Strategic Investment in Protection in Networked Systems☆

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Abstract

We study the incentives that agents have to invest in costly protection against cascading failures in networked systems. Applications include vaccination, computer security and airport security. Agents are connected through a network and can fail either intrinsically or as a result of the failure of a subset of their neighbors. We characterize the equilibrium based on an agent's failure probability and derive conditions under which equilibrium strategies are monotone in degree (i.e. in how connected an agent is on the network). We show that different kinds of applications (e.g. vaccination, airport security) lead to very different equilibrium patterns of investments in protection, with important welfare and risk implications. Our equilibrium concept is flexible enough to allow for comparative statics in terms of network properties and we show that it is also robust to the introduction of global externalities (e.g. price feedback, congestion).

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1. Introduction

Many systems of interconnected components are exposed to the risk of cascading failures. The latter arises from interdependencies or interlinkage, where the failure of a single entity (or small set of entities) can result in a cascade of failures jeopardizing the whole system. This phenomenon occurs in various kinds of systems. Well-known examples include 'black-outs' in power grids, where overload redistribution following the failure of a single component can result in a cascade of failures that ripples through the entire grid (e.g. Rosas-Casals et al. (2007), Wang et al. (2010)). The internet and computer networks also exhibit this phenomenon—one manifestation being the spread of malware (e.g. Lelarge and Bolot (2008b), Balthrop et al. (2004) and Cohen et al. (2000)). Likewise, human populations are exposed to the spread of contagious diseases[3].

While the existing work tends to be mostly descriptive, less attention has been devoted to studying the incentives to guard against the risk of cascading failures. In early 2015, a measles epidemic spread across the western part of the Unites States. It was reported that one of the causes was the unwillingness of parents to vaccinate their children (e.g. The Economist (5 February 2015), The Economist (4 February 2015), Reuters (27 August 2015)). Indeed, some people may want to avoid the perceived risks of a vaccine's side effects and free-ride on the "herd immunity" provided by the vaccination of other people. This raises the following question: what are the incentives to vaccinate against a contagious disease? The same type of question can be asked about other systems subject to the risk of cascading failures. What are the incentives to invest in computer security solutions to protect against the spread of malware? What incentives do airports have to invest in security equipment/personnel? How does the structure of interactions between individuals, computers or airports affect those incentives? The literature studying such strategic decisions is still in its early stages. Some papers worth noting are Lelarge and Bolot (2008b), Lelarge and Bolot (2008a), Lelarge and Bolot (2009), Galeotti and Rogers (2013), Dziubinski and Goyal (2014), Goyal and Vigier (2014) and Blume et al. (2011). There is also a literature on games of "interdependent security" (e.g. Heal and Kunreuther (2005), Heal et al. (2006) and Johnson et al. (2010)), but a complex networked interaction structure is generally not studied[4].

In this paper, we develop a framework to study the incentives that agents have to invest in protection against cascading failures in networked systems. A set of interconnected agents can each fail individually or as a result of a cascade of failures. Depending on the application, failure can mean a human being contracting an infectious disease, a computer being infected by a virus or an airport being exposed to a security event (e.g. a suspicious luggage or passenger being checked in or being in transit). Each agent must decide on whether to make a costly investment in protection against cascading failures. This investment can mean vaccination, investing in computer security solutions or airport security equipment, to

[3]For different applications, such as cascading risk in financial systems, see Amini et al. (2011), Acemoglu et al. (2013), Elliott et al. (2013), Thurner and Poledna (2013), Bastos-Santos et al. (2010), Boss et al. (2004) and Lorenz et al. (2009)

[4]For other examples of games played on networks, the reader is referred to Kearns (2007), Candogan et al. (2012), Bloch and Querou (2013), Bramoullé and Kranton (2007), Ballester et al. (2009), Ambrus et al. (2014) or Jackson and Zenou (2014). For some early work on the topic of vaccination in a fully mixing population, see Francis (1997) or Brito et al. (1991).

name a few important examples. Strategic decisions to invest in protection are based on an agent's intrinsic failure risk as well as on his belief about his neighbors and their probability of failure. In a complex networked system, forming such a belief can be challenging. For that reason, we study the problem with an infinite number of agents and employ a solution concept that considerably simplifies how agents reason about the network: a mean-field equilibrium (MFE). This is similar to the equilibrium concept used in Galeotti et al. (2010), Jackson and Yariv (2007) and Leduc et al. (2015). An agent simply considers a mean-field approximation of the cascading failure process. This equilibrium concept allows us to preserve the heterogeneity of the networked interaction structure (each agent can have a different degree, i.e. a different number of connections) while simplifying the computation of an equilibrium. This mean-field equilibrium concept also conveniently allows for comparative statics in terms of the network structure (as captured by the degree distribution), as well as other model parameters. This allows us to measure such things as the effect of an increase in the level of connectedness on investments in protection.

We characterize the equilibrium for two broad classes of games: (i) *games of total protection*, in which agents invest in protection against *both* their intrinsic failure risk and the failure risk of their neighbors and (ii) *games of self protection*, in which agents invest in protection *only* against their intrinsic failure risk. The first class defines a game of strategic substitutes, in which some agents free-ride on the protection provided by others. Applications covered by this class of games include vaccination or computer security. The second class defines a game of strategic complements, in which agents pool their investments in protection and this can result in coordination failures. Applications covered by this class of games include airport security.

Another of our contributions is to analyze the effect of the network structure on equilibrium behavior in those two classes of games. For example, in the case of vaccination, it is the agents who have *more* neighbors than a certain threshold who choose to vaccinate and the agents who are less connected who free-ride. The more connected agents thus bear the burden of vaccination, which can be seen as a positive outcome. In the case of airport security, on the other hand, it is agents who have *fewer* neighbors than a certain threshold who choose to invest in security equipment/personnel. Since the less connected airports are less likely to act as hubs that can transmit failures, this can be seen as an inefficient outcome. To our knowledge, we are the first to explicitly characterize such features, which are the consequence of network structure and can have important policy and welfare implications.

Finally, we study the case when the cost of protection depends on the global demand for it. For example, the price of airport security equipment or computer security solutions will increase if demand increases. It is important to understand the impact that this can have on agent's behavior as the introduction of such a global congestion externality may conflict with the local (network-related) externalities. We characterize the equilibrium after introducing this price feedback and show that the results derived previously still hold. In other words, the results obtained solely with local, network-related externalities are robust to introduction of such global effects. To the best of our knowledge, we are the first to study strategic interactions in networks in the presence of both global and local externalities.

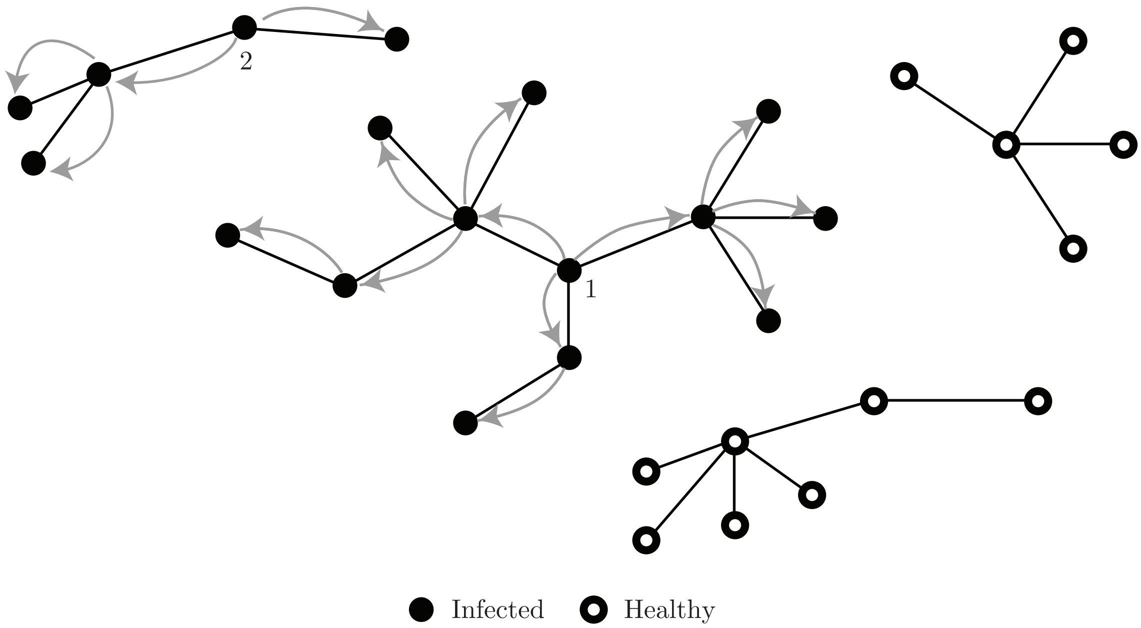


Figure 1: Example of a Contagion Cascade: individuals labeled 1 and 2 contract the disease from exogenous sources. From then on, a contagion cascade takes place in discrete steps: all their neighbors become infected. This then leads to their neighbors' neighbors to become infected and so on.

2. Cascading Failures in Networks

2.1. Overview

In this section we will discuss how cascades of failures can propagate through networks. A *cascade of failures* is defined as a process involving the subsequent failures of interconnected components. A *failure* is a general term that may represent different kinds of costly events. Let us consider, for example, the spread of a disease in a human population. Initially, some individuals get infected through exogenous sources such as livestock, mosquitos or the mutation of a pathogen. These individuals can then transmit the disease through contacts with other humans. Let us suppose that an individual is sure to catch the disease if one of his neighbors is infected. Figure 1 illustrates this process. We can see the impact of network structure on contagion. Some people lying in certain components remain healthy whereas others are infected by their neighbors. We also see that individuals with a high number of contacts tend to facilitate contagion. This is a simplified model of contagion. A more realistic model could, for example, transmit the disease only to randomly selected neighbors, depending on its virulence.

Now let us imagine that some individuals are vaccinated and therefore are not susceptible to becoming infected, neither by exogenous sources nor by contacts with other people. This will have an impact on the cascading process. Indeed, it will effectively 'cut' certain contagion channels, thereby impeding the spread of the disease. Figure 2 illustrates this. We see that the importance of the network structure becomes even more striking. In Fig. 2a), immunized individuals have been selected randomly, whereas in Fig. 2b) individuals with 4 or more contacts have been immunized. It is clear that those more connected individuals often act as hubs through which contagion can spread more easily. When these individuals are immunized, the effect of impeding the propagation of the disease tends to be much greater than when the immunized individuals are chosen at random.

In this example, the 'failure' of an individual means his becoming infected by the disease. In other applications, 'failure' can mean infection by malware. The nodes then no longer represent individuals but computers (or local subnetworks or autonomous systems). Antivirus software or other sorts of computer security solutions are means by which the spread of malware can be impeded.

We saw in the simple example of Fig. 2 that the configuration of the vaccinated nodes was crucial to impeding contagion. An important question is to study the incentives that an individual may have to become vaccinated. How does the network structure affect his decision to become vaccinated? What roles other individuals play in influencing that decision through their own vaccination behavior?

Given the range of applications, we will talk of an *investment in protection*. This refers to an investment made by a node in order to protect itself against the risk of failure. In the next section, we build a model of strategic investment in protection against cascading failures in networked systems. We will refer to nodes as *agents*, since they make decisions regarding this investment in protection. More generally, we will be interested in how the network structure and the failure propagation mechanism influence those decisions through the externalities that they generate.

