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AN ESTIMATE OF QUASI-ARITHMETIC MEAN FOR CONVEX FUNCTIONS

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Abstract. For a selfadjoint operator A on a Hilbert space H and a normalized positive linear map Φ, a quasi-arithmetic mean is defined by $\varphi^{-1}(\Phi(\varphi(A)))$ for a strictly monotone function φ. In this paper, we shall show an order relation among quasi-arithmetic means for convex functions through positive linear maps and its complementary problems, in which we use the Mond-Pečarić method for convex functions.

1 Introduction. Let Φ be a normalized positive linear map from $B(H)$ to $B(K)$, where $B(H)$ is a C*-algebra of all bounded linear operators on a Hilbert space H and the symbol I stands for the identity operator. A real valued function φ is said to be operator convex on an interval J if

$$\varphi((1-\lambda)A+\lambda B) \le (1-\lambda)\varphi(A)+\lambda\varphi(B)$$

holds for each $\lambda \in [0,1]$ and every pair of selfadjoint operators A, B in $B(H)$ with spectra in J. φ is operator concave if $-\varphi$ is operator convex. Davis-Choi-Jensen inequality [3, 1] asserts that if a real valued continuous function f is operator convex on an interval J, then

$$f(\Phi(A)) \le \Phi(f(A)) \tag{1.1}$$

for every selfadjoint operator A with the spectrum $\sigma(A) \subset J$. A real valued function φ is said to be operator monotone on an interval J if it is monotone with respect to the operator order, i.e.,

$$A \le B \quad \text{with } \sigma(A), \sigma(B) \subset J \qquad \text{implies} \qquad f(A) \le f(B).$$

To relate them, Mond-Pečarić [8] showed the following order among power means, also see [9, 10, 11]:

Theorem A. *Let A be a positive operator on a Hilbert space H. Then*

$$\Phi(A^r)^{1/r} \le \Phi(A^s)^{1/s} \tag{1.2}$$

holds for either $r \le s$, $r \notin (-1,1)$, $s \notin (-1,1)$ or $1/2 \le r \le 1 \le s$ or $r \le -1 \le s \le -1/2$.

For positive invertible operators A and B, the chaotic order $A \gg B$ is defined by $\log A \ge \log B$. In [4], Fujii, Nakamura and Takahasi introduced a chaotically quasi-arithmetic mean of positive operators A and B: For each $t \in [0,1]$

$$\varphi^{-1}((1-t)\varphi(A)+t\varphi(B))$$

for a non-constant operator monotone function φ on $(0,\infty)$ such that φ^{-1} is chaotically monotone, that is, $0 \le A \le B$ implies $\varphi^{-1}(A) \ll \varphi^{-1}(B)$. They discussed an order among this class like Cooper's classical results [2]:

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Theorem B. *If ψ is operator monotone and $\psi \circ \varphi^{-1}$ is operator convex, then*

$$\varphi^{-1}((1-t)\varphi(A) + t\varphi(B)) \ll \psi^{-1}((1-t)\psi(A) + t\psi(B)) \tag{1.3}$$

for all $t \in [0,1]$.

We want to consider orders of (1.2) and (1.3) under a more general situation. We recall that a quasi-arithmetic mean of a selfadjoint operator A is defined by

$$\varphi^{-1}(\Phi(\varphi(A)))$$

for a strictly monotone continuous function φ. Matsumoto and Tominaga [6] investigated the relation between the quasi-arithmetic mean $\varphi^{-1}(\Phi(\varphi(A))$ and $\Phi(A)$ for a convex function φ.

In this paper, we shall show an order relation among quasi-arithmetic means for convex functions through positive linear maps and its complementary problems, in which we use the Mond-Pečarić method for convex functions in [5, 7].

2 Order among quasi-arithmetic mean First of all, we shall show an order relation among quasi-arithmetic means of selfadjoint operators for convex functions. Let $C[m,M]$ be a set of all real valued continuous functions on a closed interval $[m,M]$

Theorem 1. *Let Φ be a normalized positive linear map, A a selfadjoint operator with the spectrum $\sigma(A) \subset [m,M]$ and $\varphi, \psi \in C[m,M]$ strictly monotone functions. If one of the following conditions is satisfied:*

(i) *$\psi \circ \varphi^{-1}$ is operator convex and ψ^{-1} is operator monotone,*

(i)' *$\psi \circ \varphi^{-1}$ is operator concave and $-\psi^{-1}$ is operator monotone,*

(ii) *φ^{-1} is operator convex and ψ^{-1} is operator concave,*

then

$$\varphi^{-1}(\Phi(\varphi(A))) \le \psi^{-1}(\Phi(\psi(A))). \tag{2.1}$$

Proof. (i): Since $\psi \circ \varphi^{-1}$ is operator convex, it follows from Davis-Choi-Jensen inequality (1.1) that

$$\psi \circ \varphi^{-1}(\Phi(\varphi(A))) \le \Phi(\psi \circ \varphi^{-1} \circ \varphi(A)) = \Phi(\psi(A)).$$

