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SOME BETTER BOUNDS IN CONVERSES OF THE JENSEN OPERATOR INEQUALITY

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Abstract. In this paper we study converses of a generalized Jensen's inequality for a continuous field of self-adjoint operators, a unital field of positive linear mappings and real values continuous convex functions. We obtain some better bounds than the ones calculated in a series of papers in which these inequalities are studied. As an application, we provide a refined calculation of bounds in the case of power functions.

1. Introduction

We recall some definitions and notations. Let H, K will be Hilbert spaces and $\mathscr{B}(H)$, $\mathscr{B}(K)$ will be appropriate C^*-algebras of all bounded linear operators. Let $\mathscr{A}$ be a C^*-algebra of operators on H and T be a locally compact Hausdorff space. We say that a *field* $(A_t)_{t\in T}$ of *operators* $A_t \in \mathscr{A}$ is *continuous* if the function $t \mapsto A_t$ is norm continuous on T. If μ is a bounded Radon measure on T and the function $t \mapsto \|A_t\|$ is integrable, then we can form the *Bochner integral* $\int_T A_t\, d\mu(t)$ which is the unique element in the multiplier algebra

$$\mathscr{M}(\mathscr{A}) = \{B \in \mathscr{B}(H) : AB + BA \in \mathscr{A} \text{ for every } A \in \mathscr{A}\}$$

so that

$$\varphi\left(\int_T A_t\, d\mu(t)\right) = \int_T \varphi(A_t)\, d\mu(t)$$

for every linear functional φ in the norm dual $\mathscr{A}^*$.

Let $\mathscr{A}$ and $\mathscr{B}$ be C^*-algebras on H and K respectively. A field $(\Phi_t)_{t\in T}$ of positive linear mappings $\Phi_t : \mathscr{A} \to \mathscr{B}$ is continuous if the function $t \mapsto \Phi_t(A)$ is continuous for every $A \in \mathscr{A}$. Additionally, if the field $(\Phi_t(1_H))_{t\in T}$ is integrable with

$$\int_T \Phi_t(1_H)\, d\mu(t) = 1_K,$$

we say that a *field* $(\Phi_t)_{t\in T}$ is *unital*.

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The self-adjoint operators on Hilbert spaces with its numerous applications play an important part of the theory of operators. The bounds research for self-adjoint operators is a very useful area of this theory. There is no better inequality in bounds examination than Jensen's inequality. It is an extensively used inequality in various fields of mathematics.

The starting point in an investigation of converses of Jensen's operator inequality by using the Mond-Pečarić method is given in [6]. The mentioned inequalities and their consequences have been explored in the last years [2, 3, 5]. So, F. Hansen, J. Pečarić and I. Perić gave in [3, Theorem 3.1] the following theorem.

THEOREM A. *Let* $(A_t)_{t\in T}$ *be a bounded continuous field of self-adjoint elements in a unital* C^**-algebra* $\mathscr{A}$ *with the spectra in* $[m,M]$, $m<M$, *defined on a locally compact Hausdorff space* T *equipped with a bounded Radon measure* μ, *and let* $(\Phi_t)_{t\in T}$ *be a unital field of positive linear maps* $\Phi_t:\mathscr{A}\to\mathscr{B}$ *from* $\mathscr{A}$ *to another unital* C^*-*algebra* $\mathscr{B}$. *Let* $f,g:[m,M]\to\mathbb{R}$ *and* $F:U\times V\to\mathbb{R}$ *be functions such that* $f([m,M])\subset U$, $g([m,M])\subset V$ *and* F *is bounded. If* F *is operator monotone in the first variable and* f *is convex in the interval* $[m,M]$, *then*

$$F\left[\int_T\Phi_t\left(f(A_t)\right)d\mu(t),g\left(\int_T\Phi_t(A_t)d\mu(t)\right)\right] \\ \leqslant \sup_{m\leqslant z\leqslant M} F\left[\frac{M-z}{M-m}f(m)+\frac{z-m}{M-m}f(M),g(z)\right]1_K. \tag{1.1}$$

In the dual case (when f *is concave) the opposite inequality holds in* (1.1) *with* inf *instead of* sup.

In [5] converses of a generalized Jensen operator inequality were studied for fields of positive linear mappings $(\Phi_t)_{t\in T}$ such that $\int_T\Phi_t(1_H)\,d\mu(t)=k1_K$ for some positive scalar k.

Very recently, we gave in [4, Theorem 1] a version of Jensen's operator inequality without operator convexity as follows.

THEOREM B. *Let* $(A_1,\ldots,A_n)$ *be an* n*-tuple of self-adjoint operators* $A_i\in\mathscr{B}(H)$ *with bounds* m_i *and* M_i, $m_i\leqslant M_i$, $i=1,\ldots,n$. *Let* $(\Phi_1,\ldots,\Phi_n)$ *be an* n*-tuple of positive linear mappings* $\Phi_i:\mathscr{B}(H)\to\mathscr{B}(K)$, $i=1,\ldots,n$, *such that* $\sum_{i=1}^n\Phi_i(1_H)=1_K$. *If*

$$(m_A,M_A)\cap[m_i,M_i]=\emptyset \quad \textit{for } i=1,\ldots,n, \tag{1.2}$$

where m_A *and* M_A, $m_A\leqslant M_A$, *are bounds of the self-adjoint operator* $A=\sum_{i=1}^n\Phi_i(A_i)$, *then*

$$f\left(\sum_{i=1}^n\Phi_i(A_i)\right)\leqslant\sum_{i=1}^n\Phi_i\left(f(A_i)\right) \tag{1.3}$$

holds for every continuous convex function $f:I\to\mathbb{R}$ *provided that the interval* I *contains all* m_i,M_i.

If $f:I\to\mathbb{R}$ *is concave, then the reverse inequality is valid in* (1.3).

The result from Theorem B, where a special condition is given on the bounds of the arithmetic mean of operators which appear in Jensen's inequality, has inspired us to similarly improve the Mond-Pečarić method used in [1, 2, 3, 5]. So, in this paper we give a better bound than the ones calculated in the above references, when we take into account an interval which contains the bounds of the integral arithmetic mean of operators. In the second section we give a general formulation of converses of Jensen's operator inequality. In the third and the fourth sections we give the difference and ratio type converses, respectively, and applications to provide a refinement of bounds for power functions.

2. Main results

In the following theorem we give a general form of converses of Jensen's inequality which give a better bound than the one in Theorem A.

