GLOBAL HYPOELLIPTICITY AND GLOBAL SOLVABILITY FOR VECTOR FIELDS ON COMPACT LIE GROUPS

ALEXANDRE KIRILOV, WAGNER A. A. DE MORAES, AND MICHAEL RUZHANSKY

Abstract. We present necessary and sufficient conditions to have global hypoellipticity and global solvability for a class of vector fields defined on a product of compact Lie groups. In view of Greenfield's and Wallach's conjecture, about the non-existence of globally hypoelliptic vector fields on compact manifolds different from tori, we also investigate different notions of regularity weaker than global hypoellipticity and describe completely the global hypoellipticity and global solvability of zero-order perturbations of our vector fields. We also present a class of vector fields with variable coefficients whose operators can be reduced to a normal form, and we prove that the study of the global properties of such operators is equivalent to the study of the respective properties for their normal forms.

Contents

2010 *Mathematics Subject Classification.* Primary 35R03, 43A80; Secondary 35H10, 58D25.
Key words and phrases. Compact Lie groups, Global hypoellipticity, Global solvability, Normal form, Vector fields, Low order perturbations.

1. Introduction

In this note, we study the regularity of solutions and solvability of vector fields (and their perturbations by zero order terms) on a compact Lie group G. More precisely, if $\mathcal{D}'(G)$ stands for the space of distributions on G and $P : \mathcal{D}'(G) \to \mathcal{D}'(G)$ is a first-order differential operator, we are interested in establishing conditions that ensure that u is smooth whenever Pu is smooth. We also want to identify under what conditions it is possible to guarantee that $Pu = f \in \mathcal{D}'(G)$ has a solution, in the sense of distributions. These properties are known as global hypoellipticity and global solvability, and have been widely studied in recent years, especially on the $d-$dimensional torus $\mathbb{T}^d$. See, for example, the impressive list of authors who have published articles addressing these subjects: [3], [4], [5], [6], [10], [17], [18], [19], [20], [21], [22], [23], [24], [25], [26], [29], [30] and references therein.

Even in the case of $\mathbb{T}^d$, the investigation of these global properties for vector fields is a challenging problem that still has open questions. Perhaps, the most famous and seemingly far-off question of a solution is the Greenfield's and Wallach's conjecture, which states the following: if a closed smooth orientable manifold admits a globally hypoelliptic vector field, then this manifold is $C^\infty-$diffeomorphic to a torus and this vector field is $C^\infty-$conjugated to a constant vector field whose coefficients satisfy a Diophantine condition (see [16] and [20]). S. Greenfield and N. Wallach have proved this conjecture for compact Lie groups in [20]. The conjecture it was also proved for compact manifolds of dimensions 2 and 3, and in some very particular cases, which are described by G. Forni in [16] and by L. Flaminio, G. Forni, and F. Rodriguez Hertz in [15].

Most of the studies that deal with the question of global hypoellipticity and global solvability in the torus make use of the Fourier analysis as the main tool to obtain results from conditions imposed on the symbol or on the coefficients of the operator. For example, in [19], S. Greenfield and N. Wallach use only the Fourier series in $\mathbb{T}^d$ to characterize the global hypoellipticity of a differential operator through its symbol. In that paper there appears for the first time the famous application: $L = \partial_x + a\partial_y$, $a \in \mathbb{R}$ is globally hypoelliptic in $\mathbb{T}^2$ if, and only if, a is an irrational non-Liouville number. Therefore a natural way of extending such studies to other smooth manifolds would be to consider manifolds where we have a Fourier analysis.

In this direction, based on ideas [21] and [36], J. Delgado and the third author [13] introduced on compact smooth manifolds M a notion of Fourier series for operators that commute with a fixed elliptic operator. Using these ideas, a study of global hypoellipticity for such operators was made in [11], [12], and [27]. The obvious disadvantage of this technique is that for now, it works only for operators that commute with a fixed elliptic operator.

In the particular case where the compact manifold is a Lie group G, there is a natural way of introducing a Fourier analysis into G, see for example [7], [8], [9], [14], [32], [33], [34], [35], [37]. In this paper we use the notation and results based on the book by M. Ruzhansky and V. Turunen [31] to study the global hypoellipticity and global solvability of a class of vector fields defined on Lie groups.

In the development of this project, we find natural to begin by extending the results of [19] to a product of Lie groups $G_1 \times G_2$. More precisely, if $L = X_1 + aX_2$, where X_j is a vector field on the Lie algebra $\mathfrak{g}_j$ and $a \in \mathbb{C}$, then we characterize completely the global hypoellipticity and global solvability of L, presenting necessary and sufficient conditions on the behavior of its symbol σ_L. In a subsection dedicated to examples, we recover the results obtained in [19] to the torus and present an example on $\mathbb{T}^1 \times \mathbb{S}^3$. This case was called "constant-coefficient vector field".

Given the validity of the Greenfield's and Wallach's conjecture on compact Lie groups, we introduce two different notions of global regularity weaker than the global hypoellipticity. The first one, global hypoellipticity modulo kernel, was inspired by the paper [1] of G. Araújo; and the second, $\mathcal{W}$–global hypoellipticity,

emerged naturally in the development of this study. In both cases, we also characterize the regularity of $L = X_1 + aX_2$ by analyzing the behavior of the symbol σ_L. To complement the study of the case of constant coefficients vector fields, we consider perturbations of L by low order terms, obtaining necessary and sufficient conditions to have global hypoellipticity and global solvability.

Finally, we introduce the class of variable coefficients vector fields of the form $L = X_1 + a(x_1)X_2$, where $a \in C^\infty(G_1)$ is a real-valued function. We show that the vector field of this class can be reduced to a normal form, so the study of the global properties of such operators is equivalent to the study of the respective properties of their normal forms.

The paper is organized as follows. In Section 2 we recall some classical results about Fourier analysis on compact Lie groups and fix the notation that will be used throughout the text. In Section 3 we give necessary and sufficient conditions for the global hypoellipticity and the global solvability of constant-coefficient vector fields defined on compact Lie groups and we present a class of examples. Because of the Greenfield-Wallach conjecture, in Section 4 we present two weaker notions of hypoellipticity. In Section 5 we study perturbations of vector fields by low order terms, both by constants and by functions. Finally, in Section 6 we characterize global properties of a class of perturbed vector fields with variable coefficients. We present the normal form and establish a connection between the global properties of perturbed vector fields with variable and constant coefficients.

2. Fourier analysis on compact Lie groups

In this section we recall most of the notations and preliminary results necessary for the development of this study. A very careful presentation of these concepts and the demonstration of all the results presented here can be found in the references [14] and [31].

Let G be a compact Lie group and let $\mathrm{Rep}(G)$ be the set of continuous irreducible unitary representations of G. Since G is compact, every continuous irreducible unitary representation ϕ is finite dimensional and it can be viewed as a matrix-valued function $\phi : G \to \mathbb{C}^{d_\phi \times d_\phi}$, where $d_\phi = \dim \phi$. We say that $\phi \sim \psi$ if there exists an unitary matrix $A \in \mathbb{C}^{d_\phi \times d_\phi}$ such that $A\phi(x) = \psi(x)A$, for all $x \in G$. We will denote by $\widehat{G}$ the quotient of $\mathrm{Rep}(G)$ by this equivalence relation.

For $f \in L^1(G)$ the group Fourier transform of f at $\phi \in \mathrm{Rep}(G)$ is

$$\widehat{f}(\phi) = \int_G f(x)\phi(x)^* \, dx,$$

where dx is the normalized Haar measure on G. By the Peter-Wyel theorem, we have that

$$\mathcal{B} := \left\{ \sqrt{\dim \phi}\, \phi_{ij} \,;\ \phi = (\phi_{ij})_{i,j=1}^{d_\phi}, [\phi] \in \widehat{G} \right\}, \tag{2.1}$$

is an orthonormal basis for $L^2(G)$, where we pick only one matrix unitary representation in each class of equivalence, and we may write

$$f(x) = \sum_{[\phi]\in\widehat{G}} d_\phi \mathrm{Tr}(\phi(x)\widehat{f}(\phi)).$$

Moreover, the Plancherel formula holds:

$$\|f\|_{L^2(G)} = \left(\sum_{[\phi]\in\widehat{G}} d_\phi \ \|\widehat{f}(\phi)\|_{\mathtt{HS}}^2 \right)^{\frac{1}{2}} =: \|\widehat{f}\|_{\ell^2(\widehat{G})}, \tag{2.2}$$

where

$$\|\widehat{f}(\phi)\|_{\mathtt{HS}}^2 = \mathrm{Tr}(\widehat{f}(\phi)\widehat{f}(\phi)^*) = \sum_{i,j=1}^{d_\phi} \left|\widehat{f}(\phi)_{ij}\right|^2.$$

Let $\mathcal{L}_G$ be the Laplace-Beltrami operator of G. For each $[\phi] \in \widehat{G}$, its matrix elements are eigenfunctions of $\mathcal{L}_G$ correspondent to the same eigenvalue that we will denote by $-\nu_{[\phi]}$, where $\nu_{[\phi]} \geq 0$. Thus

$$-\mathcal{L}_G \phi_{ij}(x) = \nu_{[\phi]}\phi_{ij}(x), \quad \text{for all } 1 \leq i,j \leq d_\phi,$$

and we will denote by

$$\langle \phi \rangle := \left(1 + \nu_{[\phi]}\right)^{1/2}$$

the eigenvalues of $(I - \mathcal{L}_G)^{1/2}$. We have the following estimate for the dimension of ϕ (Proposition 10.3.19 of [31]): there exists $C > 0$ such that for all $[\xi] \in \widehat{G}$ it holds

$$d_\phi \leq C\langle \phi \rangle^{\frac{\dim G}{2}}.$$

For $x \in G$, $X \in \mathfrak{g}$ and $f \in C^\infty(G)$, define

$$L_X f(x) := \frac{d}{dt} f(x \exp(tX))\bigg|_{t=0}.$$

The operator L_X is left-invariant, that is, $\pi_L(y)L_X = L_X\pi_L(y)$, for all $y \in G$. When there is no possibility of ambiguous meaning, we will write only Xf instead of L_Xf.

Let G be a compact Lie group of dimension d and $\{X_i\}_{i=1}^d$ a basis of its Lie algebra. For a multi-index $\alpha = (\alpha_1, \cdots, \alpha_d) \in \mathbb{N}_0^d$, the left-invariant differential operator of order $|\alpha|$ is

$$\partial^\alpha := Y_1 \cdots Y_{|\alpha|}, \tag{2.3}$$

with $Y_j \in \{X_i\}_{i=1}^d$, $1 \leq j \leq |\alpha|$ and $\sum\limits_{j:Y_j=X_k} 1 = \alpha_k$ for every $1 \leq k \leq d$. It means that ∂^α is a composition of left-invariant derivatives with respect to vector $X_1, \ldots, X_d$ such that each X_k enters ∂^α exactly α_k times. We do not specify in the notation ∂^α the order of vectors $X_1, \ldots, X_d$, but this will not be relevant for the arguments that we will use in this article.

We endow $C^\infty(G)$ with the usual Fréchet space topology defined by the family of seminorms $p_\alpha(f) = \max\limits_{x\in G} |\partial^\alpha f(x)|$, $\alpha \in \mathbb{N}$. Thus, the convergence on $C^\infty(G)$ is just the uniform convergence of functions and all their derivatives. As usual the space of distributions $\mathcal{D}'(G)$ is the space of all continuous linear functionals on $C^\infty(G)$. For $u \in \mathcal{D}'(G)$, we define the distribution $\partial^\alpha u$ as

$$\langle \partial^\alpha u, \psi\rangle := (-1)^{|\alpha|}\langle u, \partial^\alpha \psi\rangle,$$

for all $\psi \in C^\infty(G)$, and $\alpha \in \mathbb{N}_0^d$.

Let $P : C^\infty(G) \to C^\infty(G)$ be a continuous linear operator. The symbol of the operator P in $x \in G$ and $\phi \in \mathrm{Rep}(G)$, $\phi = (\phi_{ij})_{i,j=1}^{d_\phi}$ is

$$\sigma_P(x, \phi) := \phi(x)^*(P\phi)(x) \in \mathbb{C}^{d_\phi \times d_\phi},$$

where $(P\phi)(x)_{ij} := (P\phi_{ij})(x)$, for all $1 \leq i, j \leq d_\phi$, and we have

$$Pf(x) = \sum_{[\phi]\in\widehat{G}} \dim(\phi)\mathrm{Tr}\left(\phi(x)\sigma_P(x, \phi)\widehat{f}(\phi)\right)$$

for every $f \in C^\infty(G)$ and $x \in G$.

Notice that the last expression is independent of the choice of the representative. When $P : C^\infty(G) \to C^\infty(G)$ is a continuous linear left-invariant operator, that is $P\pi_L(y) = \pi_L(y)P$, for all $y \in G$, we have that σ_P is independent of $x \in G$ and

$$\widehat{Pf}(\phi) = \sigma_P(\phi)\widehat{f}(\phi),$$

for all $f \in C^\infty(G)$ and $[\phi] \in \widehat{G}$.

Let $X \in \mathfrak{g}$ be a vector field normalized by the norm induced by the Killing form. It is easy to see that the operator iX is symmetric on $L^2(G)$. Hence, for all $[\phi] \in \widehat{G}$ we can choose a representative ϕ such that $\sigma_{iX}(\phi)$ is a diagonal matrix, with entries $\lambda_m \in \mathbb{R}$, $1 \leq m \leq d_\phi$. By the linearity of the symbol, we obtain

$$\sigma_X(\phi)_{mn} = i\lambda_m \delta_{mn}, \quad \lambda_j \in \mathbb{R}.$$

Notice that $\{\lambda_m\}_{m=1}^{d_\phi}$ are the eigenvalues of $\sigma_{iX}(\phi)$ and then are independent of the choice of the representative, since the symbol of equivalent representations are similar matrices. Moreover, since $-(\mathcal{L}_G - X^2)$ is a positive operator and commutes with X^2,

$$|\lambda_m(\phi)| \leq \langle \phi \rangle,$$

for all $[\phi] \in \widehat{G}$ and $1 \leq m \leq d_\phi$.

Smooth functions and distributions can be characterized by their Fourier coefficients:

Proposition 2.1 (Page 759 of [7])**.** *Let G be a compact Lie group. The following three statements are equivalent:*

1. *$f \in C^\infty(G)$;*
2. *for each $N > 0$, there exists $C_N > 0$ such that*

$$\|\widehat{f}(\phi)\|_{\mathtt{HS}} \leq C_N \langle \phi \rangle^{-N},$$

   for all $[\phi] \in \widehat{G}$;
3. *for each $N > 0$, there exists $C_N > 0$ such that*

$$|\widehat{f}(\phi)_{ij}| \leq C_N \langle \phi \rangle^{-N},$$

   for all $[\phi] \in \widehat{G}$ and $1 \leq i, j \leq d_\phi$.

