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Simultaneous stabilization and trajectory tracking of underactuated mechanical systems with included actuators dynamics

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Abstract The paper deals with a novel control algorithm for simultaneous stabilization and trajectory tracking of underactuated nonlinear mechanical systems (UNMS) with included actuators dynamics. Simultaneous stabilization and trajectory tracking refer to arbitrary chosen actuated and unactuated degrees of freedom (DOF) of the system. The proposed control approach can be applied both to the second-order nonholonomic systems and the systems with input coupling, while a general model of actuators dynamics includes electrical, pneumatic, and hydraulic drives. Control law is based on linear combination of two control signals, where the first signal is designed to separately control only actuated DOF, and second to separately control only unactuated DOF. Simulation example of rotational inverted pendulum driven by electrical DC motor is presented, showing the effectiveness of the proposed approach.

Keywords Underactuated nonlinear mechanical systems · Sliding mode control · Trajectory tracking · Stabilization · Actuators dynamics

1 Introduction

Underactuated nonlinear mechanical systems (UNMS) are defined as systems in which the dimension of the configuration space exceeds that of the control input space. The difficulty of the control problem for UNMS is due to the reduced dimension of the input space. Examples of UNMS are mobile robots, cranes, airplanes, spacecrafts, missiles, underwater vehicles, surface vessels, underactuated robot manipulators, etc. In this article, we deal with two the most often used classes of UNMS, i.e., second-order nonholonomic mechanical systems and UNMS with input coupling. These two classes of UNMS are distinguished because of their often usage in different research fields. For instance, nonholonomic systems are the most used in robotics, while UNMS with input coupling are the most used in aerospace and

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naval research fields. Nonholonomic systems are systems which satisfy classical nonholonomic constraints [32]. If the system is a first-order nonholonomic system, then it satisfies generalized coordinates and velocities constraints of the form $\Phi(\mathbf{q}, \dot{\mathbf{q}}) = \mathbf{0}$, and the constraints are nonintegrable (such constraints often occur in kinematic mathematical models such are wheeled mobile robots and wheeled vehicles). If the system is a second-order nonholonomic system, then it satisfies constraints on generalized coordinates, velocities, and accelerations of the form $\Phi(\mathbf{q}, \dot{\mathbf{q}}, \ddot{\mathbf{q}}) = \mathbf{0}$ and these constraints are nonintegrable (such constraints often occur in dynamic mathematical models for underactuated manipulators, underwater vehicles, surface vessel, airplanes, space robots, etc.) [2]. If UNMS with second-order nonholonomic constraints of the form $\Phi(\mathbf{q}, \dot{\mathbf{q}}, \ddot{\mathbf{q}}) = \mathbf{0}$ are replaced by $\Phi(\mathbf{q}, \dot{\mathbf{q}}, \ddot{\mathbf{q}}) \not\equiv \mathbf{0}$, then such systems are called UNMS with input coupling (such systems often occur in dynamic mathematical models where the actuator's dynamics affects directly on more than one generalized acceleration) [18, 29].

The well-known control property of a general class[1] of UNMS is its impossibility of asymptotic stabilization around the equilibrium point using a continuous linear-time invariant feedback control law [10]. This theorem is applied to general form of underactuated manipulators [20] where the possibility of asymptotic stabilization to the equilibrium manifold is shown. General class of UNMS can be asymptotically stabilized using discontinuous [4], structural-variable [5] or time-variant [12] control feedback law. Conditions for partial and full integrability of underactuated robot manipulators are shown in [20], and for underactuated vehicles in [31].

Euler–Lagrange's approach to UNMS mathematical modeling can be found in [2, 18], with classification to holonomic and nonholonomic systems together with definitions of UNMS with input coupling. Among first ideas how to control actuated DOF only or unactuated DOF only of nonholonomic systems, the method of partial feedback linearization (PFL) [25] is given. Generalization of PFL to UNMS with input coupling is described in [18], but there is no known approach to control the arbitrary[2] DOF. In another way, generalization of PFL to arbitrary controlled DOF is described in [23], but without interests in UNMS with input coupling. In this paper, we consider modeling and control of both second-order nonholonomic systems and UNMS with input coupling. In both cases, we take actuators nonlinear dynamics into account, while modeling and control of UNMS with *included* actuators dynamics is highlighted in this work. Neglecting of actuators dynamics in UNMS controllers design can cause unstable system behavior [11]. A presented nonlinear actuators model describes a class of pneumatic, electrical, and hydraulic drives, and it offers an easy and simple mathematical way to connect to UNMS. Pneumatic and hydraulic actuators have a complicated nonlinear dynamics that is often approximated by the first- or the second-order dynamics [34]. Modeling and control of fully-actuated[3] systems with including actuator dynamics are considered in literature, using pneumatic-driven systems [24, 30], electrical-driven [13, 28] and hydraulic-driven [1] systems. After combining actuators equations with dynamic equations of UNMS, the third- and higher-order dynamic equations are obtained, what highly complicates control law derivation. A more challenging problem in control of UNMS is simultaneous stabilization and trajectory tracking of particular degrees of freedom. A unified controller for both trajectory tracking and point regulation of

[1] There are some specific examples of UNMS that can be asymptotically stabilized around equilibrium point using a continuous linear-time invariant feedback control law, but this is not possible for a general class.

[2] Arbitrary DOF means arbitrarily chosen DOF, and can be any actuated or any unactuated DOF.

[3] A dimension of a configuration space for fully-actuated systems is the same as a control input space dimension.

second-order nonholonomic chained systems is given in [16], control of UNMS with servo-constraints is presented in [7–9, 22], while control of holonomic and nonholonomic systems using sliding mode is presented by the direct Lyapunov approach [15]. Specific examples of simultaneous stabilization and/or trajectory tracking of UNMS are given in [3, 14, 27, 33].

Previous works in this area of research did not give results to: (a) control of *arbitrary* chosen DOF of UNMS with *included actuators dynamics* and (b) enveloping both the second-order *nonholonomic* systems and *UNMS with input coupling*. These generalizations from (a) and (b) are not only contributions of this paper, but also a *simultaneous* control of arbitrary chosen actuated and unactuated degrees of freedom of the system is provided.

The paper is organized as follows. Section 2 consists of a UNMS mathematical model in Euler–Lagrange (EL) and state space formalism. Here, we introduced novel ideas: how to present EL equations in a state space form avoiding matrix pseudo-inversion. A matrix pseudo-inversion is commonly used in noncollocated PFL to control just unactuated DOF. In the same section, a general mathematical model of actuators dynamics is presented. Section 3 shows development of novel control law derived as linear combination of two separated control laws, where the first control law is designed to control only actuated DOF and the second to control only unactuated DOF. Section 4 presents an illustrative example: control of rotational inverted pendulum (RIP) with a direct current electric actuator. Section 5 contains the conclusions.

2 Mathematical model

2.1 Euler–Lagrange equations

Mathematical modeling of UNMS is based on the Euler–Lagrange's approach, as given by expressions:

$$\frac{d}{dt}\frac{\partial \mathfrak{L}(\mathbf{q},\dot{\mathbf{q}})}{\partial \dot{\mathbf{q}}_a} - \frac{\partial \mathfrak{L}(\mathbf{q},\dot{\mathbf{q}})}{\partial \mathbf{q}_a} + \mathbf{F}_{\mathrm{dis}_a}(\mathbf{q},\dot{\mathbf{q}}) = \mathbf{H}_a(\mathbf{q})\mathbf{F} \tag{1}$$

$$\frac{d}{dt}\frac{\partial \mathfrak{L}(\mathbf{q},\dot{\mathbf{q}})}{\partial \dot{\mathbf{q}}_u} - \frac{\partial \mathfrak{L}(\mathbf{q},\dot{\mathbf{q}})}{\partial \mathbf{q}_u} + \mathbf{F}_{\mathrm{dis}_u}(\mathbf{q},\dot{\mathbf{q}}) = \mathbf{H}_u(\mathbf{q})\mathbf{F} \tag{2}$$

where index a refers to actuated, and index u to unactuated generalized coordinates.
Lagrangian $\mathfrak{L}(\mathbf{q},\dot{\mathbf{q}})$ is described as follows:

$$\mathfrak{L}(\mathbf{q},\dot{\mathbf{q}}) = K(\mathbf{q},\dot{\mathbf{q}}) - U(\mathbf{q}) = \frac{1}{2}\dot{\mathbf{q}}^T\mathbf{M}(\mathbf{q})\dot{\mathbf{q}} - U(\mathbf{q})$$

where $\mathbf{q} \in Q$; n dimensional vector [19]:

$$\begin{aligned}
&\mathbf{q} = col(\mathbf{q}_a, \mathbf{q}_u) \in Q_a \times Q_u\\
&\dim(Q_i) = n_i \quad i = a, u\\
&n = n_a + n_u
\end{aligned}$$

where "*col*"-column vector, dim(Q)-dimension of configuration space, and $\dim(Q_i)$-dimension of configuration subspace Q_i.
Variables $\mathbf{q}, \dot{\mathbf{q}}, \ddot{\mathbf{q}} \in \mathbb{R}^n$ represent generalized vectors of position, velocity, and acceleration, respectively; $\mathbf{F}_{\mathrm{dis}_a}(\mathbf{q},\dot{\mathbf{q}}) \in \mathbb{R}^{n_a}$ and $\mathbf{F}_{\mathrm{dis}_u}(\mathbf{q},\dot{\mathbf{q}}) \in \mathbb{R}^{n_u}$—vectors of generalized dissipation

forces/torques; $\mathbf{F} \in \mathbb{R}^{n_a}$—vectors of generalized control forces/torques; $\mathbf{H}_a(\mathbf{q}) \in \mathbb{R}^{n_a \times n_a}$ and $\mathbf{H}_u(\mathbf{q}) \in \mathbb{R}^{n_u \times n_a}$—input coupling matrices; $K(\mathbf{q}, \dot{\mathbf{q}}) \in \mathbb{R}$—kinetic energy; $U(\mathbf{q}) \in \mathbb{R}$—potential energy; $\mathbf{M}(\mathbf{q}) \in \mathbb{R}^{n \times n}$ inertia matrix.