Since ψ^{-1} is operator monotone, it follows that

$$\varphi^{-1}(\Phi(\varphi(A))) = \psi^{-1} \circ \psi \circ \varphi^{-1}(\Phi(\varphi(A)) \le \psi^{-1}(\Phi(\psi(A))),$$

which is the desired inequality (2.1).
(i)': We have (2.1) under the assumption (i)' by a similar method as in (i).
(ii): Since φ^{-1} is operator convex, it follows that

$$\varphi^{-1}(\Phi(\varphi(A))) \le \Phi(\varphi^{-1} \circ \varphi(A)) = \Phi(A).$$

Similarly, since ψ^{-1} is operator concave, we have

$$\Phi(A) \le \psi^{-1}(\Phi(\psi(A))).$$

Using two inequalities above, we have (2.1). □

Remark 2. Notice that the condition (i) is equivalent to (i)$'$ in Theorem 1: In fact, it follows that $\psi \circ \varphi^{-1}$ is operator concave if and only if $-\psi \circ \varphi^{-1}$ is operator convex, and $-\psi^{-1}$ is operator monotone if and only if $(-\psi)^{-1}$ is operator monotone.

The following corollary is a complementary result to Theorem 1.

Corollary 3. *Let Φ be a normalized positive linear map, A a selfadjoint operator with the spectrum $\sigma(A) \subset [m, M]$ and $\varphi, \psi \in C[m, M]$ strictly monotone functions. If one of the following conditions is satisfied:*

(i) $\psi \circ \varphi^{-1}$ *is operator concave and* ψ^{-1} *is operator monotone,*

(i)' $\psi \circ \varphi^{-1}$ *is operator convex and* $-\psi^{-1}$ *is operator monotone,*

(ii) φ^{-1} *is operator concave and* ψ^{-1} *is operator convex,*

then

$$\psi^{-1}(\Phi(\psi(A))) \le \varphi^{-1}(\Phi(\varphi(A))).$$

Remark 4. Theorem 1 and Corollary 3 are a generalization of (1.2) in Theorem A: In fact, if we put $\varphi(t) = t^r$ and $\psi(t) = t^s$ in Theorem 1 and $\varphi(t) = t^s$ and $\psi(t) = t^r$ in Corollary 3, then we have (1.2) in Theorem A.

3 Ratio type complementary order among quasi-arithmetic means

Let A be a positive operator on a Hilbert space H such that $mI \le A \le MI$ for some scalars $0 < m < M$, let $\varphi \in C[m, M]$ be convex and $\varphi > 0$ on $[m, M]$. By using the Mond-Pečarić method for convex functions, Mond-Pečarić [7] showed that

$$\varphi((Ax, x)) \le (\varphi(A)x, x) \le \lambda(m, M, \varphi)\ \varphi((Ax, x)) \tag{3.1}$$

holds for every unit vector $x \in H$, where

$$\lambda(m, M, \varphi) = \max\left\{\frac{1}{\varphi(t)}\left(\frac{\varphi(M) - \varphi(m)}{M - m}(t - m) + \varphi(m)\right) : t \in [m, M]\right\} > 0. \tag{3.2}$$

If φ is concave and $\varphi > 0$ on $[m, M]$, then

$$\mu(m, M, \varphi)\ \varphi((Ax, x)) \le (\varphi(A)x, x) \le \varphi((Ax, x)) \tag{3.3}$$

holds for every unit vector $x \in H$, where

$$\mu(m, M, \varphi) = \min\left\{\frac{1}{\varphi(t)}\left(\frac{\varphi(M) - \varphi(m)}{M - m}(t - m) + \varphi(m)\right) : t \in [m, M]\right\} > 0. \tag{3.4}$$

In particular, if $\varphi(t) = t^p$, then the constant $\lambda(m, M, t^p)$ (resp. $\mu(m, M, t^p)$) concides with a generalized Kantorovich constant $K(m, M, p)$ for $p \notin [0, 1]$ (resp. $p \in [0, 1]$) defined by

$$K(m, M, p) = \frac{mM^p - Mm^p}{(p-1)(M-m)}\left(\frac{p-1}{p}\frac{M^p - m^p}{mM^p - Mm^p}\right)^p \qquad \text{for any } p \in \mathbb{R},$$

also see [5, Chapter 2]. We remark that $K(m, M, 1) = \lim_{p\to 1} K(m, M, p) = 1$ and $K(m, M, 0) = \lim_{p\to 0} K(m, M, p) = 1$. We use the following notations:

$$\varphi_m = \min\{\varphi(m), \varphi(M)\} \qquad \text{and} \qquad \varphi_M = \max\{\varphi(m), \varphi(M)\} \tag{3.5}$$

for a strictly monotone function $\varphi \in C[m, M]$.

In (i) of Theorem 1, suppose that $\psi \circ \varphi^{-1}$ is operator convex. What happened if ψ^{-1} is not operator monotone? An order among quasi-arithmetic mean (2.1) doe not always holds. By using the Mond-Pečarić method, we show a complementary order to (2.1).

