Computation of Time Probability Distributions for the Occurrence of Uncertain Future Events

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Abstract

The determination of the time at which an event may take place in the future is a well-studied problem in a number of science and engineering disciplines. Indeed, for more than fifty years, researchers have tried to establish adequate methods to characterize the behaviour of dynamic systems in time and implement predictive decision-making policies. Most of these efforts intend to model the evolution in time of nonlinear dynamic systems in terms of stochastic processes; while defining the occurrence of events in terms of first-passage time problems with thresholds that could be either deterministic or probabilistic in nature. The random variable associated with the occurrence of such events has been determined in closed-form for a variety of specific continuous-time diffusion models, being most of the available literature motivated by physical phenomena. Unfortunately, literature is quite limited in terms of rigorous studies related to discrete-time stochastic processes, despite the tremendous amount of digital information that is currently being collected worldwide. In this regard, this article provides a mathematically rigorous formalization for the problem of computing the probability of occurrence of uncertain future events in both discrete- and

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continuous-time stochastic processes, by extending the notion of thresholds in first-passage time problems to a fully probabilistic notion of “uncertain events” and “uncertain hazard zones”. We focus on discrete-time applications by showing how to compute those probability measures and validate the proposed framework by comparing to the results obtained with Monte Carlo simulations; all motivated by the problem of fatigue crack growth prognosis.

Keywords: First-hitting time, First-passage time, Time of Failure probability, Remaining useful life, Fatigue crack prognosis

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1. Introduction

One of the motivations behind the use of mathematical models to characterize the evolution in time of dynamic systems is to provide the means to predict and anticipate the occurrence of possibly critical future events. Many different mathematical frameworks can be used for this purpose, and the “best” choice for a model structure will largely depend on the specific application domain. Dynamic models including an explicit characterization of uncertainty sources (e.g., those that include stochastic equations) are particularly suitable to quantify the risk associated with the occurrence of events, since they provide a rigorous mathematical framework for the computation of probability measures. In this context, the conditions that define the occurrence of *events* have been typically defined in terms of a “threshold”, so that the event of interest is always triggered when a scalar function of the system states reaches this threshold for the first time. Naturally, this implies the assumption that the requirements needed to trigger the occurrence of *events* can always be represented by a deterministic function of the system condition. As the system condition randomly evolves in time (i.e., the condition indicator is a stochastic process), a probability distribution for the threshold first hitting time is therefore induced: the *First-Passage Time* (FPT) [1–4] or *First-Hitting Time* (FHT) [5, 6] probability distribution. On the one hand, this concept is equivalent to *duration models* [7, 8] and, although understood in a different context, it is also analogous to *survival probability* in statistics [9–11]. On the other hand, in the engineering discipline of *Prognostics and Health Management* (PHM) these concepts are related to the *Remaining Useful Life* (RUL), *End-of-Life* (EoL), *Time-of-Failure* (ToF) and *Time-to-Failure* (TtF) probability distributions [12–14].

Efforts on finding analytical expressions for FPT probability distributions have been carried out on many disciplines and application domains such as in chemistry [15, 16], physics [17, 18], biology [19, 20], neurobiology [21, 22], epidemiology [23], psychology [24], finance [25, 26], economy [27, 28], reliability theory [29, 30], among others [1, 2]. Nonetheless, it is important to emphasize the fact that most of these research efforts have focused on continuous-time [31–37], rather than discrete-time systems [27, 38–40] (except the case of autoregressive models [39, 41–48]). In continuous-time systems, the FPT probability distribution constitutes the solution to particular *Stochastic Differential Equation* (SDE) with boundary conditions, which is typically solved using transformations [49–51] or on eigenfunction expansions [32, 50] (most of the times numerically approximated). Derivations of direct closed-form expressions are constrained to just a few standard cases related to Brownian motion, like in [52], and some other direct approximations [53–65]. Although it may be natural to think that events occur when some threshold or region is reached by a variable (or condition indicator) that is evolving in time, in some cases it is not straightforward to determine an appropriate value for this threshold. In this regard, and to the best of our knowledge, just a handful of contributions have aimed at incorporating the notion of random thresholds [66–71] (and solely for very specific types of stochastic processes).

1.1. Failure prognosis in the discipline of Prognostics and Health Management

Fundamental problems of interest in the modern engineering discipline of PHM are, on the one hand, the implementations of *Fault Detection and Diagnostics* (FDD) schemes and, on the other hand, the prediction of catastrophic system failures (i.e., failure prognosis). In this regard, it is noteworthy that a clear distinction should be made between the concepts of "faults" (abnormal conditions in which the systems is still operative) and catastrophic failures (which imply the total inoperatibility of a system), as it is indicated in recent and comprehensive surveys on FDD and failure prognostic approaches [72]. The concept of *hazard zone* [73] arose as an extension to the typical threshold standpoint found in FDD schemes by defining a likelihood over the state-space in regions suggesting faulty conditions. In [74, 75], there was an attempt to provide a more general insight, but it was restricted to Markov processes. Also, there was an underlying hypothesis of statistical independence in the proposed probability measures; although these measures have still proven to be useful to define a functional cost criterion for prognostic algorithm design [76]. Moreover, despite the fact that *hazard zones* are well known and accepted in the PHM community [13], the current state-of-the-art formalization of failure prognosis problem [77, 78] still defines failure events with the classical deterministic threshold approach, is restricted to events over Markov processes, lacks of mathematical demonstrations, and has led to inconsistencies when computing FPT probability distributions with methods different from those simulating complete state trajectories of systems [74].

1.2. Uncertainty characterization in Reliability-Based Optimization

The discipline of Reliability Analysis has been a precursor to PHM regarding the study of risks associated with the design and operation of engineering systems. Particularly, the contribution of Reliability-Based Optimization (RBO) [79–82] to this specific aspect is noteworthy. It is only natural, then, to scrutinize how events (or failures) have been defined in the specific context of RBO and the manner in which the concept of uncertainty have been incorporated into these definitions. As it will be shown below, RBO corresponds to a good example for the notion of thresholds-triggered events, even though the value associated with these thresholds might be random in nature and, thus, endowed with probability distributions.

RBO defines an optimization problem for the design of engineering systems under uncertainty. This problem can be formulated in different ways, although minimization of expected life time costs, and particularly corrective maintenance costs, is particularly relevant. Indeed, without loss of generality, the RBO problem can be formulated as [81]:

$$\begin{aligned} &\min_{z \in \Omega_z} \mathbb{E}\{C(z, \Phi)\} \\ s.t.\ h_i(z) \leq& 0 \quad ,\ i = 1, \dots, n_C \\ P_j(z) \leq& P_j^{tol},\ j = 1, \dots, n_P, \end{aligned} \tag{1}$$

where the objective function is the expectation of a cost $C(\cdot)$, which depends on the vector z (design variables) and a random vector Φ (uncertain parameters). The functions $h_i(\cdot)$ represent constraints to the problem, and $P_j(\cdot)$ denotes the probability of occurrence of a j-th event. Consequently,

$$P_j(z) = \mathbb{P}(g_j(z, \Phi) \leq 0) = \int_{\{\phi \in \Omega_\phi : g_j(z,\phi) \leq 0\}} p(\phi|z) d\phi, \tag{2}$$

where $p(\phi|z)$ is the joint probability density function of the random vector of uncertain parameters Φ. The function

$$g_j(z, \Phi) = B_j - r_j(z, \Phi) \tag{3}$$

is a *performance function* associated with the occurrence of a certain event. The function $r_j(z, \Phi)$ characterizes the response of the system (i.e., demand), whereas B_j corresponds to a –possibly random– threshold of maximum tolerance (i.e., capacity). Therefore, an event in RBO occurs if $g_j(z, \Phi) \leq 0$ or, equivalently, if $B_j \leq r_j(z, \Phi)$. As a result, if B_j is a random variable, the probability of occurrence for the j-th event is given by:

$$P_j(z) = \mathbb{P}(g_j(z, \Phi) \leq 0) \tag{4}$$

$$= \mathbb{P}(B_j \leq r_j(z, \Phi)) \tag{5}$$

$$= \int_{b_j \in \Omega_{B_j}} \mathbb{P}(b_j \leq r_j(z, \Phi)|b_j) p(b_j) db_j, \tag{6}$$

where $p(b_j)$ denotes a probability density function associated with a threshold of maximum tolerance b_j. As Eq. (6) indicates, the occurrence of events in RBO are triggered by situations in which a threshold is violated, although it is allowed to characterize the values of these thresholds using random variables.

