Free Logic and the Quantified Argument Calculus

Edi Pavlović*[1] and Norbert Gratzl[2]

[1]University of Helsinki, Finland, edi.pavlovic@helsinki.fi
[2]Ludwig Maximilian University of Munich, Germany, N.Gratzl@lmu.de

Abstract

The Quantified Argument Calculus (or Quarc for short) is a novel and peculiar system of quantified logic, particularly in its treatment of non-emptiness of unary predicates, as in Quarc unary predicates are never empty, and singular terms denote. Moreover, and as a consequence of this, the universally quantified formulas entail their corresponding particular ones, similar to existential import. But at the same time, Quarc eschews talk of existence entirely by having a particular quantifier instead of an existential one. To bring it back into consideration, we explicitly introduce the existence predicate, and modify the rules to make the existence assumption obvious. This, along with some modifications, leads to a version of negative free logic. A question that arises at this point, given that we are interested in free logic, is what happens when we remove the existence assumption on singular terms; here we can quite naturally choose the negative free logic framework as well. In this paper we shall therefore investigate interrelations between Quarc and free logic (especially with its negative variant), and approach these interrelations with proof-theoretic methods.

1 Introduction

Classical Logic (CL) is the most well-known approach to formal reasoning; it has its own quirks and features. Recently, a new alternative to CL has been developed. One of the reasons for developing a new alternative to CL is to provide for formal reasoning and formalization processes that are (arguably) closer to natural language. This system (or rather family of system) is called Quantified argument calculus (Quarc for short), c.f. e.g. [4], [12], [15].

In the 1960ies CL has been investigated with respect to its specific existence assumptions and the outcome has been (again a family of) free logics, where free logic is short for first order logic free of existence assumptions. Existence

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assumptions vary but the central claims are: (1) the domain of an interpretation is not empty (this is respected in the CL theorem $\exists x(x = t)$, (2) every name denotes exactly one object in the domain, and (3) the quantifiers have existential import (expressed by $\forall x E!x$).

Neither CL nor Free Logic, however, have existence assumptions on unary predicates. This is not so in Quarc – in Quarc unary predicates are never empty, and singular terms denote. But at the same time, Quarc eschews talk of existence entirely by having a particular quantifier instead of an existential one. A particular quantifier tells us that there is an instance of the unary predicate, so in this sense the said predicate is not empty. Furthermore, of this instance (at least) something can be truthfully predicated. So, in this sense it expresses that "there are" things, but is stops short of identifying this construction with the existence statement about it, and therefore remains agnostic on the claim of existence (it's possible we could say true things about non-existents, such as, for example, that there are some). To bring it back into the discussion, we explicitly introduce the existence predicate, modify the rules to make the existence assumption obvious, and introduce the rule for the new predicate. This leads to a version of negative free logic, and we investigate the versions both with and without identity.

In this paper we shall therefore investigate interrelations between Quarc and free logic (especially with its negative variant). Furthermore, this paper approaches these interrelations with proof-theoretic methods and results of [15]. In it the authors claim that the rules of quantification in (the family of logics) Quarc resemble those of free logic. The results of this paper substantiate that claim.

1.1 Quarc

Quarc is a system of quantified logic which does away with variables and unrestricted predicates, but nonetheless achieves results similar to the Predicate Calculus by employing quantifiers, applied directly to unary predicates, which then appear as arguments of other predicates (hence the name Quantified Argument Calculus), along with operators that attach to predicates and subsequently modify the mode of predication, as well as anaphors. It is in these respects, as mentioned previously, closer to natural language.

Let us note that the quantifiers in Quarc do have particular import, a fact that is expressed semantically by the condition of non-emptiness of (unary) predicates. This is in contrast to first-order predicate logic, where, as it is well known, (unary) predicates can be empty. For the purpose of investigating logics free of existence assumptions, we will in the proceeding also eliminate this condition[1], and focus on the resulting systems, labelled Quarc_B for version

[1] [13] investigates a three-valued version of Quarc that also omits this assumption. Here, however, we will remain within the confines of a two-valued system.

without identity, and Quarc$_2$ for version with it, as well as their respective sequent-calculus representations, LK-Quarc$_B$ and LK-Quarc$_2$.