Theorem 5. *Let Φ be a normalized positive linear map, A a positive operator such that $mI \le A \le MI$ for some scalars $0 < m < M$ and $\varphi, \psi \in C[m, M]$ strictly monotone functions such as $\psi > 0$ on $[m, M]$. Suppose that $\psi \circ \varphi^{-1}$ is operator convex.*
(i) *If ψ^{-1} is increasing convex (resp. decreasing convex), then*

$$\varphi^{-1}(\Phi(\varphi(A))) \le \lambda(\psi(m), \psi(M), \psi^{-1})\ \psi^{-1}(\Phi(\psi(A))). \tag{3.6}$$

$$\left(\textit{resp.} \qquad \frac{1}{\lambda(\psi(M), \psi(m), \psi^{-1})} \psi^{-1}(\Phi(\psi(A))) \le \varphi^{-1}(\Phi(\varphi(A))). \qquad \right)$$

(ii) *If ψ^{-1} is increasing concave (resp. decreasing concave), then*

$$\varphi^{-1}(\Phi(\varphi(A))) \le \frac{1}{\mu(\psi(m), \psi(M), \psi^{-1})}\ \psi^{-1}(\Phi(\psi(A))), \tag{3.7}$$

$$\left(\textit{resp.} \qquad \mu(\psi(M), \psi(m), \psi^{-1})\ \psi^{-1}(\Phi(\psi(A))) \le \varphi^{-1}(\Phi(\varphi(A))), \qquad \right)$$

where the constants $\lambda(m, M, \varphi)$ and $\mu(m, M, \varphi)$ are defined as (3.2) *and* (3.4) *respectively.*

Proof. Since $\psi \circ \varphi^{-1}$ is operator convex, we have

$$\psi \circ \varphi^{-1}(\Phi(\varphi(A)) \le \Phi(\psi \circ \varphi^{-1} \circ \varphi(A)) = \Phi(\psi(A)). \tag{3.8}$$

(i): Suppose that ψ^{-1} is increasing convex. Since φ is strictly monotone, we have $mI \le \varphi^{-1}(\Phi(\varphi(A))) \le MI$ and hence

$$0 < \psi(m)I \le \psi \circ \varphi^{-1}(\Phi(\varphi(A))) \le \psi(M)I$$

by the increase of ψ and $\psi > 0$. Since $\psi^{-1} > 0$, it follows that for each unit vector $x \in H$

$$\begin{aligned}
&(\varphi^{-1}(\Phi(\varphi(A)))x, x) = (\psi^{-1} \circ \psi \circ \varphi^{-1}(\Phi(\varphi(A)))x, x) \\
&\le \lambda(\psi(m), \psi(M), \psi^{-1})\ \psi^{-1}(\psi \circ \varphi^{-1}(\Phi(\varphi(A)))x, x) \qquad \text{by convexity of } \psi^{-1} \text{ and (3.1)} \\
&\le \lambda(\psi(m), \psi(M), \psi^{-1})\ \psi^{-1}(\Phi(\psi(A))x, x) \qquad \text{by increase of } \psi^{-1} \text{ and (3.8)} \\
&\le \lambda(\psi(m), \psi(M), \psi^{-1})\ (\psi^{-1}(\Phi(\psi(A)))x, x) \qquad \text{by convexity of } \psi^{-1} \text{ and (3.1)}
\end{aligned}$$

and hence we have the desired inequality (3.6).

Suppose that ψ^{-1} is decreasing convex. Then it follows that ψ is decreasing and $0 < \psi(M)I \le \psi(A) \le \psi(m)I$ by $\psi > 0$. Therefore, it follows that for each unit vector $x \in H$

$$\begin{aligned}
&(\varphi^{-1}(\Phi(\varphi(A)))x, x) = (\psi^{-1} \circ \psi \circ \varphi^{-1}(\Phi(\varphi(A)))x, x) \\
&\ge \psi^{-1}(\psi \circ \varphi^{-1}(\Phi(\varphi(A)))x, x) \qquad \text{by convexity of } \psi^{-1} \text{ and (3.1)} \\
&\ge \psi^{-1}(\Phi(\psi(A))x, x) \qquad \text{by decrease of } \psi^{-1} \text{ and (3.8)} \\
&\ge \frac{1}{\lambda(\psi(M), \psi(m), \psi^{-1})}\ (\psi^{-1}(\Phi(\psi(A)))x, x) \qquad \text{by convexity of } \psi^{-1} \text{ and (3.1)}
\end{aligned}$$

and hence

$$\varphi^{-1}(\Phi(\varphi(A))) \ge \frac{1}{\lambda(\psi(M), \psi(m), \psi^{-1})}\ \psi^{-1}(\Phi(\psi(A))).$$