Depending on the matrices $\mathbf{H}_a(\mathbf{q})$ and $\mathbf{H}_u(\mathbf{q})$ from (1) and (2), two classes of UNMS can be expressed:[4]

- Nonholonomic 2. order systems: $\det(\mathbf{H}_a(\mathbf{q})) \neq 0$, $\mathbf{H}_u(\mathbf{q}) \equiv \mathbf{0}$ (*R1*)
- UNMS with input coupling: $\det(\mathbf{H}_a(\mathbf{q})) \neq 0$, $\mathbf{H}_u(\mathbf{q}) \not\equiv \mathbf{0}$. (*R2*)

where $\mathbf{0}$ is zero matrix.[5] The control space dimension is less than the configuration space dimension. In the case (*R1*), it is well defined (due to the nonholonomic constraints, where $\mathbf{H}_u(\mathbf{q}) \equiv \mathbf{0}$) which generalized coordinates are actuated and which are unactuated, but in the case (*R2*) this is not explicitly defined. To achieve a proper selection for both cases, actuated and unactuated generalized coordinates have to be chosen so that they meet the requirements (*R1*) and (*R2*) of input coupling matrices $\mathbf{H}_a(\mathbf{q})$ and $\mathbf{H}_u(\mathbf{q})$. Equations (1) and (2) can be expressed as follows:

$$\mathbf{M}_{a1}(\mathbf{q})\ddot{\mathbf{q}}_a + \mathbf{M}_{a2}(\mathbf{q})\ddot{\mathbf{q}}_u + \mathbf{h}_a(\mathbf{q}, \dot{\mathbf{q}}) + \mathbf{F}_{\mathrm{dis}_a}(\mathbf{q}, \dot{\mathbf{q}}) = \mathbf{H}_a(\mathbf{q})\mathbf{F} \tag{3a}$$

$$\mathbf{M}_{u1}(\mathbf{q})\ddot{\mathbf{q}}_a + \mathbf{M}_{u2}(\mathbf{q})\ddot{\mathbf{q}}_u + \mathbf{h}_u(\mathbf{q}, \dot{\mathbf{q}}) + \mathbf{F}_{\mathrm{dis}_u}(\mathbf{q}, \dot{\mathbf{q}}) = \mathbf{H}_u(\mathbf{q})\mathbf{F} \tag{3b}$$

where $\mathbf{h}_a(\mathbf{q}, \dot{\mathbf{q}}) \in \mathbb{R}^{n_a}$ and $\mathbf{h}_u(\mathbf{q}, \dot{\mathbf{q}}) \in \mathbb{R}^{n_u}$ include Coriolis, centrifugal, and gravitational elements, while $\mathbf{M}_{ij}(\mathbf{q})$, $i \in \{a, u\}$, $j \in \{1, 2\}$ are matrices $\mathbf{M}_{a1}(\mathbf{q}) \in \mathbb{R}^{n_a \times n_a}$, $\mathbf{M}_{a2}(\mathbf{q}) \in \mathbb{R}^{n_a \times n_u}$, $\mathbf{M}_{u1}(\mathbf{q}) \in \mathbb{R}^{n_u \times n_a}$ i $\mathbf{M}_{u2}(\mathbf{q}) \in \mathbb{R}^{n_u \times n_u}$ and constitute the matrix $\mathbf{M}(\mathbf{q})$:

$$\mathbf{M}(\mathbf{q}) = \begin{bmatrix} \mathbf{M}_{a1}(\mathbf{q}) & \mathbf{M}_{a2}(\mathbf{q}) \\ \mathbf{M}_{u1}(\mathbf{q}) & \mathbf{M}_{u2}(\mathbf{q}) \end{bmatrix} \tag{4}$$

2.2 Transformations to nonlinear state space

Here, we describe a specific way of transforming EL equations to state space form which will enable us to simultaneously control arbitrary chosen actuated and unactuated DOF of the system. Novelty is in selecting actuated and unactuated generalized accelerations from (3a) and (3b) aiming to set state space vector without using matrix pseudo-inversions (matrix pseudo-inversions are usually used in the noncollocated partial feedback linearization [17, 25]). In the following, EL equations (3a) and (3b) are transformed into two equations, so that in the first there is only acceleration $\ddot{\mathbf{q}}_a$ and in the second one only acceleration $\ddot{\mathbf{q}}_u$. Both equations represent the *same* system, but explicitly expressed by different acceleration vectors. Such formalism enables direct approach to accelerations $\ddot{\mathbf{q}}_a$ and $\ddot{\mathbf{q}}_u$ through control variable $\mathbf{F}$, and enables an easy representation in the nonlinear state space, affine in control variable. Acceleration $\ddot{\mathbf{q}}_a$ is derived from (3a) and acceleration $\ddot{\mathbf{q}}_u$ is derived from (3b) as follows:

$$\ddot{\mathbf{q}}_a = -\mathbf{M}_{a1}^{-1}(\mathbf{q})\left[\mathbf{M}_{a2}(\mathbf{q})\ddot{\mathbf{q}}_u + \mathbf{h}_a(\mathbf{q}, \dot{\mathbf{q}}) + \mathbf{F}_{\mathrm{dis}_a}(\mathbf{q}, \dot{\mathbf{q}}) - \mathbf{H}_a(\mathbf{q})\mathbf{F}\right] \tag{5a}$$

$$\ddot{\mathbf{q}}_u = -\mathbf{M}_{u2}^{-1}(\mathbf{q})\left[\mathbf{M}_{u1}(\mathbf{q})\ddot{\mathbf{q}}_a + \mathbf{h}_u(\mathbf{q}, \dot{\mathbf{q}}) + \mathbf{F}_{\mathrm{dis}_u}(\mathbf{q}, \dot{\mathbf{q}}) - \mathbf{H}_u(\mathbf{q})\mathbf{F}\right] \tag{5b}$$

[4] These two classes of UNMS are distinguished because of their often usage in different research fields. For instance, nonholonomic systems are the most used in robotics, while UNMS with input coupling are the most used in aerospace and naval research fields.

[5] The expression $\mathbf{H}_u(\mathbf{q}) \not\equiv \mathbf{0}$ means that every row of the matrix $\mathbf{H}_u(\mathbf{q})$ has at least one nonzero element, for all $\mathbf{q}$.

Matrices $\mathbf{M}_{a1}(\mathbf{q}) \in \mathbb{R}^{n_a \times n_a}$ and $\mathbf{M}_{u2}(\mathbf{q}) \in \mathbb{R}^{n_u \times n_u}$ are both full ranked and square matrices, so invertible. This follows from uniform positive definiteness of the matrix $\mathbf{M}(\mathbf{q})$.
Acceleration expression from (5a) is replaced in (3b) to get (7). In the same way, acceleration expression from (5b) is replaced in (3a) to get (6). This way of explicitly acceleration selection will remove unnecessary pseudoinversion of the inertia submatrices $\mathbf{M}_{ij}(\mathbf{q})$ what will happen for any other selection of accelerations $\ddot{\mathbf{q}}_a, \ddot{\mathbf{q}}_u$.

$$\begin{aligned}
&\tilde{\mathbf{M}}_{a1}(\mathbf{q})\ddot{\mathbf{q}}_a + \tilde{\mathbf{M}}_{a2}(\mathbf{q})\big[\mathbf{h}_u(\mathbf{q},\dot{\mathbf{q}}) + \mathbf{F}_{\mathrm{dis}_u}(\mathbf{q}_u,\dot{\mathbf{q}}_u)\big] + \mathbf{h}_a(\mathbf{q},\dot{\mathbf{q}}) + \mathbf{F}_{\mathrm{dis}_a}(\mathbf{q},\dot{\mathbf{q}}) \\
&\quad = \big[\mathbf{H}_a(\mathbf{q}) + \tilde{\mathbf{M}}_{a2}(\mathbf{q})\mathbf{H}_u(\mathbf{q})\big]\mathbf{F} \qquad (6)\\
&\tilde{\mathbf{M}}_{u2}(\mathbf{q})\ddot{\mathbf{q}}_u + \tilde{\mathbf{M}}_{u1}(\mathbf{q})\big[\mathbf{h}_a(\mathbf{q},\dot{\mathbf{q}}) + \mathbf{F}_{\mathrm{dis}_a}(\mathbf{q}_a,\dot{\mathbf{q}}_a)\big] + \mathbf{h}_u(\mathbf{q},\dot{\mathbf{q}}) + \mathbf{F}_{\mathrm{dis}_u}(\mathbf{q},\dot{\mathbf{q}}) \\
&\quad = \big[\mathbf{H}_u(\mathbf{q}) + \tilde{\mathbf{M}}_{u1}(\mathbf{q})\mathbf{H}_a(\mathbf{q})\big]\mathbf{F} \qquad (7)
\end{aligned}$$

where

$$\begin{aligned}
\tilde{\mathbf{M}}_{u2}(\mathbf{q}) &= \mathbf{M}_{u2}(\mathbf{q}) - \mathbf{M}_{u1}(\mathbf{q})\mathbf{M}_{a1}^{-1}(\mathbf{q})\mathbf{M}_{a2}(\mathbf{q})\\
\tilde{\mathbf{M}}_{u1}(\mathbf{q}) &= -\mathbf{M}_{u1}(\mathbf{q})\mathbf{M}_{a1}^{-1}(\mathbf{q})\\
\tilde{\mathbf{M}}_{a1}(\mathbf{q}) &= \mathbf{M}_{a1}(\mathbf{q}) - \mathbf{M}_{a2}(\mathbf{q})\mathbf{M}_{u2}^{-1}(\mathbf{q})\mathbf{M}_{u1}(\mathbf{q})\\
\tilde{\mathbf{M}}_{a2}(\mathbf{q}) &= -\mathbf{M}_{a2}(\mathbf{q})\mathbf{M}_{u2}^{-1}(\mathbf{q})
\end{aligned}$$

while $\tilde{\mathbf{M}}_{ij}(\mathbf{q}),\ i \in \{a, u\},\ j \in \{1, 2\}$ represents matrices $\tilde{\mathbf{M}}_{u2}(\mathbf{q}) \in \mathbb{R}^{n_u \times n_u}$, $\tilde{\mathbf{M}}_{u1}(\mathbf{q}) \in \mathbb{R}^{n_u \times n_a}$, $\tilde{\mathbf{M}}_{a1}(\mathbf{q}) \in \mathbb{R}^{n_a \times n_a}$ and $\tilde{\mathbf{M}}_{a2}(\mathbf{q}) \in \mathbb{R}^{n_a \times n_u}$.
Accelerations $\ddot{\mathbf{q}}_a, \ddot{\mathbf{q}}_u$ are explicitly defined from (6) and (7) as follows:

$$\ddot{\mathbf{q}}_a = \mathbf{f}_a(\mathbf{q},\dot{\mathbf{q}}) + \mathbf{f}_{\mathrm{dis}_a}(\mathbf{q},\dot{\mathbf{q}}) + \mathbf{B}_a(\mathbf{q})\mathbf{F} \qquad (8a)$$

$$\ddot{\mathbf{q}}_u = \mathbf{f}_u(\mathbf{q},\dot{\mathbf{q}}) + \mathbf{f}_{\mathrm{dis}_u}(\mathbf{q},\dot{\mathbf{q}}) + \mathbf{B}_u(\mathbf{q})\mathbf{F} \qquad (8b)$$

where

$$\begin{aligned}
\mathbf{f}_a(\mathbf{q},\dot{\mathbf{q}}) &= -\tilde{\mathbf{M}}_{a1}^{-1}(\mathbf{q})\big\{\tilde{\mathbf{M}}_{a2}(\mathbf{q})\mathbf{h}_u(\mathbf{q},\dot{\mathbf{q}}) + \mathbf{h}_a(\mathbf{q},\dot{\mathbf{q}})\big\}\\
\mathbf{f}_u(\mathbf{q},\dot{\mathbf{q}}) &= -\tilde{\mathbf{M}}_{u2}^{-1}(\mathbf{q})\big\{\tilde{\mathbf{M}}_{u1}(\mathbf{q})\mathbf{h}_a(\mathbf{q},\dot{\mathbf{q}}) + \mathbf{h}_u(\mathbf{q},\dot{\mathbf{q}})\big\}\\
\mathbf{f}_{\mathrm{dis}_a}(\mathbf{q},\dot{\mathbf{q}}) &= -\tilde{\mathbf{M}}_{a1}^{-1}(\mathbf{q})\big\{\tilde{\mathbf{M}}_{a2}(\mathbf{q})\mathbf{F}_{\mathrm{dis}_u}(\mathbf{q},\dot{\mathbf{q}}) + \mathbf{F}_{\mathrm{dis}_a}(\mathbf{q},\dot{\mathbf{q}})\big\}\\
\mathbf{f}_{\mathrm{dis}_u}(\mathbf{q},\dot{\mathbf{q}}) &= -\tilde{\mathbf{M}}_{u2}^{-1}(\mathbf{q})\big\{\tilde{\mathbf{M}}_{u1}(\mathbf{q})\mathbf{F}_{\mathrm{dis}_a}(\mathbf{q},\dot{\mathbf{q}}) + \mathbf{F}_{\mathrm{dis}_u}(\mathbf{q},\dot{\mathbf{q}})\big\}\\
\mathbf{B}_a(\mathbf{q}) &= \tilde{\mathbf{M}}_{a1}^{-1}(\mathbf{q})\big[\mathbf{H}_a(\mathbf{q}) + \tilde{\mathbf{M}}_{a2}(\mathbf{q})\mathbf{H}_u(\mathbf{q})\big]\\
\mathbf{B}_u(\mathbf{q}) &= \tilde{\mathbf{M}}_{u2}^{-1}(\mathbf{q})\big[\mathbf{H}_u(\mathbf{q}) + \tilde{\mathbf{M}}_{u1}(\mathbf{q})\mathbf{H}_a(\mathbf{q})\big]
\end{aligned}$$

