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Well-Posedness for a System of Quadratic Derivative Nonlinear Schrödinger Equations with Radial Initial Data

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Abstract. In the present paper, we consider the Cauchy problem of the system of quadratic derivative nonlinear Schrödinger equations. This system was introduced by Colin and Colin (Differ Integral Equ 17:297–330, 2004). The first and second authors obtained some well-posedness results in the Sobolev space $H^s(\mathbb{R}^d)$. We improve these results for conditional radial initial data by rewriting the system radial form.

1. Introduction

We consider the Cauchy problem of the system of nonlinear Schrödinger equations:

$$\begin{cases} (i\partial_t + \alpha\Delta)u = -(\nabla\cdot w)v, & t>0,\ x\in\mathbb{R}^d,\\ (i\partial_t + \beta\Delta)v = -(\nabla\cdot \overline{w})u, & t>0,\ x\in\mathbb{R}^d,\\ (i\partial_t + \gamma\Delta)w = \nabla(u\cdot\overline{v}), & t>0,\ x\in\mathbb{R}^d,\\ (u,v,w)|_{t=0} = (u_0,v_0,w_0)\in (H^s(\mathbb{R}^d))^d\times(H^s(\mathbb{R}^d))^d\times(H^s(\mathbb{R}^d))^d, \end{cases} \tag{1.1}$$

where α, β, $\gamma \in \mathbb{R}\backslash\{0\}$ and the unknown functions u, v, w are d-dimensional complex vector-valued. System (1.1) was introduced by Colin and Colin in [6] as a model of laser–plasma interaction. (See also [7,8].) They also showed that the local existence of the solution of (1.1) in $H^s(\mathbb{R}^d)$ for $s > \frac{d}{2}+3$. System (1.1) is invariant under the following scaling transformation:

$$A_\lambda(t,x) = \lambda^{-1}A(\lambda^{-2}t,\lambda^{-1}x) \quad (A=(u,v,w)), \tag{1.2}$$

and the scaling critical regularity is $s_c = \frac{d}{2}-1$. We put

$$\theta := \alpha\beta\gamma\left(\frac{1}{\alpha}-\frac{1}{\beta}-\frac{1}{\gamma}\right), \quad \kappa := (\alpha-\beta)(\alpha-\gamma)(\beta+\gamma). \tag{1.3}$$

Table 1. Well-posedness (WP for short) for (1.1) proved in [15]

	$d=1$	$d=2,3$	$d\geq 4$
$\theta>0$	WP for $s\geq 0$	WP for $s\geq s_c$	WP for $s\geq s_c$
$\theta=0$	WP for $s\geq 1$	WP for $s\geq 1$	
$\kappa\neq 0$ and $\theta<0$	WP for $s\geq \frac{1}{2}$		

We note that $\kappa=0$ does not occur when $\theta\geq 0$ for α, β, $\gamma\in\mathbb{R}\backslash\{0\}$.

First, we introduce some known results for related problems. System (1.1) has quadratic nonlinear terms which contain a derivative. A derivative loss arising from the nonlinearity makes the problem difficult. In fact, Mizohata [21] considered the Schrödinger equation

$$\begin{cases} i\partial_t u-\Delta u=(b_1(x)\cdot\nabla)u, & t\in\mathbb{R},\ x\in\mathbb{R}^d,\\ u(0,x)=u_0(x), & x\in\mathbb{R}^d \end{cases}$$

and proved that the uniform bound

$$\sup_{x\in\mathbb{R}^n,\omega\in S^{n-1},R>0}\left|\mathrm{Re}\int_0^R b_1(x+r\omega)\cdot\omega \mathrm{d}r\right|<\infty$$

is a necessary condition for the $L^2(\mathbb{R}^d)$ well-posedness. Furthermore, Christ [5] proved that the flow map of the nonlinear Schrödinger equation

$$\begin{cases} i\partial_t u-\partial_x^2 u=u\partial_x u, & t\in\mathbb{R},\ x\in\mathbb{R},\\ u(0,x)=u_0(x), & x\in\mathbb{R} \end{cases} \tag{1.4}$$

is not continuous on $H^s(\mathbb{R}^d)$ for any $s\in\mathbb{R}$. From these results, it is difficult to obtain the well-posedness for quadratic derivative nonlinear Schrödinger equation in general. For the system of quadratic derivative nonlinear equations, it is known that the well-posedness holds. In [15], the first author proved the well-posedness of (1.1) in $H^s(\mathbb{R}^d)$, where s is given in Table 1.

Recently, in [16], the first and second authors have improved this result by using the generalization of the Loomis–Whitney inequality introduced in [2] and [3]. They proved the well-posedness of (1.1) in $H^s(\mathbb{R}^d)$ for $s\geq\frac{1}{2}$ if $d=2$ and $s>\frac{1}{2}$ if $d=3$, under the condition $\kappa\neq 0$ and $\theta<0$. In [15], the first author also proved that the flow map is not C^2 for $s<1$ if $\theta=0$ and for $s<\frac{1}{2}$ if $\theta<0$ and $\kappa\neq 0$. Therefore, the well-posedness obtained in [15] and [16] is optimal except the case $d=3$ and $s=\frac{1}{2}$ (which is scaling critical) as far as we use the iteration argument. In particular, the optimal regularity is far from the scaling critical regularity if $d\leq 3$ and $\theta\leq 0$.

We point out that the results in [15,16] do not contain the scattering of the solution for $d\leq 3$ under the condition $\theta=0$ (and also $\theta<0$). In [17], Ikeda, Katayama, and Sunagawa considered the system of quadratic nonlinear Schrödinger equations

$$\left(i\partial_t+\frac{1}{2m_j}\Delta\right)u_j=F_j(u,\partial_x u), \quad t>0,\ x\in\mathbb{R}^d,\ j=1,2,3, \tag{1.5}$$

under the mass resonance condition $m_1 + m_2 = m_3$ (which corresponds to the condition $\theta = 0$ for (1.1)), where $u = (u_1, u_2, u_3)$ is $\mathbb{C}^3$-valued, m_1, m_2, $m_3 \in \mathbb{R}\backslash\{0\}$, and F_j is defined by

$$\begin{cases} F_1(u, \partial_x u) = \sum_{|\alpha|,|\beta|\le 1} C_{1,\alpha,\beta}(\overline{\partial^\alpha u_2})(\partial^\beta u_3), \\ F_2(u, \partial_x u) = \sum_{|\alpha|,|\beta|\le 1} C_{1,\alpha,\beta}(\partial^\beta u_3)(\overline{\partial^\alpha u_1}), \\ F_3(u, \partial_x u) = \sum_{|\alpha|,|\beta|\le 1} C_{1,\alpha,\beta}(\partial^\alpha u_1)(\partial^\beta u_2) \end{cases} \tag{1.6}$$

with some constants $C_{1,\alpha,\beta}$, $C_{2,\alpha,\beta}$, $C_{3,\alpha,\beta} \in \mathbb{C}$. They obtained the small data global existence and the scattering of the solution to (1.5) in the weighted Sobolev space for $d = 2$ under the mass resonance condition and the null condition for the nonlinear terms (1.6). They also proved the same result for $d \ge 3$ without the null condition. In [18], Ikeda, Kishimoto, and Okamoto proved the small data global well-posedness and the scattering of the solution to (1.5) in $H^s(\mathbb{R}^d)$ for $d \ge 3$ and $s \ge s_c$ under the mass resonance condition and the null condition for the nonlinear terms (1.6). They also proved the local well-posedness in $H^s(\mathbb{R}^d)$ for $d = 1$ and $s \ge 0$, $d = 2$ and $s > s_c$, and $d = 3$ and $s \ge s_c$ under the same conditions. (The results in [15] for $d \le 3$ and $\theta = 0$ say that if the nonlinear terms do not have null condition, then $s = 1$ is optimal regularity to obtain the well-posedness by using the iteration argument.)

Recently, in [23], Sakoda and Sunagawa have considered (1.5) for $d = 2$ and $j = 1, \dots, N$ with

$$F_j(u, \partial_x u) = \sum_{|\alpha|,|\beta|\le 1} \sum_{1\le k,l\le 2N} C^{\alpha,\beta}_{j,k,l}(\partial^\alpha_x u^\#_k)(\partial^\beta_x u^\#_l), \tag{1.7}$$

where $u^\#_j = u_j$ if $j = 1, \dots, N$, and $u^\#_j = \overline{u_j}$ if $j = N + 1, \dots, 2N$. They obtained the small data global existence and the time decay estimate for the solution under some conditions for $m_1, \cdots m_N$ and the nonlinear terms (1.7), where the conditions contain (1.1) with $\theta = 0$. There exists the blow-up solutions for the system of nonlinear Schrödinger equations. Ozawa and Sunagawa [22] gave the examples of the derivative nonlinearity which causes the small data blow-up for a system of Schrödinger equations. There are also some known results for a system of nonlinear Schrödinger equations with no derivative nonlinearity [12–14].

The aim in the present paper is to improve the results in [15,16] for conditional radial initial data in $\mathbb{R}^2$ and $\mathbb{R}^3$. The radial solution to (1.1) is only trivial solution since the nonlinear terms of (1.1) are not radial form. Therefore, we rewrite (1.1) into a radial form. Here, we focus on $d = 2$. Let $\mathcal{S}(\mathbb{R}^2)$ denote the Schwartz class. If $w = (w_1, w_2) \in (\mathcal{S}(\mathbb{R}^2))^2$ satisfies

$$\xi^\perp \cdot \widehat{w}(\xi) = \xi_1 \widehat{w_2}(\xi) - \xi_2 \widehat{w_1}(\xi) = 0, \quad x^\perp \cdot w(x) = x_1 w_2(x) - x_2 w_1(x) = 0 \tag{1.8}$$

for any $\xi = (\xi_1, \xi_2) \in \mathbb{R}^2$ and $x = (x_1, x_2) \in \mathbb{R}^2$, then there exists a scalar potential $W \in C^1(\mathbb{R}^2)$ satisfying

$$\nabla W(x) = w(x), \quad {}^\forall x \in \mathbb{R}^2 \tag{1.9}$$

and

$$\frac{\partial}{\partial\vartheta}W(r\cos\vartheta, r\sin\vartheta) = 0, \quad {}^\forall(r,\vartheta) \in [0,\infty)\times[0,2\pi). \tag{1.10}$$

Indeed, if we put

$$W(x) := \int_{a_1}^{x_1} w_1(y_1, x_2)\mathrm{d}y_1 + \int_{a_2}^{x_2} w_2(a_1, y_2)\mathrm{d}y_2$$

for some a_1, $a_2 \in \mathbb{R}$, then W satisfies (1.9) by the first equality in (1.8). Furthermore, W also satisfies (1.10) by the second equality in (1.8). We note that the first equality in (1.8) is equivalent to

$$\nabla^\perp \cdot w(x) = \partial_1 w_2(x) - \partial_2 w_1(x) = 0,$$

which is the irrotational condition.

Remark 1.1. If $d = 3$, we can also obtain the radial scalar potential $W \in C^1(\mathbb{R}^3)$ of $w = (w_1, w_2, w_3) \in (\mathcal{S}(\mathbb{R}^3))^3$ by assuming the conditions

$$\xi \times \widehat{w}(\xi) = 0,\ x \times w(x) = 0 \tag{1.11}$$

instead of (1.8).

Definition 1. We say $f \in \mathcal{S}'(\mathbb{R}^d)$ is radial if it holds that

$$< f, \varphi \circ R > = < f, \varphi >$$

for any $\varphi \in \mathcal{S}(\mathbb{R}^d)$ and rotation $R : \mathbb{R}^d \to \mathbb{R}^d$.

Remark 1.2. If $f \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$, then Definition 1 is equivalent to

$$^\exists g : \mathbb{R} \to \mathbb{C} \text{ s.t. } f(x) = g(|x|), \quad \text{a.e. } x \in \mathbb{R}^d.$$

Now, we consider the system of nonlinear Schrödinger equations:

$$\begin{cases} (i\partial_t + \alpha\Delta)u = -(\Delta W)v, & t > 0,\ x \in \mathbb{R}^d, \\ (i\partial_t + \beta\Delta)v = -(\Delta \overline{W})u, & t > 0,\ x \in \mathbb{R}^d, \\ (i\partial_t + \gamma\Delta)\nabla W = \nabla(u \cdot \overline{v}), & t > 0,\ x \in \mathbb{R}^d, \\ (u, v, [W])|_{t=0} = (u_0, v_0, [W_0]) \in \mathcal{H}^s(\mathbb{R}^d) \end{cases} \tag{1.12}$$

instead of (1.1), where $d = 2$ or 3, and

$$\begin{aligned} \mathcal{H}^s(\mathbb{R}^d) &:= (H^s_{\mathrm{rad}}(\mathbb{R}^d))^d \times (H^s_{\mathrm{rad}}(\mathbb{R}^d))^d \times \widetilde{H}^{s+1}_{\mathrm{rad}}(\mathbb{R}^d), \\ H^s_{\mathrm{rad}}(\mathbb{R}^d) &:= \{f \in H^s(\mathbb{R}^d) |\ f \text{ is radial}\}, \\ \widetilde{H}^{s+1}(\mathbb{R}^d) &:= \{f \in \mathcal{S}'(\mathbb{R}^d) |\ \nabla f \in (H^s(\mathbb{R}^d))^d\}/\mathcal{N}_0, \\ \mathcal{N}_0 &:= \{f \in \mathcal{S}'(\mathbb{R}^d) |\ \nabla f = 0\}, \\ \widetilde{H}^{s+1}_{\mathrm{rad}}(\mathbb{R}^d) &:= \{[f] \in \widetilde{H}^{s+1}(\mathbb{R}^d) |\ f \text{ is radial}\}. \end{aligned}$$

The norm for an equivalent class $[f] \in \widetilde{H}^{s+1}(\mathbb{R}^d)$ is defined by

$$\|[f]\|_{\widetilde{H}^{s+1}} := \|\nabla f\|_{(H^s)^d} \sim \|f\|_{\dot{H}^{s+1}} + \|f\|_{\dot{H}^1},$$

which is well defined since $\widetilde{H}^{s+1}(\mathbb{R}^d)$ is a quotient space. System (1.12) is obtained by substituting $w = \nabla W$ and $w_0 = \nabla W_0$ in (1.1).

