Weak tracking in nonautonomous chaotic systems

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(Received 8 July 2020; accepted 22 October 2020; published 19 November 2020)

Previous studies have shown that rate-induced transitions can occur in pullback attractors of systems subject to "parameter shifts" between two asymptotically steady values of a system parameter. For cases where the attractors limit to equilibrium or periodic orbit in past and future limits of such an nonautonomous systems, these can occur as the parameter change passes through a critical rate. Such rate-induced transitions for attractors that limit to chaotic attractors in past or future limits has been less examined. In this paper, we identify a new phenomenon is associated with more complex attractors in the future limit: weak tracking, where a pullback attractor of the system limits to a proper subset of an attractor of the future limit system. We demonstrate weak tracking in a nonautonomous Rössler system, and argue there are infinitely many critical rates at each of which the pullback attracting solution of the system tracks an embedded unstable periodic orbit of the future chaotic attractor. We also state some necessary conditions that are needed for weak tracking.

DOI: 10.1103/PhysRevE.102.052210

I. INTRODUCTION

Attractors of nonautonomous (time-varying) dynamical systems that limit to autonomous systems in both past and future time can undergo rate-induced transitions. Many studies and applications of these transitions in such "parameter shift" systems assume equilibrium attractors of both limiting systems, see Refs. [1–8]. If there are nonequilibrium attractors for past and future limits these can lead to new phenomena. For example, Kaszás *et al.* [9] study the equation of the forced pendulum with time-dependent amplitude of forcing and show there is an analogy between the behavior of the pullback attractor of the nonautonomous system and the bifurcation diagram of the associated autonomous (or "frozen system"). The structure of the pullback attractor may be very complex even for parameter values where is no stable chaos. In another paper, Kaszás *et al.* [10] explain the time-dependent topology of the same system and show that it can be described using properties of pullback saddles and their unstable foliations.

Rate-induced transitions for attractors that limit to various sets in the past are discussed in Alkhayuon and Ashwin [11], where each attractor for the past limit system can be associated with a pullback attractor for the nonautonomous system. For such a system with a branch of exponentially stable attractors, Ref. [11] identify a number of rate-induced phenomena:

(i) *strong tracking*: where a pullback attractor of the system end-point tracks the branch of attractors and limit fully to the attractor of the future limit system;

(ii) *partial tipping*: where certain trajectories of a pullback attractor track the branch but other trajectories tip (i.e., limit to other attractors forward in time);

(iii) *total tipping*: where a whole pullback attractor limits forward in time to an attractor that is not included in the considered branch.

An invariant set M is called a *minimal invariant set* if it contains no proper invariant subset. Analogously, an attractor A is called a *minimal attractor* if it has no proper sub-attractors [12]. Chaotic attractors such as the Rössler attractor provide a rich source of attractors that are nonminimally invariant, as they typically contain a dense set of embedded unstable periodic orbits.

Assume we have a parameter shift system that limits forward in time to a system with a nonminimal attractor, or even a minimal attractor that is not minimally invariant. We say there is a *weak tracking*, if there is a pullback attractor for the parameter shift system that limits forward in time to one of the invariant subsets of the future limit attractor. The future limit system needs to have at least one attractor that is nonminimal invariant set, in order for the parameter shift system to exhibit weak tracking. This can be seen on applying [11, Lemma II.1] which shows that the upper forward limit of a pullback attractor must be invariant with respect to the future limit system.

In this paper we demonstrate the existence of weak tracking of pullback attractors for parameter shift systems. In Sec. II we define weak tracking for parameter shift systems. In doing so, we use the results on asymptotic behavior of parameter shift systems from Ref. [11]. Section III illustrate the phenomena of weak tracking in Rössler system [13]. We shift one bifurcation parameter of the system monotonically such that future limit system has always a chaotic Rössler attractor,

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whereas, the past limit system has an attracting equilibrium. We show that there is a dense set of critical rate at each of which the system exhibits weak tracking. Finally, we discuss and conclude in Sec. IV. In particular, we note a dimension restriction that must be satisfied for weak tracking to take place—the past limit attractor can have dimension no bigger than the stable manifold of a proper subset of the future limit attractor.