2 Free Logic

Let us start with admittedly very broad, but nonetheless instructive, explanation of what free logics are:

> A free logic is a formal system of quantification theory, with or without identity, which allows for some singular terms in some circumstances to be thought of as denoting no existing object, and in which quantifiers are invariably thought of as having existential import. [2, 148–149]

In this quote one might glimpse a connection between Quarc and free logics – the quantifiers having existential import. Of course, given that Quarc doesn't talk of existence, the connection will have to be refined in the proceeding, but this will serve as an initial point of contact – both Quarc and free logic challenge the standard commitments to existence. A more formal way to characterize (negative, as it among the many free logics will be the sole focus of this paper) free logic would be to describe it via axioms:

1. $\forall x(A \to B) \to (\forall xA \to \forall xB)$
2. $A \to \forall xA$, if x is not free in A
3. $\forall xA \to (E!t \to At)$
4. $\forall xE!x$
5. $R(t_1, \ldots, t_n) \to (E!t_1 \wedge \ldots \wedge E!t_n)$
6. $\forall x(x = x)$
7. $s = t \to (As \to At)$

On the other hand, in a sequent calculus the rules for quantifiers in free logic can be formulated as follows, following [2] and slightly simplified:

$$\frac{A[t/x], \Gamma \Rightarrow \Delta \qquad \Gamma \Rightarrow \Delta, E!t}{\forall xA, \Gamma \Rightarrow \Delta}\,\text{L}\forall \qquad\qquad \frac{E!t, \Gamma \Rightarrow \Delta, A[t/x]}{\Gamma \Rightarrow \Delta, \forall xA}\,\text{R}\forall^*$$

$$\frac{E!t, A[t/x], \Gamma \Rightarrow \Delta}{\exists xA, \Gamma \Rightarrow \Delta}\,\text{L}\exists^* \qquad\qquad \frac{\Gamma \Rightarrow \Delta, E!t \qquad \Gamma \Rightarrow \Delta, A[t/x]}{\Gamma \Rightarrow \Delta, \exists xA}\,\text{R}\exists$$

* – t does not occur below the inference line.

One can see that in all of these cases an extra requirement has been added – that of $E!t$. In the following section we will employ the same principle to transform Quarc into its free version.

3 Free Quarc

To produce the free versions of the systems LK-Quarc$_B$ and LK-Quarc$_2$ we add the new rule for the existence predicate $E!$, replace the rules for the quantifiers, and also replace one of the identity rules in the latter of the two systems. In the interest of brevity, the full systems will not be laid out here, but the reader can find both those, and a thorough discussion of their metatheoretical properties, in [15].

3.1 The base system – FQ$_B$

To transform the system LK-Quarc$_B$ (which does not contain identity) into a system of free logic FQ$_B$, we modify the quantifier rules by an explicit condition on the existence of the singular term, in the same vein as above:

$$\frac{A[a/\forall M], \Gamma \Rightarrow \Delta \qquad \Gamma \Rightarrow \Delta, aM \qquad \Gamma \Rightarrow \Delta, aE!}{A[\forall M], \Gamma \Rightarrow \Delta}\,\mathrm{L}\forall \qquad \frac{aM, aE!, \Gamma \Rightarrow \Delta, A[a/\forall M]}{\Gamma \Rightarrow \Delta, A[\forall M]}\,\mathrm{R}\forall^*$$

$$\frac{aM, aE!, A[a/\exists M], \Gamma \Rightarrow \Delta}{A[\exists M], \Gamma \Rightarrow \Delta}\,\mathrm{L}\exists^* \qquad \frac{\Gamma \Rightarrow \Delta, aM \qquad \Gamma \Rightarrow \Delta, aE! \qquad \Gamma \Rightarrow \Delta, A[a/\exists M]}{\Gamma \Rightarrow \Delta, A[\exists M]}\,\mathrm{R}\exists$$

* – a does not occur below the inference line.

