Journal of Physics: Conference Series

Non-Abelian anomalies and hadronic fluids

To cite this article: Juan L. Mañes *et al* 2019 *J. Phys.: Conf. Ser.* **1194** 012073

IOP Conf. Series: Journal of Physics: Conf. Series **1194** (2019) 012073 doi:10.1088/1742-6596/1194/1/012073

Non-Abelian anomalies and hadronic fluids

Juan L. Mañes[1], Eugenio Megías[2,3], Manuel Valle[3], Miguel Á. Vázquez-Mozo[4]

[1]Departamento de Física de la Materia Condensada, Universidad del País Vasco UPV/EHU, Apartado 644, E-48080 Bilbao, Spain
[2]Departamento de Física Atómica, Molecular y Nuclear and Instituto Carlos I de Física Teórica y Computacional, Universidad de Granada, Avenida de Fuente Nueva s/n, E-18071 Granada, Spain
[3]Departamento de Física Teórica, Universidad del País Vasco UPV/EHU, Apartado 644, E-48080 Bilbao, Spain
[4]Departamento de Física Fundamental, Universidad de Salamanca, Plaza de la Merced s/n, E-37008 Salamanca, Spain

E-mail: wmpmapaj@lg.ehu.es, emegias@ugr.es, manuel.valle@ehu.es, Miguel.Vazquez-Mozo@cern.ch

Abstract. By using differential geometry methods, we study the role of non-Abelian anomalies in relativistic fluids. We obtain closed expressions for the covariant currents derived from the Chern-Simons effective action. Our results are also applied to the Wess-Zumino-Witten action that accounts for the interaction of Goldstone bosons with external electromagnetic fields. We particularize these results to QCD with two light flavors.

1. Introduction

Quantum anomalies play an important role in the hydrodynamics of relativistic fluids. In the presence of anomalous currents coupling to external electromagnetic fields, parity is broken and new tensor structures appear in the constitutive relations associated with new transport coefficients. This is the case of the *chiral magnetic effect*, which is responsible for the generation of an electric current induced by an external magnetic field in the fluid [1], and the *chiral vortical effect*, in which the electric current is induced by a vortex [2], i.e. $\langle J^\mu\rangle_{\text{anom}} = \sigma_{\mathcal{B}}\mathscr{B}^\mu + \sigma_{\mathcal{V}}\omega^\mu + \cdots$. The corresponding susceptibilities, $\sigma_{\mathcal{B}}$ and $\sigma_{\mathcal{V}}$, don't contribute to entropy production so that they are related to non-dissipative phenomena. It is believed that these phenomena can produce observable effects in heavy ion physics [3], as well as in condensed matter systems [4]. Preservation of the second law of thermodynamics leads to a number of constraints to be satisfied by the corresponding anomalous coefficients [5]. Alternatively, these coefficients have been computed with a wide variety of methods, including kinetic theory [6, 7], Kubo formulae [8, 9, 10], diagrammatic methods [11] and fluid/gravity correspondence [12, 13, 14, 15].

Recently it has been proposed a new formalism convenient to obtain the non-dissipative part of the anomalous constitutive relations. The underlying idea is the construction of an effective action for the hydrodynamic sources on a generic stationary background [16, 17, 18, 19]. While this effective action can be constructed by solving the anomaly equations by trial and error [20], there are other strategies based on differential geometry methods. Differential geometry has revealed itself as a very powerful tool in the analysis of anomalies in quantum field theories [21], and the reason is that anomalies are originated in the topological structure of the gauge bundle, so that they are determined from topological invariant quantities. This is also the case of systems with spontaneously symmetry breaking, where the dynamics of Goldstone bosons is very much constrained by the anomalies [22, 23, 24]. A further advantage of the differential geometry approach is that it exhausts all perturbative contributions to the anomaly, which in the non-Abelian case include triangle, square and pentagon one-loop diagrams. Some recent applications of these techniques to the physics of anomalous fluids in thermal equilibrium have been presented in e.g. Refs. [25, 26, 27].

In this manuscript we will present a systematic construction of the equilibrium partition function for fluids with non-Abelian chiral anomalies using differential geometry methods along the lines of [27]. We also extend this analysis to study the hydrodynamics in presence of spontaneous symmetry breaking.

2. Equilibrium partition function formalism

We will present in this section the main ingredients of this formalism relevant to compute anomalous constitutive relations [16, 17]. Let us consider a relativistic invariant quantum field theory with a gauge connection on the manifold

$$ds^2 = G_{\mu\nu}dx^\mu dx^\nu = -e^{2\sigma(\boldsymbol{x})}(dx^0 + a_i(\boldsymbol{x})dx^i)^2 + g_{ij}(\boldsymbol{x})dx^i dx^j \,, \tag{1}$$

$$\mathcal{A} = \mathcal{A}_0(\boldsymbol{x})dx^0 + \mathcal{A}_i(\boldsymbol{x})dx^i \,. \tag{2}$$

We consider that all the fields $\{\sigma(\boldsymbol{x}), a_i(\boldsymbol{x}), g_{ij}(\boldsymbol{x}), \mathcal{A}_\mu(\boldsymbol{x})\}$ are independent of the time coordinate x^0. From the partition function of the system, one can compute the energy-momentum tensor and charged currents by performing the appropriate time independent variations, i.e.

$$\delta \log Z = \frac{1}{T_0}\int d^3x\sqrt{g_3}\left(-\frac{1}{2}T_{\mu\nu}\delta g^{\mu\nu} + J^\mu \delta\mathcal{A}_\mu\right) \,, \tag{3}$$

where $g_3 = \det(g_{ij})$. We have considered in Eq. (3) the imaginary time prescription, in which the Euclidean time direction is compactified to a circle of length $\beta = 1/T_0$. We will denote by T_0 and μ_0 the temperature and chemical potential at equilibrium, respectively. The invariance of the partition function under transformations of the time coordinate, $x^0 \to x'^0 = x^0 + \phi(\boldsymbol{x})$, demands that $\log Z$ depends only on the gauge fields through the following invariant combinations

$$A_0 = \mathcal{A}_0 \,, \qquad A_i = \mathcal{A}_i - a_i\mathcal{A}_0 \,. \tag{4}$$

