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Operations Research and Control Systems

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Research Report 2012-04
September 2012

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Abstract. The paper develops the needle variations technique in application to a class of infinite-horizon optimal control problems in which an appropriate relation between the growth rate of the solution and the growth rate of the objective function is satisfied. The optimal objective value does not need to be finite. Based on the concept of weakly overtaking optimality, we establish the normal form version of the Pontryagin maximum principle with an explicitly specified adjoint variable. A few illustrative examples are presented as well.

1. Introduction

Infinite-horizon optimal control problems arise in many fields of economics, in particular in problems of optimization of economic growth. Typically, the initial state is fixed and the terminal state (at infinity) is free in such problems, while the utility functional to be maximized is given by an improper integral on the time interval $[0, \infty)$. The infinite time-horizon gives rise to some specific and challenging mathematical features of the problems.

First, the infinite planning horizon may cause the appearance of various "pathological" phenomena in the relations of the Pontryagin maximum principle. Although the state at infinity is not constrained, such problems could be abnormal ($\psi^0 = 0$ in this case) and the "standard" transversality conditions of the form

$$\lim_{t\to\infty} \psi(t) = 0 \tag{1.1}$$

or

$$\lim_{t\to\infty} \langle \psi(t), x_*(t) \rangle = 0 \tag{1.2}$$

1991 *Mathematics Subject Classification.* Primary 49J15, 49K15, 91B62.

The first author was supported in part by the Russian Foundation for Basic Research (RFBR) Grants No 10-01-91004-ANF-a and No 13-01-00685-a.

The second author was supported by the Austrian Science Foundation (FWF) Grant No I 476-N13.

may fail. Here $x_*(\cdot)$ is an optimal trajectory and $(\psi^0, \psi(\cdot))$ is a pair of adjoint variables corresponding to the optimal pair $(x_*(\cdot), u_*(\cdot))$ according to the maximum principle. Examples demonstrating pathologies of these types are well known (see [**5, 10, 12, 14, 17**]).

Second, the utility functional given by an improper integral on the time interval $[0,\infty)$ can diverge. In such a situation the notion of optimality should be specially adopted (see the corresponding discussion in [**10**]). This creates additional difficulties in the analysis of the problems.

To our knowledge for the first time a version of the maximum principle for the infinite-horizon optimal control problem was proved in [**16**] in the case when the improper integral utility functional converges and the optimal trajectories satisfy additional boundary constraint $\lim_{t\to\infty} x_*(t) = x_1$, where x_1 is a given point in R^n. In the case when the integral utility functional not necessary converges, the maximum principle was proved in [**12**]. Both these results are formulated similarly. Their relations comprise the "core" conditions of the maximum principle (adjoint system and the maximum condition), but they do not provide any additional characterizations of the adjoint variables ψ^0 and $\psi(\cdot)$ such as normality of the problem and/or some boundary conditions for $\psi(\cdot)$ at infinity.

At the end of 1970s it was suggested in [**8**] that a normal form ($\psi^0 = 1$) version of the maximum principle involving a complementary integral condition on the adjoint variable $\psi(\cdot)$ takes place if the discount rate ρ is sufficiently large. This condition provides bounds (in appropriate L-spaces) for $\psi(\cdot)$ rather than only a condition for the asymptotics at infinity. Such a stronger "transversality" condition was proved in [**8**] for linear autonomous control systems. Then the result in [**8**] was extended in [**4, 5**] for nonlinear autonomous systems. Moreover, it was proved in [**4, 5**] that if the discount rate ρ is sufficiently large then the adjoint variable $\psi(\cdot)$ that satisfies the conditions of the maximum principle admits an explicit single-valued representation similar to the classical Cauchy formula for the solutions of systems of linear differential equations. In the linear case, this Cauchy sort representation of $\psi(\cdot)$ implies the integral "transversality" condition suggested in [**8**] and an even stronger exponential pointwise estimate for $\psi(\cdot)$ (see [**5, 6**] for more details).

The requirement for the discount rate $\rho \geq 0$ to be sufficiently large was expressed in [**4, 5, 8**] in the form of the following inequality:

$$\rho > (r+1)\lambda, \tag{1.3}$$

where $r \geq 0$ and $\lambda \in R^1$ are parameters characterizing the growth of the instantaneous utility and the trajectories of the control system, respectively (see [**4, 5, 8**] for precise definitions of the parameters r and λ). Condition (1.3) requires that the discount factor ρ "dominates" the growth parameters r and λ. That is why conditions of this type are usually referred as *dominating discount* conditions.

Recently, the results in [**4, 5**] were extended in [**3**]. In particular, the dominating discount condition was expressed in [**3**] in a more general form of convergence of an appropriate improper integral.

The approaches used in [**8**] and [**3, 4, 5**] for establishing additional characterizations of the adjoint variable $\psi(\cdot)$ are different. The approach used in [**8**] is based on methods of functional and non-smooth analysis. The method of finite-horizon approximations used in [**3, 4, 5**] is based on an appropriate *regularization* of the infinite-horizon problem, namely on its explicit approximation by a family of standard finite-horizon problems. Notice, that both approaches assume that the improper integral utility functional converges uniformly for all admissible pairs.

In contrast, the original proof of the maximum principle for the infinite-horizon problem in [**16**] is based on application of the classical needle variations technique that does not assume any uniformity in convergence of the integral utility functional. Nevertheless, the straightforward application of needle variations faces some difficulties (see discussion in [**16**, Chapter 4]) and does not provide additional conditions on the adjoint variable. Recently, the application of needle variations technique to infinite-horizon problems was revisited in [**7**] under a dominating discount condition similar to (1.3). Moreover, it is demonstrated in [**7**] that under this condition the needle variations can be applied even in the case when the objective value may be infinite. A local modification of the notion of weakly overtaking optimality (see [**10**]) is employed in this case. The result obtained in [**7**] involves the same explicit single-valued representation for the adjoint variable $\psi(\cdot)$ as in [**3, 4, 5**] but under different assumptions.