(ii): Suppose that ψ^{-1} is increasing concave. Then it follows that ψ is increasing and $0 < \psi(m)I \le \Phi(\psi(A)) \le \psi(M)I$. Hence for each unit vector $x \in H$

$$\begin{aligned}
&(\varphi^{-1}(\Phi(\varphi(A)))x, x) = (\psi^{-1} \circ \psi \circ \varphi^{-1}(\Phi(\varphi(A)))x, x) \\
&\le \psi^{-1}(\psi \circ \varphi^{-1}(\Phi(\varphi(A)))x, x) \quad \text{by concavity of } \psi^{-1} \text{ and (3.3)} \\
&\le \psi^{-1}(\Phi(\psi(A))x, x) \quad \text{by increase of } \psi^{-1} \text{ and (3.8)} \\
&\le \frac{1}{\mu(\psi(m), \psi(M), \psi^{-1})} \, (\psi^{-1}(\Phi(\psi(A)))x, x) \quad \text{by concavity of } \psi^{-1} \text{ and (3.3)}
\end{aligned}$$

and hence we have the desired inequality (3.7). In the case of decreasing concavity, we have our result by a similar method as in (i). □

Remark 6. The upper bound $\lambda(\psi(m), \psi(M), \psi^{-1})$ in (3.6) of Theorem 5 is sharp in the following sense: Define a normalized positive linear map $\Phi : M_2(\mathbb{C}) \mapsto \mathbb{C}$ by

$$\Phi(X) = \theta x_{11} + (1-\theta)x_{22} \qquad \text{for} \quad X = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \text{ with } 0 < \theta < 1$$

and put $A = \begin{pmatrix} m & 0 \\ 0 & M \end{pmatrix}$ with $M > m > 0$. Obviously $0 < mI \le A \le MI$. By definition, there exists $t^* \in [\psi(m), \psi(M)]$ such that

$$\lambda(\psi(m), \psi(M), \psi^{-1}) = \frac{1}{\psi^{-1}(t^*)} \left(\frac{M-m}{\psi(M) - \psi(m)} (t^* - \psi(m)) + m \right).$$

Put

$$\theta = \frac{\psi(M) - t^*}{\psi(M) - \psi(m)}$$

and we have $0 < \theta < 1$.

Suppose that

$$\varphi((1-\theta)M + \theta m) = (1-\theta)\varphi(M) + \theta\varphi(m).$$

Then we can show that

$$\varphi^{-1}(\Phi(\varphi(A))) = \lambda(\psi(m), \psi(M), \psi^{-1}) \; \psi^{-1}(\Phi(\psi(A))).$$

Indeed, it follows that

$$\begin{aligned}
\psi^{-1}(\Phi(\psi(A))) &= \psi^{-1}(\Phi(\begin{pmatrix} \psi(m) & 0 \\ 0 & \psi(M) \end{pmatrix})) \\
&= \psi^{-1}(\theta\psi(m) + (1-\theta)\psi(M)) \\
&= \psi^{-1}(t^*)
\end{aligned}$$

and hence

$$\begin{aligned}
\varphi^{-1}(\Phi(\varphi(A))) &= \varphi^{-1}(\theta\varphi(m) + (1-\theta)\varphi(M)) \\
&= (1-\theta)M + \theta m \\
&= \frac{(M-m)t^* + m\psi(M) - M\psi(m)}{\psi(M) - \psi(m)} \\
&= \lambda(\psi(m), \psi(M), \psi^{-1})\psi^{-1}(t^*) \\
&= \lambda(\psi(m), \psi(M), \psi^{-1}) \; \psi^{-1}(\Phi(\psi(A))).
\end{aligned}$$

The following theorem is a complementary result to (i)' of Theorem 1 under the assumption that $\psi \circ \varphi^{-1}$ is operator concave.

Theorem 7. *Let Φ be a normalized positive linear map, A a positive operator such that $mI \leq A \leq MI$ for some scalars $0 < m < M$ and $\varphi, \psi \in C[m, M]$ strictly monotone functions such as $\psi > 0$ on $[m, M]$. Suppose that $\psi \circ \varphi^{-1}$ is operator concave.*
(i) *If ψ^{-1} is decreasing concave (resp. increasing concave), then*

$$\varphi^{-1}(\Phi(\varphi(A))) \leq \frac{1}{\mu(\psi(M), \psi(m), \psi^{-1})} \ \psi^{-1}(\Phi(\psi(A))).$$

$$\left(\text{resp.} \qquad \mu(\psi(m), \psi(M), \psi^{-1}) \ \psi^{-1}(\Phi(\psi(A))) \leq \varphi^{-1}(\Phi(\varphi(A))). \qquad \right)$$

(ii) *If ψ^{-1} is decreasing convex (resp. increasing convex), then*

$$\varphi^{-1}(\Phi(\varphi(A))) \leq \lambda(\psi(M), \psi(m), \psi^{-1}) \ \psi^{-1}(\Phi(\psi(A))),$$

$$\left(\text{resp.} \qquad \frac{1}{\lambda(\psi(m), \psi(M), \psi^{-1})} \psi^{-1}(\Phi(\psi(A))) \leq \varphi^{-1}(\Phi(\varphi(A))), \qquad \right)$$

where the constants $\lambda(m, M, \varphi)$ and $\mu(m, M, \varphi)$ are defined as (3.2) *and* (3.4) *respectively.*

The following theorem is a complementary result to (ii) of Theorem 1.