This is the so-called Kaluza-Klein (KK) invariance. In particular, for a general dependence $\log Z = W(e^\sigma, A_0, a_i, A_i, g^{ij}, T_0, \mu_0)$, one gets the consistent currents and energy-momentum tensor

$$\langle J^i\rangle_{\rm cons} = \frac{T_0}{\sqrt{-G}}\frac{\delta W}{\delta A_i} \,, \qquad \langle J_0\rangle_{\rm cons} = -\frac{T_0 e^{2\sigma}}{\sqrt{-G}}\frac{\delta W}{\delta A_0} \,, \tag{5}$$

$$\langle T_0{}^i\rangle = \frac{T_0}{\sqrt{-G}}\left(\frac{\delta W}{\delta a_i} - A_0\frac{\delta W}{\delta A_i}\right) \,, \qquad \langle T_{00}\rangle = -\frac{T_0 e^{2\sigma}}{\sqrt{-G}}\frac{\delta W}{\delta \sigma} \,, \tag{6}$$

and a similar expression for $\langle T^{ij}\rangle$. This result illustrates the fact that W plays the role of a generating functional for the hydrodynamic constitutive relations. In the rest of this manuscript we will use differential geometry methods to compute the anomalous partition function, and obtain from it the currents by performing the appropriate functional derivatives. Moreover, we will show that in situations with spontaneous symmetry breaking, the covariant currents can be computed directly from the Bardeen-Zumino (BZ) term needed to renormalize the anomalous action.

3. Non-Abelian anomalies

In this section we will study the partition function and currents for non-Abelian theories by considering very general prescriptions within the differential geometry approach. We refer to [28] for full details in the computation.

3.1. The chiral anomaly

Let us consider a theory of a chiral fermion coupled to an exterrnal gauge field $\mathcal{A}_\mu^a$, with Lagrangian

$$\mathscr{L}_{\rm YM} = i\overline{\psi}\gamma^\mu\Big(\partial_\mu - it_a\mathcal{A}_\mu^a\Big)\psi\,, \tag{7}$$

where $t_a = t_a^\dagger$ are the Hermitian generators of the Lie algebra satisfying the commutation relations $[t_a, t_b] = if_{abc}t_c$. Let us introduce the Lie algebra valued one-form for the gauge fields, and the associated field strength two-forms, which are defined by

$$\mathcal{A} = -i\mathcal{A}_\mu^a t_a dx^\mu \equiv -i\mathcal{A}_\mu dx^\mu\,, \qquad \mathcal{F} = d\mathcal{A} + \mathcal{A}^2 \equiv -\frac{i}{2}\mathcal{F}_{\mu\nu}dx^\mu dx^\nu\,, \tag{8}$$

respectively, where $\mathcal{F}_{\mu\nu} = \partial_\mu\mathcal{A}_\nu - \partial_\nu\mathcal{A}_\mu - i[\mathcal{A}_\mu, \mathcal{A}_\nu]$. Finite gauge transformations are implemented by the Lie group elements $g = e^{-iu^a t_a} \equiv e^u$, thus leading to

$$\mathcal{A}_g = g^{-1}\mathcal{A}g + g^{-1}dg\,, \qquad \mathcal{F}_g = g^{-1}\mathcal{F}g\,. \tag{9}$$

The corresponding infinitesimal transformations are given by $\delta_u\mathcal{A} = du + [\mathcal{A}, u] \equiv Du$ and $\delta_u\mathcal{F} = [\mathcal{F}, u]$, where we have introduced the adjoint covariant derivative Du.

To study gauge anomalies it is convenient to work with the fermion effective action functional obtained by integrating out the fermion field

$$e^{i\Gamma[\mathcal{A}]} \equiv \int \mathscr{D}\overline{\psi}\mathscr{D}\psi\, e^{iS_{\rm YM}[\mathcal{A},\psi,\overline{\psi}]}\,. \tag{10}$$

Under a general shift $\mathcal{A}_\mu^a \to \mathcal{A}_\mu^a + \delta\mathcal{A}_\mu^a$, the first order variation of $\Gamma[\mathcal{A}]$ can be expressed as

$$\delta\Gamma[\mathcal{A}] = \int d^Dx\, \delta\mathcal{A}_\mu^a(x)\, J_a^\mu(x)_{\rm cons}\,. \tag{11}$$

The axial anomaly is given by the failure of the effective action to be invariant under axial gauge transformations, which are defined as $\delta\mathcal{A}_\mu^a = (D_\mu u_A)^a$. Plugging this into Eq. (11), one gets after integrating by parts

$$\delta_{u_A}\Gamma[\mathcal{A}] = -\int d^Dx\, u_A^a(x)\, G_a[\mathcal{A}(x)]_{\rm cons} \qquad \text{with} \qquad G_a[\mathcal{A}(x)]_{\rm cons} = D_\mu J_a^\mu(x)_{\rm cons}\,, \tag{12}$$

so that the consistent anomaly, $G_a[\mathcal{A}(x)]_{\rm cons}$, is responsible for the non-conservation of the currents. This well known result leads to important consequences for the hydrodynamics of fluids affected by gauge anomalies, as we will show below.