The goal of the present paper is to extend and strengthen the results obtained in [**7**] to general non-autonomous infinite-horizon problems without explicit discounting. The "dominating discount" condition is adopted in an "invariant" form to this case.

The paper is organized as follows. In Section 2, we state the problem and introduce the notion of optimality used in present paper. Some auxiliary results about the effect of simple needle variations on the objective value are presented in Section 3. Section 4 is devoted to the formulation and the proof of a new version of the Pontryagin maximum principle for infinite-horizon problems. In Section 5, we consider a few illustrative examples and discuss the obtained result. In particular, we demonstrate that the developed Cauchy sort single-valued characterization of the adjoint variable completes the core conditions of the maximum principle in Halkin's example [**12**] while the standard transversality conditions are inconsistent with them in this case.

2. Statement of the problem

Let G be a nonempty open convex subset of R^n and U be an arbitrary nonempty set in R^m. Let

$$f : [0,\infty) \times G \times U \mapsto R^n \quad \text{and} \quad g : [0,\infty) \times G \times U \mapsto R^1.$$

Consider the following optimal control problem (P):

$$J(x(\cdot),u(\cdot)) = \int_0^\infty g(t,x(t),u(t))\,dt \to \max\,, \tag{2.1}$$

$$\dot{x}(t) = f(t,x(t),u(t)), \qquad u(t) \in U, \tag{2.2}$$

$$x(0) = x_0.$$

Here $x_0 \in G$ is a given initial state of the system. The exact meaning of this problem will be given below.

The following will be assumed throughout the paper.

Assumption (A1): *The functions $f : [0, \infty) \times G \times U \mapsto R^n$ and $g : [0, \infty) \times G \times U \mapsto R^1$ together with their partial derivatives $f_x(\cdot, \cdot, \cdot)$ and $g_x(\cdot, \cdot, \cdot)$ are defined and locally bounded, measurable in t for every $(x, u) \in G \times U$, and continuous in (x, u) for almost every $t \in [0, \infty)$.* [1]

In what follows, we assume that the class of *admissible controls* in problem (P) consists of all measurable locally bounded functions $u : [0, \infty) \mapsto U$. Then for any initial state $x_0 \in G$ and any admissible control $u(\cdot)$ plugged in the right-hand side of the control system (2.2) we obtain the following Cauchy problem:

$$\dot{x}(t) = f(t, x(t), u(t)), \qquad x(0) = x_0. \tag{2.3}$$

Due to assumption $(A1)$, this problem has a unique solution $x(\cdot)$ (in the sense of Carathéodory) which is defined on some time interval $[0, \tau]$ with $\tau > 0$ and takes values in G (see e.g. [**11**]). This solution is uniquely extendible to a maximal interval of existence in G and is called *admissible trajectory* corresponding to the admissible control $u(\cdot)$.

If $u(\cdot)$ is an admissible control and the corresponding admissible trajectory $x(\cdot)$ exists on $[0, T]$, $T > 0$, in G, then the integral

$$J_T(x(\cdot), u(\cdot)) := \int_0^T g(t, x(t), u(t))\, dt$$

is finite. This follows from (A1), the definition of admissible control and the existence of $x(\cdot)$ on $[0, T]$.

The following notion of optimality of an admissible control $u_*(\cdot)$ goes back to Halkin [**12**] (see [**10**] for a discussion on different concepts of optimality in infinite-horizon problems).

Definition 2.1. An admissible control $u_*(\cdot)$ for which the corresponding trajectory $x_*(\cdot)$ exists on $[0, \infty)$ is finitely optimal in problem (P) if for any $T > 0$ and for an arbitrary admissible control $u(\cdot)$ such that the corresponding admissible trajectory $x(\cdot)$ is also defined on $[0, T]$ and satisfies $x(T) = x_*(T)$ it holds that

$$J_T(x_*(\cdot), u_*(\cdot)) \geq J_T(x(\cdot), u(\cdot)).$$

Notice that the finite optimality of an admissible control $u_*(\cdot)$ does not assume any boundedness of the corresponding value of the utility functional in problem (P).

Define the Hamilton-Pontryagin function $\mathcal{H} : [0, \infty) \times G \times U \times R^1 \times R^n \mapsto R^1$ for problem (P) in the usual way:

$$\mathcal{H}(t, x, u, \psi^0, \psi) = \psi^0 g(t, x, u) + \langle f(t, x, u), \psi \rangle,$$
$$t \in [0, \infty),\ x \in G,\ u \in U,\ \psi \in R^n,\ \psi^0 \in R^1.$$

In the normal case we will omit the variable $\psi^0 = 1$ and write simply $\mathcal{H}(t, x, u, \psi)$ instead of $\mathcal{H}(t, x, u, 1, \psi)$.

[1]The local boundedness of these functions of t, x and u (take $\phi(\cdot, \cdot, \cdot)$ as a representative) means that for every $T > 0$, every compact $D \subset G$ and every bounded set $V \subset U$ there exists M such that $\|\phi(t, x, u)\| \leq M$ for almost all $t \in [0, T]$, and all $x \in D$ and $u \in V$.

According to [**12**, Theorem 4.2] any finitely optimal control $u_*(\cdot)$ satisfies the following general version of the maximum principle[2].

THEOREM 2.2. *Let $u_*(\cdot)$ be a finitely optimal control in problem (P) and let $x_*(\cdot)$ be the corresponding admissible trajectory. Then there is a non-vanishing pair of adjoint variables $(\psi^0, \psi(\cdot))$ with $\psi^0 \geq 0$ and a locally absolutely continuous $\psi(\cdot) : [0, \infty) \mapsto R^n$ such that the core conditions of the maximum principle hold, i.e.,*

(i) *$\psi(\cdot)$ is a solution to the adjoint system*

$$\dot{\psi}(t) = -\mathcal{H}_x(t, x_*(t), u_*(t), \psi^0, \psi(t)),$$

(ii) *the maximum condition takes place:*

$$\mathcal{H}(t, x_*(t), u_*(t), \psi^0, \psi(t)) \stackrel{a.e.}{=} \sup_{u \in U} \mathcal{H}(t, x_*(t), u, \psi^0, \psi(t)).$$

In contrast with Definition 2.1, the next notion of optimality of an admissible control $u_*(\cdot)$ assumes that the utility functional is bounded (see [**10**, Chapter 1.5]).