In addition to the rules for quantifiers, we also supply the rule for the (negative free logic) existence predicate:

$$\frac{aE!, A[a], \Gamma \Rightarrow \Delta}{A[a], \Gamma \Rightarrow \Delta}\,\mathrm{N}E!^*$$

* – A is basic[2].

With these in place, we can demonstrate the following axioms of negative free logic. Given that the system we were expanding did not contain identity, the resulting system will likewise not contain it.

[2] In the terminology of Quarc, a basic formula corresponds to an atomic formula of PC.

Theorem 1 All of the following axioms of negative free logic are derivable in FQ_B:

1. $(\forall M_\alpha A \to \alpha B) \to (\forall MA \to \forall MB)$[3]
2. $A \to (\forall MM \wedge B)$
3. $\forall MA \to ((aE! \wedge aM) \to aA)$
4. $\forall ME!$
5. $A[a_1, \ldots, a_n] \to (a_1E! \wedge \ldots \wedge a_nE!)$

Note that while these axioms characterize negative free logic, only axioms 3-5 are specific to it (i.e. 1-2 are likewise derivable in LK-Quarc_B).

3.2 Metatheoretic properties of $\mathbf{FQ}_B$

We now turn towards establishing some metatheoretic properties of FQ_B, first and foremost being the Cut elimination theorem. As everywhere in this paper, the proof is omitted, but it is a straightforward adaptation of the one found in [15].

Theorem 2 The Cut elimination property holds of FQ_B. Namely, any sequent derivable in FQ_B is derivable without using the Cut rule.

To consider some consequences of this theorem, we first define the Subformula property, specifically its weak version (which will suffice to establish the results required in this paper).

Definition 3 (Weak subformula property) A sequent calculus system possesses the Weak subformula property just in case any formula occurring anywhere in a derivation of an endsequent is either a subformula of some formula occurring in that endsequent, or a basic formula.

It follows straightforwardly from Theorem 2 that

Corollary 4 FQ_B possesses the Weak subformula property.

From this Corollary we can further demonstrate that

Corollary 5 FQ_B is consistent.

This Corollary represents a desirable property of a logical system, but will not be of particular note going forward. Quite the opposite holds of the following one, however.

[3] This simplified formulation assumes the formula $\forall MB$ is governed by the Quantified Argument $\forall M$. Otherwise, a more involved form, namely $\forall M_\alpha M \wedge \alpha B$, must be used. Similar for the next axiom.

Corollary 6 FQ_B is a conservative extension of LK-Quarc$_B$. Namely, any derivation $\Gamma \Rightarrow \Delta$ derivable in FQ_B and such that Γ, Δ do not contain $E!$ is likewise derivable in LK-Quarc$_B$.

We will discuss these properties of FQ_B at some length in Section 4, but for the moment we will examine adding the identity predicate to the system at hand.

3.3 Adding identity – FQ_2

We add the rule for identity into the mix. Given the close connection of identity and the existence predicate in negative free logic, it should come as no surprise that the rules for the two look the same. We add this rule (it replaces the rule $=_1$ of LK-Quarc$_2$) and the rule $=_2$ to FQ_B to produce the system FQ_2.

$$\frac{a = a, A[a], \Gamma \Rightarrow \Delta}{A[a], \Gamma \Rightarrow \Delta}\ \text{N=}^*$$

* – A is basic.

With the addition of the identity rules, we can now derive the remaining axioms:

Theorem 7 In addition to those axioms mentioned in Theorem 1, the following are derivable in FQ_2:

6. $\forall M_\alpha = \alpha$, for any M
7. $s = a \to (sA \to aA)$

In addition to Theorem 7, several other results characteristic of a negative free logic are now derivable, namely

Theorem 8 Equivalence of existence and self-identity, $aE! \leftrightarrow a = a$, and indiscernibility of non-existents, $(a\neg E! \wedge b\neg E!) \to (A[a] \to A[b/a])$, are both derivable in FQ_2.