Theorem 8. *Let Φ be a normalized positive linear map, A a positive operator such that $mI \leq A \leq MI$ for some scalars $0 < m < M$ and $\varphi, \psi \in C[m, M]$ strictly monotone functions.*
(i) *If φ^{-1} is operator convex and ψ^{-1} is concave and $\psi > 0$ on $[m, M]$, then*

$$\varphi^{-1}(\Phi(\varphi(A))) \leq \frac{1}{\mu(\psi_m, \psi_M, \psi^{-1})} \ \psi^{-1}(\Phi(\psi(A))). \tag{3.9}$$

(ii) *If φ^{-1} is convex and $\varphi > 0$ on $[m, M]$, and ψ^{-1} is operator concave, then*

$$\varphi^{-1}(\Phi(\varphi(A))) \leq \lambda(\varphi_m, \varphi_M, \varphi^{-1}) \ \psi^{-1}(\Phi(\psi(A))). \tag{3.10}$$

(iii) *If φ^{-1} is convex and $\varphi > 0$ on $[m, M]$ and ψ^{-1} is concave and $\psi > 0$ on $[m, M]$, then*

$$\varphi^{-1}(\Phi(\varphi(A))) \leq \frac{\lambda(\varphi_m, \varphi_M, \varphi^{-1})}{\mu(\psi_m, \psi_M, \psi^{-1})} \ \psi^{-1}(\Phi(\psi(A))), \tag{3.11}$$

where the constants $\lambda(m, M, \varphi)$ and $\mu(m, M, \varphi)$ are defined as (3.2) *and* (3.4) *respectively.*

Proof. (i): Since a C*-algebra $C^*(A)$ generated by A and the identity operator I is abelian, it follows from Stinespring decomposition theorem [12] that Φ restricted to $C^*(A)$ admits a decomposition $\Phi(X) = V^*\pi(X)V$ for all $X \in C^*(A)$, where π is a representation of $C^*(A) \subset B(H)$, and V is an isometry from K into H. Since ψ^{-1} is monotone and $\psi > 0$, we have $0 < \psi_m I \leq \Phi(\psi(A)) \leq \psi_M I$. Since $\psi^{-1} > 0$, it follows that for each unit vector $x \in H$

$$\begin{aligned}
&(\psi^{-1}(\Phi(\psi(A)))x, x) \\
&\geq \mu(\psi_m, \psi_M, \psi^{-1}) \ \psi^{-1}(\Phi(\psi(A))x, x) \qquad \text{by concavity of } \psi^{-1} \text{ and (3.3)} \\
&= \mu(\psi_m, \psi_M, \psi^{-1}) \ \psi^{-1}(\pi(\psi(A))Vx, Vx) \\
&\geq \mu(\psi_m, \psi_M, \psi^{-1}) \ (\psi^{-1}(\pi(\psi(A)))Vx, Vx) \quad \text{by } \| Vx \| = 1 \text{ and (3.3)} \\
&= \mu(\psi_m, \psi_M, \psi^{-1}) \ (\pi(A)Vx, Vx) \\
&= \mu(\psi_m, \psi_M, \psi^{-1}) \ (\Phi(A)x, x)
\end{aligned}$$

and hence

$$\mu(\psi_m, \psi_M, \psi^{-1})\Phi(A) \le \psi^{-1}(\Phi(\psi(A))). \tag{3.12}$$

On the other hand, the operator convexity of φ^{-1} implies

$$\varphi^{-1}(\Phi(\varphi(A))) \le \Phi(A). \tag{3.13}$$

Combining two inequalities (3.12) and (3.13), we have the desired inequality (3.9).
(ii): We have (3.10) by a similar method as in (i).
(iii): We have (3.11) by combining (i) and (ii). □

The following theorem is a complementary result to (i) or (i)' of Theorem 1 under the assumption that $\psi \circ \varphi^{-1}$ is only convex or concave, respectively.

Theorem 9. *Let Φ be a normalized positive linear map, A a positive operator such that $mI \le A \le MI$ for some scalars $0 < m < M$ and $\varphi, \psi \in C[m, M]$ strictly monotone functions such that $\varphi > 0$ on $[m, M]$. If one of the following conditions is satisfied:*

(i) $\psi \circ \varphi^{-1}$ *is convex (resp. concave) and ψ^{-1} is operator monotone,*