DEFINITION 2.3. An admissible control $u_*(\cdot)$ for which the corresponding trajectory $x_*(\cdot)$ exists on $[0, \infty)$ is strongly optimal if the value $J(x_*(\cdot), u_*(\cdot))$ is finite and for any admissible control $u(\cdot)$ such that the corresponding admissible trajectory $x(\cdot)$ is defined on $[0, \infty)$ it holds that

$$J(x_*(\cdot), u_*(\cdot)) \geq \limsup_{T \to \infty} J_T(x(\cdot), u(\cdot)).$$

Clearly, the strong optimality of an admissible control implies the finite one.

Let us illustrate these two concepts of optimality with a simple example [**12**].

EXAMPLE 2.4 (Halkin, 1974). Consider the following problem $(P1)$:

$$J(x(\cdot), u(\cdot)) = \int_0^\infty (1 - x(t))u(t)\, dt \to \max, \tag{2.4}$$

$$\dot{x}(t) = (1 - x(t))u(t), \quad u(t) \in [0, 1],$$
$$x(0) = 0.$$

Set $G = R^1$. Obviously problem $(P1)$ is a particular case of problem (P). For any $T > 0$ and for an arbitrary admissible pair $(x(\cdot), u(\cdot))$ we have

$$J_T(x(\cdot), u(\cdot)) = x(T) = 1 - e^{-\int_0^T u(s)\, ds}. \tag{2.5}$$

Hence, according to Definition 2.1 all admissible pairs $(x(\cdot), u(\cdot))$ are finitely optimal in problem $(P1)$ and due to Theorem 2.2 all of them satisfy the core conditions of the maximum principle together with the corresponding adjoint variables ψ^0 and $\psi(\cdot)$. Moreover,

[2] The proof of this result in [**12**] is based on consideration of a family of auxiliary optimal control problems on finite time intervals $[0, T]$, $T > 0$, with the fixed end-points $x(0) = x_0$ and $x(T) = x_*(T)$ and then taking a limit in the conditions of the maximum principle for these problems as $T \to \infty$. It should be noted that exactly the same result can be obtained with the strightforward application of the needle variations technique. Indeed, the construction of the "initial cone" presented in [**16**, Chapter 4] uses only the property of finite optimality of the reference admissible control $u_*(\cdot)$.

it is easy to see that the adjoint variables $\psi^0 = 1$ and $\psi(t) \equiv -1$, $t \geq 0$, satisfy conditions (i) and (ii) of Theorem 2.2 together with any admissible pair in problem $(P1)$.

On the other hand, the utility functional (2.4) is bounded and the set of strongly optimal controls is nonempty in this problem. Due to (2.5) an admissible control $u_*(\cdot)$ is strongly optimal in problem $(P1)$ if and only if $\int_0^\infty u_*(s)\,ds = \infty$.

In particular, $u_*(t) \stackrel{\text{a.e.}}{=} 1$, $t \geq 0$, is a strongly optimal control and $x_*(t) \equiv 1-e^{-t}$, $t \geq 0$, is the corresponding strongly optimal trajectory. Along this optimal pair $(x_*(\cdot), u_*(\cdot))$ the adjoint system and the maximum condition are the following:

$$\dot{\psi}(t) = u_*(t)(\psi(t) + \psi^0) = \psi(t) + \psi^0 \quad \text{and} \quad \psi(t) + \psi^0 \geq 0.$$

Further, according to Theorem 2.2 either $\psi^0 = 0$ or $\psi^0 > 0$. If $\psi^0 = 0$ then $\psi(t) = \psi(0)e^t \neq 0$, $t \geq 0$, hence $\psi(t) \to \infty$, as $t \to \infty$ in this case. If $\psi^0 > 0$ then one can put $\psi^0 = 1$. In this case $\psi(t) = e^t\,(\psi(0) + 1) - 1$, $t \geq 0$, $\psi(0) \geq -1$, and either $\psi(t) \equiv -1$ or $\psi(t) \to \infty$ as $t \to \infty$. It is easy to see that both standard transversality conditions (1.1) and (1.2) fail for the strongly optimal pair $(x_*(\cdot), u_*(\cdot))$ in this example.

The above example reveals two remarkable facts. First, Example 2.4 demonstrates that the concept of finite optimality is too weak. It can happen that even in simple situations such as problem $(P1)$ this concept does not recognize strongly optimal pairs (which exist) in the set of all admissible ones. Second, Example 2.4 shows that in some cases the standard transversality conditions (1.1) and (1.2) are inconsistent with the core conditions of the maximum principle (even with condition (i) alone). Thus, in general, the complementary conditions on the adjoint variables must have a different form (if such exists).

In this paper, we use the the following (local) modification of the notion of weakly overtaking optimal control[3] [**10, 12**].

Definition 2.5. An admissible control $u_*(\cdot)$ for which the corresponding trajectory $x_*(\cdot)$ exists on $[0, \infty)$ is locally weakly overtaking optimal (LWOO) if there exists $\delta > 0$ such that for any admissible control $u(\cdot)$ satisfying

$$\text{meas}\,\{t \geq 0 : u(t) \neq u_*(t)\} \leq \delta$$

and for every $\varepsilon > 0$ and $T > 0$, one can find $T' \geq T$ such that the corresponding admissible trajectory $x(\cdot)$ is either non-extendible to $[0, T']$ in G or

$$J_{T'}(x_*(\cdot), u_*(\cdot)) \geq J_{T'}(x(\cdot), u(\cdot)) - \varepsilon.$$

[3]The weak overtaking optimality takes an intermediate place between strong optimality and finite optimality, i.e. strong optimality $\Rightarrow$ weak overtaking optimality $\Rightarrow$ finite optimality (see [**12**, Chapter 1.5.] for details). The property of local weak overtaking optimality is obviously weaker than the property of weak overtaking optimality, but it does not imply the finite optimality in general. In general case the property of local weak overtaking optimality should be compared also with a "local" version of the property of finite optimality.