2.2. Network, Failure Mechanism and Basic Informational Assumptions

In this section, we briefly describe the finite interaction setting that we will later approximate with a mean-field model.

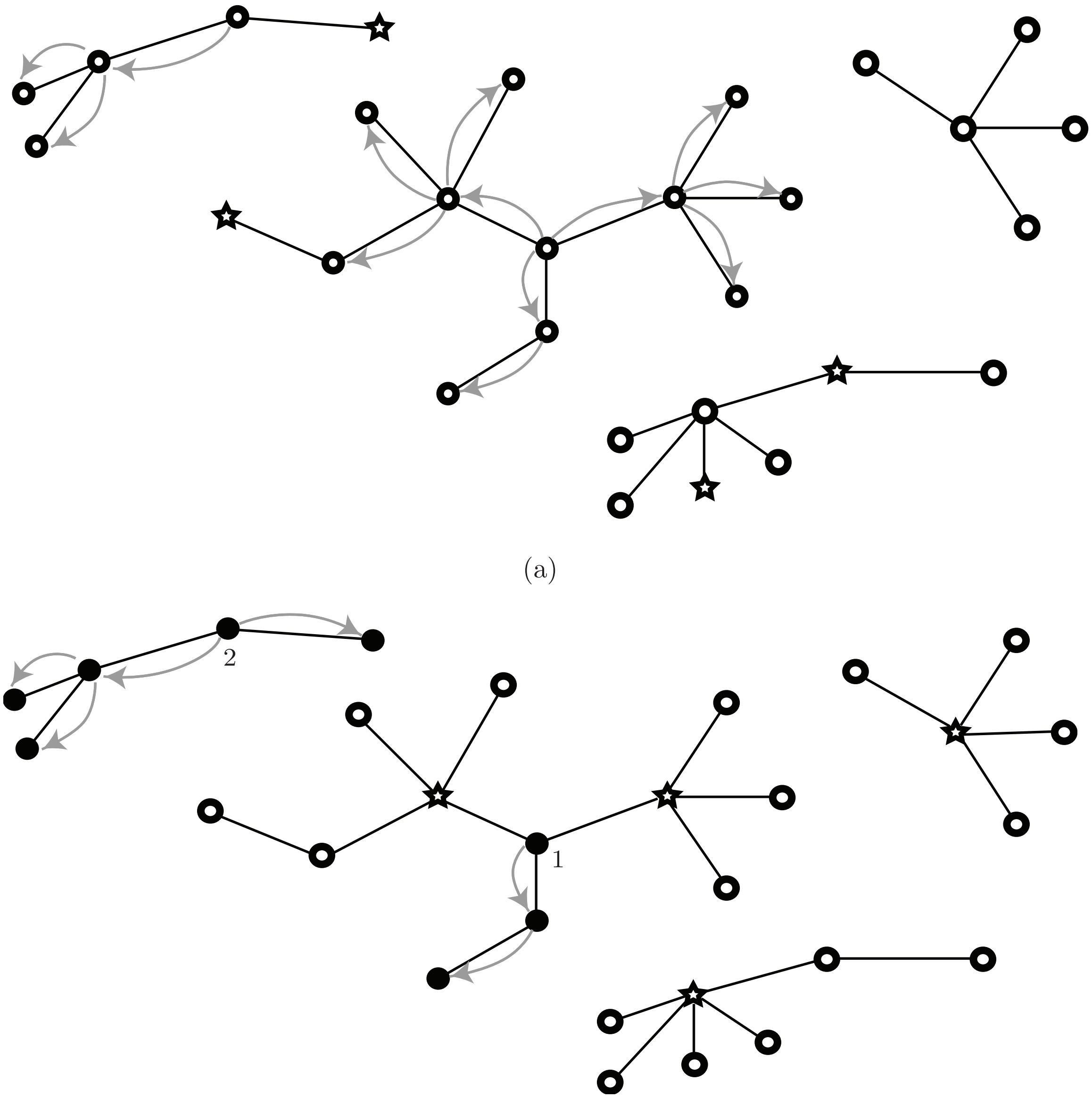
There is a finite set of agents $\mathcal{N} = \{1, 2, ..., n\}$. The connections between them are described by an *undirected* network that is represented by a symmetrical adjacency matrix $g \in \{0,1\}^{n\times n}$, with $g_{ij} = 1$ implying that i and j are connected. i can thus be affected by the failure of j and vice versa. By convention, we set $g_{ii} = 0$ for all $i \in \mathcal{N}$. The network realization g is unobservable to the agents, but it is drawn from the probability measure $P : \{0,1\}^{n\times n} \to [0,1]$ over the set of all possible networks with n nodes. We assume that P is permutation-invariant, i.e. that changing node labels does not change the measure.

Each agent i has a neighborhood $N_i(g) = \{j | g_{ij} = 1\}$. The *degree* of agent i, $d_i(g)$, is the number of i's connections, i.e. $d_i(g) = |N_i(g)|$.

We study an informational environment[5] in which agents are aware of their proclivity to interact with others, but do not know who these others will be when taking actions. Formally, this means that an agent knows only his degree d_i. For example, a bank may have a good idea of the number of financial counter-parties it has but not the number of counterparties the latter have, let alone the whole topology of the interbank system. Likewise, someone may know the number of people he interacts with, but not the number of people the latter interact with. An airport may know the number of connecting flights it has with other airports, but not the number of connecting flights other airports have.

We now introduce two definitions that capture how agents reason about cascading failures.

[5]Such an environment is similar to the one presented in Galeotti et al. (2010).



(a)

(b)

● Infected ○ Healthy, not immunized ☆ Healthy, immunized

Figure 2: Examples of Contagion Cascades in the Presence of Immunized Individuals: individuals labeled 1 and 2 contract the disease from exogenous sources. The contagion cascade then propagates. In part (a), a randomly-chosen subset of agents were vaccinated against the disease. In part (b), individuals with at least 4 contacts were vaccinated against the disease.

Definition 1. *An agent's intrinsic failure probability is denoted by* $p \in [0, 1]$.

We assume all agents can fail intrinsically with the same probability p. The interpretation of intrinsic failure depends on the application. In the context of malware, intrinsic failure means a computer becoming infected as a result of a direct hacking attack. In the context of the spread of contagious diseases, intrinsic failure means being infected by a virus through non-human sources, such as contact with livestock or insects. In the context of airport security, intrinsic failure can mean a suspicious luggage being checked in at the airport.

An agent can also fail as a result of the failure of his neighbors through a *cascading failure function* defined below.

Definition 2. *A cascading failure function* $\mathcal{H}_d : \{0, 1\}^d \to [0, 1]$ *for an agent of degree* d *takes a* d*-dimensional vector* $\vec{\alpha}_{N(g)}$ *of binary variables, each representing the state (*0 *for healthy or* 1 *for failed) of a neighbor, and returns the probability of failure of the agent.* $\mathcal{H}_d(\vec{\alpha}_{N(g)})$ *is assumed to be non-decreasing in each entry of the vector* $\vec{\alpha}_{N(g)}$.

We assume that $\mathcal{H}_d$ is permutation invariant, so that if $\vec{\alpha}'_{N(g)}$ is a permutation of $\vec{\alpha}_{N(g)}$ (both d-dimensional vectors), then $\mathcal{H}_d(\vec{\alpha}'_{N(g)}) = \mathcal{H}_d(\vec{\alpha}_{N(g)})$. $\mathcal{H}_d$ can be deterministic (in which case $\mathcal{H}_d(\vec{\alpha}_{N(g)}) \in \{0, 1\}$) or stochastic (in which case $\mathcal{H}_d(\vec{\alpha}_{N(g)}) \in [0, 1]$). Note that any two agents i and j of the same degree d have the same cascading failure function.

Thus an agent can either fail intrinsically (i.e. by himself) or as a result of the failures of a subset of his neighbors. Those neighbors who have failed may have done so intrinsically or as a result of the failure of a subset of their own neighbors.

2.3. Action Sets and Decision Time

In order to protect himself against the risk of failure, we allow an agent i to make a costly investment in *protection*. This is a *one-shot investment* that can be made *in anticipation* of a cascade of failures, which may take place in the future. This investment in protection is represented by an action a^i, which is part of a binary action set $\mathcal{A} = \{0, 1\}$. The latter represents the set of possible investments in protection against failure: $a^i = 1$ means that the agent invests in protection while $a^i = 0$ means that the agent remains unprotected. In an application to computer security, a^i can represent an investment in computer security solutions or anti-virus software. In applications to disease spread, a^i can represent vaccination, whereas in the case of airport security, a^i can represent an investment in security personnel or equipment. We assume throughout that $\mathcal{A}$ is the same for all agents. The exact effect of this action on an agent's actual failure risk will be formalized in Definition 5.

Note that this is a very convenient set-up in which to study strategic investments in protection against cascading failures. Indeed, we reduce the problem to a static game, in which agents simultaneously decide whether to invest in protection *in anticipation* of a cascade of failures that may take place in the future. Thus $\vec{\alpha}_{N(g)}$ reflects the state of each neighbor ('failed' or 'healthy') in steady state, when the cascade of failures has stabilized. This is a credible context in which to study the applications mentioned earlier. For example, flu vaccines tend to be taken once a year, in anticipation of a possible outbreak. Likewise, investments in airport security or computer security are costly, long-term decisions that are taken in anticipation of cascades of negative events. A static game, in which agents consider the steady state of a cascade of failures to estimate their own risk of failure, is thus a convenient set up to study this problem.

Normally, we would be interested in finding a Bayes-Nash equilibrium of this finite game. This however requires an agent to hold fairly sophisticated beliefs about $\vec{\alpha}_{N(g)}$, the vector of joint failures of his neighbors. An agent thus needs to formulate the joint probability distribution function of $\vec{\alpha}_{N(g)}$ by combining the measure P from which the network is drawn with the investments in protection made by other agents. This is highly non-trivial since, for example, an agent's neighbors may be interconnected and their failures may thus be correlated. This is a common problem in such network games; see, e.g. Adlakha et al. (2011), for models that face similar issues. In order to circumvent such complications and to impose a more realistic cognitive burden on the agents, we will develop a mean-field model as an approximation to this finite model. This is done the next section.

3. The Mean-Field Model

In this section, we develop a formal mean-field model as an approximation to the finite interaction setting described in the previous section. Our mean-field model is developed in two parts. First, we make two *mean-field assumptions* that simplify the decision problem faced by a single agent. Given these two assumptions, we show that the optimal response of an agent has a particularly simple structure. Next, we develop a consistency check: namely, the mean-field assumptions should arise from the strategies that agents choose. This combination of requirements—optimality and consistency—leads us to a formal definition of *mean-field equilibrium* for our game.

We make two mean-field assumptions: one to simplify how an agent reasons about the graph; and a second to simplify how an agent reasons about whether or not a neighbor will fail. Throughout this section, since we deal with a single agent, we suppress the agent index i.

First, since P is permutation invariant (cf. Section 2.2), we can define the *degree distribution* of P as the probability a node has degree d in a graph drawn according to P; we denote the degree distribution[6] by $f(d)$ for $d \geq 1$. Note that we are not interested in modeling agents of degree 0 (since they do not play a game) and we therefore always assume that $f(0) = 0$. We assume a countably infinite set of agents. An agent's degree d is drawn according to the degree distribution $f(d)$.

The first mean-field assumption formalizes the idea that agents reason about the graph structure in a simple way through the degree distribution.