3.2. The Chern-Simons effective action and the Bardeen form of the anomaly

In the following we will consider a general theory with chiral fermions, ψ_L and ψ_R, coupled to two external gauge fields described by the Lagrangian

$$\mathscr{L}_{\rm YM} = i\overline{\psi}_L\gamma^\mu\Big(\partial_\mu - it_a\mathcal{A}_{L\mu}^a\Big)\psi_L + i\overline{\psi}_R\gamma^\mu\Big(\partial_\mu - it_a\mathcal{A}_{R\mu}^a\Big)\psi_R\,. \tag{13}$$

The starting point to compute the functional $\Gamma[A_R, A_L]$ is the anomaly polynomial, which is given by

$$\mathcal{P}_n(\mathcal{F}_R, \mathcal{F}_L) = c_n\Big(\mathrm{Tr}\,\mathcal{F}_R^n - \mathrm{Tr}\,\mathcal{F}_L^n\Big)\,, \tag{14}$$

where $c_n = i^n/(n!(2\pi)^{n-1})$ is the normalization found for a Dirac operator in $2n = D+2$ dimensions. Since the field strengths satisfy the Bianchi identity $d\mathcal{F}_{R,L} = [\mathcal{F}_{R,L}, \mathcal{A}_{R,L}]$, the anomaly polynomial is closed, i.e. $d\mathcal{P}_n(\mathcal{F}_R, \mathcal{F}_L) = 0$, and thus we can write

$$\mathrm{Tr}\,\mathcal{F}_R^n - \mathrm{Tr}\,\mathcal{F}_L^n = d\omega^0_{2n-1}(\mathcal{A}_R, \mathcal{A}_L)\,, \tag{15}$$

where ω^0_{2n-1} is the Chern-Simons form. The fermion effective action is constructed as the integral of the Chern-Simons form in a $(2n-1)$-dimensional manifold

$$\Gamma[\mathcal{A}_R, \mathcal{A}_L]_{\mathrm{CS}} = c_n \int_{\mathcal{M}_{2n-1}} \omega^0_{2n-1}(\mathcal{A}_R, \mathcal{A}_L)\,. \tag{16}$$

One can easily check that the anomaly polynomial is gauge invariant, $\delta_u \mathcal{P}_n(\mathcal{F}_R, \mathcal{F}_L) = 0$. As a consequence, the gauge variation of ω^0_{2n-1} is, locally, a total differential, i.e. $\delta_u \omega^0_{2n-1}(\mathcal{A}_R, \mathcal{A}_L) = d\omega^1_{2n-2}(u, \mathcal{A}_R, \mathcal{A}_L)$, so that the gauge variation of the CS action turns out to be [1]

$$\delta_{u_A}\Gamma[\mathcal{A}_R, \mathcal{A}_L]_{\mathrm{CS}} = c_n \int_{\partial\mathcal{M}_{2n-1}} \omega^1_{2n-2}(u_A, \mathcal{A}_R, \mathcal{A}_L) = -\int_{\mathcal{M}_{2n-2}} \mathrm{Tr}\Big(u_A G[\mathcal{A}_R, \mathcal{A}_L]_{\mathrm{cons}}\Big)\,, \tag{17}$$

where the Stokes theorem has been used. Using the generalized transgression formula derived in [29] (see also [28]), the Chern-Simons form can be written as

$$\omega^0_{2n-1}(\mathcal{A}_{R,L}, \mathcal{F}_{R,L}) = n\int_0^1 dt\,\mathrm{Tr}\left(\dot{\mathcal{A}}_t \mathcal{F}_t^{n-1}\right), \tag{18}$$

where $\mathcal{A}_t$ is a one-parameter family of connections interpolating between $\mathcal{A}_0 = \mathcal{A}_L$ and $\mathcal{A}_1 = \mathcal{A}_R$, and the dot stands for derivative with respect to t. There are different choices for the family of connections, and this ambiguity amounts to adding an exact differential in

$$\omega^0_{2n-1}(\mathcal{A}_R, \mathcal{A}_L) = \omega^0_{2n-1}(\mathcal{A}_R) - \omega^0_{2n-1}(\mathcal{A}_L) + dS_{2n-2}(\mathcal{A}_R, \mathcal{A}_L)\,, \tag{19}$$

where $\omega^0_{2n-1}(\mathcal{A}_{R,L})$ is the Cherm-Simons form for a single chirality. In the following we will consider the interpolating curve $\mathcal{A}_t = (1-t)\mathcal{A}_L + t\mathcal{A}_R$ which leads to the so-called Bardeen counterterm preserving vector gauge transformations. Other choices lead to Chern-Simons effective actions that do not remain invariant under vector transformations.

For completeness, we will present here some explicit formulas in the four-dimensional case ($n = 3$). The Chern-Simons form preserving the vector Ward identity is

$$\omega^0_5(\mathcal{A}, \mathcal{F}_V, \mathcal{F}_A) = 6\,\mathrm{Tr}\left(\mathcal{A}\mathcal{F}_V^2 + \frac{1}{3}\mathcal{A}\mathcal{F}_A^2 - \frac{4}{3}\mathcal{A}^3\mathcal{F}_V + \frac{8}{15}\mathcal{A}^5\right)\,, \tag{20}$$

and the Bardeen expression for the consistent anomaly [30] is obtained from

$$\omega^1_4(u_A, \mathcal{A}, \mathcal{F}_V, \mathcal{F}_A) = 6\,\mathrm{Tr}\left\{u_A\left[\mathcal{F}_V^2 + \frac{1}{3}\mathcal{F}_A^2 - \frac{4}{3}\Big(\mathcal{A}^2\mathcal{F}_V + \mathcal{A}\mathcal{F}_V\mathcal{A} + \mathcal{F}_V\mathcal{A}^2\Big) + \frac{8}{3}\mathcal{A}^4\right]\right\}\,. \tag{21}$$

[1] We define the vector-axial gauge fields $(\mathcal{V}, \mathcal{A})$ in terms of $(\mathcal{A}_R, \mathcal{A}_L)$ by $\mathcal{A}_R \equiv \mathcal{V} + \mathcal{A}$ and $\mathcal{A}_L \equiv \mathcal{V} - \mathcal{A}$, and similarly for the vector-axial gauge transformation parameters, i.e. $u_R \equiv u_V + u_A$ and $u_L \equiv u_V - u_A$.