Notice that the expression $d(u(\cdot), u_*(\cdot)) = \text{meas}\,\{t \in [0,T] : u(t) \neq u_*(t)\}$ generates a metric in the space of the measurable functions on $[0,T]$, $T > 0$, which is suitable to use in the framework of the needle variations technique (see [**2**]).

The proof of the necessary optimality conditions in the form of the Pontryagin maximum principle in a normal form, presented in Section 4, is based on some auxiliary analysis given in the next section.

3. Auxiliary results about simple needle variations

Our analysis of problem (P) with Definition 2.5 of optimality is based on the notion of simple needle variation (see for example [**1**, Chapter 1.5.4]). Below we present some auxiliary results which evaluate the effect of simple needle variations on the objective functional.

Let $u_*(\cdot)$ be an admissible control and $x_*(\cdot)$ – the corresponding admissible trajectory, which is assumed to be defined on $[0,\infty)$.

Let us fix an arbitrary $v \in U$ and denote by $\Omega(v)$ the set of all $\tau > 0$ which are Lebesgue points of each of the measurable functions $f(\cdot, x_*(\cdot), u_*(\cdot))$, $g(\cdot, x_*(\cdot), u_*(\cdot))$, $f(\cdot, x_*(\cdot), v)$, $g(\cdot, x_*(\cdot), v)$. This means (see [**15**]) that for every $\tau \in \Omega(v)$ and each of these functions of t (take $\varphi(\cdot)$ as a representative)

$$\lim_{\alpha \to 0} \frac{1}{\alpha} \int_{\tau-\alpha}^{\tau} \varphi(t)\, dt = \varphi(\tau).$$

Note that almost every $\tau \in [0,\infty)$ belongs to $\Omega(v)$.

Let us fix an arbitrary $\tau \in \Omega(v)$. For any $0 < \alpha \leq \tau$ define

$$u_\alpha(t) := \begin{cases} u_*(t), & t \notin (\tau - \alpha, \tau], \\ v, & t \in (\tau - \alpha, \tau]. \end{cases} \tag{3.1}$$

The control $u_\alpha(\cdot)$ is called a *simple variation* of the admissible control $u_*(\cdot)$. Denote by $x_\alpha(\cdot)$ the admissible trajectory that corresponds to $u_\alpha(\cdot)$.

If α is sufficiently small then due $(A1)$ the admissible trajectory $x_\alpha(\cdot)$ is defined at least on the time interval $[0,\tau]$ ($x_\alpha(\cdot)$ coincides with $x_*(\cdot)$ on $[0, \tau - \alpha]$). Due to the property that τ is a Lebesgue point of $f(\cdot, x_*(\cdot), u_*(\cdot))$ and $f(\cdot, x_*(\cdot), v)$, we obviously have that

$$x_\alpha(\tau) - x_*(\tau) = \alpha\, [f(\tau, x_*(\tau), v) - f(\tau, x_*(\tau), u_*(\tau))] + o(\alpha), \tag{3.2}$$

where here and further $\mathrm{o}(\alpha)$ denotes a function of α that satisfies $\|\mathrm{o}(\alpha)\|/\alpha \to 0$ as $\alpha \to 0$. Note that $\mathrm{o}(\alpha)$ may depend on v and τ (which are fixed in the present consideration).

For an arbitrary $\tau \geq 0$, consider the following linear differential equation (the linearization of (2.2) along $(x_*(\cdot), u_*(\cdot))$:

$$\dot{y}(t) = f_x(t, x_*(t), u_*(t))\, y(t), \quad t \geq 0 \tag{3.3}$$

with initial condition

$$y(\tau) = y_*(\tau) := f(\tau, x_*(\tau), v) - f(\tau, x_*(\tau), u_*(\tau)). \tag{3.4}$$

Due to condition $(A1)$ the partial derivative $f_x(\cdot, x_*(\cdot), u_*(\cdot))$ is measurable and locally bounded. Hence, there is a unique (Carathéodory) solution $y_*(\cdot)$ of the Cauchy problem (3.3), (3.4) which is defined on the whole time interval $[0,\infty)$. Moreover,

$$y_*(t) = K_*(t,\tau)\, y_*(\tau), \quad t \geq 0, \tag{3.5}$$

where $K_*(\cdot,\cdot)$ is the state-transition matrix of differential system (3.3) (see [**13**]). Recall that

$$K_*(t,\tau) = Y_*(t)\, Y_*^{-1}(\tau), \quad t,\tau \geq 0, \tag{3.6}$$

where $Y_*(\cdot)$ is the fundamental matrix solution of (3.3) normalized at $t=0$. This means that the columns $\xi_i(\cdot)$, $i=1,\dots,n$, of the $(n\times n)$-matrix function $Y_*(\cdot)$ are (linearly independent) solutions of (3.3) on $[0,\infty)$ that satisfy the initial conditions

$$\xi_i^j(0) = \delta_{i,j}, \quad i,j = 1,\dots,n,$$

where

$$\delta_{i,i} = 1, \quad i=1,\dots,n, \quad \text{and} \quad \delta_{i,j} = 0, \quad i \neq j, \quad i,j=1,\dots,n.$$

Analogously, consider the fundamental matrix solution $Z_*(\cdot)$ (normalized at $t=0$) of the linear adjoint equation

$$\dot z(t) = -\left[f_x(t, x_*(t), u_*(t))\right]^* z(t).$$

Then

$$Z_*^{-1}(t) = \left[Y_*(t)\right]^*, \qquad t \geq 0. \tag{3.7}$$

The following condition is an "invariant" counterpart of the dominating discount condition introduced in [**7**] in terms of the discount rate and some parameters characterizing the growth rates of the admissible trajectories and of the instantaneous utility.