Mean-Field Assumption 1. *Each agent conjectures that the degrees of his neighbors are drawn i.i.d. according to the edge-perspective degree distribution* $\tilde{f}(d) = \frac{f(d)d}{\sum_{d\geq 1} f(d)d}$.

The expression for $\tilde{f}(d)$ in the above assumption follows from a standard calculation in graph theory (see Jackson (2008) for more details). $\tilde{f}(d)$ is the probability that a neighbor

[6]Throughout the chapter, we use the term *degree distribution* to mean *degree density.* When referring to the *cumulative distribution function (CDF)*, we will do so explicitly.

has degree d. It therefore takes into account the fact that a higher-degree node has a higher chance of being connected to any agent and thus of being his neighbor.

The second mean-field assumption addresses how an agent reasons about the failure probability of his neighbors.

Mean-Field Assumption 2. *Each agent conjectures that each of his neighbors fails with probability $\alpha \in [0, 1]$, independently across neighbors.*

In the mean-field setting, the computation of an agent's expected cascading failure probability is considerably simplified. It can be directly defined in terms of α, as seen in the following definition.

Definition 3. *For any d, let the mapping $q_d : [0, 1] \to [0, 1]$ denote a degree-d agent's mean-field cascading failure probability, i.e. $q_d(\alpha)$ is the probability that an agent of degree d will fail as a result of a cascade of failures, given that his neighbors each fail independently with probability α.*

We make the following natural assumption.

Assumption 1. *For any d, $q_d(\alpha)$ is strictly increasing and continuous in α. Moreover, we explicitly set $q_0(\alpha) = 0$ and thus an agent with no neighbors cannot fail as a result of a cascade of failures.*

The actual expression for $q_d(\alpha)$ depends on the type of cascade we are considering. We will consider only a situation when $q_d(\alpha)$ is a non-decreasing function of d. In other words, the cascading failure risk is higher when an agent has more connections.[7]

The mean-field framework allows us to make assumptions directly on $q_d(\alpha)$ instead of making them on the actual failure mechanism $\mathcal{H}_d$, defined in Section 2.2. We now give some examples.

3.0.1. Example 1

(Malware or virus spread) Let a computer be infected by a direct hacking attack with probability p. Assume that malware[8] (i.e. computer viruses) can spread from computer to computer according to a general contact process: if a neighbor is infected, then the computer will be infected with probability r. Then $q_d(\alpha) = 1 - (1 - r\alpha)^d$. Indeed, each neighbor is infected with probability α and this infection spreads independently across each edge with probability r. This contact process can also serve as a model for the spread of viruses among human populations. Note that the parameter r models the virulence or infectiousness of the process: given that a neighbor is infected, r is the probability[9] that he will infect the agent.

[7] In the appendix, we also study the case where $q_d(\alpha)$ is a decreasing function of d. This can model a form of diversification of failure risk across neighbors.

[8] For models developed specifically for malware, see Lelarge and Bolot (2008b), Lelarge and Bolot (2008a) or Lelarge and Bolot (2009).

[9] In Fig. 1 and Fig. 2, r was assumed to be 1 for simplicity of exposure.

3.0.2. Example 2

(Airport security risk) The contact process of Example 1 can also be applied to airport security, where failure can mean a security event, i.e. a failure to stop a suspicious luggage from getting on a flight. The agents represent airports, and the edges linking them represent flights. The suspicious luggage can then cascade, i.e. travel to one or more other airports, exposing them to security events.

In the next two sections we develop both the optimal response of an agent to the environment described by the mean-field assumptions, as well as the consistency check that α should satisfy given the strategic choices of the agents.

3.1. Optimal Response

Again, we assume a binary action set $\mathcal{A} = \{0, 1\}$, where $a = 1$ means an investment in protection against failure while $a = 0$ means no investment in protection against failure. Since an agent of degree d either fails intrinsically with probability p or in a cascade with probability q_d, we can define his *total probability of failure* as follows.

Definition 4 (Total probability of failure)**.** *The total probability of failure of an agent of degree d is*

$$\beta_d = p + (1 - p)q_d \tag{1}$$

We study a static setting, in which agents make decisions simultaneously, *in anticipation* of a cascade of failures that may happen in the future. Therefore each agent is in state *healthy* when he chooses an action $a \in \mathcal{A}$ representing a costly investment in protection against failure. We now describe how this action affects an agent's failure probability.

Definition 5. *Let the mapping $\mathcal{B} : [0, 1] \times [0, 1] \times \mathcal{A} \to [0, 1]$ denote the effective failure probability of an agent. We assume that $\mathcal{B}(p, q_d, a)$ is continuous in all arguments, increasing in p, linear and increasing in q_d and that it is decreasing and convex in a.*

Thus, $\mathcal{B}(p, q_d, a)$ is the total failure probability of an agent (defined in (1)) when he has invested a in protection against failure. Note that this definition allows this action to operate separately on p and q_d, as will be seen in Section 4. We can now state an agent's expected utility function, which will capture his decision problem.

Definition 6. *A mean-field strategy $\mu : \mathbb{N}^+ \to [0, 1]$ is a scalar-valued function that specifies, for every $d > 0$, the probability that an agent of degree d invests in protection. We denote by $\mathcal{M}$ the set of all mean-field strategies.*

Note that $\mathcal{M} = [0, 1]^\infty$, the space of $[0, 1]$-valued sequences. Throughout, we endow $\mathcal{M}$ with the product topology and $[0, 1]$ with the Euclidean topology.

A degree-d agent's expected utility function is now given by

$$U_d(a, \alpha) = -V \cdot \mathcal{B}(p, q_d(\alpha), a) - C \cdot a \tag{2}$$

where $C > 0$ is the cost of investing in protection, $V > 0$ is the value that is lost in the event of failure and $\mathcal{B}(\cdot, \cdot, \cdot)$ is the *expected effective failure probability* (cf. Definition 5).

This utility function thus captures the tradeoff between the expected loss $V \cdot \mathcal{B}(p, q_d, a)$ and the cost[10] C of investing in protection. Notice again that an agent's expected utility depends on the actions of others only through the cascading failure probability $q_d(\alpha)$, since they will affect the probability of failure α of a randomly-picked neighbor. Note also that the expected utility function $U_d(\cdot, \cdot)$ depends on the agent's degree d but not on his identity i. Therefore, any two agents i and j who have the same degree have the same utility function. From the assumptions on $\mathcal{B}$, U_d is continuous in all arguments and concave in a. An agent is risk-neutral and will thus maximize this expected utility function by choosing the appropriate action a.

It is now straightforward to solve for the optimal strategy of an agent of degree d: an agent invests in protection, does not invest, or is indifferent if $U_d(1, \alpha)$ is greater than, less than, or equal to $U_d(0, \alpha)$, respectively. We thus have the following definition.

Definition 7. *Let $\mathcal{S}_d(\alpha) \subset [0,1]$ denote the set of optimal responses for a degree-d agent given α; i.e.:*

$$\begin{aligned} U_d(1, \alpha) > U_d(0, \alpha) &\implies \mathcal{S}_d(\alpha) = \{1\}; \\ U_d(1, \alpha) < U_d(0, \alpha) &\implies \mathcal{S}_d(\alpha) = \{0\}; \\ U_d(1, \alpha) = U_d(0, \alpha) &\implies \mathcal{S}_d(\alpha) = [0,1]. \end{aligned}$$

Let $\mathcal{S}(\alpha) \subset \mathcal{M}$ denote the set of optimal mean-field strategies given α; i.e.,

$$\mathcal{S}(\alpha) = \prod_{d \geq 1} \mathcal{S}_d(\alpha).$$

Note that at least one optimal response always exists and is essentially uniquely defined, except at those degrees where an agent is indifferent.

3.2. Consistency

We will now develop a consistency check that guarantees that a randomly-picked neighbor's failure probability α is consistent with the mean-field strategy μ played by the population.

Definition 8. *Let the function $\mathcal{T} : \mathcal{M} \to [0,1]$ be defined as*

$$\mathcal{T}(\mu) = \{\alpha : \mathcal{F}(\mu, \alpha) = \alpha\} \tag{3}$$

where $\mathcal{F} : \mathcal{M} \times [0,1] \to [0,1]$ is the function

$$\mathcal{F}(\mu, \alpha) = \sum_{d \geq 1} \tilde{f}(d) \mathcal{B}(p, q_{d-1}(\alpha), \mu(d)) \tag{4}$$

[10]The cost of investing in protection may represent the price of airport security equipment or computer security solutions. It may also represent the possible side-effects that may be associated with a vaccine (e.g. The Economist (4 February 2015)).

In the above definition, $\mathcal{T}(\mu)$ represents the mean-field failure probability of a randomly-picked neighbor, given that strategy μ is played by other agents. It is important to note that for some μ, $\mathcal{T}(\mu)$ is a fixed-point of $\mathcal{F}(\mu, \alpha)$ in α. $\mathcal{F}(\mu, \alpha)$ is the failure probability of a randomly-picked neighbor given that agents play strategy μ and this neighbor's other neighbors fail with probability α. A fixed point $\alpha = \mathcal{F}(\mu, \alpha)$ ensures that α is the same across all agents.

Note that an agent does not internalize the effect of his own failure on others when forming his belief about the failure risk of a neighbor. Hence the presence of $q_{d-1}(\alpha)$ on the right-hand side of (4) instead of $q_d(\alpha)$: the cascading failure risk of a given neighbor of degree d is only due to his $d-1$ other neighbors.

To guarantee that $\mathcal{T}(\mu)$ is unique (i.e. that (4) has a unique fixed point), we make the following assumption on $\mathcal{F}(\mu, \alpha)$:

Assumption 2. *For any $\mu \in \mathcal{M}$, $\mathcal{F}(\mu, \alpha)$ has a unique fixed point in α.*

Note that Assumption 2 is not particularly stringent. This is easy to verify in the contact process models of Example 1 and Example 2.

3.3. Mean-Field Equilibrium

We now formally define the equilibrium concept and state our first theorem.

Definition 9 (Mean-field equilibrium)**.** *A mean-field strategy μ^* constitutes a mean-field equilibrium (MFE) if $\mu^* \in \mathcal{S}(\mathcal{T}(\mu^*))$.*

This equilibrium definition ensures that the mean-field assumptions arise from the optimal strategies chosen by agents. Also note that to any MFE μ^*, there corresponds a unique equilibrium neighbor failure probability $\alpha^* = \mathcal{T}(\mu^*)$.

Theorem 1 (Existence)**.** *There exists a mean-field equilibrium.*

The computation of this equilibrium is considerably simpler than the computation of a Bayes-Nash equilibrium in a finite game. In fact, α^* is obtained from a one-dimensional fixed-point equation resulting from the composition of $\mathcal{T}$ and $\mathcal{S}$, i.e. $\alpha^* = \mathcal{T}(\mathcal{S}(\alpha^*))$. μ^* is then found from the map $\mathcal{S}(\alpha^*)$ (cf. Definition 7). In the finite game with n agents, μ would be a $(n-1)$-dimensional object and so would the equilibrium fixed-point condition[11]. However, it is worth mentioning that existence no longer follows from a standard Nash argument as it would in a finite game, since we now have a countably infinite set of agents. Theorem 1 is thus a non-trivial result.

4. Characterizing MFE

In this section, we will study two classes of games in which agents make decisions to invest in protection. We will start with games of *total protection*, in which an agent's investment decreases his total risk of failure. We will then proceed with games of *self protection*, in which an agent's investment in protection only protects him against his own intrinsic risk of failure.