3.3. Currents and anomaly inflow

The currents induced by $\Gamma[\mathcal{A}_R, \mathcal{A}_L]_{\text{CS}}$ are obtained by performing a general variation of the gauge field $\delta\mathcal{A} = B$, where B is an infinitesimal Lie-algebra valued one-form. [2] The variation $\delta_B\omega^0_{2n-1}(\mathcal{A})$ can be efficiently computed with the help of the generalized transgression formula by considering a family of connections $\mathcal{A}_t = \mathcal{A} + tB$ interpolating between $\mathcal{A}$ and $\mathcal{A} + B$. The results obtained can be expressed in the form

$$\delta_B\Gamma[\mathcal{A}]_{\text{CS}} = \int_{\mathcal{M}_{2n-1}} \text{Tr}\,(B\mathcal{J}_{\text{bulk}}) - \int_{\mathcal{M}_{2n-2}} \text{Tr}\,(B\mathcal{J}_{\text{BZ}})\,, \tag{22}$$

where $\mathcal{J}_{\text{bulk}} = nc_n\mathcal{F}^{n-1}$ is the dual form of the bulk current, while $\mathcal{J}_{\text{BZ}}$ is the dual form of the Bardeen-Zumino current. Let us consider now a gauge variation of the gauge field, i.e. $B = \delta_u\mathcal{A}$. Then, after integrating by parts in the rhs of Eq. (22) and using the Stokes theorem, one finds

$$\delta_u\Gamma[\mathcal{A}]_{\text{CS}} = \int_{\mathcal{M}_{2n-2}} \text{Tr}\left[u(\mathcal{J}_{\text{bulk}} + D\mathcal{J}_{\text{BZ}})\right] = -\int_{\mathcal{M}_{2n-2}} \text{Tr}\left(u\, G[\mathcal{A}]_{\text{cons}}\right), \tag{23}$$

where we have used Eq. (17). Finally, this implies

$$\mathcal{J}_{\text{bulk}}\Big|_{\mathcal{M}_{2n-2}} = -\left(G[\mathcal{A}]_{\text{cons}} + D\mathcal{J}_{\text{BZ}}\right) = -G[\mathcal{A}]_{\text{cov}}\,, \tag{24}$$

where $G[\mathcal{A}]_{\text{cov}}$ is the covariant anomaly. This identification of the bulk current with (minus) the covariant anomaly is in agreement with the anomaly inflow mechanism [27]. In the four-dimensional case the dual forms of the bulk and BZ currents read

$$\mathcal{J}^V_{\text{bulk}} = -\frac{i}{24\pi^2}\left(\mathcal{F}_V^2 + \mathcal{F}_A^2\right), \qquad \mathcal{J}^A_{\text{bulk}} = -\frac{i}{24\pi^2}\left(\mathcal{F}_V\mathcal{F}_A + \mathcal{F}_A\mathcal{F}_V\right), \tag{25}$$

and

$$\mathcal{J}^V_{\text{BZ}} = \frac{i}{12\pi^2}\left[3(\mathcal{F}_V\mathcal{A} + \mathcal{A}\mathcal{F}_V) - 4\mathcal{A}^3\right], \qquad \mathcal{J}^A_{\text{BZ}} = \frac{i}{12\pi^2}\left(\mathcal{F}_A\mathcal{A} + \mathcal{A}\mathcal{F}_A\right), \tag{26}$$

respectively. Let us mention at this point that the invariant polynomial is unique and so are the bulk currents and the covariant anomalies, i.e. they are not affected by renormalization ambiguities.

3.4. Dimensional reduction

Using the previous results, we can compute the partition function of anomalous hydrodynamics. To this end, one should take all the fields to be time-independent, compactify the Euclidean time direction to a circle, and finally perform a dimensional reduction of the Chern-Simons action. In the language of differential forms, the one-form of the gauge field, $\mathcal{A}(\boldsymbol{x}) = \mathcal{A}_\mu(\boldsymbol{x})dx^\mu$, can be decomposed into KK-invariant quantities

$$\mathcal{A}(\boldsymbol{x}) = \mathcal{A}_0(\boldsymbol{x})\theta(\boldsymbol{x}) + \boldsymbol{A}(\boldsymbol{x})\,, \qquad \text{where} \qquad \theta(\boldsymbol{x}) = dx^0 + a(\boldsymbol{x}) \quad \text{and} \quad a(\boldsymbol{x}) = a_i(\boldsymbol{x})dx^i\,, \tag{27}$$

while $\boldsymbol{A}(\boldsymbol{x}) = A_i(\boldsymbol{x})dx^i$. Taking into account that $\theta^2 = 0$, it turns out that the only terms relevant for the anomalous partition function are those which are linear in θ. We will skip here the technical details and just discuss the result. The Chern-Simons effective action $\Gamma[\mathcal{A}_0, \boldsymbol{A}]_{\text{CS}}$ naturally splits into two terms [27]

$$i\Gamma[\mathcal{A}_0, \boldsymbol{A}]_{\text{CS}} = W[\mathcal{A}_0, \boldsymbol{A}]_{\text{inv}} + W[\mathcal{A}_0, \boldsymbol{A}]_{\text{anom}}\,, \tag{28}$$

[2] In the following, it is understood that an expression is valid either for $(\mathcal{A}_R, \mathcal{F}_R)$ or $(\mathcal{A}_L, \mathcal{F}_L)$ where the chirality of the fields has not been explicitly indicated.