THEOREM 2.1. *Let* $(A_t)_{t\in T}$ *be a bounded continuous field of self-adjoint elements in a unital* C^**-algebra* $\mathscr{A}$ *with the spectra in* $[m,M]$, $m<M$, *defined on a locally compact Hausdorff space* T *equipped with a bounded Radon measure* μ, *and let* $(\Phi_t)_{t\in T}$ *be a unital field of positive linear maps* $\Phi_t:\mathscr{A}\to\mathscr{B}$ *from* $\mathscr{A}$ *to another unital* C^*-*algebra* $\mathscr{B}$. *Let* m_A *and* M_A, $m_A\leqslant M_A$, *be the bounds of the self-adjoint operator* $A=\int_T\Phi_t(A_t)\,d\mu(t)$ *and* $f:[a,b]\to\mathbb{R}$, $g:[m_A,M_A]\to\mathbb{R}$, $F:U\times V\to\mathbb{R}$, *where* $f([a,b])\subseteq U$, $g([m_A,M_A])\subseteq V$ *and* F *be bounded.*

If f *is convex and* F *is operator monotone in the first variable, then*

$$F\left[\int_T\Phi_t(f(A_t))\,d\mu(t)\,,\,g\left(\int_T\Phi_t(A_t)\,d\mu(t)\right)\right]\leqslant C_1\,1_K\leqslant C\,1_K, \tag{2.1}$$

where constants $C_1\equiv C_1(F,f,g,m,M,m_A,M_A)$ *and* $C\equiv C(F,f,g,m,M)$ *are*

$$\begin{aligned}
C_1 &:= \sup_{m_A\leqslant z\leqslant M_A}\left\{F\left[\frac{M-z}{M-m}f(m)+\frac{z-m}{M-m}f(M),g(z)\right]\right\}\\
&= \sup_{\frac{M-M_A}{M-m}\leqslant p\leqslant\frac{M-m_A}{M-m}}\left\{F[pf(m)+(1-p)f(M)\,,\,g(pm+(1-p)M)]\right\},\\
C &:= \sup_{m\leqslant z\leqslant M}\left\{F\left[\frac{M-z}{M-m}f(m)+\frac{z-m}{M-m}f(M),g(z)\right]\right\}\\
&= \sup_{0\leqslant p\leqslant 1}\left\{F[pf(m)+(1-p)f(M)\,,\,g(pm+(1-p)M)]\right\}.
\end{aligned}$$

If f *is concave, then the reverse inequality is valid in* (2.1) *with* inf *instead of* sup *in bounds* C_1 *and* C.

Proof. We only prove the convex case. Since $m\Phi_t(1_H)\leqslant\Phi_t(A_t)\leqslant M\Phi_t(1_H)$ and $\int_T\Phi_t(1_H)d\mu(t)=1_K$, then $m1_K\leqslant\int_T\Phi_t(A_t)d\mu(t)\leqslant M1_K$. Next, since m_A and M_A, are the bounds of the operator $\int_T\Phi_t(A_t)\,d\mu(t)$ it follows that $[m_A,M_A]\subseteq[m,M]$.

By using convexity of f and functional calculus, we obtain

$$\begin{aligned}\int_T \Phi_t(f(A_t))\,d\mu(t) &\leqslant \int_T \Phi_t\left(\frac{M1_H - A_t}{M-m} f(m) + \frac{A_t - m1_H}{M-m} f(M)\right) d\mu(t)\\ &= \frac{M1_K - \int_T \Phi_t(A_t)}{M-m} f(m) + \frac{\int_T \Phi_t(A_t) - m1_K}{M-m} f(M).\end{aligned}$$

Using operator monotonicity of $u \mapsto F(u,v)$ and boundedness of F, it follows

$$\begin{aligned}&F\left[\int_T \Phi_t(f(A_t))\,d\mu(t)\,,\, g\left(\int_T \Phi_t(A_t)\,d\mu(t)\right)\right]\\ &\leqslant F\left[\frac{M1_K - \int_T \Phi_t(A_t)}{M-m} f(m) + \frac{\int_T \Phi_t(A_t) - m1_K}{M-m} f(M)\,,\, g\left(\int_T \Phi_t(A_t)\,d\mu(t)\right)\right]\\ &\leqslant \sup_{m_A \leqslant z \leqslant M_A} \left\{F\left[\frac{M-z}{M-m} f(m) + \frac{z-m}{M-m} f(M), g(z)\right]\right\} 1_K\\ &\leqslant \sup_{m \leqslant z \leqslant M} \left\{F\left[\frac{M-z}{M-m} f(m) + \frac{z-m}{M-m} f(M), g(z)\right]\right\} 1_K. \qquad \square\end{aligned}$$

REMARK 2.2. We can obtain an inequality similar to the one in Theorem 2.1 in the case when $(\Phi_t)_{t\in T}$ is a non-unit field of positive linear mappings, i.e. when $\int_T \Phi_t(\mathbf{1})\,d\mu(t) = k\mathbf{1}$ for some positive scalar k. Then,

$$\begin{aligned}&F\left[\int_T \Phi_t(f(A_t))\,d\mu(t)\,,\, g\left(\int_T \Phi_t(A_t)\,d\mu(t)\right)\right]\\ &\leqslant \sup_{km_A \leqslant z \leqslant kM_A} \left\{F\left[\frac{kM-z}{M-m} f(m) + \frac{z-km}{M-m} f(M)\,,\, g(z)\right]\right\} 1_K\\ &\leqslant \sup_{km \leqslant z \leqslant kM} \left\{F\left[\frac{kM-z}{M-m} f(m) + \frac{z-km}{M-m} f(M)\,,\, g(z)\right]\right\} 1_K.\end{aligned}$$

This means that we obtain a better upper bound than the one given in [5, Theorem 2.3].

3. Difference type converse inequalities

We recall that the following generalization of Jensen's inequality holds (see [3, Theorem 1]). Let $(A_t)_{t\in T}$ and $(\Phi_t)_{t\in T}$ be as in Theorem 2.1. If f is an operator convex function on $[m,M]$ and $\alpha g \leqslant f$ on $[m,M]$ for some function g and real number α, then

$$0 \leqslant \int_T \Phi_t\left(f(A_t)\right) d\mu(t) - \alpha g\left(\int_T \Phi_t(A_t) d\mu(t)\right). \tag{3.1}$$

In this section we consider the difference type converses of the above inequality.

For convenience we introduce some abbreviations. Let $f : [m,M] \to \mathbb{R}$, $m < M$, be a convex or a concave function. We denote a linear function through $(m, f(m))$ and

$(M, f(M))$ by $f^{cho}_{[m,M]}$, i.e.

$$f^{cho}_{[m,M]}(z) = \frac{M-z}{M-m} f(m) + \frac{z-m}{M-m} f(M), \qquad z \in \mathbb{R}$$

and the slope and the intercept by k_f and l_f, respectively, i.e.

$$k_f = \frac{f(M)-f(m)}{M-m} \qquad \text{and} \qquad l_f = \frac{Mf(m)-mf(M)}{M-m}.$$

The following Theorem 3.1 and Corollary 3.2 are refinements of [1, Theorem 2.4].