1.3. Structure of the article and main contributions

In this paper we provide a mathematically rigorous formalization for the time of occurrence of uncertain future events, characterized over both discrete- and continuous-time stochastic processes by extending the classical deterministic threshold crossing standpoint to a probabilistic notion of *uncertain event*, equivalent to that of *uncertain hazard zone* in PHM [73]. Particularly, Section 2 presents explicit semi-closed expressions for the associated probability measures, which are derived and demonstrated using Probability Theory. In addition to these theoretical contributions, in Section 3 we present a friendly explanation of the practical implications related to these theoretical concepts, using for this purpose a case study based on the problem of crack growth prognostics (discrete-time). Results obtained using the proposed semi-closed expressions for the probability of failure are validated and compared with those obtained by using Monte Carlo simulations. Finally, in Section 4 we summarize the main conclusions of this research effort.

2. Occurrence Probability of Uncertain Future Events

Let us consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a measurable space $(\mathbb{X}, \Sigma)$. Also, let $X : \mathbb{T} \cup \{0\} \times \Omega \rightarrow \mathbb{X}$, $\mathbb{T} \in \{\mathbb{N}, \mathbb{R}_+\}$, be a stochastic process; and F_{X_τ} denote the respective probability measure in $(\mathbb{X}, \Sigma)$ induced by X_τ, $\tau \in \mathbb{T}$.

Definition 1. ***[Uncertain Event Process & Likelihood]*** *Let $\mathcal{E}$ denote an event of interest. An* ***uncertain event process*** *is defined as a function $E : \mathbb{T} \times \mathbb{X} \rightarrow \{\mathcal{E}, \mathcal{E}^c\}$ such that*

$$\mathbb{P}(E_\tau = \mathcal{E}) = \int_\Omega \mathbb{P}(E_\tau = \mathcal{E}, X_\tau(\omega)) d\mathbb{P}(\omega) \tag{7}$$

$$= \int_{\mathbb{X}_\tau} \mathbb{P}(E_\tau = \mathcal{E} | x_\tau) dF_{X_\tau}(x_\tau), \ \forall \tau \in \mathbb{T}. \tag{8}$$

Additionally, we define an ***uncertain event likelihood function*** *as $\mathbb{P}(E_\tau = \mathcal{E}|x) : \mathbb{T} \times \mathbb{X} \rightarrow [0, 1]$.*

Thus, an event process $\{E_\tau\}_{\tau\in\mathbb{T}}$ describes a random variable evolving in time associated with the occurrence of an uncertain event whose statistics are subjected only to those of a stochastic process $\{X_\tau\}_{\tau\in\mathbb{T}\cup\{0\}}$ evaluated at the same time instant. Indeed, given $i, j \in \mathbb{T}$, $i \neq j$, we have

$$\mathbb{P}(\{E_i = \mathcal{E}\}, \{E_j = \mathcal{E}\}|\{X_\tau\}_{\tau\in\mathbb{T}}) = \mathbb{P}(E_i = \mathcal{E}|X_i)\mathbb{P}(E_j = \mathcal{E}|X_j). \tag{9}$$

In general, E_τ is independent of any other variable as long as it is conditioned on X_τ, $\forall \tau \in \mathbb{T}$. This is a quite important property that will be used later.

Remark 1. ***[Time-variant definition of uncertain events]*** *The time index τ of the binary random variable E_τ, in the definition of uncertain event likelihood functions (see Definition 1), denotes a time dependence associated with the concept of uncertain event. For this reason, the definition of the probability $\mathbb{P}(E_\tau = \mathcal{E}|x)$ uses "x" as conditioning argument instead of "x_τ"; i.e., the system state "x" solely corresponds to an argument in the time-dependant likelihood function. In other words, the time dependency of the uncertain event likelihood function determines the manner in which the definition of the event of interest changes over time (see Section 3.2 for an example of a time-invariant definition of uncertain event likelihood function).*

Remark 2. ***[Particular case: Threshold]*** *According to Definition 1, determining $\mathbb{P}(E_\tau = \mathcal{E}|x)$ as a function of $x \in \mathbb{X}$ corresponds exactly to a probabilistic description of the occurrence of an uncertain event $\mathcal{E}$. By assuming $\mathbb{X} = \mathbb{R}$, for example, and defining $\mathbb{P}(E_\tau = \mathcal{E}|x) = \mathbb{1}_{\mathbb{X}_\mathcal{E}}(x)$, with $\mathbb{X}_\mathcal{E} = \{x \in \mathbb{R} : x > c, c \in \mathbb{R}\}$, we get the classical threshold crossing event setting studied so far in the literature, where $E_\tau(x)$ conditional to a fixed $x \in \mathbb{R}$ is no longer a random variable:*

$$E_\tau(x) = \begin{cases} \mathcal{E}, & x \in \mathbb{X}_\mathcal{E} \\ \mathcal{E}^c, & \sim . \end{cases} \tag{10}$$

Remark 3. ***[Hazard zone]*** *The uncertain event likelihood function $\mathbb{P}(E_\tau = \mathcal{E}|x)$ as a function of $x \in \mathbb{X}$ is* ***exactly*** *what is understood as hazard zone [73] in the discipline of PHM.*

Now, let us introduce a formal definition for the first time of occurrence of an *uncertain event*.

Definition 2. ***[First Event Time]*** *Let* $\{X_\tau\}_{\tau \in \mathbb{T}\cup\{0\}}$ *be a stochastic process and* $\{E_\tau\}_{\tau\in\mathbb{T}}$ *be an event process, respectively. The first time of occurrence of an event* $\mathcal{E}$ *after a time instant* $\tau_p \in \mathbb{T}\cup\{0\}$ *is defined as*

$$\tau_{\mathcal{E}}(\tau_p) := \inf\{\tau \in \mathbb{T} : \{\tau > \tau_p\} \wedge \{E_\tau = \mathcal{E}\}\}. \tag{11}$$

With these few definitions, the probability distribution associated to $\tau_{\mathcal{E}}$ can be mathematically formalized in a general way for both discrete- and continuous-time stochastic processes as follows.

2.1. Discrete-Time Stochastic Processes

Let $\{X_k\}_{k\in\mathbb{N}\cup\{0\}}$ be a stochastic process and $\{E_k\}_{k\in\mathbb{N}}$ be an uncertain event process. The probability mass function associated to $\tau_{\mathcal{E}} = \tau_{\mathcal{E}}(k_p)$, $k_p \in \mathbb{N}\cup\{0\}$, can be obtained using an expression with the same structure of *survival probability*, as shown below:

$$\mathbb{P}(\tau_{\mathcal{E}} = k) := \mathbb{P}\left(\{E_k = \mathcal{E}\}, \{E_j = \mathcal{E}^c\}_{j=k_p}^{k-1}\right) \tag{12}$$

$$= \mathbb{P}\left(E_k = \mathcal{E}\middle|\{E_j = \mathcal{E}^c\}_{j=k_p}^{k-1}\right)\mathbb{P}\left(\{E_j = \mathcal{E}^c\}_{j=k_p}^{k-1}\right) \tag{13}$$

$$\vdots$$

$$= \mathbb{P}\left(E_k = \mathcal{E}\middle|\{E_j = \mathcal{E}^c\}_{j=k_p}^{k-1}\right)\prod_{j=k_p+1}^{k-1}\mathbb{P}\left(E_j = \mathcal{E}^c\middle|\{E_i = \mathcal{E}^c\}_{i=k_p}^{j-1}\right)\cancelto{1}{\mathbb{P}\left(E_{k_p} = \mathcal{E}^c\right)} \tag{14}$$

$$= \mathbb{P}\left(E_k = \mathcal{E}|\tau_{\mathcal{E}} \geq k\right)\prod_{j=k_p+1}^{k-1}\mathbb{P}\left(E_j = \mathcal{E}^c|\tau_{\mathcal{E}} \geq j\right) \tag{15}$$

Alternatively, there is a recursive way to express $\mathbb{P}(\tau_{\mathcal{E}} = k)$ according to [27], which is developed below but under the generalized notion of *uncertain event* presented in Definition 1. If an event occurs at time k, it implies that $k_p < \tau_{\mathcal{E}} \leq k$, and by the *Law of Total Probability* it can be obtained

$$\mathbb{P}(E_k = \mathcal{E}) = \sum_{j=k_p+1}^{k}\mathbb{P}(E_k = \mathcal{E}|\tau_{\mathcal{E}} = j)\mathbb{P}(\tau_{\mathcal{E}} = j) \tag{16}$$

$$= \mathbb{P}(E_k = \mathcal{E}|\tau_{\mathcal{E}} = k)\mathbb{P}(\tau_{\mathcal{E}} = k) + \sum_{j=k_p+1}^{k-1} \mathbb{P}(E_k = \mathcal{E}|\tau_{\mathcal{E}} = j)\mathbb{P}(\tau_{\mathcal{E}} = j) \tag{17}$$

Thus, provided $\mathbb{P}(E_k = \mathcal{E}|\tau_{\mathcal{E}} = k) = 1$, it yields

$$\mathbb{P}(\tau_{\mathcal{E}} = k) = \mathbb{P}(E_k = \mathcal{E}) - \sum_{j=k_p+1}^{k-1} \mathbb{P}(E_k = \mathcal{E}|\tau_{\mathcal{E}} = j)\mathbb{P}(\tau_{\mathcal{E}} = j) \tag{18}$$

Let us prove now the equivalence of both probability distributions presented above with the following lemma.