where $W[\mathcal{A}_0, \boldsymbol{A}]_{\rm inv}$ is non-local from the viewpoint of a $(2n-2)$-dim Euclidean manifold, and it is manifestly invariant under time-independent gauge transformations, so that it does not contribute to the gauge anomaly. Consequently, the second piece $W[\mathcal{A}_0, \boldsymbol{A}]_{\rm anom}$ completely accounts for the gauge non-invariance of the partition function, and its gauge variation reproduces the consistent anomaly. Using Eq. (22), this means that the summation of the boundary currents induced by these terms [3]

$$\delta_B W[\mathcal{A}_0, \boldsymbol{A}]_{\rm inv} = \int_{S^{2n-3}} \mathrm{Tr}\Big(B\mathcal{X}\Big) + \cdots\,, \quad \delta_B W[\mathcal{A}_0, \boldsymbol{A}]_{\rm anom} = \int_{S^{2n-3}} \mathrm{Tr}\Big(B\mathcal{J}_{\rm cons}\Big) + \cdots\,, \tag{29}$$

should be equal to (minus) the BZ current, i.e. $\mathcal{X} + \mathcal{J}_{\rm cons} = -\mathcal{J}_{\rm BZ}$. Consequently, the physical current induced by $W[\mathcal{A}_0, \boldsymbol{A}]_{\rm inv}$ is precisely (minus) the covariant current, i.e. $\mathcal{X} = -(\mathcal{J}_{\rm cons} + \mathcal{J}_{\rm BZ}) = -\mathcal{J}_{\rm cov}$. This result implies that one can compute the covariant currents directly from the invariant part of the action, bypassing the need to compute the anomalous partition function, $W[\mathcal{A}_0, \boldsymbol{A}]_{\rm anom}$. Using this procedure, the anomalous covariant current in a stationary background reads (cf. [27, 28])

$$\mathcal{J}_{0,\,\rm cov} = 0\,, \qquad\qquad \boldsymbol{J}_{\rm cov} = T_0 \frac{\delta}{\delta \boldsymbol{F}} W[\mathcal{A}_0, \boldsymbol{F}, da]_{\rm inv}\,. \tag{30}$$

The results for the covariant currents in four-dimensional spacetime are

$$\begin{aligned} \boldsymbol{J}_{V,\,\rm cov} &= -\frac{i}{4\pi^2}\Big[\mathcal{V}_0\boldsymbol{F}_A + \boldsymbol{F}_A\mathcal{V}_0 + \mathcal{A}_0\boldsymbol{F}_V + \boldsymbol{F}_V\mathcal{A}_0 + da(\mathcal{V}_0\mathcal{A}_0 + \mathcal{A}_0\mathcal{V}_0)\Big]\,, \\ \boldsymbol{J}_{A,\,\rm cov} &= -\frac{i}{4\pi^2}\Big[\mathcal{V}_0\boldsymbol{F}_V + \boldsymbol{F}_V\mathcal{V}_0 + \mathcal{A}_0\boldsymbol{F}_A + \boldsymbol{F}_A\mathcal{A}_0 + da(\mathcal{V}_0^2 + \mathcal{A}_0^2)\Big]\,. \end{aligned} \tag{31}$$

The fact that the covariant currents are obtained from $W_{\rm inv}$ implies that they are the same independently of the counterterm of the theory.

3.5. Currents in QCD with 2 flavors

While these results are valid for a non-Abelian theory with symmetry group $\mathrm{U}(N_f)\times\mathrm{U}(N_f)$, it would be interesting to illustrate these techniques by considering the case of QCD with $N_f = 2$. The maximal number of chemical potentials to be consistently introduced corresponds to the dimension of the Cartan subalgebra. Then, we consider the background given by a combination of $t_0 = \frac{1}{2}\mathbb{1}_{2\times 2}$ and $t_3 = \frac{1}{2}\sigma_3$, where σ_i are the Pauli matrices, i.e.

$$V(\boldsymbol{x}) = V_0(\boldsymbol{x})\,t_0 + V_3(\boldsymbol{x})\,t_3\,, \qquad A = A_0\,t_0\,, \qquad \boldsymbol{A}(\boldsymbol{x}) = 0\,. \tag{32}$$

V and A are one-forms, so that $V_a = \mathcal{V}_{0\,a}\,dx^0 + V_{i\,a}\,dx^i$ where $a = 0, 3$ are flavor group indices (and similarly for A). Since all external fields lie on the Cartan subalgebra, their field strengths are particularly simple: $\boldsymbol{F}_V = dV_0\,t_0 + dV_3\,t_3$ and $\boldsymbol{F}_A = 0$. In addition to the equilibrium velocity field $u_\mu = -e^\sigma(1, a_i)$, we can define the equilibrium baryonic, isospin and axial chemical potentials, as

$$\mu_0 = \mathcal{V}_{0\,0}\,e^{-\sigma}\,, \qquad \mu_3 = \mathcal{V}_{0\,3}\,e^{-\sigma}\,, \qquad \mu_5 = \mathcal{A}_{0\,0}\,e^{-\sigma}\,, \tag{33}$$

where μ_5 controls the chiral imbalance of the system [31, 32]. We assume that μ_5 is constant, so that we are just interested in the one-derivative terms from the vectorial part of the background. Let us define the charge matrix for two light flavors with electric charges $+\frac{2}{3}e$ and $-\frac{1}{3}e$, as

$$Q = \begin{pmatrix} \frac{2}{3} & 0 \\ 0 & -\frac{1}{3} \end{pmatrix} = \frac{1}{3}t_0 + t_3\,. \tag{34}$$

[3] In these expressions the dots stand for bulk contributions.

IOP Conf. Series: Journal of Physics: Conf. Series **1194** (2019) 012073 doi:10.1088/1742-6596/1194/1/012073

Then the electromagnetic current writes

$$J^\mu_{\mathrm{em,\,cov}} = e\overline{\psi}\gamma^\mu Q\psi = \frac{e}{3}J^\mu_{0,\,\mathrm{cov}} + eJ^\mu_{3,\,\mathrm{cov}}\,, \qquad \text{where} \qquad J^\mu_{a,\,\mathrm{cov}} = \overline{\psi}\gamma^\mu t_a\psi\,. \tag{35}$$