THEOREM 3.1. *Let $(A_t)_{t\in T}$ be a bounded continuous field of self-adjoint elements in a unital C^*-algebra $\mathscr{A}$ with the spectra in $[m,M]$, $m<M$, defined on a locally compact Hausdorff space T equipped with a bounded Radon measure μ, and let $(\Phi_t)_{t\in T}$ be a unital field of positive linear maps $\Phi_t : \mathscr{A} \to \mathscr{B}$ from $\mathscr{A}$ to another unital C^*-algebra $\mathscr{B}$. Let m_A and M_A, $m_A \leqslant M_A$, be the bounds of $A = \int_T \Phi_t(A_t)\, d\mu(t)$ and $f : [m,M] \to \mathbb{R}$, $g : [m_A, M_A] \to \mathbb{R}$ be continuous functions.*

If f is convex, then

$$\int_T \Phi_t(f(A_t))\, d\mu(t) - \alpha g\left(\int_T \Phi_t(A_t)\, d\mu(t)\right) \leqslant \max_{m_A \leqslant z \leqslant M_A} \left\{k_f z + l_f - \alpha g(z)\right\} 1_K \tag{3.2}$$

holds and the bound in RHS of (3.2) exists for any m, M, m_A and M_A.

If f is concave, then the reverse inequality with min *instead of* max *is valid in (3.2). The bound in RHS of this inequality exists for any m, M, m_A and M_A.*

Proof. We put $F(u,v) = u - \alpha v$, $\alpha \in \mathbb{R}$ in Theorem 2.1. A function $z \mapsto k_f z + l_f - \alpha g(z)$ is continuous on $[m_A, m_A]$, so the global extremes exist. $\square$

In the following corollary, we give a way of determining the bounds placed in Theorem 3.1.

COROLLARY 3.2. *Let $(A_t)_{t\in T}$, $(\Phi_t)_{t\in T}$, A, f and g be as in Theorem 3.1.*

***(i)** Let $\alpha \leqslant 0$. If f is convex and g is convex, then*

$$\int_T \Phi_t(f(A_t))\, d\mu(t) - \alpha g\left(\int_T \Phi_t(A_t)\, d\mu(t)\right) \leqslant C_\alpha 1_K \tag{3.3}$$

holds with

$$C_\alpha = \max\left\{ f^{cho}_{[m,M]}(m_A) - \alpha g(m_A),\, f^{cho}_{[m,M]}(M_A) - \alpha g(M_A) \right\}. \tag{3.4}$$

But, if f is convex and g is concave, then the inequality (3.3) holds with

$$C_\alpha = \begin{cases} f^{cho}_{[m,M]}(m_A) - \alpha g(m_A) & \text{if } \alpha g'_-(z) \geqslant k_f \text{ for every } z \in (m_A, M_A), \\ f^{cho}_{[m,M]}(z_0) - \alpha g(z_0) & \text{if } \alpha g'_-(z_0) \leqslant k_f \leqslant \alpha g'_+(z_0) \text{ for some } z_0 \in (m_A, M_A), \\ f^{cho}_{[m,M]}(M_A) - \alpha g(M_A) & \text{if } \alpha g'_+(z) \leqslant k_f \text{ for every } z \in (m_A, M_A). \end{cases} \tag{3.5}$$

If f is concave and g is convex, then

$$c_\alpha 1_K \leqslant \int_T \Phi_t(f(A_t))\,d\mu(t) - \alpha g\left(\int_T \Phi_t(A_t)\,d\mu(t)\right) \tag{3.6}$$

holds with c_α which equals the right side in (3.5) *with reverse inequality signs.*

But, if f is concave and g is concave, then the inequality (3.6) *holds with c_α which equals the right side in* (3.4) *with* min *instead of* max.

(ii) *Let $\alpha \geqslant 0$.*

If f is convex and g is convex, then the inequality (3.3) *holds with C_α defined by* (3.5)*. But if f is convex and g is concave, then* (3.3) *holds with C_α defined by* (3.4)*.*

If f is concave and g is convex, then the inequality (3.6) *holds with c_α which equals the right side in* (3.4) *with* min *instead of* max*. But, if f is concave and g is concave, then* (3.6) *holds with c_α which equals the right side in* (3.5) *with reverse inequality signs.*

Proof. ***(i)*** : We only prove the cases when f is convex. If g is convex (resp. concave) we apply Proposition 5.2 (resp. Proposition 5.1) on the convex (resp. concave) function $h_\alpha = f^{cho}_{[m,M]}(z) - \alpha g(z)$, and get (3.4) (resp. (3.5)).

In the remaining cases the proof is essentially the same as in the above cases. □

Corollary 3.2 applied on the functions $f(z) = z^p$ and $g(z) = z^q$ gives the following corollary, which is a refinement of [1, Corollary 2.6].

COROLLARY 3.3. *Let $(A_t)_{t\in T}$, $(\Phi_t)_{t\in T}$ and A be as in Theorem* 3.1*, and additionally let operators A_t be strictly positive with the spectra in $[m,M]$, where $0 < m < M$.*

(i) *Let $\alpha \leqslant 0$.*

If $p,q \in (-\infty,0] \cup [1,\infty)$, then

$$\int_T \Phi_t(A_t^p)\,d\mu(t) - \alpha\left(\int_T \Phi_t(A_t)\,d\mu(t)\right)^q \leqslant C_\alpha^\star 1_K \tag{3.7}$$

holds with

$$C_\alpha^\star = \max\left\{k_{t^p} m_A + l_{t^p} - \alpha m_A^q,\, k_{t^p} M_A + l_{t^p} - \alpha M_A^q\right\}. \tag{3.8}$$

If $p \in (-\infty,0)$ and $q \in (0,1)$, then the inequality (3.7) *holds with*

$$C_\alpha^\star = \begin{cases} k_{t^p}\, m_A + l_{t^p} - \alpha\, m_A^q & \text{if } (\alpha\, q/k_{t^p})^{1/(1-q)} \leqslant m_A, \\ l_{t^p} + \alpha(q-1)\,(\alpha\, q/k_{t^p})^{q/(1-q)} & \text{if } m_A \leqslant (\alpha\, q/k_{t^p})^{1/(1-q)} \leqslant M_A, \\ k_{t^p}\, M_A + l_{t^p} - \alpha\, M_A^q & \text{if } (\alpha\, q/k_{t^p})^{1/(1-q)} \geqslant M_A. \end{cases} \tag{3.9}$$

If $p \in (0,1)$ and $q \in (-\infty,0)$, then

$$c_\alpha^\star 1_K \leqslant \int_T \Phi_t(A_t^p)\,d\mu(t) - \alpha\left(\int_T \Phi_t(A_t)\,d\mu(t)\right)^q \tag{3.10}$$

holds with $c_\alpha^\star$ *which equals the right side in* (3.9).

If $p,q \in [0,1]$, *then the inequality* (3.10) *holds with* $c_\alpha^\star$ *which equals the right side in* (3.8) *with* min *instead of* max.

(ii) *Let* $\alpha \geqslant 0$.

If $p,q \in (-\infty,0)\cup(1,\infty)$, *then* (3.7) *holds with* $C_\alpha^\star$ *defined by* (3.9). *But, if* $p \in (-\infty,0]\cup[1,+\infty)$ *and* $q \in [0,1]$, *then* (3.7) *holds with* $C_\alpha^\star$ *defined by* (3.8).

If $p \in [0,1]$ *and* $q \in (-\infty,0]\cup[1,\infty)$, *then* (3.10) *holds with* $c_\alpha^\star$ *which equals the right side in* (3.8) *with* min *instead of* max. *But, if* $p \in (0,1)$ *and* $q \in (0,1)$, *then* (3.10) *holds with* $c_\alpha^\star$ *which equals the right side in* (3.9).