[11] The reader is referred to Chapter 3 of Leduc (2014) for a more detailed treatment of this finite game.

4.1. Games of Total Protection

We start with the following definition.

Definition 10 (Games of total protection)**.** *In a game of total protection, the mean-field effective failure probability has the following form*

$$\mathcal{B}(p, q_d(\alpha), a) = \Big(p + (1-p)q_d(\alpha)\Big) \cdot (1 - ka) \tag{5}$$

for some $k \in [0, 1]$ *and*

$$\mathcal{F}(\mu, \alpha) = \sum_{d \geq 1} \tilde{f}(d) \Big(p + (1-p)q_{d-1}(\alpha)\Big) \cdot (1 - k\mu(d)) \tag{6}$$

In games of *total protection*, as can be seen in (5), an agent's investment in protection decreases his total probability of failure $p + (1-p)q_d(\alpha)$. The parameter k governs the effectiveness of the investment in protection. The higher k, the more an investment in protection reduces the failure probability.

Examples of models covered by this class are malware and the investment in anti-virus or computer security solutions or the spread of contagious diseases and the decision to vaccinate. The action protects both against the intrinsic failure risk and the cascading failure risk. Vaccination, for example, protects against both the risk of being infected by non-human and human sources. It is also the case for standard anti-virus software, which protects against the risk of being infected by malware through connections with other computers.

Games of total protection are *submodular*. In other words, they are of *strategic substitutes*: the more other agents invest in protection (the lower α), the less an agent has an incentive to invest in protection. A nice property of games of total protection is that they have a unique equilibrium. This leads to the following theorem.

Theorem 2 (Uniqueness)**.** *In a game of total protection, the mean-field equilibrium* μ^* *is unique.*

To further characterize the nature of this equilibrium, we introduce some definitions.

Definition 11 (Upper-threshold strategy)**.** *A mean-field strategy* μ *is an upper-threshold strategy if there exists* $d_U \in \mathbb{N}^+ \bigcup \{\infty\}$*, such that:*

$$\begin{aligned} d < d_U &\implies \mu(d) = 0; \\ d > d_U &\implies \mu(d) = 1. \end{aligned}$$

Thus, under an upper-threshold strategy, agents with degrees *above* a certain threshold invest in protection whereas agents with degrees *below* that threshold do not invest.

Definition 12 (Lower-threshold strategy)**.** *A mean-field strategy* μ *is a lower-threshold strategy if there exists* $d_L \in \mathbb{N}^+ \bigcup \{\infty\}$*, such that:*

$$\begin{aligned} d > d_L &\implies \mu(d) = 0; \\ d < d_L &\implies \mu(d) = 1. \end{aligned}$$

Likewise, under a lower-threshold strategy, agents with degrees *below* a certain threshold invest in protection whereas agents with degrees *above* that threshold do not invest.

Note that the definitions above do not place any restriction on the strategies *at* the thresholds d_L and d_U themselves; we allow randomization at these thresholds. The next result shows that any equilibrium can be characterized by a threshold strategy.

Theorem 3. *In a game of total protection, the unique MFE μ^* is an upper-threshold equilibrium. That is, μ^* is an upper-threshold mean-field strategy.*

The intuition behind this result is that, higher-degree agents are more exposed to cascading failures than lower-degree agents, thus making an investment in *total* protection relatively more rewarding.

The implications of this theorem are important as higher-degree agents are more *central*. Indeed, in a mean-field interaction setting, degree centrality is an appropriate measure of centrality. This result can thus be seen as a satisfactory outcome since more central agents have higher incentives to internalize the risk they impose on the system. In equilibrium, the total cost of protection is thus born by those who have a maximal effect on decreasing $\alpha = \mathcal{T}(\mu)$. For example, in the case of malware, agents with a higher level of interaction (higher degree) have a higher incentive to invest in computer security (i.e. anti-virus software). The same principle applies in the case of human-born viruses: individuals who interact more have a higher incentive to get vaccinated.

4.1.1. Comparing MFE to Nash equilibrium

We compare the equilibrium outcome in the mean-field setting analyzed so far with the equilibrium outcomes in a finite model, where agents have full knowledge of the network topology. We will see that the mean-field model allows for the advantage of selecting a unique equilibrium outcome and of eliminating implausible equilibria that could arise under full information. In Fig. 3, we illustrate the difference between the MFE and Nash equilibria in the particular cases of star and circular networks.[12] In Fig. 3a), we can see that on a star network, there is a unique upper-threshold MFE, in which the center agent invests in protection while the periphery agents free ride. This is because the higher-degree agent is exposed to the potential failure of four neighbors, whereas the periphery agents are exposed to the failure of only one neighbor. In Fig. 3b), on the other hand, we see that there are many possible Nash equilibria that do not depend on an agent's degree, but rather on an agent's particular position in the network. We include two of them: one in which the periphery agents free ride and one in which the center agent free rides. In the case of a circular network, we see in Fig. 3c) that the unique MFE is a symmetric mixed strategy. In fact all agents have degree 2 and thus are exposed to the same ex-ante cascade risk. They therefore all play the same mixed strategy. In Fig. 3d), on the other hand, we see that there are again multiple Nash equilibria that do not depend on an agent's degree, but rather on an agent's particular position in the network. These different Nash equilibria do not have a meaningful difference, but add to the complexity of the equilibrium set. In fact, computing

[12]The mean-field model assumes a countably infinite number of agents, but it is still useful to analyze the behavior of MFEs on finite star and circular networks, as in Figs. 3 and 4. Those stars and circles can be understood as smaller components of a larger, possibly infinite network.

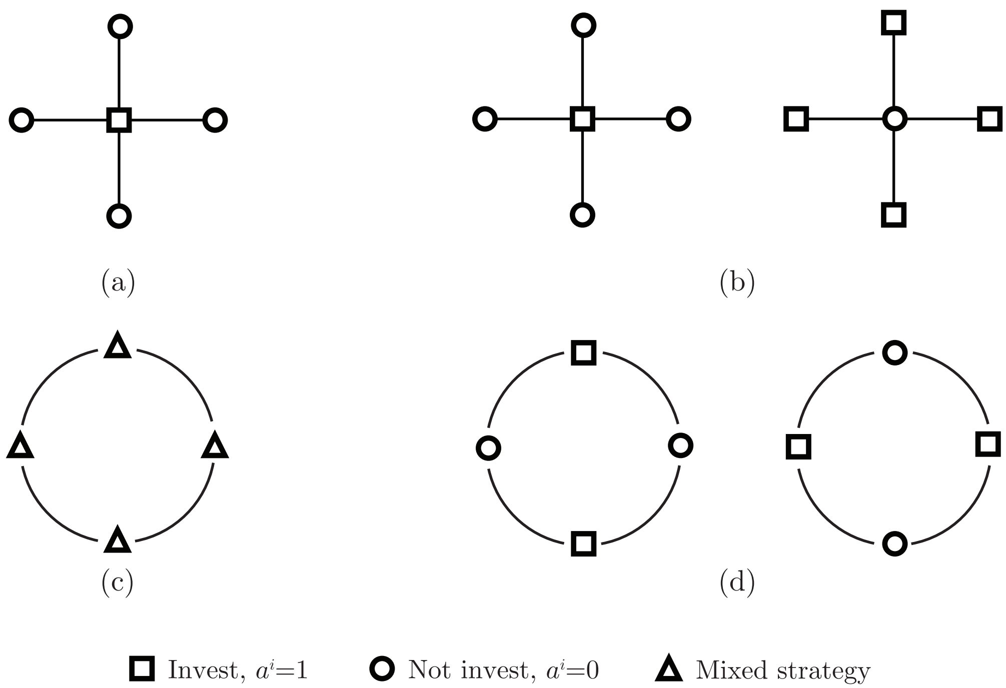


Figure 3: Comparison of MFE and Nash Equilibria in a Game of Total Protection on Star and Circular Networks. *The cascade process is assumed to follow a contact process as in Example* 1. *Model parameters are* $C = 0.6$, $k = 1$, $r = 1$, $p = 0.5$. *For the star network, the neighbor degree distribution is assumed to be* $\tilde{f}(1) = 0.5$ *and* $\tilde{f}(4) = 0.5$. *For the circular networks, the neighbor degree distribution is assumed to be* $\tilde{f}(2) = 1$. *(a) shows the unique MFE on a star network. (b) shows two of the possible Nash equilibria on a star network. (c) shows the unique MFE on a circular network. (d) shows two of the possible Nash equilibria on a circular network.*

Nash equilibria on a network with full information can pose computational challenges (see for example, Kearns (2007)).

In the next section, we study the other class of games: Games of *self protection*.

4.2. Games of Self Protection

We start with the following definition.

Definition 13 (Games of self protection)**.** *In a game of self protection, the mean-field effective failure probability has the following form*

$$\mathcal{B}(p, q_d(\alpha), a) = p \cdot (1 - ka) + (1 - p \cdot (1 - ka)) \cdot q_d(\alpha) \tag{7}$$

for some $k \in [0, 1]$ *and*

$$\mathcal{F}(\mu, \alpha) = \sum_{d \geq 1} \tilde{f}(d) \Big(p \cdot (1 - k\mu(d)) + \big(1 - p \cdot (1 - k\mu(d))\big) \cdot q_{d-1}(\alpha) \Big) \tag{8}$$

In games of *self protection*, as can be seen in (7), an agent's investment in protection only decreases his intrinsic probability of failure p. It has no effect on his cascading failure probability $q_d(\alpha)$. Again, the parameter k governs the effectiveness of the investment in protection corresponding to the action a.

Examples of applications covered by this class of games include airport security when luggage/passengers are only scanned at the originating airport. Airports then otherwise rely on each other's provision of security for transiting passengers/luggage.

Games of *self protection* are *supermodular*. In other words, they are of *strategic complements*: the more other agents invest in protection (the lower α), the more an agent has an incentive to invest in protection. Since games of self protection are effectively coordination games, there can be multiple equilibria. The next result shows that any equilibrium can be characterized by a threshold strategy. Moreover, the thresholds are reversed when compared to games total protection (cf. Theorem 3).

Theorem 4. *In a game of self protection, any MFE* μ^* *is a lower-threshold equilibrium. That is,* μ^* *is a lower-threshold mean-field strategy.*

Higher cascade risk thus leads to *lower* incentives to invest in protection. This is because an agent remains exposed to the failure risk of others irrespectively of whether she invests in protection. An investment in protection thus has lower returns as the cascading failure risk increases. An agent's cascading failure risk *increases* in degree, and thus higher-degree agents invest *less* in protection than lower-degree agents. The intuition is that higher-degree agents are more exposed to cascading failure risk than lower-degree agents, thus making an investment in their own *self* protection relatively less rewarding.

The fact that, in games of self-protection, the incentives are reversed has important implications. In fact, the more central (higher-degree) agents have a lesser incentive to invest in protection even though they are more vulnerable *and* more dangerous, i.e. they are hubs through which cascading failures can spread. More central agents thus have lower incentives to internalize the risk they impose on the system, pointing to an inefficient outcome. Moreover, in equilibrium, the total cost of protection is born by lower-degree agents: those who have the smallest effect on decreasing $\alpha = \mathcal{T}(\mu)$.

For example, in the model of airport security risk described in Example 2, an airport that interacts with a high number of other airports has smaller incentives to invest in its own security, since it remains exposed to a high risk of being hit by an event coming from a connecting flight. This, as before, is assuming that the passengers/luggage are only inspected at their point of origin and not at points of transit. The next proposition formalizes this implication by stating that, in equilibrium, the failure risk of an agent is monotone in degree.