3.4 Metatheoretical properties of FQ_2

Not much needs to be added here, as the results of this section closely resemble their counterparts in Section 3.2. It is straightforward to show that

Theorem 9 Cut elimination property holds for FQ_2.

And from this it again follows that

Corollary 10 FQ_2 possesses the Weak subformula property.

And again,

Corollary 11 FQ_2 is consistent.

An interesting consequence of Corollary 10 is

Corollary 12 $a = a$ is not derivable in FQ_2.

This corollary is of course, combined with Axiom 6 ($\forall x(x = x)$), characteristic of the way negative free logic treats the truth of self-identity sentences.

4 Comparing Quarc and Free Quarc

We have already seen that FQ_B is a conservative extension of LK-Quarc$_B$. Now we move on to compare their respective versions containing identity, FQ_2 and LK-Quarc$_2$. Given the equivalence of existence and self-identity in FQ_2, (Theorem 8), it will suffice that we observe the $E!$-free segment of FQ_2, FQ_2^*. The result we obtain in this case is that

Theorem 13 FQ_2^* is a proper subset of LK-Quarc$_2$, $FQ_2^* \subset$ LK-Quarc$_2$. Namely, every rule of FQ_2^* is admissible in LK-Quarc$_2$, but (Corollary 12), $a = a$ is not derivable in FQ_2^*.

This result should not come as a great surprise – in general, free logic is a restriction on classical logic. In this particular case, if we compare the differing identity rules in LK-Quarc$_2$ and FQ_2, respectively:

$$\frac{a = a, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta} =_1 \qquad \frac{a = a, A[a], \Gamma' \Rightarrow \Delta}{A[a], \Gamma' \Rightarrow \Delta} \text{N=}^*$$

* – A is basic.

We can see that the latter is really just a special case of the former – specifically, when Γ stands for the list of formulas $A[a], \Gamma'$. By placing a limitation on the rules of the system, we likewise limit the output of the said system.

But now it should strike us as most peculiar that the same situation did not occur in the case of LK-Quarc$_B$ and FQ_B. Much like the rules for identity above, the quantifier rules of FQ_B impose a limitation on the corresponding rules of LK-Quarc$_B$. And yet, we experience no loss of power (Corollary 6) – in fact, the only change has to do with the change in vocabulary that results from adding the existence predicate $E!$. This anomaly bears closer scrutiny.

4.1 Comparison of $\mathbf{FQ}_B$ and LK-$\mathbf{Quarc}_B$

As mentioned when we first introduced free version of Quarc, we produce it by means of an additional restriction on the rules. This, however, as we have just seen, does not result in a loss of expressive power. Normally, the most notable formula that becomes underivable in free logic is unrestricted specification, $\forall xAx \rightarrow At$. Instead, we have the weaker, restricted specification (Axiom 3). The corresponding version of unrestricted specification in Quarc would be:

Definition 14 (Unrestricted specification) $\forall MA \rightarrow (aM \rightarrow aA)$

This formula can be obtained even in the free version LK-Quarc$_B$:

Theorem 15 $\forall MA \rightarrow (aM \rightarrow aA)$ is derivable in FQ$_B$.

The formula follows from Axiom 3 (restricted specification, $\forall MA \rightarrow ((aE! \wedge aM) \rightarrow aA))$ and an instance of Axiom 5, $aM \rightarrow aE!$. The latter is what explains this anomaly – in negative free logic, there is a close connection between the truth of atomic sentences and existence, expressed by Axiom 5 (and the absence of the formula $a = a$). But in FQ$_B$, we only added the condition, in the appropriate place, depending on the rule, that $aE!$ (thus allowing for the derivation of some formulas containing the new predicate), but the aM requirement was already present in the non-free rule, i.e. the quantification was already restricted, and precisely in a manner that precludes the derivation of that formula which the free logic avoids. This demonstrates the point raised in [15],

Observation 16 The quantification rules of Quarc, even on the non-free version, have a "free flavor", or a structural resemblance to those of free logic.