Using Theorem 3.1 and Corollary 3.2 with $g = f$ and $\alpha = 1$ we have the following theorem.

THEOREM 3.4. *Let* $(A_t)_{t\in T}$ *be a bounded continuous field of self-adjoint elements in a unital* C^**-algebra* $\mathscr{A}$ *with the spectra in* $[m,M]$, $m < M$, *defined on a locally compact Hausdorff space* T *equipped with a bounded Radon measure* μ, *and let* $(\Phi_t)_{t\in T}$ *be a unital field of positive linear maps* $\Phi_t : \mathscr{A} \to \mathscr{B}$ *from* $\mathscr{A}$ *to another unital* C^*-*algebra* $\mathscr{B}$. *Let* m_A *and* M_A, $m_A \leqslant M_A$, *be the bounds of* $A = \int_T \Phi_t(A_t)\,d\mu(t)$ *and* $f:[m,M]\to\mathbb{R}$ *be a continuous function.*

If f *is convex, then*

$$0 \leqslant \int_T \Phi_t(f(A_t))\,d\mu(t) - f\left(\int_T \Phi_t(A_t)\,d\mu(t)\right) \leqslant \max_{m_A\leqslant z\leqslant M_A}\left\{f^{cho}_{[m,M]}(z) - f(z)\right\} 1_K \tag{3.11}$$

holds and the bound in RHS of (3.11) *exists for any* m, M, m_A *and* M_A.

The value of the constant

$$\bar{C} \equiv \bar{C}(f,m,M,m_A,M_A) := \max_{m_A\leqslant z\leqslant M_A}\left\{f^{cho}_{[m,M]}(z) - f(z)\right\}$$

can be determined as follows

$$\bar{C} = \begin{cases} f^{cho}_{[m,M]}(m_A) - f(m_A) & \text{if } f'_-(z) \geqslant k_f \text{ for every } z\in(m_A,M_A),\\ f^{cho}_{[m,M]}(z_0) - f(z_0) & \text{if } g'_-(z_0) \leqslant k_f \leqslant g'_+(z_0) \text{ for some } z_0\in(m_A,M_A),\\ f^{cho}_{[m,M]}(M_A) - f(M_A) & \text{if } g'_+(z) \leqslant k_f \text{ for every } z\in(m_A,M_A). \end{cases} \tag{3.12}$$

If f *is concave, then the reverse inequality with* min *instead of* max *is valid in* (3.11). *The bound in this inequality exists for any* m, M, m_A *and* M_A. *The value of the constant*

$$\bar{c} \equiv \bar{c}(f,m,M,m_A,M_A) := \min_{m_A\leqslant z\leqslant M_A}\left\{f^{cho}_{[m,M]}(z) - f(z)\right\}$$

can be determined as in the right side in (3.12) *with reverse inequality signs.*

If f is a strictly convex differentiable function on $[m_A, M_A]$, then we obtain the following corollary of Theorem 3.4. This is a refinement of [1, Corollary 2.16].

COROLLARY 3.5. *Let* $(A_t)_{t\in T}$, $(\Phi_t)_{t\in T}$ *and* A *be as in Theorem* 3.4. *Let* $f:[m,M]\to\mathbb{R}$ *be a continuous function. If* f *is strictly convex differentiable on* $[m_A,M_A]$, *then*

$$0\leqslant \int_T \Phi_t(f(A_t))\,d\mu(t) - f\left(\int_T \Phi_t(A_t)\,d\mu(t)\right) \leqslant \left(k_f z_0 + l_f - f(z_0)\right)1_K, \tag{3.13}$$

where

$$z_0=\begin{cases} m_A & \text{if } f'(m_A)\geqslant k_f,\\ f'^{-1}\left(k_f\right) & \text{if } f'(m_A)\leqslant k_f\leqslant f'(M_A),\\ M_A & \text{if } f'(M_A)\leqslant k_f. \end{cases} \tag{3.14}$$

The global upper bound is $C(m,M,f)=k_f\bar{z}_0+l_f-f(\bar{z}_0)$, *where* $\bar{z}_0=(f')^{-1}(k_f)\in(m,M)$. *The upper bound in RHS of* (3.13) *is better than the global upper bound provided that either* $f'(m_A)\geqslant k_f$ *or* $f'(M_A)\leqslant k_f$.

In the dual case, when f *is strictly concave differentiable on* $[m_A,M_A]$, *then the reverse inequality is valid in* (3.13)*, with* z_0 *which equals the right side in* (3.14) *with reverse inequality signs. The global lower bound is defined as the global upper bound in the convex case. The lower bound in the reverse inequality in* (3.13) *is better than the global lower bound provided that either* $f'(m_A)\leqslant k_f$ *or* $f'(M_A)\geqslant k_f$.

Proof. We only prove the cases when f is strictly convex differentiable on $[m_A,M_A]$. The inequality (3.13) follows from Theorem 3.4 by using the differential calculus. Since $h(z)=k_f z+l_f-f(z)$ is a continuous strictly concave function on $[m,M]$, then there is exactly one point $z_0\in[m,M]$ which achieves the global maximum. If neither of these points is in the interval $[m_A,M_A]$, then the global maximum in $[m_A,M_A]$ is less than the global maximum in $[m,M]$. □

Using Corollary 3.3 with $q=p$, $\alpha=1$ or applying Corollary 3.5 we have the following corollary, which is a refinement of [1, Corollary 2.18].

COROLLARY 3.6. *Let* $(A_t)_{t\in T}$, $(\Phi_t)_{t\in T}$ *and* A *be as in Theorem* 3.4*, and additionally let operators* A_t *be strictly positive with the spectra in* $[m,M]$, *where* $0<m<M$. *Then*

$$0\leqslant \int_T \Phi_t(A_t^p)\,d\mu(t) - \left(\int_T \Phi_t(A_t)\,d\mu(t)\right)^p \leqslant \bar{C}(m_A,M_A,m,M,p)1_K \leqslant C(m,M,p)1_K,$$

for $p\notin(0,1)$, *and*

$$0\geqslant \int_T \Phi_t(A_t^p)\,d\mu(t) - \left(\int_T \Phi_t(A_t)\,d\mu(t)\right)^p \geqslant \bar{c}(m_A,M_A,m,M,p)1_K \geqslant C(m,M,p)1_K,$$

for $p\in(0,1)$, *where*

$$\bar{C}(m_A,M_A,m,M,p)=\begin{cases} k_{t^p}m_A+l_{t^p}-m_A^p & \text{if } pm_A^{p-1}\geqslant k_{t^p},\\ C(m,M,p) & \text{if } pm_A^{p-1}\leqslant k_{t^p}\leqslant pM_A^{p-1},\\ k_{t^p}M_A+l_{t^p}-M_A^p & \text{if } pM_A^{p-1}\leqslant k_{t^p}, \end{cases} \tag{3.15}$$

and $\bar{c}(m_A, M_A, m, M, p)$ *equals the right side in* (3.15) *with reverse inequality signs.* $C(m,M,p)$ *is the known constant Kantorovich type for difference (see i.e.* [1, §2.7]*):*

$$C(m,M,p) = (p-1)\left(\frac{M^p - m^p}{p(M-m)}\right)^{1/(p-1)} + \frac{Mm^p - mM^p}{M-m}, \qquad \text{for } p \in \mathbb{R}.$$

4. Ratio type converse inequalities

In this section we consider the ratio type converses of Jensen's inequality. The following Theorem 4.1 and Corollary 4.3 are refinements of [1, Theorem 2.9].