Lemma 1. *Let* $\{X_k\}_{k\in\mathbb{N}\cup\{0\}}$ *be a stochastic process and* $\{E_k\}_{k\in\mathbb{N}}$ *be an event process, respectively. Let also* $\tau_{\mathcal{E}} = \tau_{\mathcal{E}}(k_p)$, $k_p \in \mathbb{N}\cup\{0\}$. *The mapping* $\mathbb{P}(\tau_{\mathcal{E}} = \cdot) : \mathbb{N} \to [0,1]$ *can be either defined as*

$$\mathbb{P}(\tau_{\mathcal{E}} = k) := \mathbb{P}\left(E_k = \mathcal{E}|\tau_{\mathcal{E}} \geq k\right) \prod_{j=k_p+1}^{k-1} \mathbb{P}\left(E_j = \mathcal{E}^c|\tau_{\mathcal{E}} \geq j\right) \tag{19}$$

as well as

$$\mathbb{P}(\tau_{\mathcal{E}} = k) := \mathbb{P}(E_k = \mathcal{E}) - \sum_{j=k_p+1}^{k-1} \mathbb{P}(E_k = \mathcal{E}|\tau_{\mathcal{E}} = j)\mathbb{P}(\tau_{\mathcal{E}} = j). \tag{20}$$

Proof.

Using Eq. (20), Eq. (19) can be obtained as shown below

$$\begin{aligned}
\mathbb{P}(\tau_{\mathcal{E}} = k) &= \mathbb{P}(E_k = \mathcal{E}) - \sum_{j=k_p+1}^{k-1} \mathbb{P}(E_k = \mathcal{E}|\tau_{\mathcal{E}} = j)\mathbb{P}(\tau_{\mathcal{E}} = j) && (21)\\
&= \mathbb{P}(E_k = \mathcal{E}) - \sum_{j=k_p+1}^{k-1} \mathbb{P}(\tau_{\mathcal{E}} = j|E_k = \mathcal{E})\mathbb{P}(E_k = \mathcal{E}) && (22)\\
&= \mathbb{P}(E_k = \mathcal{E}) \left(1 - \sum_{j=k_p+1}^{k-1} \mathbb{P}(\tau_{\mathcal{E}} = j|E_k = \mathcal{E})\right) && (23)\\
&= \mathbb{P}(E_k = \mathcal{E}) \left(1 - \mathbb{P}(\tau_{\mathcal{E}} < k|E_k = \mathcal{E})\right) && (24)\\
&= \mathbb{P}(E_k = \mathcal{E})\mathbb{P}(\tau_{\mathcal{E}} \geq k|E_k = \mathcal{E}) && (25)\\
&= \mathbb{P}(E_k = \mathcal{E}|\tau_{\mathcal{E}} \geq k)\mathbb{P}(\tau_{\mathcal{E}} \geq k) && (26)
\end{aligned}$$

However, note that

$$\begin{aligned}
\mathbb{P}(\tau_{\mathcal{E}} \geq k) &= 1 - \mathbb{P}(\tau_{\mathcal{E}} < k) && (27)\\
&= 1 - \mathbb{P}(\tau_{\mathcal{E}} < k-1) - \mathbb{P}(\tau_{\mathcal{E}} = k-1) && (28)\\
&= \mathbb{P}(\tau_{\mathcal{E}} \geq k-1) - \mathbb{P}(\tau_{\mathcal{E}} = k-1) && (29)\\
&= \mathbb{P}\left(\{E_j = \mathcal{E}^c\}_{j=k_p}^{k-2}\right) - \mathbb{P}\left(\{E_{k-1} = \mathcal{E}\}, \{E_j = \mathcal{E}^c\}_{j=k_p}^{k-2}\right) && (30)\\
&= \mathbb{P}\left(\{E_j = \mathcal{E}^c\}_{j=k_p}^{k-2}\right) - \mathbb{P}\left(E_{k-1} = \mathcal{E} \middle| \{E_j = \mathcal{E}^c\}_{j=k_p}^{k-2}\right) \mathbb{P}\left(\{E_j = \mathcal{E}^c\}_{j=k_p}^{k-2}\right) && (31)\\
&= \mathbb{P}\left(\{E_j = \mathcal{E}^c\}_{j=k_p}^{k-2}\right)\left(1 - \mathbb{P}\left(E_{k-1} = \mathcal{E} \middle| \{E_j = \mathcal{E}^c\}_{j=k_p}^{k-2}\right)\right) && (32)\\
&= \mathbb{P}(\tau_{\mathcal{E}} \geq k-1)\mathbb{P}\left(E_{k-1} = \mathcal{E}^c \middle| \tau_{\mathcal{E}} \geq k-1\right) && (33)
\end{aligned}$$

By iterating this result, it yields

$$\mathbb{P}(\tau_{\mathcal{E}} \geq k) = \prod_{j=k_p+1}^{k-1} \mathbb{P}\left(E_j = \mathcal{E}^c | \tau_{\mathcal{E}} \geq j\right) \quad (34)$$

□

Before presenting the previous results in a formal theorem, please note the following. Since each E_k depends on X_k, using the property illustrated with Eq. (9) and the concept of *uncertain event likelihood function* introduced in Definition 1, it can be obtained:

$$\begin{aligned}
\mathbb{P}(\tau_{\mathcal{E}} = k) &= \mathbb{P}\left(E_k = \mathcal{E} | \tau_{\mathcal{E}} \geq k\right) \prod_{j=k_p+1}^{k-1} \mathbb{P}\left(E_j = \mathcal{E}^c | \tau_{\mathcal{E}} \geq j\right) && (35)\\
&= \int_{\mathbb{X}_{k_p+1:k}} \mathbb{P}\left(E_k = \mathcal{E} | \{\tau_{\mathcal{E}} \geq k\}, x_{k_p+1:k}\right) \prod_{j=k_p+1}^{k-1} \mathbb{P}\left(E_j = \mathcal{E}^c | \{\tau_{\mathcal{E}} \geq j\}, x_{k_p+1:k}\right) dF_{X_{k_p+1:k}}(x_{k_p+1:k}) && (36)\\
&= \int_{\mathbb{X}_{k_p+1:k}} \mathbb{P}\left(E_k = \mathcal{E} | x_k\right) \prod_{j=k_p+1}^{k-1} \mathbb{P}\left(E_j = \mathcal{E}^c | x_j\right) dF_{X_{k_p+1:k}}(x_{k_p+1:k}) && (37)\\
&= \int_{\mathbb{X}_{k_p+1:k}} \mathbb{P}\left(E_k = \mathcal{E} | x_k\right) \prod_{j=k_p+1}^{k-1} \left(1 - \mathbb{P}\left(E_j = \mathcal{E} | x_j\right)\right) dF_{X_{k_p+1:k}}(x_{k_p+1:k}). && (38)
\end{aligned}$$

This expression is used later in Section 3.2 to implement a procedure to compute these probabilities based on numeric methods.