and $\mathbf{f}_a(\mathbf{q},\dot{\mathbf{q}}),\ \mathbf{f}_{\mathrm{dis}_a}(\mathbf{q},\dot{\mathbf{q}}) \in \mathbb{R}^{n_a}$, $\mathbf{f}_u(\mathbf{q},\ \dot{\mathbf{q}}), \mathbf{f}_{\mathrm{dis}_u}(\mathbf{q},\dot{\mathbf{q}}) \in \mathbb{R}^{n_u}$, $\mathbf{B}_a(\mathbf{q}) \in \mathbb{R}^{n_a \times n_a}$, $\mathbf{B}_u(\mathbf{q}) \in \mathbb{R}^{n_u \times n_a}$.
Both matrices $\tilde{\mathbf{M}}_{a1}(\mathbf{q})$ and $\tilde{\mathbf{M}}_{u2}(\mathbf{q})$ are square and invertible, i.e., have full ranks. Proof for full rank of $\tilde{\mathbf{M}}_{a1}(\mathbf{q})$ is given in (32), while for $\tilde{\mathbf{M}}_{u2}(\mathbf{q})$ is given in (33), in Appendix A.

Mechanical system can be easily transformed from (8a) and (8b) in nonlinear state space form, affine in control variable,[6] as follows:

$$\dot{\mathbf{x}}_q = \underbrace{\begin{bmatrix} \mathbf{x}_{qa2} \\ \mathbf{x}_{qu2} \\ \mathbf{f}_a(\mathbf{x}_q) + \mathbf{f}_{\mathrm{dis}_a}(\mathbf{x}_q) \\ \mathbf{f}_u(\mathbf{x}_q) + \mathbf{f}_{\mathrm{dis}_u}(\mathbf{x}_q) \end{bmatrix}}_{\mathbf{f}_q(\mathbf{x}_q)} + \underbrace{\begin{bmatrix} \mathbf{0}_{n_a \times n_a} \\ \mathbf{0}_{n_u \times n_a} \\ \mathbf{B}_a(\mathbf{x}_q) \\ \mathbf{B}_u(\mathbf{x}_q) \end{bmatrix}}_{\mathbf{B}_q(\mathbf{x}_q)} \mathbf{F} \tag{9}$$

Vector function $\mathbf{f}_q(\mathbf{x}_q) \in \mathbb{R}^{2n}$ and matrix function $\mathbf{B}_q(\mathbf{x}_q) \in \mathbb{R}^{2n \times n_a}$ are defined in expression (9).
Space variables vector $\mathbf{x}_q \in \mathbb{R}^{2n}$ of mechanical system is defined as

$$\mathbf{x}_q = \begin{bmatrix} \mathbf{q} \\ \dot{\mathbf{q}} \end{bmatrix} = \begin{bmatrix} \mathbf{q}_a \\ \mathbf{q}_u \\ \dot{\mathbf{q}}_a \\ \dot{\mathbf{q}}_u \end{bmatrix} = \begin{bmatrix} \mathbf{x}_{qa1} \\ \mathbf{x}_{qu1} \\ \mathbf{x}_{qa2} \\ \mathbf{x}_{qu2} \end{bmatrix} = \begin{bmatrix} \mathbf{x}_{q1} \\ \mathbf{x}_{q2} \end{bmatrix} \tag{10}$$

2.3 Actuator model

Mathematical model of actuators connected to UNMS is described as follows:

$$\mathbf{F} = \boldsymbol{\gamma}_{ac}(\mathbf{x}_{ac}, \mathbf{x}_{qa1}, \mathbf{x}_{qa2}) - \mathbf{M}_{ac}\dot{\mathbf{x}}_{qa2} - \mathbf{F}_{\mathrm{dis}_{ac1}}(\mathbf{x}_{qa1}, \mathbf{x}_{qa2}) \tag{11a}$$

$$\dot{\mathbf{x}}_{ac} = \mathbf{f}_{ac}(\mathbf{x}_{ac}, \mathbf{x}_{qa1}, \mathbf{x}_{qa2}) + \mathbf{g}_{ac}(\mathbf{x}_{ac})\mathbf{u} \tag{11b}$$

where $\boldsymbol{\gamma}_{ac}(\mathbf{x}_{ac}, \mathbf{x}_{qa1}, \mathbf{x}_{qa2}) \in \mathbb{R}^{n_a}$ is a vector of nonlinear functions,[7] $\mathbf{M}_{ac} > \mathbf{0} \in \mathbb{R}^{n_a \times n_a}$ is a constant diagonal matrix of actuators' inertias, $\mathbf{F}_{\mathrm{dis}_{ac1}}(\mathbf{x}_{qa1}, \mathbf{x}_{qa2}) \in \mathbb{R}^{n_a}$ is a dissipation vector, $\mathbf{g}_{ac}(\mathbf{x}_{ac}) \in \mathbb{R}^{(\sum_{k=1}^{n_a} n_{ac}^k) \times n_a}$ is an input matrix, $\mathbf{f}_{ac}(\cdot) \in \mathbb{R}^{\sum_{k=1}^{n_a} n_{ac}^k}$ is a nonlinear vector function and $\mathbf{u} \in \mathbb{R}^{n_a}$ is actuators' input vector. The actuators' state space vector $\mathbf{x}_{ac} \in X_{ac}$ is defined as follows:

$$\mathbf{x}_{ac} = col\left(\mathbf{x}_{ac}^1, \mathbf{x}_{ac}^2, \ldots, \mathbf{x}_{ac}^k\right) \in X_{ac}^1 \times X_{ac}^2 \times \cdots \times X_{ac}^k$$

$$\dim\left(X_{ac}^k\right) = n_{ac}^k \quad k = 1, 2, \ldots, n_a$$

$$\dim(X_{ac}) = \sum_{k=1}^{n_a} n_{ac}^k$$

where: superscript "k" denotes kth actuator, "*col*"-column vector, $\dim(X_{ac})$-dimension of actuators' state space, and $\dim(X_{ac}^k)$-dimension of actuators' state subspace X_{ac}^k.
An expanded form of (11b) is given by

$$\begin{bmatrix} \dot{\mathbf{x}}_{ac}^1 \\ \vdots \\ \dot{\mathbf{x}}_{ac}^k \end{bmatrix} = \begin{bmatrix} \mathbf{f}_{ac}^1(\mathbf{x}_{ac}^1, \mathbf{x}_{qa1}^1, \mathbf{x}_{qa2}^1) \\ \vdots \\ \mathbf{f}_{ac}^k(\mathbf{x}_{ac}^k, \mathbf{x}_{qa1}^k, \mathbf{x}_{qa2}^k) \end{bmatrix} + \begin{bmatrix} \mathbf{g}_{ac}^1(\mathbf{x}_{ac}^1) & \ldots & \mathbf{0} \\ \vdots & \ddots & \vdots \\ \mathbf{0} & \ldots & \mathbf{g}_{ac}^k(\mathbf{x}_{ac}^k) \end{bmatrix} \begin{bmatrix} u^1 \\ \vdots \\ u^k \end{bmatrix}$$

[6] In general, the system described as affine in control variable $\mathbf{u}$ has the form $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \mathbf{B}(\mathbf{x})\mathbf{u}$. Vector function $\mathbf{f}(\mathbf{x})$ represents the drift of the system. If $\mathbf{f}(\mathbf{x}) = \mathbf{0}$, then system is driftless.

[7] For convenience, in the remainder the vector function $\boldsymbol{\gamma}_{ac}(\mathbf{x}_{ac}, \mathbf{x}_{qa1}, \mathbf{x}_{qa2})$ will be used in shorter forms $\boldsymbol{\gamma}_{ac}(\mathbf{x}_{ac}, \mathbf{x}_q)$ or $\boldsymbol{\gamma}_{ac}$, to reduce the length of further expressions.

with definitions for k-th actuator: $\dot{\mathbf{x}}_{ac}^k \in \mathbb{R}^{n_{ac}^k}$ denotes time derivative of the state space vector $\mathbf{x}_{ac}^k$, $\mathbf{f}_{ac}^k(\mathbf{x}_{ac}^k, \mathbf{x}_{qa1}^k, \mathbf{x}_{qa2}^k) \in \mathbb{R}^{n_{ac}^k}$ denotes a drift vector function, $\mathbf{g}_{ac}^k(\mathbf{x}_{ac}^k) \in \mathbb{R}^{n_{ac}^k}$ denotes an input matrix and u^k is a scalar input.

Note that the nonlinear functions in (11a)–(11b) depend on[8] the UNMS state variables $\mathbf{x}_{qa1}^k, \mathbf{x}_{qa2}^k$. These dependencies enable a simple way of connecting the actuators' and the UNMS' mathematical models.

Here, we also define a relative degree of actuators, according to the system's output (11a). In words, a relative degree is a number that shows how many time derivatives of (11a) is needed so that an input variable $\mathbf{u}$ from (11b) appear in it. An example of a pneumatic actuator with proportional spool valve is given in Appendix C.