Assumption (A2): *There exist a number $\gamma > 0$ and a nonnegative integrable function $\lambda : [0,\infty) \mapsto R^1$ such that for every $\zeta \in G$ with $\|\zeta - x_0\| < \gamma$ equation* (2.3) *with $u(\cdot) = u_*(\cdot)$ and initial condition $x(0) = \zeta$ (instead of $x(0) = x_0$) has a solution $x(\zeta;\cdot)$ on $[0,\infty)$ in G, and*

$$\max_{\theta\in[x(\zeta;t),x_*(t)]} \left|\langle g_x(t,\theta,u_*(t)), x(\zeta;t) - x_*(t)\rangle\right| \overset{a.e.}{\leq} \|\zeta - x_0\|\lambda(t).$$

Here $[x(\zeta;t), x_*(t)] = \operatorname{co}\{x(\zeta;t), x_*(t)\}$ denotes the line segment with vertices $x(\zeta;t)$ and $x_*(t)$.

Lemma 3.1. *Let $(A2)$ be satisfied. Then the following estimation holds:*

$$\left\| \left[Y_*(t)\right]^* g_x(t, x_*(t), u_*(t)) \right\| \leq \sqrt{n}\lambda(t) \quad \textit{for a.e.} \quad t \geq 0. \tag{3.8}$$

Proof. Define $\zeta_i \in R^n$ as the vector with components $\zeta_i^j = \delta_{i,j}$, $i,j = 1,\dots n$. Due to (A2) for every $\alpha \in (0,\gamma)$, the solution $x(x_0 + \alpha\zeta_i;\cdot)$ of equation (2.3) with $u(\cdot) = u_*(\cdot)$ and initial condition $x(0) = x_0 + \alpha\zeta_i$ exists on $[0,\infty)$ and

$$\left|\langle g_x(t, x_*(t), u_*(t)),\, x(x_0 + \alpha\zeta_i; t) - x_*(t)\rangle\right| \overset{a.e.}{\leq} \alpha\lambda(t). \tag{3.9}$$

Due to the theorem on differentiation of the solution of a differential equation with respect to the initial conditions (see e.g. Chapter 2.5.6 in [**1**]) we get the following equality:

$$x(x_0 + \alpha\zeta_i; t) = x_*(t) + \alpha\xi_i(t) + \mathrm{o}_i(\alpha, t), \quad i = 1, \ldots, n, \quad t \geq 0.$$

Here the vector functions $\xi_i(\cdot)$, $i = 1, \ldots, n$, are columns of $Y_*(\cdot)$ and for any $i = 1, \ldots, n$ we have $\|\mathrm{o}_i(\alpha, t)\|/\alpha \to 0$ as $\alpha \to 0$, uniformly with respect to t on any finite time interval $[0, T]$, $T > 0$. Then in view of (3.9), we get

$$\left|\langle g_x(t, x_*(t), u_*(t)), \xi_i(t) + \frac{\mathrm{o}_i(\alpha, t)}{\alpha}\rangle\right| \overset{a.e.}{\leq} \lambda(t), \quad i = 1, \ldots, n, \quad t \geq 0.$$

Passing to the limit with $\alpha \to 0$ in the last inequality for a.e. $t \geq 0$ and $i = 1, \ldots, n$, we get

$$\left|\langle g_x(t, x_*(t), u_*(t)), \xi_i(t)\rangle\right| \overset{a.e.}{\leq} \lambda(t), \quad i = 1, \ldots, n, \quad t \geq 0.$$

This implies (3.8). □

Due to (3.7) and Lemma 3.1, condition ($A2$) implies that for any $t \geq 0$ the integral

$$I_*(t) = \int_t^\infty [Z_*(s)]^{-1} g_x(s, x_*(s), u_*(s))\, ds \tag{3.10}$$

converges absolutely. Hence, we can define a locally absolutely continuous function $\psi : [0, \infty) \mapsto R^n$ as follows:

$$\psi(t) = Z_*(t) I_*(t), \qquad t \geq 0. \tag{3.11}$$

By a direct differentiation, we verify that the so defined function $\psi(\cdot)$ satisfies on $[0, \infty)$ the adjoint system

$$\dot{\psi}(t) = -\mathcal{H}_x(t, x_*(t), u_*(t), \psi(t)).$$

(Remind that in the case $\psi^0 = 1$ we omit this variable in the Hamilton-Pontryagin function.)

The following lemma provides the key tool for proving the maximum principle in the next section.

Lemma 3.2. *Let condition ($A2$) be satisfied. Then for arbitrarily fixed $v \in U$ and $\tau \in \Omega(v)$ there is an $\alpha_0 > 0$ such that for all $\alpha \in (0, \alpha_0]$ the trajectory $x_\alpha(\cdot)$ corresponding to the simple variation $u_\alpha(\cdot)$ (see (3.1)) is defined on the whole time interval $[0, \infty)$. Moreover, for fixed τ and v as above there exist a constant $c \geq 0$ and a function $\sigma : (0, \alpha_0] \times [\tau, \infty) \mapsto [0, \infty)$ with $\lim_{\alpha\to 0} \sigma(\alpha, t) \to 0$ for any fixed $t \geq \tau$, such that for every $\alpha \in (0, \alpha_0]$ and $T > \tau$*

$$\frac{J_T(x_\alpha(\cdot), u_\alpha(\cdot)) - J_T(x_*(\cdot), u_*(\cdot))}{\alpha}$$

$$= \mathcal{H}(\tau, x_*(\tau), v, \psi(\tau)) - \mathcal{H}(\tau, x_*(\tau), u_*(\tau), \psi(\tau)) + \eta(\alpha, T), \tag{3.12}$$

where the function $\eta(\alpha, T)$ satisfies the following inequality for every $\tilde{T} \in [\tau, T]$:

$$|\eta(\alpha, T)| \leq \sigma(\alpha, \tilde{T}) + c\int_{\tilde{T}}^\infty \lambda(t)\, dt. \tag{3.13}$$