Theorem 1. ***[First Event Time in Stochastic Processes]*** *Let $\{X_k\}_{k\in\mathbb{N}\cup\{0\}}$ be a stochastic process and $\{E_k\}_{k\in\mathbb{N}}$ be an uncertain event process, respectively. If the first time of occurrence of the event $\mathcal{E}$, $\tau_{\mathcal{E}} = \tau_{\mathcal{E}}(k_p)$, with $k_p \in \mathbb{N}\cup\{0\}$, is such that $\tau_{\mathcal{E}} < +\infty$, $\mathbb{P}-a.s.$, then the mapping $\mathbb{P}(\tau_{\mathcal{E}} = \cdot) : \mathbb{N} \to [0,1]$ exists and is well-defined in terms of its uncertain event likelihood function as:*

$$\mathbb{P}(\tau_{\mathcal{E}} = k) := \int_{\mathbb{X}_{k_p+1:k}} \mathbb{P}\left(E_k = \mathcal{E}|x_k\right) \prod_{j=k_p+1}^{k-1} \big(1 - \mathbb{P}\left(E_j = \mathcal{E}|x_j\right)\big) dF_{X_{k_p+1:k}}(x_{k_p+1:k}). \tag{39}$$

Therefore,

$$\mathbb{P}_{\mathcal{E}}(A) = \sum_{k\in A} \mathbb{P}(\tau_{\mathcal{E}} = k),\ \forall A \in 2^{\mathbb{N}}, \tag{40}$$

is a probability measure that defines the probability space $(\mathbb{N}, 2^{\mathbb{N}}, \mathbb{P}_{\mathcal{E}})$. Indeed, the following conditions hold:

1) $\mathbb{P}_{\mathcal{E}}(\mathbb{N}) = 1$.

2) $0 \le \mathbb{P}_{\mathcal{E}}(A) \le 1$, $\forall A \in 2^{\mathbb{N}}$.

3) $\mathbb{P}_{\mathcal{E}}(\cup_{k\in\mathbb{N}} A_k) = \sum_{k\in\mathbb{N}} \mathbb{P}_{\mathcal{E}}(A_k)$, $\forall \{A_k \in 2^{\mathbb{N}}\}_{k\in\mathbb{N}}$, with $A_i \cap A_j = \phi$, $\forall i \neq j$.

Proof.

1) Let us define $\{A_k\}_{k\in\mathbb{N}}$, $A_k = \{1, \ldots, k\}$, such that $A_k \nearrow \mathbb{N}$

$$\mathbb{P}_{\mathcal{E}}(A_k) = \sum_{j=1}^{k} \mathbb{P}(\tau_{\mathcal{E}} = j) = \mathbb{P}(\tau_{\mathcal{E}} < k+1) \tag{41}$$

$$\Rightarrow \lim_{k\to+\infty} \mathbb{P}_{\mathcal{E}}(A_k) = \lim_{k\to+\infty} \mathbb{P}(\tau_{\mathcal{E}} < k+1) \tag{42}$$

$$\Rightarrow \mathbb{P}_{\mathcal{E}}(\mathbb{N}) = \mathbb{P}(\tau_{\mathcal{E}} < +\infty) = 1, \tag{43}$$

due to the continuity property of probability measures and because $\tau_{\mathcal{E}} < +\infty$, $\mathbb{P}-a.s.$

2) By definition, because

$$0 \le \mathbb{P}(\tau_{\mathcal{E}} = k),\ \forall k \in \mathbb{N} \ \Rightarrow\ 0 \le \mathbb{P}_{\mathcal{E}}(A),\ \forall A \in 2^{\mathbb{N}}, \tag{44}$$

and, on the other hand,

$$A \subseteq \mathbb{N} \;\Rightarrow\; \sum_{k\in A} \mathbb{P}(\tau_{\mathcal{E}} = k) \leq \sum_{k\in\mathbb{N}} \mathbb{P}(\tau_{\mathcal{E}} = k) \tag{45}$$

$$\Leftrightarrow\; \mathbb{P}_{\mathcal{E}}(A) \leq \mathbb{P}_{\mathcal{E}}(\mathbb{N}) = 1,\; \forall A \in 2^{\mathbb{N}}. \tag{46}$$

3) Let $\{A_k \in 2^{\mathbb{N}}\}_{k\in\mathbb{N}}$ such that $A_i \cap A_j = \phi$, $\forall i \neq j$. By definition,

$$\mathbb{P}_{\mathcal{E}}\left(\cup_{k\in\mathbb{N}} A_k\right) = \sum_{j\in\cup_{k\in\mathbb{N}} A_k} \mathbb{P}(\tau_{\mathcal{E}} = j) \tag{47}$$

$$= \sum_{k\in\mathbb{N}} \sum_{j\in A_k} \mathbb{P}(\tau_{\mathcal{E}} = j) \tag{48}$$

$$= \sum_{k\in\mathbb{N}} \mathbb{P}_{\mathcal{E}}(A_k) \tag{49}$$

□

2.2. Continuous-Time Stochastic Processes

Let $\{X_t\}_{t\in\mathbb{R}_+\cup\{0\}}$ be a stochastic process and $\{E_t\}_{t\in\mathbb{R}_+}$ be an uncertain event process. By definition, if there was a probability density function associated to $\tau_{\mathcal{E}} = \tau_{\mathcal{E}}(t_p)$, $t_p \in \mathbb{R}_+ \cup \{0\}$, then it could be obtained using an expression with the same structure of *survival probability*, as shown below:

$$p(\tau_{\mathcal{E}} = t) := \mathbb{P}\left(\{E_t = \mathcal{E}\}, \{E_\tau = \mathcal{E}^c\}_{\tau\in(t_p,t)}\right) \tag{50}$$

$$= \mathbb{P}\left(\{E_t = \mathcal{E}\}, \{\tau_{\mathcal{E}} \geq t\}\right) \tag{51}$$

$$= \mathbb{P}\left(E_t = \mathcal{E} | \tau_{\mathcal{E}} \geq t\right) \mathbb{P}\left(\tau_{\mathcal{E}} \geq t\right) \tag{52}$$

with $\mathbb{P}(\tau_{\mathcal{E}} \geq t) = 1 - \mathbb{P}(\tau_{\mathcal{E}} < t)$. Let $\mathcal{B}(\mathbb{R}_+)$ and $\lambda = \lambda(\mathbb{R}_+)$ denote the Borel σ-algebra and Lebesgue measure in $\mathbb{R}_+$, respectively. Let also $F_{\tau_{\mathcal{E}}}(t) = \mathbb{P}(\tau_{\mathcal{E}} \leq t)$ be a probability measure in the measurable space $(\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+))$ such that $F_{\tau_{\mathcal{E}}} << \lambda$. According to the Theorem of Radon-Nikodym, there is a probability density function $p(\tau_{\mathcal{E}} = t) := \frac{dF_{\tau_{\mathcal{E}}}}{d\lambda}(t)$, $t \in \mathbb{R}_+$, such that

$$\mathbb{P}(\tau_{\mathcal{E}} < t) = \int_{(t_p,t)} dF_{\tau_{\mathcal{E}}}(\tau) \tag{53}$$

$$= \int_{(t_p,t)} p(\tau_{\mathcal{E}} = \tau) d\tau \tag{54}$$

$$= \int_{(t_p,t)} \mathbb{P}\left(E_\tau = \mathcal{E} | \tau_\mathcal{E} \geq \tau\right) \mathbb{P}\left(\tau_\mathcal{E} \geq \tau\right) d\tau \tag{55}$$

Due to the existence of the aforementioned probability density function, $\mathbb{P}(\tau_\mathcal{E} \geq t)$ must be differentiable

$$\Rightarrow \frac{d}{dt}\mathbb{P}(\tau_\mathcal{E} \geq t) = -\mathbb{P}\left(E_t = \mathcal{E} | \tau_\mathcal{E} \geq t\right) \mathbb{P}\left(\tau_\mathcal{E} \geq t\right), \tag{56}$$

because $\mathbb{P}\left(E_{t_p} = \mathcal{E} | \tau_\mathcal{E} \geq t_p\right) = \mathbb{P}\left(E_{t_p} = \mathcal{E}\right) = 0$ (at the beginning it was stated that $\tau_\mathcal{E} > t_p$). Integrating over time,

$$-\int_{(t_p,t)} \mathbb{P}\left(E_\tau = \mathcal{E} | \tau_\mathcal{E} \geq \tau\right) d\tau = \int_{(t_p,t)} \frac{1}{\mathbb{P}\left(\tau_\mathcal{E} \geq \tau\right)} \frac{d}{d\tau}\mathbb{P}\left(\tau_\mathcal{E} \geq \tau\right) d\tau \tag{57}$$

$$= \int_{(t_p,t)} \frac{d}{d\tau} \log \mathbb{P}\left(\tau_\mathcal{E} \geq \tau\right) d\tau \tag{58}$$

$$= \log \mathbb{P}\left(\tau_\mathcal{E} \geq t\right) - \cancelto{0}{\log \mathbb{P}\left(\tau_\mathcal{E} \geq t_p\right)} \tag{59}$$