Similarly, the baryonic and isospin currents write

$$J^\mu_{\mathrm{bar,\,cov}} = \frac{2}{3}J^\mu_{0,\,\mathrm{cov}}\,, \qquad J^\mu_{\mathrm{iso,\,cov}} = J^\mu_{3,\,\mathrm{cov}}\,, \tag{36}$$

respectively. Notice that these currents fulfill the Gell-Mann–Nishijima (GMN) relation $J^\mu_{\mathrm{em}} = \frac{e}{2}J^\mu_{\mathrm{bar}} + eJ^\mu_{\mathrm{iso}}$. The expectation values of the currents $\langle J^\mu_{a,\,\mathrm{cov}}\rangle$ are given by the first equation in (31) after making the replacements $(\mathcal{V}_{0\,0}, V_{i\,0}) = \frac{e}{3}(\mathbb{V}_0, \mathbb{V}_i)$ and $(\mathcal{V}_{0\,3}, V_{i\,3}) = e(\mathbb{V}_0, \mathbb{V}_i)$. [4] Then, we find the following expressions for the currents, written in terms of the chemical potential

$$\langle J^\mu_{\mathrm{em,\,cov}}\rangle = -\frac{5e^2N_c}{36\pi^2}\mu_5\epsilon^{\mu\nu\lambda\rho}u_\nu\partial_\lambda\mathscr{V}_\rho\,, \qquad \langle J^\mu_{\mathrm{iso,\,cov}}\rangle = -\frac{eN_c}{8\pi^2}\mu_5\epsilon^{\mu\nu\lambda\rho}u_\nu\partial_\lambda\mathscr{V}_\rho\,, \tag{37}$$

where N_c is the number of colors. To arrive at these out-of-equilibrium results we have performed a Lorentz covariantization of the equilibrium currents. The baryonic current is obtained from the GMN relation. Note that we find contributions from the chiral magnetic effect, and from the first equation in (37) we can read the chiral magnetic conductivity

$$\sigma_{\mathcal{B}} = \frac{5e^2N_c}{36\pi^2}\mu_5\,. \tag{38}$$

These results for the covariant currents are in agreement with the alternative method consisting in computing the consistent currents from the functional derivative of the anomalous partition function and adding the BZ currents, cf. Refs. [20, 33].

4. Partition function in presence of spontaneous symmetry breaking

In this section we will consider physical situations in which the symmetries are spontaneously broken, either total or partially. This is the case of chiral flavor symmetry in QCD, which is broken down to its vector subgroup, or the U(1) global phase in superfluids. A consequence of the spontaneous symmetry breaking is the appearance of Goldstone bosons that can couple to external gauge fields and contribute to the anomaly.

4.1. The Wess-Zumino-Witten partition function

The Wess-Zumino-Witten (WZW) partition function is used to describe the effects of the anomaly when the symmetry is spontaneously broken. It accounts for the anomaly-induced interactions between the external gauge fields $\mathcal{A}$ and the Goldstone bosons ξ^a [34, 35, 36, 37]. An immediate application of the non-Abelian case is the study of QCD in the confined phase, leading to the prediction of relevant properties of hadronic fluids interacting with external electromagnetic fields. The WZW action admits a very simple expression in terms of the CS action of Eq. (16) [21, 22, 23, 24, 38]

$$\Gamma[\mathcal{A}, g]_{\mathrm{WZW}} = \Gamma[\mathcal{A}]_{\mathrm{CS}} - \Gamma[\mathcal{A}_g]_{\mathrm{CS}}\,, \tag{39}$$

where $\mathcal{A}_g$ is the gauge transformed field with the group element $g \equiv \exp(-i\xi^a t_a)$, cf. Eq. (9). The dimensional reduction of the WZW action can be performed by following the same procedure

[4] We denote the physical electromagnetic potential by $\mathscr{V}_\mu$, and its KK invariant form by $\mathbb{V}_\mu$, i.e. $\mathbb{V}_0 = \mathscr{V}_0$ and $\mathbb{V}_i = \mathscr{V}_i - a_i\mathscr{V}_0$. The physical magnetic field is then $\mathscr{B}^i = \epsilon^{ijk}\partial_j\mathscr{V}_k$.

explained in Sec. 3.4. The result is that the invariant part cancels and it is written only in terms of the local anomalous part. It is easy to check that $\mathcal{A}_g$ is gauge invariant, and thus the gauge variation of the WZW action gives precisely the consistent anomaly

$$\delta_u \Gamma[\mathcal{A}, g]_{\mathrm{WZW}} = \delta_u \Gamma[\mathcal{A}]_{\mathrm{CS}} = -\int_{\mathcal{M}_{2n-2}} \mathrm{Tr}\left(u\, G[\mathcal{A}(x)]_{\mathrm{cons}}\right). \tag{40}$$

Unlike the Chern-Simons action, the WZW effective action does not induce any bulk current. Instead, under a general variation $\delta\mathcal{A} = B$ the WZW action induces a local consistent current on the boundary $\mathcal{M}_{2n-2}$ given by

$$\delta_B \Gamma[\mathcal{A}, g]_{\mathrm{WZW}} = \int_{\mathcal{M}_{2n-2}} \mathrm{Tr}\left\{B\Big[g\, \mathcal{J}(\mathcal{A}_g)_{\mathrm{BZ}}\, g^{-1} - \mathcal{J}(\mathcal{A})_{\mathrm{BZ}}\Big]\right\}. \tag{41}$$

The first term can be identified with the covariant current, and we conclude

$$\mathcal{J}(\mathcal{A}, g)_{\mathrm{cov}} = g\, \mathcal{J}(\mathcal{A}_g)_{\mathrm{BZ}}\, g^{-1} = \mathcal{J}(\mathcal{A} + dgg^{-1})_{\mathrm{BZ}}\,. \tag{42}$$

This connection between the covariant current and the BZ current provides the most efficient computational method in the presence of spontaneous symmetry breaking, bypassing the need to use the WZW action.