THEOREM 4.1. *Let* $(A_t)_{t\in T}$ *be a bounded continuous field of self-adjoint elements in a unital* C^**-algebra* $\mathscr{A}$ *with the spectra in* $[m,M]$, $m < M$, *defined on a locally compact Hausdorff space* T *equipped with a bounded Radon measure* μ, *and let* $(\Phi_t)_{t\in T}$ *be a unital field of positive linear maps* $\Phi_t : \mathscr{A} \to \mathscr{B}$ *from* $\mathscr{A}$ *to another unital* C^*-*algebra* $\mathscr{B}$. *Let* m_A *and* M_A, $m_A \leqslant M_A$, *be the bounds of* $A = \int_T \Phi_t(A_t)\,d\mu(t)$ *and* $f : [m,M] \to \mathbb{R}$ *be a continuous function and* $g : [m_A, M_A] \to \mathbb{R}$ *be a strictly positive continuous function.*

If f *is convex, then*

$$\int_T \Phi_t(f(A_t))\,d\mu(t) \leqslant \max_{m_A \leqslant z \leqslant M_A} \left\{\frac{k_f z + l_f}{g(z)}\right\} g\left(\int_T \Phi_t(A_t)\,d\mu(t)\right) \tag{4.1}$$

holds and the bound in RHS of (4.1) *exists for any* m, M, m_A *and* M_A.

If f *is concave, then the reverse inequality with* min *instead of* max *is valid in* (4.1)*. The bound in RHS of this inequality exists for any* m, M, m_A *and* M_A.

Proof. We put $F(u,v) = v^{-\frac{1}{2}} u v^{-\frac{1}{2}}$ in Theorem 2.1.

A function $z \mapsto \frac{k_f z + l_f}{g(z)}$ is continuous on $[m_A, m_A]$, so the global extremes exist. □

REMARK 4.2. If f is convex and g is strictly negative on $[m_A, M_A]$, then the inequality with min instead of max is valid in (4.1). If f is concave and g is strictly negative on $[m_A, M_A]$, then the reverse inequality is valid in (4.1).

In the following corollary, we give a way of determining the bounds placed in Theorem 4.1.

COROLLARY 4.3. *Let* $(A_t)_{t\in T}$, $(\Phi_t)_{t\in T}$, A, f *and* g *be as in Theorem* 4.1. *Additionally, let* $f^{cho}_{[m,M]}$ *and* g *be strictly positive on* $[m_A, M_A]$.

If f *is convex and* g *is convex, then*

$$\int_T \Phi_t(f(A_t))\,d\mu(t) \leqslant C\,g\left(\int_T \Phi_t(A_t)\,d\mu(t)\right) \tag{4.2}$$

holds with

$$C=\begin{cases}\frac{f^{cho}_{[m,M]}(m_A)}{g(m_A)} & \text{if } g'_-(z)\geqslant \frac{k_f g(z)}{k_f z+l_f} \text{ for every } z\in(m_A,M_A),\\ \frac{f^{cho}_{[m,M]}(z_0)}{g(z_0)} & \text{if } g'_-(z_0)\leqslant \frac{k_f g(z_0)}{k_f z_0+l_f}\leqslant g'_+(z_0) \text{ for some } z_0\in(m_A,M_A),\\ \frac{f^{cho}_{[m,M]}(M_A)}{g(M_A)} & \text{if } g'_+(z)\leqslant \frac{k_f g(z)}{k_f z+l_f} \text{ for every } z\in(m_A,M_A).\end{cases} \tag{4.3}$$

If f is convex and g is concave, then the inequality (4.2) *holds with*

$$C=\max\left\{\frac{f^{cho}_{[m,M]}(m_A)}{g(m_A)},\frac{f^{cho}_{[m,M]}(M_A)}{g(M_A)}\right\}. \tag{4.4}$$

If f is concave and g is convex, then

$$\int_T \Phi_t(f(A_t))\,d\mu(t)\geqslant c\,g\left(\int_T \Phi_t(A_t)\,d\mu(t)\right) \tag{4.5}$$

holds with c which equals the right side in (4.4) *with* min *instead of* max.

If f is concave and g is concave, then the inequality (4.5) *holds with c which equals the right side in* (4.3) *with reverse inequality signs.*

Proof. We only prove the cases when f is convex. If g is convex (resp. concave) we apply Proposition 5.3 (resp. Proposition 5.5) on the ratio function $h(z)=\frac{f^{cho}_{[m,M]}(z)}{g(z)}$ with the convex (resp. concave) denominator g, and so we get (4.3) (resp. (4.4)). □

Corollary 4.3 applied on the functions $f(z)=z^p$ and $g(z)=z^q$ gives the following corollary, which is a refinement of [1, Corollary 2.11].

COROLLARY 4.4. *Let $(A_t)_{t\in T}$, $(\Phi_t)_{t\in T}$ and A be as in Theorem* 4.1, *and additionally let operators A_t be strictly positive with the spectra in $[m,M]$, where $0<m<M$.*

If $p,q\in(-\infty,0)\cup(1,\infty)$, then

$$\int_T \Phi_t(A_t^p)\,d\mu(t)\leqslant C^\star\left(\int_T \Phi_t(A_t)\,d\mu(t)\right)^q \tag{4.6}$$

holds with

$$C^\star=\begin{cases}\frac{k_{t^p}\,m_A+l_{t^p}}{m_A^q} & \text{if } \frac{q}{1-q}\frac{l_{t^p}}{k_{t^p}}\leqslant m_A,\\ \frac{l_{t^p}}{1-q}\left(\frac{1-q}{q}\frac{k_{t^p}}{l_{t^p}}\right)^q & \text{if } m_A\leqslant \frac{q}{1-q}\frac{l_{t^p}}{k_{t^p}}\leqslant M_A,\\ \frac{k_{t^p}\,M_A+l_{t^p}}{M_A^q} & \text{if } \frac{q}{1-q}\frac{l_{t^p}}{k_{t^p}}\geqslant M_A.\end{cases} \tag{4.7}$$

If $p\in(-\infty,0]\cup[1,\infty)$ and $q\in[0,1]$, then the inequality (4.6) *holds with*

$$C^\star=\max\left\{\frac{k_{t^p}m_A+l_{t^p}}{m_A^q},\frac{k_{t^p}M_A+l_{t^p}}{M_A^q}\right\}. \tag{4.8}$$

If $p \in [0,1]$ *and* $q \in (-\infty,0] \cup [1,\infty)$*, then*

$$\int_T \Phi_t(A_t^p)\, d\mu(t) \geqslant c^\star \left(\int_T \Phi_t(A_t)\, d\mu(t) \right)^q \tag{4.9}$$

holds with c_α *which equals the right side in* (4.8) *with* min *instead of* max*.*

If $p,q \in (0,1)$*, then the inequality* (4.9) *holds with* $c^\star$ *which equals the right side in* (4.7)*.*

Using Theorem 4.1, Proposition 5.4 and 5.6 with $g = f$ we have the following theorem.