Moreover, the following three statements are equivalent:

4. *$u \in \mathcal{D}'(G)$;*
5. *there exist C, $N > 0$ such that*

$$\|\widehat{u}(\phi)\|_{\mathtt{HS}} \leq C \langle \phi \rangle^{N},$$

   for all $[\phi] \in \widehat{G}$;

6. there exist C, $N > 0$ such that

$$|\widehat{u}(\phi)_{ij}| \leq C\langle\phi\rangle^N,$$

for all $[\phi] \in \widehat{G}$ and $1 \leq i, j \leq d_\phi$.

Let G_1 and G_2 be compact Lie groups and set $G = G_1 \times G_2$. Given $f \in L^1(G)$ and $\xi \in \mathrm{Rep}(G_1)$, the partial Fourier coefficient of f with respect to the first variable is defined by

$$\widehat{f}(\xi, x_2) = \int_{G_1} f(x_1, x_2)\, \xi(x_1)^*\, dx_1 \in \mathbb{C}^{d_\xi \times d_\xi}, \quad x_2 \in G_2,$$

with components

$$\widehat{f}(\xi, x_2)_{mn} = \int_{G_1} f(x_1, x_2)\, \overline{\xi(x_1)_{nm}}\, dx_1, \quad 1 \leq m, n \leq d_\xi.$$

Analogously we define the partial Fourier coefficient of f with respect to the second variable. Notice that, by definition, $\widehat{f}(\xi, \cdot\,)_{mn} \in C^\infty(G_2)$ and $\widehat{f}(\cdot\,, \eta)_{rs} \in C^\infty(G_1)$.

Let $u \in \mathcal{D}'(G)$, $\xi \in \mathrm{Rep}(G_1)$ and $1 \leq m, n \leq d_\xi$. The mn-component of the partial Fourier coefficient of u with respect to the first variable is the linear functional defined by

$$\begin{array}{rccl} \widehat{u}(\xi, \cdot\,)_{mn}: & C^\infty(G_2) & \longrightarrow & \mathbb{C} \\ & \psi & \longmapsto & \langle \widehat{u}(\xi, \cdot\,)_{mn}, \psi\rangle := \left\langle u, \overline{\xi_{nm}} \times \psi \right\rangle_G. \end{array}$$

In a similar way, for $\eta \in \mathrm{Rep}(G_2)$ and $1 \leq r, s \leq d_\eta$, we define the rs-component of the partial Fourier coefficient of u with respect to the second variable. It is easy to see that $\widehat{u}(\xi, \cdot\,)_{mn} \in \mathcal{D}'(G_2)$ and $\widehat{u}(\cdot\,, \eta)_{rs} \in \mathcal{D}'(G_1)$.

Notice that

$$\widehat{\widehat{u}}(\xi, \eta)_{mn_{rs}} = \widehat{\widehat{u}}(\xi, \eta)_{rs_{mn}} = \widehat{u}(\xi \otimes \eta)_{ij},$$

with $i = d_\eta(m-1) + r$ and $j = d_\eta(n-1) + s$, whenever $u \in C^\infty(G)$ or $u \in \mathcal{D}'(G)$.

Finally, we have the following characterization of smooth functions and distributions on $G = G_1 \times G_2$:

Proposition 2.2 (Theorems 3.3 and 3.4 of [28])**.** *Let G_1 and G_2 be compact Lie groups, and set $G = G_1 \times G_2$. The following three statements are equivalent:*

1. $f \in C^\infty(G)$;

2. *for each $N > 0$, there exists $C_N > 0$ such that*

$$\|\widehat{\widehat{f}}(\xi,\eta)\|_{\mathtt{HS}} \leq C_N(\langle\xi\rangle + \langle\eta\rangle)^{-N},$$

for all $[\xi] \in \widehat{G_1}$ and $[\eta] \in \widehat{G_2}$;

3. *for each $N > 0$, there exists $C_N > 0$ such that*

$$\left|\widehat{\widehat{f}}(\xi,\eta)_{mn_{rs}}\right| \leq C_N(\langle\xi\rangle + \langle\eta\rangle)^{-N},$$

for all $[\xi] \in \widehat{G_1}$, $[\eta] \in \widehat{G_2}$, $1 \leq m, n \leq d_\xi$, and $1 \leq r, s \leq d_\eta$.

Moreover, the following statements are equivalent:

4. *$u \in \mathcal{D}'(G)$;*
5. *there exists C, $N > 0$ such that*

$$\|\widehat{\widehat{u}}(\xi,\eta)\|_{\mathtt{HS}} \leq C(\langle\xi\rangle + \langle\eta\rangle)^{N},$$

for all $[\xi] \in \widehat{G_1}$ and $[\eta] \in \widehat{G_2}$;

6. *there exists C, $N > 0$ such that*

$$\left|\widehat{\widehat{u}}(\xi,\eta)_{mn_{rs}}\right| \leq C(\langle\xi\rangle + \langle\eta\rangle)^{N},$$

for all $[\xi] \in \widehat{G_1}$, $[\eta] \in \widehat{G_2}$, $1 \leq m, n \leq d_\xi$, and $1 \leq r, s \leq d_\eta$.

3. Constant coefficient vector fields

Let G_1 and G_2 be compact Lie groups, $G := G_1 \times G_2$, and consider the linear operator $L : C^\infty(G) \to C^\infty(G)$ defined by

$$L := X_1 + aX_2,$$

where $X_1 \in \mathfrak{g}_1$, $X_2 \in \mathfrak{g}_2$ and $a \in \mathbb{C}$. Thus, for each $u \in C^\infty(G)$ we have

$$\begin{aligned} Lu(x_1,x_2) &:= X_1u(x_1,x_2) + aX_2u(x_1,x_2) \\ &:= \frac{d}{dt}u(x_1\exp(tX_1),x_2)\bigg|_{t=0} + a\frac{d}{ds}u(x_1,x_2\exp(sX_2))\bigg|_{s=0}. \end{aligned}$$

The operator L extends to distributions in a natural way, that is, if $u \in \mathcal{D}'(G)$, then

$$\langle Lu,\varphi\rangle_G := -\langle u, L\varphi\rangle_G, \quad \varphi \in C^\infty(G).$$

In this section we present necessary and sufficient conditions for the vector field L to be globally hypoelliptic and to be globally solvable. We also present examples recovering known results in the torus and presenting an example in $\mathbb{T}^1 \times \mathbb{S}^3$.

3.1. **Global hypoellipticity.**

Definition 3.1. *Let G be a compact Lie group. We say that an operator $P : \mathcal{D}'(G) \to \mathcal{D}'(G)$ is globally hypoelliptic if the conditions $u \in \mathcal{D}'(G)$ and $Pu \in C^\infty(G)$ imply that $u \in C^\infty(G)$.*

Consider the equation

$$Lu(x_1, x_2) = X_1 u(x_1, x_2) + aX_2 u(x_1, x_2) = f(x_1, x_2),$$

where $f \in C^\infty(G)$. For each $[\xi] \in \widehat{G_1}$, we can choose a representative $\xi \in \mathrm{Rep}(G_1)$ such that

$$\sigma_{X_1}(\xi)_{mn} = i\lambda_m(\xi)\delta_{mn}, \quad 1 \leq m, n \leq d_\xi,$$

where $\lambda_m(\xi) \in \mathbb{R}$ for all $[\xi] \in \widehat{G_1}$ and $1 \leq m \leq d_\xi$. Similarly, for each $[\eta] \in \widehat{G_2}$, we can choose a representative $\eta \in \mathrm{Rep}(G_2)$ such that

$$\sigma_{X_2}(\eta)_{rs} = i\mu_r(\eta)\delta_{rs}, \quad 1 \leq r, s \leq d_\eta,$$

where $\mu_r(\eta) \in \mathbb{R}$ for all $[\eta] \in \widehat{G_2}$ and $1 \leq r \leq d_\eta$.

Suppose that $u \in C^\infty(G)$. Thus, taking the partial Fourier coefficient with respect to the first variable at $x_2 \in G_2$ we obtain

$$\begin{aligned}
\widehat{f}(\xi, x_2) &= \widehat{Lu}(\xi, x_2) \\
&= \int_{G_1} Lu(x_1, x_2)\xi(x_1)^* \, dx_1 \\
&= \int_{G_1} X_1 u(x_1, x_2)\xi(x_1)^* \, dx_1 + a\int_{G_1} X_2 u(x_1, x_2)\xi(x_1)^* \, dx_1 \\
&= \widehat{X_1 u}(\xi, x_2) + aX_2 \int_{G_1} u(x_1, x_2)\xi(x_1)^* \, dx_1 \\
&= \sigma_{X_1}(\xi)\widehat{u}(\xi, x_2) + aX_2\widehat{u}(\xi, x_2).
\end{aligned}$$

Hence, for each $x_2 \in G_2$, $\widehat{f}(\xi, x_2) \in \mathbb{C}^{d_\xi \times d_\xi}$ and

$$\widehat{f}(\xi, x_2)_{mn} = i\lambda_m(\xi)\widehat{u}(\xi, x_2)_{mn} + aX_2\widehat{u}(\xi, x_2)_{mn}, \quad 1 \leq m, n \leq d_\xi.$$

The details about partial Fourier series can be found in [28]. Now, taking the Fourier coefficient of $\widehat{f}(\xi,\cdot)_{mn}$ with respect to the second variable, we obtain

$$\begin{aligned}\widehat{\widehat{f}}(\xi,\eta)_{mn} &= \int_{G_2} \widehat{f}(\xi,x_2)_{mn}\eta(x_2)^*\,dx_2\\ &= \int_{G_2} (i\lambda_m(\xi)\widehat{u}(\xi,x_2)_{mn} + aX_2\widehat{u}(\xi,x_2)_{mn})\eta(x_2)^*\,dx_2\\ &= i\lambda_m(\xi)\int_{G_2}\widehat{u}(\xi,x_2)_{mn}\eta(x_2)^*\,dx_2 + a\int_{G_2} X_2\widehat{u}(\xi,x_2)_{mn}\eta(x_2)^*\,dx_2\\ &= i\lambda_m(\xi)\,\widehat{\widehat{u}}(\xi,\eta)_{mn} + a\sigma_{X_2}(\eta)\,\widehat{\widehat{u}}(\xi,\eta)_{mn}.\end{aligned}$$

Thus, $\widehat{\widehat{f}}(\xi,\eta)_{mn}\in\mathbb{C}^{d_\eta\times d_\eta}$ and

$$\widehat{\widehat{f}}(\xi,\eta)_{mn_{rs}} = i(\lambda_m(\xi)+a\mu_r(\eta))\,\widehat{\widehat{u}}(\xi,\eta)_{mn_{rs}},\quad 1\le r,s\le d_\eta.$$

From this we can conclude that

$$\widehat{\widehat{f}}(\xi,\eta)_{mn_{rs}} = 0,\ \text{whenever } \lambda_m(\xi)+a\mu_r(\eta)=0. \tag{3.1}$$

Moreover, if $\lambda_m(\xi)+a\mu_r(\eta)\neq 0$, then

$$\widehat{\widehat{u}}(\xi,\eta)_{mn_{rs}} = \frac{1}{i(\lambda_m(\xi)+a\mu_r(\eta))}\,\widehat{\widehat{f}}(\xi,\eta)_{mn_{rs}}. \tag{3.2}$$

We begin by presenting the following necessary condition for global hypoellipticity of the vector field $L=X_1+aX_2$.

Proposition 3.2. *Suppose that the set*

(3.3)

$$\mathcal{N} = \{([\xi],[\eta])\in\widehat{G_1}\times\widehat{G_2};\ \lambda_m(\xi)+a\mu_r(\eta)=0,\ \textit{for some } 1\le m\le d_\xi, 1\le r\le d_\eta\}$$

has infinitely many elements. Then there exists $u\in\mathcal{D}'(G)\setminus C^\infty(G)$ such that

$$Lu = 0.$$

In particular, L is not globally hypoelliptic.

Proof. Consider the sequence

$$\widehat{\widehat{u}}(\xi,\eta)_{mn_{rs}} = \begin{cases} 1, & \text{if } \lambda_m(\xi)+a\mu_r(\eta)=0,\\ 0, & \text{otherwise}\end{cases}$$

Notice that for any $[\xi]\in\widehat{G_1}$, $[\eta]\in\widehat{G_2}$, $1\le m,n\le d_\xi$ and $1\le r,s\le d_\eta$ we have

$$|\,\widehat{\widehat{u}}(\xi,\eta)_{mn_{rs}}|\le\langle\xi\rangle+\langle\eta\rangle.$$

Thus by the characterization of distributions by Fourier coefficients (Theorem 2.2) we conclude that $u \in \mathcal{D}'(G)$, where

$$u = \sum_{[\xi]\in\widehat{G_1}} \sum_{[\eta]\in\widehat{G_2}} d_\xi d_\eta \sum_{m,n=1}^{d_\xi} \sum_{r,s=1}^{d_\eta} \widehat{\widehat{u}}(\xi,\eta)_{mn_{rs}} \xi_{nm} \eta_{sr}.$$

Since there exist infinitely many representations such that $\widehat{\widehat{u}}(\xi,\eta)_{mn_{rs}} = 1$, it follows that $u \notin C^\infty(G)$. Furthermore, we have

$$\widehat{\widehat{Lu}}(\xi,\eta)_{mn_{rs}} = i(\lambda_m(\xi) + a\mu_r(\eta))\,\widehat{\widehat{u}}(\xi,\eta)_{mn_{rs}} = 0,$$

for all $[\xi] \in \widehat{G_1},\ [\eta] \in \widehat{G_2}, 1 \le m \le d_\xi, 1 \le r \le d_\eta$. Then, by Plancherel formula (2.2), we conclude that $Lu = 0$. □

Theorem 3.3. *The operator $L = X_1 + aX_2$ is globally hypoelliptic if and only if the following conditions are satisfied:*

1. *The set*

$$\mathcal{N} = \{([\xi],[\eta]) \in \widehat{G_1} \times \widehat{G_2};\ \lambda_m(\xi) + a\mu_r(\eta) = 0,\ \text{for some } 1 \le m \le d_\xi, 1 \le r \le d_\eta\}$$

   is finite.

2. *$\exists C,\ M > 0$ such that*

$$|\lambda_m(\xi) + a\mu_r(\eta)| \ge C(\langle\xi\rangle + \langle\eta\rangle)^{-M}, \tag{3.4}$$

   for all $[\xi] \in \widehat{G_1},\ [\eta] \in \widehat{G_2},\ 1 \le m \le d_\xi,\ 1 \le r \le d_\eta$, whenever $\lambda_m(\xi) + a\mu_r(\eta) \neq 0$.