(i)' $\psi \circ \varphi^{-1}$ *is concave (resp. convex) and $-\psi^{-1}$ is operator monotone,*

then

$$\psi^{-1}(\Phi(\psi(A))) \le \tilde{\lambda}(\varphi_m, \varphi_M, \psi \circ \varphi^{-1}, \psi^{-1})\ \varphi^{-1}(\Phi(\varphi(A))). \tag{3.14}$$

$$\left(resp. \qquad \psi^{-1}(\Phi(\psi(A))) \ge \tilde{\mu}(\varphi_m, \varphi_M, \psi \circ \varphi^{-1}, \psi^{-1})\ \varphi^{-1}(\Phi(\varphi(A))). \qquad \right)$$

where

$$\tilde{\lambda}(m, M, \varphi, \psi) = \max\left\{\frac{1}{\psi \circ \varphi(t)} \cdot \psi\left(\frac{\varphi(M) - \varphi(m)}{M - m}(t - m) + \varphi(m)\right) : t \in [m, M]\right\},$$

$$\tilde{\mu}(m, M, \varphi, \psi) = \min\left\{\frac{1}{\psi \circ \varphi(t)} \cdot \psi\left(\frac{\varphi(M) - \varphi(m)}{M - m}(t - m) + \varphi(m)\right) : t \in [m, M]\right\}.$$

Proof. (i): We will prove only the convex case. Since the inequality

$$f(z) \le \frac{f(M) - f(m)}{M - m}\ (z - m) + f(m), \quad z \in [m, M]$$

holds for every convex function $f \in \mathcal{C}[m, M]$, then we have that inequality

$$f(\varphi(t)) \le \frac{f(\varphi_M) - f(\varphi_m)}{\varphi_M - \varphi_m}\ (\varphi(t) - \varphi_m) + f(\varphi_m), \quad t \in [m, M]$$

holds for every convex function $f \in \mathcal{C}[\varphi_m, \varphi_M]$. Then for a convex function $\psi \circ \varphi^{-1} \in \mathcal{C}[\varphi_m, \varphi_M]$, we obtain

$$\psi(t) \le \frac{\psi(\varphi^{-1}(\varphi_M)) - \psi(\varphi^{-1}(\varphi_m))}{\varphi_M - \varphi_m}\ (\varphi(t) - \varphi_m) + \psi(\varphi^{-1}(\varphi_m)), \quad t \in [m, M].$$

Using the functional calculus and applying a normalized positive linear map Φ, we obtain that

$$\Phi(\psi(A)) \le \frac{\psi(M) - \psi(m)}{\varphi(M) - \varphi(m)}\ (\Phi(\varphi(A)) - \varphi_m I) + \psi(\varphi^{-1}(\varphi_m))I$$

holds for every operator A such that $0 < mI \le A \le MI$. Applying an operator monotone function ψ^{-1}, it follows

$$\psi^{-1}(\Phi(\psi(A))) \le \psi^{-1}\left(\frac{\psi(M)-\psi(m)}{\varphi(M)-\varphi(m)}\ (\Phi(\varphi(A)) - \varphi_m I) + \psi(\varphi^{-1}(\varphi_m))I\right).$$

Using that $0 < \varphi_m I \le \Phi(\varphi(A)) \le \varphi_M I$, we obtain

$$\begin{aligned} & \psi^{-1}(\Phi(\psi(A))) \\ \le\ & \max_{\varphi_m \le t \le \varphi_M} \left\{ \frac{1}{\varphi^{-1}(t)} \cdot \psi^{-1}\left(\frac{\psi(M)-\psi(m)}{\varphi(M)-\varphi(m)}\ (t-\varphi_m) + \psi(\varphi^{-1}(\varphi_m))\right)\right\}\ \varphi^{-1}(\Phi(\varphi(A))) \\ =\ & \tilde{\lambda}(\varphi_m, \varphi_M, \psi\circ\varphi^{-1}, \psi^{-1})\ \varphi^{-1}(\Phi(\varphi(A))) \end{aligned}$$

and hence we have the desired inequality (3.14).

In the case (i)', the proof is essentially same as in the previous case. □

Remark 10. The upper bound $\tilde{\lambda}(\varphi_m, \varphi_M, \psi\circ\varphi^{-1}, \psi^{-1})$ in (3.14) of Theorem 9 is sharp in the sense that for any strictly monotone functions ψ and φ there exist a positive operator A and a positive linear map Φ such that the equality holds in (3.14).

It is obvious that

$$\begin{aligned} & \tilde{\lambda}(\varphi_m, \varphi_M, \psi\circ\varphi^{-1}, \psi^{-1}) \\ =\ & \max_{\varphi_m \le t \le \varphi_M} \left\{ \frac{1}{\varphi^{-1}(t)} \cdot \psi^{-1}\left(\frac{\psi(M)-\psi(m)}{\varphi(M)-\varphi(m)}\ (t-\varphi_m) + \psi(\varphi^{-1}(\varphi_m))\right)\right\} \\ =\ & \max_{0\le\theta\le1} \left\{ \frac{\psi^{-1}\left(\theta\psi(M) + (1-\theta)\psi(m)\right)}{\varphi^{-1}\left(\theta\varphi(M) + (1-\theta)\varphi(m)\right)} \right\}. \end{aligned}$$