Definition 2. We say $(u, v, [W]) \in C([0,T]; \mathcal{H}^s(\mathbb{R}^d))$ is a solution to (1.12) if

$$u(t) = e^{it\alpha\Delta}u_0 + i\int_0^t e^{i(t-t')\alpha\Delta}(\Delta W(t'))v(t')\mathrm{d}t' \quad \text{in } (H^s(\mathbb{R}^d))^d,$$

$$v(t) = e^{it\beta\Delta}v_0 + i\int_0^t e^{i(t-t')\beta\Delta}(\Delta\overline{W(t')})v(t')\mathrm{d}t' \quad \text{in } (H^s(\mathbb{R}^d))^d,$$

$$\nabla W(t) = e^{it\gamma\Delta}\nabla W_0 - i\int_0^t e^{i(t-t')\gamma\Delta}\nabla(u(t')\cdot\overline{v(t')})\mathrm{d}t' \quad \text{in } H^s(\mathbb{R}^d)$$

hold for any $t \in [0,T]$. This definition does not depend on how we choose a representative W.

Now, we give the main results in this paper.

Theorem 1.1. *Assume $\kappa \neq 0$.*

(i) *Let $d = 2$. Assume that $s \geq \frac{1}{2}$ if $\theta = 0$ and $s > 0$ if $\theta < 0$. Then, (1.12) is locally well posed in $\mathcal{H}^s(\mathbb{R}^2)$.*

(ii) *Let $d = 3$. Assume that $\theta \leq 0$ and $s \geq \frac{1}{2}$. Then, (1.12) is locally well posed in $\mathcal{H}^s(\mathbb{R}^3)$.*

(iii) *Let $d = 3$. Assume that $\theta \leq 0$ and $s \geq \frac{1}{2}$. Then, (1.12) is globally well posed in $\mathcal{H}^s(\mathbb{R}^3)$ for small data. Furthermore, the solution scatters in $\mathcal{H}^s(\mathbb{R}^3)$.*

Remark 1.3. $s = 0$ for $d = 2$, and $s = \frac{1}{2}$ for $d = 3$ are scaling critical regularity for (1.1).

We obtain the following.

Theorem 1.2. *Let $d = 2$ and $\theta = 0$. Then, the flow map of (1.12) is not C^2 in $\mathcal{H}^s(\mathbb{R}^2)$ for $s < \frac{1}{2}$.*

Remark 1.4. Theorem 1.2 says that the well-posedness in Theorem 1.1 for $\theta = 0$ is optimal as far as we use the iteration argument.

Remark 1.5. It is interesting that the result for 2D radial initial data is better than that for 1D initial data. Actually, the optimal regularity for 1D initial data is $s = 1$ if $\theta = 0$, and $s = \frac{1}{2}$ if $\theta < 0$ and $\kappa \neq 0$, which are larger than the optimal regularity for 2D radial initial data. The reason is the following. We use the angular decomposition, and each angular localized term has a better property. For radial functions, the angular localized bound leads to an estimate for the original functions. (See (2.15).)

We note that if $\nabla W_0 = w_0$ holds and $(u, v, [W])$ is a solution to (1.12) with $(u, v, [W])|_{t=0} = (u_0, v_0, [W_0]) \in \mathcal{H}^s(\mathbb{R}^d)$, then $(u, v, \nabla W)$ is a solution to (1.1) with $(u, v, \nabla W)|_{t=0} = (u_0, v_0, w_0) \in (H^s_{\mathrm{rad}}(\mathbb{R}^d))^d \times (H^s_{\mathrm{rad}}(\mathbb{R}^d))^d \times H^s(\mathbb{R}^d)$. The existence of a scalar potential $W_0 \in \widetilde{H}^{s+1}_{\mathrm{rad}}(\mathbb{R}^d)$ will be proved for $w_0 \in \mathcal{A}^s(\mathbb{R}^d)$ with $s > \frac{1}{2}$ (see Proposition 3.2), where

$$\mathcal{A}^s(\mathbb{R}^2) := \left\{ f = (f_1, f_2) \in (H^s(\mathbb{R}^2))^2 \middle|\, f \text{ satisfies (1.8) a.e.} x, \xi \in \mathbb{R}^2 \right\},$$

$$\mathcal{A}^s(\mathbb{R}^3) := \left\{ f = (f_1, f_2, f_3) \in (H^s(\mathbb{R}^3))^3 \middle|\, f \text{ satisfies(1.11) a.e.} x, \xi \in \mathbb{R}^3 \right\}.$$

Therefore, we obtain the following.

Theorem 1.3. *Let $d = 2$ or 3. Assume that $\theta = 0$ and $s > \frac{1}{2}$. Then,* (1.1) *is locally well posed in $(H^s_{\mathrm{rad}}(\mathbb{R}^d))^d \times (H^s_{\mathrm{rad}}(\mathbb{R}^d))^d \times \mathcal{A}^s(\mathbb{R}^d)$.*

Remark 1.6. For $d = 3$, Theorem 1.1 can be obtained by almost the same way as in [15]. In Proposition 4.4 (i) of [15], the author used the Strichartz estimate

$$\|e^{it\Delta}P_N u_0\|_{L^q_t L^r_x(\mathbb{R}\times\mathbb{R}^d)} \lesssim \|P_N u_0\|_{L^2}$$

and

$$\left| N_{\max}\int_0^T\int_{\mathbb{R}^d}(P_{N_1}u_1)(P_{N_2}u_2)(P_{N_3}u_3)\mathrm{d}x\mathrm{d}t\right| \lesssim N_{\max}^{s_c}\prod_{j=1}^3\|P_{N_j}u_j\|_{L^q_t L^r_x}$$

with an admissible pair $(q, r) = (3, \frac{6d}{3d-4})$ for $d \geq 4$. But this trilinear estimate does not hold for $d = 3$. This is the reason why the well-posedness in $H^{s_c}(\mathbb{R}^3)$ could not be obtained in [15]. For the radial function $u_0 \in L^2(\mathbb{R}^3)$, it is known that the improved Strichartz estimate ([24], Corollary 6.2)

$$\|e^{it\Delta}P_N u_0\|_{L^3_{t,x}(\mathbb{R}\times\mathbb{R}^3)} \lesssim N^{-\frac{1}{6}}\|P_N u_0\|_{L^2}.$$

It holds that

$$\left| N_{\max}\int_0^T\int_{\mathbb{R}^3}(P_{N_1}u_1)(P_{N_2}u_2)(P_{N_3}u_3)\mathrm{d}x\mathrm{d}t\right| \lesssim N_{\max}^{\frac{1}{2}}\prod_{j=1}^3 N_j^{\frac{1}{6}}\|P_{N_j}u_j\|_{L^3_{t,x}}$$

for $N_1 \sim N_2 \sim N_3 \geq 1$. Therefore, for $d = 3$, we can obtain the same estimate in Proposition 4.4 (i). Because of such reason, we omit more detail of the proof for $d = 3$ and only consider $d = 2$ in the following sections.

Notation. We denote the spatial Fourier transform by $\widehat{\cdot}$ or $\mathcal{F}_x$, the Fourier transform in time by $\mathcal{F}_t$ and the Fourier transform in all variables by $\widetilde{\cdot}$ or $\mathcal{F}_{tx}$. For $\sigma \in \mathbb{R}$, the free evolution $e^{it\sigma\Delta}$ on L^2 is given as a Fourier multiplier

$$\mathcal{F}_x[e^{it\sigma\Delta}f](\xi) = e^{-it\sigma|\xi|^2}\widehat{f}(\xi).$$

We will use $A \lesssim B$ to denote an estimate of the form $A \leq CB$ for some constant C and write $A \sim B$ to mean $A \lesssim B$ and $B \lesssim A$. We will use the convention that capital letters denote dyadic numbers, e.g. $N = 2^n$ for $n \in \mathbb{N}_0 := \mathbb{N}\cup\{0\}$, and for a dyadic summation, we write $\sum_N a_N := \sum_{n\in\mathbb{N}_0} a_{2^n}$ and $\sum_{N\geq M} a_N := \sum_{n\in\mathbb{N}_0, 2^n\geq M} a_{2^n}$ for brevity. Let $\chi \in C_0^\infty((-2, 2))$ be an even, non-negative function such that $\chi(t) = 1$ for $|t| \leq 1$. We define $\psi(t) := \chi(t) - \chi(2t)$, $\psi_1(t) := \chi(t)$, and $\psi_N(t) := \psi(N^{-1}t)$ for $N \geq 2$. Then, $\sum_N \psi_N(t) = 1$. We define frequency and modulation projections

$$\widehat{P_N u}(\xi) := \psi_N(\xi)\widehat{u}(\xi),\ \widetilde{Q_L^\sigma u}(\tau,\xi) := \psi_L(\tau + \sigma|\xi|^2)\widetilde{u}(\tau,\xi).$$

Furthermore, we define $Q^\sigma_{\geq M} := \sum_{L\geq M} Q^\sigma_L$ and $Q_{<M} := Id - Q_{\geq M}$.

The rest of this paper is planned as follows. In Section 2, we will give the bilinear estimates which will be used to prove the well-posedness. In Sect. 3, we will give the proof of Theorems 1.1 and 1.3. In Sect. 4, we will give the proof of Theorem 1.2.

2. Bilinear Estimates

In this section, we prove the bilinear estimates. First, we define the radial condition for time–space function.

Definition 3. We say $u \in \mathcal{S}'(\mathbb{R}_t \times \mathbb{R}_x^2)$ is radial with respect to x if it holds that

$$< u, \varphi_R >=< u, \varphi >$$

for any $\varphi \in \mathcal{S}(\mathbb{R}_t \times \mathbb{R}_x^2)$ and rotation $R : \mathbb{R}^2 \to \mathbb{R}^2$, where $\varphi_R \in \mathcal{S}(\mathbb{R}_t \times \mathbb{R}_x^2)$ is defined by $\varphi_R(t,x) = \varphi(t, R(x))$.

Next, we define the Fourier restriction norm, which was introduced by Bourgain in [4].

Definition 4. Let $s \in \mathbb{R}$, $b \in \mathbb{R}$, $\sigma \in \mathbb{R}\backslash\{0\}$.

(i) We define $X_\sigma^{s,b} := \{u \in \mathcal{S}'(\mathbb{R}_t \times \mathbb{R}_x^2) |\ \|u\|_{X_\sigma^{s,b}} < \infty\}$, where

$$\|u\|_{X_\sigma^{s,b}} := \|\langle\xi\rangle^s \langle\tau + \sigma|\xi|^2\rangle^b \widetilde{u}(\tau,\xi)\|_{L^2_{\tau\xi}} \sim \left(\sum_{N\geq 1}\sum_{L\geq 1} N^{2s}L^{2b}\|Q_L^\sigma P_N u\|_{L^2}^2\right)^{\frac{1}{2}}.$$

(ii) We define $\widetilde{X}_\sigma^{s+1,b} := \{u \in \mathcal{S}'(\mathbb{R}_t \times \mathbb{R}_x^2) |\ \nabla u \in X_\sigma^{s,b}\}/\mathcal{N}$ with the norm

$$\|[u]\|_{\widetilde{X}_\sigma^{s+1,b}} := \|\nabla u\|_{X_\sigma^{s,b}},$$

where $\mathcal{N} := \{u \in \mathcal{S}'(\mathbb{R}_t \times \mathbb{R}_x^2) |\ \nabla u = 0\}$.

(iii) We define

$$X_{\sigma,\mathrm{rad}}^{s,b} := \{u \in X_\sigma^{s,b} |\ u \text{ is radial with respect to } x\},$$
$$\widetilde{X}_{\sigma,\mathrm{rad}}^{s,b} := \{[u] \in \widetilde{X}_\sigma^{s+1,b} |\ u \text{ is radial with respect to } x\}.$$

We put

$$\widetilde{\theta} := \sigma_1\sigma_2\sigma_3\left(\frac{1}{\sigma_1} + \frac{1}{\sigma_2} + \frac{1}{\sigma_3}\right), \quad \widetilde{\kappa} := (\sigma_1+\sigma_2)(\sigma_2+\sigma_3)(\sigma_3+\sigma_1).$$

We note that if $(\sigma_1,\sigma_2,\sigma_3) \in \{(\beta,\gamma,-\alpha),(-\gamma,\alpha,-\beta),(\alpha,-\beta,-\gamma)\}$, then it hold that $\widetilde{\theta} = \theta$ and $|\widetilde{\kappa}| = |\kappa|$.

The following bilinear estimate plays a central role to show Theorem 1.1.