THEOREM 4.5. *Let* $(A_t)_{t\in T}$ *be a bounded continuous field of self-adjoint elements in a unital* C^**-algebra* $\mathscr{A}$ *with the spectra in* $[m,M]$*,* $m < M$*, defined on a locally compact Hausdorff space* T *equipped with a bounded Radon measure* μ*, and let* $(\Phi_t)_{t\in T}$ *be a unital field of positive linear maps* $\Phi_t : \mathscr{A} \to \mathscr{B}$ *from* $\mathscr{A}$ *to another unital* C^*-*algebra* $\mathscr{B}$*. Let* m_A *and* M_A*,* $m_A \leqslant M_A$*, be the bounds of* $A = \int_T \Phi_t(A_t)\, d\mu(t)$*. let* $f : [m,M] \to \mathbb{R}$ *be a continuous function and strictly positive on* $[m_A, M_A]$*.*

If f *is convex, then*

$$\int_T \Phi_t(f(A_t))\, d\mu(t) \leqslant \max_{m_A \leqslant z \leqslant M_A} \left\{ \frac{f^{cho}_{[m,M]}(z)}{f(z)} \right\} f\left(\int_T \Phi_t(A_t)\, d\mu(t) \right) \tag{4.10}$$

holds and the bound in RHS of (4.10) *exists for any* m, M, m_A *and* M_A*.*

The value of the constant

$$\bar{C} \equiv \bar{C}(f,m,M,m_A,M_A) := \max_{m_A \leqslant z \leqslant M_A} \left\{ \frac{f^{cho}_{[m,M]}(z)}{f(z)} \right\}$$

can be determined as follows:

$$\bar{C} = \begin{cases} \frac{f^{cho}_{[m,M]}(m_A)}{f(m_A)} & \text{if } f'_-(z) \geqslant \frac{k_f f(z)}{k_f z + l_f} \text{ for every } z \in (m_A, M_A), \\ \frac{f^{cho}_{[m,M]}(z_0)}{f(z_0)} & \text{if } f'_-(z_0) \leqslant \frac{k_f f(z_0)}{k_f z_0 + l_f} \leqslant f'_+(z_0) \text{ for some } z_0 \in (m_A, M_A), \\ \frac{f^{cho}_{[m,M]}(M_A)}{f(M_A)} & \text{if } f'_+(z) \leqslant \frac{k_f f(z)}{k_f z + l_f} \text{ for every } z \in (m_A, M_A). \end{cases} \tag{4.11}$$

If f *is concave, then the reverse inequality with* min *instead of* max *is valid in* (4.10)*. The bound in this inequality exists for any* m, M, m_A *and* M_A*. The value of the constant*

$$\bar{c} \equiv \bar{c}(f,m,M,m_A,M_A) := \min_{m_A \leqslant z \leqslant M_A} \left\{ \frac{f^{cho}_{[m,M]}(z)}{f(z)} \right\}$$

can be determined as in the right side in (4.10) *with reverse inequality signs.*

REMARK 4.6. If f is convex and strictly negative on $[m_A, M_A]$, then the inequality with min instead of max is valid in (4.10). If f is concave and strictly negative on $[m_A, M_A]$, then the reverse inequality is valid in (4.10).

If f is a strictly convex differentiable function on $[m_A, M_A]$, then we obtain the following corollary of Theorem 4.5. This is a refinement of [1, Corollary 2.10].

COROLLARY 4.7. *Let* $(A_t)_{t\in T}$, $(\Phi_t)_{t\in T}$ *and* A *be as in Theorem* 4.5. *Let* $f : [m,M] \to \mathbb{R}$ *be a continuous function and* $f(m), f(M) > 0$. *If* f *is strictly positive and strictly convex twice differentiable on* $[m_A, M_A]$, *then*

$$\int_T \Phi_t(f(A_t))\,d\mu(t) \leqslant \left(\frac{k_f z_0 + l_f}{f(z_0)}\right) f\left(\int_T \Phi_t(A_t)\,d\mu(t)\right), \tag{4.12}$$

where $z_0 \in (m_A, M_A)$ *is defined as the unique solution of* $k_f f(z) = (k_f z + l_f) f'(z)$ *provided* $(k_f m_A + l_f) f'(m_A)/f(m_A) \leqslant k_f \leqslant (k_f M_A + l_f) f'(M_A)/f(M_A)$, *otherwise* z_0 *is defined as* m_A *or* M_A *provided* $k_f \leqslant (k_f m_A + l_f) f'(m_A)/f(m_A)$ *or* $k_f \geqslant (k_f M_A + l_f) f'(M_A)/f(M_A)$, *respectively.*

The global upper bound is $C(m,M,f) = (k_f \bar{z}_0 + l_f)/f(\bar{z}_0)$, *where* $\bar{z}_0 \in (m,M)$ *is defined as the unique solution of* $k_f f(z) = (k_f z + l_f) f'(z)$. *The upper bound in RHS of* (4.12) *is better than the global upper bound provided that either* $k_f \leqslant (k_f m_A + l_f) f'(m_A)/f(m_A)$ *or* $k_f \geqslant (k_f M_A + l_f) f'(M_A)/f(M_A)$.

In the dual case, when f *is positive and strictly concave differentiable on* $[m_A, M_A]$, *then the reverse inequality is valid in* (4.12), *with* z_0 *is defined as in* (4.12) *with reverse inequality signs. The global lower bound is defined as the global upper bound in the convex case. The lower bound in the reverse inequality in* (4.12) *is better than the global lower bound provided that either* $k_f \geqslant (k_f m_A + l_f) f'(m_A)/f(m_A)$ *or* $k_f \leqslant (k_f M_A + l_f) f'(M_A)/f(M_A)$.

Proof. We only prove the cases when f is strictly convex differentiable on $[m_A, M_A]$. The inequality (4.12) follows from Theorem 4.5 by using the differential calculus.

Next, we put $h(z) = (k_f z + l_f)/f(z)$. Then $h'(z) = H(z)/f(z)^2$, where $H(z) = k_f f(z) - (k_f z + l_f) f'(z)$. Due to the strict convexity of f on $[m_A, M_A]$ and since $f(m), f(M) > 0$, it follows that $H'(z) = -(k_f z + l_f) f''(z) < 0$. Hence $H(z)$ is decreasing on $[m_A, M_A]$. If $H(m_A)H(M_A) \leqslant 0$, then the minimum value of the function h on $[m_A, M_A]$ is attained in z_0 which is the unique solution of the equation $H(z) = 0$. Otherwise, if $H(m_A)H(M_A) \geqslant 0$, then this minimum value is attained in m_A or M_A according to $H(m_A) \leqslant 0$ or $H(M_A) \geqslant 0$.