II. ASYMPTOTIC BEHAVIOR OF PARAMETER SHIFT SYSTEMS

A *parameter shift system* [14] is a nonautonomous differential equation of the form:

$$\dot{x} = f[x, \Lambda(rt)], \tag{1}$$

where $x \in \mathbb{R}^n$, $t, r \in \mathbb{R}$, $\Lambda : \mathbb{R} \to \mathbb{R}$ and f is at least C^1 in both arguments. For some λ_- and $\lambda_+ \in \mathbb{R}$ with $\lambda_- < \lambda_+$, the parameter shift Λ satisfies (i) $\Lambda(\tau) \in (\lambda_-, \lambda_+)$ for all $\tau \in \mathbb{R}$, (ii) $\lim_{\tau\to\pm\infty} \Lambda(\tau) = \lambda_\pm$, and (iii) $\lim_{\tau\to\pm\infty} d\Lambda/d\tau = 0$. We denote the solution process of (1) with $x(s) = x_0$ by $\Phi(t, s, x_0) := x(t)$. One can understand much of the behavior of system Eq. (1) by studying the associated autonomous (or frozen) system, which is given by

$$\dot{x} = f(x, \lambda), \tag{2}$$

where λ is time-independent and denote the flow of Eq. (2) by $\phi_\lambda(t, x_0) := x(t)$, where $x(0) = x_0$.

We say a set valued function $\mathcal{M} = \{M_t\}_{t\in\mathbb{R}}$ of $t \in \mathbb{R}$ is a *nonautonomous set* for Eq. (1) if M_t is nonempty for all $t \in \mathbb{R}$ [15]. Moreover, $\mathcal{M}$ is called Φ-invariant if $\Phi(t, s, M_s) = M_t$ for all $t, s \in \mathbb{R}$. We say that $\mathcal{M}$ has a property p if and only if M_t has p for all $t \in \mathbb{R}$.

To study the asymptotic behavior of nonautonomous sets, note there are several different notions of limit for set valued sequences [16]. More precisely, for a nonautonomous set $\mathcal{M} = \{M_t\}_{t\in\mathbb{R}}$ [17] one can define the upper forward limit $(M_{+\infty})$ and the upper backward limit $(M_{-\infty})$ of $\mathcal{M}$ as follows:

$$M_{+\infty} := \limsup_{t\to\infty} M_t = \bigcap_{\tau>0} \overline{\bigcup_{t\geqslant\tau} M_t},$$

$$M_{-\infty} := \limsup_{t\to-\infty} M_t = \bigcap_{\tau>0} \overline{\bigcup_{t\leqslant\tau} M_t}.$$

We focus on these upper limits (rather than lower limits) as they capture the asymptotic behavior in maximal sense.

Furthermore, we denote the set of asymptotically stable attractors of Eq. (2) that are parametrized by λ by $\mathcal{X}_{\rm as}$. The set of all exponentially stable attractors $\mathcal{X}_{\rm stab}$ is a subset of $\mathcal{X}_{\rm as}$. We call the boundary of $\mathcal{X}_{\rm stab}$, $\overline{\mathcal{X}_{\rm stab}} \setminus \mathcal{X}_{\rm stab}$, set of bifurcations. One can think of them as subsets of $\mathbb{R}^n \times [\lambda_-, \lambda_+]$. A continuous set valued function $A(\lambda) \in \mathcal{X}_{\rm as}$, for all $\lambda \in [\lambda_-, \lambda_+]$, is called a *stable path*. If $A(\lambda) \in \mathcal{X}_{\rm stab}$, for all $\lambda \in [\lambda_-, \lambda_+]$, and its stability is independent of λ, in the sense that the exponential rate of converging to $A(\lambda)$ is independent of λ then we say the path is *uniformly stable*, for more details see Ref. [11]. A uniformly stable path is called a *stable branch* [11]. Note that a stable path can include a several stable branches joined at bifurcation points, for an example of a stable path that continues bifurcation points, see Sec. III.

A. Weak tracking of pullback attractors

We define local pullback attractors as in Ref. [11]. Suppose that Φ is a process on $\mathbb{R}^n$. A compact and Φ-invariant nonautonomous set $\mathcal{A}$ is called *local pullback attractor* if there exists an open set U that contains the upper backward limit of $\mathcal{A}$ and satisfies

$$\lim_{s\to-\infty} d[\Phi(t, s, U), A_t] = 0,$$

for all $t \in \mathbb{R}$, where d is Hausdorff semi-distance.

Theorem II.2 shows that for each asymptotically stable attractor A_- for the past limit system there is a local pullback attractor for Eq. (1) whose upper backward limit is contained in A_-. This pullback attractor depends on the parameter shift Λ, the rate r as well as the attractor of the past limit system A_-. Therefore, we denote the pullback attractor by $\mathcal{A}^{[\Lambda,r,A_-]}$ and it consists of t-fibres that are defined as

$$A_t^{[\Lambda,r,A_-]} := \bigcap_{\tau>0} \overline{\bigcup_{s\leqslant\tau} \Phi[t, s, \mathcal{N}_\eta(A_-)]} \tag{3}$$

for some $\eta > 0$. Note that if A_- is an equilibrium, then Ref. [14, Theorem 2.2] shows that the pullback attractor is a single trajectory or so-called *pullback attracting solution*.