Proposition 1 (Risk)**.** *Let $a_d \in \mu^*(d)$. In a game of self protection the equilibrium mean-field effective failure probability $\mathcal{B}(p, q_d(\alpha^*), a_d)$ is non-decreasing in d.*

The next result is also a consequence of the monotonicity of equilibrium strategies stated in Theorem 4.

Proposition 2 (Welfare I)**.** *Let $a_d \in \mu^*(d)$. In a game of self protection, the equilibrium expected utility $U_d(a_d, \alpha^*)$ is non-increasing in d.*

In games of self-protection, agents effectively *pool* their investments in protection and, as said earlier, there can be multiple equilibria. These equilibria can however be ordered by level of investment. Suppose there are m possible equilibria. Then, they can be ordered in the following way

$$\mu_1^* \preceq \mu_2^* \preceq ... \preceq \mu_m^*.$$

Since (8) is decreasing in μ, it follows that $\alpha_1^* \geq \alpha_2^* ... \geq \alpha_m^*$.

We then have a second welfare result.

Proposition 3 (Welfare II)**.** *In a game of self protection, let $\mu_k^* \preceq \mu_l^*$ be two equilibria ordered by level of investment. Then μ_l^* weakly Pareto-dominates μ_k^*.*

This result is not trivial. It effectively states that in the high investment equilibrium, the decrease in risk resulting from higher investments outweighs the cost of those investments. This is due to the positive externality stemming from the effect of pooled investments in protection, which reduce all agents' failure risk.

We can focus our attention on the minimum-investment equilibrium $(\underline{\mu}^*, \bar{\alpha}^*)$ and the maximum-investment equilibrium $(\bar{\mu}^*, \underline{\alpha}^*)$. In the former, $\bar{\alpha}^* = \mathcal{T}(\underline{\mu}^*)$ is actually maximal since agents invest least, while in the latter, $\underline{\alpha}^* = \mathcal{T}(\bar{\mu}^*)$ is actually minimal since agents invest most. From Proposition 3, agents playing the minimum-investment equilibrium can be thus considered a coordination failure.

4.2.1. Comparing MFE to Nash equilibrium

We again compare the equilibrium outcomes in the mean-field setting with the equilibrium outcomes in a finite model, where agents have full knowledge of the network topology. In Fig. 4, we illustrate the difference between MFE and Nash equilibria in the particular case of a star network. In Fig. 4a), we see that there are three possible MFE's: a low-investment equilibrium, in which nobody invests in protection; an intermediate-investment equilibrium, in which the center agent does not invest while the periphery agents invest; and a high-investment equilibrium, in which all agents invest. All equilibria are lower-threshold strategies (the first and third ones being trivially so). Since from Proposition 3, a lower-investment equilibrium is Pareto dominated by a higher-investment equilibrium, the first

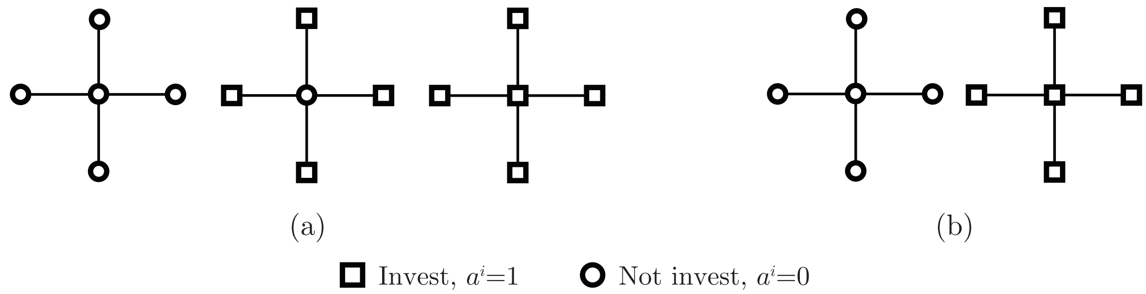


Figure 4: Comparison of MFE and Nash Equilibria in a Game of Self Protection on Star Networks. *The cascade process is assumed to follow a contact process as in Example* 1. *Model parameters are* $C = 0.4$, $k = 1$, $r = 1$, $p = 0.5$. *The neighbor degree distribution is assumed to be* $\tilde{f}(1) = 0.86$ *and* $\tilde{f}(4) = 0.14$. *(a) shows the three MFEs. (b) shows the two pure strategy Nash equilibria.*

two equilibria can be regarded as coordination failures. In Fig. 4b), we see that there are two pure strategy Nash equilibria corresponding to the no-investment equilibrium and the full-investment equilibrium. Again, we see that equilibrium behavior is not related to an agent's degree.

In Fig. 5, we illustrate Theorems 3 and 4 on a complex network. We see how the upper (resp. lower) threshold nature of MFE's in games of total (resp. self) protection affects the spread of cascading failures differently.

4.3. Comparative Statics

An advantage of the mean-field setting is that we can relate equilibrium behavior to network properties as captured by the edge-perspective degree distribution $\tilde{f}(d)$. We can then ask questions such as "does a higher level of connectedness[13] increase or decrease the incentives to invest in protection?" This is examined in the next two propositions.

Proposition 4. *Let* $\underline{\mu}^*$ *and* $\bar{\mu}^*$ *be the minimum- and maximum-investment equilibria in a game of self protection, when the edge-perspective degree distribution is* $\tilde{f}$. *Then, a first-order distributional shift* $\tilde{f}' \succ \tilde{f}$ *results in* $\underline{\mu}'^* \preceq \underline{\mu}^*$ *and* $\bar{\mu}'^* \preceq \bar{\mu}^*$ *and thus in* $\bar{\alpha}'^* \geq \bar{\alpha}^*$ *and* $\underline{\alpha}'^* \geq \underline{\alpha}^*$.

Thus in a game of self protection, a higher level of connectedness leads to *lower* incentives to invest in protection: each of the new maximum- and minimum-investment equilibria are weakly dominated by the corresponding equilibria in the less connected network. The intuition behind this result is that an agent is more likely to be connected to a high-degree neighbor (high-risk and unprotected). This increases the agent's cascading failure risk and therefore lowers the incentive to invest in *self* protection. We note that in equilibrium, the corresponding neighbor failure probabilities are larger, i.e. $\bar{\alpha}'^* \geq \bar{\alpha}^*$ and $\underline{\alpha}'^* \geq \underline{\alpha}^*$.

When cascading failures follow a contact process as in Example 1 (cf. Section 3), it is interesting to study the effect of a change in the infectiousness parameter r on mean-field equilibria. The following two propositions illustrate that a change in r has opposite effects, depending on whether the game is one of self protection or total protection.

[13]Note that by a higher level of connectedness, we mean an edge-perspective degree distribution placing higher mass on higher-degree nodes. We do not mean the presence of short paths between any two nodes.

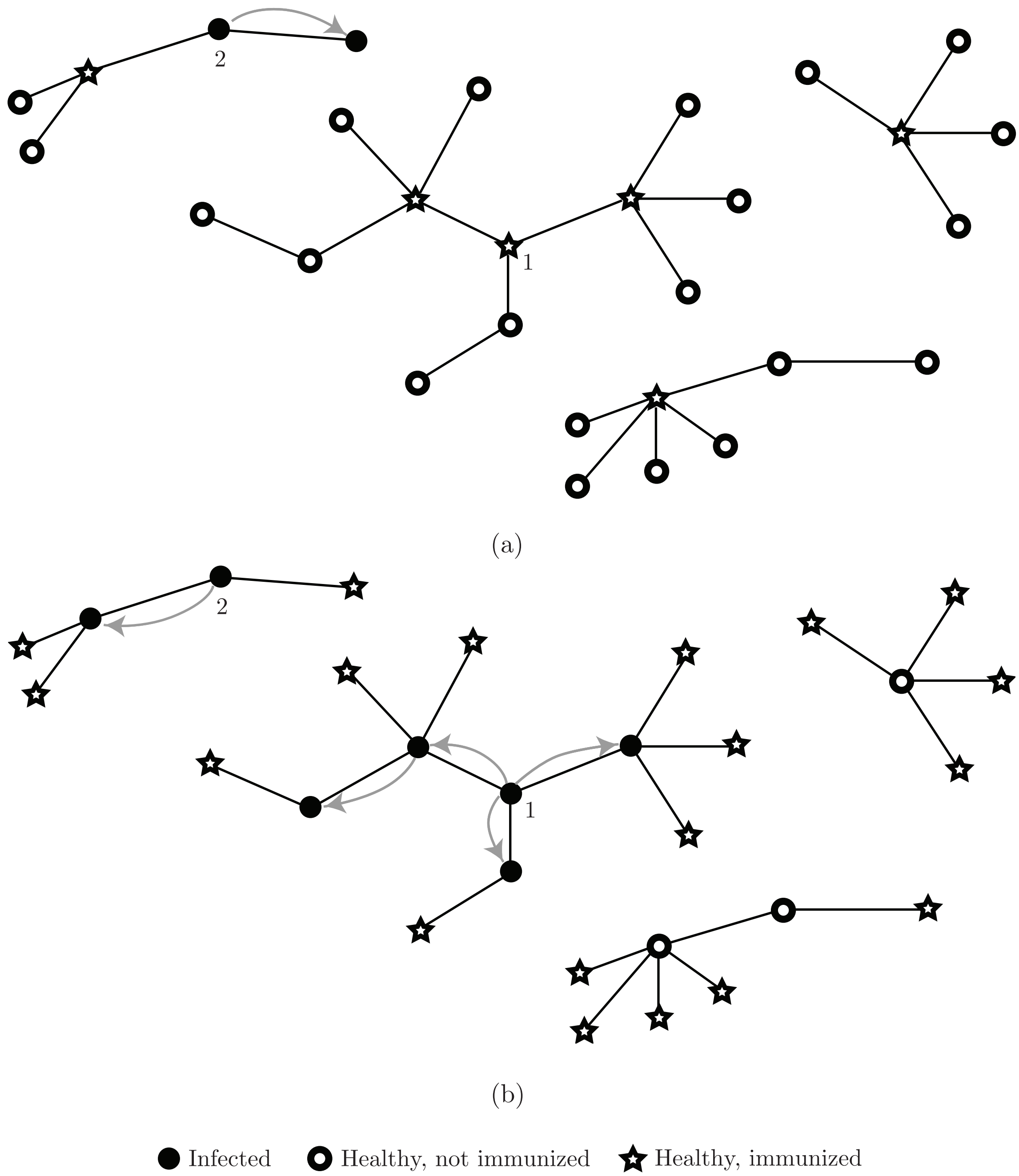


Figure 5: Illustration of Theorems 3 and 4 on a complex network with the cascading process of Fig. 1: possible equilibrium strategies in (a) a game of total protection and (b) a game of self protection. In (a), we see that the upper-threshold strategy insulates contagion hubs whereas in (b) we see that the lower-threshold strategy insulates periphery nodes and leaves contagion hubs vulnerable.