Since $h(z) = (k_f z + l_f)/f(z)$ is a continuous function on $[m,M]$, then the global maximum in $[m_A, M_A]$ is less than the global maximum in $[m,M]$. □

Using Corollary 4.4 with $q = p$ or applying Corollary 4.7 we have the following corollary, which is a refinement of [1, Corollary 2.12].

COROLLARY 4.8. *Let $(A_t)_{t\in T}$, $(\Phi_t)_{t\in T}$ and A be as in Theorem 4.5, and additionally let operators A_t be strictly positive with the spectra in $[m,M]$, where $0<m<M$. Then*

$$\begin{aligned}&\int_T \Phi_t(A_t^p)\,d\mu(t)\\ &\leqslant \bar{K}(m_A,M_A,m,M,p)\left(\int_T \Phi_t(A_t)\,d\mu(t)\right)^p \leqslant K(m,M,p)\left(\int_T \Phi_t(A_t)\,d\mu(t)\right)^p,\end{aligned}$$

for $p\notin(0,1)$, and

$$\begin{aligned}&\int_T \Phi_t(A_t^p)\,d\mu(t)\\ &\geqslant \bar{k}(m_A,M_A,m,M,p)\left(\int_T \Phi_t(A_t)\,d\mu(t)\right)^p \geqslant K(m,M,p)\left(\int_T \Phi_t(A_t)\,d\mu(t)\right)^p,\end{aligned}$$

for $p\in(0,1)$, where

$$\bar{K}(m_A,M_A,m,M,p)=\begin{cases}\frac{k_{t^p}\,m_A+l_{t^p}}{m_A^p} & \text{if } p\,l_{t^p}/m_A \geqslant (1-p)\,k_{t^p},\\ K(m,M,p) & \text{if } p\,l_{t^p}/m_A < (1-p)\,k_{t^p} < p\,l_{t^p}/M_A,\\ \frac{k_{t^p}\,M_A+l_{t^p}}{M_A^p} & \text{if } p\,l_{t^p}/M_A \leqslant (1-p)\,k_{t^p},\end{cases} \tag{4.13}$$

and $\bar{k}(m_A,M_A,m,M,p)$ equals the right side in (4.13) with reverse inequality signs. $K(m,M,p)$ is the known Kantorovich constant (see i.e. [1, §2.7]):

$$K(m,M,p)=K(m,M,p):=\frac{mM^p-Mm^p}{(p-1)(M-m)}\left(\frac{p-1}{p}\frac{M^p-m^p}{mM^p-Mm^p}\right)^p,\ \textit{for } p\in\mathbb{R}.$$

REMARK 4.9. We can obtain inequalities similar to the ones in Section §3 and §4 in the case when $(\Phi_t)_{t\in T}$ is a field of positive linear mappings such that $\int_T \Phi_t(\mathbf{1})\,d\mu(t) = k\mathbf{1}$ for some positive scalar k. The details are left to the interested reader.

5. Calculating the extreme values

In this section we give the calculation of extreme values of a difference or ratio function $y=h(z)$, of a linear function $y=kx+l$ and a continuous convex or concave function $y=g(x)$ on a closed interval. The basic facts about the convex and concave functions can be found e.g. in books [7, 6].

We first examine two cases for the difference.

PROPOSITION 5.1. *Let $g:[a,b]\to\mathbb{R}$ be a continuous function and let $h(z)=kz+l-g(z)$ be a difference function. If g is convex, then*

$$\min_{a\leqslant z\leqslant b} h(z)=\min\{h(a),\,h(b)\} \tag{5.1}$$

and

$$\max_{a\leqslant z\leqslant b} h(z) = \begin{cases} h(a) & \text{if } g'_-(z) \geqslant k \text{ for every } z\in(a,b), \\ h(z_0) & \text{if } g'_-(z_0) \leqslant k \leqslant g'_+(z_0) \text{ for some } z_0\in(a,b), \\ h(b) & \text{if } g'_+(z) \leqslant k \text{ for every } z\in(a,b). \end{cases} \tag{5.2}$$

Additionally, if g is strictly convex and h is not monotone, then a unique number $z_0 \in (a,b)$ exists so that

$$h(z_0) = \max_{a\leqslant z\leqslant b} h(z). \tag{5.3}$$

Proof. A function $y = h(z)$ is continuously concave because it is the sum of two continuous concave functions $y = kz + l$ and $y = -g(z)$. Since a function h is lower bounded by the chord line through endpoints $P_a(a,h(a))$ and $P_b(b,h(b))$, then (5.1) holds. Next, (5.2) follows from the global maximum property for concave functions. With additional assumptions the equality (5.3) follows from the strict concavity of h. □

PROPOSITION 5.2. *Let $g : [a,b] \to \mathbb{R}$ be a continuous function and let $h(z) = kz + l - g(z)$ be a difference function. If g is concave, then*

$$\max_{a\leqslant z\leqslant b} h(z) = \max\{h(a), h(b)\}$$

and

$$\min_{a\leqslant z\leqslant b} h(z) = \begin{cases} h(a) & \text{if } g'_-(z) \leqslant k \text{ for every } z\in(a,b), \\ h(z_0) & \text{if } g'_+(z_0) \leqslant k \leqslant g'_-(z_0) \text{ for some } z_0\in(a,b), \\ h(b) & \text{if } g'_+(z) \geqslant k \text{ for every } z\in(a,b). \end{cases}$$

Additionally, if g is strictly concave and h is not monotone, then a unique number $z_0 \in (a,b)$ exists so that

$$h(z_0) = \min_{a\leqslant z\leqslant b} h(z).$$

Proof. The proof is essentially the same as the one in Proposition 5.1. □

We now examine four cases for the ratio.

PROPOSITION 5.3. *Let $g : [a,b] \to \mathbb{R}$ be either a strictly positive or strictly negative continuous function and let $h(z) = \frac{kz+l}{g(z)}$ be a ratio function with strictly positive numerator. If g is convex, then*

$$\min_{a\leqslant z\leqslant b} h(z) = \min\{h(a), h(b)\} \tag{5.4}$$

and

$$\max_{a\leqslant z\leqslant b} h(z) = \begin{cases} h(a) & \text{if } g'_-(z) \geqslant \frac{kg(z)}{kz+l} \text{ for every } z\in(a,b), \\ h(z_0) & \text{if } g'_-(z_0) \leqslant \frac{kg(z_0)}{kz_0+l} \leqslant g'_+(z_0) \text{ for some } z_0\in(a,b), \\ h(b) & \text{if } g'_+(z) \leqslant \frac{kg(z)}{kz+l} \text{ for every } z\in(a,b). \end{cases} \tag{5.5}$$

