\,N^2 + 14562\,N - 8648 \tag{G.2.52}$$

$$P_{2,7} = 135\,N^6 + 3189\,N^5 + 23673\,N^4 + 69447\,N^3 + 74574\,N^2 + 55014\,N + 8968 \tag{G.2.53}$$

$$P_{2,8} = 2835\,N^5 + 17493\,N^4 + 21549\,N^3 - 19317\,N^2 - 10044\,N + 15464 \tag{G.2.54}$$

$$\begin{aligned}
P_{2,9} = {} & 10035\,N^8 + 100587\,N^7 + 320580\,N^6 + 201774\,N^5 - 613761\,N^4 - 529313\,N^3 + 665098\,N^2 + 243952\,N \\
& - 359952
\end{aligned} \tag{G.2.55}$$

$$P_{2,10} = 1131\,N^7 + 10440\,N^6 + 29667\,N^5 + 13182\,N^4 - 54074\,N^3 - 45886\,N^2 + 22026\,N + 24264 \tag{G.2.56}$$

$$P_{2,11} = 2280\,N^6 + 24384\,N^5 + 95505\,N^4 + 166002\,N^3 + 120739\,N^2 + 22034\,N - 11694 \tag{G.2.57}$$

For $p = [3,1,1,1]$ the operator is

$$S^{BPS}_{[3,1,1,1]} = \frac{1}{18\sqrt{2P_3}} \big(-3N(N-2)(N-3)P_{3,1}\mathcal{O}_{[5,1]}$$

$$+\,6(N-3)P_{3,2}\left[3(N+1)(N-2)\mathcal{O}_{[4,2],1}+5\,(N+3)(N-2)\mathcal{O}_{[3,3]}-12\sqrt{2}(N+3)(N+1)\mathcal{O}_{[3,2,1],2}\right]$$
$$+\,3(N-3)P_{3,3}\left[3\,(N-2)\mathcal{O}_{[4,2],2}+8\sqrt{2}(N+3)\mathcal{O}_{[3,2,1],3}\right]$$
$$+\,(N-3)\left[2\sqrt{30}NP_{3,4}\mathcal{O}_{[5,1],1}-\sqrt{6}NP_{3,5}\mathcal{O}_{[5,1],2}-6\,(N+3)P_{3,6}\mathcal{O}_{[3,2,1],1}\right]$$
$$+\,2\sqrt{30}P_{3,7}\mathcal{O}_{[4,1,1],1}-\sqrt{6}P_{3,8}\mathcal{O}_{[4,1,1],2}$$
$$-3(N+2)\left[10\,(N+3)(N-3)P_{3,9}\mathcal{O}_{[2,2,2]}-6P_{3,10}\mathcal{O}_{[2,2,1,1],1}-3P_{3,11}\mathcal{O}_{[2,2,1,1],2}+P_{3,12}\mathcal{O}_{[2,1,1,1,1]}\right]\Bigg) \tag{G.2.58}$$

where the normalisation and coefficient polynomials are

$$\begin{aligned}P_3 &= 93476025\,N^{16}+1612393695\,N^{15}+11013446394\,N^{14}+34526289987\,N^{13}+29660697936\,N^{12}\\&\quad-98498965581\,N^{11}-203072674968\,N^{10}+154945270125\,N^{9}+449766055695\,N^{8}\\&\quad-624364696710\,N^{7}-1246035300318\,N^{6}+1119952316004\,N^{5}+1953728842580\,N^{4}\\&\quad-1114329042600\,N^{3}-1086753482680\,N^{2}+1691309503680\,N+1297828640736\end{aligned} \tag{G.2.59}$$

$$P_{3,1}=7155\,N^5+22752\,N^4-21231\,N^3-76512\,N^2+21066\,N+63020 \tag{G.2.60}$$

$$P_{3,2}=270\,N^5+1728\,N^4+1287\,N^3-6762\,N^2-4278\,N+9380 \tag{G.2.61}$$

$$P_{3,3}=2025\,N^6+14460\,N^5+19239\,N^4-46512\,N^3-80274\,N^2+42292\,N+71520 \tag{G.2.62}$$

$$P_{3,4}=2295\,N^6+16458\,N^5+22254\,N^4-51987\,N^3-91314\,N^2+47394\,N+80900 \tag{G.2.63}$$

$$P_{3,5}=24435\,N^6+175044\,N^5+235749\,N^4-555432\,N^3-971334\,N^2+506028\,N+861760 \tag{G.2.64}$$

$$P_{3,6}=135\,N^6+1524\,N^5+4881\,N^4+2712\,N^3-8046\,N^2-1476\,N+3520 \tag{G.2.65}$$

$$\begin{aligned}P_{3,7}&=18630\,N^8+168027\,N^7+436488\,N^6-22071\,N^5-1221552\,N^4-330750\,N^3+1226756\,N^2\\&\quad-644796\,N-1298232\end{aligned} \tag{G.2.66}$$

$$\begin{aligned}P_{3,8}&=37260\,N^8+400629\,N^7+1309611\,N^6+387381\,N^5-4795443\,N^4-5190456\,N^3+4201270\,N^2\\&\quad+6554016\,N+1298232\end{aligned} \tag{G.2.67}$$

$$P_{3,9}=135\,N^5+609\,N^4-1257\,N^3-9444\,N^2-12438\,N-5860 \tag{G.2.68}$$

$$P_{3,10}=525\,N^6+3840\,N^5+6681\,N^4-8262\,N^3-49724\,N^2-91728\,N-41832 \tag{G.2.69}$$

$$P_{3,11}=6825\,N^6+61005\,N^5+175743\,N^4+113439\,N^3-265222\,N^2-407694\,N-160596 \tag{G.2.70}$$

$$P_{3,12}=22575\,N^6+198375\,N^5+553953\,N^4+307269\,N^3-994562\,N^2-1589994\,N-649116 \tag{G.2.71}$$