2.4 Model of UNMS with actuators

A model of the mechanical system with actuators is derived after joining (9) and (11a)–(11b), as follows:

$$\overbrace{\begin{bmatrix} \dot{\mathbf{x}}_{qa1} \\ \dot{\mathbf{x}}_{qu1} \\ \dot{\mathbf{x}}_{qa2} \\ \dot{\mathbf{x}}_{qu2} \\ \dot{\mathbf{x}}_{ac} \end{bmatrix}}^{\dot{\mathbf{x}}} = \overbrace{\begin{bmatrix} \mathbf{x}_{qa2} \\ \mathbf{x}_{qu2} \\ \mathbf{f}_3(\mathbf{x}_q) + \mathbf{f}_{\mathrm{dis}_3}(\mathbf{x}_q) + \mathbf{B}_3(\mathbf{x}_q)\boldsymbol{\gamma}_{ac} \\ \mathbf{f}_4(\mathbf{x}_q) + \mathbf{f}_{\mathrm{dis}_4}(\mathbf{x}_q) + \mathbf{B}_4(\mathbf{x}_q)\boldsymbol{\gamma}_{ac} \\ \mathbf{f}_{ac}(\mathbf{x}_{ac}, \mathbf{x}_{qa1}, \mathbf{x}_{qa2}) \end{bmatrix}}^{\mathbf{f}(\mathbf{x})} + \overbrace{\begin{bmatrix} \mathbf{0}_{n_a \times n_a} \\ \mathbf{0}_{n_u \times n_a} \\ \mathbf{0}_{n_a \times n_a} \\ \mathbf{0}_{n_u \times n_a} \\ \mathbf{g}_{ac}(\mathbf{x}_{ac}) \end{bmatrix}}^{\mathbf{g}(\mathbf{x})} \mathbf{u} \tag{12}$$

Vector's and matrix's elements of (12) are defined in the following expressions:

$$\begin{aligned} \mathbf{f}_3(\mathbf{x}_q) &= \tilde{\mathbf{B}}_a^{-1}(\mathbf{x}_q)\mathbf{f}_a(\mathbf{x}_q) \\ \mathbf{f}_{\mathrm{dis}_3}(\mathbf{x}_q) &= \tilde{\mathbf{B}}_a^{-1}(\mathbf{x}_q)\mathbf{f}_{\mathrm{dis}_a}(\mathbf{x}_q) - \mathbf{B}_3(\mathbf{x}_q)\mathbf{F}_{\mathrm{dis}_{ac1}}(\mathbf{x}_{qa1}, \mathbf{x}_{qa2}) \\ \mathbf{B}_3(\mathbf{x}_q) &= \tilde{\mathbf{B}}_a^{-1}(\mathbf{x}_q)\mathbf{B}_a(\mathbf{x}_q) \\ \mathbf{f}_4(\mathbf{x}_q) &= \mathbf{f}_u(\mathbf{x}_q) - \mathbf{B}_u(\mathbf{x}_q)\mathbf{M}_{ac}\mathbf{f}_3(\mathbf{x}_q) \\ \mathbf{f}_{\mathrm{dis}_4}(\mathbf{x}_q) &= \mathbf{f}_{\mathrm{dis}_u}(\mathbf{x}_q) - \mathbf{B}_u(\mathbf{x}_q)\mathbf{F}_{\mathrm{dis}_{ac1}}(\mathbf{x}_{qa1}, \mathbf{x}_{qa2}) - \mathbf{B}_u(\mathbf{x}_q)\mathbf{M}_{ac}\mathbf{f}_{\mathrm{dis}_3}(\mathbf{x}_q) \\ \mathbf{B}_4(\mathbf{x}_q) &= \mathbf{B}_u(\mathbf{x}_q) - \mathbf{B}_u(\mathbf{x}_q)\mathbf{M}_{ac}\mathbf{B}_3(\mathbf{x}_q) \end{aligned} \tag{13}$$

where $\tilde{\mathbf{B}}_a(\mathbf{x}_q) = \mathbf{I} + \mathbf{B}_a(\mathbf{x}_q)\mathbf{M}_{ac}$ is an invertible square matrix and $\mathbf{I} \in \mathbb{R}^{n_a \times n_a}$ is an identity matrix.

3 Control law synthesis

In this section, we present derivation of the control law that is defined as a linear combination of two separate controls. Every separated control is based on the sliding mode approach.

[8]This is due to a rigid connection (with a transmission ratio N) between actuators and UNMS. More precisely, this means that the state variables of the actuators, which connect the actuators with the UNMS, are identified with some state variables of the mechanical system. For instance, if DC electric motor drives kth generalized coordinate with a transmission ratio N, then a relation between rotor's angular velocity ω and generalized velocity $\mathbf{x}_{qa2}^k$ of UMNS is $\mathbf{x}_{qa2}^k = \frac{1}{N}\omega$.

First, we define the output vector $\mathbf{y} \in \mathbb{R}^n$ of the system (9) that consists of generalized coordinates:

$$\mathbf{y} = \begin{bmatrix} \mathbf{x}_{qa1} \\ \mathbf{x}_{qu1} \end{bmatrix} \equiv \mathbf{q} = \begin{bmatrix} \mathbf{q}_a \\ \mathbf{q}_u \end{bmatrix} \tag{14}$$

The output vector $\mathbf{y}$, the desired output vector $\mathbf{y}_d \in \mathbb{R}^n$ and the control error $\mathbf{e} \in \mathbb{R}^n$ are defined by the following form:

$$\mathbf{y}_d = \begin{bmatrix} (\mathbf{x}_{qa1})_d \\ (\mathbf{x}_{qu1})_d \end{bmatrix} \qquad \mathbf{e} = \mathbf{y} - \mathbf{y}_d = \begin{bmatrix} \mathbf{e}_a \\ \mathbf{e}_u \end{bmatrix} \tag{15}$$

First time derivatives $\dot{\mathbf{y}}$, $\dot{\mathbf{y}}_d$ and $\dot{\mathbf{e}}$ are defined as

$$\dot{\mathbf{y}} = \begin{bmatrix} \mathbf{x}_{qa2} \\ \mathbf{x}_{qu2} \end{bmatrix} \qquad \dot{\mathbf{y}}_d = \begin{bmatrix} (\dot{\mathbf{x}}_{qa1})_d \\ (\dot{\mathbf{x}}_{qu1})_d \end{bmatrix} \qquad \dot{\mathbf{e}} = \dot{\mathbf{y}} - \dot{\mathbf{y}}_d = \begin{bmatrix} \dot{\mathbf{e}}_a \\ \dot{\mathbf{e}}_u \end{bmatrix} \tag{16}$$

Second time derivatives $\ddot{\mathbf{y}}$, $\ddot{\mathbf{y}}_d$ and $\ddot{\mathbf{e}}$ are defined as

$$\begin{gathered} \ddot{\mathbf{y}} = \begin{bmatrix} \mathbf{f}_3(\mathbf{x}_q) + \mathbf{f}_{\mathrm{dis}_3}(\mathbf{x}_q) + \mathbf{B}_3(\mathbf{x}_q)\boldsymbol{\gamma}_{ac} \\ \mathbf{f}_4(\mathbf{x}_q) + \mathbf{f}_{\mathrm{dis}_4}(\mathbf{x}_q) + \mathbf{B}_4(\mathbf{x}_q)\boldsymbol{\gamma}_{ac} \end{bmatrix} \qquad \ddot{\mathbf{y}}_d = \begin{bmatrix} (\ddot{\mathbf{x}}_{qa1})_d \\ (\ddot{\mathbf{x}}_{qu1})_d \end{bmatrix} \\ \ddot{\mathbf{e}} = \ddot{\mathbf{y}} - \ddot{\mathbf{y}}_d = \begin{bmatrix} \ddot{\mathbf{e}}_a \\ \ddot{\mathbf{e}}_u \end{bmatrix} = \begin{bmatrix} \mathbf{f}_3(\mathbf{x}_q) + \mathbf{f}_{\mathrm{dis}_3}(\mathbf{x}_q) + \mathbf{B}_3(\mathbf{x}_q)\boldsymbol{\gamma}_{ac} - (\ddot{\mathbf{x}}_{qa1})_d \\ \mathbf{f}_4(\mathbf{x}_q) + \mathbf{f}_{\mathrm{dis}_4}(\mathbf{x}_q) + \mathbf{B}_4(\mathbf{x}_q)\boldsymbol{\gamma}_{ac} - (\ddot{\mathbf{x}}_{qu1})_d \end{bmatrix} \end{gathered} \tag{17}$$

Third time derivatives $\dddot{\mathbf{y}}$, $\dddot{\mathbf{y}}_d$ and $\dddot{\mathbf{e}}$ are defined as

$$\dddot{\mathbf{y}} = \begin{bmatrix} \boldsymbol{\Upsilon}_3 + \mathbf{B}_3(\mathbf{x}_q)\dot{\boldsymbol{\gamma}}_{ac} \\ \boldsymbol{\Upsilon}_4 + \mathbf{B}_4(\mathbf{x}_q)\dot{\boldsymbol{\gamma}}_{ac} \end{bmatrix} \qquad \dddot{\mathbf{y}}_d = \begin{bmatrix} (\dddot{\mathbf{x}}_{qa1})_d \\ (\dddot{\mathbf{x}}_{qu1})_d \end{bmatrix} \tag{18}$$

$$\dddot{\mathbf{e}} = \dddot{\mathbf{y}} - \dddot{\mathbf{y}}_d = \begin{bmatrix} \boldsymbol{\Upsilon}_3 + \mathbf{B}_3(\mathbf{x}_q)\dot{\boldsymbol{\gamma}}_{ac} - (\dddot{\mathbf{x}}_{qa1})_d \\ \boldsymbol{\Upsilon}_3 + \mathbf{B}_4(\mathbf{x}_q)\dot{\boldsymbol{\gamma}}_{ac} - (\dddot{\mathbf{x}}_{qu1})_d \end{bmatrix} = \begin{bmatrix} \dddot{\mathbf{e}}_a \\ \dddot{\mathbf{e}}_u \end{bmatrix}$$

where are

$$\boldsymbol{\Upsilon}_3 = \dot{\mathbf{f}}_3(\mathbf{x}_q) + \dot{\mathbf{f}}_{\mathrm{dis}_3}(\mathbf{x}_q) + \dot{\mathbf{B}}_3(\mathbf{x}_q)\boldsymbol{\gamma}_{ac}$$

$$\boldsymbol{\Upsilon}_4 = \dot{\mathbf{f}}_4(\mathbf{x}_q) + \dot{\mathbf{f}}_{\mathrm{dis}_4}(\mathbf{x}_q) + \dot{\mathbf{B}}_4(\mathbf{x}_q)\boldsymbol{\gamma}_{ac}$$

The sliding variable $\mathbf{s} \in \mathbb{R}^n$ is defined by the following expression:

$$\mathbf{s} = \begin{bmatrix} \mathbf{s}_a \\ \mathbf{s}_u \end{bmatrix} = \ddot{\mathbf{e}} + \boldsymbol{\lambda}_1\dot{\mathbf{e}} + \boldsymbol{\lambda}_2\mathbf{e} \tag{19}$$

where $\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2 \succ \mathbf{0}$ represent parameters matrices. The sliding variable vector $\mathbf{s}$ consists of two vector functions $\mathbf{s}_a \in \mathbb{R}^{n_a}$ and $\mathbf{s}_u \in \mathbb{R}^{n_u}$. The positive definite diagonal matrices $\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2 \in \mathbb{R}^{n \times n}$ have the following form:

$$\boldsymbol{\lambda}_1 = \begin{bmatrix} \boldsymbol{\lambda}_{1a} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\lambda}_{1u} \end{bmatrix} \qquad \boldsymbol{\lambda}_2 = \begin{bmatrix} \boldsymbol{\lambda}_{2a} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\lambda}_{2u} \end{bmatrix} \tag{20}$$

where $\boldsymbol{\lambda}_{1a}, \boldsymbol{\lambda}_{2a} \in \mathbb{R}^{n_a \times n_a}$, $\boldsymbol{\lambda}_{1u}, \boldsymbol{\lambda}_{2u} \in \mathbb{R}^{n_u \times n_u}$ are positive diagonal matrices.