For applications to hadronic fluids, we are interested in the case $\mathcal{G} \times \mathcal{G} \to \mathcal{G}$, where the symmetry is broken down to the diagonal subgroup of vector gauge transformation. One just have to make the replacements $\mathcal{A} \to (\mathcal{A}_R, \mathcal{A}_L)$ and $g \to (e, U)$ in the previous relations, where e is the identity element and $U = \exp(2i\xi^a t_a)$. In that case the WZW action takes the form

$$\Gamma[\mathcal{A}_R, \mathcal{A}_L, U]_{\mathrm{WZW}} = \Gamma[\mathcal{A}_R, \mathcal{A}_L]_{\mathrm{CS}} - \Gamma[\mathcal{A}_R, \mathcal{A}_L^U]_{\mathrm{CS}}\,, \tag{43}$$

where $\mathcal{A}_L^U = U^{-1}\mathcal{A}_L U + U^{-1}dU$ and $\Gamma[\mathcal{A}_R, \mathcal{A}_L]_{\mathrm{CS}}$ is given by Eq. (16). Using Eq. (42), the covariant currents can be obtained as

$$\mathcal{J}^R(\mathcal{A}_R, \mathcal{A}_L, U)_{\mathrm{cov}} = \mathcal{J}^R(\mathcal{A}_R, \mathcal{A}_L^U)_{\mathrm{BZ}}\,, \qquad \mathcal{J}^L(\mathcal{A}_R, \mathcal{A}_L, U)_{\mathrm{cov}} = U\, \mathcal{J}^L(\mathcal{A}_R, \mathcal{A}_L^U)_{\mathrm{BZ}}\, U^{-1}\,. \tag{44}$$

These relations, and their particularization to stationary backgrounds, can be used to obtain the covariant currents for the hydrodynamics of hadronic fluids without computing the anomalous WZW action. We have explicitly checked that the result for the covariant currents of hadronic fluids are the same by using either the direct method of Eq. (44), or the anomalous WZW action. We will present some partial results in the next subsection.

4.2. Hadronic fluids

Let us particularize some of the previous results to QCD with $N_f = 2$. We will consider the spontaneous breaking of the axial symmetry $\mathrm{U}(2)_L \times \mathrm{U}(2)_R \to \mathrm{U}(2)_V$. The matrix of Goldstone bosons

$$U(\xi) = \exp\Big(iX(\xi)\Big) \qquad \text{with} \qquad X(\xi) := 2\sum_{a=1}^{3} \xi^a t_a = \frac{\sqrt{2}}{f_\pi}\begin{pmatrix} \frac{1}{\sqrt{2}}\pi^0 & \pi^+ \\ \pi^- & -\frac{1}{\sqrt{2}}\pi^0 \end{pmatrix}, \tag{45}$$

includes three Goldstone bosons from the broken $\mathrm{SU}(2)_A$ symmetry. $\{\pi^0, \pi^\pm\}$ are the conventionally normalized Goldstone boson fields, and $f_\pi \approx 92\,\mathrm{MeV}$ is the pion decay constant.

We will assume that the fourth Goldstone boson ξ^0 is absent, as the $\mathrm{U}(1)_A$ symmetry is violated by non-perturbative effects. Then, the WZW effective action writes

$$T_0 W = \int d^3x \left[\frac{e^2 N_c}{12\pi^2 f_\pi} \mathbb{V}_0 \partial_i \pi^0 \mathbb{B}^i - \frac{ie\mu_5 N_c}{12\pi^2 f_\pi^2} \left(\pi^- \partial_j \pi^+ - \pi^+ \partial_j \pi^- - 2ie\pi^- \pi^+ \mathbb{V}_j \right) \mathbb{B}^j + \mathcal{O}(\pi^3) \right] , \tag{46}$$

where $\mathbb{B}^i = \epsilon^{ijk} \partial_j \mathbb{V}_k$ is the KK-invariant magnetic field. To obtain this result we have expanded the action in powers of the pion fields. The first term on the right-hand side of Eq. (46), linear in the pion field, is responsible for the electromagnetic decay of the neutral pion, $\pi^0 \to 2\gamma$,

$$\mathscr{L}_{\mathrm{eff}} \supset \frac{e^2 N_c}{96\pi^2 f_\pi} \pi^0 \epsilon^{\mu\nu\lambda\rho} \mathscr{F}_{\mu\nu} \mathscr{F}_{\lambda\rho} , \tag{47}$$

where $\mathscr{F}_{\mu\nu} = \partial_\mu \mathscr{V}_\nu - \partial_\nu \mathscr{V}_\mu$ is the electromagnetic field strength. The quadratic term in Eq. (46) also agrees with the known form of the parity-odd couplings obtained from the WZW action in the presence of chiral imbalance (see e.g. Ref. [32]). Finally, the expectation values of the covariant currents can be obtained either from the WZW effective action or from the BZ current by using Eq. (44). We will leave this analysis for Ref. [33].

5. Conclusions

In this work we have used differential geometry methods to carry out a systematic construction of partition functions and currents for non-dissipative effects in relativistic fluids in presence of non-Abelian anomalies. In particular, our analysis can be applied to study transport phenomena induced by external magnetic fields and vortices. We have studied in detail the covariant currents induced by the chiral anomaly, and we have shown that they can be determined solely from the gauge invariant piece of the Chern-Simons effective action. As a consequence, the covariant currents are not affected by renormalization ambiguities, as these only appear in the anomalous part of the action due to the local counterterms.

We have extended our study to theories with spontaneous symmetry breaking, as this can lead to relevant information about the hydrodynamics of the corresponding Goldstone bosons interacting with external fields. We have found that the Bardeen-Zumino current fully determines the covariant currents, so that in this case it is not demanding to compute the anomalous partition function to obtain the hydrodynamical constitutive relations. These techniques have been illustrated by considering the case of QCD with two light flavors and, in particular, we have obtained results for the chiral magnetic effect. Our findings are also in agreement with previous results in the literature in the presence of chiral imbalance.

Finally, let us stress that the techniques presented in this work can be applied to a wide variety of physical situations, ranging from the study of other sectors of the Standard Model of particle physics, to superfluids [39, 40] and condensed matter systems affected by triangle anomalies [41, 42].

Acknowledgments

This work has been supported by Plan Nacional de Altas Energías Spanish MINECO grants FPA2015-64041-C2-1-P, FPA2015-64041-C2-2-P, and by Basque Government grant IT979-16. The research of E.M. is also supported by Spanish MINEICO and European FEDER funds grant FIS2017-85053-C2-1-P, Junta de Andalucía grant FQM-225, as well as by Spanish MINEICO Ramón y Cajal Program grant RYC-2016-20678, and by Universidad del País Vasco UPV/EHU, Bilbao, Spain, through a Visiting Professor appointment. M.A.V.-M. gratefully acknowledges the hospitality of the KEK Theory Center and the Department of Theoretical Physics of the University of the Basque Country during the early stages of this work.