For $p=[2,2,2]$ the operator is

$$S^{BPS}_{[2,2,2]}=\frac{1}{36\sqrt{P_4}}\Big(30\sqrt{2}N(N-2)P_{4,1}\mathcal{O}_{[5,1]}$$
$$-\,15\sqrt{2}P_{4,2}\left[3(N+1)(N-2)\mathcal{O}_{[4,2],1}+5(N+3)(N-2)\mathcal{O}_{[3,3]}-12\sqrt{2}(N+3)(N+1)\mathcal{O}_{[3,2,1],2}\right]$$
$$+\,30\sqrt{2}P_{4,3}\left[3(N-2)\mathcal{O}_{[4,2],2}+8\sqrt{2}(N+3)\mathcal{O}_{[3,2,1],3}\right]+10\sqrt{15}NP_{4,4}\mathcal{O}_{[5,1],1}-20\sqrt{3}NP_{4,5}\mathcal{O}_{[5,1],2}$$
$$+\,60\sqrt{2}(N+3)P_{4,6}\mathcal{O}_{[3,2,1],1}+10\sqrt{3}N\left[\sqrt{5}P_{4,7}\mathcal{O}_{[4,1,1],1}+2P_{4,8}\mathcal{O}_{[4,1,1],2}\right]+3\sqrt{2}P_{4,9}\mathcal{O}_{[2,2,2]}$$

$$+45\sqrt{2}P_{4,10}\mathcal{O}_{\yng(2,2,1,1),1} - 90\sqrt{2}P_{4,11}\mathcal{O}_{\yng(2,2,1,1),2} - 30\sqrt{2}NP_{4,12}\mathcal{O}_{\yng(2,1,1,1,1)}\Big) \tag{G.2.72}$$

where the normalisation and coefficient polynomials are

$$\begin{aligned}P_4 = {} & 149226300\,N^{14} + 2094533640\,N^{13} + 10660893948\,N^{12} + 20470965300\,N^{11} - 2209082715\,N^{10} \\ & - 23656646682\,N^9 + 108969897216\,N^8 + 185022077310\,N^7 - 186235972937\,N^6 \\ & - 216166001512\,N^5 + 413959581308\,N^4 + 246958572128\,N^3 - 287690109584\,N^2 \\ & - 143358681600\,N + 276161485248\end{aligned} \tag{G.2.73}$$

$$P_{4,1} = 135\,N^5 - 18\,N^4 - 15\,N^3 - 1022\,N^2 - 76\,N - 816 \tag{G.2.74}$$

$$P_{4,2} = 864\,N^5 + 2430\,N^4 - 5973\,N^3 - 6119\,N^2 + 17876\,N - 12228 \tag{G.2.75}$$

$$P_{4,3} = 243\,N^6 + 924\,N^5 - 1152\,N^4 - 3670\,N^3 + 5307\,N^2 + 2256\,N - 2988 \tag{G.2.76}$$

$$P_{4,4} = 108\,N^6 + 402\,N^5 - 1065\,N^4 - 2588\,N^3 + 9471\,N^2 + 3376\,N + 276 \tag{G.2.77}$$

$$P_{4,5} = 945\,N^6 + 3576\,N^5 - 5586\,N^4 - 16186\,N^3 + 34863\,N^2 + 13520\,N - 8412 \tag{G.2.78}$$

$$P_{4,6} = 1971\,N^6 + 7512\,N^5 - 8238\,N^4 - 27854\,N^3 + 28821\,N^2 + 13552\,N - 27444 \tag{G.2.79}$$

$$P_{4,7} = 906\,N^5 + 7485\,N^4 + 18394\,N^3 + 9099\,N^2 - 14000\,N - 13164 \tag{G.2.80}$$

$$P_{4,8} = 285\,N^5 + 5955\,N^4 + 34507\,N^3 + 63369\,N^2 + 19738\,N + 7536 \tag{G.2.81}$$

$$P_{4,9} = 48060\,N^7 + 429834\,N^6 + 1227525\,N^5 + 919710\,N^4 - 762363\,N^3 - 208286\,N^2 + 1345500\,N + 946584 \tag{G.2.82}$$

$$P_{4,10} = 3702\,N^6 + 29949\,N^5 + 68544\,N^4 + 7491\,N^3 - 79610\,N^2 + 45780\,N + 78984 \tag{G.2.83}$$

$$P_{4,11} = 699\,N^6 + 5163\,N^5 + 8795\,N^4 - 9599\,N^3 - 20952\,N^2 + 21736\,N + 26328 \tag{G.2.84}$$

$$P_{4,12} = 1605\,N^5 + 14460\,N^4 + 42159\,N^3 + 36288\,N^2 - 16754\,N - 19428 \tag{G.2.85}$$

For $p = [3,2,1]$ there is a two-dimensional multiplicity space. The first operator is

$$\begin{aligned}S^{BPS}_{[3,2,1],1} = {} & \frac{1}{45\sqrt{P_5}}\Big(30\sqrt{10}(N-2)NP_{5,1}\mathcal{O}_{\yng(5,1)} - 6\sqrt{10}(N-2)P_{5,2}\left[3(N+1)\mathcal{O}_{\yng(4,2),1} + 5(N+3)\mathcal{O}_{\yng(3,3)}\right] \\ & - 18\sqrt{10}(N-2)P_{5,3}\mathcal{O}_{\yng(4,2),2} - 20\sqrt{3}N\left[P_{5,4}\mathcal{O}_{\yng(4,1,1),1} - \sqrt{5}P_{5,5}\mathcal{O}_{\yng(4,1,1),2}\right] - 12\sqrt{10}P_{5,6}\mathcal{O}_{\yng(3,2,1),1} \\ & - 6\sqrt{5}P_{5,7}\mathcal{O}_{\yng(3,2,1),2} + 3\sqrt{5}P_{5,8}\mathcal{O}_{\yng(3,2,1),3} - 20\sqrt{3}N\left[P_{5,9}\mathcal{O}_{\yng(3,1,1,1),1} - \sqrt{5}P_{5,10}\mathcal{O}_{\yng(3,1,1,1),2}\right] \\ & -6\sqrt{10}(N+2)P_{5,11}\left[5\mathcal{O}_{\yng(2,2,2)} + 3\mathcal{O}_{\yng(2,2,1,1),1}\right] - 6\sqrt{10}(N+2)\left[3P_{5,12}\mathcal{O}_{\yng(2,2,1,1),2} - 5NP_{5,13}\mathcal{O}_{\yng(2,1,1,1,1)}\right]\Big)\end{aligned} \tag{G.2.86}$$

where the normalisation and coefficient polynomials are

$$\begin{aligned}P_5 = {} & 1329483780\,N^{14} + 13761404280\,N^{13} + 47552297508\,N^{12} + 41944792356\,N^{11} - 43156801080\,N^{10} \\ & + 23239162764\,N^9 - 47497601127\,N^8 - 299164340106\,N^7 + 683116078397\,N^6 \\ & + 45647911732\,N^5 - 883683643044\,N^4 + 341394177280\,N^3 + 617090703216\,N^2 \\ & - 378227252672\,N + 179121262144\end{aligned} \tag{G.2.87}$$

$$P_{5,1} = 3708\,N^5 + 2172\,N^4 - 19509\,N^3 + 17427\,N^2 + 9416\,N - 13724 \tag{G.2.88}$$

$$P_{5,2} = 1656\,N^5 + 5619\,N^4 - 10194\,N^3 - 12393\,N^2 + 36194\,N - 24704 \tag{G.2.89}$$