The time derivative of the sliding variable is given by the following expression:

$$\dot{\mathbf{s}} = \begin{bmatrix} \dot{\mathbf{s}}_a \\ \dot{\mathbf{s}}_u \end{bmatrix} = \dddot{\mathbf{e}} + \boldsymbol{\lambda}_1 \ddot{\mathbf{e}} + \boldsymbol{\lambda}_2 \dot{\mathbf{e}} = \begin{bmatrix} \dddot{\mathbf{e}}_a + \boldsymbol{\lambda}_{1a} \ddot{\mathbf{e}}_a + \boldsymbol{\lambda}_{2a} \dot{\mathbf{e}}_a \\ \dddot{\mathbf{e}}_u + \boldsymbol{\lambda}_{1u} \ddot{\mathbf{e}}_u + \boldsymbol{\lambda}_{2u} \dot{\mathbf{e}}_u \end{bmatrix} \tag{21}$$

i.e., in expanded form after implementation $\dot{\mathbf{e}}$, $\ddot{\mathbf{e}}$, $\dddot{\mathbf{e}}$ from (16), (17), (18) into (21), as follows:

$$\begin{bmatrix} \dot{\mathbf{s}}_a \\ \dot{\mathbf{s}}_u \end{bmatrix} = \begin{bmatrix} \boldsymbol{\Gamma}_3 + \mathbf{B}_3(\mathbf{x}_q)\dot{\boldsymbol{\gamma}}_{ac} \\ \boldsymbol{\Gamma}_4 + \mathbf{B}_4(\mathbf{x}_q)\dot{\boldsymbol{\gamma}}_{ac} \end{bmatrix} \tag{22}$$

where

$$\boldsymbol{\Gamma}_3 = \boldsymbol{\Upsilon}_3 - (\dddot{\mathbf{x}}_{qa1})_d + \boldsymbol{\lambda}_{1a} \ddot{\mathbf{e}}_a + \boldsymbol{\lambda}_{2a} \dot{\mathbf{e}}_a$$

$$\boldsymbol{\Gamma}_4 = \boldsymbol{\Upsilon}_4 - (\dddot{\mathbf{x}}_{qu1})_d + \boldsymbol{\lambda}_{1u} \ddot{\mathbf{e}}_u + \boldsymbol{\lambda}_{2u} \dot{\mathbf{e}}_u$$

The vector function $\dot{\boldsymbol{\gamma}}_{ac}$ contains the input vector $\mathbf{u}$ as follows (detailed procedure is given in Appendix B):

$$\dot{\boldsymbol{\gamma}}_{ac} = \frac{\partial \boldsymbol{\gamma}_{ac}}{\partial \mathbf{x}} \mathbf{f}(\mathbf{x}) + \frac{\partial \boldsymbol{\gamma}_{ac}}{\partial \mathbf{x}_{ac}} \mathbf{g}_{ac}(\mathbf{x}_{ac})\mathbf{u} \tag{23}$$

It is important to note that $\boldsymbol{\gamma}_{ac}$ is not necessarily a function of all elements of the state vector $\mathbf{x}_{ac}$. In the case where some element of $\mathbf{x}_{ac}$ does not appear in $\boldsymbol{\gamma}_{ac}$ but its time derivative has an input variable $\mathbf{u}$, then it is needed to compute higher derivatives of $\boldsymbol{\gamma}_{ac}$ to reach the input vector $\mathbf{u}$, as is shown in Appendix C.

Finally, we propose control law that controls all degrees of freedom, as given by the following equation:

$$\mathbf{u} = \boldsymbol{\varphi}_a (\mathbf{u}_{eq})_a + \boldsymbol{\varphi}_u (\mathbf{u}_{eq})_u \tag{24}$$

where $\boldsymbol{\varphi}_a$, $\boldsymbol{\varphi}_u \in \mathbb{R}^{n_a \times n_a}$ are diagonal matrices, used for control law distribution of actuated $\mathbf{x}_{qa1}$ and unactuated $\mathbf{x}_{qu1}$ degrees of freedom. Control vector $(\mathbf{u}_{eq})_a \in \mathbb{R}^{n_a}$ is responsible only for stabilization and trajectory tracking of actuated degrees of freedom $\mathbf{x}_{qa1}$ of UNMS, while control vector $(\mathbf{u}_{eq})_u \in \mathbb{R}^{n_a}$ is responsible only for stabilization and trajectory tracking of unactuated degrees of freedom $\mathbf{x}_{qu1}$ of UNMS. Definition of separated (i.e., equivalent) control laws $(\mathbf{u}_{eq})_a$ and $(\mathbf{u}_{eq})_u$ for simultaneous stabilization and trajectory tracking follows from (22), (23) and is presented by following expressions:

$$(\mathbf{u}_{eq})_a = -\boldsymbol{\Phi}_3^{-1} \left[\boldsymbol{\Gamma}_3 + \mathbf{B}_3(\mathbf{x}_q) \frac{\partial \boldsymbol{\gamma}_{ac}}{\partial \mathbf{x}} \mathbf{f}(\mathbf{x}) + \boldsymbol{\chi}_a(\mathbf{s}_a) \right] \tag{25}$$

$$(\mathbf{u}_{eq})_u = -\boldsymbol{\Phi}_4^{+} \left[\boldsymbol{\Gamma}_4 + \mathbf{B}_4(\mathbf{x}_q) \frac{\partial \boldsymbol{\gamma}_{ac}}{\partial \mathbf{x}} \mathbf{f}(\mathbf{x}) + \boldsymbol{\chi}_u(\mathbf{s}_u) \right] \tag{26}$$

$$\boldsymbol{\Phi}_3 = \mathbf{B}_3(\mathbf{x}_q) \frac{\partial \boldsymbol{\gamma}_{ac}}{\partial \mathbf{x}_{ac}} \mathbf{g}_{ac}(\mathbf{x}_{ac}) \qquad \boldsymbol{\Phi}_4 = \mathbf{B}_4(\mathbf{x}_q) \frac{\partial \boldsymbol{\gamma}_{ac}}{\partial \mathbf{x}_{ac}} \mathbf{g}_{ac}(\mathbf{x}_{ac})$$

while $\boldsymbol{\Phi}_4^{+}$ is the pseudoinversion of the matrix[9] $\boldsymbol{\Phi}_4$, and $\boldsymbol{\chi}_a(\mathbf{s}_a)$, $\boldsymbol{\chi}_u(\mathbf{s}_u)$ represent functions of convergence. The function $\boldsymbol{\chi}_a(\mathbf{s}_a)$ ensures that the control $(\mathbf{u}_{eq})_a$ steers the sliding variable $\mathbf{s}_a$ to the sliding surface $\mathbf{s}_a = \mathbf{0}$, on which $\mathbf{e}_a \to \mathbf{0}$, $t \to \infty$. The function $\boldsymbol{\chi}_u(\mathbf{s}_u)$ ensures

[9]For the right Moore–Penrose pseudoinversion $\boldsymbol{\Phi}_4^{+}$ of the matrix $\boldsymbol{\Phi}_4$ it holds: $\boldsymbol{\Phi}_4 \boldsymbol{\Phi}_4^{+} = \mathbf{I}$, $\mathbf{I} \in \mathbb{R}^{n_u \times n_u}$ identity matrix, $\boldsymbol{\Phi}_4^{+} = \boldsymbol{\Phi}_4^{T} (\boldsymbol{\Phi}_4 \boldsymbol{\Phi}_4^{T})^{-1}$.

that the control $(\mathbf{u}_{eq})_u$ steers the sliding variable $\mathbf{s}_u$ to the sliding surface $\mathbf{s}_u = \mathbf{0}$, on which $\mathbf{e}_u \to \mathbf{0}$, $t \to \infty$.
The vector function $\boldsymbol{\chi}(\mathbf{s})$ can be signum function, tangens hyperbolic function, arcus tangens or some other function that steers dynamic system to reference trajectories [5]. To prove stability of the system for each separated control law, we use a scalar Lyapunov functions $V_i = \frac{\mathbf{s}_i^T \mathbf{s}_i}{2} > 0$, $i = \{a, b\}$. The control law $(\mathbf{u}_{eq})_i$, $i = \{a, b\}$ used for separated control of actuated generalized coordinates makes the system stable if the following conditions are valid: $V_i > 0$, $\dot{V}_i = \mathbf{s}_i^T \dot{\mathbf{s}}_i \leq 0$. To show the system's stability by controlling only actuated coordinates, we replaced $\mathbf{u}$ from top equation in (22) by $(\mathbf{u}_{eq})_a$ from (25) getting $\dot{V}_a = \mathbf{s}_a^T \dot{\mathbf{s}}_a = -\mathbf{s}_a^T \boldsymbol{\chi}_a(\mathbf{s}_a) \leq 0$, $\forall \mathbf{s}_a$. Also, we replaced $\mathbf{u}$ from bottom equation in (22) by $(\mathbf{u}_{eq})_u$ from (26) getting $\dot{V}_u = \mathbf{s}_u^T \dot{\mathbf{s}}_u = -\mathbf{s}_u^T \boldsymbol{\chi}_u(\mathbf{s}_u) \leq 0$, $\forall \mathbf{s}_u$. Stability conditions for using linear combination of separated control laws (24) will be addressed in future work.

4 Simulations

Efficiency of the novel control law is demonstrated on simulation example of rotational inverted pendulum (RIP) which represents an example of second-order nonholonomic UNMS.

The model has two DOF and one control signal, which means that system is underactuated. There is one actuator (DC electric motor) that drives the base link of RIP. This system is linear controllable in unstable equilibrium, which means that system is stabilizable using linear feedback time-invariant controller starting close to unstable equilibrium. Here, the simultaneous stabilization and trajectory tracking using proposed control law with initial conditions that are not close enough for usage of linear controllers will be shown.

Euler–Lagrange's approach for modeling of rotational inverted pendulum [26] yields the following expressions:

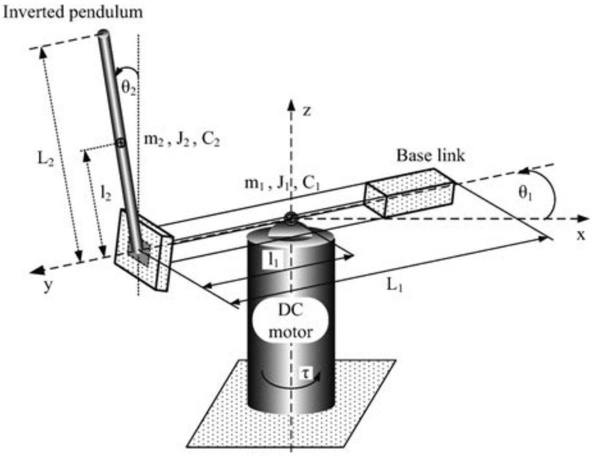
$$
\begin{aligned}
h_1 \ddot{q}_a + h_2 \cos(q_u) \ddot{q}_u - h_2 \sin(q_u) \dot{q}_u^2 + C_1 \dot{q}_a &= \tau \\
h_2 \cos(q_u) \ddot{q}_a + h_3 \ddot{q}_u + h_4 \sin(q_u) + C_2 \dot{q}_u &= 0
\end{aligned}
\tag{27}
$$

where $h_1 = J_1 + m_2 L_1^2$, $h_2 = m_2 L_1 l_2$, $h_3 = J_2 + m_2 l_2^2$, $h_4 = -m_2 l_2 g$. Constants $J_1, J_2, l_1, l_2, L_1, L_2, C_1, C_2, m_1, m_2, g$ represent in the following order: center of mass inertia moment, link distance from center of mass, link length, coefficient of viscous friction in joints, link masses, and gravitation acceleration, with their values.[10] Generalized coordinates $q_a = \theta_1$, $q_u = \theta_2$ represent base link angle (actuated link) and angle of inverted pendulum (unactuated link) as shown in Fig. 1. DC motor moment τ is the actuator output, i.e., mechanical system input.