Proof. ($\Longleftarrow$) Suppose that $Lu = f \in C^\infty(G)$ for some $u \in \mathcal{D}'(G)$. Let us prove that $u \in C^\infty(G)$. Since the set $\mathcal{N}$ is finite, there exists $C > 0$ such that

$$|\,\widehat{\widehat{u}}(\xi,\eta)_{mn_{rs}}| \le C,$$

for all $([\xi],[\eta]) \in \mathcal{N},\ 1 \le m,n \le d_\xi,\ 1 \le r,s \le d_\eta$. Let $N \in \mathbb{N}$. Then, for $([\xi],[\eta]) \in \mathcal{N}$, we have

$$\begin{aligned} |\,\widehat{\widehat{u}}(\xi,\eta)_{mn_{rs}}| &\le C(\langle\xi\rangle + \langle\eta\rangle)^N (\langle\xi\rangle + \langle\eta\rangle)^{-N} \\ &\le C'_N(\langle\xi\rangle + \langle\eta\rangle)^{-N} \end{aligned}$$

where $C'_N = \max_{([\xi],[\eta])\in\mathcal{N}} \{C(\langle\xi\rangle + \langle\eta\rangle)^N\}$. On the other hand, if $([\xi],[\eta]) \notin \mathcal{N}$, by (3.2) and (3.4) we obtain

$$\begin{aligned} |\widehat{\widehat{u}}(\xi,\eta)_{mn_{rs}}| &= \frac{1}{|\lambda_m(\xi) + a\mu_r(\eta)|} |\widehat{\widehat{f}}(\xi,\eta)_{mn_{rs}}| \\ &\leq C^{-1}(\langle\xi\rangle + \langle\eta\rangle)^M |\widehat{\widehat{f}}(\xi,\eta)_{mn_{rs}}| \end{aligned}$$

Since $f \in C^\infty(G)$, there exists $C_{N+M} > 0$ such that

$$|\widehat{\widehat{f}}(\xi,\eta)_{mn_{rs}}| \leq C_{N+M}(\langle\xi\rangle + \langle\eta\rangle)^{-(N+M)}$$

Thus,

$$\begin{aligned} |\widehat{\widehat{u}}(\xi,\eta)_{mn_{rs}}| &\leq C^{-1}C_{N+M}(\langle\xi\rangle + \langle\eta\rangle)^M(\langle\xi\rangle + \langle\eta\rangle)^{-(N+M)} \\ &= C''_N(\langle\xi\rangle + \langle\eta\rangle)^{-N}, \end{aligned}$$

where $C''_N = C^{-1}C_{N+M}$. Hence, if $([\xi],[\eta]) \notin \mathcal{N}$ we conclude that

$$|\widehat{\widehat{u}}(\xi,\eta)_{mn_{rs}}| \leq C''_N(\langle\xi\rangle + \langle\eta\rangle)^{-N}.$$

Setting $C_N := \max\{C'_N, C''_N\}$, we have

$$|\widehat{\widehat{u}}(\xi,\eta)_{mn_{rs}}| \leq C_N(\langle\xi\rangle + \langle\eta\rangle)^{-N},$$

for all $[\xi] \in \widehat{G_1}$, $[\eta] \in \widehat{G_2}$. Therefore by Theorem 2.2 we conclude that $u \in C^\infty(G)$.

$(\implies)$ Let us prove the result by contradiction. If the condition 1 were not satisfied, by Proposition 3.2, there would be $u \in \mathcal{D}'(G)\backslash C^\infty(G)$ such that $Lu = 0$, contradicting the hypothesis of global hypoellipticity of L. So, let us assume that Condition 2 is not satisfied, then for every $M \in \mathbb{N}$, we choose $[\xi_M] \in \widehat{G_1}$ and $[\eta_M] \in \widehat{G_2}$ such that

$$0 < |\lambda_m(\xi_M) + a\mu_r(\eta_M)| \leq (\langle\xi_M\rangle + \langle\eta_M\rangle)^{-M}, \tag{3.5}$$

for some $1 \leq m \leq d_{\xi_M}$ and $1 \leq r \leq d_{\eta_M}$.

Let $\mathcal{A} = \{([\xi_j],[\eta_j])\}_{j\in\mathbb{N}}$. It is easy to see that $\mathcal{A}$ has infinitely many elements. Define

$$\widehat{\widehat{u}}(\xi,\eta)_{mn_{rs}} = \begin{cases} 1, & \text{if } ([\xi],[\eta]) = ([\xi_j],[\eta_j]) \text{ for some } j \in \mathbb{N} \text{ and (3.5) is satisfied,} \\ 0, & \text{otherwise.} \end{cases}$$

In this way, $u \in \mathcal{D}'(G)\backslash C^\infty(G)$. Let us show that $Lu = f \in C^\infty(G)$.

If $([\xi],[\eta]) \notin \mathcal{A}$, then $|\widehat{\widehat{f}}(\xi,\eta)_{mn_{rs}}| = 0$. Moreover, for every $M \in \mathbb{N}$, we have

$$\begin{aligned}|\widehat{\widehat{f}}(\xi_M,\eta_M)_{mn_{rs}}| &= |\lambda_m(\xi_M) + a\mu_r(\eta_M)||\widehat{\widehat{u}}(\xi_M,\eta_M)_{mn_{rs}}| \\ &\leq (\langle\xi_M\rangle + \langle\eta_M\rangle)^{-M}\end{aligned}$$

for every element of $\mathcal{A}$.

Fix $N > 0$. If $M > N$, then

$$|\widehat{\widehat{f}}(\xi_M,\eta_M)_{mn_{rs}}| \leq (\langle\xi_M\rangle + \langle\eta_M\rangle)^{-M} \leq (\langle\xi_M\rangle + \langle\eta_M\rangle)^{-N}.$$

For $M \leq N$ we have

$$\begin{aligned}\left|\widehat{\widehat{f}}(\xi_M,\eta_M)_{mn_{rs}}\right| &= \left|\widehat{\widehat{f}}(\xi_M,\eta_M)_{mn_{rs}}\right| (\langle\xi_M\rangle + \langle\eta_M\rangle)^N(\langle\xi_M\rangle + \langle\eta_M\rangle)^{-N} \\ &\leq C'_N(\langle\xi_M\rangle + \langle\eta_M\rangle)^{-N}.\end{aligned}$$

where $C'_N := \max\limits_{M\leq N}\left\{|\widehat{\widehat{f}}(\xi_M,\eta_M)_{mn_{rs}}|(\langle\xi_M\rangle + \langle\eta_M\rangle)^N\right\}$. For $C_N = \max\{C'_N, 1\}$ we obtain

$$|\widehat{\widehat{f}}(\xi,\eta)_{mn_{rs}}| \leq C_N(\langle\xi\rangle + \langle\eta\rangle)^{-N},$$

for all $[\xi] \in \widehat{G_1}$, $[\eta] \in \widehat{G_2}$, $1 \leq m,n \leq d_\xi$, $1 \leq r,s \leq d_\eta$. Therefore $f \in C^\infty(G)$, which contradicts the assumption that L is globally hypoelliptic. □

3.2. **Global solvability.** In the literature there are several notions of the solvability of an operator, mainly depending on the functional environment in which one is working and what one intends to study. So the first step here is to define precisely what we mean by the global solvability.

Given a function (or distribution) f defined on G, assume that $u \in \mathcal{D}'(G)$ is a solution of $Lu = f$. By taking the partial Fourier coefficient with respect to x_1 and x_2 separately, and following the same procedure of the last subsection, we obtain from (3.1) that

$$\lambda_m(\xi) + a\mu_r(\eta) = 0 \Longrightarrow \widehat{\widehat{f}}(\xi,\eta)_{mn_{rs}} = 0.$$

Therefore, let us consider the following set

$$\mathcal{K} := \{w \in \mathcal{D}'(G);\ \widehat{\widehat{w}}(\xi,\eta)_{mn_{rs}} = 0,\ \text{whenever } \lambda_m(\xi) + a\mu_r(\eta) = 0\}.$$

If $f \notin \mathcal{K}$, then there is no $u \in \mathcal{D}'(G)$ such that $Lu = f$. We call the elements of $\mathcal{K}$ of admissible functions (distributions) for the solvability of L.

Definition 3.4. *We say that the operator L is globally solvable if $L(\mathcal{D}'(G)) = \mathcal{K}$.*

Theorem 3.5. *The operator $L = X_1 + aX_2$ is globally solvable if and only if there exist C, $M > 0$ such that*

$$|\lambda_m(\xi) + a\mu_r(\eta)| \geq C(\langle\xi\rangle + \langle\eta\rangle)^{-M}, \tag{3.6}$$

for all $[\xi] \in \widehat{G_1}$, $[\eta] \in \widehat{G_2}$, $1 \leq m \leq d_\xi$, $1 \leq r \leq d_\eta$ whenever $\lambda_m(\xi) + a\mu_r(\eta) \neq 0$.

Proof. ($\Longleftarrow$) For each $f \in \mathcal{K}$ define

(3.7)

$$\widehat{\widehat{u}}(\xi,\eta)_{mn_{rs}} = \begin{cases} 0, & \text{if } \lambda_m(\xi) + a\mu_r(\eta) = 0, \\ -i(\lambda_m(\xi) + a\mu_r(\eta))^{-1}\,\widehat{\widehat{f}}(\xi,\eta)_{mn_{rs}}, & \text{otherwise.} \end{cases}$$

Let us show that $\{\widehat{\widehat{u}}(\xi,\eta)_{mn_{rs}}\}$ is the sequence of Fourier coefficient of an element $u \in \mathcal{D}'(G)$. Since $f \in \mathcal{D}'(G)$, there exists $N \in \mathbb{N}$ and $C > 0$ such that

$$|\widehat{\widehat{f}}(\xi,\eta)_{mn_{rs}}| \leq C(\langle\xi\rangle + \langle\eta\rangle)^{N},$$

for all $[\xi] \in \widehat{G_1}$, $[\eta] \in \widehat{G_2}$, $1 \leq m \leq d_\xi$, $1 \leq r \leq d_\eta$. So

$$\begin{aligned} |\widehat{\widehat{u}}(\xi,\eta)_{mn_{rs}}| &= |\lambda_m(\xi) + a\mu_r(\eta)|^{-1}|\widehat{\widehat{f}}(\xi,\eta)_{mn_{rs}}| \\ &\leq C(\langle\xi\rangle + \langle\eta\rangle)^{M}|\widehat{\widehat{f}}(\xi,\eta)_{mn_{rs}}| \\ &\leq C(\langle\xi\rangle + \langle\eta\rangle)^{N+M} \end{aligned} \tag{3.8}$$

Therefore $u \in \mathcal{D}'(G)$ and $Lu = f$.

($\Longrightarrow$) Let us proceed by contradiction by constructing an element $f \in \mathcal{K}$ such that there is no $u \in \mathcal{D}'(G)$ satisfying $Lu = f$.

If (3.6) is not satisfied, for each $M \in \mathbb{N}$, there exists $[\xi_M] \in \widehat{G_1}$ and $[\eta_M] \in \widehat{G_2}$ such that

$$0 < |\lambda_{\tilde{m}}(\xi_M) + a\mu_{\tilde{r}}(\eta_M)| < (\langle\xi_M\rangle + \langle\eta_M\rangle)^{-M}, \tag{3.9}$$

for some $1 \leq \tilde{m} \leq d_{\xi_M}$ and $1 \leq \tilde{r} \leq d_{\eta_M}$. We can suppose that $\langle\xi_M\rangle + \langle\eta_M\rangle \leq \langle\xi_N\rangle + \langle\eta_N\rangle$ when $M \leq N$. Let $\mathcal{A} = \{([\xi_j],[\eta_j])\}_{j\in\mathbb{N}}$. Consider $f \in \mathcal{K}$ defined by

$$\widehat{\widehat{f}}(\xi,\eta)_{mn_{rs}} = \begin{cases} 1, & \text{if } ([\xi],[\eta]) = ([\xi_j],[\eta_j]) \text{ for some } j \in \mathbb{N} \text{ and (3.9) is satisfied,} \\ 0, & \text{otherwise.} \end{cases}$$

Suppose that there exits $u \in \mathcal{D}'(G)$ such that $Lu = f$. In this way, its Fourier coefficients must satisfy

$$i(\lambda_m(\xi) + a\mu_r(\eta))\,\widehat{\widehat{u}}(\xi,\eta)_{mn_{rs}} = \widehat{\widehat{f}}(\xi,\eta)_{mn_{rs}}.$$

So

$$|\widehat{\widehat{u}}(\xi_M,\eta_M)_{\tilde{m}1_{\tilde{r}1}}| = |\lambda_{\tilde{m}}(\xi_M)+a\mu_{\tilde{r}}(\eta_M)|^{-1}||\widehat{\widehat{f}}(\xi_M,\eta_M)_{\tilde{m}1_{\tilde{r}1}}|$$
$$> (\langle\xi_M\rangle+\langle\eta_M\rangle)^M,$$

where $\tilde{m}$ and $\tilde{r}$ are coefficients that satisfy (3.9). Thus

$$\|\widehat{\widehat{u}}(\xi_M,\eta_M)\|_{\mathtt{HS}} > (\langle\xi_M\rangle+\langle\eta_M\rangle)^M,$$

for all $M>0$, which contradicts the fact that $u\in\mathcal{D}'(G)$. Therefore there does not exist $u\in\mathcal{D}'(G)$ such that $Lu=f$. □

Notice that the estimate for the global solvability in the statement of the last theorem is exactly the same as one of the conditions to obtain global hypoellipticity announced in (3.4), thus we have the following corollary.

Corollary 3.6. *If L is globally hypoelliptic, then L is globally solvable.*

A more detailed analysis of the last proof shows that it is possible to obtain a better control on the Fourier coefficients of u when f is smooth, more precisely, we have the following result.

Proposition 3.7. *If L is globally solvable and $f\in\mathcal{K}\cap C^\infty(G)$, then there exists $u\in C^\infty(G)$ such that $Lu=f$.*

Proof. Let $f\in\mathcal{K}\cap C^\infty(G)$ and define u as in (3.7). Since L is globally solvable, we have (3.6), and then by (3.8)

$$|\widehat{\widehat{u}}(\xi,\eta)_{mn_{rs}}| = |\lambda_m(\xi)+a\mu_r(\eta)|^{-1}|\widehat{\widehat{f}}(\xi,\eta)_{mn_{rs}}|$$
$$\leq C(\langle\xi\rangle+\langle\eta\rangle)^M|\widehat{\widehat{f}}(\xi,\eta)_{mn_{rs}}|.$$

In view of the smoothness of f, for every $N>0$ there exists $C_N>0$ such that

$$|\widehat{\widehat{f}}(\xi,\eta)_{mn_{rs}}|\leq C_N(\langle\xi\rangle+\langle\eta\rangle)^{-N},$$

for all $[\xi]\in\widehat{G_1}$, $[\eta]\in\widehat{G_2}$, $1\leq m\leq d_\xi$, $1\leq r\leq d_\eta$. Hence

$$|\widehat{\widehat{u}}(\xi,\eta)_{mn_{rs}}|\leq C(\langle\xi\rangle+\langle\eta\rangle)^M|\widehat{\widehat{f}}(\xi,\eta)_{mn_{rs}}|\leq C_{N+M}(\langle\xi\rangle+\langle\eta\rangle)^{-N}.$$

Therefore $u\in C^\infty(G)$ and $Lu=f$. □

Remark 3.8. *The proposition above says that we can obtain a smooth solution for $Lu = f$ in the case where L is globally solvable and f is a smooth admissible function. Notice that this does not mean that L is globally hypoelliptic.*

3.3. **Examples.** In this section we recover some classical examples of S. Greenfield and N. Wallach (see [19]), on the global hypoellipticity and global solvability in tori ($\mathbb{T}^2$ and $\mathbb{T}^d$) and present a class of examples in $\mathbb{T}^1 \times \mathbb{S}^3$.