$$P_{5,3} = 5076\,N^6 + 23490\,N^5 - 14985\,N^4 - 85957\,N^3 + 116006\,N^2 + 32240\,N - 75024 \tag{G.2.90}$$

$$P_{5,4} = 11808\,N^6 + 54255\,N^5 - 34545\,N^4 - 194501\,N^3 + 255813\,N^2 + 75970\,N - 174752 \tag{G.2.91}$$

$$P_{5,5} = 12492\,N^6 + 57498\,N^5 - 36627\,N^4 - 207175\,N^3 + 274254\,N^2 + 80120\,N - 184816 \tag{G.2.92}$$

$$P_{5,6} = 6462\,N^7 + 45960\,N^6 + 66285\,N^5 - 117979\,N^4 - 124298\,N^3 + 345680\,N^2 + 16132\,N - 99960 \tag{G.2.93}$$

$$P_{5,7} = 16326\,N^7 + 99666\,N^6 + 144600\,N^5 - 54717\,N^4 - 52871\,N^3 + 89526\,N^2 + 116\,N + 579336 \tag{G.2.94}$$

$$P_{5,8} = 165978\,N^7 + 1065798\,N^6 + 1496280\,N^5 - 1235731\,N^4 - 1949513\,N^3 + 8498\,N^2 + 679228\,N + 177528 \tag{G.2.95}$$

$$P_{5,9} = 14061\,N^5 + 85773\,N^4 + 110389\,N^3 - 97565\,N^2 - 111014\,N + 4008 \tag{G.2.96}$$

$$P_{5,10} = 15126\,N^5 + 66642\,N^4 - 35467\,N^3 - 293389\,N^2 + 81536\,N + 220812 \tag{G.2.97}$$

$$P_{5,11} = 1137\,N^5 + 5721\,N^4 - 3097\,N^3 - 35915\,N^2 - 21618\,N - 12024 \tag{G.2.98}$$

$$P_{5,12} = 6462\,N^5 + 40026\,N^4 + 56743\,N^3 - 30825\,N^2 - 44698\,N + 8016 \tag{G.2.99}$$

$$P_{5,13} = 4332\,N^4 + 26304\,N^3 + 32807\,N^2 - 32861\,N - 35466 \tag{G.2.100}$$

The second operator is

$$\begin{aligned} S^{BPS}_{[3,2,1],2} = \frac{1}{18\sqrt{P_6}} \Big(& -3\,(N-2)NP_{6,1}\mathcal{O}_{[5,1]} - 3(N-2)P_{6,2}\left[3\,(N+1)\mathcal{O}_{[4,2],1} + 5\,(N+3)\mathcal{O}_{[3,3]}\right] \\ & + 9(N-2)P_{6,3}\mathcal{O}_{[4,2],2} + \sqrt{6}N\left[\sqrt{5}P_{6,4}\mathcal{O}_{[4,1,1],1} - P_{6,5}\mathcal{O}_{[4,1,1],2}\right] - 6P_{6,6}\mathcal{O}_{[3,2,1],1} \\ & + 12\,\sqrt{2}P_{6,7}\mathcal{O}_{[3,2,1],2} - 3P_{6,8}\left[8\,\sqrt{2}(N-1)\mathcal{O}_{[3,2,1],3} - 3(N+2)\mathcal{O}_{[2,2,1,1],2}\right] + \sqrt{30}NP_{6,9}\mathcal{O}_{[3,1,1,1],1} \\ & -\sqrt{6}NP_{6,10}\mathcal{O}_{[3,1,1,1],2} + 3(N+2)P_{6,11}\left[5\mathcal{O}_{[2,2,2]} + 3\mathcal{O}_{[2,2,1,1],1}\right] - 3N(N+2)P_{6,12}\mathcal{O}_{[2,1,1,1,1]} \Big) \end{aligned} \tag{G.2.101}$$

where the normalisation and coefficient polynomials are

$$\begin{aligned} P_6 = {} & 433202580\,N^{14} + 4164719976\,N^{13} + 11536183026\,N^{12} - 111051000\,N^{11} - 29053464768\,N^{10} \\ & + 10364014080\,N^9 + 31360792437\,N^8 - 51088773768\,N^7 + 37140544622\,N^6 - 29831349568\,N^5 \\ & + 55411748788\,N^4 - 79360524160\,N^3 + 66216685440\,N^2 - 31168716800\,N + 6758052160 \end{aligned} \tag{G.2.102}$$

$$P_{6,1} = 864\,N^5 + 7734\,N^4 - 29931\,N^3 + 34329\,N^2 - 2260\,N - 8780 \tag{G.2.103}$$

$$P_{6,2} = 5724\,N^5 + 7509\,N^4 - 30912\,N^3 + 37572\,N^2 - 28648\,N + 10144 \tag{G.2.104}$$

$$P_{6,3} = 4104\,N^6 + 12552\,N^5 - 15267\,N^4 - 24025\,N^3 + 50968\,N^2 - 18276\,N - 4944 \tag{G.2.105}$$

$$P_{6,4} = 2484\,N^6 + 11871\,N^5 - 7131\,N^4 - 54710\,N^3 + 93012\,N^2 - 18048\,N - 20032 \tag{G.2.106}$$

$$P_{6,5} = 22248\,N^6 + 85140\,N^5 - 74325\,N^4 - 290915\,N^3 + 524952\,N^2 - 127020\,N - 94960 \tag{G.2.107}$$

$$P_{6,6} = 54810\,N^7 + 273690\,N^6 + 107331\,N^5 - 533633\,N^4 + 371678\,N^3 - 35050\,N^2 - 241112\,N + 130440 \tag{G.2.108}$$

$$P_{6,7} = 7830\,N^7 + 45693\,N^6 + 36684\,N^5 - 104202\,N^4 - 1453\,N^3 + 78398\,N^2 - 91900\,N + 46392 \tag{G.2.109}$$

$$P_{6,8} = 516\,N^5 + 5496\,N^4 + 12187\,N^3 - 4448\,N^2 - 12124\,N + 4416 \tag{G.2.110}$$

$$P_{6,9} = 3717\,N^5 + 23415\,N^4 + 29524\,N^3 - 26804\,N^2 - 15272\,N + 2208 \tag{G.2.111}$$

$$P_{6,10} = 16416\,N^5 + 68640\,N^4 - 93347\,N^3 - 401726\,N^2 + 197500\,N + 132600 \tag{G.2.112}$$

$$P_{6,11} = 2685\,N^5 + 12423\,N^4 + 5150\,N^3 - 17908\,N^2 + 8976\,N - 6624 \tag{G.2.113}$$

$$P_{6,12} = 6918\,N^4 + 41334\,N^3 + 46861\,N^2 - 49160\,N - 18420 \tag{G.2.114}$$