Relation between general UNMS model (3a), (3b) and RIP model (27) is given as follows: $\mathbf{M}_{a1}(\mathbf{q}) = h_1$, $\mathbf{M}_{u1}(\mathbf{q}) = h_2 \cos(q_u)$, $\mathbf{M}_{a2}(\mathbf{q}) = h_2 \cos(q_u)$, $\mathbf{M}_{u2}(\mathbf{q}) = h_3$, $\mathbf{h}_a(\mathbf{q}, \dot{\mathbf{q}}) = -h_2 \sin(q_u) \dot{q}_u^2$, $\mathbf{h}_u(\mathbf{q}, \dot{\mathbf{q}}) = h_4 \sin(q_u)$, $\mathbf{F}_{\mathrm{dis}_a}(\mathbf{q}, \dot{\mathbf{q}}) = C_1 \dot{q}_a$, $\mathbf{F}_{\mathrm{dis}_u}(\mathbf{q}, \dot{\mathbf{q}}) = C_2 \dot{q}_u$, $\mathbf{H}_a(\mathbf{q}) = 1$, $\mathbf{H}_u(\mathbf{q}) = 0$, $\mathbf{F} = \tau$. Transformations of EL equations into state space form are described in Sects. 2.2 and 2.4. State variables are $x_{qa1} = q_a$, $x_{qu1} = q_u$, with time derivatives $x_{qa2} = \dot{q}_a$, $x_{qu2} = \dot{q}_u$. RIP equations (27) are presented in state space form (12) with functions defined

[10] $m_1 = 0.83$ kg, $L_1 = 0.6$ m, $J_1 = 0.00208$ kg·m^{-2}, $m_2 = 0.1$ kg, $L_2 = 0.3$ m, $J_2 = 0.001$ kg·m^{-2}, $g = 9.81$ m·s^{-2}, $l_1 = 0.3$ m, $l_2 = 0.1$ m.

Fig. 1 Rotational inverted pendulum



by following expressions:

$$
\begin{aligned}
f_3(x_q) &= \frac{h_2 \sin(x_{qu1})(h_3 x_{qu2}^2 + h_4 \cos(x_{qu1}))}{h_1 h_3 - h_2^2 \cos^2(x_{qu1})} \\
f_{\mathrm{dis}_3}(x_q) &= \frac{h_2 \cos(x_{qu1}) C_2 x_{qu2} - h_3 C_1 x_{qa2}}{h_1 h_3 - h_2^2 \cos^2(x_{qu1})} \\
B_3(x_q) &= \frac{h_3}{h_1 h_3 - h_2^2 \cos^2(x_{qu1})} \\
f_4(x_q) &= -\frac{\sin(x_{qu1})(h_2^2 \cos(x_{qu1}) x_{qu2}^2 + h_1 h_4)}{h_1 h_3 - h_2^2 \cos^2(x_{qu1})} \\
f_{\mathrm{dis}_4}(x_q) &= \frac{h_2 \cos(x_{qu1}) C_1 x_{qa2} - h_1 C_2 x_{qu2}}{h_1 h_3 - h_2^2 \cos^2(x_{qu1})} \\
B_4(x_q) &= \frac{h_2 \cos(x_{qu1})}{h_2^2 \cos^2(x_{qu1}) - h_1 h_3}
\end{aligned}
\tag{28}
$$

Mathematical model of DC electric motor with influence of rotor's coil inductance is described as

$$L_a \dot{x}_{ac} + R_a x_{ac} + K_v \omega = u \tag{29}$$

$$F = \underbrace{K_t x_{ac}}_{\gamma_{ac}} \tag{30}$$

where[11] x_{ac}, R_a, L_a, K_a, K_t, u represent armature current, armature resistance, armature inductance, back emf constant, motor torque constant, and the voltage input applied to armature circuit, respectively. Rotor's angular velocity ω is connected with generalized veloc-

[11] Numerical values of DC motor parameters are: $K_t = 1.68$ N·m·A^{-1}, $K_v = 0.168$ V·s, $R_a = 28.6$ Ω, $L_a = 0.01$ H.

ity of base link as $\omega = N x_{qa2}$. Here, transmission ratio N between actuator and base link is taken $N = 1$.
In relation to (11a) and (11b), we have $M_{ac} = 0$ and $F_{\text{dis}_{ac1}} = 0$.

Connection between actuator's and UNMS's dynamics gives the following expressions:

$$\begin{aligned}
\dot{x}_{qa1} &= x_{qa2} \\
\dot{x}_{qu1} &= x_{qu2} \\
\dot{x}_{qa2} &= f_3(x_q) + f_{\text{dis}_3}(x_q) + B_3(x_q)\gamma_{ac} \\
\dot{x}_{qu2} &= f_4(x_q) + f_{\text{dis}_4}(x_q) + B_4(x_q)\gamma_{ac} \\
\dot{x}_{ac} &= -h_5 x_{ac} - h_6 x_{qa2} + h_7 u \\
y &= \begin{bmatrix} x_{qa1} \\ x_{qu1} \end{bmatrix}
\end{aligned} \tag{31}$$

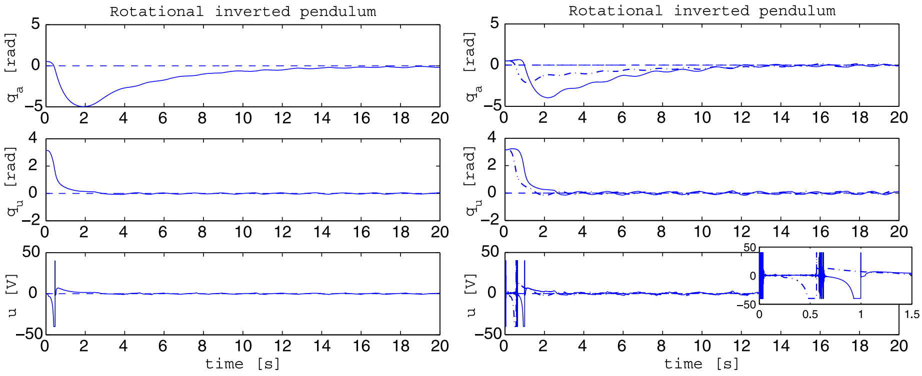
where $h_5 = \frac{R_a}{L_a}$, $h_6 = \frac{K_v}{L_a}$, $h_7 = \frac{1}{L_a}$.

To show the efficiency of the proposed control law on this example, both links can be either stabilized/positioned or can follow some time-dependent trajectory at the same time. This gives 8 possibilities. We present 3 of them. First, stabilization of both links simultaneously, but with same initial positions to illustrate a power of the proposed nonlinear control law, as shown in the Fig. 2. Second, a combination of stabilization and trajectory tracking, where the Fig. 3(a) illustrates a stabilization of unactuated link while the actuated link follows time-dependent trajectory, and the Fig. 3(b) illustrates a stabilization of the actuated link while unactuated link follows time-dependent trajectory. Also, a robustness of the control law was verified by two different simulations on the Fig. 3(b), first by increasing both links' masses for 30%, and second, if measurement noise appears on q_a. The reference trajectories in the Figs. 2 and 3 are shown by dashed lines. The graphs at the top of the figures represent base link—actuated coordinate q_a, the graphs in the middle of the figures represent free moving link – unactuated coordinate q_u, and the graphs at the bottom of the figures represent input control signal u.

4.1 Case 1: RIP stabilization

This case shows stabilization of both DOF, i.e. $(x_{qu1})_d = 0$ rad, and $(x_{qa1})_d = 0$ rad. Graphs in the Fig. 2(a) show response of the RIP without viscous friction[12] and initial pendulum deflection equals π (180°), where a linear control law can not be used for stabilization. Control signal was in saturation ±40 V for short time, see Fig. 2(a) and time about 0.5 seconds, but that did not disable the process of stabilization. Graphs in the Fig. 2(b) show cases in which $(- \cdot -)$ lines represent a case when masses of both links were increased for 30%, while full lines represent a case when measurement noise appears on q_a. Increasing masses, the control signal becomes more oscillating, inducing oscillations in both links. When a measurement noise appears then control signal becomes very oscillating with high values, which can easily destroy an actuator. In such a case, filtering of noise is needed. Measurement noise is a random signal with amplitude ±1 milli-radians.

[12] Sliding variable: $\lambda_{a1} = 40$, $\lambda_{u1} = 10$, $\lambda_{a2} = 10$, $\lambda_{u2} = 10$, Initial conditions: $x_{qa1}(0) = 0.5$ rad, $x_{qu1}(0) = 0.8$ rad, Control law: $\varphi_a = -0.4$, $\varphi_u = 1.4$, $\alpha_1 = 250$, $\alpha_2 = 250$, $\chi(s_a) = \alpha_1 \tanh(s_a)$, $\chi(s_u) = \alpha_2 \tanh(s_u)$, Viscous friction: $C_1 = 0$ N·m·s, $C_2 = 0$ N·m·s, Reference trajectory: $(x_{qa1})_d = 0$ rad, $(x_{qu1})_d = 0$ rad.



(a) (b)

Fig. 2 Stabilization of both actuated and unactuated link. Figure (**a**): $x_{qa1}(0) = 0.5$ rad, $x_{qu1}(0) = 3.14$ rad, $C_1 = C_2 = 0$ N·m·s, saturation ±40. Figure (**b**): $(- \cdot -)$ *lines* represent a case when masses of both links were increased for 30%, while *full lines* represent a case when measurement noise appears on q_a. Measurement noise is a random signal with amplitude ±1 milli-radians. *Dashed lines* represent reference/desired trajectories

4.2 Case 2: RIP stabilization and trajectory tracking

This case shows stabilization of one DOF simultaneously with trajectory tracking of other DOF. Figure 3(a) illustrates[13] a stabilization of unactuated link $(x_{qu1})_d = 0$ rad while the actuated link follows time-dependent trajectory $(x_{qa1})_d = \sin(t) + 0.5\sin(1.5t)$ rad. Also, Fig. 3(b) illustrates[14] a stabilization of the actuated link $(x_{qa1})_d = 0$ while unactuated link follows time-dependent trajectory $(x_{qu1})_d = 0.2\sin(t) + 0.4\sin(1.5t)$ rad. Control signal was in saturation ±20 V for short time (see Fig. 3(a) and time about 0.1 seconds), but that did not destabilize the system. Tracking errors of generalized coordinates q_a, q_u are given in the Figs. 4(a) and 4(b). It can be observed in Fig. 4(b) that some tracking error will stay permanent. It happens due to the imposed behavior $(q_a)_d \equiv (x_{qa1})_d$, $(q_u)_d \equiv (x_{qu1})_d$ on both links at the same time.

5 Conclusion

Control algorithm for simultaneous stabilization and trajectory tracking of underactuated nonlinear mechanical systems with included actuators dynamics has been presented. Simple and intuitive control law enables simultaneous stabilization and trajectory tracking of arbitrary chosen actuated and unactuated degrees of freedom of the UNMS. Novel unified controller for underactuated systems, including both the second-order nonholonomic systems and the systems with input coupling, shows very promising results on simulations. Future work will consider the Lyapunov-based stability analysis with an aim to provide exact controller tuning rules.

[13] Same as in footnote 12, except: Sliding variable: $\lambda_{a1} = 5$, $\lambda_{u1} = 10$, $\lambda_{a2} = 5$, $\lambda_{u2} = 20$, Control law: $\alpha_1 = 250$, $\alpha_2 = 200$, Reference trajectory: $(x_{qa1})_d = \sin(t) + 0.5\sin(1.5t)$ rad, $(x_{qu1})_d = 0$ rad.

[14] Same as in footnote 12, except: Sliding variable: $\lambda_{a1} = 3$, $\lambda_{u1} = 10$, $\lambda_{a2} = 5$, $\lambda_{u2} = 10$, Control law: $\alpha_1 = 250$, $\alpha_2 = 250$, Reference trajectory: $(x_{qa1})_d = 0$ rad, $(x_{qu1})_d = 0.2\sin(t) + 0.4\sin(1.5t)$ rad.