Example 3.9. $G = \mathbb{T}^2$

Set $G_1 = G_2 = \mathbb{T}^1$, where $\mathbb{T}^1 = \mathbb{R}/2\pi\mathbb{Z}$. Since $\mathbb{T}^1$ is abelian, the irreducible unitary representations of $\mathbb{T}^1$ are unidimensional. Moreover the dual $\widehat{\mathbb{T}^1}$ can be identified to $\mathbb{Z}$. For each $k \in \mathbb{Z}$, the function $e_k : \mathbb{T}^1 \to \mathcal{U}(\mathbb{C})$ defined by

$$e_k(t) := e^{itk}$$

is an element of $\widehat{\mathbb{T}^1}$ and

$$\widehat{\mathbb{T}^1} \cong \{e_k\}_{k\in\mathbb{Z}}.$$

The Haar measure on $\mathbb{T}^1$ is the normalized Lebesgue measure and

$$\langle k\rangle := \langle e_k\rangle = \sqrt{1+k^2}.$$

Let $a \in \mathbb{C}$ and consider the operator

$$L = \partial_t + a\partial_x, \quad (t,x) \in \mathbb{T}^1 \times \mathbb{T}^1.$$

Notice that

$$\sigma_{\partial_t}(e_k) = e_k(t)^*(\partial_t e_k)(t) = e^{-itk}(ike^{itk}) = ik,$$

that is, $\lambda(e_k) = k$, for all $k \in \mathbb{Z}$. Thus, if $Lu = f$, then

$$\widehat{\widehat{f}}(k,\ell) = i(k + a\ell)\ \widehat{\widehat{u}}(k,\ell).$$

In this case,

$$\mathcal{N} = \{(k,\ell) \in \mathbb{Z}^2;\ k + a\ell = 0\}.$$

By Theorem 3.3, L is globally hypoelliptic if and only if $\mathcal{N}$ is finite and there exist $C, M > 0$ such that

$$|k + a\ell| \geq C(\langle k\rangle + \langle \ell\rangle)^{-M}$$

for all $(k,\ell) \in \mathbb{Z}^2$, whenever $k + a\ell \neq 0$. For $(k,\ell) \neq (0,0)$, we have

$$|k| + |\ell| \leq \langle k \rangle + \langle \ell \rangle \leq 3(|k| + |\ell|),$$

then the second condition of the Theorem 3.3 becomes

$$|k + a\ell| \geq C(|k| + |\ell|)^{-M} \tag{3.10}$$

for all $(k,\ell) \in \mathbb{Z}^2$, whenever $k + a\ell \neq 0$.

Notice that $\mathcal{N}$ is an infinity set if and only if $a \in \mathbb{Q}$. Moreover, if $a \notin \mathbb{Q}$, then $\mathcal{N} = \{(0,0)\}$. Suppose that $\mathrm{Im}(a) \neq 0$. If $\ell \neq 0$, then

$$|k + a\ell| \geq |\mathrm{Im}(a)||\ell| \geq |\mathrm{Im}(a)|(|k| + |\ell|)^{-1}.$$

If $\ell = 0$, we have $k \neq 0$ and

$$|k + a\ell| = |k| \geq (|k| + |\ell|)^{-1}.$$

Take $C = \min\{1, |\mathrm{Im}(a)|\}$. Then

$$|k + a\ell| \geq C(|k| + |\ell|)^{-1},$$

for all $(k,\ell) \in \mathbb{Z}^2 \setminus \{(0,0)\}$. Therefore, if $\mathrm{Im}(a) \neq 0$ then L is globally hypoelliptic.

Suppose now that $\mathrm{Im}(a) = 0$. We recall that an irrational number a is called a Liouville number if it can be approximated by rational numbers to any order. That is, for every positive integer N there is $K > 0$ and infinitely many integer pairs (k,ℓ) so that

$$\left| a - \frac{k}{\ell} \right| < \frac{K}{\ell^N}.$$

Notice that the inequality (3.10) is satisfied if and only if a is an irrational non-Liouville number.

We conclude that $L = \partial_t + a\partial_x$ is globally hypoelliptic if and only if $\mathrm{Im}(a) \neq 0$ or a is an irrational non-Liouville number.

For solvability we need to analyze the condition 2 of the Theorem 3.3 when $a \in \mathbb{Q}$. Suppose that $a = \frac{p}{q}$, $p \in \mathbb{Z}$ and $q \in \mathbb{N}$. We have

$$|k + a\ell| = \left| k + \frac{p}{q}\ell \right| = \frac{1}{q}|qk + p\ell| \geq \frac{1}{q} \geq \frac{1}{q}(|k| + |\ell|)^{-1},$$

for all $(k,\ell) \in \mathbb{Z}^2$, whenever $qk + p\ell \neq 0$.

Therefore, $L = \partial_t + a\partial_x$ is globally solvable if and only if $\mathrm{Im}(a) \neq 0$, or $a \in \mathbb{Q}$, or a is an irrational non-Liouville number.

Example 3.10. $G = \mathbb{T}^d$

From the above example we can extend the analysis for operators defined on $\mathbb{T}^d$. Let

$$L = \sum_{j=1}^{d} a_j \partial_{t_j}, \quad a_j \in \mathbb{C}$$

If $Lu = f$, then

$$\widehat{f}(k_1, \cdots, k_d) = i \left(\sum_{j=1}^{d} a_j k_j \right) \widehat{u}(k_1, \cdots, k_d).$$

The set $\mathcal{N}$ is

$$\mathcal{N} = \left\{ k \in \mathbb{Z}^d; \sum_{j=1}^{d} a_j k_j = 0 \right\},$$

and by Theorem 3.3, L is globally hypoelliptic if and only if $\mathcal{N}$ is finite and there exists $C, M > 0$ such that

$$\left| \sum_{j=1}^{d} a_j k_j \right| \geq C \left(\sum_{j=1}^{d} |k_j| \right)^{-M},$$

for all $k \in \mathbb{Z}^d$ whenever $\sum_{j=1}^{d} a_j k_j \neq 0$.

For instance, if some $a_j = 0$, then the set $\mathcal{N}$ is infinity, which implies that L is not globally hypoelliptic. It is easy to see that if all $a_j \in \mathbb{Q}$, then L is globally solvable, even if some of $a_j = 0$.

If $a_j = 1$ for $j = 1, \cdots, d-1$ and $\mathrm{Im}(a_d) \neq 0$, than L is globally hypoelliptic. The same is true if we consider a_d being an irrational non-Liouville number.

Example 3.11. $G = \mathbb{T}^1 \times \mathbb{S}^3$

Let $\widehat{\mathbb{S}^3}$ be the unitary dual of $\mathbb{S}^3$, that is, $\widehat{\mathbb{S}^3}$ consists of equivalence classes $[\mathsf{t}^\ell]$ of continuous irreducible unitary representations $\mathsf{t}^\ell : \mathbb{S}^3 \to \mathbb{C}^{(2\ell+1)\times(2\ell+1)}$, $\ell \in \frac{1}{2}\mathbb{N}_0$, of matrix-valued functions satisfying $\mathsf{t}^\ell(xy) = \mathsf{t}^\ell(x)\mathsf{t}^\ell(y)$ and $\mathsf{t}^\ell(x)^* = \mathsf{t}^\ell(x)^{-1}$ for all $x, y \in \mathbb{S}^3$. We will use the standard convention of enumerating the matrix elements t^ℓ_{mn} of t^ℓ using indices m, n ranging between $-\ell$ to ℓ with step one, i.e. we have $-\ell \leq m, n \leq \ell$ with $\ell - m, \ell - n \in \mathbb{N}_0$. For $\ell \in \frac{1}{2}\mathbb{N}_0$ we have

$$\langle \ell \rangle := \langle \mathsf{t}^\ell \rangle = \sqrt{1 + \ell(\ell+1)}.$$

The details about the Fourier analysis on $\mathbb{S}^3$ can be found in Chapter 11 of [31].

Let X be a smooth vector field on $\mathbb{S}^3$ and $a \in \mathbb{C}$. Consider the following operator defined on $\mathbb{T}^1 \times \mathbb{S}^3$:

$$L = \partial_t + aX.$$

Using rotation on $\mathbb{S}^3$, without loss of generality, we may assume that the vector field X has the symbol

$$\sigma_X(\ell)_{mn} = im\delta_{mn}, \quad \ell \in \tfrac{1}{2}\mathbb{N}_0,\ -\ell \leq m, n \leq \ell,\ \ell - m, \ell - n \in \mathbb{N}_0,$$

with δ_{mn} standing for the Kronecker's delta (see [31], [33], and [34]). Hence, if $Lu = f$, then

$$\widehat{\widehat{f}}(k,\ell)_{mn} = i(k + am)\,\widehat{\widehat{u}}(k,\ell)_{mn},$$

where $k \in \mathbb{Z}$, $\ell \in \frac{1}{2}\mathbb{N}_0$, $-\ell \leq m, n \leq \ell$ and $\ell - m, \ell - n \in \mathbb{N}_0$. In this case,

$$\mathcal{N} = \{(k,\ell) \in \mathbb{Z} \times \tfrac{1}{2}\mathbb{N}_0;\ k + am = 0,\ \text{for some}\ -\ell \leq m \leq \ell, \ell - m \in \mathbb{N}_0\}.$$

By Theorem 3.3, L is globally hypoelliptic if and only if $\mathcal{N}$ is finite and there exist $C, M > 0$ such that

$$|k + am| \geq C(\langle k\rangle + \langle \ell\rangle)^{-M} \tag{3.11}$$

for all $(k,\ell) \in \mathbb{Z} \times \frac{1}{2}\mathbb{N}_0$, $-\ell \leq m \leq \ell$, $\ell - m \in \mathbb{N}_0$ whenever $k + am \neq 0$. For $\ell \in \frac{1}{2}\mathbb{N}_0$, we have

$$\frac{1}{\sqrt{2}}(1 + \ell) \leq \langle \mathrm{t}^\ell\rangle \leq 1 + \ell$$

and we can write (3.11) as

$$|k + am| \geq C(|k| + 1 + \ell)^{-M}$$

for all $(k,\ell) \in \mathbb{Z} \times \frac{1}{2}\mathbb{N}_0$, $-\ell \leq m \leq \ell$, $\ell - m \in \mathbb{N}_0$ whenever $k + am \neq 0$.

Notice that $(0,\ell) \in \mathcal{N}$, for all $\ell \in \mathbb{N}_0$, so $\mathcal{N}$ has infinitely many elements and then L is not globally hypoelliptic for any $a \in \mathbb{C}$.

The analysis of the global solvability of L is similar to the $\mathbb{T}^2$ case and we have that L is globally solvable if and only if $\mathrm{Im}(a) \neq 0$, or $a \in \mathbb{Q}$, or a is an irrational non-Liouville number.

4. Weaker notions of hypoellipticity

All the known examples of globally hypoelliptic vector fields are set on tori. Actually, in 1973, S. Greenfield and N. Wallach proposed the following conjecture.

Conjecture 4.1 (Greenfield-Wallach)**.** *If a closed, connected, orientable manifold M admits a globally hypoelliptic vector field X, then M is diffeomorphic to a torus and X is smoothly conjugate to a constant Diophantine vector field.*

In [16], G. Forni showed the equivalence between this conjecture and Katok's conjecture, about the existence of C^∞–cohomology free smooth vector fields on closed, connected, orientable smooth manifolds. From this equivalence we will show that on compact connected Lie groups the set $\mathcal{N}$ defined in (3.3) contains only the trivial representation. First, let us define what is a C^∞–cohomology free vector field.

Definition 4.2. *Let M be a closed, connected, orientable smooth manifold. A smooth vector field X on M is C^∞–cohomology free if for all $f \in C^\infty(M)$ there exists a constant $c(f) \in \mathbb{C}$ and $u \in C^\infty(M)$ such that*

$$Xu = f - c(f).$$

Theorem 4.3. *[G. Forni [16]] Let X be a smooth vector field on a closed connected manifold M. Then X is C^∞–cohomology free if and only if X is globally hypoelliptic.*

Proposition 4.4. *If G is a compact connected Lie group and L is globally hypoelliptic, then $\mathcal{N}$ has only one element.*

Proof. Notice that for the trivial representations $\mathbb{1}_{G_1}$ and $\mathbb{1}_{G_2}$ we have $\lambda_1(\mathbb{1}_{G_1}) = \mu_1(\mathbb{1}_{G_2}) = 0$, so $\mathcal{N} \neq \varnothing$. Suppose that there exists a non-trivial representation such that

$$\lambda_m(\xi) + a\mu_r(\eta) = 0.$$

for some $1 \leq m \leq d_\xi$, $1 \leq r \leq d_\eta$. Let $f = \xi_{1m} \times \eta_{1r} \in C^\infty(G)$, so

$$\begin{aligned}\widehat{\widehat{f}}(\xi,\eta)_{m1_{r1}} &= \int_{G_1}\int_{G_2} f(x_1,x_2)\overline{\xi(x_1)_{1m}}\,\overline{\eta(x_2)_{1r}}dx_2dx_1\\ &= \int_{G_1}\int_{G_2} \xi(x_1)_{1m}\eta(x_2)_{1r}\overline{\xi(x_1)_{1m}}\,\overline{\eta(x_2)_{1r}}dx_2dx_1\\ &= \int_{G_1} |\xi(x_1)_{1m}|^2\,dx_1\int_{G_2}|\eta(x_2)_{1r}|^2\,dx_2\\ &= (d_\xi d_\eta)^{-1}.\end{aligned}$$

Since L is globally hypoelliptic, by Theorem 4.3 L is C^∞–cohomology free, then there exists $u\in C^\infty(G)$ such that

$$Lu = f - f_0,$$

where $f_0 = \int_G f\,d\mu_G$. We have

$$\widehat{\widehat{Lu}}(\xi,\eta)_{m1_{r1}} = i(\lambda_m(\xi)) + a\mu_r(\eta))\,\widehat{\widehat{u}}(\xi,\eta)_{m1_{r1}} = 0,$$

which implies that

$$\widehat{\widehat{f-f_0}}(\xi,\eta)_{m1_{r1}} = 0.$$

Since $\xi\otimes\eta$ is not the trivial representation, by (2.1) we have $\widehat{\widehat{f}}_0(\xi,\eta)_{m1_{r1}} = 0$, so

$$\widehat{\widehat{f}}(\xi,\eta)_{m1_{r1}} = 0,$$

which is a contradiction because $\widehat{\widehat{f}}(\xi,\eta)_{m1_{r1}} = (d_\xi d_\eta)^{-1}$. Therefore $\mathcal{N}$ contains only the trivial representation. □

In view of Example 3.11 and Proposition 4.4, the following question naturally arises:

Question 4.1. *Does there exist a compact Lie group $G\neq\mathbb{T}^d$ such that there exists $X\in\mathfrak{g}$ such that $\sigma_X(\phi)$ is singular for only finitely many $[\phi]\in\widehat{G}$, that is, the set*

$$\mathcal{Z} = \{[\phi]\in\widehat{G};\ \lambda_m(\phi) = 0,\ \textit{for some } 1\leq m\leq d_\phi\}$$

is finite, where $\sigma_X(\phi)_{mn} = i\lambda_m(\phi)\delta_{mn}$?