For $p = [4,1,1]$ the operator is

$$\begin{aligned} S^{BPS}_{[4,1,1]} = \frac{1}{6\sqrt{P_7}} \Big(& (N-2)NP_{7,1}\mathcal{O}_{[5,1]} - (N-2)P_{7,2}\left[3\,(N+1)\mathcal{O}_{[4,2],1} + 5\,(N+3)\mathcal{O}_{[3,3]}\right] \\ & - 3(N-2)P_{7,3}\mathcal{O}_{[4,2],2} + \sqrt{30}P_{7,4}\mathcal{O}_{[4,1,1],1} + \sqrt{6}P_{7,5}\mathcal{O}_{[4,1,1],2} - 12(N+3)\left[P_{7,6}\mathcal{O}_{[3,2,1],1} + \sqrt{2}P_{7,7}\mathcal{O}_{[3,2,1],2}\right] \\ & + (N+3)\left[2P_{7,8}\left\{8\sqrt{2}(N-1)\mathcal{O}_{[3,2,1],3} - 3(N+2)\mathcal{O}_{[2,2,1,1],2}\right\} - \sqrt{30}NP_{7,9}\mathcal{O}_{[3,1,1,1],1} + 2\sqrt{6}P_{7,10}\mathcal{O}_{[3,1,1,1],2}\right] \\ & -(N+3)(N+2)P_{7,11}\left[5\mathcal{O}_{[2,2,2]} + 3\mathcal{O}_{[2,2,1,1],1}\right] + 2\,(N+3)(N+2)P_{7,12}\mathcal{O}_{[2,1,1,1,1]} \Big) \end{aligned} \tag{G.2.115}$$

where the normalisation and coefficient polynomials are

$$\begin{aligned} P_7 = {} & 1691280\,N^{14} + 14469840\,N^{13} + 34933194\,N^{12} - 15345720\,N^{11} - 97734483\,N^{10} + 108829584\,N^9 \\ & + 94236018\,N^8 - 365252412\,N^7 + 332214736\,N^6 + 23494544\,N^5 - 188670784\,N^4 + 59358800\,N^3 \\ & + 76067360\,N^2 - 55528000\,N + 47136640 \end{aligned} \tag{G.2.116}$$

$$P_{7,1} = 2052\,N^5 + 2592\,N^4 - 7293\,N^3 + 4232\,N^2 + 1320\,N - 2240 \tag{G.2.117}$$

$$P_{7,2} = 108\,N^5 + 477\,N^4 - 348\,N^3 - 770\,N^2 + 1254\,N - 772 \tag{G.2.118}$$

$$P_{7,3} = 612\,N^6 + 3210\,N^5 + 939\,N^4 - 7568\,N^3 + 5760\,N^2 + 692\,N - 2472 \tag{G.2.119}$$

$$P_{7,4} = 1296\,N^7 + 6029\,N^6 + 1473\,N^5 - 11350\,N^4 + 9484\,N^3 - 4114\,N^2 - 3428\,N + 3440 \tag{G.2.120}$$

$$P_{7,5} = 648\,N^7 + 4186\,N^6 + 1473\,N^5 - 12472\,N^4 + 8280\,N^3 + 6672\,N^2 - 4760\,N - 3440 \tag{G.2.121}$$

$$P_{7,6} = 13\,N^5 - 7\,N^4 + 84\,N^3 - 129\,N^2 + 222\,N - 40 \tag{G.2.122}$$

$$P_{7,7} = 15\,N^5 + 86\,N^4 - 194\,N^3 + 278\,N^2 - 178\,N + 284 \tag{G.2.123}$$

$$P_{7,8} = 69\,N^4 + 222\,N^3 + 112\,N^2 - 193\,N - 62 \tag{G.2.124}$$

$$P_{7,9} = 97\,N^4 + 316\,N^3 + 62\,N^2 - 306\,N - 164 \tag{G.2.125}$$

$$P_{7,10} = 263\,N^5 + 410\,N^4 - 1216\,N^3 - 953\,N^2 + 1060\,N + 860 \tag{G.2.126}$$

$$P_{7,11} = 15\,N^4 + 60\,N^3 - 262\,N^2 - 146\,N - 244 \tag{G.2.127}$$

$$P_{7,12} = 222\,N^4 + 726\,N^3 + 74\,N^2 - 725\,N - 430 \tag{G.2.128}$$

For $p = [3, 3]$ the operator is

$$
\begin{aligned}
S^{BPS}_{[3,3]} = \frac{1}{18\sqrt{2P_8}} \Big(& -60NP_{8,1}\mathcal{O}_{\tiny\yng(5,1)} + 9P_{8,2}\mathcal{O}_{\tiny\yng(4,2),1} - 18P_{8,3}\mathcal{O}_{\tiny\yng(4,2),2} - \sqrt{30}NP_{8,4}\mathcal{O}_{\tiny\yng(4,1,1),1} \\
& - 10\sqrt{6}NP_{8,5}\mathcal{O}_{\tiny\yng(4,1,1),2} - 3P_{8,6}\mathcal{O}_{\tiny\yng(3,3)} + 12P_{8,7}\mathcal{O}_{\tiny\yng(3,2,1),1} - 12\sqrt{2}P_{8,8}\mathcal{O}_{\tiny\yng(3,2,1),2} \\
& - 6P_{8,9}\left[8\sqrt{2}(N-1)\mathcal{O}_{\tiny\yng(3,2,1),3} - 3(N+2)\mathcal{O}_{\tiny\yng(2,2,1,1),2}\right] + \sqrt{6}N\left[\sqrt{5}P_{8,10}\mathcal{O}_{\tiny\yng(3,1,1,1),1} - 10P_{8,11}\mathcal{O}_{\tiny\yng(3,1,1,1),2}\right] \\
& -3(N+2)P_{8,12}\left[5\mathcal{O}_{\tiny\yng(2,2,2)} + 3\mathcal{O}_{\tiny\yng(2,2,1,1),1}\right] - 60N(N+2)P_{8,13}\mathcal{O}_{\tiny\yng(2,1,1,1,1)}\Big) \qquad \text{(G.2.129)}
\end{aligned}
$$

where the normalisation and coefficient polynomials are

$$
\begin{aligned}
P_8 = {} & 64152\,N^{12} + 209952\,N^{11} - 137241\,N^{10} - 640440\,N^9 + 908640\,N^8 - 322236\,N^7 - 116124\,N^6 \\
& - 675864\,N^5 + 2362028\,N^4 - 3013280\,N^3 + 2221520\,N^2 - 926400\,N + 177280 \qquad \text{(G.2.130)}
\end{aligned}
$$

$$P_{8,1} = 27\,N^5 - 54\,N^4 + 39\,N^3 + 7\,N^2 - 36\,N + 20 \qquad \text{(G.2.131)}$$