Proposition 5. *Let $\underline{\mu}^*$ and $\bar{\mu}^*$ be the minimum- and maximum-investment equilibria in a game of self protection with cascading failures following a contact process with infectiousness parameter r, as in Example 1. Then, $r' > r$ results in $\underline{\mu}'^* \preceq \underline{\mu}^*$ and $\bar{\mu}'^* \preceq \bar{\mu}^*$ and thus in $\bar{\alpha}'^* \geq \bar{\alpha}^*$ and $\underline{\alpha}'^* \geq \underline{\alpha}^*$.*

Thus in a game of self protection, when cascading failures follow a contact process as in Example 1, a higher level of infectiousness creates *lower* incentives for agents to invest in protection: the initial increase in α caused by higher infectiousness causes an even greater increase in α as a result of strategic interactions. The situation is very different in a game of total protection, as shown in the next result.

Proposition 6. *Let μ^* be the unique equilibrium in a game of total protection with cascading failures following a contact process with infectiousness parameter r, as in Example 1. Then, an increase $r' > r$ in infectiousness results in an equilibrium μ'^* with $\mu'^* \succeq \mu^*$ and $r'\alpha'^* \geq r\alpha^*$.*

In a game of total protection, a higher level of infectiousness creates *higher* incentives for agents to invest in protection. This investment in protection is however not enough to counter the increase in $r\alpha$ caused by a higher level of infectiousness. This is because agents free-ride on the protection provided by others and thus an increase in $r\alpha$ cannot be completely compensated. The next result examines the effect of an increase in the parameter k, which governs the extent of the protection resulting from an investment.

Proposition 7. *Let $\underline{\mu}^*$ and $\bar{\mu}^*$ be the minimum- and maximum-investment equilibria in a game of self protection with parameter k. Then, $k' > k$ results in $\underline{\mu}'^* \succeq \underline{\mu}^*$ and $\bar{\mu}'^* \succeq \bar{\mu}^*$ and thus in $\bar{\alpha}'^* \leq \bar{\alpha}^*$ and $\underline{\alpha}'^* \leq \underline{\alpha}^*$.*

Thus in a game of self protection, an increase in the protection associated with an investment results in a higher investment.

4.4. Effect of Global Feedback

We have only examined the case where the only network effect is of a local nature. That is, a utility function depends on others only through the failure probability of one's neighbors. In reality, global effects might also influence an agent's utility. For instance, prices of vaccines, computer security solutions or airport security equipment might be affected by the global demand for them. Likewise, if protection is provided under the form of insurance[14], the insurance premium might depend on the overall failure level in the population, which itself depends on the overall level of investment in protection.

In this section, we introduce such global effects to the model developed in the previous sections. We focus on global feedback through the cost of protection, which can take the form of a price to be paid.

We will introduce the following function, which maps a mean-field strategy μ to the corresponding probability that a randomly-picked agent invests in protection:

[14]See, for example, Reuters (12 October 2015): "Cyber insurance premiums rocket after high-profile attacks". Oct 12, 2015. Reuters. The reader may also see Johnson et al. (2011) and Lelarge and Bolot (2009) for some work on insurance provision.

Definition 14. *Let the function* $\mathcal{W} : \mathcal{M} \to [0, 1]$ *be defined as:*

$$\mathcal{W}(\mu) = \sum_{d \geq 1} f(d)\mu(d) \tag{9}$$

Thus to each mean-field strategy μ corresponds a fraction $\omega = \mathcal{W}(\mu)$ of agents who invest in protection. Furthermore, it is easy to notice that this function $\mathcal{W}$ increases in μ.

4.4.1. Global Feedback Through Price

We will explore the set up in which the cost of protection is influenced by the global demand for it. Namely, when the cost of protection depends monotonically on total demand: $C_g = C \cdot g(\omega)$, where $g(\cdot)$ is either an increasing or decreasing continuous function of the total fraction of people $\omega = \mathcal{W}(\mu)$ willing to invest in protection. In the following examples, we outline two situations that can be modeled by the function $g(\cdot)$.

Example 1 ($g(\cdot)$ increasing). This case corresponds to the situation where the product is scarce or there are global congestion effects. For instance, a vaccine might be produced in limited quantity and thus, the more people demand it, the harder it may be to obtain it, which will have an increasing effect on price.

Example 2 ($g(\cdot)$ decreasing). This corresponds to the case of economies of scale. For instance, a new airport security technology might require significant initial R & D investments. Producing it in large numbers may thus lead to a lower cost per unit, which may lower the price.

We will slightly modify a degree-d agent's expected utility function in order to introduce the global feedback effect:

$$U_d(a, \alpha, \omega) = -V \cdot \mathcal{B}(p, q_d(\alpha), a) - C \cdot g(\omega) \cdot a \tag{10}$$

Note that the mean-field cascading failure probability $q_d(\alpha)$ does not depend explicitly on the global fraction of agents who invest in protection, as it is solely driven by local effects. It is also important to mention that the introduction of a global externality does not affect $\mathcal{T}(\mu)$, as defined earlier in (3). The latter function was defined to be the failure probability α of a randomly-picked neighbor, which does not depend explicitly on the total fraction of agents investing in protection ω.

We will now modify the optimality condition in order to ensure that this fraction ω arises in equilibrium. We can redefine the set of optimal responses as follows:

Definition 15. *Let* $\mathcal{S}_d(\alpha, \omega) \subset [0, 1]$ *denote the set of optimal responses for a degree-d agent given* α *and* ω*; i.e.:*

$$\begin{aligned}
U_d(1, \alpha, \omega) > U_d(0, \alpha, \omega) &\implies \mathcal{S}_d(\alpha, \omega) = \{1\}; \\
U_d(1, \alpha, \omega) < U_d(0, \alpha, \omega) &\implies \mathcal{S}_d(\alpha, \omega) = \{0\}; \\
U_d(1, \alpha, \omega) = U_d(0, \alpha, \omega) &\implies \mathcal{S}_d(\alpha, \omega) = [0, 1].
\end{aligned}$$

Let $\mathcal{S}(\alpha, \omega) \subset \mathcal{M}$ denote the set of optimal mean-field strategies given α and ω; i.e.,

$$\mathcal{S}(\alpha, \omega) = \prod_{d \geq 1} \mathcal{S}_d(\alpha, \omega).$$

We now only need to slightly modify the equilibrium condition:

Definition 16 (Mean-field equilibrium with global effects)**.** *A mean-field strategy μ^* constitutes a mean-field equilibrium (MFE) if $\mu^* \in \mathcal{S}(\mathcal{T}(\mu^*), \mathcal{W}(\mu^*))$.*

4.4.2. Equilibrium Characterization with Global Feedback

It turns out that under both global and local network effects, the main results that were stated in the previous sections of the paper still hold. We thus state the following more general existence result.

Theorem 5 (Existence)**.** *There exists a mean-field equilibrium in the game with global price feedback.*

Under both global and local externalities, the equilibrium condition now involves a two-dimensional fixed-point equation (in α and ω), as opposed to a one-dimensional fixed point in α only (cf. Section 3.3). Moreover, it turns out that the threshold nature of the equilibria, derived earlier in the paper (cf. Theorems 3 and 4), still holds in the presence of a global externality. This more general result is formalized in the following theorem.

Theorem 6 (Threshold Strategies)**.** *The threshold characterization of equilibria is robust to the introduction of a global price feedback. In the presence of both global and local externalities, the equilibrium is of: (1) an upper-threshold nature for a game of total protection; (2) a lower-threshold nature for a game of self protection.*

Moreover, it turns out that the equilibrium uniqueness result for games of total protection (cf. Theorem 2) also holds, provided that $g(\cdot)$ is increasing. This is formalized below.

Theorem 7 (Uniqueness)**.** *In a game of total protection with global price feedback, the mean-field equilibrium μ^* is unique if $g(\cdot)$ is an increasing function.*

As before, there can be multiple equilibria for games of self protection.

5. Conclusion

In this paper, we developed a framework to study the strategic investment in protection against cascading failures in networked systems. Agents connected through a network can fail either intrinsically or as a result of a cascade of failures which may cause their neighbors to fail. To impose a realistic cognitive burden on agents and to ease the computation and characterization of equilibria, we developed a mean-field model in which agents choose whether to invest in costly protection against failure. The assumption of independence across neighbors (local tree-like independence) allows for a tractable way to express an agent's expected cascading failure probability.

We studied two broad classes of games covering a wide range of applications. We showed that equilibrium strategies are monotone in degree (i.e. in the number of neighbors an agent has on the network) and that this monotonicity is reversed depending on whether (i) an investment in protection insulates an agent against the risk of failure of his neighbors (games of total protection) or (ii) only against his own intrinsic risk of failure (games of self protection). The first case covers the important examples of vaccination and computer security solutions, in which it is the *more* connected agents who have higher incentives to invest in protection. The second case, on the other hand, covers airport security, in which it is the *less* connected agents who have higher incentives to invest in protection. Our analysis reveals that it is the nature of strategic interactions (strategic substitutes/complements), combined with a network structure that leads to such strikingly different equilibrium behavior in each case, with important implications for the system's resilience to cascading failures.

More generally, the research presented in this paper contributes to advancing the work on systemic risk that involves strategic decisions within a networked system of agents. The current literature is generally descriptive and most often does not involve strategic decisions on the part of the agents.

Appendix A. Extensions

Appendix A.1. Continuous Action Sets

We have dealt with a binary action set $\mathcal{A} = \{0, 1\}$. It is however straightforward to extend our analysis to continuous action sets, e.g. $\mathcal{A} = [0, 1]$. This can model an investment in security that decreases the risk of failure in a proportional manner. All the monotonicity results expressed in the paper hold in such a setting as well.

Appendix B. Proofs

Theorem 1. Note that we endow $[0, 1]$ with the Euclidean topology.

For any $\alpha \in [0, 1]$ define the correspondence Φ by $\Phi(\alpha) = \mathcal{T}(\mathcal{S}(\alpha))$. Any fixed point α^* of Φ, with the corresponding $\mu^* \in \mathcal{S}(\alpha^*)$ such that $\mathcal{T}(\mu^*) = \alpha^*$ constitute a MFE. We thus need to show that the correspondence Φ has a fixed point. We employ Kakutani's fixed point theorem on the composite map $\Phi(\alpha) = \mathcal{T}(\mathcal{S}(\alpha))$.

Kakutani's fixed point theorem requires that Φ have a compact domain, which is trivial since $[0, 1]$ is compact. Further, $\Phi(\alpha)$ must be nonempty; again, this is straightforward, since both $\mathcal{S}$ and $\mathcal{T}$ have nonempty image.

Next, we show that $\Phi(\alpha)$ has a closed graph. We first show that $\mathcal{S}$ has a closed graph, when we endow the set of mean-field strategies with the product topology on $[0, 1]^\infty$. This follows easily: if $\alpha_n \to \alpha$, and $\mu_n \to \mu$, where $\mu_n \in \mathcal{S}(\alpha_n)$ for all n, then $\mu_n(d) \to \mu(d)$ for all d. Since $U_d(1, \alpha)$ and $U_d(0, \alpha)$ are continuous, it follows that $\mu(d) \in \mathcal{S}_d(\alpha)$, so $\mathcal{S}$ has a closed graph. Note also that with the product topology on the space of mean-field strategies, $\mathcal{T}$ is continuous: if $\mu_n \to \mu$, then $\mathcal{T}(\mu_n) \to \mathcal{T}(\mu)$ by the bounded convergence theorem.