Proof. As argued above, for all sufficiently small $\alpha > 0$ the trajectory $x_\alpha(\cdot)$ corresponding to $u_\alpha(\cdot)$ exists at least on $[0,\tau]$ (and equals $x_*(t)$ for $t \in [0, \tau - \alpha]$) and from (3.2) we have that $\|x_*(\tau) - x_\alpha(\tau)\| \le c'\,\alpha$ with some constant c'. Consider the Cauchy problem

$$\dot{x}(t) = f(t, x(t), u_*(t)), \qquad x(\tau) = x_\alpha(\tau). \tag{3.14}$$

Due to the continuous dependence of the solution of a differential equation on the initial condition (see e.g. Chapter 2.5.5 in [**1**]), there is a sufficiently small $\alpha_0 > 0$ such that for all $\alpha \in (0, \alpha_0]$, the solution $\tilde{x}_\alpha(\cdot)$ of (3.14) exists on $[0,\tau]$ and $\|\tilde{x}_\alpha(0) - x_*(0)\| \le \gamma$. Then the first part of (A2) implies that the solution $\tilde{x}_\alpha(\cdot)$ exists in G on $[0,\infty)$. Thus for all $\alpha \in (0, \alpha_0]$, the solution $x_\alpha(\cdot)$ also exists on $[0,\infty)$, since $x_\alpha(t) = \tilde{x}_\alpha(t)$ for $t \ge \tau$.

Due to the theorem on differentiability of the solution of a differential equation with respect to the initial conditions (see e.g. Chapter 2.5.6 in [**1**]), the following representation holds:

$$\tilde{x}_\alpha(t) = x_*(t) + \alpha y_*(t) + \mathrm{o}(\alpha, t), \quad t \ge 0, \tag{3.15}$$

where $y_*(\cdot)$ is the solution of the Cauchy problem (3.3), (3.4). Here $\|\mathrm{o}(\alpha,t)\|/\alpha \to 0$ as $\alpha \to 0$ and the convergence is uniform in t on every finite interval $[\tau, T]$, $T > \tau$.

Let us prove that for any sufficiently small $\alpha > 0$ the following estimate holds:

$$\max{}_{\theta \in [x_\alpha(t), x_*(t)]} \left| \left\langle g_x(t, \theta, u_*(t)), y_*(t) + \frac{\mathrm{o}(\alpha,t)}{\alpha} \right\rangle \right| \overset{a.e.}{\le} c_1\,\lambda(t), \quad t \ge \tau, \tag{3.16}$$

where $c_1 \ge 0$ is independent of α and t.

Due to $(A2)$,

$$\max{}_{\theta \in [\tilde{x}_\alpha(t), x_*(t)]} \left| \langle g_x(t, \theta, u_*(t)), \tilde{x}_\alpha(t) - x_*(t) \rangle \right| \overset{a.e.}{\le} \|\tilde{x}_\alpha(0) - x_*(0)\| \lambda(t), \quad t \ge 0.$$

Then using (3.15), we obtain that

$$\max{}_{\theta \in [\tilde{x}_\alpha(t), x_*(t)]} \left| \langle g_x(t, \theta, u_*(t)), \alpha y_*(t) + o(\alpha, t) \rangle \right| \overset{a.e.}{\le} \|\alpha y_*(0) + o(\alpha, 0)\| \lambda(t).$$

Choosing $c_1 \ge \|y_*(0)\| + 1$, dividing by α and taking into account that $\tilde{x}_\alpha(t) = x_\alpha(t)$ for $t \ge \tau$, we obtain (3.16).

Now, using that $\tau \in \Omega(v)$ and (3.15) (where $\tilde{x}_\alpha(t) = x_\alpha(t)$ for $t \geq \tau$) for all $\alpha \in (0, \alpha_0]$, we get

$$
\begin{aligned}
(3.17) \qquad & \frac{1}{\alpha}\Big[J_T(x_\alpha(\cdot), u_\alpha(\cdot)) - J_T(x_*(\cdot), u_*(\cdot))\Big] \\
= \; & \frac{1}{\alpha}\int_{\tau-\alpha}^{\tau} [g(t, x_\alpha(t), v) - g(t, x_*(t), u_*(t))]\, dt \\
& + \frac{1}{\alpha}\int_{\tau}^{T} [g(t, x_\alpha(t), u_*(t)) - g(t, x_*(t), u_*(t))]\, dt \\
= \; & g(\tau, x_*(\tau), v) - g(\tau, x_*(\tau), u_*(\tau)) + \frac{\mathrm{o}(\alpha)}{\alpha} \\
& + \int_{\tau}^{T} \left\langle \int_0^1 g_x(t, x_*(t) + s(x_\alpha(t) - x_*(t)), u_*(t))\, ds,\; y_*(t) + \frac{\mathrm{o}(\alpha, t)}{\alpha} \right\rangle dt.
\end{aligned}
$$

On the other hand, according to (3.5), (3.6), (3.7), (3.4), (3.10) and (3.11)

$$
\begin{aligned}
& \int_{\tau}^{\infty} \langle g_x(t, x_*(t), u_*(t)),\; y_*(t)\rangle\, dt \\
& = \left\langle Z_*(\tau) \int_{\tau}^{\infty} \Big[Z_*(t)\Big]^{-1} g_x(t, x_*(t), u_*(t))\, dt,\; f(\tau, x_*(\tau), v) - f(\tau, x_*(\tau), u_*(\tau)) \right\rangle \\
& = \langle \psi(\tau), f(\tau, x_*(\tau), v) - f(\tau, x_*(\tau), u_*(\tau))\rangle.
\end{aligned}
$$

Using this equality in (3.17) we obtain (3.12) with

$$
\begin{aligned}
\eta(\alpha, T) \; := \; & \int_{\tau}^{T} \left\langle \int_0^1 g_x(t, x_*(t) + s(x_\alpha(t) - x_*(t)), u_*(t))\, ds,\; y_*(t) + \frac{\mathrm{o}(\alpha, t)}{\alpha} \right\rangle dt \\
& - \int_{\tau}^{\infty} \langle g_x(t, x_*(t), u_*(t)),\; y_*(t))\rangle\, dt + \frac{\mathrm{o}(\alpha)}{\alpha}.
\end{aligned}
$$