Proposition 2.1. *Let σ_1, σ_2, $\sigma_3 \in \mathbb{R}\backslash\{0\}$ satisfy $\widetilde{\kappa} \neq 0$. Let $s \geq \frac{1}{2}$ if $\widetilde{\theta} = 0$ and $s > 0$ if $\widetilde{\theta} < 0$. Then there exists $b' \in (0, \frac{1}{2})$ and $C > 0$ such that*

$$\||\nabla|(uv)\|_{X_{-\sigma_3}^{s,-b'}} \leq C\|u\|_{X_{\sigma_1}^{s,b'}}\|v\|_{X_{\sigma_2}^{s,b'}}, \tag{2.1}$$
$$\|(\Delta U)v\|_{X_{-\sigma_3}^{s,-b'}} \leq C(\|\partial_1 U\|_{X_{\sigma_1}^{s,b'}} + \|\partial_2 U\|_{X_{\sigma_1}^{s,b'}})\|v\|_{X_{\sigma_2}^{s,b'}} \tag{2.2}$$

hold for any $u \in X_{\sigma_1,\mathrm{rad}}^{s,b'}$, $v \in X_{\sigma_2,\mathrm{rad}}^{s,b'}$, and $[U] \in \widetilde{X}_{\sigma_1,\mathrm{rad}}^{s+1,b'}$.

Remark 2.1. Since $\|\partial_1(uv)\|_{X^{s,-b'}_{-\sigma_3}} + \|\partial_2(uv)\|_{X^{s,-b'}_{-\sigma_3}} \sim \||\nabla|(uv)\|_{X^{s,-b'}_{-\sigma_3}}$, (2.1) implies

$$\|\partial_1(uv)\|_{X^{s,-b'}_{-\sigma_3}} + \|\partial_2(uv)\|_{X^{s,-b'}_{-\sigma_3}} \leq C\|u\|_{X^{s,b'}_{\sigma_1}}\|v\|_{X^{s,b'}_{\sigma_2}}.$$

To prove Proposition 2.1, we first give the Strichartz estimate.

Proposition 2.2. (Strichartz estimate (cf. [11,19])). *Let $\sigma \in \mathbb{R}\backslash\{0\}$ and (p,q) be an admissible pair of exponents for the 2D Schrödinger equation, i.e. $p > 2$, $\frac{1}{p} + \frac{1}{q} = \frac{1}{2}$. Then, we have*

$$\|e^{it\sigma\Delta}\varphi\|_{L^p_t L^q_x(\mathbb{R}\times\mathbb{R}^2)} \lesssim \|\varphi\|_{L^2_x(\mathbb{R}^2)}.$$

for any $\varphi \in L^2(\mathbb{R}^2)$.

The Strichartz estimate implies the following. (See the proof of Lemma 2.3 in [10].)

Corollary 2.3. *Let $L \in 2^{\mathbb{N}_0}$, $\sigma \in \mathbb{R}\backslash\{0\}$, and (p,q) be an admissible pair of exponents for the Schrödinger equation. Then, we have*

$$\|Q^{\sigma}_L u\|_{L^p_t L^q_x} \lesssim L^{\frac{1}{2}}\|Q^{\sigma}_L u\|_{L^2_{tx}}. \tag{2.3}$$

for any $u \in L^2(\mathbb{R}\times\mathbb{R}^2)$.

Next, we give the bilinear Strichartz estimate.

Proposition 2.4. *We assume that σ_1, $\sigma_2 \in \mathbb{R}\backslash\{0\}$ satisfy $\sigma_1 + \sigma_2 \neq 0$. For any dyadic numbers N_1, N_2, $N_3 \in 2^{\mathbb{N}_0}$ and L_1, $L_2 \in 2^{\mathbb{N}_0}$, we have*

$$\begin{aligned} &\|P_{N_3}(Q^{\sigma_1}_{L_1}P_{N_1}u_1 \cdot Q^{\sigma_2}_{L_2}P_{N_2}u_2)\|_{L^2_{tx}(\mathbb{R}\times\mathbb{R}^2)} \\ &\quad\lesssim \left(\frac{N_{\min}}{N_{\max}}\right)^{\frac{1}{2}} L_1^{\frac{1}{2}}L_2^{\frac{1}{2}}\|Q^{\sigma_1}_{L_1}P_{N_1}u_1\|_{L^2_{tx}(\mathbb{R}\times\mathbb{R}^2)}\|Q^{\sigma_2}_{L_2}P_{N_2}u_2\|_{L^2_{tx}(\mathbb{R}\times\mathbb{R}^2)}, \end{aligned} \tag{2.4}$$

where $N_{\min} = \min_{1\leq i\leq 3} N_i$, $N_{\max} = \max_{1\leq i\leq 3} N_i$.

Proposition 2.4 can be obtained by the same way as Lemma 1 in [9]. (See also Lemma 3.1 in [15].)

Corollary 2.5. *Let $b' \in (\frac{1}{4}, \frac{1}{2})$, and σ_1, $\sigma_2 \in \mathbb{R}\backslash\{0\}$ satisfy $\sigma_1 + \sigma_2 \neq 0$, We put $\delta = \frac{1}{2} - b'$. For any dyadic numbers N_1, N_2, $N_3 \in 2^{\mathbb{N}_0}$ and L_1, $L_2 \in 2^{\mathbb{N}_0}$, we have*

$$\begin{aligned} &\|P_{N_3}(Q^{\sigma_1}_{L_1}P_{N_1}u_1 \cdot Q^{\sigma_2}_{L_2}P_{N_2}u_2)\|_{L^2_{tx}(\mathbb{R}\times\mathbb{R}^2)} \\ &\quad\lesssim N_{\min}^{4\delta}\left(\frac{N_{\min}}{N_{\max}}\right)^{\frac{1}{2}-2\delta} L_1^{b'}L_2^{b'}\|Q^{\sigma_1}_{L_1}P_{N_1}u_1\|_{L^2_{tx}(\mathbb{R}\times\mathbb{R}^2)}\|Q^{\sigma_2}_{L_2}P_{N_2}u_2\|_{L^2_{tx}(\mathbb{R}\times\mathbb{R}^2)}. \end{aligned} \tag{2.5}$$

The proof is given in Corollary 2.5 in [16].

2.1. The Estimates for Low Modulation

In this subsection, we assume that $L_{\max} \ll N_{\max}^2$.

Lemma 2.6. *We assume that* $\sigma_1, \sigma_2, \sigma_3 \in \mathbb{R} \setminus \{0\}$ *satisfy* $\widetilde{\kappa} \neq 0$ *and* (τ_1, ξ_1), (τ_2, ξ_2), $(\tau_3, \xi_3) \in \mathbb{R} \times \mathbb{R}^2$ *satisfy* $\tau_1 + \tau_2 + \tau_3 = 0$, $\xi_1 + \xi_2 + \xi_3 = 0$. *If* $\max_{1 \leq j \leq 3} |\tau_j + \sigma_j |\xi_j|^2| \ll \max_{1 \leq j \leq 3} |\xi_j|^2$, *then we have*

$$|\xi_1| \sim |\xi_2| \sim |\xi_3|.$$

Since the above lemma is the contrapositive of the following lemma which was utilized in [15], we omit the proof.

Lemma 2.7. (Lemma 4.1 in [15]) *We assume that* $\sigma_1, \sigma_2, \sigma_3 \in \mathbb{R} \setminus \{0\}$ *satisfy* $\widetilde{\kappa} \neq 0$ *and* (τ_1, ξ_1), (τ_2, ξ_2), $(\tau_3, \xi_3) \in \mathbb{R} \times \mathbb{R}^2$ *satisfy* $\tau_1 + \tau_2 + \tau_3 = 0$, $\xi_1 + \xi_2 + \xi_3 = 0$. *If there exist* $1 \leq i, j \leq 3$ *such that* $|\xi_i| \ll |\xi_j|$, *then we have*

$$\max_{1 \leq j \leq 3} |\tau_j + \sigma_j |\xi_j|^2| \gtrsim \max_{1 \leq j \leq 3} |\xi_j|^2. \tag{2.6}$$

Lemma 2.6 suggests that if $\max_{1 \leq j \leq 3} |\tau_j + \sigma_j |\xi_j|^2| \ll \max_{1 \leq j \leq 3} |\xi_j|^2$ then we can assume

$$\max_{1 \leq j \leq 3} |\tau_j + \sigma_j |\xi_j|^2| \ll \min_{1 \leq j \leq 3} |\xi_j|^2. \tag{2.7}$$

We first introduce the angular frequency localization operators which were utilized in [1].

Definition 5 [1]. We define the angular decomposition of $\mathbb{R}^2$ in frequency. We define a partition of unity in $\mathbb{R}$,

$$1 = \sum_{j \in \mathbb{Z}} \omega_j, \quad \omega_j(s) = \psi(s - j) \left(\sum_{k \in \mathbb{Z}} \psi(s - k) \right)^{-1}.$$

For a dyadic number $A \geq 64$, we also define a partition of unity on the unit circle,

$$1 = \sum_{j=0}^{A-1} \omega_j^A, \quad \omega_j^A(\vartheta) = \omega_j \left(\frac{A\vartheta}{\pi} \right) + \omega_{j-A} \left(\frac{A\vartheta}{\pi} \right).$$

We observe that ω_j^A is supported in

$$\Theta_j^A = \left[\frac{\pi}{A}(j-2),\ \frac{\pi}{A}(j+2) \right] \cup \left[-\pi + \frac{\pi}{A}(j-2),\ -\pi + \frac{\pi}{A}(j+2) \right].$$

We now define the angular frequency localization operators R_j^A,

$$\mathcal{F}_x(R_j^A f)(\xi) = \omega_j^A(\vartheta) \mathcal{F}_x f(\xi), \quad \text{where } \xi = |\xi|(\cos\vartheta, \sin\vartheta).$$

For any function $u : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{C}$, $(t, x) \mapsto u(t, x)$, we set $(R_j^A u)(t, x) = (R_j^A u(t, \cdot))(x)$. This operator localizes function in frequency to the set

$$\mathfrak{D}_j^A = \left\{ (\tau, |\xi| \cos\vartheta, |\xi| \sin\vartheta) \in \mathbb{R} \times \mathbb{R}^2 \,|\, \vartheta \in \Theta_j^A \right\}.$$

Immediately, we can see

$$u = \sum_{j=0}^{A-1} R_j^A u.$$

The next lemma will be used to obtain Proposition 2.1 for the case $\widetilde{\theta} = 0$.

Lemma 2.8. *Let N, L_1, L_2, L_3, $A \in 2^{\mathbb{N}_0}$. We assume that σ_1, σ_2, $\sigma_3 \in \mathbb{R}\backslash\{0\}$ satisfy $\widetilde{\theta} = 0$ and (τ_1, ξ_1), (τ_2, ξ_2), $(\tau_3, \xi_3) \in \mathbb{R} \times \mathbb{R}^2$ satisfy $\tau_1 + \tau_2 + \tau_3 = 0$, $\xi_1 + \xi_2 + \xi_3 = 0$, $|\xi_i| \sim N_i$, $|\tau_i + \sigma_i|\xi_i|^2| \sim L_i$, and $(\tau_i, \xi_i) \in \mathfrak{D}_{j_i}^A$ $(i = 1, 2, 3)$ for some j_1, j_2, $j_3 \in \{0, 1, \ldots, A-1\}$. If $N_1 \sim N_2 \sim N_3$, $L_{\max} := \max_{1 \leq i \leq 3} L_i \leq N_{\max}^2 A^{-2}$, and $A \gg 1$ hold, then we have* $\min\{|j_1 - j_2|, |A - (j_1 - j_2)|\} \lesssim 1$, $\min\{|j_2 - j_3|, |A - (j_2 - j_3)|\} \lesssim 1$, *and* $\min\{|j_1 - j_3|, |A - (j_1 - j_3)|\} \lesssim 1$.