To complete the proof that Φ has a closed graph, suppose that $\alpha_n \to \alpha$, and that $\alpha'_n \to \alpha'$, where $\alpha'_n \in \Phi(\alpha_n)$ for all n. Choose $\mu_n \in \mathcal{S}(\alpha_n)$ such that $\mathcal{T}(\mu_n) = \alpha'_n$ for all n. By Tychonoff's theorem, $[0, 1]^\infty$ is compact in the product topology; so taking subsequences if necessary, we can assume that μ_n converges to a limit μ. Since $\mathcal{S}$ has a closed graph, we know $\mu \in \mathcal{S}(\alpha)$. Finally, since $\mathcal{T}$ is continuous, we know that $\mathcal{T}(\mu) = \alpha'$. Thus $\alpha' \in \Phi(\alpha)$, as required.

Finally, we show that the image of Φ is convex. Let $\alpha_1, \alpha_2 \in \Phi(\alpha)$, and choose $\mu_1, \mu_2 \in \mathcal{S}(\alpha)$ such that $\alpha_1 = \mathcal{T}(\mu_1)$ and $\alpha_2 = \mathcal{T}(\mu_2)$. Since $\mathcal{F}$ is continuous in μ and since $\mathcal{T}$ is unique (this follows from Assumption 2), then $\mathcal{T}$ is continuous in μ. Now since $\mathcal{S}(\alpha)$ is convex, it follows that for any $\delta \in (0, 1)$,

$$\begin{aligned} \delta\mathcal{T}(\mu_1) + (1-\delta)\mathcal{T}(\mu_2) &\in [\min_{\mu\in\mathcal{S}(\alpha)} \mathcal{T}(\mu), \max_{\mu\in\mathcal{S}(\alpha)} \mathcal{T}(\mu)] \\ &= \Phi(\alpha) \end{aligned}$$

and thus $\delta\alpha_1 + (1-\delta)\alpha_2 \in \Phi(\alpha)$—as required.

By Kakutani's fixed point theorem, Φ possesses a fixed point α^*. Letting $\mu^* \in \mathcal{S}(\alpha^*)$ be such that $\mathcal{T}(\mu^*) = \alpha^*$, we conclude that μ^* is an MFE. □ □

Theorem 2. Consider the incremental expected utility for an agent of degree d, i.e.

$$\begin{aligned} \Delta U_d(\alpha) &= U_d(1, \alpha) - U_d(0, \alpha) \\ &= -V \cdot (p + (1-p)q_d(\alpha))(1-k) - C - (-V \cdot (p + (1-p)q_d(\alpha)) \\ &= V \cdot \Big(p + (1-p)q_d(\alpha)\Big)k - C \end{aligned} \tag{B.1}$$

We prove the theorem in a sequence of steps:

Step 1: For all $d \geq 1$, $\Delta U_d(\alpha)$ is strictly increasing in $\alpha \in [0, 1]$. This follows directly from Assumption 1.

Step 2: For all $d \geq 1$, and $\alpha' > \alpha$, $\mathcal{S}_d(\alpha') \succeq \mathcal{S}_d(\alpha)$.[15] This follows immediately from Step 1 and the definition of $\mathcal{S}_d$ in Definition 7.

Step 3: If μ', μ are mean-field strategies such that $\mu'(d) \geq \mu(d)$, then $\mathcal{T}(\mu') \leq \mathcal{T}(\mu)$. This follows from the fact that $\mathcal{F}(\mu, \alpha)$ (cf. (6) in Definition 10) is non-increasing in μ and that it is also continuous in both μ and α. Thus the unique fixed point $\bar{\alpha} = \mathcal{F}(\mu, \bar{\alpha})$ is non-increasing in μ. Therefore, $\mathcal{T}(\mu') \leq \mathcal{T}(\mu)$.

Step 4: Completing the proof. So now suppose that there are two mean-field equilibria (μ^*, α^*) and (μ'^*, α'^*), with $\alpha'^* > \alpha^*$. By Step 2, since $\mu^* \in \mathcal{S}(\alpha^*)$ and $\mu'^* \in \mathcal{S}(\alpha'^*)$, we have $\mu'^*(d) \geq \mu^*(d)$. By Step 3, we have $\alpha^* = \mathcal{T}(\mu^*) \geq \mathcal{T}(\mu'^*) = \alpha'^*$, a contradiction. Thus the α^* in any MFE must be unique, as required.

It then follows from the threshold nature of the equilibrium strategy μ^* (cf. Theorem 3) that to α^*, there corresponds a unique $\mu^* \in \mathcal{S}(\alpha^*)$ such that $\alpha^* = \mathcal{T}(\mu^*)$. □ □

Theorem 3. Consider now, $\Delta U_d(\alpha)$ as a function of the continuous variable d over the connected support $[1, \infty)$. From (B.1), we can write

$$\Delta U_d(\alpha) = V \cdot \Big(p + (1-p)q_d(\alpha)\Big)k - C$$

When $q_d(\alpha)$ is non-decreasing in d, for any $\alpha \in (0, 1)$, $\Delta U_d(\alpha)$ is a non-decreasing function of d. It follows that the inverse image of $(-\infty, 0)$ is $\emptyset$ if $\Delta U_1(\alpha) > 0$ or an interval $[1, x)$

[15] Here the set relation $A \preceq B$ means that for all $x \in A$ and $y \in B$, $x \leq y$.

where $x \geq 1$ otherwise. The integers in such intervals (i.e. $\emptyset \bigcap \mathbb{N}^+$ or $[1, x) \bigcap \mathbb{N}^+$) represent the degrees of agents for whom not investing in protection is a strict best response, i.e. $\{d : \mathcal{S}_d(\alpha) = \{0\}\}$. It follows that the degrees of agents for whom investing in protection is a strict best response (i.e. $\{d : \mathcal{S}_d(\alpha) = \{1\}\}$) are located at the rightmost extremity of the degree support.

Thus we may write $\mu(d) = 1$, for all $d > d_U$ and $\mu(d) = 0$, for all $d < d_U$. This is valid for any best-responding strategy μ and it is therefore valid for the equilibrium strategy μ^*. □ □

Theorem 4. In a game of self protection, consider now $\Delta U_d(\alpha)$ as a function of the continuous variable d over the connected support $[1, \infty)$. From (7) and (2), we can write

$$\begin{aligned} \Delta U_d(\alpha) &= U_d(1, \alpha) - U_d(0, \alpha) \\ &= -V \cdot (p(1-k) + (1 - p(1-k))q_d(\alpha)) - C + V \cdot (p + (1-p)q_d(\alpha)) \\ &= V \cdot (pk - pkq_d(\alpha)) - C \end{aligned}$$

When $q_d(\alpha)$ is non-decreasing in d, for any $\alpha \in (0, 1)$, $\Delta U_d(\alpha)$ is a non-increasing function of d. It follows that the inverse image of $(-\infty, 0)$ is an interval $[1, \infty)$ if $\Delta U_1(\alpha) < 0$ or an interval (x, ∞) where $x \geq 1$ otherwise. The integers in such intervals (i.e. $[1, \infty) \bigcap \mathbb{N}^+$ or $(x, \infty) \bigcap \mathbb{N}^+$) represent the degrees of agents for whom not investing in protection is a strict best response, i.e. $\{d : \mathcal{S}_d(\alpha) = \{0\}\}$. It follows that the degrees of agents for whom investing in protection is a strict best response (i.e. $\{d : \mathcal{S}_d(\alpha) = \{1\}\}$) are located at the leftmost extremity of the degree support.

Thus we may write $\mu(d) = 1$, for all $d < d_L$ and $\mu(d) = 0$, for all $d > d_L$. This is valid for any best-responding strategy μ and it is therefore valid for the equilibrium strategy μ^*. □ □

Proposition 1. From Theorem 4, when $q_d(\alpha)$ is non-decreasing in d, the equilibrium strategy μ^* is non-increasing in d and $q_d(\alpha^*)$ is non-decreasing in d. From (7), it thus follows that for $a_d \in \mu^*(d)$, $\mathcal{B}(p, q_d(\alpha^*), a_d)$ is non-decreasing in d (since $\mathcal{B}$ is non-decreasing in $q_d(\alpha^*)$ and non-increasing in a_d). □ □

Proposition 2. Note that from Theorem 4, $q_d(\alpha^*)$ is non-decreasing in d. Thus for $d' > d$, $a_{d'} \in \mu^*(d')$ and $a_d \in \mu^*(d)$, we have

$$U_d(a_d, \alpha^*) \geq U_d(a_{d'}, \alpha^*) \geq U_{d'}(a_{d'}, \alpha^*) \tag{B.2}$$

where the first inequality follows from $a_d \in \mu^*(d)$, while the second inequality follows from $q_d(\alpha^*)$ being non-decreasing in d. Thus $U_d(a_d, \alpha^*)$ is non-increasing in d. □ □

Proposition 3. Let $\mu_l^* \in \mathcal{S}(\alpha_l^*)$ and $\mu_k^* \in \mathcal{S}(\alpha_k^*)$. Then for any d,

$$U_d(a_l, \alpha_l^*) \geq U_d(a_k, \alpha_l^*) \geq U_d(a_k, \alpha_k^*) \tag{B.3}$$

where $a_l \in \mu_l^*(d)$ and $a_k \in \mu_k^*(d)$.

The first inequality follows from a_l being a best response to α_l^* (i.e. $a_l \in \mu_l^*(d)$) for an agent of degree d. The second inequality follows from U_d being decreasing in α^*.

Since (B.3) holds for any d, all agents have expected utility that is weakly greater in the higher-investment equilibrium μ_l^*. We therefore conclude that μ_l^* weakly Pareto-dominates μ_k^*. □ □

Proposition 4. Let $\mathcal{F}'(\mu, \alpha)$ and $\mathcal{F}(\mu, \alpha)$ denote (8) under $\tilde{f}'$ and $\tilde{f}$ respectively. We know from Theorem 4 that in a game of self-protection, $q_d(\alpha)$ is non-decreasing in d and any equilibrium strategy is a lower-threshold strategy. We therefore only need to consider such strategies. It then follows from (8) that given any lower-threshold strategy μ, $\mathcal{F}'(\mu, \alpha) \geq \mathcal{F}(\mu, \alpha)$ for all $\alpha \in [0, 1]$. Since under the assumptions, (8) has a single fixed point in α and we conclude that $\mathcal{T}'(\mu) \geq \mathcal{T}(\mu)$, where $\mathcal{T}'(\mu)$ and $\mathcal{T}(\mu)$ denote the correspondence (3) under $\tilde{f}'$ and $\tilde{f}$ respectively.

It then follows that

$$\begin{aligned} \Phi'(\alpha) &= \mathcal{T}'(\mathcal{S}(\alpha)) \\ &\succeq \mathcal{T}(\mathcal{S}(\alpha)) \\ &= \Phi(\alpha) \end{aligned}$$

It therefore follows that $\underline{\alpha}'^* = min\{\alpha : \alpha = \Phi'(\alpha)\} \geq min\{\alpha : \alpha = \Phi(\alpha)\} = \underline{\alpha}^*$ and that $\bar{\alpha}'^* = max\{\alpha : \alpha = \Phi'(\alpha)\} \geq max\{\alpha : \alpha = \Phi(\alpha)\} = \bar{\alpha}^*$.