$$P_{8,2} = 540\,N^6 + 765\,N^5 - 2364\,N^4 + 2098\,N^3 - 686\,N^2 - 592\,N + 488 \qquad \text{(G.2.132)}$$

$$P_{8,3} = 90\,N^6 + 75\,N^5 - 198\,N^4 + 156\,N^3 - 202\,N^2 + 216\,N - 104 \qquad \text{(G.2.133)}$$

$$P_{8,4} = 255\,N^4 - 78\,N^3 - 310\,N^2 + 218\,N - 28 \qquad \text{(G.2.134)}$$

$$P_{8,5} = 69\,N^4 - 156\,N^3 + 52\,N^2 + 178\,N - 164 \qquad \text{(G.2.135)}$$

$$P_{8,6} = 1188\,N^6 + 3519\,N^5 - 2868\,N^4 - 5066\,N^3 + 9654\,N^2 - 8560\,N + 3000 \qquad \text{(G.2.136)}$$

$$P_{8,7} = 87\,N^5 + 81\,N^4 - 302\,N^3 + 396\,N^2 - 214\,N + 60 \qquad \text{(G.2.137)}$$

$$P_{8,8} = 99\,N^5 - 12\,N^4 - 486\,N^3 + 706\,N^2 - 640\,N + 168 \qquad \text{(G.2.138)}$$

$$P_{8,9} = 21\,N^4 + 48\,N^3 - 36\,N^2 - 2\,N - 12 \qquad \text{(G.2.139)}$$

$$P_{8,10} = 51\,N^4 + 138\,N^3 - 86\,N^2 - 142\,N - 12 \qquad \text{(G.2.140)}$$

$$P_{8,11} = 33\,N^4 + 48\,N^3 - 164\,N^2 + 2\,N + 156 \qquad \text{(G.2.141)}$$

$$P_{8,12} = 33\,N^4 + 54\,N^3 - 58\,N^2 + 134\,N - 36 \qquad \text{(G.2.142)}$$

$$P_{8,13} = 3\,N^3 + 9\,N^2 - 5\,N - 14 \qquad \text{(G.2.143)}$$

For $p = [4, 2]$ the operator is

$$
\begin{aligned}
S^{BPS}_{[4,2]} = \frac{1}{6\sqrt{3P_9}} \Big(& -\sqrt{2}P_{9,1}\mathcal{O}_{\tiny\yng(5,1)} - 6\sqrt{2}P_{9,2}\mathcal{O}_{\tiny\yng(4,2),1} - 3\sqrt{2}P_{9,3}\mathcal{O}_{\tiny\yng(4,2),2} - 4\sqrt{15}NP_{9,4}\mathcal{O}_{\tiny\yng(4,1,1),1} \\
& - 2\sqrt{3}P_{9,5}\mathcal{O}_{\tiny\yng(4,1,1),2} + (N+3)\Big[-10\sqrt{2}(N-1)P_{9,6}\mathcal{O}_{\tiny\yng(3,3)} + 6\sqrt{2}P_{9,7}\left\{N\mathcal{O}_{\tiny\yng(3,2,1),1} + 2\sqrt{2}\mathcal{O}_{\tiny\yng(3,2,1),2}\right\} \\
& - \sqrt{2}P_{9,8}\left\{8\sqrt{2}(N-1)\mathcal{O}_{\tiny\yng(3,2,1),3} - 3(N+2)\mathcal{O}_{\tiny\yng(2,2,1,1),2}\right\} + 4\sqrt{15}(N+1)(N-1)N\mathcal{O}_{\tiny\yng(3,1,1,1),1} \\
& -2\sqrt{3}P_{9,9}\mathcal{O}_{\tiny\yng(3,1,1,1),2} - \sqrt{2}(N+2)\left\{10\mathcal{O}_{\tiny\yng(2,2,2)} + 6\mathcal{O}_{\tiny\yng(2,2,1,1),1} + P_{9,10}\mathcal{O}_{\tiny\yng(2,1,1,1,1)}\right\}\Big]\Big) \qquad \text{(G.2.144)}
\end{aligned}
$$

where the normalisation and coefficient polynomials are

$$P_9 = 297\,N^{10} + 378\,N^8 - 1260\,N^7 + 390\,N^6 + 1080\,N^5 - 1256\,N^4 + 640\,N^3 + 760\,N^2 - 1920\,N + 1440 \tag{G.2.145}$$

$$P_{9,1} = 81\,N^5 - 129\,N^4 + 51\,N^3 + 76\,N^2 - 130\,N + 60 \tag{G.2.146}$$

$$P_{9,2} = 9\,N^5 + 3\,N^4 - 3\,N^3 - 13\,N^2 + 13\,N - 6 \tag{G.2.147}$$

$$P_{9,3} = \left(9\,N^3 + 13\,N^2 - 13\,N + 6\right)\left(3\,N^2 - 2\right) \tag{G.2.148}$$

$$P_{9,4} = 5\,N^3 - 2\,N^2 - 5\,N + 3 \tag{G.2.149}$$

$$P_{9,5} = 29\,N^4 - 65\,N^3 + 24\,N^2 + 70\,N - 60 \tag{G.2.150}$$

$$P_{9,6} = 3\,N^2 - 3\,N + 2 \tag{G.2.151}$$

$$P_{9,7} = N^2 - 2\,N + 2 \tag{G.2.152}$$

$$P_{9,8} = 3\,N^2 - 2 \tag{G.2.153}$$

$$P_{9,9} = 11\,N^3 - 18\,N^2 - 10\,N + 20 \tag{G.2.154}$$

$$P_{9,10} = 9\,N^2 - 10 \tag{G.2.155}$$

G.2.3 Norms of operators with multiplicity

As explained in section 7.4.7, for Λ, p with $\mathcal{M}_{\Lambda,p} \geq 1$, the BPS norms of the operators are dependent on the choice of basis for the multiplicity space. In that section, we described a process to extract norm-like functions of N that characterise the multiplicity space and are independent of the choice of basis.

In the $\Lambda = [4,2]$ sector, there are two partitions $p = [2,2,1,1]$ and $[3,2,1]$ with $\mathcal{M}_{\Lambda,p} = 2$. For the first of these, we go through the process described in section 7.4.7 in some detail, while for the second we only give the results.