For a uniformly exponentially stable branch $A(\lambda)$ that contains an attractor of the past limit system $A_- := A(\lambda_-)$ and for sufficiently small positive r, Ref. [11, Theorem III.1] proves that the pullback attractor (3) end-point tracks the branch $A(\lambda)$.

This tracking is not guaranteed for large values of $r > 0$ or where a stable branch is weakened to a stable path. Rate-induced transitions take place when this tracking breaks. Reference [11, Definition III.1] defines different rate-induced transitions between partial tipping, total tipping and invisible tipping. Here we present a new phenomenon we call *weak tracking* that can also lead to transitions.

Definition 1. Suppose that $[A(\lambda), \lambda] \subset \mathcal{X}_{\rm as}$ is a path of asymptotically stable attractors for $\lambda \in [\lambda_-, \lambda_+]$. Define $A_\pm := A(\lambda_\pm)$ and consider the pullback attractor $\mathcal{A}^{[\Lambda,r,A_-]}$ with past limit $A_{-\infty}^{[\Lambda,r,A_-]}$ that is contained in A_-. We say there is *strong tracking* for system (1) from A_- for some Λ and $r > 0$ if $A_{+\infty}^{[\Lambda,r,A_-]} = A_+$. We say there is *weak tracking* if $A_{+\infty}^{[\Lambda,r,A_-]} \subsetneq A_+$.

Lemma II.1 from Ref. [11] shows that the upper forward limit $A_{+\infty}^{[\Lambda,r,A_-]}$ is invariant with respect to the future limit system. Consequently, to exhibit weak tracking the future limit system needs to have an attractor with a proper invariant subset. As an example of this behavior we consider the Rössler system [13] with embedded unstable periodic orbits that can be the upper forward limit of the pullback attractor for some positive r.

III. WEAK TRACKING FOR NONAUTONOMOUS RÖSSLER SYSTEM

The Rössler system [13] is one of the simplest systems of ODEs that can have chaotic attractors. This has only one

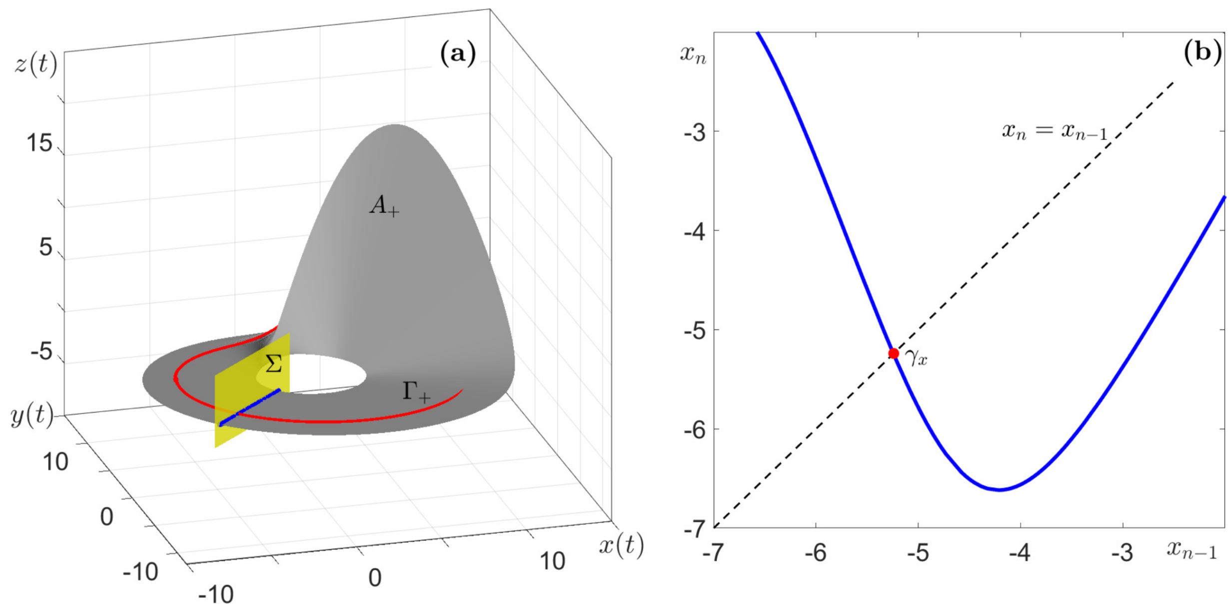

FIG. 1. In (a), the Rössler attractor for parameter values $a = b = 0.2$ and $c = 5.7$. This also shows the period-one unstable periodic orbit Γ_+, and Poincaré section Σ defined as $x(t) - y(t) = 0$. In panel (b) we plot the the projection of the x-component of the return map of Rössler system. Assuming that a trajectory $[x(t), y(t), z(t)]$ intersects with Σ at $t = t_n$ for $n = 1, 2, ...,$ we define $x_n = x(t_n)$. (γ_x, γ_z) represents the intersection of the periodic orbit Γ_+ with the section Σ.