IOP Conf. Series: Journal of Physics: Conf. Series **1194** (2019) 012073 doi:10.1088/1742-6596/1194/1/012073

References

[1] Fukushima K, Kharzeev D E and Warringa H J 2008 *Phys. Rev.* **D78** 074033 (*Preprint* 0808.3382)
[2] Son D T and Surowka P 2009 *Phys. Rev. Lett.* **103** 191601 (*Preprint* 0906.5044)
[3] Keren-Zur B and Oz Y 2010 *JHEP* **06** 006 (*Preprint* 1002.0804)
[4] Gooth J *et al.* 2017 *Nature* **547** 324–327 (*Preprint* 1703.10682)
[5] Kharzeev D E and Yee H U 2011 *Phys. Rev.* **D84** 045025 (*Preprint* 1105.6360)
[6] Stephanov M A and Yin Y 2012 *Phys. Rev. Lett.* **109** 162001 (*Preprint* 1207.0747)
[7] Huang A, Jiang Y, Shi S, Liao J and Zhuang P 2018 *Phys. Lett.* **B777** 177–183 (*Preprint* 1703.08856)
[8] Kharzeev D E and Warringa H J 2009 *Phys. Rev.* **D80** 034028 (*Preprint* 0907.5007)
[9] Landsteiner K, Megías E and Pena-Benitez F 2011 *Phys. Rev. Lett.* **107** 021601 (*Preprint* 1103.5006)
[10] Landsteiner K, Megías E and Pena-Benitez F 2013 *Lect. Notes Phys.* **871** 433–468 (*Preprint* 1207.5808)
[11] Manes J L and Valle M 2013 *JHEP* **01** 008 (*Preprint* 1211.0876)
[12] Bhattacharyya S, Hubeny V E, Minwalla S and Rangamani M 2008 *JHEP* **02** 045 (*Preprint* 0712.2456)
[13] Erdmenger J, Haack M, Kaminski M and Yarom A 2009 *JHEP* **01** 055 (*Preprint* 0809.2488)
[14] Landsteiner K, Megías E, Melgar L and Pena-Benitez F 2011 *JHEP* **09** 121 (*Preprint* 1107.0368)
[15] Megías E and Pena-Benitez F 2013 *JHEP* **05** 115 (*Preprint* 1304.5529)
[16] Banerjee N, Bhattacharya J, Bhattacharyya S, Jain S, Minwalla S and Sharma T 2012 *JHEP* **09** 046 (*Preprint* 1203.3544)
[17] Jensen K, Kovtun P and Ritz A 2013 *JHEP* **10** 186 (*Preprint* 1307.3234)
[18] Bhattacharyya S 2014 *JHEP* **08** 165 (*Preprint* 1312.0220)
[19] Megías E and Valle M 2014 *JHEP* **11** 005 (*Preprint* 1408.0165)
[20] Mañes J L, Megías E, Valle M and Vázquez-Mozo M A 2018 *EPJ Web Conf.* **192** 00022
[21] Zumino B 1983 Chiral Anomalies and Differential Geometry *"Relativity, groups and topology"* (The Netherlands: Elsevier)
[22] Wess J and Zumino B 1971 *Phys. Lett.* **37B** 95–97
[23] Witten E 1983 *Nucl. Phys.* **B223** 422–432
[24] Mañes J L 1985 *Nucl. Phys.* **B250** 369–384
[25] Jensen K, Loganayagam R and Yarom A 2013 *JHEP* **02** 088 (*Preprint* 1207.5824)
[26] Haehl F M, Loganayagam R and Rangamani M 2014 *JHEP* **03** 034 (*Preprint* 1312.0610)
[27] Jensen K, Loganayagam R and Yarom A 2014 *JHEP* **05** 134 (*Preprint* 1310.7024)
[28] Mañes J L, Megías E, Valle M and Vázquez-Mozo M A 2018 *JHEP* **11** 076 (*Preprint* 1806.07647)
[29] Manes J, Stora R and Zumino B 1985 *Commun. Math. Phys.* **102** 157
[30] Bardeen W A 1969 *Phys. Rev.* **184** 1848–1857
[31] Gatto R and Ruggieri M 2012 *Phys. Rev.* **D85** 054013 (*Preprint* 1110.4904)
[32] Andrianov A, Andrianov V and Espriu D 2017 *EPJ Web Conf.* **138** 01007
[33] Mañes J L, Megías E, Valle M and Vázquez-Mozo M A 2018 Anomalous currents of nuclear matter in the chiral limit, *to appear*
[34] Kaiser R 2001 *Phys. Rev.* **D63** 076010 (*Preprint* hep-ph/0011377)
[35] Son D T and Stephanov M A 2008 *Phys. Rev.* **D77** 014021 (*Preprint* 0710.1084)
[36] Fukushima K and Mameda K 2012 *Phys. Rev.* **D86** 071501 (*Preprint* 1206.3128)
[37] Brauner T and Kadam S 2017 *JHEP* **03** 015 (*Preprint* 1701.06793)
[38] Chu C S, Ho P M and Zumino B 1996 *Nucl. Phys.* **B475** 484–504 (*Preprint* hep-th/9602093)
[39] Lin S 2012 *Phys. Rev.* **D85** 045015 (*Preprint* 1112.3215)
[40] Hoyos C, Kim B S and Oz Y 2014 *JHEP* **10** 127 (*Preprint* 1404.7507)
[41] Basar G, Kharzeev D E and Yee H U 2014 *Phys. Rev.* **B89** 035142 (*Preprint* 1305.6338)
[42] Landsteiner K 2014 *Phys. Rev.* **B89** 075124 (*Preprint* 1306.4932)