We begin by renormalising the BPS operators to have norm 1 in the physical inner product as given in (7.4.71). For $p = [2,2,1,1]$, this replaces P_1 and P_2 in the expansions (G.2.33) and (G.2.45) with

$$\widehat{P}_1 = 3(N+3)(N+2)(N+1)^2N^2(N-1)(N-2)(N-3)Q_{mult}Q_1 \tag{G.2.156}$$

$$\widehat{P}_2 = 3(N+1)N^2(N-1)(N-2)(N-3)Q_{mult}Q_2 \tag{G.2.157}$$

where Q_1 and Q_2 are defined in (G.2.181) and (G.2.182) and

$$Q_{mult} = 2145\,N^8 + 21570\,N^7 + 69156\,N^6 + 44856\,N^5 - 130747\,N^4 - 117106\,N^3 + 138802\,N^2 + 53280\,N - 75456 \tag{G.2.158}$$

after normalising, the S_n inner product matrix can be calculated, and is given by

$$A_{[2,2,1,1]} = \begin{pmatrix} \left\langle \widehat{S}^{BPS}_{[2,2,1,1],1} | \widehat{S}^{BPS}_{[2,2,1,1],1} \right\rangle_{S_n} & \left\langle \widehat{S}^{BPS}_{[2,2,1,1],1} | \widehat{S}^{BPS}_{[2,2,1,1],2} \right\rangle_{S_n} \\ \left\langle \widehat{S}^{BPS}_{[2,2,1,1],2} | \widehat{S}^{BPS}_{[2,2,1,1],1} \right\rangle_{S_n} & \left\langle \widehat{S}^{BPS}_{[2,2,1,1],2} | \widehat{S}^{BPS}_{[2,2,1,1],2} \right\rangle_{S_n} \end{pmatrix} = \begin{pmatrix} \frac{A_{1,1}}{\widehat{P}_1} & \frac{A_{1,2}}{\sqrt{\widehat{P}_1 \widehat{P}_2}} \\ \frac{A_{1,2}}{\sqrt{\widehat{P}_1 \widehat{P}_2}} & \frac{A_{2,2}}{\widehat{P}_2} \end{pmatrix} \tag{G.2.159}$$

where

$$\begin{aligned} A_{1,1} = {} & 1254825\, N^{16} + 25236900\, N^{15} + 212913135\, N^{14} + 949347864\, N^{13} + 2265287922\, N^{12} + 2296326096\, N^{11} \\ & - 483268806\, N^{10} - 64991400\, N^{9} + 7717590681\, N^{8} + 4250132076\, N^{7} - 14563157385\, N^{6} \\ & - 5596987632\, N^{5} + 20300164460\, N^{4} + 5660498272\, N^{3} - 5514459136\, N^{2} + 14594125824\, N \\ & + 12396386304 \end{aligned} \tag{G.2.160}$$

$$\begin{aligned} A_{1,2} = {} & 2\sqrt{5}\big(394875\, N^{15} + 7400484\, N^{14} + 57527991\, N^{13} + 231664914\, N^{12} + 476892396\, N^{11} + 249273666\, N^{10} \\ & - 1301445666\, N^{9} - 4474130634\, N^{8} - 7919982621\, N^{7} - 8401406142\, N^{6} - 6257132757\, N^{5} \\ & - 4801800696\, N^{4} - 1575438250\, N^{3} - 1395294808\, N^{2} - 4205573568\, N - 1295069184\big) \end{aligned} \tag{G.2.161}$$

$$\begin{aligned} A_{2,2} = {} & 64575225\, N^{16} + 1221543180\, N^{15} + 9292923450\, N^{14} + 34312809600\, N^{13} + 49747071546\, N^{12} \\ & - 49520811024\, N^{11} - 212528733480\, N^{10} + 81502221096\, N^{9} + 872883407025\, N^{8} + 609873915684\, N^{7} \\ & - 949480261506\, N^{6} - 778095650280\, N^{5} + 986491220724\, N^{4} + 591265527264\, N^{3} - 532623199736\, N^{2} \\ & - 150593123520\, N + 181872634752 \end{aligned} \tag{G.2.162}$$

We now take the trace of $A_{[2,2,1,1]}$, divide by $\mathcal{M}_{\Lambda,p}$, and take the reciprocal. This gives the first $p = [2,2,1,1]$ invariant

$$\frac{2}{\mathrm{Tr} A_{[2,2,1,1]}} = \frac{2(N+3)(N+2)(N+1)^2 N^2 (N-1)(N-2)(N-3) Q_1 Q_2}{D_1} \tag{G.2.163}$$

where the denominator is

$$\begin{aligned} D_1 = {} & 3913650\, N^{16} + 78795855\, N^{15} + 656781957\, N^{14} + 2811679470\, N^{13} + 5818416030\, N^{12} + 1501757316\, N^{11} \\ & - 15672370512\, N^{10} - 14255947158\, N^{9} + 42286367112\, N^{8} + 71992040249\, N^{7} - 32371301901\, N^{6} \\ & - 121059621624\, N^{5} - 22843286488\, N^{4} + 77152295508\, N^{3} + 42542435352\, N^{2} + 5036467584\, N \\ & + 2255817600 \end{aligned} \tag{G.2.164}$$

We can also consider the trace of $A^2_{[2,2,1,1]}$. This leads to the second invariant

$$\sqrt{\frac{2}{\mathrm{Tr} A^2_{[2,2,1,1]}}} = \frac{\sqrt{2}(N+3)(N+2)(N+1)^2 N^2 (N-1)(N-2)(N-3) Q_1 Q_2}{\sqrt{D_2}} \tag{G.2.165}$$

where the denominator is

$$D_2 = 7658328161250\, N^{32} + 308379397920750\, N^{31} + 5678040590961075\, N^{30} + 62861883407800200\, N^{29}$$

$$
\begin{aligned}
&+ 461553133569402069\, N^{28} + 2323880655128992368\, N^{27} + 7893896923770889320\, N^{26} \\
&+ 16200841926037924512\, N^{25} + 9738474984510581700\, N^{24} - 43140893567922372492\, N^{23} \\
&- 100830809456338189482\, N^{22} + 66300678032545590264\, N^{21} + 576422366985618028290\, N^{20} \\
&+ 587496624365125252152\, N^{19} - 1266939757691694906384\, N^{18} - 3370314414344723267400\, N^{17} \\
&- 1442277915561708579 0\, N^{16} + 8873284172309294711934\, N^{15} + 9228283693975324117807\, N^{14} \\
&- 8309143471774592802944\, N^{13} - 21871661389590847910159\, N^{12} - 3725069874701998817592\, N^{11} \\
&+ 25451491117140266214976\, N^{10} + 18757146605106723110568\, N^{9} - 15395309506022451870416\, N^{8} \\
&- 22339442519546818907728\, N^{7} + 3985689055612424950064\, N^{6} + 16657691069689910952704\, N^{5} \\
&+ 3604888800092578331072\, N^{4} - 4775351642112978422784\, N^{3} - 8269668856322537472 0\, N^{2} \\
&+ 2740871464097166655488\, N + 1006239182315089379328
\end{aligned} \tag{G.2.166}
$$