nonlinear term and the system is given by

$$\begin{aligned} \dot{x} &= -y - z, \\ \dot{y} &= x + ay, \\ \dot{z} &= b + z(x - c). \end{aligned} \tag{4}$$

There are many choices of parameters $a, b,$ and c that give chaotic attractors [18–20]. We use as default $a = b = 0.2$ and $c = 5.7$ [13], which give a chaotic attractor as shown in Fig. 1(a).

We fix $b = 0.2$ and $c = 5.7$ throughout and analyze the bifurcations of Eq. (4) as a varies between asymptotic values of $a_\pm$ as $t \to \pm\infty$. This "frozen" system has equilibria at

$$(x_{1,2}, y_{1,2}, z_{1,2}) = \frac{c \pm \sqrt{c^2 - 4ab}}{2a}(a, -1, 1).$$

The equilibrium p_1 is asymptotically stable for any negative a and bifurcates to stable periodic orbit at supercritical Hopf bifurcation point $a_{\mathrm{HB}} \approx 0.005978$. Soon after Hopf bifurcation, the resulting stable periodic orbit exhibits period doubling at $a_{\mathrm{PD}} = 0.1096$, and a period doubling cascade as a increases until the system exhibit chaotic behavior at $a \approx 0.155$.

To examine weak tracking, we shift a from a_- to a_+ for some $a_-, a_+ \in \mathbb{R}$. Namely,

$$a(rt) = \frac{\Delta}{2}\left[\tanh\left(\frac{\Delta rt}{2}\right) + 1\right] - a_-,$$

where $\Delta = a_+ - a_-$, $r > 0$ and a_- (a_+) are the minimum (maximum) value of the parameter shift a. Throughout this paper we fix $a_+ = -a_- = 0.2$. We can write the resulting Rössler system with parameter shift $a(t)$ as

$$\begin{aligned} \dot{x} &= -y - z, \\ \dot{y} &= x + y\,a(rt), \\ \dot{z} &= b + z(x - c). \end{aligned} \tag{5}$$

The past limit system of Eq. (5) has a hyperbolic stable equilibrium, $Z_- = \frac{c - \sqrt{c^2 - 4ba_-}}{2a_-}(a_-, -1, 1) \approx (-0.007, 0.0351, -0.0351)$. The future limit system, however, has a chaotic attractor A_+ that is the typical Rössler attractor in Fig. 1(a).

According to Ref. [14, Theorem 2.2], for any $r > 0$ system Eq. (5) must have a pullback attracting solution $\mathcal{A}^{[a,r,Z_-]}$ that limits to Z_-, backward in time. Moreover, One can show that for almost every small enough $r > 0$, the upper forward limit of the pullback attractor $\mathcal{A}^{[a,r,Z_-]}$ is the whole chaotic attractor A_+. Nevertheless, there is a set of isolated values of $r > 0$ that allow $\mathcal{A}^{[a,r,Z_-]}$ to end up tracking one of the unstable periodic orbits that are densely embedded in A_+. In this paper, we consider the period-one periodic orbit Γ_+, in particular, see Fig. 1. However, similar arguments can be made for any unstable periodic orbits contained in A_+.

A. Piecewise linear shift

To show that there are values of r such that $\mathcal{A}^{[a,r,Z_-]}$ limits to Γ_+ as $t \to \infty$ we approximate the parameter shift $a(rt)$ by the following piecewise linear function $\hat{a}(rt)$:

$$\hat{a}(s) = \begin{cases} a_- & s \in (-\infty, -\tau), \\ (\Delta s + a_+ + a_-)/2 & s \in [-\tau, \tau], \\ a_+ & s \in (\tau, \infty), \end{cases}$$

where $\tau = [\log(\Delta - \delta) - \log(\delta)]/\Delta$, for small enough $\delta > 0$, note that at time $\pm\tau$ the value of a is δ-close to the upper