Proof. Because $0 = \widetilde{\theta} = \sigma_1\sigma_2\sigma_3(\frac{1}{\sigma_1} + \frac{1}{\sigma_2} + \frac{1}{\sigma_3}) = \sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1$, we have

$$(\sigma_1 + \sigma_3)(\sigma_2 + \sigma_3) = \sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1 + \sigma_3^2 = \sigma_3^2 > 0.$$

We put $p := \operatorname{sgn}(\sigma_1 + \sigma_3) = \operatorname{sgn}(\sigma_2 + \sigma_3)$, $q := \operatorname{sgn}(\sigma_3)$. Let $\angle(\xi_1, \xi_2) \in [0, \pi]$ denote the smaller angle between ξ_1 and ξ_2. Since

$$\frac{|\sigma_1 + \sigma_3|^{\frac{1}{2}}|\sigma_2 + \sigma_3|^{\frac{1}{2}}}{|\sigma_3|} = \sqrt{1 + \frac{\sigma_1\sigma_2\sigma_3}{\sigma_3^2}\left(\frac{1}{\sigma_1} + \frac{1}{\sigma_2} + \frac{1}{\sigma_3}\right)} = 1,$$

we have

$$\begin{aligned}
N_{\max}^2 A^{-2} &\geq L_{\max} \\
&\gtrsim |\sigma_1|\xi_1|^2 + \sigma_2|\xi_2|^2 + \sigma_3|\xi_1 + \xi_2|^2| \\
&= |(\sigma_1 + \sigma_3)|\xi_1|^2 + (\sigma_2 + \sigma_3)|\xi_2|^2 + 2\sigma_3|\xi_1||\xi_2|\cos\angle(\xi_1, \xi_2)| \\
&= |p(|\sigma_1 + \sigma_3|^{\frac{1}{2}}|\xi_1| - |\sigma_2 + \sigma_3|^{\frac{1}{2}}|\xi_2|)^2 \\
&\quad + 2|\xi_1||\xi_2|(p|\sigma_1 + \sigma_3|^{\frac{1}{2}}|\sigma_2 + \sigma_3|^{\frac{1}{2}} + q|\sigma_3|\cos\angle(\xi_1, \xi_2))| \\
&= |(|\sigma_1 + \sigma_3|^{\frac{1}{2}}|\xi_1| - |\sigma_2 + \sigma_3|^{\frac{1}{2}}|\xi_2|)^2 + 2|\sigma_3||\xi_1||\xi_2|(1 + pq\cos\angle(\xi_1, \xi_2))| \\
&\geq 2|\sigma_3||\xi_1||\xi_2|(1 + pq\cos\angle(\xi_1, \xi_2)).
\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}
1 - \cos\angle(\xi_1, \xi_2) &\lesssim A^{-2} \quad \text{if } (\sigma_1 + \sigma_3)\sigma_3 < 0, \\
1 + \cos\angle(\xi_1, \xi_2) &\lesssim A^{-2} \quad \text{if } (\sigma_1 + \sigma_3)\sigma_3 > 0.
\end{aligned}$$

This implies

$$\angle(\xi_1, \xi_2) \lesssim A^{-1} \text{ or } \pi - \angle(\xi_1, \xi_2) \lesssim A^{-1}.$$

Therefore, we get $\min\{|j_1 - j_2|, |A - (j_1 - j_2)|\} \lesssim 1$. By the same argument, we also get $\min\{|j_2 - j_3|, |A - (j_2 - j_3)|\} \lesssim 1$ and $\min\{|j_1 - j_3|, |A - (j_1 - j_3)|\}$ $\lesssim 1$. □

Now we introduce the necessary bilinear estimates to obtain Proposition 2.1 for the case $\widetilde{\theta} < 0$.

Theorem 2.1. (Theorem 2.8 in [16]) *We assume that* $\sigma_1, \sigma_2, \sigma_3 \in \mathbb{R}\backslash\{0\}$ *satisfy* $\widetilde{\kappa} \neq 0$ *and* $\widetilde{\theta} < 0$. *Let* $L_{\max} := \max_{1\leq j\leq 3}(L_1, L_2, L_3) \ll |\widetilde{\theta}|N_{\min}^2$, $A \geq 64$, *and* $|j_1 - j_2| \lesssim 1$. *Then the following estimates hold:*

$$\begin{aligned}
&\|Q_{L_3}^{-\sigma_3} P_{N_3}(R_{j_1}^A Q_{L_1}^{\sigma_1} P_{N_1} u_1 \cdot R_{j_2}^A Q_{L_2}^{\sigma_2} P_{N_2} u_2)\|_{L^2_{tx}} \\
&\quad \lesssim A^{-\frac{1}{2}} L_1^{\frac{1}{2}} L_2^{\frac{1}{2}} \|R_{j_1}^A Q_{L_1}^{\sigma_1} P_{N_1} u_1\|_{L^2_{tx}} \|R_{j_2}^A Q_{L_2}^{\sigma_2} P_{N_2} u_2\|_{L^2_{tx}}, \qquad (2.8)\\
&\|R_{j_1}^A Q_{L_1}^{-\sigma_1} P_{N_1}(R_{j_2}^A Q_{L_2}^{\sigma_2} P_{N_2} u_2 \cdot Q_{L_3}^{\sigma_3} P_{N_3} u_3)\|_{L^2_{tx}} \\
&\quad \lesssim A^{-\frac{1}{2}} L_2^{\frac{1}{2}} L_3^{\frac{1}{2}} \|R_{j_2}^A Q_{L_2}^{\sigma_2} P_{N_2} u_2\|_{L^2_{tx}} \|Q_{L_3}^{\sigma_3} P_{N_3} u_3\|_{L^2_{tx}}, \qquad (2.9)\\
&\|R_{j_2}^A Q_{L_2}^{-\sigma_2} P_{N_2}(Q_{L_3}^{\sigma_3} P_{N_3} u_3 \cdot R_{j_1}^A Q_{L_1}^{\sigma_1} P_{N_1} u_1)\|_{L^2_{tx}} \\
&\quad \lesssim A^{-\frac{1}{2}} L_3^{\frac{1}{2}} L_1^{\frac{1}{2}} \|Q_{L_3}^{\sigma_3} P_{N_3} u_3\|_{L^2_{tx}} \|R_{j_1}^A Q_{L_1}^{\sigma_1} P_{N_1} u_1\|_{L^2_{tx}}. \qquad (2.10)
\end{aligned}$$

Proposition 2.9. (Proposition 2.9 in [16]) *We assume that* $\sigma_1, \sigma_2, \sigma_3 \in \mathbb{R}\backslash\{0\}$ *satisfy* $\widetilde{\kappa} \neq 0$ *and* $\widetilde{\theta} < 0$. *Let* $L_{\max} \ll |\widetilde{\theta}|N_{\min}^2$ *and* $64 \leq A \leq N_{\max}$, $16 \leq |j_1 - j_2| \leq 32$. *Then the following estimate holds:*

$$\begin{aligned}
&\|Q_{L_3}^{-\sigma_3} P_{N_3}(R_{j_1}^A Q_{L_1}^{\sigma_1} P_{N_1} u_1 \cdot R_{j_2}^A Q_{L_2}^{\sigma_2} P_{N_2} u_2)\|_{L^2_{tx}} \\
&\quad \lesssim A^{\frac{1}{2}} N_1^{-1} L_1^{\frac{1}{2}} L_2^{\frac{1}{2}} L_3^{\frac{1}{2}} \|R_{j_1}^A Q_{L_1}^{\sigma_1} P_{N_1} u_1\|_{L^2_{tx}} \|R_{j_2}^A Q_{L_2}^{\sigma_2} P_{N_2} u_2\|_{L^2_{tx}}.
\end{aligned} \qquad (2.11)$$

2.2. Proof of Proposition 2.1

By the duality argument, we have

$$\begin{aligned}
\||\nabla|(uv)\|_{X^{s,-b'}_{-\sigma_3}} &\lesssim \sup_{\|w\|_{X^{-s,b'}_{\sigma_3}}=1} \left|\int |\nabla|(uv) w \mathrm{d}x\mathrm{d}t\right|, \\
\|(\Delta U)v\|_{X^{s,-b'}_{-\sigma_3}} &\lesssim \sup_{\|w\|_{X^{-s,b'}_{\sigma_3}}=1} \left|\int (\Delta U) v w \mathrm{d}x\mathrm{d}t\right| \\
&\leq \sup_{\|w\|_{X^{-s,b'}_{\sigma_3}}=1} \left(\left|\int \partial_1(\partial_1 U) v w \mathrm{d}x\mathrm{d}t\right| + \left|\int \partial_2(\partial_2 U) v w \mathrm{d}x\mathrm{d}t\right|\right),
\end{aligned}$$

where we used $(Q_{L_3}^{-\sigma_3} f, \overline{g})_{L^2_{tx}} = (f, \overline{Q_{L_3}^{\sigma_3} g})_{L^2_{tx}}$. Since $|\nabla|(uv)$ and $(\Delta U)v$ are radial with respect to x, we can assume w is also radial with respect to x. Therefore, to obtain (2.1), it suffices to show that

$$\begin{aligned}
&\sum_{N_1,N_2,N_3\geq 1} \sum_{L_1,L_2,L_3\geq 1} N_{\max} \left|\int u_{N_1,L_1} v_{N_2,L_2} w_{N_3,L_3} \mathrm{d}x\mathrm{d}t\right| \\
&\quad \lesssim \|u\|_{X^{s,b'}_{\sigma_1}} \|v\|_{X^{s,b'}_{\sigma_2}} \|w\|_{X^{-s,b'}_{\sigma_3}}
\end{aligned} \qquad (2.12)$$

for the radial functions u, v, and w, where we put

$$u_{N_1,L_1} := Q_{L_1}^{\sigma_1} P_{N_1} u,\ v_{N_2,L_2} := Q_{L_2}^{\sigma_2} P_{N_2} v,\ w_{N_3,L_3} := Q_{L_3}^{\sigma_3} P_{N_3} w$$

and used $(Q_{L_3}^{-\sigma_3} f, \overline{g})_{L^2_{tx}} = (f, \overline{Q_{L_3}^{\sigma_3} g})_{L^2_{tx}}$. By Plancherel's theorem, we have

$$\left| \int u_{N_1,L_1} v_{N_2,L_2} w_{N_3,L_3} \mathrm{d}x\mathrm{d}t \right| \sim \left| \int_{\substack{\xi_1+\xi_2+\xi_3=0 \\ \tau_1+\tau_2+\tau_3=0}} \mathcal{F}_{tx}[u_{N_1,L_1}](\tau_1,\xi_1) \mathcal{F}_{tx}[v_{N_2,L_2}](\tau_2,\xi_2) \mathcal{F}_{tx}[w_{N_3,L_3}](\tau_3,\xi_3) \right|.$$

We only consider the case $N_1 \lesssim N_2 \sim N_3$, because the remaining cases $N_2 \lesssim N_3 \sim N_1$ and $N_3 \lesssim N_1 \sim N_2$ can be shown similarly. It suffices to show that

$$\begin{aligned} &N_2 \left| \int u_{N_1,L_1} v_{N_2,L_2} w_{N_3,L_3} \mathrm{d}x\mathrm{d}t \right| \\ &\quad \lesssim \left(\frac{N_1}{N_2} \right)^{\epsilon} N_1^s (L_1 L_2 L_3)^c \|u_{N_1,L_1}\|_{L^2_{tx}} \|v_{N_2,L_2}\|_{L^2_{tx}} \|w_{N_3,L_3}\|_{L^2_{tx}} \end{aligned} \tag{2.13}$$

for some $b' \in (0, \frac{1}{2})$, $c \in (0, b')$, and $\epsilon > 0$. Indeed, from (2.13) and the Cauchy–Schwarz inequality, we obtain

$$\begin{aligned} &\sum_{N_1 \lesssim N_2 \sim N_3} \sum_{L_1,L_2,L_3 \geq 1} N_2 \left| \int u_{N_1,L_1} v_{N_2,L_2} w_{N_3,L_3} \mathrm{d}x\mathrm{d}t \right| \\ &\lesssim \sum_{N_1 \lesssim N_2 \sim N_3} \sum_{L_1,L_2,L_3 \geq 1} \left(\frac{N_1}{N_2} \right)^{\epsilon} N_1^s (L_1 L_2 L_3)^c \|u_{N_1,L_1}\|_{L^2_{tx}} \|v_{N_2,L_2}\|_{L^2_{tx}} \|w_{N_3,L_3}\|_{L^2_{tx}} \\ &\lesssim \sum_{N_3} \sum_{N_2 \sim N_3} \left(\sum_{N_1 \lesssim N_2} N_1^{s+\varepsilon} N_2^{-\varepsilon} \sum_{L_1 \geq 1} L_1^c \|u_{N_1,L_1}\|_{L^2_{tx}} \right) \\ &\quad \times \left(N_2^s \sum_{L_2 \geq 1} L_2^{-(b'-c)} L_2^{b'} \|v_{N_2,L_2}\|_{L^2_{tx}} \right) \left(N_3^{-s} \sum_{L_3 \geq 1} L_3^{-(b'-c)} L_3^{b'} \|w_{N_3,L_3}\|_{L^2_{tx}} \right) \\ &\lesssim \|u\|_{X^{s,b'}_{\sigma_1}} \|v\|_{X^{s,b'}_{\sigma_2}} \|w\|_{X^{-s,b'}_{\sigma_3}}. \end{aligned}$$

We put $L_{\max} := \max_{1 \leq j \leq 3}(L_1, L_2, L_3)$.

Case 1 High modulation, $L_{\max} \gtrsim N_{\max}^2$

In this case, the radial condition is not needed. We assume $L_1 \gtrsim N_{\max}^2 \sim N_2^2$. By the Cauchy–Schwarz inequality and (2.5), we have

$$\begin{aligned} &\left| \int u_{N_1,L_1} v_{N_2,L_2} w_{N_3,L_3} \mathrm{d}x\mathrm{d}t \right| \\ &\quad \lesssim \|u_{N_1,L_1}\|_{L^2_{tx}} \|P_{N_1}(v_{N_2,L_2} w_{N_3,L_3})\|_{L^2_{tx}} \\ &\quad \lesssim N_1^{4\delta} \left(\frac{N_1}{N_2} \right)^{\frac{1}{2}-2\delta} L_2^c L_3^c \|u_{N_1,L_1}\|_{L^2_{tx}} \|v_{N_2,L_2}\|_{L^2_{tx}} \|w_{N_3,L_3}\|_{L^2_{tx}}, \end{aligned}$$

where $\delta := \frac{1}{2} - c$. Therefore, we obtain

$$\begin{aligned} &N_2 \left| \int u_{N_1,L_1} v_{N_2,L_2} w_{N_3,L_3} \mathrm{d}x\mathrm{d}t \right| \\ &\quad \lesssim N_1^{\frac{1}{2}+2\delta} N_2^{\frac{1}{2}-2c+2\delta} (L_1 L_2 L_3)^c \|u_{N_1,L_1}\|_{L^2_{tx}} \|v_{N_2,L_2}\|_{L^2_{tx}} \|w_{N_3,L_3}\|_{L^2_{tx}}. \end{aligned}$$

Thus, it suffices to show that

$$N_1^{\frac{1}{2}+2\delta}N_2^{\frac{1}{2}-2c+2\delta} \lesssim \left(\frac{N_1}{N_2}\right)^{\epsilon} N_1^s. \tag{2.14}$$

Since $\delta = \frac{1}{2} - c$, we have

$$\begin{aligned} N_1^{\frac{1}{2}+2\delta}N_2^{\frac{1}{2}-2c+2\delta} &= N_1^{\frac{3}{2}-2c}N_2^{\frac{3}{2}-4c} \\ &\sim N_1^{3-6c-s}\left(\frac{N_1}{N_2}\right)^{4c-\frac{3}{2}} N_1^s. \end{aligned}$$

Therefore, by choosing b' and c as $\max\{\frac{3-s}{6}, \frac{3}{8}\} < c < b' < \frac{1}{2}$ for $s > 0$, we get (2.14).