The answer to this question is a way to obtain an alternative proof for the Greenfield-Wallach conjecture on compact Lie groups

In view of the validity of the Greenfield-Wallach conjecture, the study of the global hypoellipticity of vector fields defined on closed manifolds is restricted to

tori. However, the study of the regularity of solutions of such vector fields is yet an interesting subject. For this reason, in this section we will make some considerations looking to weaken the usual concept of the global hypoellipticity and introduce what we will call global hypoellipticity modulo kernel and global $\mathcal{W}$-hypoellipticity.

4.1. **Global hypoellipticity modulo kernel.**

First, assuming that the set $\mathcal{N}$ has infinitely many elements, we will show that to reduce the range of the operator does not help us to obtain a weaker version of global hypoellipticity.

Proposition 4.5. *Suppose that $\mathcal{N}$ has infinitely many elements. Then there is no subset $\mathcal{A} \subseteq C^\infty(G)$ that satisfies the condition: $u \in \mathcal{D}'(G)$ and $Lu \in \mathcal{A}$ imply that $u \in C^\infty(G)$.*

Proof. Assume that there exists a subset $\mathcal{A} \subseteq C^\infty(G)$ that satisfies the property above. Let $u \in \mathcal{D}'(G)$ such that $Lu \in \mathcal{A}$, then $u \in C^\infty(G)$. By Proposition 3.2 there exists an element $v \in \ker L$ such that $v \in \mathcal{D}'(G)\backslash C^\infty(G)$. Since $v \in \ker L$, we have $L(u+v) = Lu \in \mathcal{A}$, which implies that $u + v \in C^\infty(G)$. Therefore $v = (u+v) - u \in C^\infty(G)$, a contradiction. □

In view of Proposition 4.5 we give the following definition:

Definition 4.6. *We say that an operator $P : \mathcal{D}'(G) \to \mathcal{D}'(G)$ is globally hypoelliptic modulo* $\ker P$ *if the conditions $u \in \mathcal{D}'(G)$ and $Pu \in C^\infty(G)$ imply that there exists $v \in C^\infty(G)$ such that $u - v \in \ker P$.*

Clearly, global hypoellipticity implies global hypoellipticity modulo kernel. Our main result here is the equivalence of the concepts of global hypoellipticity modulo kernel and global solvability for constant coefficient vector fields.

Proposition 4.7. *The operator $L = X_1 + aX_2$ is globally hypoelliptic modulo* $\ker L$ *if and only if L is globally solvable.*

Proof. ($\Longrightarrow$) Suppose that L is not globally solvable. Then by Theorem 3.5, for every $M \in \mathbb{N}$, choose $[\xi_M] \in \widehat{G_1}$ and $[\eta_M] \in \widehat{G_2}$ such that

$$0 < |\lambda_m(\xi_M) + a\mu_r(\eta_M)| \leq (\langle \xi_M \rangle + \langle \eta_M \rangle)^{-M},$$

for some $1 \leq m \leq d_{\xi_M}$ and $1 \leq r \leq d_{\eta_M}$. Using the same construction of the proof of Theorem 3.3, we find a $u \in \mathcal{D}'(G) \setminus C^\infty(G)$ such that $Lu = f \in C^\infty(G)$. Notice that if $u - v \in \ker L$, for some $v \in C^\infty(G)$, then

$$i(\lambda_m(\xi) + a\mu_r(\eta))\widehat{\widehat{u-v}}(\xi,\eta)_{mn_{rs}} = 0,$$

for all $[\xi] \in \widehat{G_1}$, $[\eta] \in \widehat{G_2}$, $1 \leq m, n \leq d_\xi$, $1 \leq r, s \leq d_\eta$, which implies that

$$\lambda_m(\xi) + a\mu_r(\eta) \neq 0 \implies \widehat{\widehat{u}}(\xi,\eta)_{mn_{rs}} = \widehat{\widehat{v}}(\xi,\eta)_{mn_{rs}}.$$

Since $\widehat{\widehat{u}}(\xi_M,\eta_M)_{mn_{rs}} = 1$, we conclude that $v \notin C^\infty(G)$, so L is not globally hypoelliptic modulo $\ker L$.

($\Longleftarrow$) Let $u \in \mathcal{D}'(G)$ such that $Lu = f \in C^\infty(G)$. Notice that $f \in \mathcal{K} \cap C^\infty(G)$ and by Proposition 3.7 there exists $v \in C^\infty(G)$ such that $Lv = f$. Therefore $u - v \in \ker L$ and then L is globally hypoelliptic modulo $\ker L$. □

Example 4.8. *Let $G = \mathbb{T}^1 \times \mathbb{S}^3$. In Example 3.11 we saw that the operator $L = \partial_t + X$ is not globally hypoelliptic but it is globally solvable. By Proposition 4.7, we conclude that even not being globally hypoelliptic, the operator L is globally hypoelliptic modulo kernel.*

4.2. **$\mathcal{W}$–global hypoellipticity.** In the light of Proposition 4.5, our next notion of hypoellipticity is based on the reduction of the domain of the operator.

Definition 4.9. *Let $\mathcal{W}$ be a subset of $\mathcal{D}'(G)$. We say that an operator $P : \mathcal{D}'(G) \to \mathcal{D}'(G)$ is $\mathcal{W}$-globally hypoelliptic if the conditions $u \in \mathcal{W}$ and $Pu \in C^\infty(G)$ imply that $u \in C^\infty(G)$.*

Observe that an operator P is always $C^\infty(G)$–globally hypoelliptic, and to say that P is $\mathcal{D}'(G)$-globally hypoelliptic means that P is globally hypoelliptic.

Example 4.10. *Let $L = X_1 + aX_2$ and set*

$$\mathcal{K} := \{u \in \mathcal{D}'(G);\ \widehat{\widehat{u}}(\xi,\eta)_{mn_{rs}} = 0, \text{whenever } \lambda_m(\xi) + a\mu_r(\eta) = 0\}.$$

If L is globally solvable, then L is $\mathcal{K}$-globally hypoelliptic.

Indeed, by the characterization of the global solvability (Theorem 3.5), there exist C, $M > 0$ such that

$$|\lambda_m(\xi) + a\mu_r(\eta)| \geq C(\langle\xi\rangle + \langle\eta\rangle)^{-M},$$

for all $[\xi] \in \widehat{G_1},\ [\eta] \in \widehat{G_2},\ 1 \leq m \leq d_\xi,\ 1 \leq r \leq d_\eta$, *whenever* $\lambda_m(\xi) + a\mu_r(\eta) \neq 0$.

Let $u \in \mathcal{K}$ *such that* $Lu = f \in C^\infty(G)$. *We know that*

$$\widehat{\widehat{f}}(\xi,\eta)_{mn_{rs}} = i(\lambda_m(\xi) + a\mu_r(\eta))\,\widehat{\widehat{u}}(\xi,\eta)_{mn_{rs}}.$$

If $\lambda_m(\xi) + a\mu_r(\eta) = 0$ *then* $\widehat{\widehat{u}}(\xi,\eta)_{mn_{rs}} = 0$.

If $\lambda_m(\xi) + a\mu_r(\eta) \neq 0$, *we have*

$$|\,\widehat{\widehat{u}}(\xi,\eta)_{mn_{rs}}| = \frac{1}{|\lambda_m(\xi) + a\mu_r(\eta)|}|\,\widehat{\widehat{f}}(\xi,\eta)_{mn_{rs}}| \leq C(\langle\xi\rangle + \langle\eta\rangle)^M|\,\widehat{\widehat{f}}(\xi,\eta)_{mn_{rs}}|.$$

Therefore $u \in C^\infty(G)$.

Proposition 4.11. *If* $\mathcal{W}_1 \subseteq \mathcal{W}_2$ *and* L *is* $\mathcal{W}_2$*–globally hypoelliptic, then* L *is* $\mathcal{W}_1$*–globally hypoelliptic.*

Proof. Let $u \in \mathcal{W}_1$ such that $Lu \in C^\infty(G)$. As $\mathcal{W}_1 \subseteq \mathcal{W}_2$, we have $u \in \mathcal{W}_2$ and since L is $\mathcal{W}_2$–globally hypoelliptic, $u \in C^\infty(G)$. Therefore L is $\mathcal{W}_1$–globally hypoelliptic. □

Corollary 4.12. *If* L *is globally solvable, then* L *is* $L(\mathcal{D}'(G))$*–globally hypoelliptic, where* $L(\mathcal{D}'(G))$ *denotes the image of* L.

Proof. If $f \in L(\mathcal{D}'(G))$, then there exists $u \in \mathcal{D}'(G)$ such that $Lu = f$, which implies that $f \in \mathcal{K}$, so $L(\mathcal{D}'(G)) \subset \mathcal{K}$. Since L is $\mathcal{K}$–globally hypoelliptic (Example 4.10), by Proposition 4.11 we conclude that L is $L(\mathcal{D}'(G))-$hypoelliptic. □

Corollary 4.13. *Suppose that* L *is globally solvable. If there exists* $k \in \mathbb{N}$ *such that* $L^k u \in C^\infty(G)$, *then* $Lu \in C^\infty(G)$.

Proof. Suppose that there exists $k \in \mathbb{N}$ such that $L^k u \in C^\infty(G)$. Since $v = L^{k-1}u \in L(\mathcal{D}'(G))$ and $Lv \in C^\infty(G)$, we have, by the $L(\mathcal{D}'(G))$–global hypoellipticity of L, that $v \in L^{k-1}u \in C^\infty(G)$. We can continue this process to conclude that $Lu \in C^\infty(G)$. □

If L is globally solvable, the previous corollary says that if $Lu \notin C^\infty(G)$, then $L^k u \notin C^\infty(G)$ for all $k \in \mathbb{N}$.

Let

$$\mathcal{M} := \{u \in \mathcal{D}'(G); \forall N \in \mathbb{N}, \exists C_N > 0; \|\,\widehat{\widehat{u}}(\xi,\eta)\|_{\mathtt{HS}} \leq C_N(\langle\xi\rangle + \langle\eta\rangle)^{-N}, ([\xi],[\eta]) \in \mathcal{N}\}.$$

Notice that $C^\infty(G) \subsetneq \mathcal{M}$.

Theorem 4.14. *If L is globally solvable, then L is $\mathcal{M}$–globally hypoelliptic.*

Proof. Let $u \in \mathcal{M}$ such that $Lu \in C^\infty(G)$. We know that

$$\widehat{\widehat{Lu}}(\xi,\eta)_{mn_{rs}} = i(\lambda_m(\xi) + a\mu_r(\eta))\,\widehat{\widehat{u}}(\xi,\eta)_{mn_{rs}},$$

for all $[\xi] \in \widehat{G_1}$, $[\eta] \in \widehat{G_2}, 1 \leq m \leq d_\xi, 1 \leq r \leq d_\eta$. If $([\xi],[\eta]) \notin \mathcal{N}$, then $\lambda_m(\xi) + a\mu_r(\eta) \neq 0$ and

$$\widehat{\widehat{u}}(\xi,\eta)_{mn_{rs}} = \frac{1}{i(\lambda_m(\xi) + a\mu_r(\eta))}\widehat{\widehat{Lu}}(\xi,\eta)_{mn_{rs}}.$$

Proceeding similarly as in the proof of Theorem 3.3, it can be proved that for every $N \in \mathbb{N}$, there exists $C'_N > 0$ such that

$$\|\,\widehat{\widehat{u}}(\xi,\eta)\|_{\mathtt{HS}} \leq C'_N(\langle\xi\rangle + \langle\eta\rangle)^{-N},$$

for all $([\xi],[\eta]) \notin \mathcal{N}$. Since $u \in \mathcal{M}$, we can conclude that for every $N \in \mathbb{N}$, there exists $K_N > 0$ such that

$$\|\,\widehat{\widehat{u}}(\xi,\eta)\|_{\mathtt{HS}} \leq K_N(\langle\xi\rangle + \langle\eta\rangle)^{-N},$$

for all $([\xi],[\eta]) \in \widehat{G_1} \times \widehat{G_2}$. Therefore $u \in C^\infty(G)$. □

5. Low order perturbations

In view of the Greenfield-Wallach conjecture, a way to obtain examples of globally hypoelliptic first order differential operators defined on compact Lie groups other than the torus is to consider perturbations of vector fields by low order terms.

We start by considering the case where q is a constant, next we will consider perturbations by functions $q \in C^\infty(G)$. This approach was inspired by the reference [2] of A. Bergamasco. In both situations, perturbations by constant and functions, we characterize the global hypoellipticity and the global solvability.

5.1. **Perturbations by constants.** Let G be a compact Lie group, $X \in \mathfrak{g}$ and $q \in \mathbb{C}$. Define the operator
$L_q : C^\infty(G) \to C^\infty(G)$ as

$$L_q u := Xu + qu, \quad u \in C^\infty(G).$$

We can extend L_q to $\mathcal{D}'(G)$ as

$$\langle L_q u, \varphi\rangle := -\langle u, X\varphi\rangle + \langle u, q\varphi\rangle = -\langle u, L_{-q}\varphi\rangle, \quad u \in \mathcal{D}'(G),\ \varphi \in C^\infty(G). \tag{5.1}$$

If $L_q u = f \in C^\infty(G)$, the Fourier coefficient of f can be obtained as

$$\widehat{f}(\xi) = \widehat{L_q u}(\xi) = \widehat{Xu}(\xi) + \widehat{qu}(\xi) = \sigma_X(\xi)\widehat{u}(\xi) + q\widehat{u}(\xi),$$

for all $[\xi] \in \widehat{G}$. So

$$\widehat{f}(\xi)_{mn} = i\lambda_m(\xi)\widehat{u}(\xi)_{mn} + q\widehat{u}(\xi)_{mn} = i(\lambda_m(\xi) - iq)\widehat{u}(\xi)_{mn},$$

for all $[\xi] \in \widehat{G}$, $1 \leq m, n \leq d_\xi$.

From this we conclude that

$$\widehat{f}(\xi)_{mn} = 0, \text{ whenever } \lambda_m(\xi) - iq = 0.$$

In addition, if $\lambda_m(\xi) - iq \neq 0$, then

$$\widehat{u}(\xi)_{mn} = \frac{1}{i(\lambda_m(\xi) - iq)}\widehat{f}(\xi)_{mn}.$$

Thus, we obtain the following characterization for the global hypoellipticity and solvability of L_q which is similar to the vector fields case and so its proof will be omitted.