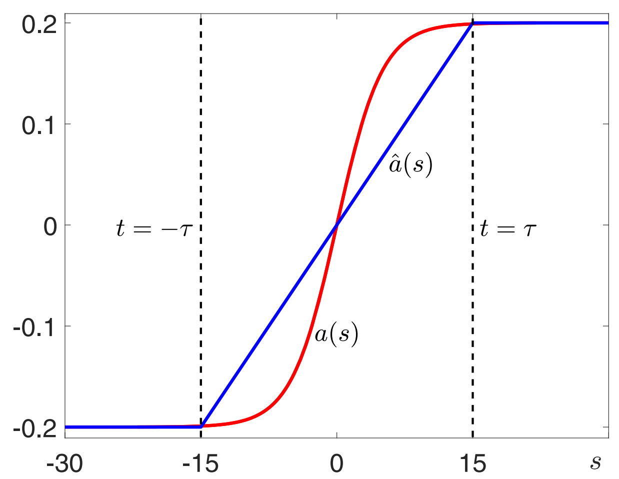

FIG. 2. The parameter shift $a(s)$ and the piecewise linear approximation $\hat{a}(s)$ vs. time, for $a_+ = -a_- = 0.2$ and $\delta = 0.001$.

and lower limits. i.e., $a(\tau) = a_+ - \delta$ and $a(-\tau) = a_- + \delta$, see Fig. 2.

The fact that $\hat{a}$ is fixed for any $t > \tau$, allows us to consider A_+ as an attractor for the system rather than just the upper forward limit of the pullback attractor $A_t^{[a,r,Z_-]}$. We embed a Poincaré section Σ parametrized by (x, z) with $x \leqslant 0$, as

$$\{(x, x, z) : (x, z) \in \Sigma\} \subset \mathbb{R}^3,$$

and consider t^*, which is any real value that satisfies (i) $t^* \geqslant \tau$ and (ii) $A_{t^*}^{[a,r,Z_-]} \in \Sigma$, i.e., $A_{t^*}^{[\hat{a},r,Z_-]}$ is a point in Σ.

Note that, the intersection of Γ_+ with Σ is a fixed point γ for the return map. If $r_c > 0$ is chosen such that $A_{t^*}^{[\hat{a},r,Z_-]}$ is one of the preimages of γ, then the the upper forward limit of $\mathcal{A}^{[\hat{a},r,Z_-]}$ is Γ_+ and r_c is a critical rate for weak tracking.

B. Density of critical rates: Numerical evidence

To investigate weak tracking for system Eq. (5), with the smooth parameter shift $a(rt)$, we use a shooting method as follows:

(i) We approximate the pullback attractor $\mathcal{A}^{[a,r,Z_-]}$ by integrating Eq. (5), subject to an initial condition Z_{init} fairly close to Z_-. Namely, we choose $Z_{\mathrm{init}} = (-0.007, 0.035, -0.035)$ and the integration time is from -30 to T.

(ii) The point pullback attractor can be given as $\mathcal{A}^{[a,r,Z_-]} = [\tilde{x}^r(t), \tilde{y}^r(t), \tilde{z}^r(t)]$, where $t \in [-30, T]$.

(iii) Recall that the Poincaré section Σ is parametrized by (x, z) with $x \leqslant 0$, as

$$\{(x, x, z) : (x, z) \in \Sigma\} \subset \mathbb{R}^3.$$

(iv) Assume that $\mathcal{A}^{[a,r,Z_-]}$ intersects Σ at times $t_n \leqslant T$ for $n = 1, 2, ..., N$, $N \in \mathbb{N}$ and $t_{n-1} < t_n$.

(v) Consider the final intersection point $[\tilde{x}^r(t_N), \tilde{z}^r(t_N)] \in \Sigma$. We approximate a signed distance from the stable manifold of γ by the following real valued "gap function"

$$\eta(r) := \frac{\{[\tilde{x}^r(t_N), \tilde{z}^r(t_N)] - \gamma\} v_s^T}{v_s v_s^T},$$

where $\gamma = (\gamma_x, \gamma_z) \in \Sigma$ is the fixed point of Rössler return map, see Fig. 1, and v_s is stable eigenvector of γ for the return map. Note that $\eta(r)$ also depends on T, b, c, $a_{\min}$, $a_{\max}$, and Z_{init}. However, here we only consider variation of r.

(vi) By analogy to Sec. III A, whenever $\eta(r_c) \approx 0$ the pullback attractor $\mathcal{A}^{[a,r,Z_-]}$ intersects the stable manifold of Γ_+, which gives the desired EtoP connection, see Fig. 3. In other words, $\mathcal{A}^{[a,r,Z_-]}$ weakly tracks A_+ at $r = r_c$. The method is illustrated schematically in Fig. 4.