Case 2: Low modulation, $L_{\max} \ll N_{\max}^2$

By Lemma 2.6, we can assume $N_1 \sim N_2 \sim N_3$ thanks to $L_{\max} \ll N_{\max}^2$. We assume $L_{\max} = L_3$ for simplicity. The other cases can be treated similarly.

∘ **The case** $\widetilde{\theta} = 0$

Let $A := L_{\max}^{-\frac{1}{2}}N_{\max} \sim L_3^{-\frac{1}{2}}N_1$. We decompose $\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3$ as follows:

$$\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3 = \bigcup_{0\le j_1,j_2,j_3\le A-1} \mathfrak{D}_{j_1}^A \times \mathfrak{D}_{j_2}^A \times \mathfrak{D}_{j_3}^A.$$

Since $L_{\max} \le N_{\max}^2(L_{\max}^{-\frac{1}{2}}N_{\max})^{-2} = N_{\max}^2 A^{-2}$, by Lemma 2.8, we can write

$$\begin{aligned} &\left|\int u_{N_1,L_1}v_{N_2,L_2}w_{N_3,L_3}\mathrm{d}x\mathrm{d}t\right| \\ &\quad\le \sum_{j_1=0}^{A-1}\sum_{j_2\in J(j_1)}\sum_{j_3\in J(j_1)}\left|\int u_{N_1,L_1,j_1}v_{N_2,L_2,j_2}w_{N_3,L_3,j_3}\mathrm{d}x\mathrm{d}t\right| \end{aligned}$$

with $u_{N_1,L_1,j_1} := R_{j_1}^A u_{N_1,L_1}$, $v_{N_2,L_2,j_2} := R_{j_2}^A v_{N_2,L_2}$ and $w_{N_3,L_3,j_3} := R_{j_3}^A v_{N_3,L_3}$, where

$$J(j_1) := \{j \in \{0,1,\ldots,A-1\} \,|\, \min\{|j_1 - j|, |A - (j_1 - j)|\} \lesssim 1\}.$$

We note that $\#J(j_1) \lesssim 1$. By using the Hölder inequality and Corollary 2.3 with $p = q = 4$, we get

$$\begin{aligned} &\sum_{j_1=0}^{A-1}\sum_{j_2\in J(j_1)}\sum_{j_3\in J(j_1)}\left|\int u_{N_1,L_1,j_1}v_{N_2,L_2,j_2}w_{N_3,L_3,j_3}\mathrm{d}x\mathrm{d}t\right| \\ &\quad\lesssim \sum_{j_1=0}^{A-1}\sum_{j_2\in J(j_1)}\sum_{j_3\in J(j_1)} \|u_{N_1,L_1,j_1}\|_{L_{tx}^4}\|v_{N_2,L_2,j_2}\|_{L_{tx}^4}\|w_{N_3,L_3,j_3}\|_{L_{tx}^2} \\ &\quad\lesssim AL_1^{\frac{1}{2}}L_2^{\frac{1}{2}}\sup_{j_1}\|u_{N_1,L_1,j_1}\|_{L_{tx}^2}\sup_{j_2}\|v_{N_2,L_2,j_2}\|_{L_{tx}^2}\sup_{j_3}\|w_{N_3,L_3,j_3}\|_{L_{tx}^2}. \end{aligned}$$

Since u, v, and w are radial respect to x, we have

$$\begin{aligned} &\|u_{N_1,L_1,j_1}\|_{L_{tx}^2} \lesssim A^{-\frac{1}{2}}\|u_{N_1,L_1}\|_{L_{tx}^2},\ \ \|v_{N_2,L_2,j_2}\|_{L_{tx}^2} \lesssim A^{-\frac{1}{2}}\|v_{N_2,L_2}\|_{L_{tx}^2}, \\ &\|w_{N_3,L_3,j_3}\|_{L_{tx}^2} \lesssim A^{-\frac{1}{2}}\|w_{N_3,L_3}\|_{L_{tx}^2}. \end{aligned} \tag{2.15}$$

Therefore, we obtain

$$
\begin{aligned}
&N_2 \left| \int u_{N_1,L_1} v_{N_2,L_2} w_{N_3,L_3} \mathrm{d}x\mathrm{d}t \right| \\
&\lesssim N_2 A^{-\frac{1}{2}} L_1^{\frac{1}{2}} L_2^{\frac{1}{2}} \|u_{N_1,L_1}\|_{L^2_{tx}} \|v_{N_2,L_2}\|_{L^2_{tx}} \|w_{N_3,L_3}\|_{L^2_{tx}} \\
&\sim N_1^{\frac{1}{2}} L_1^{\frac{1}{2}} L_2^{\frac{1}{2}} L_3^{\frac{1}{4}} \|u_{N_1,L_1}\|_{L^2_{tx}} \|v_{N_2,L_2}\|_{L^2_{tx}} \|w_{N_3,L_3}\|_{L^2_{tx}} \\
&\lesssim N_1^{\frac{1}{2}} (L_1 L_2 L_3)^{\frac{5}{12}} \|u_{N_1,L_1}\|_{L^2_{tx}} \|v_{N_2,L_2}\|_{L^2_{tx}} \|w_{N_3,L_3}\|_{L^2_{tx}}.
\end{aligned}
$$

This estimate gives the desired estimate (2.13) for $s \geq \frac{1}{2}$ by choosing b' and c as $\frac{5}{12} \leq c < b' < \frac{1}{2}$.

○ **The case** $\widetilde{\theta} < 0$

We decompose $\mathbb{R}^3 \times \mathbb{R}^3$ as follows:

$$
\mathbb{R}^3 \times \mathbb{R}^3 = \left(\bigcup_{\substack{0 \leq j_1, j_2 \leq N_1 - 1 \\ |j_1 - j_2| \leq 16}} \mathfrak{D}_{j_1}^{N_1} \times \mathfrak{D}_{j_2}^{N_1} \right) \cup \left(\bigcup_{64 \leq A \leq N_1} \bigcup_{\substack{0 \leq j_1, j_2 \leq A-1 \\ 16 \leq |j_1 - j_2| \leq 32}} \mathfrak{D}_{j_1}^{A} \times \mathfrak{D}_{j_2}^{A} \right).
$$

We can write

$$
\begin{aligned}
&\left| \int u_{N_1,L_1} v_{N_2,L_2} w_{N_3,L_3} \mathrm{d}x\mathrm{d}t \right| \\
&\leq \sum_{\substack{A = N_1 \\ 0 \leq j_1, j_2 \leq N_1 - 1 \\ |j_1 - j_2| \leq 16}} \sum_{j_3 \in J(j_1)} \left| \int u_{N_1,L_1,j_1} v_{N_2,L_2,j_2} w_{N_3,L_3,j_3} \mathrm{d}x\mathrm{d}t \right| \\
&\quad + \sum_{64 \leq A \leq N_1} \sum_{\substack{0 \leq j_1, j_2 \leq A-1 \\ 16 \leq |j_1 - j_2| \leq 32}} \sum_{j_3 \in J(j_1)} \left| \int u_{N_1,L_1,j_1} v_{N_2,L_2,j_2} w_{N_3,L_3,j_3} \mathrm{d}x\mathrm{d}t \right| .
\end{aligned}
$$

For the former term, by using the Hölder inequality, Theorem 2.1, and (2.15), we get

$$
\begin{aligned}
&\sum_{\substack{A = N_1 \\ 0 \leq j_1, j_2 \leq N_1 - 1 \\ |j_1 - j_2| \leq 16}} \sum_{j_3 \in J(j_1)} \left| \int u_{N_1,L_1,j_1} v_{N_2,L_2,j_2} w_{N_3,L_3,j_3} \mathrm{d}x\mathrm{d}t \right| \\
&\lesssim \sum_{\substack{A = N_1 \\ 0 \leq j_1, j_2 \leq N_1 - 1 \\ |j_1 - j_2| \leq 16}} \|Q_{L_3}^{-\sigma_3} P_{N_3} (u_{N_1,L_1,j_1} v_{N_2,L_2,j_2})\|_{L^2_{tx}} \sum_{j_3 \in J(j_1)} \|w_{N_3,L_3,j_3}\|_{L^2_{tx}} \\
&\lesssim N_1^{-1} L_1^{\frac{1}{2}} L_2^{\frac{1}{2}} \|w_{N_3,L_3}\|_{L^2_{tx}} \sum_{\substack{A = N_1 \\ 0 \leq j_1, j_2 \leq N_1 - 1 \\ |j_1 - j_2| \leq 16}} \|u_{N_1,L_1,j_1}\|_{L^2_{tx}} \|v_{N_2,L_2,j_2}\|_{L^2_{tx}} \\
&\lesssim N_1^{-1} L_1^{\frac{1}{2}} L_2^{\frac{1}{2}} \|u_{N_1,L_1}\|_{L^2_{tx}} \|v_{N_2,L_2}\|_{L^2_{tx}} \|w_{N_3,L_3}\|_{L^2_{tx}} \\
&\lesssim N_1^{-1} (L_1 L_2 L_3)^{\frac{1}{3}} \|u_{N_1,L_1}\|_{L^2_{tx}} \|v_{N_2,L_2}\|_{L^2_{tx}} \|w_{N_3,L_3}\|_{L^2_{tx}}.
\end{aligned}
$$

For the latter term, by using Proposition 2.9, (2.15), and $L_1L_2L_3 \lesssim N_1^6$ that we get

$$\begin{aligned}
&\sum_{64\le A\le N_1}\sum_{\substack{0\le j_1,j_2\le A-1\\ 16\le|j_1-j_2|\le 32}}\sum_{j_3\in J(j_1)}\left|\int u_{N_1,L_1,j_1}v_{N_2,L_2,j_2}w_{N_3,L_3,j_3}\mathrm{d}x\mathrm{d}t\right|\\
&\lesssim \sum_{64\le A\le N_1}\sum_{\substack{0\le j_1,j_2\le A-1\\ 16\le|j_1-j_2|\le 32}}\|Q_{L_3}^{-\sigma_3}P_{N_3}(u_{N_1,L_1,j_1}v_{N_2,L_2,j_2})\|_{L^2_{tx}}\sum_{j_3\in J(j_1)}\|w_{N_3,L_3,j_3}\|_{L^2_{tx}}\\
&\lesssim \|w_{N_3,L_3}\|_{L^2_{tx}}\sum_{64\le A\le N_1}N_1^{-1}(L_1L_2L_3)^{\frac12}\sum_{\substack{0\le j_1,j_2\le A-1\\ 16\le|j_1-j_2|\le 32}}\|u_{N_1,L_1,j_1}\|_{L^2_{tx}}\|v_{N_2,L_2,j_2}\|_{L^2_{tx}}\\
&\lesssim (\log N_1)N_1^{-1}(L_1L_2L_3)^{\frac12}\|u_{N_1,L_1}\|_{L^2_{tx}}\|v_{N_2,L_2}\|_{L^2_{tx}}\|w_{N_3,L_3}\|_{L^2_{tx}}\\
&\lesssim (\log N_1)N_1^{2-6c}(L_1L_2L_3)^{c}\|u_{N_1,L_1}\|_{L^2_{tx}}\|v_{N_2,L_2}\|_{L^2_{tx}}\|w_{N_3,L_3}\|_{L^2_{tx}}.
\end{aligned}$$

The above two estimates give the desired estimate (2.13) for $s>0$ by choosing b' and c as $\max\{\frac{3-s}{6},\frac13\}<c<b'<\frac12$. □

3. Proof of the Well-Posedness

In this section, we prove Theorems 1.1 and 1.3. For a Banach space H and $r>0$, we define $B_r(H):=\{f\in H\,|\,\|f\|_H\le r\}$. Furthermore, we define $\mathcal{X}_T^{s,b}$ as

$$\mathcal{X}_T^{s}:=(X^{s,b}_{\alpha,\mathrm{rad},T})^2\times(X^{s,b}_{\beta,\mathrm{rad},T})^2\times\widetilde{X}^{s+1,b}_{\gamma,\mathrm{rad},T},$$

where $X^{s,b}_{\alpha,\mathrm{rad},T}$ and $X^{s,b}_{\beta,\mathrm{rad},T}$ are the time localized spaces defined by

$$X^{s,b}_{\sigma,\mathrm{rad},T}:=\left\{u|_{[0,T]}\big|\ u\in X^{s,b}_{\sigma,\mathrm{rad}}\right\}$$

with the norm

$$\|u\|_{X^{s,b}_{\sigma,T}}:=\inf\left\{\|v\|_{X^{s,b}_{\sigma,T}}\big|\ v\in X^{s,b}_{\sigma,\mathrm{rad}},\ v|_{[0,T]}=u|_{[0,T]}\right\}.$$

Also, $\widetilde{X}^{s+1,b}_{\gamma,\mathrm{rad},T}$ is defined by the same way. Now, we restate Theorem 1.1 for $d=2$ more precisely.