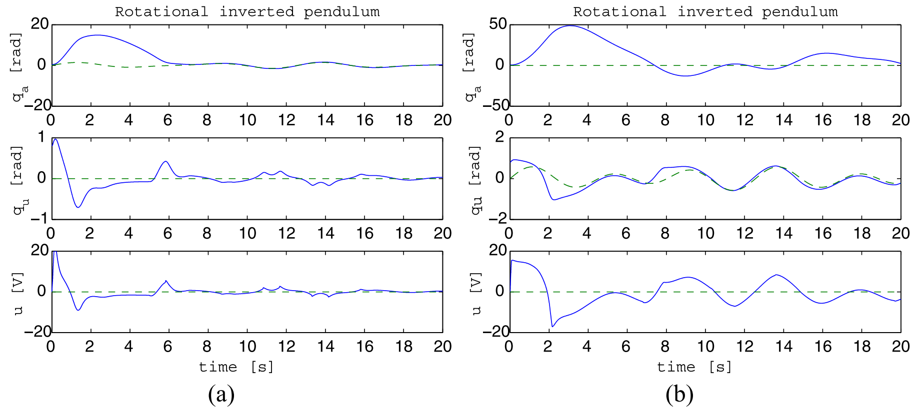


Fig. 3 Figure (**a**) shows trajectory tracking of actuated link simultaneously with stabilization of unactuated link where $x_{qa1}(0) = 0.5$ rad, $x_{qu1}(0) = 0.8$ rad (about 46°). Figure (**b**) shows stabilization of actuated link simultaneously with trajectory tracking of unactuated link where $x_{qa1}(0) = 0.5$ rad, $x_{qu1}(0) = 0.8$ rad. The *dashed lines* represent reference/desired trajectories

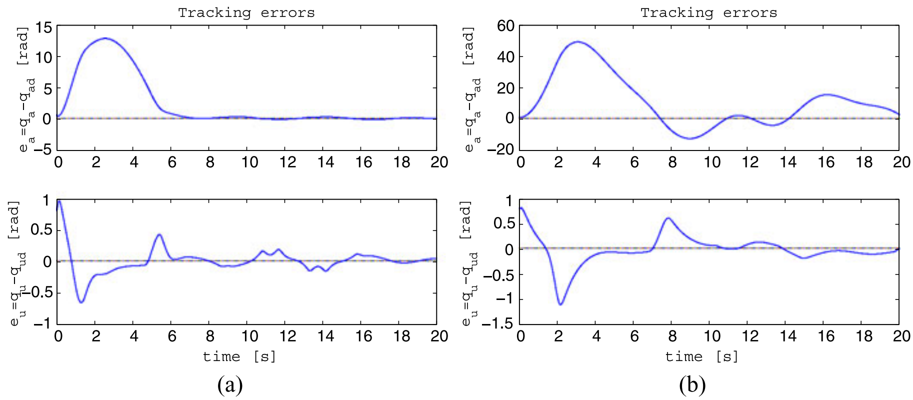


Fig. 4 Figure (**a**) shows tracking errors of generalized coordinates q_a, q_u given in Fig. 3(a), while fig. (**b**) shows tracking errors of generalized coordinates q_a, q_u given in Fig. 3(b)

Appendix A

A proof of the matrix inversion for matrix $\tilde{\mathbf{M}}_{a1}(\mathbf{q})$, is derived in the similar way as given in [25], and is given by the following expression:

$$\begin{bmatrix} \mathbf{M}_{a1}(\mathbf{q}) & \mathbf{M}_{a2}(\mathbf{q}) \\ \mathbf{M}_{u1}(\mathbf{q}) & \mathbf{M}_{u2}(\mathbf{q}) \end{bmatrix} \begin{bmatrix} \mathbf{I}_{n_a \times n_a} \\ -\mathbf{M}_{u2}^{-1}(\mathbf{q})\mathbf{M}_{u1}(\mathbf{q}) \end{bmatrix} = \begin{bmatrix} \tilde{\mathbf{M}}_{a1}(\mathbf{q}) \\ \mathbf{0}_{n_u \times n_a} \end{bmatrix} \tag{32}$$

where $\mathbf{I}_{n_a \times n_a} \in \mathbb{R}^{n_a \times n_a}$ is identity matrix, and $\mathbf{0}_{n_u \times n_a} \in \mathbb{R}^{n_u \times n_a}$ is zero matrix. Since the matrix $\mathbf{M}(\mathbf{q})$ is positive definite matrix, and matrix

$$\begin{bmatrix} \mathbf{I}_{n_a \times n_a} \\ -\mathbf{M}_{u2}^{-1}(\mathbf{q})\mathbf{M}_{u1}(\mathbf{q}) \end{bmatrix}$$

has full column rank n_a, it follows that matrix $\tilde{\mathbf{M}}_{a1}(\mathbf{q})$ has full rank n_a, i.e. it is invertible.

A proof of matrix inversion for matrix $\tilde{\mathbf{M}}_{u2}(\mathbf{q})$, is derived in the similar way as given in [25], and is given by the following expression:

$$\begin{bmatrix} \mathbf{M}_{a1}(\mathbf{q}) & \mathbf{M}_{a2}(\mathbf{q}) \\ \mathbf{M}_{u1}(\mathbf{q}) & \mathbf{M}_{u2}(\mathbf{q}) \end{bmatrix} \begin{bmatrix} -\mathbf{M}_{a1}^{-1}(\mathbf{q})\mathbf{M}_{a2}(\mathbf{q}) \\ \mathbf{I}_{n_u \times n_u} \end{bmatrix} = \begin{bmatrix} \mathbf{0}_{n_a \times n_u} \\ \tilde{\mathbf{M}}_{u2}(\mathbf{q}) \end{bmatrix} \tag{33}$$

Since the inertia matrix $\mathbf{M}(\mathbf{q})$ is positive definite matrix, and matrix

$$\begin{bmatrix} -\mathbf{M}_{a1}^{-1}(\mathbf{q})\mathbf{M}_{a2}(\mathbf{q}) \\ \mathbf{I}_{n_u \times n_u} \end{bmatrix}$$

has full column rank n_u, it follows that matrix $\tilde{\mathbf{M}}_{u2}(\mathbf{q})$ has full rank n_u, i.e., it is invertible.

Appendix B

Relation between the function $\dot{\boldsymbol{\gamma}}_{ac}(\mathbf{x}_{ac}, \mathbf{x}_q)$ and the input vector $\mathbf{u}$ using expressions from (9) and (12) is given by

$$\begin{aligned} \dot{\boldsymbol{\gamma}}_{ac}(\mathbf{x}_{ac}, \mathbf{x}_q) &= \frac{\partial \boldsymbol{\gamma}_{ac}}{\partial \mathbf{x}_{ac}} \dot{\mathbf{x}}_{ac} + \frac{\partial \boldsymbol{\gamma}_{ac}}{\partial \mathbf{x}_q} \dot{\mathbf{x}}_q \\ &= \frac{\partial \boldsymbol{\gamma}_{ac}}{\partial \mathbf{x}_{ac}} \left[\mathbf{f}_{ac}(\mathbf{x}_q, \mathbf{x}_{ac}) + \mathbf{g}_{ac}(\mathbf{x}_{ac})\mathbf{u} \right] + \frac{\partial \boldsymbol{\gamma}_{ac}}{\partial \mathbf{x}_q} \dot{\mathbf{x}}_q \\ &= \frac{\partial \boldsymbol{\gamma}_{ac}}{\partial \mathbf{x}_{ac}} \mathbf{f}_{ac}(\mathbf{x}_q, \mathbf{x}_{ac}) + \frac{\partial \boldsymbol{\gamma}_{ac}}{\partial \mathbf{x}_q} \dot{\mathbf{x}}_q + \frac{\partial \boldsymbol{\gamma}_{ac}}{\partial \mathbf{x}_{ac}} \mathbf{g}_{ac}(\mathbf{x}_{ac})\mathbf{u} \\ &= \frac{\partial \boldsymbol{\gamma}_{ac}}{\partial \mathbf{x}} \mathbf{f}(\mathbf{x}) + \frac{\partial \boldsymbol{\gamma}_{ac}}{\partial \mathbf{x}_{ac}} \mathbf{g}_{ac}(\mathbf{x}_{ac})\mathbf{u} \end{aligned} \tag{34}$$

where $\boldsymbol{\gamma}_{ac}(\mathbf{x}_{ac}, \mathbf{x}_q) = [\gamma_{ac}^1(\mathbf{x}_{ac}, \mathbf{x}_q), \gamma_{ac}^2(\mathbf{x}_{ac}, \mathbf{x}_q), \ldots, \gamma_{ac}^k(\mathbf{x}_{ac}, \mathbf{x}_q)]^T$ is a vector function, at $k = n_a$, while $\gamma_{ac}^k(\mathbf{x}_{ac}, \mathbf{x}_q)$ is a scalar function. The matrix of partial derivatives $\frac{\partial \boldsymbol{\gamma}_{ac}(\mathbf{x}_{ac}, \mathbf{x}_q)}{\partial \mathbf{x}_{ac}}$ is defined by the following expression:

$$\frac{\partial \boldsymbol{\gamma}_{ac}(\mathbf{x}_{ac}, \mathbf{x}_q)}{\partial \mathbf{x}_{ac}} = \begin{bmatrix} \frac{\partial \gamma_{ac}^1(\mathbf{x}_{ac}, \mathbf{x}_q)}{\partial \mathbf{x}_{ac}^1} & \frac{\partial \gamma_{ac}^1(\mathbf{x}_{ac}, \mathbf{x}_q)}{\partial \mathbf{x}_{ac}^2} & \cdots & \frac{\partial \gamma_{ac}^1(\mathbf{x}_{ac}, \mathbf{x}_q)}{\partial \mathbf{x}_{ac}^k} \\ \frac{\partial \gamma_{ac}^2(\mathbf{x}_{ac}, \mathbf{x}_q)}{\partial \mathbf{x}_{ac}^1} & \frac{\partial \gamma_{ac}^2(\mathbf{x}_{ac}, \mathbf{x}_q)}{\partial \mathbf{x}_{ac}^2} & \cdots & \frac{\partial \gamma_{ac}^2(\mathbf{x}_{ac}, \mathbf{x}_q)}{\partial \mathbf{x}_{ac}^k} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial \gamma_{ac}^k(\mathbf{x}_{ac}, \mathbf{x}_q)}{\partial \mathbf{x}_{ac}^1} & \frac{\partial \gamma_{ac}^k(\mathbf{x}_{ac}, \mathbf{x}_q)}{\partial \mathbf{x}_{ac}^2} & \cdots & \frac{\partial \gamma_{ac}^k(\mathbf{x}_{ac}, \mathbf{x}_q)}{\partial \mathbf{x}_{ac}^k} \end{bmatrix} \tag{35}$$

where, for instance, the matrix element in (35) is in position (1,2) defined as shown:

$$\frac{\partial \gamma_{ac}^1(\mathbf{x}_{ac}, \mathbf{x}_q)}{\partial \mathbf{x}_{ac}^2} = \frac{\partial \gamma_{ac}^1(\mathbf{x}_{ac}, \mathbf{x}_q)}{\partial x_{ac1}^2} + \frac{\partial \gamma_{ac}^1(\mathbf{x}_{ac}, \mathbf{x}_q)}{\partial x_{ac2}^2} + \cdots + \frac{\partial \gamma_{ac}^1(\mathbf{x}_{ac}, \mathbf{x}_q)}{\partial x_{acj}^2} \tag{36}$$

Here, $j = n_{ac}^2$, i.e. j represents a number of the state space variables of the actuator 2, while the number 2 from $j = n_{ac}^2$ denotes a mark for actuator and does not mean squaring of n_{ac}.