Thus,

$$\Rightarrow \mathbb{P}\left(\tau_\mathcal{E} \geq t\right) = e^{-\int_{t_p}^{t} \mathbb{P}\left(E_\tau = \mathcal{E} | \tau_\mathcal{E} \geq \tau\right) d\tau}. \tag{60}$$

Before presenting the previous results in a formal theorem, please note the following. Since each E_t depends on X_t, using the property illustrated with Eq. (9) and the concept of *uncertain event likelihood function* introduced in Definition 1, it can be obtained:

$$p(\tau_\mathcal{E} = t) = \mathbb{P}\left(E_t = \mathcal{E} | \tau_\mathcal{E} \geq t\right) e^{-\int_{t_p}^{t} \mathbb{P}\left(E_\tau = \mathcal{E} | \tau_\mathcal{E} \geq \tau\right) d\tau}$$
$$= \int_{\mathbb{X}_{(t_p,t]}} \mathbb{P}\left(E_t = \mathcal{E} | \{\tau_\mathcal{E} \geq t\}, x_{(t_p,t]}\right) e^{-\int_{t_p}^{t} \mathbb{P}\left(E_\tau = \mathcal{E} | \{\tau_\mathcal{E} \geq \tau\}, x_{(t_p,t]}\right) d\tau} dF_{\mathbb{X}_{(t_p,t]}}(x_{(t_p,t]}) \tag{61}$$

$$= \int_{\mathbb{X}_{(t_p,t]}} \mathbb{P}\left(E_t = \mathcal{E} | x_t\right) e^{-\int_{t_p}^{t} \mathbb{P}\left(E_\tau = \mathcal{E} | x_\tau\right) d\tau} dF_{\mathbb{X}_{(t_p,t]}}(x_{(t_p,t]}). \tag{62}$$

Theorem 2. ***[First Event Time in Stochastic Processes]*** *Let $\mathcal{B}(\mathbb{R}_+)$ and $\lambda = \lambda(\mathbb{R}_+)$ denote the Borel σ-algebra and Lebesgue measure in $\mathbb{R}_+$, respectively. Let also $\{X_t\}_{t\in\mathbb{R}_+\cup\{0\}}$ be a stochastic process and $\{E_t\}_{t\in\mathbb{R}_+}$ be an uncertain event process. If the first time of occurrence of the event $\mathcal{E}$, $\tau_\mathcal{E} = \tau_\mathcal{E}(t_p)$, with*

$t_p \in \mathbb{R}_+ \cup \{0\}$, is such that $\tau_{\mathcal{E}} < +\infty$, $\mathbb{P}-a.s.$, and $F_{\tau_{\mathcal{E}}} << \lambda$, then the mapping $p(\tau_{\mathcal{E}} = \cdot) : \mathbb{R}_+ \to [0,1]$ exists and is well-defined in terms of its uncertain event likelihood function as

$$\mathbb{P}(\tau_{\mathcal{E}} = t) := \int_{\mathbb{X}_{(t_p,t]}} \mathbb{P}\left(E_t = \mathcal{E}|x_t\right) e^{-\int_{t_p}^{t} \mathbb{P}(E_\tau = \mathcal{E}|x_\tau) d\tau} dF_{\mathbb{X}_{(t_p,t]}}(x_{(t_p,t]}). \tag{63}$$

Therefore,

$$\mathbb{P}_{\mathcal{E}}(B) = \int_B dF_{\tau_{\mathcal{E}}}(\tau) = \int_B p(\tau_{\mathcal{E}} = \tau) d\tau, \ \forall B \in \mathcal{B}(\mathbb{R}_+), \tag{64}$$

is a probability measure defining a probability space $(\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+), \mathbb{P}_{\mathcal{E}})$. Indeed, the following conditions hold:

1) $\mathbb{P}_{\mathcal{E}}(\mathbb{R}_+) = 1$.

2) $0 \le \mathbb{P}_{\mathcal{E}}(B) \le 1$, $\forall B \in \mathcal{B}(\mathbb{R}_+)$.

3) $\mathbb{P}_{\mathcal{E}}(\cup_{k\in\mathbb{N}} B_k) = \sum_{k\in\mathbb{N}} \mathbb{P}_{\mathcal{E}}(B_k)$, $\forall \{B_k \in \mathcal{B}(\mathbb{R}_+)\}_{k\in\mathbb{N}}$, *with* $B_i \cap B_j = \phi$, $\forall i \neq j$.

Proof.

1) Let us define $B_t = (0,t)$, $t \in \mathbb{R}_+$, such that $B_t \nearrow \mathbb{R}_+$

$$\mathbb{P}_{\mathcal{E}}(B_t) = \int_{(0,t)} p(\tau_{\mathcal{E}} = \tau) d\tau = \mathbb{P}(\tau_{\mathcal{E}} < t) \tag{65}$$

$$\Rightarrow \lim_{t\to+\infty} \mathbb{P}_{\mathcal{E}}(B_t) = \lim_{t\to+\infty} \mathbb{P}(\tau_{\mathcal{E}} < t) \tag{66}$$

$$\Rightarrow \mathbb{P}_{\mathcal{E}}(\mathbb{R}_+) = \mathbb{P}(\tau_{\mathcal{E}} < +\infty) = 1, \tag{67}$$

due to the continuity property of probability measures and because $\tau_{\mathcal{E}} < +\infty$, $\mathbb{P} - a.s.$

2) By definition, because

$$0 \le \mathbb{P}(\tau_{\mathcal{E}} = t), \ \forall t \in \mathbb{R}_+ \ \Rightarrow \ 0 \le \mathbb{P}_{\mathcal{E}}(B), \ \forall B \in \mathcal{B}(\mathbb{R}_+), \tag{68}$$

and, on the other hand,

$$B \subseteq \mathbb{R}_+ \ \Rightarrow \ \int_B p(\tau_{\mathcal{E}} = \tau) d\tau \le \int_{\mathbb{R}_+} p(\tau_{\mathcal{E}} = \tau) d\tau \tag{69}$$

$$\Leftrightarrow \ \mathbb{P}_{\mathcal{E}}(B) \le \mathbb{P}_{\mathcal{E}}(\mathbb{R}_+) = 1, \ \forall B \in \mathcal{B}(\mathbb{R}_+). \tag{70}$$

3) Let $\{B_k \in \mathcal{B}(\mathbb{R}_+)\}_{k\in\mathbb{N}}$ such that $B_i \cap B_j = \phi$, $\forall i \neq j$. By definition,

$$\mathbb{P}_{\mathcal{E}}(\cup_{k\in\mathbb{N}} B_k) = \int_{\cup_{k\in\mathbb{N}} B_k} p(\tau_{\mathcal{E}} = \tau) d\tau \tag{71}$$

$$= \sum_{k\in\mathbb{N}} \int_{B_k} p(\tau_{\mathcal{E}} = \tau) d\tau \tag{72}$$

$$= \sum_{k\in\mathbb{N}} \mathbb{P}_{\mathcal{E}}(B_k) \tag{73}$$

□

3. Uncertain Event Prognosis in Practice

This section aims at facilitating the transition between theory and practice by providing a friendly interpretation of theorems that constitute the true contribution of this article. In addition, a case study inspired on the problem of fatigue crack growth is used to illustrate the application of these concepts in failure prognostics.

3.1. A Friendly Interpretation of Theorems 1 and 2

Since theoretical implications of Theorems 1 and 2 are completely analogous, we will focus on providing adequate interpretations for Theorem 1, which characterizes the probability of future events in discrete-time dynamic systems.

For this purpose, let us assume that it is intended to perform critical event prognostics at a time k_p. The first element to consider is a proper representation for the dynamic system that characterizes the future evolution of the condition indicator of interest. This representation corresponds to the stochastic process $\{X_k\}_{k\in\mathbb{N}\cup\{0\}}$, where the variable k is a time index (seconds, minutes, hours, cycles of operation) that takes values in the set of natural numbers. System dynamics in typical real-world applications are commonly characterised using Markov processes (i.e. future is independent of the past, conditional on the present), which leads to the state-space model:

$$x_{k+1} = f(x_k, u_k, \omega_k), \tag{74}$$

where u_k denotes an exogenous system input and ω_k is a random vector that accounts for model uncertainty, a.k.a. the *process noise*. It is important to note, however, that the Markovian assumption was used in this section solely for illustrative purposes since, as can be verified in Theorems 1 and 2, there are no restrictions on the stochastic process that could be used to describe the dynamics of the system.