This point is further strengthened in the following section, when we discuss free logic in relation to quantified arguments.

5 Empty Predicates

Given that quantified arguments containing predicates feature in the same syntactic roles as names in Quarc, it has two different sets of existence assumptions – those of non-emptiness of names, and also of predicates. As noted in the introduction, we will be dropping both of those in this paper.

In this section, we restate the axioms to talk not of individuals, referred to by constants (or singular arguments in the terminology of Quarc), but of "some M's", captured by unary predicates. In what [15] refer to as *full Quarc*, these are required to be non-empty, but both systems under consideration here, LK-Quarc$_B$ and LK-Quarc$_2$, omit that requirement.

It should be noted that restricted specification as applied to predicates instead of names, $\forall MA \rightarrow \exists MA$, is not valid in either of those systems [16], and therefore neither is it so in FQ_B (by Corollary 6), nor in FQ_2 (by Theorem 13). This checks off the first requirement on being able to describe a system as a (negative) free logic. As importantly, all the axioms must likewise hold, and this is in fact the case, restated for "some M's":

Theorem 17 All of the following axioms are derivable in FQ_B [FQ_2]:

1. $(\forall M_\alpha A \rightarrow \alpha B) \rightarrow (\forall MA \rightarrow \forall MB)$
2. $A \rightarrow (\forall MM \wedge B)$
3. $\forall MA \rightarrow (\exists ME! \rightarrow \exists MA)$
4. $\forall ME!$
5. $\exists MP \rightarrow \exists ME!$
6. [$\forall M_\alpha = \alpha$, for any M]
7. [$\exists M = \exists P \rightarrow (\exists MA \rightarrow \exists PA)$]

So, both FQ_B and FQ_2 are free not just with respect to non-empty names, but also non-empty predicates. That this feature transfers back to LK-Quarc$_B$ and LK-Quarc$_2$ can be demonstrated by restating the axioms without the existence predicate $E!$. We are able to do this, when talking about some M's, since (given the close connection between unary predicates and existence predicate), for some M to exist, $E!M$, is for it to be some unary predicate, namely M, $\exists MM$:

Lemma 18 $\exists ME! \leftrightarrow \exists MM$

The left-to-right direction is obtained by a simple use of R$\exists$ and then L$\exists$, and the right-to-left direction is an instance of the Axiom 5 from the Theorem 17.

It follows from Theorem 17 and Lemma 18, again using Corollary 6 and Theorem 13 that

Theorem 19 All of the following axioms are derivable in LK-Quarc$_B$ [LK-Quarc$_2$]:

1. $(\forall M_\alpha A \rightarrow \alpha B) \rightarrow (\forall MA \rightarrow \forall MB)$
2. $A \rightarrow (\forall MM \wedge B)$
3. $\forall MA \rightarrow (\exists MM \rightarrow \exists MA)$
4. $(\forall M_\beta M \wedge (\forall M_\alpha M \wedge A[\alpha, \beta])) \rightarrow (\forall M_\alpha M \wedge (\forall M_\beta M \wedge A[\alpha, \beta]))$ [$\forall M = \exists M$]
5. $\exists MP \rightarrow \exists MM$
6. [$\forall M_\alpha = \alpha$, for any M]
7. [$\exists M = \exists P \rightarrow (\exists MA \rightarrow \exists PA)$]

A note on Axiom 4 – for LK-Quarc$_2$, a very elegant axiom is available (as elsewhere, written within square brackets). However, since it requires the identity predicate, we cannot use it for LK-Quarc$_B$. Instead, what we do here is emulate the Permutation Principle [6] in Quarc.

We can now strengthen the Observation 16 and conclude that

Observation 20 The quantification rules of Quarc, even on the non-free version and with respect to both emptiness of names, as well as that of unary predicates, bear a structural resemblance to those of free logic, specifically negative free logic.

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