Thus, $\underline{\mu}'^* = \mathcal{S}(\bar{\alpha}'^*) \preceq \mathcal{S}(\bar{\alpha}^*) = \underline{\mu}^*$ and $\bar{\mu}'^* = \mathcal{S}(\underline{\alpha}'^*) \preceq \mathcal{S}(\underline{\alpha}^*) = \bar{\mu}^*$. □ □

Proposition 5. Let $\mathcal{F}'(\mu, \alpha)$ and $\mathcal{F}(\mu, \alpha)$ denote (8) under r' and r respectively. In the case of the contact process described in Example 1, $q_d'(\alpha) > q_d(\alpha)$ for all $\alpha \in [0, 1]$, $d > 0$. It then follows from (8) that given any strategy μ, $\mathcal{F}'(\mu, \alpha) \geq \mathcal{F}(\mu, \alpha)$ for all $\alpha \in [0, 1]$. Since under the assumptions, (8) has a single fixed point, we conclude that $\mathcal{T}'(\mu) \geq \mathcal{T}(\mu)$, where $\mathcal{T}'(\mu)$ and $\mathcal{T}(\mu)$ denote the correspondence (3) under r' and r respectively.

It then follows that

$$\begin{aligned} \Phi'(\alpha) &= \mathcal{T}'(\mathcal{S}(\alpha)) \\ &\succeq \mathcal{T}(\mathcal{S}(\alpha)) \\ &= \Phi(\alpha) \end{aligned}$$

It therefore follows that $\underline{\alpha}'^* = min\{\alpha : \alpha = \Phi'(\alpha)\} \geq min\{\alpha : \alpha = \Phi(\alpha)\} = \underline{\alpha}^*$ and that $\bar{\alpha}'^* = max\{\alpha : \alpha = \Phi'(\alpha)\} \geq max\{\alpha : \alpha = \Phi(\alpha)\} = \bar{\alpha}^*$.

Thus, $\underline{\mu}'^* = \mathcal{S}(\bar{\alpha}'^*) \preceq \mathcal{S}(\bar{\alpha}^*) = \underline{\mu}^*$ and $\bar{\mu}'^* = \mathcal{S}(\underline{\alpha}'^*) \preceq \mathcal{S}(\underline{\alpha}^*) = \bar{\mu}^*$. □ □

Proposition 6. We prove by contradiction. Suppose $r'\alpha'^* < r\alpha^*$. Then $\mathcal{S}'(\alpha'^*) \preceq \mathcal{S}(\alpha^*)$ and thus $\mu'^* \leq \mu^*$. Since $\mathcal{F}'(\mu, \alpha) \geq \mathcal{F}(\mu, \alpha)$ for any $\mu \in \mathcal{M}$ and $\alpha \in [0, 1]$ and since $\mathcal{F}'$ and $\mathcal{F}$ are decreasing in μ, we have that $\mathcal{F}'(\mu'^*, \alpha) \geq \mathcal{F}(\mu^*, \alpha)$ for any $\alpha \in [0, 1]$. Therefore,

$$\begin{aligned} \alpha'^* &= \mathcal{T}'(\mu'^*) \\ &\geq \mathcal{T}(\mu^*) \\ &= \alpha^* \end{aligned}$$

and thus, since $r' > r$, we have that $r'\alpha'^* > r\alpha^*$, a contradiction. We conclude that $r'\alpha'^* \geq r\alpha^*$.

It then follows that $\mathcal{S}'(\alpha'^*) \succeq \mathcal{S}(\alpha^*)$ and thus $\mu'^* \succeq \mu^*$. This completes the proof. □ □

Proposition 7. Let $\mathcal{F}'(\mu, \alpha)$ and $\mathcal{F}(\mu, \alpha)$ denote (8) under k' and k respectively. It follows from (8) that given any strategy μ, $\mathcal{F}'(\mu, \alpha) \leq \mathcal{F}(\mu, \alpha)$ for all $\alpha \in [0, 1]$. Since under the assumptions, (8) has a single fixed point, we conclude that $\mathcal{T}'(\mu) \leq \mathcal{T}(\mu)$, where $\mathcal{T}'(\mu)$ and $\mathcal{T}(\mu)$ denote the correspondence (3) under k' and k respectively.

It then follows that

$$\begin{aligned} \Phi'(\alpha) &= \mathcal{T}'(\mathcal{S}(\alpha)) \\ &\preceq \mathcal{T}(\mathcal{S}(\alpha)) \\ &= \Phi(\alpha) \end{aligned}$$

It therefore follows that $\underline{\alpha}'^* = min\{\alpha : \alpha = \Phi'(\alpha)\} \leq min\{\alpha : \alpha = \Phi(\alpha)\} = \underline{\alpha}^*$ and that $\bar{\alpha}'^* = max\{\alpha : \alpha = \Phi'(\alpha)\} \leq max\{\alpha : \alpha = \Phi(\alpha)\} = \bar{\alpha}^*$.

Thus, $\underline{\mu}'^* = \mathcal{S}(\bar{\alpha}'^*) \succeq \mathcal{S}(\bar{\alpha}^*) = \underline{\mu}^*$ and $\bar{\mu}'^* = \mathcal{S}(\underline{\alpha}'^*) \succeq \mathcal{S}(\underline{\alpha}^*) = \bar{\mu}^*$. □

□

Theorem 5. The proof is analogous to that of Theorem 1, with only minor modifications.

Denote the function $\mathfrak{T}(\mu) = (\mathcal{T}(\mu), \mathcal{W}(\mu))$ and let the correspondence Ψ be such that $\Psi(\alpha, \omega) = \mathfrak{T}(\mathcal{S}(\alpha, \omega))$, with the correspondence $\mathcal{S}(\alpha, \omega)$ defined as in Definition 15.

First, note that Ψ still has a compact domain $[0, 1] \times [0, 1]$ and a nonempty image.

Furthermore, it is also simple to show that Ψ has a closed graph. First, note that $\mathcal{S}(\alpha, \omega)$ has a closed graph when we endow the set of mean-field strategies with the product topology on $[0, 1]^\infty$. Indeed, choose any $(\alpha_n, \omega_n) \to (\alpha, \omega)$ and $\mu_n \to \mu$ such that $\mu_n \in \mathcal{S}(\alpha_n, \omega_n)$. Then $\mu_n(d) \to \mu(d)$ for any d and by the continuity of $U_d(1, \alpha, \omega)$ and $U_d(0, \alpha, \omega)$, it follows that $\mu(d) \in \mathcal{S}_d(\alpha, \omega)$. Thus, $\mathcal{S}$ has a closed graph. Note also that with the product topology on the space of mean-field strategies, $\mathfrak{T}$ is continuous: by the bounded convergence theorem, both $\mathcal{T}(\mu_n) \to \mathcal{T}(\mu)$ and $\mathcal{W}(\mu_n) \to \mathcal{W}(\mu)$ and therefore it is also true that $\mathfrak{T}(\mu_n) \to \mathfrak{T}(\mu)$. We now only need to consider the sequences $(\alpha_n, \omega_n) \to (\alpha, \omega)$ and $(\alpha'_n, \omega'_n) \to (\alpha', \omega')$ where $(\alpha'_n, \omega'_n) \in \Psi(\alpha_n, \omega_n)$. By choosing $\mu_n \in \mathcal{S}(\alpha_n, \omega_n)$ such that $\mathcal{T}(\mu_n) = \alpha'_n$ and $\mathcal{W}(\mu_n) = \omega'_n$, and by the same argument as in the proof of Theorem 1, we can conclude that $(\alpha', \omega') \in \Psi(\alpha, \omega)$, as desired.

Finally, the image of Ψ is convex. Indeed, $\mathfrak{T}(\mu)$ is continuous in μ. Furthermore, $\mathcal{S}(\alpha, \omega)$ is convex (which follows from convexity of $\mathcal{S}_d(\alpha, \omega)$ for any d). Convexity of the image of Ψ thus follows from an argument analogous to that presented in the proof of Theorem 1.

By Kakutani's fixed point theorem, Ψ has a fixed point (α^*, ω^*). Letting $\mu^* \in \mathcal{S}(\alpha^*, \omega^*)$ be such that $\mathfrak{T}(\mu^*) = (\alpha^*, \omega^*)$, we conclude that μ^* is an MFE. □ □

Theorem 6. Note that the incremental expected utilities for an agent of degree d in games of total and self protection are respectively:

$$\Delta U_d(\alpha, \omega) = V \cdot (p + (1 - p) q_d(\alpha))\, k - C g(\omega) \tag{B.4}$$

and

$$\Delta U_d(\alpha, \omega) = V \cdot (pk - pk q_d(\alpha)) - C g(\omega) \tag{B.5}$$

It is obvious that for any given α and ω, these functions preserve the properties (i.e. monotonicity in d) that were discussed in the proofs of Theorems 3 and 4. The threshold nature of equilibria are thus maintained. □ □

Theorem 7. For a game of total protection, the incremental expected utility for an agent of degree d is

$$\Delta U_d(\alpha, \omega) = V \cdot (p + (1-p)q_d(\alpha))\, k - C \cdot g(\omega) \tag{B.6}$$

We will consider the case when $g(\omega)$ is an increasing function. As in the proof of Theorem 2, we will conduct the analysis in 4 steps.

Step 1: For all $d \geq 1$, $\Delta U_d(\alpha, \omega)$ is strictly increasing in $\alpha \in [0, 1]$ and strictly decreasing in $\omega \in [0, 1]$.

Step 2: Notice that α and ω are moving $\Delta U_d(\alpha, \omega)$ in opposite directions. Hence, if both α and ω increase, we cannot conclude anything about the change in $\mathcal{S}_d(\alpha, \omega)$. However for $\alpha' > \alpha$ and $\omega' > \omega$ it holds $\mathcal{S}_d(\alpha', \omega) \succeq \mathcal{S}_d(\alpha, \omega')$.

Step 3: For any mean-field strategies μ', μ such that $\mu'(d) \geq \mu(d), \forall d \geq 1$, then $\mathcal{W}(\mu') \geq \mathcal{W}(\mu)$ and $\mathcal{T}(\mu') \leq \mathcal{T}(\mu)$. As we have noted before, the global externality does not have a direct impact on $\mathcal{F}(\mu, \alpha)$ and thus the behavior of $\mathcal{T}(\mu)$ remains as in step 3 of the proof of Theorem 2.

Step 4: Suppose that there are two mean-field equilibria $(\mu^*, \alpha^*, \omega^*)$ and $(\mu'^*, \alpha'^*, \omega'^*)$. Without loss of generality assume that $\alpha'^* > \alpha^*$. We need to consider two cases. First, if $\omega'^* \leq \omega^*$, then it is true that $\mathcal{S}(\alpha'^*, \omega'^*) \succeq \mathcal{S}(\alpha^*, \omega^*)$. As $\mu^* \in \mathcal{S}(\alpha^*, \omega^*)$ and $\mu'^* \in \mathcal{S}(\alpha'^*, \omega'^*)$, then it follows that $\mu'^* \succeq \mu^*$. However that leads to the contradiction: $\alpha^* = \mathcal{T}(\mu^*) \geq \mathcal{T}(\mu'^*) = \alpha'^*$. Finally consider the case of $\omega'^* > \omega^*$. By Theorem 6, due to the threshold nature of the equilibrium, the equilibrium strategies can be ordered as either $\mu'^* \succeq \mu^*$ or $\mu'^* \preceq \mu^*$. If $\mu'^* \preceq \mu^*$ then $\omega^* = \mathcal{W}(\mu^*) \geq \mathcal{W}(\mu'^*) = \omega'^*$, which is a contradiction. If $\mu'^* \succeq \mu^*$, it follows that $\alpha^* = \mathcal{T}(\mu^*) \geq \mathcal{T}(\mu'^*) = \alpha'^*$ and we arrive at a contradiction.

Thus, we showed that in a game of total protection with both local and global externalities (with $g(\cdot)$ increasing), any MFE must be unique. □

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