The MATLAB code used for the shooting method is provided in GitHub [31]. The function $\eta(r)$ is as smooth as the state variables of Eq. (4), i.e., it is at least C^1. Consequently, one can numerically approximate its roots, and hence the critical rates of weak tracking, using a root-finding algorithm such as Newton-Raphson method. Figure 3 shows that system Eq. (4) exhibit weak tracking at two different critical rates.

We point out two difficulties in our numerical approach to approximate the rates of weak tracking: First, there is a large delay in Hopf bifurcation that forces us to choose fairly large integration time T in our calculations, increasing the computational cost. Delay in dynamic bifurcations is common and not easy to avoid. For a system with linearly changing time-dependent parameter with slope r, dynamic Hopf bifurcation may have a delay time proportional to $1/r$ before fast escape from the curve of unstable equilibria occurs [21,22]. More details on dynamic bifurcations and their delay can be found in Ref. [23, Chapter 2]. Second, Figure 5 shows that $\eta(r)$ is smooth with respect to r for a particular range of r, which is [0.9,1]. However, there is no guarantee that $\eta(r)$ is smooth or even continuous for finite T. The definition of $\eta(r)$ depends on the maximum intersection time which in turn depends on the integration time T. Nevertheless, it seems that T can be chosen to give a smooth $\eta(r)$ for any range of r.

Our numerical investigation suggests that there are infinitely many critical rates that give weak tracking for Eq. (4). In Fig. 5 we plotted $\eta(r)$ against $0.9 \leqslant r \leqslant 1$, for different values of $T = 125$, 135, 145 and 155. The results show that as T increases, the number of roots of $\eta(r)$ increases rapidly.

Despite the fact that other periodic orbits are embedded in A_+, even for just one periodic orbit Γ_+, our numerical investigation suggests there are infinitely many critical rates that give weak tracking. In fact, we believe that the set of all critical rates r_c is dense in some interval.

IV. DISCUSSION

We study the well-known Rössler system Eq. (5) with parameter shift, as a tool to illustrate a new rate-induced phenomenon that we term "weak tracking." We monotonically shift the bifurcation parameter a such that the system has an equilibrium attractor for the past limit system and chaotic attractor for the future limit system. We then show that there are isolated critical rates at each of which the pullback attractor solution of the system ends up tracking an embedded saddle periodic orbit in the future chaotic attractor. We use a numerical approach, based on shooting method and carefully chosen Poincaré section, to approximate these critical rates.

For the nonautonomous Rössler system Eq. (5) with a parameter shift from stable equilibrium to chaos, we suggest

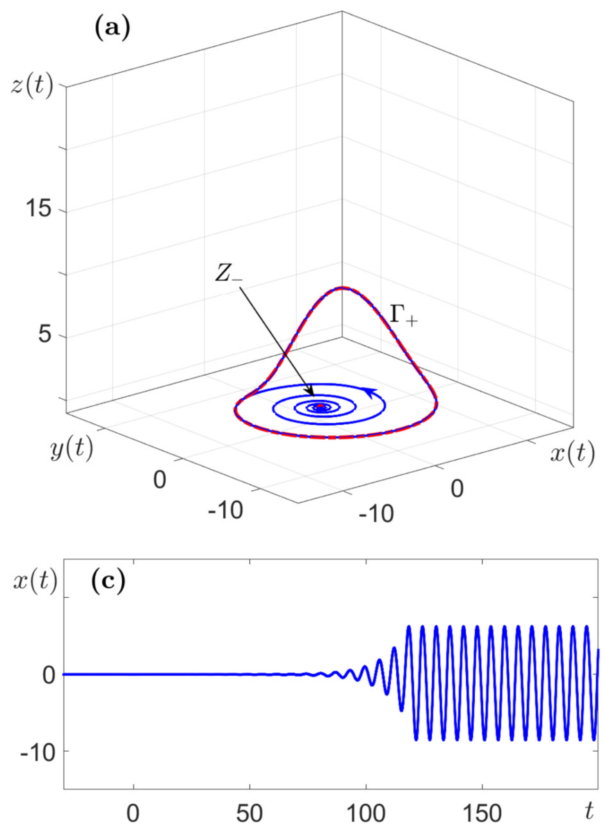

FIG. 3. Two examples of weak tracking (EtoP connection) for Eq. (5). The parameters are $b = a_{\max} = -a_{\min} = 0.2$, $c = 5.7$ and $T = 150$. (a) and (c) show the EtoP connection at $r = 0.9202212159423$, (b) and (d) show the connection at $r = 0.995651959127$.

there is a dense set of critical rates that give weak tracking. We give an argument below that this is the case if the system has piecewise linear forcing instead of smooth parameter shift and provide in Fig. 5 numerical evidence of the existence of the dense set of critical rates for smooth parameter shift.