Since a function $f(\theta) = \frac{\psi^{-1}(\theta\psi(M)+(1-\theta)\psi(m))}{\varphi^{-1}(\theta\varphi_M+(1-\theta)\varphi_m)}$ is continuous on $[0,1]$, there exists $\theta^* \in [0,1]$ such that

$$\tilde{\lambda}(\varphi_m, \varphi_M, \psi\circ\varphi^{-1}, \psi^{-1}) = \frac{\psi^{-1}\left(\theta^*\psi(M) + (1-\theta^*)\psi(m)\right)}{\varphi^{-1}\left(\theta^*\varphi(M) + (1-\theta^*)\varphi(m)\right)}.$$

Let Φ and A be as in Remark 6. Then the equality

$$\psi^{-1}(\Phi(\psi(A))) = \tilde{\lambda}(\varphi_m, \varphi_M, \psi\circ\varphi^{-1}, \psi^{-1})\ \varphi^{-1}(\Phi(\varphi(A)))$$

holds. Indeed,

$$\begin{aligned} \psi^{-1}(\Phi(\psi(A))) &= \psi^{-1}\left(\Phi(\begin{pmatrix} \psi(m) & 0 \\ 0 & \psi(M) \end{pmatrix})\right) \\ &= \frac{\psi^{-1}\left((1-\theta^*)\psi(m) + \theta^*\psi(M)\right)}{\varphi^{-1}\left((1-\theta^*)\varphi(m) + \theta^*\varphi(M)\right)} \cdot \varphi^{-1}\left((1-\theta^*)\varphi(m) + \theta^*\varphi(M)\right) \\ &= \tilde{\lambda}(\varphi_m, \varphi_M, \psi\circ\varphi^{-1}, \psi^{-1})\ \varphi^{-1}(\Phi(\varphi(A))). \end{aligned}$$

4 Difference type complementary order among quasi-arithmetic means Let A be a selfadjoint operator on a Hilbert space H such that $mI \le A \le MI$ for some scalars $m < M$, let $\varphi \in C[m, M]$ be a convex function. By using the Mond-Pečarić method for convex functions, Mond-Pečarić [7] showed that

$$\varphi((Ax, x)) \le (\varphi(A)x, x) \le \varphi((Ax, x)) + \nu(m, M, \varphi)$$

holds for every unit vector $x \in H$, where

$$\nu(m,M,\varphi) = \max\left\{\frac{\varphi(M)-\varphi(m)}{M-m}(t-m)+\varphi(m)-\varphi(t) : t\in[m,M]\right\} \geq 0. \tag{4.1}$$

If φ is concave on $[m,M]$, then

$$\xi(m,M,\varphi)+\varphi((Ax,x)) \leq (\varphi(A)x,x) \leq \varphi((Ax,x))$$

holds for every unit vector $x \in H$, where

$$\xi(m,M,\varphi) = \min\left\{\frac{\varphi(M)-\varphi(m)}{M-m}(t-m)+\varphi(m)-\varphi(t) : t\in[m,M]\right\} \geq 0. \tag{4.2}$$

In particular, if $\varphi(t)=t^p$, then the constant $\nu(m,M,t^p)$ (resp. $\xi(m,M,t^p)$) coincides with a generalized Kantorovich constant for the difference $C(m,M,p)$ for $p \notin [0,1]$ (resp. $p\in[0,1]$) defined by

$$C(m,M,p) = (p-1)\left(\frac{1}{p}\frac{M^p-m^p}{M-m}\right)^{\frac{p}{p-1}} + \frac{Mm^p-mM^p}{M-m} \qquad \text{for any } p\in\mathbb{R},$$

also see [5, Chapter 2]. We remark that $C(m,M,1)=\lim_{p\to 1} C(m,M,p)=0$.

Similarly as in the previous section, we can obtain the complementary order to (2.1) for the difference case. When $\psi\circ\varphi^{-1}$ is operator convex and ψ^{-1} is not operator monotone, we obtain the following theorem corresponding to Theorem 5.

Theorem 11. *Let Φ be a normalized positive linear map, A a selfadjoint operator such that $mI \leq A \leq MI$ for some scalars $m<M$ and $\varphi,\psi\in C[m,M]$ strictly monotone functions.*

(I) *Suppose that $\psi\circ\varphi^{-1}$ is operator convex.*

(i) If ψ^{-1} is increasing convex (resp. decreasing convex), then

$$\varphi^{-1}(\Phi(\varphi(A))) \leq \psi^{-1}(\Phi(\psi(A))) + \nu(\psi(m),\psi(M),\psi^{-1}). \tag{4.3}$$

$$\left(\textit{resp.} \quad \psi^{-1}(\Phi(\psi(A))) - \nu(\psi(M),\psi(m),\psi^{-1}) \leq \varphi^{-1}(\Phi(\varphi(A))). \right)$$

(ii) If ψ^{-1} is increasing concave (resp. decreasing concave), then

$$\varphi^{-1}(\Phi(\varphi(A))) \leq \psi^{-1}(\Phi(\psi(A))) - \xi(\psi(m),\psi(M),\psi^{-1}).$$

$$\left(\textit{resp.} \quad \xi(\psi(M),\psi(m),\psi^{-1}) + \psi^{-1}(\Phi(\psi(A))) \leq \varphi^{-1}(\Phi(\varphi(A))). \right)$$