Theorem 3.1. *Let $s\ge\frac12$ if $\theta=0$ and $s>0$ if $\theta<0$. For any $r>0$ and for all initial data $(u_0,v_0,[W_0])\in B_r(\mathcal{H}^s(\mathbb{R}^2))$, there exist $T=T(r)>0$ and a solution $(u,v,[W])\in\mathcal{X}_T^{s,b}$ to system (1.12) on $[0,T]$ for suitable $b>\frac12$. Such solution is unique in $B_R(\mathcal{X}_T^s)$ for some $R>0$. Moreover, the flow map*

$$S:B_r(\mathcal{H}^s(\mathbb{R}^2))\ni(u_0,v_0,[W_0])\mapsto(u,v,[W])\in\mathcal{X}_T^s$$

is Lipschitz continuous.

Remark 3.1. Since $X_T^{s,b}\hookrightarrow C([0,T];H^s(\mathbb{R}^2))$ holds for $b>\frac12$, we have $\mathcal{X}_T^{s,b}\hookrightarrow C([0,T];\mathcal{H}^s(\mathbb{R}^2))$.

To prove Theorem 3.1, we give the linear estimate.

Proposition 3.1. *Let $s\in\mathbb{R}$, $\sigma\in\mathbb{R}\backslash\{0\}$, $b\in(\frac12,1]$, $b'\in[0,1-b]$ and $0<T\le1$.*

(1) *There exists* $C_1 > 0$ *such that for any* $\varphi \in H^s(\mathbb{R}^2)$*, we have*

$$\|e^{it\sigma\Delta}\varphi\|_{X^{s,b}_{\sigma,T}} \le C_1\|\varphi\|_{H^s}.$$

(2) *There exists* $C_2 > 0$ *such that for any* $F \in X^{s,-b'}_{\sigma,T}$*, we have*

$$\left\|\int_0^t e^{i(t-t')\sigma\Delta}F(t')\,dt'\right\|_{X^{s,b}_{\sigma,T}} \le C_2T^{1-b'-b}\|F\|_{X^{s,-b'}_{\sigma,T}}.$$

(3) *There exists* $C_3 > 0$ *such that for any* $u \in X^{s,b}_{\sigma,T}$*, we have*

$$\|u\|_{X^{s,b'}_{\sigma,T}} \le C_3T^{b-b'}\|u\|_{X^{s,b}_{\sigma,T}}.$$

For the proof of Proposition 3.1, see Lemma 2.1 and 3.1 in [10]. We define the map $\Phi(u,v,[W]) = (\Phi^{(1)}_{\alpha,u_0}([W],v), \Phi^{(1)}_{\beta,v_0}([\overline{W}],u), [\Phi^{(2)}_{\gamma,[W_0]}(u,\overline{v})])$ as

$$\Phi^{(1)}_{\sigma,\varphi}([f],g)(t) := e^{it\sigma\Delta}\varphi - i\int_0^t e^{i(t-t')\sigma\Delta}(\Delta f(t'))g(t')\mathrm{d}t',$$
$$\Phi^{(2)}_{\sigma,[\varphi]}(f,g)(t) := e^{it\sigma\Delta}\varphi + i\int_0^t e^{i(t-t')\sigma\Delta}(f(t')\cdot g(t'))\mathrm{d}t'.$$

To prove the existence of the solution of (1.1), we prove that Φ is a contraction map on $B_R(\mathcal{X}^s_T)$ for some $R>0$ and $T>0$. For a vector-valued function $f=(f_1,f_2)$, $\|f\|_{H^s}$ and $\|f\|_{X^{s,b}_T}$ denote $\|f_1\|_{H^s}+\|f_2\|_{H^s}$ and $\|f_1\|_{X^{s,b}_T}+\|f_2\|_{X^{s,b}_T}$, respectively.

Proof of Theorem 3.1. We choose $b>\frac{1}{2}$ as $b=1-b'$, where b' is as in Proposition 2.1. Let $(u_0,\ v_0,\ [W_0]) \in B_r(\mathcal{H}^s(\mathbb{R}^2))$ be given. By Proposition 2.1 with $(\sigma_1,\sigma_2,\sigma_3) \in \{(\beta,\gamma,-\alpha),(-\gamma,\alpha,-\beta),(\alpha,-\beta,-\gamma)\}$ and Proposition 3.1 with $\sigma \in \{\alpha,\beta,\gamma\}$, there exist constants $C_1,\ C_2,\ C_3 > 0$ such that for any $(u,v,[W]) \in B_R(\mathcal{X}^s_T)$, we have

$$\begin{aligned}
\|\Phi^{(1)}_{\alpha,u_0}([W],v)\|_{X^{s,b}_{\alpha,T}} &\le C_1\|u_0\|_{H^s} + CC_2C_3^2T^{4b-2}\|[W]\|_{\widetilde{X}^{s+1,b}_{\gamma,T}}\|v\|_{X^{s,b}_{\beta,T}}\\
&\le C_1r + CC_2C_3^2T^{4b-2}R^2,\\
\|\Phi^{(1)}_{\beta,v_0}([\overline{W}],u)\|_{X^{s,b}_{\beta,T}} &\le C_1\|v_0\|_{H^s} + CC_2C_3^2T^{4b-2}\|[W]\|_{\widetilde{X}^{s+1,b}_{\gamma,T}}\|u\|_{X^{s,b}_{\alpha,T}}\\
&\le C_1r + CC_2C_3^2T^{4b-2}R^2,\\
\|[\Phi^{(2)}_{\gamma,[W_0]}(u,\overline{v})]\|_{\widetilde{X}^{s+1,b}_{\gamma,T}} &\le C_1\|[W_0]\|_{\widetilde{H}^{s+1}} + CC_2C_3^2T^{4b-2}\|u\|_{X^{s,b}_{\alpha,T}}\|v\|_{X^{s,b}_{\beta,T}}\\
&\le C_1r + CC_2C_3^2T^{4b-2}R^2.
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\|\Phi^{(1)}_{\alpha,u_0}([W],v)-\Phi^{(1)}_{\alpha,u_0}([W'],v')\|_{X^{s,b}_{\alpha,T}}\\
&\quad\le CC_2C_3^2T^{4b-2}R\left(\|[W]-[W']\|_{\widetilde{X}^{s+1,b}_{\gamma,T}}+\|v-v'\|_{X^{s,b}_{\beta,T}}\right),\\
&\|\Phi^{(1)}_{\beta,v_0}([\overline{W}],u)-\Phi^{(1)}_{\beta,v_0}([\overline{W'}],u')\|_{X^{s,b}_{\beta,T}}\\
&\quad\le CC_2C_3^2T^{4b-2}R\left(\|[W]-[W']\|_{\widetilde{X}^{s+1,b}_{\gamma,T}}+\|u-u'\|_{X^{s,b}_{\alpha,T}}\right),\\
&\|[\Phi^{(2)}_{\gamma,[W_0]}(u,\overline{v})]-[\Phi^{(2)}_{\gamma,[W_0]}(u',\overline{v'})]\|_{\widetilde{X}^{s+1,b}_{\gamma,T}}\\
&\quad\le CC_2C_3^2T^{4b-2}R\left(\|u-u'\|_{X^{s,b}_{\alpha,T}}+\|v-v'\|_{X^{s,b}_{\beta,T}}\right).
\end{aligned}$$

Therefore, if we choose $R>0$ and $T>0$ as

$$R=6C_1r,\ CC_2C_3^2T^{4b-2}R\le\frac{1}{4}$$

then Φ is a contraction map on $B_R(\mathcal{X}^s_T)$. This implies the existence of the solution of system (1.1) and the uniqueness in the ball $B_R(\mathcal{X}^s_T)$. The Lipschitz continuity of the flow map is also proved by similar argument. □

Next, to prove Theorem 1.3, we justify the existence of a scalar potential of $w\in(H^s(\mathbb{R}^2))^2$. Let $\mathcal{F}_1$ and $\mathcal{F}_2$ denote the Fourier transform with respect to the first component and the second component, respectively. We note that $\mathcal{F}_1^{-1}\mathcal{F}_2^{-1}=\mathcal{F}_2^{-1}\mathcal{F}_1^{-1}=\mathcal{F}_x^{-1}$ (and also $\mathcal{F}_1\mathcal{F}_2=\mathcal{F}_2\mathcal{F}_1=\mathcal{F}_x$) holds on $L^2(\mathbb{R}^2)$.

Proposition 3.2. *Let $s>\frac{1}{2}$ and $w=(w_1,w_2)\in(H^s(\mathbb{R}^2))^2$. If w_1 and w_2 satisfy*

$$\xi_2\widehat{w_1}(\xi)-\xi_1\widehat{w_2}(\xi)=0\quad a.e.\xi=(\xi_1,\xi_2)\in\mathbb{R}^2,$$

then there exists $W\in L^1_{\text{loc}}(\mathbb{R}^2)$ $(\subset\mathcal{S}'(\mathbb{R}^2))$ such that

$$\nabla W(x)=w(x)\ a.e.x=(x_1,x_2)\in\mathbb{R}^2.$$

To obtain Proposition 3.2, we use the next lemma.

Lemma 3.3. *Let $s>\frac{1}{2}$. If $f\in H^s(\mathbb{R}^2)$, then it hold that*

$$\mathcal{F}_1[f](\cdot,x_2)\in L^1(\mathbb{R})\quad\text{a.e. }x_2\in\mathbb{R},\qquad\mathcal{F}_2[f](x_1,\cdot)\in L^1(\mathbb{R})\quad\text{a.e. }x_1\in\mathbb{R}.$$

Proof. By the Cauchy–Schwarz inequality and Plancherel's theorem, we have

$$\begin{aligned}
\left\|\|\mathcal{F}_1[f](\xi_1,x_2)\|_{L^1_{\xi_1}}\right\|_{L^2_{x_2}}&\le\left\|\|\langle\xi_1\rangle^{-s}\|_{L^2_{\xi_1}}\|\langle\xi_1\rangle^s\mathcal{F}_1[f](\xi_1,x_2)\|_{L^2_{\xi_1}}\right\|_{L^2_{x_2}}\\
&\lesssim\|\langle\xi_1\rangle^s\widehat{f}(\xi_1,\xi_2)\|_{L^2_\xi}\\
&\lesssim\|f\|_{H^s}<\infty
\end{aligned}$$

for $s>\frac{1}{2}$. Therefore, we obtain

$$\|\mathcal{F}_1[f](\xi_1,x_2)\|_{L^1_{\xi_1}}<\infty\quad\text{a.e. }x_2\in\mathbb{R}.$$

Similarly, we have

$$\|\mathcal{F}_2[f](x_1,\xi_2)\|_{L^1_{\xi_2}}<\infty\quad\text{a.e. }x_1\in\mathbb{R}.$$

□

Proof of Proposition 3.2. We put

$$W(x) := \int_{a_1}^{x_1} w_1(y_1, x_2)\mathrm{d}y_1 + \int_{a_2}^{x_2} w_2(a_1, y_2)\mathrm{d}y_2 =: W_1(x) + W_2(x)$$

for some a_1, $a_2 \in \mathbb{R}$. By $w \in L^2(\mathbb{R}^2)$, we have $W \in L^1_{\mathrm{loc}}(\mathbb{R}^2)$. Hence, it remains to show that $\nabla W = w$. Since

$$\partial_1 W_1(x) = w_1(x), \quad \partial_1 W_2(x) = 0, \quad \partial_2 W_2(x) = w_2(a_1, x_2)$$

hold for almost all $x = (x_1, x_2) \in \mathbb{R}^2$, it suffices to show

$$\partial_2 W_1(x) = w_2(x) - w_2(a_1, x_2) \quad \text{a.e. } x = (x_1, x_2) \in \mathbb{R}^2. \tag{3.1}$$