Let $\tilde{T}$ be any number between τ and T. Define

$$
\begin{aligned}
\sigma(\alpha, \tilde{T}) \; := \; & \Bigg| \int_{\tau}^{\tilde{T}} \left\langle \int_0^1 g_x(t, x_*(t) + s(x_\alpha(t) - x_*(t)), u_*(t))\, ds,\; y_*(t) + \frac{\mathrm{o}(\alpha, t)}{\alpha} \right\rangle dt \\
& - \int_{\tau}^{\tilde{T}} \langle g_x(t, x_*(t), u_*(t)),\; y_*(t))\rangle\, dt + \frac{\mathrm{o}(\alpha)}{\alpha} \Bigg|.
\end{aligned}
$$

Due to (A1), we apparently have for fixed $\tilde{T}$ that $\sigma(\alpha, \tilde{T}) \to 0$ as $\alpha \to 0$. Moreover, due to (3.16) we have

$$
\left| \int_{\tilde{T}}^{T} \left\langle \int_0^1 g_x(t, x_*(t) + s(x_\alpha(t) - x_*(t)), u_*(t))\, ds,\; y_*(t) + \frac{\mathrm{o}(\alpha, t)}{\alpha} \right\rangle dt \right| \leq c_1 \int_{\tilde{T}}^{\infty} \lambda(t)\, dt.
$$

Moreover,

$$
\begin{aligned}
&\left| \int_{\tilde{T}}^{\infty} \langle g_x(t, x_*(t), u_*(t)),\ y_*(t)) \rangle \, dt \right| \\
&= \left| \left\langle Z_*(\tau) \int_{\tilde{T}}^{\infty} \Big[Z_*(t) \Big]^{-1} g_x(t, x_*(t), u_*(t)) \, dt,\ f(\tau, x_*(\tau), v) - f(\tau, x_*(\tau), u_*(\tau)) \right\rangle \right| \\
&\leq \| Z_*(\tau) \| \left\| \int_{\tilde{T}}^{\infty} [Y_*(t)]^* \, g_x(t, x_*(t), u_*(t)) \, dt \right\| \, \| f(\tau, x_*(\tau), v) - f(\tau, x_*(\tau), u_*(\tau)) \| \\
&\leq c_2 \int_{\tilde{T}}^{\infty} \lambda(t) \, dt,
\end{aligned}
$$

where in the last inequality we use Lemma 3.1.

Combining the above two inequalities and the definition of $\sigma(\alpha, \tilde{T})$, we obtain (3.13) with $c := c_1 + c_2$. □

4. Maximum principle

This section presents the main result in the paper – a version of the Pontryagin maximum principle for the non-autonomous infinite-horizon problem (P) with Definition 2.5 of optimality.

THEOREM 4.1. *Let $u_*(\cdot)$ be an admissible LWOO control and let $x_*(\cdot)$ be the corresponding trajectory. Assume that $(A2)$ holds. Then the vector function $\psi : [0, \infty) \mapsto R^n$ defined by* (3.11) *is (locally) absolutely continuous and satisfies the core conditions of the normal form maximum principle, i.e.,*

(i) *$\psi(\cdot)$ is a solution to the adjoint system*

$$\dot{\psi}(t) = -\mathcal{H}_x(t, x_*(t), u_*(t), \psi(t)),$$

(ii) *the maximum condition takes place:*

$$\mathcal{H}(t, x_*(t), u_*(t), \psi(t)) \stackrel{a.e.}{=} \sup_{u \in U} \mathcal{H}(t, x_*(t), u, \psi(t)).$$

PROOF. Due to $(A1)$, $(A2)$, the vector function $\psi : [0, \infty) \mapsto R^n$ defined by (3.11) is locally absolutely continuous and satisfies condition (i). We shall prove condition (ii) by using simple needle variations of the control $u_*(\cdot)$.

Let us fix an arbitrary $v \in U$. As in the preceding section, denote by $\Omega(v)$ the set of all $\tau > 0$ which are Lebesgue points of each of the measurable functions $f(\cdot, x_*(\cdot), u_*(\cdot))$, $g(\cdot, x_*(\cdot), u_*(\cdot))$, $f(\cdot, x_*(\cdot), v)$, $g(\cdot, x_*(\cdot), v)$. Let us fix an arbitrary $\tau \in \Omega(v)$ (notice that $[0, \infty) \setminus \Omega(v)$ is of measure zero).

Let $\alpha_0 > 0$ and c be the numbers from Lemma 3.2. Let $u_\alpha(\cdot)$ be defines as in (3.1). According to Lemma 3.2 the corresponding trajectory $x_\alpha(\cdot)$ is defined on $[0, \infty)$.

Let us fix an arbitrary number $\varepsilon_0 > 0$ and also the number $\tilde{T} > \tau$ in such a way that $\int_{\tilde{T}}^{\infty} \lambda(t) \, dt \leq \varepsilon_0$. According to Definition 2.5 for every $\alpha \in (0, \alpha_0] \cap (0, \delta]$, for $\varepsilon := \alpha^2$, and for the number $T = \tilde{T}$ there exists $T_\alpha \geq \tilde{T}$ such that

$$J_{T_\alpha}(x_\alpha(\cdot), u_\alpha(\cdot)) - J_{T_\alpha}(x_*(\cdot), u_*(\cdot)) \leq \alpha^2.$$

Then from (3.12) we obtain that

$$\mathcal{H}(\tau, x_*(\tau), v, \psi(\tau)) - \mathcal{H}(\tau, x_*(\tau), u_*(\tau), \psi(\tau)) \leq \alpha - \eta(\alpha, T_\alpha).$$

Since $\tilde{T} \in [\tau, T_\alpha]$, we obtain from (3.13)

$$\begin{aligned}\mathcal{H}(\tau, x_*(\tau), v, \psi(\tau)) - \mathcal{H}(\tau, x_*(\tau), u_*(\tau), \psi(\tau)) &\leq \alpha + \sigma(\alpha, \tilde{T}) + c\int_{\tilde{T}}^{\infty} \lambda(t)\, dt \\ &\leq \alpha + \sigma(\alpha, \tilde{T}) + c\varepsilon_0.\end{aligned}$$