Theorem 5.1. *The operator $L_q = X + q$ is globally hypoelliptic if and only if the following conditions are satisfied:*

1. *The set*

$$\mathcal{N} = \{[\xi] \in \widehat{G};\ \lambda_m(\xi) - iq = 0 \text{ for some } 1 \leq m \leq d_\xi\}$$

is finite.

2. *$\exists C$, $M > 0$ such that*

$$|\lambda_m(\xi) - iq| \geq C\langle\xi\rangle^{-M}, \tag{5.2}$$

for all $[\xi] \in \widehat{G}$, $1 \leq m \leq d_\xi$ whenever $\lambda_m(\xi) + iq \neq 0$.

Let $\mathcal{K}_q := \{w \in \mathcal{D}'(G);\ \widehat{w}(\xi)_{mn} = 0$ whenever $\lambda_m(\xi) - iq = 0$, for all $1 \leq m, n \leq d_\xi, 1 \leq r, s \leq d_\eta\}$.

Definition 5.2. *We say that L_q is globally solvable if $L_q(\mathcal{D}'(G)) = \mathcal{K}_q$.*

Theorem 5.3. *The operator $L_q = X + q$ is globally solvable if and only if the condition (5.2) is satisfied, that is, $\exists C$, $M > 0$ such that*

$$|\lambda_m(\xi) - iq| \geq C\langle\xi\rangle^{-M},$$

for all $[\xi] \in \widehat{G},\ 1 \leq m \leq d_\xi$ *whenever* $\lambda_m(\xi) + iq \neq 0$.

Corollary 5.4. *If* L_q *is globally hypoelliptic, then* L_q *is globally solvable.*

Recall the definition of global hypoellipticity modulo kernel given in Section 4. The proof of the next result is similar to Proposition 4.7 and its proof will be omitted.

Proposition 5.5. *The operator* L_q *is globally hypoelliptic modulo* $\ker L_q$ *if and only if* L_q *is globally solvable.*

Example 5.6. *Let* $G = \mathbb{S}^3$ *and consider*

$$L_q = X + q, \quad q \in \mathbb{C}.$$

As in Example 3.11 we can assume that the vector field X *defined on* $\mathbb{S}^3$ *has the symbol*

$$\sigma(\ell)_{mn} = im\delta_{mn}, \quad \ell \in \tfrac{1}{2}\mathbb{N}_0,\ -\ell \leq m, n \leq \ell, \ell - m, \ell - n \in \mathbb{N}_0.$$

In this case,

$$\mathcal{N} = \{\ell \in \tfrac{1}{2}\mathbb{N}_0; m - iq = 0,\ \text{for some}\ -\ell \leq m \leq \ell,\ \ell - m \in \mathbb{N}_0\}$$

and the inequality (5.2) *becomes*

$$|m - iq| \geq C(1 + \ell)^{-M}, \tag{5.3}$$

for all $\ell \in \frac{1}{2}\mathbb{N}_0$, $-\ell \leq m \leq \ell$, $\ell - m \in \mathbb{N}_0$ *whenever* $m - iq \neq 0$, *for some* $C, M > 0$.

We have $\mathcal{N} = \varnothing$ *for all* $q \notin i\frac{1}{2}\mathbb{Z}$ *and* $\mathcal{N}$ *has infinitely many elements otherwise. Notice that the condition* (5.3) *is always satisfied. Therefore* L_q *is globally hypoelliptic if and only if* $q \notin i\frac{1}{2}\mathbb{Z}$ *and* L_q *is globally solvable for all* $q \in \mathbb{C}$.

For examples on the torus, we refer the reader to Proposition 4.1 of [2] where the author presented conditions to obtain global hypoellipticity of perturbed operators by constants and constructed some enlightening examples.

5.2. **Perturbations by functions.** In this section we are concerned with the operator $L_q := X+q$, considering now $q \in C^\infty(G)$. The idea is to establish a connection between the global hypoellipticity of L_q and $L_{q_0} = X + q_0$, where q_0 is the average of q in G.

Lemma 5.7. *For any $\varphi \in C^\infty(G)$ we have*

$$Xe^{\varphi} = (X\varphi)e^{\varphi}.$$

Proof. Let $x \in G$, then

$$\begin{aligned}(Xe^{\varphi})(x) &= X\sum_{k=0}^{\infty}\frac{\varphi(x)^k}{k!} = \sum_{k=0}^{\infty}\frac{X\varphi(x)^k}{k!} = \sum_{k=1}^{\infty}\frac{k\varphi(x)^{k-1}(X\varphi)(x)}{k!}\\ &= (X\varphi)(x)\sum_{k=1}^{\infty}\frac{\varphi(x)^{k-1}}{(k-1)!} = (X\varphi)(x)e^{\varphi(x)}.\end{aligned}$$

□

Let $L_q : C^\infty(G) \to C^\infty(G)$ be defined by

$$L_q u = Xu(x) + qu, \quad u \in C^\infty(G).$$

We can extend L_q to $\mathcal{D}'(G)$ as in (5.1).

Proposition 5.8. *Given $q \in C^\infty(G)$, assume that there exists $Q \in C^\infty(G)$ such that $XQ = q - q_0$, where $q_0 = \int_G q(x)\,dx$. Then*

1. *$L_q \circ e^{-Q} = e^{-Q} \circ L_{q_0}$, in both $C^\infty(G)$ and in $\mathcal{D}'(G)$;*
2. *L_q is globally hypoelliptic if and only if L_{q_0} is globally hypoelliptic;*
3. *L_q is globally hypoelliptic modulo $\ker L_q$ if and only if L_{q_0} is globally hypoelliptic modulo $\ker L_{q_0}$.*

Proof. 1. Let $u \in C^\infty(G)$. Then

$$\begin{aligned}(L_q \circ e^{-Q})u &= L_q(e^{-Q}u)\\ &= X(e^{-Q}u) + qe^{-Q}u = (Xe^{-Q})u + e^{-Q}Xu + qe^{-Q}u\\ &= (-XQ)e^{-Q}u + e^{-Q}Xu + qe^{-Q}u\\ &= -(q-q_0)e^{-Q}u + e^{-Q}Xu + qe^{-Q}u\\ &= e^{-Q}(Xu + q_0u)\\ &= (e^{-Q} \circ L_{q_0})u.\end{aligned}$$

The same is true when we have $u \in \mathcal{D}'(G)$.

2. Suppose that L_q is globally hypoelliptic. If $L_{q_0}u = f \in C^\infty(G)$ for some $u \in \mathcal{D}'(G)$, then $e^{-Q}L_{q_0}u = e^{-Q}f \in C^\infty(G)$. Since $e^{-Q} \circ L_{q_0} = L_q \circ e^{-Q}$, we have $L_q(e^{-Q}u) \in C^\infty(G)$ and, by the global hypoellipticity of L_q we have $e^{-Q}u \in C^\infty(G)$, which implies that $u \in C^\infty(G)$ and then L_{q_0} is globally hypoelliptic.

Assume now that L_{q_0} is globally hypoelliptic. If $L_q u = f \in C^\infty(G)$ for some $u \in \mathcal{D}'(G)$, we can write $L_q(e^{-Q}e^{Q}u) = f \in C^\infty(G)$. By the fact that $L_q \circ e^{-Q} = e^{-Q} \circ L_{q_0}$ we obtain $e^{-Q}L_{q_0}(e^{Q}u) = f$, that is, $L_{q_0}(e^{Q}u) = e^{Q}f \in C^\infty(G)$. By the global hypoellipticity of L_{q_0} we have that $e^{Q}u \in C^\infty(G)$ and then $u \in C^\infty(G)$.

3. The proof is analogous to the item 2. □

Now assume that $L_q u = f \in \mathcal{D}'(G)$ for some $u \in \mathcal{D}'(G)$. We may write $u = e^{-Q}(e^{Q}u)$, so $L_q(e^{-Q}(e^{Q}u)) = f$. By Proposition 5.8, we have $e^{-Q}L_{q_0}e^{Q}u = f$, that is,

$$L_{q_0}e^{Q}u = e^{Q}f.$$

This implies that $e^{Q}f \in \mathcal{K}_{q_0}$.

Definition 5.9. *We say that the operator L_q is globally solvable if:*

1. *there is Q such that $XQ = q - q_0$, where $q_0 = \int_G q(x)\,dx$; and*
2. *$L_q(\mathcal{D}'(G)) = \mathcal{J}_q$, where*

$$\mathcal{J}_q := \{v \in \mathcal{D}'(G);\ e^{Q}v \in \mathcal{K}_{q_0}\}.$$

Proposition 5.10. *L_q is globally solvable if and only if L_{q_0} is globally solvable.*

Proof. Assume that L_q is globally solvable and let $f \in \mathcal{K}_{q_0}$. Let us show that there exists $u \in \mathcal{D}'(G)$ such that $L_{q_0}u = f$. We can write $f = e^{Q}e^{-Q}f$, so $e^{-Q}f \in \mathcal{J}_q$. Since L_q is globally solvable, there exists $v \in \mathcal{D}'(G)$ such that $L_q v = e^{-Q}f$. We can write $v = e^{-Q}e^{Q}v$ and then $L_q(e^{-Q}e^{Q}v) = e^{-Q}f$. By Proposition 5.8, we have

$$e^{-Q}L_{q_0}e^{Q}v = L_q(e^{-Q}e^{Q}v) = e^{-Q}f,$$

that is, $L_{q_0}e^{Q}v = f$.

Suppose now that L_{q_0} is globally solvable and let $f \in \mathcal{J}_q$. By the definition of $\mathcal{J}_q$, we have $e^{Q}f \in \mathcal{K}_{q_0}$ and by the global solvability of L_{q_0}, there exists $u \in \mathcal{D}'(G)$ such that $L_{q_0}u = e^{Q}f$, that is, $e^{-Q}L_{q_0}u = f$. By Proposition 5.8, we get $L_q e^{-Q}u = f$.

□

Corollary 5.11. *If L_q is globally hypoelliptic then L_q is globally solvable.*

Proof. Suppose that L_q is globally hypoelliptic. By Proposition 5.8 the operator L_{q_0} is globally hypoelliptic, so by Corollary 5.4, L_{q_0} is globally solvable. Finally, by Proposition 5.10, we conclude that L_q is globally solvable. □

Corollary 5.12. *L_q is globally hypoelliptic modulo* $\ker L_q$ *if and only if L_q is globally solvable.*

Example 5.13. *Let $G = \mathbb{T}^2$ and consider*

$$L_q = \partial_t + \partial_x + q(t,x),$$

where $q(t,x) = \sin(t+x)$. For $Q(t,x) = -\frac{1}{2}\cos(t+x)$ we have $(\partial_t + \partial_x)Q(t,x) = q(t,x) - q_0$, where $q_0 = 0$. Since $L_{q_0} = \partial_t + \partial_x$ is not globally hypoelliptic, we conclude by Proposition 5.8 that L_q is not globally hypoelliptic. On the other hand, the operator L_{q_0} is globally solvable, so that L_q is globally solvable.

For $q(t,x) = \sin(t+x) + 1$, we have $q_0 = 1$ and by Theorem 5.1 we have that L_{q_0} is globally hypoelliptic and then L_q is globally hypoelliptic, which implies that L_q is also globally solvable.

Example 5.14. *Let $G = \mathbb{S}^3$. We can identify $\mathbb{S}^3$ with* $\mathrm{SU}(2)$*, and the Euler's angle parametrization of* $\mathrm{SU}(2)$ *is given by*

$$x(\phi,\theta,\psi) = \begin{pmatrix} \cos\left(\frac{\theta}{2}\right) e^{i(\phi+\psi)/2} & i\sin\left(\frac{\theta}{2}\right) e^{i(\phi-\psi)/2} \\ i\sin\left(\frac{\theta}{2}\right) e^{-i(\phi-\psi)/2} & \cos\left(\frac{\theta}{2}\right) e^{-i(\phi+\psi)/2} \end{pmatrix} \in \mathrm{SU}(2),$$

where $0 \leq \phi < 2\pi$, $0 \leq \theta \leq \pi$ and $-2\pi \leq \psi < 2\pi$. Hence, the trace function on $\mathrm{SU}(2)$ *in Euler's angles is given by*

$$\mathrm{tr}(x(\phi,\theta,\psi)) = 2\cos\left(\tfrac{\theta}{2}\right)\cos\left(\tfrac{\phi+\psi}{2}\right).$$

Consider the operator

$$L_q = X + q(x),$$

where X is the same vector field from Example 3.11 and $q(x) = h(x) + \sqrt{2}$, where $h : \mathbb{S}^3 \to \mathbb{C}$ is expressed in Euler's angles by

$$h(x(\phi,\theta,\psi)) = -\cos\left(\tfrac{\theta}{2}\right)\sin\left(\tfrac{\phi+\psi}{2}\right). \tag{5.4}$$

The operator X in Euler's angles is the operator ∂_ψ and then we have

$$X\,\mathrm{tr}(x) = q(x) - \sqrt{2}.$$

By Example 5.6, the operator $L_{q_0} = X + \sqrt{2}$ is globally hypoelliptic, then by Proposition 5.8 we have that L_q is globally hypoelliptic, which implies that L_q is also globally solvable.

For $q(x) = h(x) + \frac{3}{2}i$ we have by Example 3.11 that the operator $L_{q_0} = X + \frac{3}{2}i$ is not globally hypoelliptic, then L_q is not globally hypoelliptic. On the other hand, the operator L_{q_0} is globally solvable, which implies that L_q is globally solvable and globally hypoelliptic modulo kernel.

6. A class of vector fields with variable coefficients

Let G_1 and G_2 be compact Lie groups, and set $G = G_1 \times G_2$. In this section we will characterize the global hypoellipticity and the global solvability for operators in the form

$$L_{aq} = X_1 + a(x_1)X_2 + q(x_1, x_2),$$

where $X_1 \in \mathfrak{g}_1$, $X_2 \in \mathfrak{g}_2$, $a \in C^\infty(G_1)$ is a real-valued function, and $q \in C^\infty(G)$. First, let us consider the case where $q \equiv 0$.