For $p = [3, 2, 1]$, the same process produces

$$\frac{2}{\mathrm{Tr}A_{[3,2,1]}} = \frac{2(N+2)(N+1)N^2(N-1)(N-2)Q_4Q_5}{E_1} \tag{G.2.167}$$

$$\sqrt{\frac{2}{\mathrm{Tr}A^2_{[3,2,1]}}} = \frac{\sqrt{2}(N+2)(N+1)N^2(N-1)(N-2)Q_4Q_5}{\sqrt{E_2}} \tag{G.2.168}$$

where the denominators are

$$
\begin{aligned}
E_1 = {} & 41812200\, N^{14} + 448198920\, N^{13} + 1563219648\, N^{12} + 1093147920\, N^{11} - 3204936072\, N^{10} \\
& - 1375066305\, N^{9} + 4730520504\, N^{8} - 3314823954\, N^{7} + 4335640504\, N^{6} - 6084970\, N^{5} \\
& - 10209076192\, N^{4} + 9690911824\, N^{3} - 2443216896\, N^{2} + 3777810528\, N - 538272768
\end{aligned} \tag{G.2.169}
$$

$$
\begin{aligned}
E_2 = {} & 874130034420000\, N^{28} + 18740182882824000\, N^{27} + 166345754746996800\, N^{26} \\
& + 753788224097235360\, N^{25} + 1608498415010610504\, N^{24} + 181361766700128024\, N^{23} \\
& - 5390210561323512672\, N^{22} - 4416942361725000252\, N^{21} + 13381736971853528568\, N^{20} \\
& + 14451638301852715944\, N^{19} - 26479800963850159935\, N^{18} - 31944440045187411534\, N^{17} \\
& + 47031196210114852566\, N^{16} + 73597499966176725312\, N^{15} - 112834178522277863808\, N^{14} \\
& - 122423268066628273308\, N^{13} + 309535922049432602720\, N^{12} + 108895335882008823 44\, N^{11} \\
& - 425645164341054775804\, N^{10} + 196537079346722192144\, N^{9} + 367648860348492413280\, N^{8} \\
& - 423712842979380230656\, N^{7} + 67969647225996116864\, N^{6} + 101859033408413821440\, N^{5} \\
& - 27177241919312392192\, N^{4} - 38998356711672686592\, N^{3} + 48696174595572179968\, N^{2} \\
& - 18571148044644937728\, N + 3609255644969508864
\end{aligned} \tag{G.2.170}
$$

G.2.4 Norms of BPS operators

The physical norms of the BPS operators can be understood as characteristic functions of the pair Λ, p and should be reproducible from stringy physics on the other side of the AdS/CFT duality.

We give the norms for each of the BPS operators in the $\Lambda = [4,2]$ sector. For $p = [2,2,1,1]$ and $[3,2,1]$, we reproduce the invariants derived in the previous subsection in order to compare with other operators.

$$\left|S^{BPS}_{[2,1,1,1,1]}\right|^2 = \frac{(N+4)(N+3)(N+2)(N+1)N^2(N-1)^2(N-2)(N-3)(N-4)Q_1}{P_0} \tag{G.2.171}$$

$$\frac{2}{\mathrm{Tr}A_{[2,2,1,1]}} = \frac{2(N+3)(N+2)(N+1)^2N^2(N-1)(N-2)(N-3)Q_1Q_2}{D_1} \tag{G.2.172}$$

$$\sqrt{\frac{2}{\mathrm{Tr}A^2_{[2,2,1,1]}}} = \frac{\sqrt{2}(N+3)(N+2)(N+1)^2N^2(N-1)(N-2)(N-3)Q_1Q_2}{\sqrt{D_2}} \tag{G.2.173}$$

$$\left|S^{BPS}_{[3,1,1,1]}\right|^2 = \frac{(N+2)(N+1)N(N-1)(N-2)(N-3)Q_2Q_3}{2\,P_3} \tag{G.2.174}$$

$$\left|S^{BPS}_{[2,2,2]}\right|^2 = \frac{(N+1)N^2(N-1)(N-2)Q_3Q_4}{P_4} \tag{G.2.175}$$

$$\frac{2}{\mathrm{Tr}A_{[3,2,1]}} = \frac{2(N+2)(N+1)N^2(N-1)(N-2)Q_4Q_5}{E_1} \tag{G.2.176}$$

$$\sqrt{\frac{2}{\mathrm{Tr}A^2_{[3,2,1]}}} = \frac{\sqrt{2}(N+2)(N+1)N^2(N-1)(N-2)Q_4Q_5}{\sqrt{E_2}} \tag{G.2.177}$$

$$\left|S^{BPS}_{[4,1,1]}\right|^2 = \frac{(N+3)(N+2)(N+1)N(N-1)(N-2)Q_5Q_6}{P_7} \tag{G.2.178}$$

$$\left|S^{BPS}_{[3,3]}\right|^2 = \frac{(N+2)(N+1)N^2(N-1)Q_6Q_7}{P_8} \tag{G.2.179}$$

$$\left|S^{BPS}_{[4,2]}\right|^2 = \frac{(N+3)(N+2)(N+1)N^4(N-1)\left(3\,N^2-2\right)Q_7}{P_9} \tag{G.2.180}$$

where the polynomials in the denominator have been defined in previous subsections and the polynomials in the numerator are

$$Q_1 = 195\,N^5 + 1149\,N^4 + 687\,N^3 - 3927\,N^2 - 1552\,N + 4448 \tag{G.2.181}$$

$$Q_2 = 10035\,N^8 + 94914\,N^7 + 264876\,N^6 + 17268\,N^5 - 819309\,N^4 - 487830\,N^3 + 780722\,N^2 + 189568\,N - 432744 \tag{G.2.182}$$

$$Q_3 = 18630\,N^8 + 160677\,N^7 + 371643\,N^6 - 204495\,N^5 - 1326729\,N^4 - 15804\,N^3 + 1726178\,N^2 - 442368\,N - 1298232 \tag{G.2.183}$$

$$Q_4 = 8010\,N^7 + 56214\,N^6 + 79800\,N^5 - 132315\,N^4 - 158273\,N^3 + 296994\,N^2 + 33500\,N - 171336 \tag{G.2.184}$$

$$Q_5 = 2610\,N^7 + 12546\,N^6 + 3213\,N^5 - 25152\,N^4 + 20228\,N^3 - 5238\,N^2 - 8000\,N + 5160 \tag{G.2.185}$$

$$Q_6 = 648\,N^7 + 2772\,N^6 + 51\,N^5 - 5484\,N^4 + 5438\,N^3 - 2026\,N^2 - 2000\,N + 1720 \tag{G.2.186}$$

$$Q_7 = 99\,N^6 + 162\,N^5 - 324\,N^4 + 102\,N^3 + 152\,N^2 - 260\,N + 120 \tag{G.2.187}$$

Comparing these norms with those in sections G.1.2 and 7.4.7, we see a general pattern in the numerators. They typically contain a product of linear factors along with (in general) two complicated Q polynomials. These Q polynomials appear in two consecutive norms.