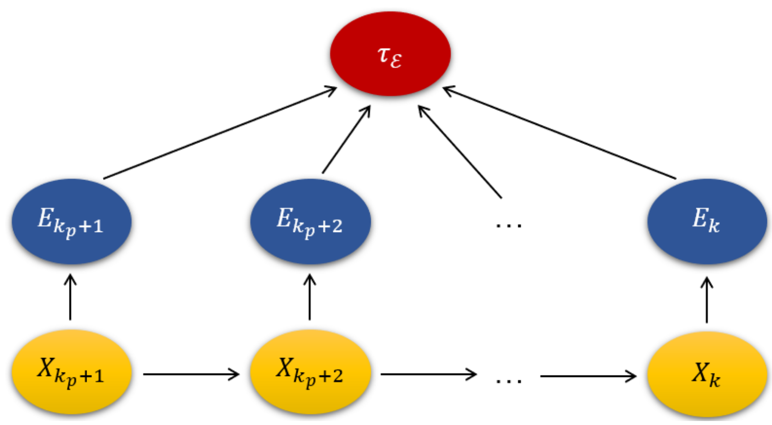

Figure 1: Illustration of statistical dependency relationships in the dynamic system described by Eq. (74). The statistics of $\tau_{\mathcal{E}}$ depend on $\{E_k\}_{k>k_p}$, and this dependency is clearly expressed in Eq. (12) (or Eq. (50) in the case of continuous-time systems).

The next step that is required to use Theorem 1 is to characterize the *uncertain event process* $\{E_k\}_{k\in\mathbb{N}}$. Naturally, this characterization depends on which event $\mathcal{E}$ is sought to be prognosticated. As it is illustrated in Fig. 1, at each future time instant k, $k > k_p$, the binary random variable E_k indicates whether the event $\mathcal{E}$ has occurred or not, solely depending on the system condition indicator X_k. This is one of the main concerns that should be settled by the designer of the prognostic algorithm: *How can the dependency of E_k on X_k be determined? What does $\mathcal{E}$ has to do with this dependency?* It is important to remark that the event $\mathcal{E}$ must be, in the first place, *qualitatively* defined; for example:

$$\mathcal{E} = \text{“Critical system failure”}.$$

The latter would imply that the occurrence of the $\mathcal{E}$ at a time k corresponds to a binary random variable E_k. This variable, in contrast to $\mathcal{E}$, must be defined in terms of a *quantitative* description given by an *uncertain event likelihood function* $\mathbb{P}(E_k = \mathcal{E}|x)$ for each $k > k_p$. Following the aforementioned example, if the system state were to be a one-dimensional fault indicator (scalar value), "Critical system failure" might be declared once the system state reaches an upper threshold $\bar{x}$. With this definition, we have

$$\mathbb{P}(E_k = \mathcal{E}|x) = \mathbb{P}(E_k = \text{“Critical system failure”}|x) = \begin{cases} 1, & x \geq \bar{x} \\ 0, & \sim . \end{cases} \tag{75}$$

However, one may wonder what happens if there this upper threshold is not absolutely and accurately known? In other words, what happens if the upper threshold is "uncertain"? The definition of *uncertain event likelihood function* allows us to incorporate uncertainty in the widely accepted "threshold" concept, leading to the notion of "*uncertain events*". Please refer to Remark 2 and Section 3.2 for more insights on this line of thought.

Now, the final aim of the prognostic algorithm is to characterize the probability distribution of the random variable $\tau_{\mathcal{E}}$, which denotes the first occurrence time of the event $\mathcal{E}$ in the future. Having defined the uncertain event likelihood function $\mathbb{P}(E_k = \mathcal{E}|x)$ for each $k > k_p$, it is straightforward to apply Eq. (39) of Theorem 1 (or Eq. (63) of Theorem 2 in the continuous time case) to probabilistically characterize $\tau_{\mathcal{E}}$:

$$\mathbb{P}(\tau_{\mathcal{E}} = k) = \int_{\mathbb{X}_{k_p+1:k}} \mathbb{P}\left(E_k = \mathcal{E}|x_k\right) \prod_{j=k_p+1}^{k-1} \left(1 - \mathbb{P}\left(E_j = \mathcal{E}|x_j\right)\right) dF_{X_{k_p+1:k}}(x_{k_p+1:k}) \tag{76}$$

This expression, however, can often be rewritten in terms the probability density $p(x_{k_p+1:k})$ as:

$$\mathbb{P}(\tau_{\mathcal{E}} = k) = \int_{\mathbb{X}_{k_p+1:k}} \mathbb{P}\left(E_k = \mathcal{E}|x_k\right) \prod_{j=k_p+1}^{k-1} \left(1 - \mathbb{P}\left(E_j = \mathcal{E}|x_j\right)\right) p(x_{k_p+1:k}) dx_{k_p+1:k}. \tag{77}$$

The use of the infinitesimal term "$dF_{X_{k_p+1:k}}(x_{k_p+1:k})$" is a mathematical technicality due to the use of Lebesgue integration, since it is possible to rewrite

$\mathbb{P}(\tau_{\mathcal{E}} = k)$ in terms of Riemann integration, except when the probability density $p(x_{k_p+1:k})$ does not exist, which is rather rare in practice.

Finally, it is noteworthy that $\mathbb{P}(\tau_{\mathcal{E}} = \cdot)$ exists only if $\tau_{\mathcal{E}} < +\infty$, $\mathbb{P} - a.s.$ The latter implies that "*the first event $\mathcal{E}$ must occur in a finite time k, $k > k_p$, almost surely or with probability 1*". This requirement in practice can be understood as "*the event $\mathcal{E}$ must occur at some future time*". If this condition does not hold, $\mathbb{P}(\tau_{\mathcal{E}} = \cdot)$ could still exist, but without fulfilling the axiom of probability $\sum_{k\in\mathbb{N}} \mathbb{P}(\tau_{\mathcal{E}} = k) = 1$.

3.2. Application to Fatigue Crack Prognosis

The theoretical contributions presented in Section 2 include their corresponding mathematical demonstrations and thus, in our humble opinion, do not need further validation. Nonetheless, in this section the authors intend to illustrate how these abstract mathematical statements can be used to solve a practical engineering application: the characterization of Time-of-Failure probability distributions in the context of failure prognostics problem; and more specifically, the problem of fatigue crack growth prognosis. For this purpose, a simplified stochastic degradation model is used to describe the growth of a fatigue crack in a test coupon as a function of loading cycles. The event $\mathcal{E}$ of interest corresponds to critical failures that may occur in mechanical systems with components that undergo fatigue crack processes, though it may not be clear that a specific crack lengths could trigger these events. The problem is addressed using the concept of *uncertain event* and is compared to the case of classical threshold-crossing-based events (i.e., critical failure always occurs when the crack length exceeds a *known* specific value). In all these cases, probability distributions for the first time of occurrence of the event are shown so as to develop a further discussion.

3.2.1. Crack Growth Model

In order to illustrate both the problem and the implementation of the concept of *uncertain events*, we have chosen to use the simplified discrete-time crack growth model presented in [83]. It is of paramount importance to emphasize the fact that the aim of this application example is to show how the presented conceptual contributions can be applied, rather than contributing to the state-of-the-art in terms of topic of crack length prognostics. More information about the specifics of fatigue crack growth in alloy test coupons can be found in [83].

According to the mathematical notation introduced in Section 2, the crack length can be described by a stochastic process $\{X_k\}_{k\in\mathbb{N}\cup\{0\}}$. Note that the indexing variable k usually denotes time, and more specifically in this case, it denotes a *cycle number*. The material undergoes compression and decompression instances. In addition, and provided that a length can only adopt positive values, we have $\mathbb{X} = \mathbb{R}_+$ and $\Sigma = \mathcal{B}(\mathbb{R}_+)$ (Borel sets in $\mathbb{R}_+$). The crack length is described in arbitrary units by the following discrete-time model:

$$x_{k+1} = x_k + e^{\omega_k} C(\beta\sqrt{x_k})^n, \tag{78}$$

where $\omega_k \sim \mathcal{N}(0, \sigma_w^2)$ is a random variable depicting white Gaussian noise, and C, β and n are fixed constants. All the model parameters values are summarized in Table 1.

	C	β	n	σ_w^2
Values	0.005	1	1.3	2.98

Table 1: Model parameters and their values.