Additionally, if g is strictly convex and h is not monotone, then a unique number $z_0 \in (a,b)$ exists so that

$$h(z_0) = \max_{a \leqslant z \leqslant b} h(z). \tag{5.6}$$

Proof. Maximum value: A function $y = h(z)$ is continuous on $[a,b]$ because it is the ratio of two continuous functions. Then there exists $z_0 \in [a,b]$ such that $h(z_0) = \max_{a \leqslant z \leqslant b} h(z)$. Also, since g is convex, then $g'_-(z)$ and $g'_+)z)$ exist and $g'_-(z) \leqslant g'_+(z)$ on (a,b). Then h'_- and h'_+ exist and

$$h'_{\mp}(z) = \frac{k\,g(z) - (kz+l)\,g'_{\mp}(z)}{(g(z))^2}.$$

First we observe the case when h is not monotone on $[a,b]$. Then there exists $z_0 \in (a,b)$ such that $h(z_0) = \max_{a \leqslant z \leqslant b} h(z)$. So for every $z \in (a,b)$ we have

$$\begin{aligned}
&(kz+l)/g(z) \leqslant (kz_0+l)/g(z_0) && \text{(because } h(z_0) \text{ is maximum)},\\
&(kz+l)\,g(z_0) \leqslant (kz+l)\,g(z) + k\,g(z)\,(z_0 - z) && \text{(because } g > 0 \text{ or } g < 0),\\
&(kz+l)\,\mu_g(z)\,(z_0 - z) \leqslant (kz+l)\,(g(z_0) - g(z)) \leqslant k\,g(z)\,(z_0 - z) && \\
& && \text{(because } g \text{ is convex)},
\end{aligned}$$

$$g'_-(z) \leqslant \mu_g(z) \leqslant \frac{k\,g(z)}{kz+l} \text{ for } a < z < z_0 \text{ and } g'_+(z) \geqslant \mu_g(z) \geqslant \frac{k\,g(z)}{kz+l} \text{ for } b > z > z_0,$$

where $\mu_g(z)$ is a subdifferential of the function g in z, i.e. $\mu_g(z) \in [g'_-(z), g'_+(z)]$. So

$$h'_-(z) \geqslant 0 \quad \text{for } a < z < z_0 \qquad \text{and} \qquad h'_+(z) \leqslant 0 \quad \text{for } b > z > z_0. \tag{5.7}$$

It follows that for each number z_0 at which the function h has the global maximum on $[a,b]$ the conditione $g'_-(z_0) \leqslant \frac{kg(z_0)}{kz_0+l} \leqslant g'_+(z_0)$ is valid.

In the case when h is monotonically decreasing on $[a,b]$, we have $\max_{a \leqslant z \leqslant b} h(z) = h(a)$ and $h'_-(z) \leqslant 0$ for all $z \in (a,b)$, which imply that $g'_-(z) \geqslant \frac{kg(z)}{kz+l}$ for every $z \in (a,b)$. In the same way we can observe the case when h is monotonically increasing.

With additional assumptions it follows by using (5.7) that the function h is strictly increasing on $(a, z_0]$ and strictly decreasing on $[z_0, b)$. Hence the equality (5.6) is valid.

Minimum value: There does not exist $z_0 \in (a,b)$ at which the function h has the global minimum. Indeed, if h is not a monotone function on $[a,b]$, it follows by using (5.7) that h is increasing on $(a, \bar{z}_0]$ and decreasing on $[\bar{z}_0, b)$, where $\bar{z}_0 \in (a,b)$ is the point at which the function h has the global maximum. It follows that the function h does not have a global minimum on (a,b), and consequently (5.4) is valid. □

Similarly to Proposition 5.3 we obtain the following result.

PROPOSITION 5.4. *Let $g : [a,b] \to \mathbb{R}$ be either a strictly positive or strictly negative continuous function and let $h(z) = \frac{kz+l}{g(z)}$ be a ratio function with a strictly negative* *numerator. If g is convex, then the equality* (5.4) *is valid with* max *instead of* min, *and the equality* (5.5) *is valid with* min *instead of* max.

Additionally, if g is strictly convex and h is not monotone, then the equality (5.6) *is valid with* min *instead of* max.

PROPOSITION 5.5. *Let $g:[a,b]\to\mathbb{R}$ be either a strictly positive or strictly negative continuous function and let $h(z)=\frac{kz+l}{g(z)}$ be a ratio function with a strictly positive numerator. If g is concave, then*

$$\max_{a\leqslant z\leqslant b} h(z)=\max\{h(a),h(b)\}. \tag{5.8}$$

and

$$\min_{a\leqslant z\leqslant b} h(z)=\begin{cases} h(a) & \text{if } g'_-(z)\leqslant \frac{kg(z)}{kz+l} \text{ for every } z\in(a,b),\\ h(z_0) & \text{if } g'_+(z_0)\leqslant \frac{kg(z_0)}{kz_0+l}\leqslant g'_-(z_0) \text{ for some } z_0\in(a,b),\\ h(b) & \text{if } g'_+(z)\geqslant \frac{kg(z)}{kz+l} \text{ for every } z\in(a,b).\end{cases} \tag{5.9}$$

Additionally, if g is strictly concave and h is not monotone, then a unique number $z_0\in(a,b)$ exists so that

$$h(z_0)=\min_{a\leqslant z\leqslant b} h(z). \tag{5.10}$$

Proof. The proof is the same as the one in Proposition 5.3. □

Similarly to the above proposition we obtain the following result.

PROPOSITION 5.6. *Let $g:[a,b]\to\mathbb{R}$ be either a strictly positive or strictly negative continuous function and let $h(z)=\frac{kz+l}{g(z)}$ be a ratio function with a strictly negative numerator. If g is concave, then the equality* (5.8) *is valid with* min *instead of* max, *and the equality* (5.9) *is valid with* max *instead of* min.

Additionally, if g is strictly concave and h is not monotone, then the equality (5.10) *is valid with* max *instead of* min.

REFERENCES

[1] T. FURUTA, J. MIĆIĆ HOT, J. PEČARIĆ, Y. SEO, *Mond-Pečarić Method in Operator Inequalities*, Element, Zagreb, 2005.

[2] F. HANSEN, J. PEČARIĆ, I. PERIĆ, *A generalization of discrete Jensen's operator inequality and its converses*, Hungarian-Croatian mathematical workshop, Debrecen, 2005.

[3] F. HANSEN, J. PEČARIĆ AND I. PERIĆ, *Jensen's operator inequality and its converses*, Math. Scand. **100** (2007), 61–73.

[4] J. MIĆIĆ, Z. PAVIĆ AND J. PEČARIĆ, *Jensen's inequality for operators without operator convexity*, Linear Algebra Appl. **434** (2011), 1228–1237.

[5] J. MIĆIĆ, J. PEČARIĆ AND Y. SEO, *Converses of Jensen's operator inequality*, Oper. and Matr. **4** (2010), 385–403.

[6] J. E. PEČARIĆ, F. PROSCHAN, Y. L. TONG, *Convex Functions, Partial Orderings, and Statistical Applications*, Academic Press, San Diego, 1992.
[7] C. P. NICULESCU, L. E. PERSSON, *Convex Functions and Their Applications*, Canadian Mathematical Society, 2006.

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