In (G.2.158) we saw that the Q polynomials appear in consecutive norms even in the non-physical multiplicity space. This suggests they are an artefact of the orthogonalisation process.

The linear factors are more interesting. Their presence is partially implied by SEP-compatibility, but there are generally more factors than would be sufficient for this purpose. The function f_p, defined in (2.3.20), that gives the free field norms, is a product of linear factors, and we can compare this with those found in the numerators of weak coupling BPS norms. In all but two ($p = [2,2,2]$ and $[3,3]$) of the examples we have calculated, the numerators contain f_p, while some partitions have considerably more factors. It would be interesting to enumerate the linear factors that appear in the numerator for general Λ, p.

G.3 $\Lambda = [3,3]$ sector

The final example we give here is the BPS basis for the $\Lambda = [3,3]$ sector at field content $(3,3)$.

Throughout this section we will work with $\Lambda = [3,3]$ and $M_\Lambda = \begin{array}{|c|c|c|}\hline 1&1&1\\\hline 2&2&2\\\hline\end{array}$, so we will suppress this index in operator labels.

For each BPS operator, we will first present it as a sum over the free field basis (3.6.20) and then as a sum over symmetrised traces and commutator traces, for which we use the covariant bases discussed in section 7.3.2. The covariant symmetrised trace basis is

$$
\begin{aligned}
t_{[3,2,1]} &= \mathrm{Tr}X^3\mathrm{Tr}Y\mathrm{Tr}Y^2 - 2\mathrm{Tr}X^2Y\mathrm{Tr}XY\mathrm{Tr}Y - \mathrm{Tr}X\mathrm{Tr}X^2Y\mathrm{Tr}Y^2 \\
&\quad + \mathrm{Tr}X^2\mathrm{Tr}XY^2\mathrm{Tr}Y + 2\mathrm{Tr}X\mathrm{Tr}XY\mathrm{Tr}XY^2 - \mathrm{Tr}X\mathrm{Tr}X^2\mathrm{Tr}Y^3 \qquad \text{(G.3.1)} \\
t_{[3,1,1,1]} &= \mathrm{Tr}X^3\left(\mathrm{Tr}Y\right)^3 - 3\mathrm{Tr}X\mathrm{Tr}X^2Y\left(\mathrm{Tr}Y\right)^2 + 3\left(\mathrm{Tr}X\right)^2\mathrm{Tr}XY^2\mathrm{Tr}Y \\
&\quad + \left(\mathrm{Tr}X\right)^3\mathrm{Tr}Y^3 \qquad \text{(G.3.2)}
\end{aligned}
$$

and the covariant commutator trace basis is

$$
\begin{aligned}
c_{[6]} &= \mathrm{Tr}X^2YXY^2 - \mathrm{Tr}X^2Y^2XY = \mathrm{Tr}X^2Y[X,Y]Y \qquad \text{(G.3.3)} \\
c_{[5,1]} &= \mathrm{Tr}X^3Y^2\mathrm{Tr}Y - \mathrm{Tr}X^2YXY\mathrm{Tr}Y - \mathrm{Tr}X\mathrm{Tr}X^2Y^3 + \mathrm{Tr}X\mathrm{Tr}XYXY^2 \\
&= \mathrm{Tr}X^2[X,Y]Y\mathrm{Tr}Y - \mathrm{Tr}X\mathrm{Tr}X[X,Y]Y^2 \qquad \text{(G.3.4)}
\end{aligned}
$$

For these two bases, the partition label describes the cycle structure of the multi-traces.

The free field operators can be written in terms of symmetrised and commutator traces

$$O_{\yng(4,1,1)} = \frac{\sqrt{3}}{36}\left(3t_{[3,2,1]} + t_{[3,1,1,1]} + 6c_{[5,1]}\right) \tag{G.3.5}$$

$$O_{\yng(3,2,1),\,\mathrm{even}} = \frac{\sqrt{3}}{18}\left(t_{[3,1,1,1]} - 3c_{[5,1]}\right) \tag{G.3.6}$$

$$O_{\yng(3,2,1),\,\mathrm{odd}} = -\frac{1}{\sqrt{2}}c_{[6]} \tag{G.3.7}$$

$$O_{\yng(3,1,1,1)} = \frac{\sqrt{3}}{36}\left(-3t_{[3,2,1]} + t_{[3,1,1,1]} + 6c_{[5,1]}\right) \tag{G.3.8}$$

The odd/even labels for the $R = [3,2,1]$ multiplicity come from the odd/even permutations used to produce the respective traces. All other zero coupling operators are defined uniquely by Λ and R.

The BPS operators are

$$S^{BPS}_{[3,1,1,1]} = \frac{1}{\sqrt{3P_1}}\left(-N(N-3)O_{\yng(4,1,1)} + (N+3)(N-3)O_{\yng(3,2,1),\,\mathrm{even}} + 2(N+3)NO_{\yng(3,1,1,1)}\right) \tag{G.3.9}$$

$$= \frac{1}{12\sqrt{P_1}}\left(-3N(N+1)t_{[3,2,1]} + (N^2+3N-6)t_{[3,1,1,1]} + 18(N+1)c_{[5,1]}\right) \tag{G.3.10}$$

$$S^{BPS}_{[3,2,1]} = \frac{1}{\sqrt{P_2}}\left((N-1)O_{\yng(4,1,1)} + (N+1)O_{\yng(3,2,1),\,\mathrm{even}} - O_{\yng(3,1,1,1)}\right) \tag{G.3.11}$$

$$= \frac{1}{4\sqrt{3P_2}}\left(Nt_{[3,2,1]} + Nt_{[3,1,1,1]} - 6c_{[5,1]}\right) \tag{G.3.12}$$

where the normalisation polynomials are

$$P_1 = 2N^4 + 6N^3 + 9N^2 + 27 \qquad P_2 = 2N^2 + 3 \tag{G.3.13}$$

The norms of the BPS operators are

$$\left|S^{BPS}_{[3,1,1,1]}\right|^2 = \frac{(N+3)(N+2)(N+1)N^2(N-1)(N-2)(N-3)Q}{P_1} \tag{G.3.14}$$

$$\left|S^{BPS}_{[3,2,1]}\right|^2 = \frac{(N+2)(N+1)N^2(N-1)(N-2)Q}{P_2} \tag{G.3.15}$$

where

$$Q = 2\,N^2 + 3\,N - 3 \tag{G.3.16}$$

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