Let $h \in \mathbb{R}$. Since $\mathcal{F}_1[w_1](\cdot, x_2) \in L^1(\mathbb{R})$ a.e. $x_2 \in \mathbb{R}$ by Lemma 3.3, we have

$$\begin{aligned}
&\frac{W_1(x_1, x_2 + h) - W_1(x_1, x_2)}{h} \\
&\quad = \frac{1}{h} \int_{a_1}^{x_1} (w_1(y_1, x_2 + h) - w_1(y_1, x_2))\, \mathrm{d}y_1 \\
&\quad = \frac{1}{h} \int_{a_1}^{x_1} \left(\int_{\mathbb{R}} (\mathcal{F}_1[w_1](\xi_1, x_2 + h) - \mathcal{F}_1[w_1](\xi_1, x_2))\, e^{i\xi_1 y_1} \mathrm{d}\xi_1 \right) \mathrm{d}y_1 \\
&\quad = \frac{1}{h} \int_{\mathbb{R}} (\mathcal{F}_1[w_1](\xi_1, x_2 + h) - \mathcal{F}_1[w_1](\xi_1, x_2)) \left(\int_{a_1}^{x_1} e^{i\xi_1 y_1} \mathrm{d}y_1 \right) \mathrm{d}\xi_1 \\
&\quad = \frac{1}{h} \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \widehat{w_1}(\xi_1, \xi_2) e^{i\xi_2 x_2} (e^{i\xi_2 h} - 1) \mathrm{d}\xi_2 \right) \frac{e^{i\xi_1 x_1} - e^{i\xi_1 a_1}}{i\xi_1} \mathrm{d}\xi_1 =: I_h
\end{aligned}$$

by Fubini's theorem. We put $\mathcal{F}_{12}^{-1} := \mathcal{F}_1^{-1}\mathcal{F}_2^{-1}$, $\mathcal{F}_{21}^{-1} := \mathcal{F}_2^{-1}\mathcal{F}_1^{-1}$. By using $\xi_2 \widehat{w_1} = \xi_1 \widehat{w_2}$ and $\mathcal{F}_{12}^{-1} = \mathcal{F}_{21}^{-1}$, we have

$$\begin{aligned}
I_h &= \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \widehat{w_2}(\xi_1, \xi_2) \frac{e^{i\xi_2 h} - 1}{i\xi_2 h} e^{i\xi_2 x_2} \mathrm{d}\xi_2 \right) (e^{i\xi_1 x_1} - e^{i\xi_1 a_1}) \mathrm{d}\xi_1 \\
&= \mathcal{F}_{12}^{-1} \left[\widehat{w_2}(\xi_1, \xi_2) \frac{e^{i\xi_2 h} - 1}{i\xi_2 h} \right] (x_1, x_2) - \mathcal{F}_{12}^{-1} \left[\widehat{w_2}(\xi_1, \xi_2) \frac{e^{i\xi_2 h} - 1}{i\xi_2 h} \right] (a_1, x_2) \\
&= \mathcal{F}_{21}^{-1} \left[\widehat{w_2}(\xi_1, \xi_2) \frac{e^{i\xi_2 h} - 1}{i\xi_2 h} \right] (x_1, x_2) - \mathcal{F}_{21}^{-1} \left[\widehat{w_2}(\xi_1, \xi_2) \frac{e^{i\xi_2 h} - 1}{i\xi_2 h} \right] (a_1, x_2) \\
&= \int_{\mathbb{R}} (\mathcal{F}_2[w_2](x_1, \xi_2) - \mathcal{F}_2[w_2](a_1, \xi_2))\, \frac{e^{i\xi_2 h} - 1}{i\xi_2 h} e^{i\xi_2 x_2} \mathrm{d}\xi_2.
\end{aligned}$$

Since $\mathcal{F}_2[w_2](x_1, \cdot) \in L^1(\mathbb{R})$ a.e. $x_1 \in \mathbb{R}$ by Lemma 3.3, we have

$$\begin{aligned}
\lim_{h \to 0} I_h &= \int_{\mathbb{R}} (\mathcal{F}_2[w_2](x_1, \xi_2) - \mathcal{F}_2[w_2](a_1, \xi_2))\, e^{i\xi_2 x_2} \mathrm{d}\xi_2 \\
&= w_2(x_1, x_2) - w_2(a_1, x_2)
\end{aligned}$$

by Lebesgue's dominant convergence theorem. Therefore, we obtain (3.1). □

Remark 3.2. In the proof of Proposition 3.2, we also used

$$\left|\frac{e^{i\xi_2 h}-1}{i\xi_2 h}\right| \le \sup_{z\in\mathbb{R}}\left(\left|\frac{\cos z-1}{z}\right|+\left|\frac{\sin z}{z}\right|\right)<\infty.$$

This implies

$$\widehat{w_2}(\xi_1,\xi_2)\frac{e^{i\xi_2 h}-1}{i\xi_2 h}\in L^2_\xi(\mathbb{R}^2)$$

and

$$\mathcal{F}_2[w_2](x_1,\xi_2)\frac{e^{i\xi_2 h}-1}{i\xi_2 h}\in L^1_{\xi_2}(\mathbb{R}) \quad \text{a.e. } x_1\in\mathbb{R}.$$

Remark 3.3. If $w=(w_1,w_2)\in(H^s(\mathbb{R}^2))^2$ for $s>\frac{1}{2}$ satisfies

$$x_2 w_1(x)-x_1 w_2(x)=0, \quad \text{a.e. } x\in\mathbb{R}^2$$

additionally in Proposition 3.2, then $W\in L^1_{\text{loc}}(\mathbb{R}^2)$ given in the proof of Proposition 3.2 is radial. Indeed, this condition with $\nabla W(x)=w(x)$ yields (1.10).

Remark 3.4. For $s\le\frac{1}{2}$, we do not know whether there exists a scalar potential of $w\in(H^s(\mathbb{R}^2))^2$ or not. But we point out that if $s<\frac{1}{2}$, then the 1D delta function appears in $\partial_2 w_1-\partial_1 w_2$ for some $w\in(H^s(\mathbb{R}^2))^2$. Then, the irrotational condition does not make sense for pointwise.

Next, we prove that $\mathcal{A}^s(\mathbb{R}^2)$ is a Banach space.

Proposition 3.4. *For $s\ge 0$, $\mathcal{A}^s(\mathbb{R}^2)$ is a closed subspace of $(H^s(\mathbb{R}^2))^2$.*

Proof. Let $f^{(n)}=(f_1^{(n)},f_2^{(n)})\in\mathcal{A}^s(\mathbb{R}^2)$ $(n=1,2,3,\dots)$ and $f=(f_1,f_2)\in(H^s(\mathbb{R}^2))^2$. Assume that $f^{(n)}$ convergences to f in $(H^s(\mathbb{R}^2))^2$ as $n\to\infty$. We prove $f\in A^s(\mathbb{R}^2)$; namely, f satisfies (1.8). By the triangle inequality, we have

$$\begin{aligned}&\left\|\frac{x_2}{\langle x\rangle}f_1-\frac{x_1}{\langle x\rangle}f_2\right\|_{L^2}\\ &\quad\le\left\|\frac{x_2}{\langle x\rangle}f_1-\frac{x_2}{\langle x\rangle}f_1^{(n)}\right\|_{L^2}+\left\|\frac{x_2}{\langle x\rangle}f_1^{(n)}-\frac{x_1}{\langle x\rangle}f_2^{(n)}\right\|_{L^2}+\left\|\frac{x_1}{\langle x\rangle}f_2^{(n)}-\frac{x_1}{\langle x\rangle}f_2\right\|_{L^2}\\ &\quad\le\|f_1-f_1^{(n)}\|_{L^2}+\|x_2 f_1^{(n)}-x_1 f_2^{(n)}\|_{L^2}+\|f_2^{(n)}-f_2\|_{L^2}.\end{aligned}$$

Since $f^{(n)}$ satisfies (1.8) and $f^{(n)}\to f$ in $(L^2(\mathbb{R}^2))^2$ as $n\to\infty$, we obtain

$$\|x_2 f_1^{(n)}-x_1 f_2^{(n)}\|_{L^2}=0, \quad \|f_1-f_1^{(n)}\|_{L^2}+\|f_2^{(n)}-f_2\|_{L^2}\to 0\ (n\to\infty).$$

Therefore, we get

$$\left\|\frac{x_2}{\langle x\rangle}f_1-\frac{x_1}{\langle x\rangle}f_2\right\|_{L^2}=0.$$

It implies $x_2 f_1(x)-x_1 f_2(x)=0$ a.e. $x\in\mathbb{R}^2$. Similarly, we obtain $\xi_2\widehat{f_1}(\xi)-\xi_1\widehat{f_2}(\xi)=0$ a.e. $\xi\in\mathbb{R}^2$. □

Proof of Theorem 1.3. Let $(u_0, v_0, w_0) \in B_r((H^s_{\mathrm{rad}}(\mathbb{R}^2))^2 \times (H^s_{\mathrm{rad}}(\mathbb{R}^2))^2 \times \mathcal{A}^s(\mathbb{R}^2))$ be given. We first prove the existence of solution to (1.1). Since w_0 satisfies (1.8), by Proposition 3.2, there exists $[W_0] \in \widetilde{H}^{s+1}_{\mathrm{rad}}$ such that $\nabla W_0 = w_0$. From Theorem 1.1, there exists $T > 0$ and a solution $(u, v, [W]) \in \mathcal{X}^s_T$ to (1.12) with $(u, v, [W])|_{t=0} = (u_0, v_0, [W_0])$. Since

$$\|[W_0]\|_{\widetilde{H}^{s+1}} = \|w_0\|_{H^s} \leq r,$$

the existence time T is decided by r. We put $w = \nabla W$. Then, $w \in X^{s,b}_{\gamma,T}$ satisfying

$$\|w\|_{X^{s,b}_{\gamma,T}} = \|[W]\|_{\widetilde{X}^{s+1,b}_{\gamma,T}} \leq R,$$

where R is as in the proof of Theorem 1.1, and (u, v, w) satisfies (1.1) since $\Delta W = \nabla \cdot w$. Furthermore, we have

$$\partial_1 w_2 - \partial_2 w_1 = \partial_1(\partial_2 W) - \partial_2(\partial_1 W) = 0$$

and

$$x_1 w_2 - x_2 w_1 = (x_1\partial_2 - x_2\partial_1)W = 0$$

because W is radial with respect to x. Therefore, $w(t) \in \mathcal{A}^s(\mathbb{R}^2)$ for any $t \in [0, T]$.

Next, we prove the uniqueness of the solution in $B_R(\mathcal{Y}^{s,b}_T)$, where

$$\mathcal{Y}^{s,b}_T := (X^{s,b}_{\alpha,\mathrm{rad},T})^2 \times (X^{s,b}_{\beta,\mathrm{rad},T})^2 \times Y^{s,b}_{\gamma,T},$$
$$Y^{s,b}_{\gamma,T} := \{w = (w_1, w_2) \in (X^{s,b}_{\gamma,T})^2 | w(t) \text{ satisfies (1.8)for any } t \in [0, T]\}.$$

Let $(u^{(1)}, v^{(1)}, w^{(1)})$, $(u^{(2)}, v^{(2)}, w^{(2)}) \in B_R(\mathcal{Y}^{s,b}_T)$ are solution to (1.1) with initial data (u_0, v_0, w_0). Then by Proposition 3.2, there exists $[W^{(1)}]$, $[W^{(2)}] \in \widetilde{X}^{s+1,b}_{\gamma,\mathrm{rad},T}$ such that $w^{(1)} = \nabla W^{(1)}$, $w^{(2)} = \nabla W^{(2)}$. By substituting $w^{(j)} = \nabla W^{(j)}$ in both sides of the integral form of (1.1), $(u^{(j)}, v^{(j)}, W^{(j)})$ $(j = 1, 2)$ satisfy

$$u^{(j)}(t) = e^{it\alpha\Delta}u_0 + i\int_0^t e^{i(t-t')\alpha\Delta}(\Delta W^{(j)}(t'))u^{(j)}(t')\mathrm{d}t' \quad \text{in } (H^s(\mathbb{R}^2))^2,$$
$$v^{(j)}(t) = e^{it\beta\Delta}v_0 + i\int_0^t e^{i(t-t')\beta\Delta}(\Delta \overline{W^{(j)}(t')})v^{(j)}(t')\mathrm{d}t' \quad \text{in } (H^s(\mathbb{R}^2))^2,$$
$$\nabla W^{(j)}(t) = e^{it\gamma\Delta}w_0 - i\int_0^t e^{i(t-t')\gamma\Delta}\nabla(u^{(j)}(t') \cdot \overline{v^{(j)}(t')})\mathrm{d}t' \quad \text{in } H^s(\mathbb{R}^2).$$

Therefore, by the same argument as in the proof of Theorem 1.1, we have

$$\|u^{(1)} - u^{(2)}\|_{X^{s,b}_{\alpha,T}} \leq \frac{1}{4}\left(\|w^{(1)} - w^{(2)}\|_{X^{s,b}_{\gamma,T}} + \|v^{(1)} - v^{(2)}\|_{X^{s,b}_{\beta,T}}\right)$$
$$\|v^{(1)} - v^{(2)}\|_{X^{s,b}_{\beta,T}} \leq \frac{1}{4}\left(\|w^{(1)} - w^{(2)}\|_{X^{s,b}_{\gamma,T}} + \|u^{(1)} - u^{(2)}\|_{X^{s,b}_{\alpha,T}}\right)$$
$$\|w^{(1)} - w^{(2)}\|_{X^{s,b}_{\gamma,T}} \leq \frac{1}{4}\left(\|u^{(1)} - u^{(2)}\|_{X^{s,b}_{\alpha,T}} + \|v^{(1)} - v^{(2)}\|_{X^{s,b}_{\beta,T}}\right)$$

since $w^{(1)} - w^{(2)} = \nabla(W^{(1)} - W^{(2)})$. This implies

$$(u^{(1)}, v^{(1)}, w^{(1)}) = (u^{(2)}, v^{(2)}, w^{(2)}) \text{ on } [0, T].$$

The continuous dependence on initial data can be obtained by the similar argument. □

4. The Lack of the Twice Differentiability of the Flow Map

The following proposition implies Theorem 1.2.