3.2.2. Uncertain Event Definition

As stated in Definition 1, and assuming the existence of a probability density that characterizes the crack length at each cycle given by the model described in Eq. (78), the statistics of an uncertain event $\mathcal{E}$ (in this case, critical failures in mechanical systems with components that undergo fatigue crack processes) are determined by the definition of $\mathbb{P}(E_k = \mathcal{E}|x)$ (see Eq. (8)), which describes how likely is that event $\mathcal{E}$ occurs at a particular time instant k given that the crack length is x. Without loss of generality, let us assume that the critical failure events of interest can be associated with crack lengths of approximately

$\bar{x} = 100$. Thus, we may define the uncertain event likelihood function

$$\mathbb{P}(E_k = \mathcal{E}|x) = \frac{1}{1 + e^{-\alpha(x-\bar{x})}},\ \alpha > 0,\ \forall k \in \mathbb{N}. \tag{79}$$

to provide a characterization of the uncertainty related to the occurrence of critical failure events in terms of the condition of the test coupon. Moreover, by using this critical failure likelihood, it is still possible to go back to a threshold-based failure characterization (see Remark 2) by simply studying the limit

$$\lim_{\alpha \to +\infty} \frac{1}{1 + e^{-\alpha(x-\bar{x})}} = \mathbb{1}_{\{x \in \mathbb{R}: x > \bar{x}\}}(x). \tag{80}$$

Remark 4. ***[Example of time-invariant uncertain event likelihood function]***
Eq. (79) corresponds to an example of a time-invariant uncertain event likelihood function $\mathbb{P}(E_k = \mathcal{E}|x)$ (see Remark 1).

Remark 5. ***[How to define the uncertain event likelihood function?]***
The uncertain event likelihood function that characterizes the uncertainty of the failure event can be built either using post-mortem statistical analysis or expert knowledge. The post-mortem statistical analysis requires the availability of run-to-failure data that could be used to reconstruct the trajectory of system condition indicators prior to the failure event. Values of condition indicators at the recorded failure events can be used to build a non-parametric likelihood function (in other words, to build an empirical joint probability mass function for condition indicators at the moment in which the system failed). Alternatively, it is always possible to adjust the parameters of a known function to fit the data. The use of expert knowledge would give rise to an epistemic source of uncertainty, where the uncertain event likelihood function is adjusted according to an expert's criteria. Bayesian approaches can always be used to fuse prior expert knowledge with scarce run-to-failure data.

It is important to note that the shortcomings associated with these procedures are equivalent to those that one would face when trying to establish a threshold for the failure indicator. The introduction of uncertain event prognosis, however, *provides a solid theoretical framework where the concept of uncertain event is properly recognized and characterized. Both researchers and practitioners can use this theoretical framework to safely explore different methods to define these likelihood functions according to the specific challenges they are facing.*

3.2.3. Method of Monte Carlo Simulations

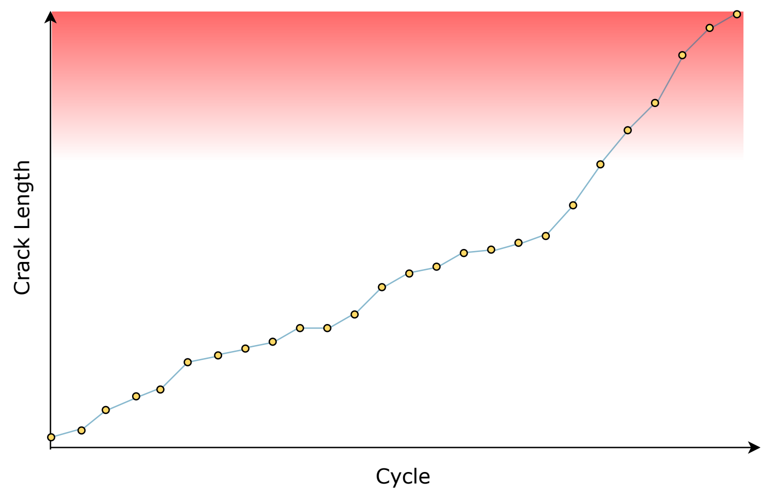

Figure 2: Single realization of the discrete-time stochastic process associated to crack length growth. The red color illustrates the magnitude of the uncertain event likelihood function $\mathbb{P}(E_k = \mathcal{E}|x)$; the greater the opacity, the greater the likelihood (see Definition 1). In contrast, the classical approach to first event time prediction would not had shown a color hue, but an abrupt and discontinuous change from color white to red.

The notion of *uncertain event* incorporates a new degree of freedom for uncertainty characterization. In order to show how this new uncertainty source may impact event predictions, hereby we study its effect on the probability distribution associated to $\tau_{\mathcal{E}}$. Monte Carlo simulations are employed below to perform the required computation given their capacity to calculate expectations with arbitrary accuracy by simply increasing the number of simulations, denoted as $N \in \mathbb{N}$. Besides, let $x^{(i)}_{k_p+1:k} = \{x^{(i)}_j\}_{j=k_p+1}^{k}$ denote the i-th realization of the stochastic process simulated from cycle k_p up to cycle k (see Fig. 2), described by the crack growth model (Markov process) of Eq. (78), with $i \in \{1, \ldots, N\}$,

$N >> 1$. According to Theorem 1, the probability $\mathbb{P}(\tau_{\mathcal{E}} = k)$, with $\tau_{\mathcal{E}} = \tau_{\mathcal{E}}(k_p)$, can be approximated as:

$$\mathbb{P}(\tau_{\mathcal{E}} = k) = \int_{\mathbb{X}_{k_p+1:k}} \mathbb{P}\left(E_k = \mathcal{E}|x_k\right) \prod_{j=k_p+1}^{k-1} \left(1 - \mathbb{P}\left(E_j = \mathcal{E}|x_j\right)\right) dF_{X_{k_p+1:k}}(x_{k_p+1:k}) \tag{81}$$

$$= \int_{\mathbb{X}_{k_p+1:k}} \mathbb{P}\left(E_k = \mathcal{E}|x_k\right) \prod_{j=k_p+1}^{k-1} \left(1 - \mathbb{P}\left(E_j = \mathcal{E}|x_j\right)\right) p(x_{k_p+1:k}) dx_{k_p+1:k} \tag{82}$$

$$\approx \int_{\mathbb{X}_{k_p+1:k}} \mathbb{P}\left(E_k = \mathcal{E}|x_k\right) \prod_{j=k_p+1}^{k-1} \left(1 - \mathbb{P}\left(E_j = \mathcal{E}|x_j\right)\right) \left(\frac{1}{N}\sum_{i=1}^{N} \delta_{x^{(i)}_{k_p+1:k}}(x_{k_p+1:k})\right) dx_{k_p+1:k} \tag{83}$$

$$= \frac{1}{N}\sum_{i=1}^{N} \mathbb{P}\left(E_k = \mathcal{E}|x_k^{(i)}\right) \prod_{j=k_p+1}^{k-1} \left(1 - \mathbb{P}\left(E_k = \mathcal{E}|x_j^{(i)}\right)\right) \tag{84}$$

As depicted in Fig. 2, each realization of the stochastic process describing the evolution of the crack growth throughout usage cycles (the i-th for example), determines a likelihood for the occurrence of the uncertain event (material futility in this case). In this example, the figure illustrates the magnitude of the uncertain event likelihood function of Eq. (8) in terms of an hue over the crack length space with colors that go from a clear white to red progressively. The smoothness of this changing color depends, in this case, of the parameter α included in Eq. (8). In order to study the impact of this parameter on the probability mass distribution $\mathbb{P}(\tau_{\mathcal{E}} = \cdot)$, we explore different values, which are shown in Table 2.

	α_1	α_2	α_3	α_4	$\alpha_{+\infty}$
Values	0.1	0.3	1.0	3.3	$\alpha \rightarrow +\infty$

Table 2: Values considered for the parameter α in the definition of the uncertain event likelihood function $\mathbb{P}(E_k = \mathcal{E}|x)$ shown in Eq. (79).

For clarity purposes, Fig. 3 illustrates how the uncertain event likelihood function looks like for the aforementioned values for the parameter α. A special case denoted by $\alpha_{+\infty}$ is also considered, which corresponds to the standard notion of threshold crossing, highly explored in the literature taking place when $\alpha \to +\infty$ (see Eq. (80)).

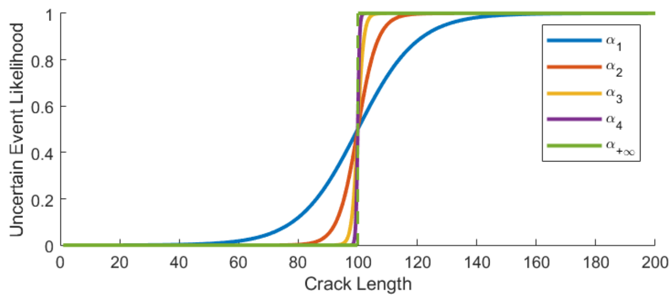

Figure 3: Uncertain event likelihood $\mathbb{P}(E_k = \mathcal{E}|x)$ as a function of the crack length $x \in \mathbb{R}_+$ for different values of the parameter α in Eq. (79). The parameter value $\alpha_{+\infty}$ depicts the behaviour of the function when $\alpha \to +\infty$ (see Eq. (80)). The less the value of the parameter α, the higher the uncertainty about an specific crack length depicting material futility.