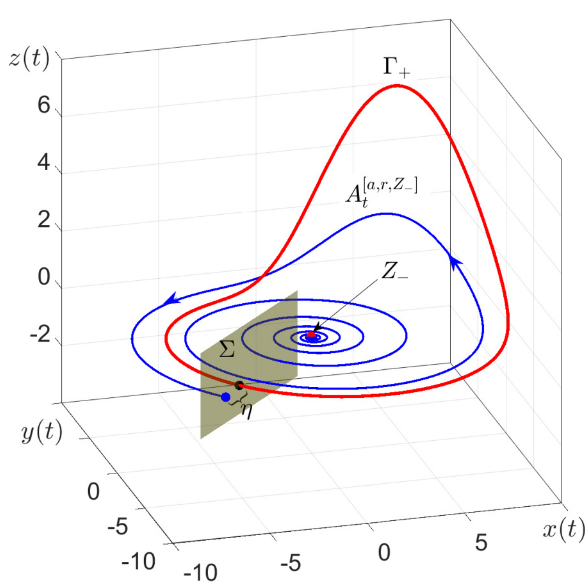

FIG. 4. A schematic diagram showing the shooting method we use to find the the connection between Z_- and Γ_+ for Eq. (5), see the GitHub repository [31] for animated version of this figure.

Although our example considers a specific choice of parameters, the necessary ingredients for weak tracking are present in a wide range of the parameter space of the nonautonomous Rössler system. These ingredients are simply (i) a hyperbolic attracting equilibrium for the past limit system (ii) a chaotic (nonminimally invariant) attractor for the future limit system and (iii) a rate dependent shift in parameters that means for certain rates the pullback attractor gets "caught" in unstable dynamics within the chaos.

More precisely, in order for the parameter shift system Eq. (1) to exhibit weak tracking along a branch of attractors $A(\lambda)$ from a past limit attractor A_- to A_+, it is clear that the future limit system must have a proper invariant subset S_+ (in our case we consider $S_+ = \Gamma_+$) of the future limit attractor A_+, and the pullback attractor with past limit A_- must "fit in" to S_+. If we consider Eq. (1), then weak tracking corresponds to existence of a pullback attractor $A_t^{[\Lambda,r,A_-]}$ with backward limit A_- and forward limit S_+. This will only be possible if the dimension of A_- is small enough with respect to that of S_+. For an eventually constant parameter shift such as in Fig. 2, note that $A_t^{[\Lambda,r,A_-]} = A_-$ as long as t is sufficiently negative, and as nonautonomous time evolution will be a diffeomorphism between any two finite times, i.e., $A_t^{[\Lambda,r,A_-]}$ is diffeomorphic to A_- for all finite t. Hence, in this eventually constant case a necessary condition for $A_t^{[\Lambda,r,A_-]}$ to limit to S_+ is that $A_t^{[\Lambda,r,A_-]} \subset W^s(S_+)$ for sufficiently large t where W^s is the stable set for the future limit flow. Hence,