Proposition 4.1. *Let $d = 2$ and $0 < T \ll 1$. Assume $\theta = 0$ and $s < \frac{1}{2}$. For every $C > 0$, there exist f, $g \in H^s_{\mathrm{rad}}(\mathbb{R}^2)$ such that*

$$\sup_{0\le t\le T} \left\| \int_0^t e^{i(t-t')\gamma\Delta} \nabla \left((e^{it'\alpha\Delta} f)(\overline{e^{it'\beta\Delta} g}) \right) dt' \right\|_{H^s} \ge C \|f\|_{H^s} \|g\|_{H^s}. \quad (4.1)$$

Proof. Let $N \gg 1$ and $p := \frac{\gamma}{\alpha-\gamma}$ $(\ne 0)$. We note that p is well defined since $\theta = 0$ implies $\kappa \ne 0$ for α, β, $\gamma \in \mathbb{R}\backslash\{0\}$. For simplicity, we assume $p > 0$. Put

$$D_1 := \{\xi \in \mathbb{R}^2 |\ N \le |\xi| \le N+1\}, \quad D_2 := \{\xi \in \mathbb{R}^2 |\ p^{-1}N \le |\xi| \le p^{-1}N + 1\},$$
$$D := \{\xi \in \mathbb{R}^2 |\ (1+p^{-1})N + 1 \le |\xi| \le (1+p^{-1})N + 1 + 2^{-10}\}.$$

We define the functions f and g as

$$\widehat{f}(\xi) := N^{-s-\frac{1}{2}} \mathbf{1}_{D_1}(\xi), \quad \widehat{g}(\xi) := N^{-s-\frac{1}{2}} \mathbf{1}_{D_2}(\xi).$$

Clearly, we have $\|f\|_{H^s} \sim \|g\|_{H^s} \sim 1$ and f, g are radial. For $\xi = (\xi_1, \xi_2) \in \mathbb{R}^2$ and $\eta = (\eta_1, \eta_2) \in \mathbb{R}^2$, we define

$$\begin{aligned}
\Phi(\xi, \eta) &:= \alpha|\eta|^2 - \beta|\xi - \eta|^2 - \gamma|\xi|^2 \\
&= (\alpha - \gamma)|\eta - p(\xi - \eta)|^2 \\
&= (\alpha - \gamma) \left\{ (\eta_1 - p(\xi_1 - \eta_1))^2 + (\eta_2 - p(\xi_2 - \eta_2))^2 \right\}
\end{aligned}$$

because $\theta = 0$ implies $\frac{\beta+\gamma}{\alpha-\gamma} = -\left(\frac{\gamma}{\alpha-\gamma}\right)^2$. We will show

$$\sup_{0\le t\le T} \left\| \int_0^t e^{i(t-t')\gamma\Delta} \nabla \left((e^{it'\alpha\Delta} f)(\overline{e^{it'\beta\Delta} g}) \right) \mathrm{d}t' \right\|_{H^s} \gtrsim N^{-s+\frac{1}{2}}.$$

We calculate that

$$\begin{aligned}
&\left\| \int_0^t e^{i(t-t')\gamma\Delta} \nabla \left((e^{it'\alpha\Delta} f)(\overline{e^{it'\beta\Delta} g}) \right) \mathrm{d}t' \right\|_{H^s} \\
&\gtrsim N^{-s} \left\| \mathbf{1}_D(\xi) \int_0^t \int_{\mathbb{R}^2} e^{-it'\Phi(\xi,\eta)} \mathbf{1}_{D_1}(\eta) \mathbf{1}_{D_2}(\xi - \eta) \mathrm{d}\eta \right\|_{L^2_\xi} \\
&\ge N^{-s} \left\| \mathbf{1}_D(\xi) \int_0^t \int_{\mathbb{R}^2} \cos(t'\Phi(\xi,\eta)) \mathbf{1}_{D_1}(\eta) \mathbf{1}_{D_2}(\xi - \eta) \mathrm{d}\eta \right\|_{L^2_\xi} \\
&=: N^{-s} \|F(\xi)\|_{L^2_\xi}.
\end{aligned}$$

Let $R:\mathbb{R}^2\to\mathbb{R}^2$ be a rotation operator. Since $\Phi(\xi,\eta)=\Phi(R\xi,R\eta)$ and $\mathbf{1}_D$, $\mathbf{1}_{D_1}$, $\mathbf{1}_{D_2}$ are radial, we can see

$$\begin{aligned} F(\xi) &= \mathbf{1}_D(\xi)\int_{\mathbb{R}^2}\frac{\sin(t\Phi(\xi,\eta))}{\Phi(\xi,\eta)}\mathbf{1}_{D_1}(\eta)\mathbf{1}_{D_2}(\xi-\eta)\mathrm{d}\eta \\ &= \mathbf{1}_D(R\xi)\int_{\mathbb{R}^2}\frac{\sin(t\Phi(R\xi,R\eta))}{\Phi(R\xi,R\eta)}\mathbf{1}_{D_1}(R\eta)\mathbf{1}_{D_2}(R\xi-R\eta)\mathrm{d}\eta \\ &= \mathbf{1}_D(R\xi)\int_{\mathbb{R}^2}\frac{\sin(t\Phi(R\xi,\eta))}{\Phi(R\xi,\eta)}\mathbf{1}_{D_1}(\eta)\mathbf{1}_{D_2}(R\xi-\eta)\mathrm{d}\eta \\ &= F(R\xi). \end{aligned}$$

It implies that F is radial. Therefore, there exists $G:\mathbb{R}\to\mathbb{R}$ such that $F(\xi)=G(|\xi|)$. We note that

$$\|F(\xi)\|_{L^2_\xi}=\|G(r)r^{\frac{1}{2}}\|_{L^2((0,\infty))}\gtrsim N^{\frac{1}{2}}\inf_{r>0}|G(r)|=N^{\frac{1}{2}}\inf_{(\xi_1,0)\in D}|F(\xi_1,0)|$$

since $\mathrm{supp}G\subset[(1+p^{-1})N+1,(1+p^{-1})N+1+2^{-10}]$. Hence, it suffices to show that

$$|F(\xi_c)|\gtrsim t^{\frac{1}{2}} \tag{4.2}$$

for any $c\in[0,2^{-10}]$ and some $0\le t\le T$, where $\xi_c:=(\xi_{c1},0)\in\mathbb{R}^2$, $\xi_{c1}:=(1+p^{-1})N+1+c$. Simple calculation gives

$$\Phi(\xi_c,\eta)=(\alpha-\gamma)\left\{((1+p)(\eta_1-N)-p(1+c))^2+(1+p)^2\eta_2^2\right\}. \tag{4.3}$$

We also observe that

$$\begin{aligned} &\mathbf{1}_{D_1}(\eta)\mathbf{1}_{D_2}(\xi_c-\eta)\neq 0 \\ &\quad\implies \eta_1\le N+1 \text{ and } \xi_{c1}-\eta_1\le p^{-1}N+1 \\ &\quad\implies N+c\le\eta_1\le N+1. \end{aligned}$$

Let $\epsilon>0$ be small. We define a new set E as

$$E:=D_1\cap\{\eta=(\eta_1,\eta_2)\in\mathbb{R}^2|\ N+c\le\eta_1\le N+1\},$$

and we decompose E into four sets:

$$\begin{aligned} E_1 &= \left\{\xi_{c1}-\sqrt{(p^{-1}N+1)^2-N^{2\epsilon}}\le\eta_1<\sqrt{(N+1)^2-N^{2\epsilon}},\ |\eta_2|\le N^\epsilon\right\}, \\ E_2 &= \{N+c\le\eta_1<\xi_{c1}-\sqrt{(p^{-1}N+1)^2-N^{2\epsilon}},\ |\eta_2|\le N^\epsilon\}\cap E, \\ E_3 &= \{\sqrt{(N+1)^2-N^{2\epsilon}}\le\eta_1\le N+1,\ |\eta_2|\le N^\epsilon\}\cap E, \\ E_4 &= \{N^\epsilon<|\eta_2|\}\cap E. \end{aligned}$$

We can easily show that $E_i\cap E_j=\emptyset$ if $i\neq j$. Furthermore, we can obtain $E_1\subset E$ and

$$\mathbf{1}_{D_1}(\eta)\mathbf{1}_{D_2}(\xi_c-\eta)=1$$

for any $\eta \in E_1$. We observe that

$$|F(\xi_c)| \geq \left| \int_{\mathbb{R}^2} \frac{\sin(t\Phi(\xi_c,\eta))}{\Phi(\xi_c,\eta)} \mathbf{1}_{E_1}(\eta) \mathrm{d}\eta \right| - \sum_{j=2}^{4} \int_{\mathbb{R}^2} \left| \frac{\sin(t\Phi(\xi_c,\eta))}{\Phi(\xi_c,\eta)} \right| \mathbf{1}_{E_j}(\eta) \mathrm{d}\eta$$
$$=: I_1 - \sum_{j=2}^{4} I_j.$$

We first consider I_1. Let

$$c' := p^{-1}N + 1 - \sqrt{(p^{-1}N+1)^2 - N^{2\epsilon}}, \quad c'' := N + 1 - \sqrt{(N+1)^2 - N^{2\epsilon}}.$$

Obviously, it holds $c' \sim c'' \sim N^{-1+2\epsilon}$. We calculate that

$$I_1 = 2 \left| \int_{N+c'+c''}^{N+1-c''} \left(\int_0^{N^\epsilon} \frac{\sin(t\Phi(\xi_c,\eta))}{\Phi(\xi_c,\eta)} \mathrm{d}\eta_2 \right) \mathrm{d}\eta_1 \right|$$
$$= \frac{2}{(1+p)|\alpha-\gamma|} \left| \int_{N+c'+c''}^{N+1-c''} \left(\int_0^{(1+p)N^\epsilon} \frac{\sin(\tau(q(\eta_1)+\eta_2^2))}{q(\eta_1)+\eta_2^2} \mathrm{d}\eta_2 \right) \mathrm{d}\eta_1 \right|,$$

where $\tau := |\alpha-\gamma|t$ and $q(\eta_1) := ((1+p)(\eta_1 - N) - p(1+c))^2$. Therefore, if we obtain

$$\inf_{\eta_1 \in [N+c'+c'', N+1-c'']} \int_0^{(1+p)N^\epsilon} \frac{\sin(\tau(q(\eta_1)+\eta_2^2))}{q(\eta_1)+\eta_2^2} \mathrm{d}\eta_2 \gtrsim t^{\frac{1}{2}}, \tag{4.4}$$

then we get $I_1 \gtrsim t^{\frac{1}{2}}$. Let $t > 0$ be small. We fix $\eta_1 \in [N+c'+c'', N+1-c'']$ and write $q(\eta_1) = q$ for simplicity. Clearly, we have $0 \leq q \lesssim 1$. We easily verify that if we restrict η_2 as $0 \leq \eta_2 \leq \sqrt{\pi\tau^{-1} - q}$, then we have $\sin(\tau(q+\eta_2^2)) \geq 0$ and $\frac{\sin(\tau(q+\eta_2^2))}{q+\eta_2^2}$ is monotone decreasing. Similarly, if $\sqrt{\pi\tau^{-1}-q} \leq \eta_2 \leq \sqrt{2\pi\tau^{-1}-q}$, then we see $\sin(\tau(q+\eta_2^2)) \leq 0$. We calculate

$$\int_0^{\sqrt{2\pi\tau^{-1}-q}} \frac{\sin(\tau(q+\eta_2^2))}{q+\eta_2^2} \mathrm{d}\eta_2$$
$$\geq \int_0^{\sqrt{\pi\tau^{-1}-q}} \frac{\sin(\tau(q+\eta_2^2))}{q+\eta_2^2} \mathrm{d}\eta_2 - \int_{\sqrt{\pi\tau^{-1}-q}}^{\sqrt{2\pi\tau^{-1}-q}} \frac{1}{q+\eta_2^2} \mathrm{d}\eta_2$$
$$\geq \frac{2\tau}{\pi} \int_0^{\sqrt{\pi(2\tau)^{-1}-q}} \mathrm{d}\eta_2 - \frac{\tau}{\pi} \int_{\sqrt{\pi\tau^{-1}-q}}^{\sqrt{2\pi\tau^{-1}-q}} \mathrm{d}\eta_2$$
$$= \frac{\tau}{\pi} \left(2\sqrt{\pi(2\tau)^{-1}-q} - \sqrt{2\pi\tau^{-1}-q} + \sqrt{\pi\tau^{-1}-q} \right)$$
$$\gtrsim t^{\frac{1}{2}}.$$

The last estimate is verified by the smallness of $\tau = |\alpha - \gamma|t$. We also see

$$\int_{\sqrt{2n\pi\tau^{-1}-q}}^{\sqrt{2(n+1)\pi\tau^{-1}-q}} \frac{\sin(\tau(q+\eta_2^2))}{q+\eta_2^2} \mathrm{d}\eta_2 \gtrsim \frac{t^{\frac{1}{2}}}{n^2}$$

for any $n \in \mathbb{N}$. Therefore, we obtain (4.4).

Next, we consider I_2, I_3, and I_4. Since $|E_2|$, $|E_3| \lesssim N^{-1+3\epsilon}$, we easily observe that

$$I_2 + I_3 \lesssim tN^{-1+3\epsilon}.$$

For I_4, we observe that

$$I_4 = \int_{E_4} \left| \frac{\sin(t\Phi(\xi_c,\eta))}{\Phi(\xi_c,\eta)} \right| d\eta \lesssim \int_{N+c}^{N+1} \left(\int_{N^\epsilon}^{\infty} \frac{1}{\eta_2^2} d\eta_2 \right) d\eta_1 \lesssim N^{-\epsilon}.$$

By the above argument, we obtain

$$|F(\xi_c)| \geq I_1 - \sum_{j=2}^{4} I_j \gtrsim t^{\frac{1}{2}} - tN^{-1+3\epsilon} + N^{-\epsilon}.$$

If we choose $N \gg 1$ satisfying $N^{-\epsilon} \ll T$, then for any $t \in [0,T]$ with $N^{-\epsilon} \ll t$, we have (4.2). □

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