(II) *Suppose that $\psi\circ\varphi^{-1}$ is operator concave.*

(i)' If ψ^{-1} is decreasing concave (resp. increasing concave), then

$$\varphi^{-1}(\Phi(\varphi(A))) \leq \psi^{-1}(\Phi(\psi(A))) - \xi(\psi(M),\psi(m),\psi^{-1}).$$

$$\left(\textit{resp.} \quad \xi(\psi(m),\psi(M),\psi^{-1}) + \psi^{-1}(\Phi(\psi(A))) \leq \varphi^{-1}(\Phi(\varphi(A))). \right)$$

(ii)' If ψ^{-1} is decreasing convex (resp. increasing convex), then

$$\varphi^{-1}(\Phi(\varphi(A))) \leq \psi^{-1}(\Phi(\psi(A))) + \nu(\psi(M),\psi(m),\psi^{-1}),$$

$$\left(\textit{resp.} \quad \psi^{-1}(\Phi(\psi(A))) - \nu(\psi(m),\psi(M),\psi^{-1}) \leq \varphi^{-1}(\Phi(\varphi(A))), \right)$$

where the constants $\nu(m,M,\varphi)$ *and* $\xi(m,M,\varphi)$ *are defined as* (4.1) *and* (4.2) *respectively.*

The proof of this theorem is quite similar to one of Theorem 5 and we omit it.

Remark 12. The inequalities in Theorem 11 are sharp in the sense of Remark 6. In (4.3), there exists $\theta^* \in [0,1]$ such that

$$\begin{aligned}\nu(\psi(m),\psi(M),\psi^{-1}) &= \theta^* M + (1-\theta^*)m - \psi^{-1}\left(\theta^*\psi(M) + (1-\theta^*)\psi(m)\right)\\ &= \max_{0\le\theta\le1}\left\{\theta M + (1-\theta)m - \psi^{-1}\left(\theta\psi(M) + (1-\theta)\psi(m)\right)\right\},\end{aligned}$$

since

$$\begin{aligned}&\max_{\psi(m)\le t\le\psi(M)}\left\{\frac{M-m}{\psi(M)-\psi(m)}\,(t-\psi(m))+m) - \psi^{-1}(t)\right\}\\ =\;&\max_{0\le\theta\le1}\left\{\theta M + (1-\theta)m - \psi^{-1}\left(\theta\psi(M) + (1-\theta)\psi(m)\right)\right\}.\end{aligned}$$

Let Φ, A and φ be as in Remark 6. Then the equality

$$\varphi^{-1}(\Phi(\varphi(A))) = \psi^{-1}(\Phi(\psi(A))) + \nu(\psi(m),\psi(M),\psi^{-1})$$

holds. Indeed,

$$\begin{aligned}\varphi^{-1}(\Phi(\varphi(A))) &= \varphi^{-1}(\theta^*\varphi(m) + (1-\theta^*)\varphi(M))\\ &= \theta^* m + (1-\theta^*)M\\ &= \psi^{-1}(\theta^*\psi(m) + (1-\theta^*)\psi(M)) + \nu(\psi(m),\psi(M),\psi^{-1})\\ &= \psi^{-1}(\Phi(\psi(A))) + \nu(\psi(m),\psi(M),\psi^{-1}).\end{aligned}$$

Remark 13. If we put $\varphi(t) = t^r$ and $\psi(t) = t^s$ in inequalities involving the complementary order among quasi-arithmetic means given in Section 3 and 4, we obtain the same bound as in [5, Theorem 4.4]. For instance, using Theorem 9, we obtain that

$$\Phi(A^s)^{1/s} \le \max_{0\le\theta\le1}\left\{\frac{\sqrt[r]{(\theta M^r + (1-\theta)m^r)}}{\sqrt[s]{(\theta M^s + (1-\theta)m^s)}}\right\}\Phi(A^r)^{1/r} = \Delta(h,r,s)\Phi(A^r)^{1/r}$$

holds for $r \le s$, $s \ge 1$ or $r \le s \le -1$, where $\Delta(h,r,s)$ is the generalized Specht ratio defined by (see [5, (2.97)])

$$\Delta(h,r,s) = \left\{\frac{r(h^s-h^r)}{(s-r)(h^r-1)}\right\}^{\frac{1}{s}}\left\{\frac{s(h^r-h^s)}{(r-s)(h^s-1)}\right\}^{-\frac{1}{r}}, \quad h = \frac{M}{m}.$$

Indeed, a function $f(\theta) := \sqrt[r]{(\theta M^r + (1-\theta)m^r)}/\sqrt[s]{(\theta M^s + (1-\theta)m^s)}$ has one stationary point

$$\theta_0 = \frac{r(h^s-1) - s(h^r-1)}{(s-r)(h^r-1)(h^s-1)}$$

and we have

$$\max_{0\le\theta\le1} f(\theta) = f(\theta_0) = \Delta(h,s,r).$$

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