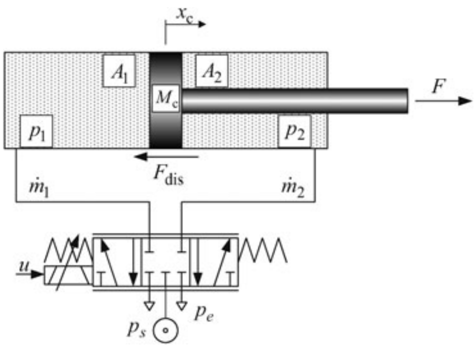


Fig. 5 Schematic representation of the pneumatic cylinder-valve system

Appendix C

A pneumatic actuator with proportional spool valve consists of a pneumatic cylinder with piston and a proportional spool valve, as shown in Fig. 5. A mathematical model of the pneumatic actuator [6, 21] consists of model of the piston dynamics, model of the fluid pressure dynamics and model of the proportional valve spool dynamics, while for kth actuator is a vector of the state space variables defined as follows:

$$\mathbf{x}_{ac}^k = \begin{bmatrix} x_{ac1}^k \\ x_{ac2}^k \\ x_{ac3}^k \\ x_{ac4}^k \end{bmatrix} = \begin{bmatrix} p_1 \\ p_2 \\ x_s \\ \dot{x}_s \end{bmatrix} \tag{37}$$

where state variables represent: p_1—air pressure in the left cylinder chamber, p_2—air pressure in the right cylinder chamber, x_s—spool displacement and $\dot{x}_s$—spool velocity. In modeling, this actuator has two additional state variables, x_c—cylinder piston displacement, and $\dot{x}_c$—cylinder piston velocity, but these variables are replaced by the state variables $(\mathbf{x}_{qa1}^k, \mathbf{x}_{qa2}^k)$ of the mechanical system. This replacement is a consequence of an assumption of the rigid connection between the actuator and mechanical system, as explained in Sect. 2.3. For that case, variables x_c i $\dot{x}_c$ do not belong to the vector $\mathbf{x}_{ac}^k$.

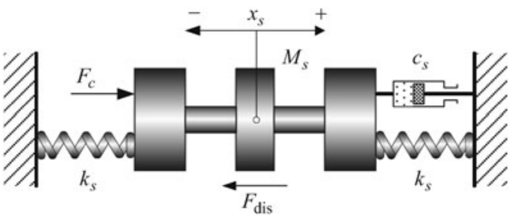
C.1 Cylinder piston dynamics

A cylinder piston dynamics is described by ordinary second-order differential equation, as shown by the following expression:

$$F^k = \underbrace{A_1 p_1 - A_2 p_2}_{\gamma_{ac}^k(\mathbf{x}_{ac}^k, \mathbf{x}_{qa1}^k, \mathbf{x}_{qa2}^k)} - \underbrace{M_c}_{M_{ac}^k} \underbrace{\ddot{q}_a^k}_{\dot{x}_{qa2}^k} - \underbrace{F_{\text{dis}_{ac1}}(q_a^k, \dot{q}_a^k)}_{F_{\text{dis}_{ac1}}^k(\mathbf{x}_{qa1}^k, \mathbf{x}_{qa2}^k)} \tag{38}$$

where M_c, A_1, A_2 are positive constants, representing mass of the piston, active surface area of the piston for chambers 1 and 2, respectively. The expression $F_{\text{dis}_{ac1}}(x_c, \dot{x}_c)$ represents friction force (which is in Fig. 5 denoted as F_{dis}) dependent on the state variables of the cylinder. The expression $A_1 p_1 - A_2 p_2$ represents a force acting on the piston and caused by air pressures p_1 and p_2, while F^k is a force produced by a kth pneumatic actuator. A rigid connection between state variables of the actuator and mechanical system is described by expressions $x_c = q_a^k$ and $\dot{x}_c = \dot{q}_a^k$. For this case, a transmission ratio is $N = 1$. An underlined representation in the equation (38) is a state space form, where state variables are $x_{ac1} = p_1$ and $x_{ac2} = p_2$.

Fig. 6 Schematic representation of valve spool dynamics



C.2 Fluid pressure dynamics

Air pressures in cylinders' chambers are defined as follows:

$$\begin{aligned}
\overbrace{\dot{p}_1}^{\dot{x}_{ac1}} &= \overbrace{\frac{\kappa RT}{V_D + V_0 + A_1 q_a^k}\dot{m}_1(x_s, p_1) - \frac{\kappa A_1}{V_D + V_0 + A_1 q_a^k} p_1 \dot{q}_a^k}^{f_{ac1}^k(\mathbf{x}_{ac}^k, \mathbf{x}_{qa1}^k, \mathbf{x}_{qa2}^k)} \\
\underbrace{\dot{p}_2}_{\dot{x}_{ac2}} &= \underbrace{\frac{\kappa RT}{V_D + V_0 - A_1 q_a^k}\dot{m}_2(x_s, p_2) + \frac{\kappa A_2}{V_D + V_0 - A_2 q_a^k} p_2 \dot{q}_a^k}_{f_{ac2}^k(\mathbf{x}_{ac}^k, \mathbf{x}_{qa1}^k, \mathbf{x}_{qa2}^k)}
\end{aligned} \tag{39}$$

where index 1 refers to the left chamber and index 2 refers to the right chamber of the cylinder.
Parameters κ, R, T, V_D, V_0, A_1, A_2 are constants greater the zero, representing the specific heat ratio, ideal gas constant, temperature, inactive cylinder volume at the and of stroke and admission port, half of the active cylinder volume, and the piston effective area, respectively. Variables p_i and $\dot{m}_i$, for $i \in \{1, 2\}$ represent air pressures and mass flow rates in cylinder chambers. Mass flows $\dot{m}_i$, for $i \in \{1, 2\}$ are expressed as functions of pressures and spool displacement x_s (see Fig. 6), and described by following equations:

$$\dot{m}_1(x_s, p_1) = \begin{cases} C_f A_v(x_s) C_1 \frac{p_s}{\sqrt{T}} & \text{if } \frac{p_1}{p_s} \le p_{cr} \\ C_f A_v(x_s) C_2 \frac{p_s}{\sqrt{T}} (\frac{p_1}{p_s})^{\frac{1}{\kappa}} \sqrt{1 - (\frac{p_1}{p_s})^{\frac{\kappa-1}{\kappa}}} & \text{if } \frac{p_1}{p_s} > p_{cr} \end{cases} \tag{40}$$

$$\dot{m}_2(x_s, p_2) = \begin{cases} C_f A_v(x_s) C_1 \frac{p_2}{\sqrt{T}} & \text{if } \frac{p_e}{p_2} \le p_{cr} \\ C_f A_v(x_s) C_2 \frac{p_2}{\sqrt{T}} (\frac{p_e}{p_2})^{\frac{1}{\kappa}} \sqrt{1 - (\frac{p_e}{p_2})^{\frac{\kappa-1}{\kappa}}} & \text{if } \frac{p_e}{p_2} > p_{cr} \end{cases} \tag{41}$$

where: C_f, C_1 and C_2 are constants greater the zero, $A_v(x_s)$ is the valve effective area for input/exhaust paths which depends on spool displacement, p_s is the pressure in the air reservoir, p_e is the exhaust pressure, p_{cr} is critical pressure value.

C.3 Valve spool dynamics

$$M_s \ddot{x}_s + c_s \dot{x}_s + F_{\mathrm{dis}_{ac3}}(x_s, \dot{x}_s) + 2k_s x_s = F_c \tag{42}$$

where M_s, k_s, c_s are constants greater the zero, representing the spool coil assembly mass, the spool springs constant, and the viscous friction coefficient, respectively. The expression $F_{\mathrm{dis}_{ac3}}(x_s, \dot{x}_s)$ represents the friction force (which is in Fig. 6 denoted as F_{dis}) dependent on the valve's state variables. The input F_c represents the force produced by the coil, and it can be linearly described as a function of electric current $F_c = K_{fc} i_c$ or voltage. The equation (42) can be expressed in a state space form as follows:

$$\dot{x}_{ac3} = \overbrace{x_{ac4}}^{f^k_{ac3}(\mathbf{x}^k_{ac},\mathbf{x}^k_{qa1},\mathbf{x}^k_{qa2})} \tag{43}$$

$$\dot{x}_{ac4} = \underbrace{-\frac{F_{\mathrm{dis}_{ac3}}(x_{ac3}, x_{ac4}) + c_s x_{ac4}}{M_s} - \frac{2k_s}{M_s} x_{ac3}}_{f^k_{ac4}(\mathbf{x}^k_{ac},\mathbf{x}^k_{qa1},\mathbf{x}^k_{qa2})} + \underbrace{\frac{K_{fc}}{M_s}}_{g^k_{ac4}(\mathbf{x}^k_{ac})} \underbrace{i_c}_{u^k} \tag{44}$$

where state variables are $x_{ac3} = x_s$ and $x_{ac4} = \dot{x}_s$.

The whole model of the pneumatic actuator (39), (43), (44), and (38), with general form presented in Sect. 2.3, is given by the following expressions:

$$\overbrace{\begin{bmatrix} \dot{x}_{ac1} \\ \dot{x}_{ac2} \\ \dot{x}_{ac3} \\ \dot{x}_{ac4} \end{bmatrix}}^{\dot{\mathbf{x}}^k_{ac}} = \overbrace{\begin{bmatrix} f^k_{ac1}(\mathbf{x}^k_{ac}, \mathbf{x}^k_{qa1}, \mathbf{x}^k_{qa2}) \\ f^k_{ac2}(\mathbf{x}^k_{ac}, \mathbf{x}^k_{qa1}, \mathbf{x}^k_{qa2}) \\ f^k_{ac3}(\mathbf{x}^k_{ac}, \mathbf{x}^k_{qa1}, \mathbf{x}^k_{qa2}) \\ f^k_{ac4}(\mathbf{x}^k_{ac}, \mathbf{x}^k_{qa1}, \mathbf{x}^k_{qa2}) \end{bmatrix}}^{\mathbf{f}^k_{ac}(\mathbf{x}^k_{ac},\mathbf{x}^k_{qa1},\mathbf{x}^k_{qa2})} + \overbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \\ g^k_{ac4}(\mathbf{x}^k_{ac}) \end{bmatrix}}^{\mathbf{g}^k_{ac}(\mathbf{x}^k_{ac})} u^k$$
$$F^k = \gamma^k_{ac}\left(\mathbf{x}^k_{ac}, \mathbf{x}^k_{qa1}, \mathbf{x}^k_{qa2}\right) - M^k_{ac}\dot{x}^k_{qa2} - F^k_{\mathrm{dis}_{ac1}}\left(\mathbf{x}^k_{qa1}, \mathbf{x}^k_{qa2}\right) \tag{45}$$

Note: A relative degree of this pneumatic actuator, according to the output F^k, is $r = 3$. A first time derivation of γ^k_{ac} contains x_s, a second time derivative of γ^k_{ac} contains $\dot{x}_s$, while a third time derivative of γ^k_{ac} contains $\ddot{x}_s$ and it contains the input u^k according to (44).

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