Remark 6. ***[Simulation of $\tau_{\mathcal{E}}$ by definition]*** *Researchers within the PHM community often compute the probability distribution of $\tau_{\mathcal{E}}$ by definition, using for this purpose Monte Carlo simulations. Assuming that system failure can be characterized using a deterministic threshold (in other words, assuming $\alpha_{+\infty}$ for this particular case study), this methodology would be equivalent to simulate N (with N being a natural number large enough to reach convergence) possible future trajectories of the system state to characterize the probability of failure. Indeed, considering the definition of $\tau_{\mathcal{E}}$ (see Definition 2), an histogram of failure times can be made by simply counting the number of times that the simulated system state trajectories hit the failure threshold for the first time. There is a consensus among members of the PHM community regarding the fact that such histogram converges to the probability distribution of $\tau_{\mathcal{E}}$ when $N \to +\infty$. With this in mind, it is possible to notice that Eq. (84) describes exactly this procedure when the uncertain event likelihood function $\mathbb{P}(E_k = \mathcal{E}|x)$ describes a failure threshold (i.e. when it is an indicator function), as explained in Remark 2. In this regard, the semi-closed analytical expressions for $\mathbb{P}(\tau_{\mathcal{E}} = \cdot)$ provided in Theorems 1 and 2 of this article formalize this procedure analytically and furthermore, extend them to more general cases than a failure threshold, where there is uncertainty regarding how events are triggered.*

3.2.4. Simulation Results

Let us consider that predictions begin at the cycle number $k_p = 100$, at which an initial crack is detected and whose length is negligible (considered as $x_{k_p} = e^{-10}$ for simulations) and, additionally, a cycle number $k_h = 1000$ at which simulations are stopped. Fig. 4 shows an example of how it would look like to simulate one hundred random crack growth trajectories. However, the Monte Carlo method described in Section 3.2.3 to approximate $\mathbb{P}(\tau_{\mathcal{E}} = k)$ with $k \in \mathbb{N}$ requires the amount of simulations to be such that $N \to +\infty$, which is not feasible in practice, but good approximations can be obtained when N is a "sufficiently large" (where "sufficiently large" depends on dimension of the state vector, uncertainty sources, complexity of the model, among others). In

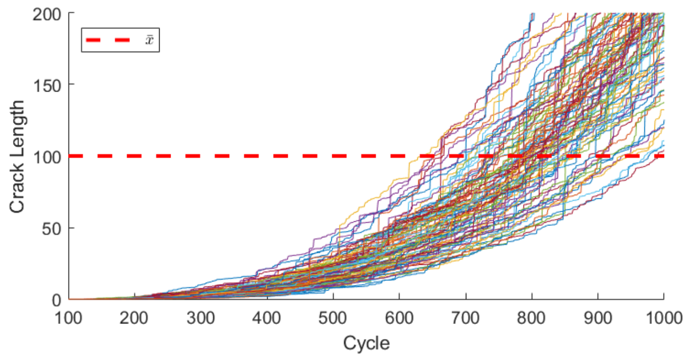

Figure 4: Example of 100 realizations of the crack growth model. The dashed horizontal line depicts a crack length of around $\bar{x} = 100$ at which the material would become useless, though there is uncertainty about it (see Section 3.2.2).

this regard, the results of performing a total amount of $N = 10^7$ Monte Carlo simulations for each of the α parameters explained in Section 3.2.3 are shown in Table 3 and Fig. 5. Higher values of N were discarded as they produce negligible effects on the results.

	α_1	α_2	α_3	α_4	$\alpha_{+\infty}$
$\mathbb{E}\{\tau_{\mathcal{E}}\}$	660.8835	766.3128	783.6094	786.7342	787.4333
$\mathbb{S}td\{\tau_{\mathcal{E}}\}$	102.6699	82.0342	82.7552	82.9145	82.9521
$\sum_{k=k_p}^{k_h} \mathbb{P}(\tau_{\mathcal{E}} = k)$	1.0000	0.9988	0.9970	0.9964	0.9962

Table 3: Results in terms of expected values, standard deviations and probability mass within a cycle span between k_p and k_h. The information is provided for each of the values considered for the parameter α in the definition of the uncertain event likelihood $\mathbb{P}(E_k = \mathcal{E}|x)$ of Eq. (79), which are shown in Table 2.

The probability distributions for $\tau_{\mathcal{E}}$ depicted in Fig. 5 are quite illustrative regarding how uncertainty on the relationship between actual crack lengths and the occurrence of critical failures may be expressed in terms of $\tau_{\mathcal{E}}$ statistics. As the shape of the failure probability distributions is similar to a Gaussian bell, the expected values and standard deviations presented in Table 3 condense

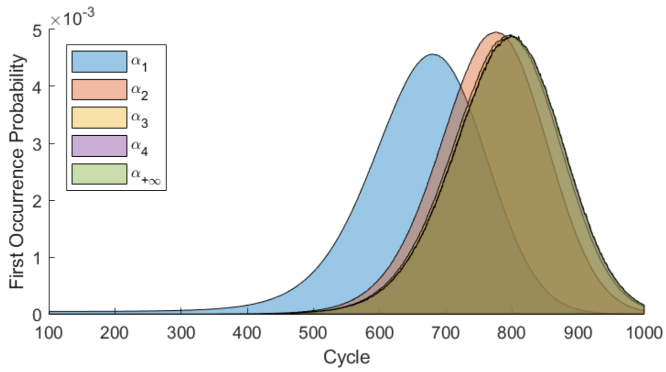

Figure 5: Computation of time probability distributions for the first occurrence of uncertain future events under different definitions of uncertain event likelihood function $\mathbb{P}(E_k = \mathcal{E}|x)$, which varies according to the different values of the parameter α (see Eq. (79)) shown in Table 2. The parameter value $\alpha_{+\infty}$ depicts the behaviour of the function when $\alpha \to +\infty$ (see Eq. (80)). The less the value of the parameter α, the higher the uncertainty about the cycle at which critical cracks could lead to critical failures.

roughly all their information required to properly analyze the results.

The current standard approach of threshold crossing found in the literature is exactly represented by $\alpha_{+\infty}$. By taking it as point of comparison, it is straightforward to note from the expected values that, as α decreases, the probability distributions of $\tau_{\mathcal{E}}$ are shifted to left. In parallel, the standard deviations increase, spreading probabilities over a wider cycle span. This behaviour is naturally produced by any uncertainty source suggesting probability of earlier events. Indeed, the definition of *uncertain event likelihood function* (see Section 3.2.2) suggests that critical failures in mechanical components are likely to occur for crack lengths lower to $\bar{x}$, which is considered as threshold in the case of $\alpha_{+\infty}$. This means that it is probably to experience critical failures in a smaller amount of loading cycles, which explains the behaviour of the expected values in Table 3. The standard deviations, on the other hand, are obtained just as an outcome of incorporating a new uncertainty source in the study. Finally, the similar results obtained with α_3, α_4 and $\alpha_{+\infty}$ are consistent with the fact that their uncertain event likelihood functions are strongly similar as well, as shown in Fig. 3.

4. Conclusion

For more than fifty years, researchers from several disciplines have approached the problem of predicting the time of occurrence of events in the future. For this reason, they have explored this idea assuming a wide variety of types of stochastic process. However, the common approach has always been to trigger an event once a particular threshold or specific zone in a higher dimensional space, is reached. The underlying reason is mainly based on an aiming at achieving closed-form mathematical expressions. In this regard, uncertainty on this threshold or higher dimensional zone has been addressed just for a reduced quantity of stochastic processes.

In this paper it has been introduced a new notion of uncertain event that generalizes the standard way of event definition for predicting its first time of occurrence in the future. Although this idea is not new, one of the greatest contributions presented in this paper is the formalization of this concept throughout a rigorous approach from Probability Theory. Moreover, the concept of *hazard zone* known in the discipline of *Prognostics and Health Management* has finally got formalized as well. On the other hand, the second –and no less important– contribution is to show its straightforward applicability with a simple example of fatigue crack growth, where practical guidelines and implications of the new concepts introduced have been provided and discussed.

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