6.1. **Normal form.** Let

$$L_a = X_1 + a(x_1)X_2,$$

where $X_1 \in \mathfrak{g}_1$, $X_2 \in \mathfrak{g}_2$ and $a \in C^\infty(G_1)$ is a real-valued function. If $L_a u = f \in C^\infty(G)$, taking the partial Fourier coefficients with respect to the second variable, we obtain

$$\widehat{L_a u}(x_1, \eta)_{rs} = X_1 \widehat{u}(x_1, \eta)_{rs} + i\mu_r(\eta) a(x_1) \widehat{u}(x_1, \eta)_{rs} = \widehat{f}(x_1, \eta)_{rs},$$

for all $[\eta] \in \widehat{G}_1$, $1 \leq r, s \leq d_\eta$. The idea now is to find $\varphi(\,\cdot\,, \eta)_{rs} \neq 0$ such that

$$v(x_1, \eta)_{rs} = \varphi(x_1, \eta)_{rs} \widehat{u}(x_1, \eta)_{rs}$$

satisfies

$$X_1 v(x_1, \eta)_{rs} + i\mu_r(\eta) a_0 v(x_1, \eta)_{rs} = \varphi(x_1, \eta)_{rs} \widehat{f}(x_1, \eta)_{rs} := g(x_1, \eta)_{rs},$$

for all $[\eta] \in \widehat{G}_1$, $1 \le r, s \le d_\eta$, for some $a_0 \in \mathbb{R}$. So

$$\begin{aligned}
\varphi(x_1,\eta)_{rs}\widehat{f}(x_1,\eta)_{rs} &= X_1(\varphi(x_1,\eta)_{rs}\widehat{u}(x_1,\eta)_{rs}) + i\mu_r(\eta)a_0\varphi(x_1,\eta)_{rs}\widehat{u}(x_1,\eta)_{rs}\\
&= X_1(\varphi(x_1,\eta)_{rs})\widehat{u}(x_1,\eta)_{rs}\\
&\quad + \varphi(x_1,\eta)_{rs}(X_1\widehat{u}(x_1,\eta)_{rs}) + i\mu_r(\eta)a_0\varphi(x_1,\eta)_{rs}\widehat{u}(x_1,\eta)_{rs}\\
&= X_1(\varphi(x_1,\eta)_{rs})\widehat{u}(x_1,\eta)_{rs}\\
&\quad + \varphi(x_1,\eta)_{rs}\big((X_1\widehat{u}(x_1,\eta)_{rs}) + i\mu_r(\eta)a(x_1)\widehat{u}(x_1,\eta)_{rs}\big)\\
&\quad - i\mu_r(\eta)(a(x_1)-a_0)\varphi(x_1,\eta)_{rs}\widehat{u}(x_1,\eta)_{rs}\\
&= X_1(\varphi(x_1,\eta)_{rs})\widehat{u}(x_1,\eta)_{rs} + \varphi(x_1,\eta)_{rs}\widehat{f}(x_1,\eta)_{rs}\\
&\quad - i\mu_r(\eta)(a(x_1)-a_0)\varphi(x_1,\eta)_{rs}\widehat{u}(x_1,\eta)_{rs}.
\end{aligned}$$

Thus, if $\widehat{u}(x_1,\eta)_{rs} \neq 0$, we have

$$X_1\varphi(x_1,\eta)_{rs} = i\mu_r(\eta)(a(x_1)-a_0)\varphi(x_1,\eta)_{rs}. \tag{6.1}$$

Suppose that there exists $A \in C^\infty(G_1)$ such that

$$X_1A(x_1) = a(x_1) - a_0. \tag{6.2}$$

We can assume that A is a real-valued smooth function. Notice that

$$\begin{aligned}
\int_{G_1} X_1A(x_1)\,dx_1 &= \int_{G_1} \frac{d}{ds}A(x_1\exp(sX_1))\Big|_{s=0}\,dx_1\\
&= \frac{d}{ds}\int_{G_1} A(x_1\exp(sX_1))\,dx_1\Big|_{s=0}\\
&= \frac{d}{ds}\int_{G_1} A(x_1)\,dx_1\Big|_{s=0} = 0.
\end{aligned}$$

So

$$0 = \int_{G_1} X_1A(x_1)\,dx_1 = \int_{G_1}(a(x_1)-a_0)\,dx_1.$$

Therefore $a_0 = \displaystyle\int_{G_1} a(x_1)\,dx_1$ and the equation (6.1) becomes

$$X_1\varphi(x_1,\eta)_{rs} = i\mu_r(\eta)(X_1A)(x_1)\varphi(x_1,\eta)_{rs}, \tag{6.3}$$

and by Lemma 5.7, the function

$$\varphi(x_1,\eta)_{rs} = e^{i\mu_r(\eta)A(x_1)}$$

is a solution of (6.3).

Remark 6.1. *When G_1 is the one-dimensional torus, the operator $X_1 = \partial_t$ is globally solvable and $a - a_0$ belongs to the set of admissible functions, therefore the assumption over the existence of such function A satisfying (6.2) is verified, for any $a \in C^\infty(G_1)$. However, for other compact Lie groups, including higher-dimensional torus and the sphere $\mathbb{S}^3$, it is not difficult to construct examples of a function a for which there is no A satisfying (6.2).*

Define the operator Ψ_a as

$$\Psi_a u(x_1, x_2) := \sum_{[\eta]\in\widehat{G_2}} d_\eta \sum_{r,s=1}^{d_\eta} e^{i\mu_r(\eta)A(x_1)} \widehat{u}(x_1,\eta)_{rs}\, \eta_{sr}(x_2). \tag{6.4}$$

The next lemma is a technical result necessary to show that the operator Ψ_a is well-defined.

Lemma 6.2. *Let G be a compact group, $f \in C^\infty(G)$, and $z \in \mathbb{C}$ with $|z| \geq 1$. Let $\{Y_1, \cdots, Y_d\}$ be a basis for $\mathfrak{g}$. Then for all $\beta \in \mathbb{N}_0^d$, there exists $C_\beta > 0$ such that*

$$|\partial^\beta e^{zf(x)}| \leq C_\beta |z|^{|\beta|} e^{\mathrm{Re}(zf(x))}, \quad \forall x \in G, \tag{6.5}$$

with ∂^β as in (2.3).

Proof. Let us proceed by induction on $|\beta|$.

For $|\beta| = 0$, we have

$$|\partial^\beta e^{zf(x)}| = |e^{zf(x)}| = e^{\mathrm{Re}(zf(x))}.$$

Suppose now that (6.5) holds for every $\gamma \in \mathbb{N}_0^d$ with $|\gamma| \leq k$ and let $\beta \in \mathbb{N}_0^d$ with $|\beta| = k+1$. We can write $\beta = \gamma + e_j$, for some $j = 1, \cdots, d$ and $|\gamma| = k$. So

$$\begin{aligned}|\partial^\beta e^{zf(x)}| = |\partial^\gamma Y_j e^{zf(x)}| = |\partial^\gamma (zY_j f(x) e^{zf(x)})| &\leq |z| \sum_{\gamma'+\gamma''=\gamma} |\partial^{\gamma'} Y_j f(x)|\, |\partial^{\gamma''} e^{zf(x)}| \\ &\leq C_\beta |z|^{|\beta|} e^{\mathrm{Re}(zf(x))}.\end{aligned}$$

□

Remark 6.3. *We have a similar result for the case where $|z| \leq 1$. In this case, the power of $|z|$ on the estimate (6.5) is equal to 1 for every $\beta \in \mathbb{N}_0$, i.e., for all $\beta \in \mathbb{N}_0^d$ there exists C_β such that*

$$|\partial^\beta e^{zf(x)}| \leq C_\beta |z| e^{\mathrm{Re}(zf(x))}, \quad \forall x \in G.$$

Proposition 6.4. *The operator Ψ_a defined in (6.4) is an automorphism of $C^\infty(G)$ and of $\mathcal{D}'(G)$.*

Proof. To demonstrate this proposition we will use results about partial Fourier series developed on [28].

First of all, notice that Ψ_{-a} is the inverse of Ψ_a, therefore we only need to prove that $\Psi_a(C^\infty(G)) = C^\infty(G)$ and $\Psi_a(\mathcal{D}'(G)) = \mathcal{D}'(G)$.

Let $\beta \in \mathbb{N}_0$ and $u \in C^\infty(G)$. We will show that $\Psi_a u \in C^\infty(G)$. Notice that $\widehat{\Psi_a u}(x_1,\eta)_{rs} = e^{i\mu_r(\eta)A(x_1)}\widehat{u}(x_1,\eta)_{rs}$ and $\mu_r(\eta)A(x_1) \in \mathbb{R}$, for all $[\eta] \in \widehat{G_2}$, $1 \le r \le d_\eta$ and $x_1 \in G_1$. Using (6.5) we obtain

$$\begin{aligned}
|\partial^\beta \widehat{\Psi_a u}(x_1,\eta)_{rs}| &= |\partial^\beta (e^{i\mu_r(\eta)A(x_1)}\widehat{u}(x_1,\eta)_{rs})| \\
&= \left| \sum_{\beta'+\beta''=\beta} \partial^{\beta'} e^{i\mu_r(\eta)A(x_1)} \partial^{\beta''}\widehat{u}(x_1,\eta)_{rs} \right| \\
&\le \sum_{\beta'+\beta''=\beta} \left|\partial^{\beta'} e^{i\mu_r(\eta)A(x_1)}\right| \left|\partial^{\beta''}\widehat{u}(x_1,\eta)_{rs}\right| \\
&\le \sum_{\beta'+\beta''=\beta} C_{\beta'} |\mu_r(\eta)|^{|\beta'|} \left|\partial^{\beta''}\widehat{u}(x_1,\eta)_{rs}\right|.
\end{aligned}$$

Since $u \in C^\infty(G)$ and $|\mu_r(\eta)| \le \langle\eta\rangle$, it is easy to see that given $N > 0$, there exists $C_{\beta N}$ such that

$$|\partial^\beta \widehat{\Psi_a u}(x_1,\eta)_{rs}| \le C_{\beta N}\langle\eta\rangle^{-N}.$$

Therefore $\Psi_a u \in C^\infty(G)$. The distribution case is analogous. □

Proposition 6.5. *Let $a \in C^\infty(G_1)$, $a_0 := \int_{G_1} a(x_1)\,dx_1$, and consider the operator $L_{a_0} = X_1 + a_0 X_2$. Assume that there exists $A \in C^\infty(G_1)$ such that $X_1 A = a - a_0$. Then we have*

$$L_{a_0} \circ \Psi_a = \Psi_a \circ L_a$$

in both $C^\infty(G)$ and in $\mathcal{D}'(G)$, where Ψ_a is given in (6.4).

Proof. Let us show that for any $u \in C^\infty(G)$ we have

$$\widehat{L_{a_0}(\Psi_a u)}(x_1,\eta)_{rs} = \widehat{\Psi_a(L_a u)}(x_1,\eta)_{rs},$$

for all $x_1 \in G_1$, $[\eta] \in \widehat{G}_2$, $1 \le r,s \le d_\eta$.

Indeed,

$$\begin{aligned}\widehat{L_{a_0}(\Psi_a u)}(x_1,\eta)_{rs} &= \widehat{X_1\Psi_a u}(x_1,\eta)_{rs} + a_0\widehat{X_2\Psi_a u}(x_1,\eta)_{rs}\\ &= X_1\widehat{\Psi_a u}(x_1,\eta)_{rs} + i\mu_r(\eta)a_0\widehat{\Psi_a u}(x_1,\eta)_{rs}\\ &= X_1(e^{i\mu_r(\eta)A(x_1)}\widehat{u}(x_1,\eta)_{rs}) + i\mu_r(\eta)a_0e^{i\mu_r(\eta)A(x_1)}\widehat{u}(x_1,\eta)_{rs}\\ &= (X_1e^{i\mu_r(\eta)A(x_1)})\widehat{u}(x_1,\eta)_{rs} + e^{i\mu_r(\eta)A(x_1)}(X_1\widehat{u}(x_1,\eta)_{rs})\\ &\quad + i\mu_r(\eta)a_0e^{i\mu_r(\eta)A(x_1)}\widehat{u}(x_1,\eta)_{rs}.\end{aligned}$$

By (6.3) we have

$$\begin{aligned}\widehat{L_{a_0}(\Psi_a u)}(x_1,\eta)_{rs} &= i\mu_r(\eta)(a(x_1)-a_0)e^{i\mu_r(\eta)A(x_1)}\widehat{u}(x_1,\eta)_{rs}\\ &\quad + e^{i\mu_r(\eta)A(x_1)}(X_1\widehat{u}(x_1,\eta)_{rs})\\ &\quad + i\mu_r(\eta)a_0e^{i\mu_r(\eta)A(x_1)}\widehat{u}(x_1,\eta)_{rs}\\ &= e^{i\mu_r(\eta)A(x_1)}(X_1\widehat{u}(x_1,\eta)_{rs} + i\mu_r(\eta)a(x_1)\widehat{u}(x_1,\eta)_{rs})\\ &= e^{i\mu_r(\eta)A(x_1)}\widehat{L_a u}(x_1,\eta)_{rs}\\ &= \widehat{\Psi_a(L_a u)}(x_1,\eta)_{rs}.\end{aligned}$$

The same is true when $u\in\mathcal{D}'(G)$. □

6.2. **Global properties.** Recall that the operator L_{a_0} is globally solvable when $L_{a_0}(\mathcal{D}'(G)) = \mathcal{K}_{a_0}$, where

$$\begin{aligned}\mathcal{K}_{a_0} := \{w\in\mathcal{D}'(G);\, \widehat{\widehat{w}}(\xi,\eta)_{mn_{rs}} = 0 \text{ whenever } \lambda_m(\xi)+a_0\mu_r(\eta)=0,\\ \text{for all } 1\le m,n\le d_\xi, 1\le r,s\le d_\eta\}.\end{aligned}$$

We will say that L_a is globally solvable if $L_a(\mathcal{D}'(G)) = \mathcal{J}_a$, where

$$\mathcal{J}_a := \{v\in\mathcal{D}'(G);\, \Psi_{-a}v\in\mathcal{K}_{a_0}\}.$$

Proposition 6.6. *The operator L_a is globally hypoelliptic (resp. globally hypoelliptic modulo* $\ker L_a$*) if and only if L_{a_0} is globally hypoelliptic (resp. globally hypoelliptic modulo* $\ker L_{a_0}$*). Similarly, the operator L_a is globally solvable if and only if L_{a_0} is globally solvable.*

Proof. The proof is analogous to the demonstration of Propositions 5.8 and 5.10. □

Corollary 6.7. *If L_a is globally hypoelliptic, then L_a is globally solvable.*

Corollary 6.8. *The operator L_a is globally hypoelliptic modulo* $\ker L_a$ *if and only if L_a is globally solvable.*

Example 6.9. *Let $G = \mathbb{T}^1 \times G_2$, where G_2 is a compact Lie group and $a \in C^\infty(\mathbb{T}^1)$ a real-valued function. Let*

$$L_a := \partial_t + a(t)X_2,$$

where $X_2 \in \mathfrak{g}_2$,

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} a(t)\,dt,$$

and define

$$A(t) = \int_0^t a(s)\,ds - a_0 t.$$

Notice that $\partial_t A(t) = a(t) - a_0$, for all $t \in \mathbb{T}^1$. Taking the Fourier coefficients with respect to the second variable, we obtain

$$\partial_t \widehat{u}(t,\eta)_{rs} + ia(t)\mu_r(\eta)\widehat{u}(t,\eta)_{rs} = \widehat{f}(t,\eta)_{rs}.$$

Define the operator Ψ_a as

$$\Psi_a u(t,x_2) := \sum_{[\eta]\in\widehat{G_2}} d_\eta \sum_{r,s=1}^{d_\eta} e^{i\mu_r(\eta)A(t)}\widehat{u}(t,\eta)_{rs}\,\eta_{sr}(x_2).$$

By Proposition 6.4, the operator Ψ_a is an automorphism of $C^\infty(G)$ and of $\mathcal{D}'(G)$ and it holds

$$\Psi_a \circ L = L_{a_0} \circ \Psi_a.$$

Thus, the operator L_a is globally hypoelliptic if and only if L_{a_0} is globally hypoelliptic. Moreover, the operator L_a is globally solvable if and only if L_{a_0} is globally solvable.