$$\dim(A_-) = \dim\left(A_t^{[\Lambda,r,A_-]}\right) \leqslant \dim[W^s(S_+)]$$

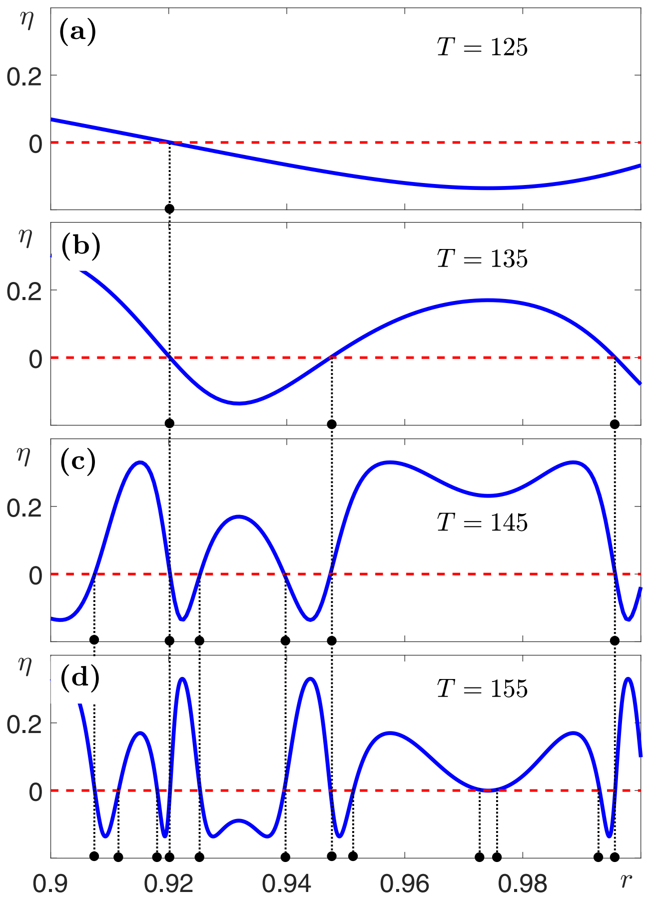

FIG. 5. Graphs of η for increasing integration time T to add additional intersections of Σ: roots correspond to connections from A_- to the periodic orbit Γ_+. (a) $T = 125$, (b) $T = 135$, (c) $T = 145$, and (d) $T = 155$. It can be seen that additional zeros of η (corresponding to critical rates that give weak tracking) appear as T increases. The parameter values are $b = a_{\max} = -a_{\min} = 0.2$ and $c = 5.7$.

[where $\dim(A)$ represents Hausdorff dimension of A]. Hence, weak tracking require

$$\dim(A_-) \leqslant \dim[W^s(S_+)], \tag{6}$$

which means in particular if $\dim(A_-) > \dim(S_+)$ then a connection is not possible.

Moreover, note that for large enough t, the set $A_t^{[\Lambda,r,A_-]}$ will, in the generic case, vary nontrivially with r. Any interaction between this and $W^s(S_+)$ will typically be transverse on varying r: this argues that values of r where there is weak tracking are isolates. Density of $W^s(S_+)$ within the basin $\mathcal{B}(A_+)$ of A_+, with respect to the future limit flow, would imply the density of a set of critical rates giving weak tracking to this S_+.

For example, if A_- is an equilibrium or periodic orbit, then it is possible to have weak tracking to a periodic orbit S_+ contained in A_+ a chaotic attractor. If A_- is chaotic, then weak tracking will only be possible to an invariant set S_+ with dimension greater than A_-. A similar result will presumably apply more generally, even if the shift is not eventually constant. In this case the condition for weak tracking will be in terms of a condition for existence of a connection from A_- to S_+ for the extended autonomous system.

Parameter shift systems such as Eq. (1), and asymptotic autonomous systems more generally, have a rich tipping behavior. Reference [11] gives an example of a system with pullback attractor that exhibit partial rate-dependent tipping, where an entire subset of the pullback attractor tracks different quasi-static attractor than it would be for other rates of shift, while the rest of the pullback attractor still tracks the associated quasi-static attractor. This behavior can still be produced in Rössler system with a suitable parameter shift that shifts the chaotic attractor partially out of its basin of attraction.

More precisely, suppose we have a parameter shift $\Lambda(rt)$ that limits to $\lambda_\pm$ forward and backward in time respectively, such that the attractors for the future and the past limit systems, $A_\pm$, are nonequilibrium attractors. Reference [24] shows that partial tipping is possible, for some values of r, if

$$A_- \not\subset \mathcal{B}(A_+). \tag{7}$$

Besides the phenomena illustrated in Ref. [11], nonautonomous systems with nonequilibrium attractors may exhibit other transitions. For example, systems that have attractors with fractal basin boundaries may exhibit fractality-induced tipping [25] due to the high complexity of the basin not because of the well-known tipping mechanisms presented in Ref. [1]. Basins of attraction with fractal boundary are very common in physical systems, and can cause a high uncertainty when it comes to predicting the final state of a trajectory. We refer to Ref. [26] for further details. Fractal boundaries can result from crossings of the stable and unstable manifold of an invariant set that is embedded in basin boundary.

Fractality may also be a sign of the presence of *transient chaos* [27]. One phenomenon that can lead to transient chaos is a *boundary crisis* [28,29], where the attractor intersects its basin boundary and leaks out. If the time dependent parameter passes through a region where there is a crisis, then the system exhibit *attractor hopping* behavior [30], which may led to partial or even total tipping.

ACKNOWLEDGMENTS

H.A.'s research is funded by Enterprise Ireland and Laya Healthcare, Grant Agreement No. 20190771. P.A.'s research was partially supported by the CRITICS Innovative Training Network, funded by the European Union's Horizon 2020 research and innovation programme under the Marie Sklodowska-Curie Grant Agreement No. 643073. Both authors gratefully acknowledge the insightful comments by Jan Sieber, Roberto Barrio, Damian Smug, Paul Richie, and Sebastian Wieczorek.

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