For instance, for $a(t) = \sin(t) + \sqrt{2}$, we have $a_0 = \sqrt{2}$ and $A(t) = -\cos(t)$. Take $G_2 = \mathbb{T}^1$. We know by Example 3.9 that the operator

$$L_{a_0} = \partial_t + \sqrt{2}\partial_x$$

is globally hypoelliptic, because $\sqrt{2}$ is an irrational non-Liouville number. Hence,

$$L_a = \partial_t + (\sin(t) + \sqrt{2})\partial_x$$

is globally hypoelliptic and then globally solvable.

Take now $G_2 = \mathbb{S}^3$. By Example 3.11, we know that

$$L_{a_0} = \partial_t + \sqrt{2}X$$

is not globally hypoelliptic but it is globally solvable. Therefore

$$L_a = \partial_t + (\sin(t) - \sqrt{2})X$$

is not globally hypoelliptic and it is globally solvable.

Remark 6.10. *We had supposed that given a function $a \in C^\infty(G_1)$ there exists a function $A \in C^\infty(G_1)$ and $a_0 \in \mathbb{R}$ such that $X_1 A = a - a_0$, that is, X_1 is C^∞–cohomology free on G_1 (see Definition 4.2).*

Conjecture 6.11 (Katok)**.** *If a closed, connected, orientable manifold M admits a C^∞–cohomology free vector field X, then M is diffeomorphic to a torus and X is smoothly conjugate to a Diophantine vector field.*

As mentioned in Section 4, G. Forni has proved in [16] *that this conjecture is equivalent to the Greenfield-Wallach conjecture 4.1 and in view of its validity on compact Lie groups, it was necessary to add in the hypothesis the existence of such A satisfying $X_1 A = a - a_0$. Otherwise, the above results would be valid only for the case where G_1 is a torus.*

6.3. **Perturbations.** We can now combine what was made in Section 5.1 to study the operator

$$L_{aq} = X_1 + a(x_1)X_2 + q(x_1, x_2),$$

where $X_1 \in \mathfrak{g}_1$, $X_2 \in \mathfrak{g}_2$, $a \in C^\infty(G_1)$ is a real-valued function, and $q \in C^\infty(G)$. Furthermore, we will assume that there exists $Q \in C^\infty(G)$ satisfying

$$(X_1 + a(x_1)X_2)Q = q - q_0.$$

By Proposition 5.8 we have

$$L_{aq} \circ e^{-Q} = e^{-Q} \circ L_{aq_0},$$

where $L_{aq_0} = X_1 + a(x_1)X_2 + q_0$.

It follows from Proposition 6.5 that

$$L_{a_0 q_0} \circ \Psi_a = \Psi_a \circ L_{aq_0},$$

where $L_{a_0q_0} = X_1 + a_0X_2 + q_0$. Thus,

$$L_{aq} \circ e^{-Q} \circ \Psi_a = e^{-Q} \circ L_{aq_0} \circ \Psi_a = e^{-Q} \circ \Psi_a \circ L_{a_0q_0}.$$

We say that L_{aq} is globally solvable if $L_{aq}(\mathcal{D}'(G)) = \mathcal{J}_{aq}$, where

$$\mathcal{J}_{aq} := \{v \in \mathcal{D}'(G);\ \Psi_{-a}e^{Q}v \in \mathcal{K}_{a_0q_0}\}$$

and

$$\mathcal{K}_{a_0q_0} := \{w \in \mathcal{D}'(G); \widehat{\widehat{w}}(\xi,\eta)_{mn_{rs}} = 0 \text{ whenever } \lambda_m(\xi) + a_0\mu_r(\eta) - iq_0 = 0, \\ \text{for all } 1 \leq m,n \leq d_\xi, 1 \leq r,s \leq d_\eta\}.$$

The next results are consequences of what was done previously.

Proposition 6.12. *The operator L_{aq} is globally hypoelliptic (resp. globally hypoelliptic modulo $\ker L_{aq}$) if and only if $L_{a_0q_0}$ is globally hypoelliptic (resp. globally hypoelliptic modulo $\ker L_{a_0q_0}$). Similarly, the operator L_{aq} is globally solvable if and only if $L_{a_0q_0}$ is globally solvable.*

Corollary 6.13. *If L_{aq} is globally hypoelliptic, then $L_{a_0q_0}$ is globally solvable.*

Corollary 6.14. *The operator L_{aq} is hypoelliptic modulo $\ker L_{aq}$ if and only if $L_{a_0q_0}$ is globally solvable.*

Example 6.15. *Let $G = \mathbb{T}^1 \times \mathbb{S}^3$ and $X \in \mathfrak{s}^3$ as in the Example 3.11. Let $a(t) = \sin(t) + \sqrt{2}$ and $q(t,x) = \cos(t) + (\sin(t) + \sqrt{2})h(x) + 1$, with h as in Example 5.14. Here, $a_0 = \sqrt{2}$ and $q_0 = 1$. Notice that the function $Q(t,x) = \sin(t) + \mathrm{tr}(x)$ satisfies $(\partial_t + a(t)X)Q(t,x) = q(t,x)$. By Theorem 5.1 the operator*

$$L_{a_0q_0} = \partial_t + \sqrt{2}X + 1$$

is globally hypoelliptic (see Example 3.11) and then the operator

$$L_{aq} = \partial_t + (\sin(t) + \sqrt{2})X + (\cos(t) + (\sin(t) + \sqrt{2})h(x) + 1)$$

is globally hypoelliptic, which implies that L_{aq} is globally solvable.

For $q(t,x) = \cos(t) + (\sin(t) + \sqrt{2})h(x) + i$, the operator

$$L_{a_0q_0} = \partial_t + \sqrt{2}X + i$$

is not globally hypoelliptic (see Example 5.6) and then the operator

$$L_{aq} = \partial_t + (\sin(t) + \sqrt{2})X + (\cos(t) + (\sin(t) + \sqrt{2})h(x) + i)$$

is not globally hypoelliptic. However, since $\sqrt{2}$ *is an irrational non-Liouville number, the operator* $L_{a_0q_0}$ *is globally solvable, which implies that* L_{aq} *is globally solvable.*

Acknowledgments

The authors would like to thank Ricardo Paleari da Silva for comments and suggestions.

This study was financed in part by the Coordenação de Aperfeiçoamento de Pessoal de Nível Superior - Brasil (CAPES) - Finance Code 001. The last author was also supported by the FWO Odysseus 1 grant G.0H94.18N: Analysis and Partial Differential Equations, by the Leverhulme Grant RPG-2017-151, and by EPSRC Grant EP/R003025/1.

References

1. G. Araújo, *Global regularity and solvability of left-invariant differential systems on compact lie groups*, Annals of Global Analysis and Geometry (2019).
2. A. P. Bergamasco, *Perturbations of globally hypoelliptic operators*, J. Differential Equations **114** (1994), no. 2, 513–526. MR 1303037
3. ———, *Remarks about global analytic hypoellipticity*, Trans. Amer. Math. Soc. **351** (1999), no. 10, 4113–4126. MR 1603878
4. A. P. Bergamasco, P. D. Cordaro, and P. A. Malagutti, *Globally hypoelliptic systems of vector fields*, J. Funct. Anal. **114** (1993), no. 2, 267–285. MR 1223704
5. A. P. Bergamasco, P. D. Cordaro, and G. Petronilho, *Global solvability for a class of complex vector fields on the two-torus*, Comm. Partial Differential Equations **29** (2004), no. 5-6, 785–819. MR 2059148
6. W. Chen and M. Y. Chi, *Hypoelliptic vector fields and almost periodic motions on the torus* T^n, Comm. Partial Differential Equations **25** (2000), no. 1-2, 337–354. MR 1737551
7. A. Dasgupta and M. Ruzhansky, *Gevrey functions and ultradistributions on compact Lie groups and homogeneous spaces*, Bull. Sci. Math. **138** (2014), no. 6, 756–782. MR 3251455
8. ———, *Eigenfunction expansions of ultradifferentiable functions and ultradistributions*, Trans. Amer. Math. Soc. **368** (2016), no. 12, 8481–8498. MR 3551578
9. ———, *Eigenfunction expansions of ultradifferentiable functions and ultradistributions. II. Tensor representations*, Trans. Amer. Math. Soc. Ser. B **5** (2018), 81–101. MR 3798588

10. F. de Ávila Silva, R. B. Gonzalez, A. Kirilov, and C. de Medeira, *Global hypoellipticity for a class of pseudo-differential operators on the torus*, Journal of Fourier Analysis and Applications (2018).
11. F. de Ávila Silva, T. Gramchev, and A. Kirilov, *Global hypoellipticity for first-order operators on closed smooth manifolds*, J. Anal. Math. **135** (2018), no. 2, 527–573. MR 3829609
12. Fernando de Ávila Silva and Alexandre Kirilov, *Perturbations of globally hypoelliptic operators on closed manifolds*, J. Spectr. Theory **9** (2019), no. 3, 825–855. MR 4003543
13. J. Delgado and M. Ruzhansky, *Fourier multipliers, symbols, and nuclearity on compact manifolds*, J. Anal. Math. **135** (2018), no. 2, 757–800. MR 3829616
14. V. Fischer and M. Ruzhansky, *Quantization on nilpotent Lie groups*, Progress in Mathematics, vol. 314, Birkhäuser/Springer, [Cham], 2016. MR 3469687
15. L. Flaminio, G. Forni, and F. Rodriguez Hertz, *Invariant distributions for homogeneous flows and affine transformations*, J. Mod. Dyn. **10** (2016), 33–79. MR 3485931
16. G. Forni, *On the Greenfield-Wallach and Katok conjectures in dimension three*, Geometric and probabilistic structures in dynamics, Contemp. Math., vol. 469, Amer. Math. Soc., Providence, RI, 2008, pp. 197–213. MR 2478471
17. R. B. Gonzalez, *On certain non-hypoelliptic vector fields with finite-codimensional range on the three-torus*, Ann. Mat. Pura Appl. (4) **197** (2018), no. 1, 61–77. MR 3747521
18. T. Gramchev, P. Popivanov, and M. Yoshino, *Global solvability and hypoellipticity on the torus for a class of differential operators with variable coefficients*, Proc. Japan Acad. Ser. A Math. Sci. **68** (1992), no. 3, 53–57. MR 1167986
19. S. J. Greenfield and N. R. Wallach, *Global hypoellipticity and Liouville numbers*, Proc. Amer. Math. Soc. **31** (1972), 112–114. MR 0296508
20. ———, *Globally hypoelliptic vector fields*, Topology **12** (1973), 247–254. MR 0320502
21. ———, *Remarks on global hypoellipticity*, Trans. Amer. Math. Soc. **183** (1973), 153–164. MR 0400313
22. A. A. Himonas, *Global analytic and Gevrey hypoellipticity of sublaplacians under Diophantine conditions*, Proc. Amer. Math. Soc. **129** (2001), no. 7, 2061–2067. MR 1825918
23. A. A. Himonas and G. Petronilho, *On Gevrey regularity of globally C^∞ hypoelliptic operators*, J. Differential Equations **207** (2004), no. 2, 267–284. MR 2102665
24. ———, *On C^∞ and Gevrey regularity of sublaplacians*, Trans. Amer. Math. Soc. **358** (2006), no. 11, 4809–4820. MR 2231873
25. J. Hounie, *Globally hypoelliptic and globally solvable first-order evolution equations*, Trans. Amer. Math. Soc. **252** (1979), 233–248. MR 534120
26. ———, *Globally hypoelliptic vector fields on compact surfaces*, Comm. Partial Differential Equations **7** (1982), no. 4, 343–370. MR 652813
27. Alexandre Kirilov and Wagner A. A. de Moraes, *Global hypoellipticity for strongly invariant operators*, J. Math. Anal. Appl. **486** (2020), no. 1, 123878, 14. MR 4057004

28. Alexandre Kirilov, Wagner A. A. de Moraes, and Michael Ruzhansky, *Partial Fourier series on compact Lie groups*, Bull. Sci. Math. **160** (2020), 102853, 27. MR 4075513
29. G. Petronilho, *Global hypoellipticity, global solvability and normal form for a class of real vector fields on a torus and application*, Trans. Amer. Math. Soc. **363** (2011), no. 12, 6337–6349. MR 2833557
30. G. Petronilho and S. L. Zani, *Global s-solvability and global s-hypoellipticity for certain perturbations of zero order of systems of constant real vector fields*, J. Differential Equations **244** (2008), no. 9, 2372–2403. MR 2413846
31. M. Ruzhansky and V. Turunen, *Pseudo-differential operators and symmetries*, Pseudo-Differential Operators. Theory and Applications, vol. 2, Birkhäuser Verlag, Basel, 2010, Background analysis and advanced topics. MR 2567604
32. ———, *Quantization of pseudo-differential operators on the torus*, J. Fourier Anal. Appl. **16** (2010), no. 6, 943–982. MR 2737765
33. ———, *Global quantization of pseudo-differential operators on compact Lie groups,* SU(2)*, 3-sphere, and homogeneous spaces*, Int. Math. Res. Not. IMRN (2013), no. 11, 2439–2496. MR 3065085
34. M. Ruzhansky, V. Turunen, and J. Wirth, *Hörmander class of pseudo-differential operators on compact Lie groups and global hypoellipticity*, J. Fourier Anal. Appl. **20** (2014), no. 3, 476–499. MR 3217484
35. M. Ruzhansky and J. Wirth, *On multipliers on compact lie groups*, Functional Analysis and Its Applications **47** (2013), no. 1, 72–75.
36. R. T. Seeley, *Integro-differential operators on vector bundles*, Trans. Amer. Math. Soc. **117** (1965), 167–204. MR 0173174
37. M. E. Taylor, *Fourier series on compact Lie groups*, Proc. Amer. Math. Soc. **19** (1968), 1103–1105. MR 0231135

Universidade Federal do Paraná, Departamento de Matemática, C.P.19096, CEP 81531-990, Curitiba, Brazil

Email address: akirilov@ufpr.br

Universidade Federal do Paraná, Programa de Pós-Graduação de Matemática, C.P.19096, CEP 81531-990, Curitiba, Brazil

Email address: wagneramat@gmail.com

Ghent University, Department of Mathematics: Analysis, Logic and Discrete Mathematics, Ghent, Belgium and Queen Mary University of London, School of Mathematical Sciences, London, United Kingdom

Email address: Michael.Ruzhansky@ugent.be