Passing to the limit with $\alpha \to 0$ and then taking into account that ε_0 was arbitrarily chosen, we obtain that

$$\mathcal{H}(\tau, x_*(\tau), u_*(\tau), \psi(\tau)) \geq \mathcal{H}(\tau, x_*(\tau), v, \psi(\tau)).$$

For the fixed $v \in U$, the last inequality holds for every $\tau \in \Omega(v)$. Let U^d be a countable and dense subset of U. From the above inequality we have

$$\mathcal{H}(t, x_*(t), u_*(t), \psi(t)) \geq \mathcal{H}(t, x_*(t), v, \psi(t)) \quad \text{for every } v \in U^d$$

and for every $t \in \cap_{v \in U^d}\Omega(v)$, that is, for almost every t. Due to the continuity of the Hamiltonian with respect to u, the last inequality implies condition (ii). $\square$

5. Discussions

In this section we demonstrate some advantages of the main result of this paper compared with previously known results.

1. Let us return to the problem $(P1)$ considered in Example 2.4. Obviously, condition $(A1)$ is satisfied. Let $u_*(\cdot)$ be an arbitrary admissible control. Then $x_*(t) = 1 - e^{-\int_0^t u_*(s)\, ds}$, $t \geq 0$, is the corresponding admissible trajectory. Further, let $x(\zeta; \cdot)$ be a solution of equation (2.4) with $u(\cdot) = u_*(\cdot)$ and initial condition $x(0) = \zeta \in R^1$ (instead of $x(0) = x_0 = 0$). Then $x(\zeta; \cdot)$ is defined on $[0, \infty)$ and

$$x(\zeta; t) = 1 - (1 - \zeta)e^{-\int_0^t u_*(s)\, ds} \quad \text{for all} \quad t \geq 0.$$

In this example $g(t, x, u) = (1 - x)u$, $x \in G = R^1$, $u \in U = [0, 1]$. Hence, for any $\theta \in R^1$ we have $g_x(t, \theta, u_*(t)) = -u_*(t)$ for a.e. $t \geq 0$. Thus, we get

$$\max_{\theta \in [x(\zeta;t), x_*(t)]} |g_x(t, \theta, u_*(t))\, (x(\zeta; t) - x_*(t))| \overset{a.e.}{=} |\zeta - x_0|\lambda(t),$$

where

$$\lambda(t) = u_*(t)e^{-\int_0^t u_*(s)\, ds} \quad \text{for all} \quad t \geq 0.$$

The function $\lambda(\cdot)$ is integrable on $[0, \infty)$. Hence, condition $(A2)$ is also satisfied in problem $(P1)$ (with an arbitrary $\gamma > 0$).

Thus, due to Theorem 4.1 any LWOO control $u_*(\cdot)$ in problem $(P1)$ satisfies the core conditions of the maximum principle with adjoint variables $\psi^0 = 1$ and

$$\psi(t) = Z_*(t)I_*(t) = e^{\int_0^t u_*(s)\,ds}\Big[\lim_{T\to\infty} e^{-\int_0^T u_*(s)\,ds} - e^{-\int_0^t u_*(s)\,ds}\Big], \quad t \ge 0. \tag{5.1}$$

In problem $(P1)$, the maximum condition takes the following form:

$$u_*(t) \overset{a.e.}{=} \begin{cases} 0, & \text{if} \quad \psi(t) < -1, \\ \bar{u} \in [0,1], & \text{if} \quad \psi(t) = -1, \\ 1, & \text{if} \quad \psi(t) > -1. \end{cases} \tag{5.2}$$

Two cases are possible: either $\int_0^\infty u_*(t)\,dt = \infty$ or $\int_0^\infty u_*(t)\,dt < \infty$.

In the first case, due to (5.1) we have $\psi(t) = -1$ for a.e. $t \ge 0$, that agrees with the maximum condition (5.2).

In the second case, due to (5.1) we have $\psi(t) > -1$, $t \ge 0$, and due to the maximum condition (5.2) we get $u_*(t) = 1$ for a.e. $t \ge 0$, which contradicts the assumption $\int_0^\infty u_*(t)\,dt < \infty$.

Hence, only admissible controls $u_*(\cdot)$ such that $\int_0^\infty u_*(t)\,dt = \infty$ together with the unique adjoint variable $\psi(t) \equiv -1$, $t \ge 0$, satisfies all conditions of Theorem 4.1. It follows from (2.5) that indeed all such admissible controls are the only strongly optimal (and hence they are LWOO controls) in problem $(P1)$.

From this we conclude that condition (5.1) is the "right" complementary condition to the core conditions of the maximum principle in problem $(P1)$ while the standard transversality conditions (1.1) and (1.2) are inconsistent with them in this case.

Notice that the stationarity condition

$$H(t, x_*(t), \psi(t)) := \sup_{u\in U} \mathcal{H}(t, x_*(t), u, \psi(t)) \to 0 \quad \text{as} \quad t \to \infty,$$

suggested in [**14**] for strongly optimal admissible pairs in autonomous problem (P) with possible discounting (see [**3, 5, 14**] for details) provides no useful information in this example. Indeed, $H(t, x(t), -1) \equiv 0$, $t \ge 0$, along any admissible trajectory $x(\cdot)$ in problem $(P1)$.

Notice also that [**3**, Theorem 4] contains the same as in Theorem 4.1 explicit single-valued characterization (3.11) of the adjoint variable $\psi(\cdot)$, but this result is not applicable here because the utility functional (2.4) does not satisfy the uniform estimate (A3) in [**3**].

2. Typical models of optimal economic growth (see for example [**9**]) are formulated as infinite-horizon optimal control problems $(\tilde{P})$ with explicit discounting:

$$J(x(\cdot), u(\cdot)) = \int_0^\infty e^{-\rho t}\tilde{g}(t, x(t), u(t))\,dt \to \max, \tag{5.3}$$

$$\dot{x}(t) = f(t, x(t), u(t)), \qquad u(t) \in U,$$

$$x(0) = x_0.$$

Here $\rho \in R^1$ is a "discount" rate (which could be even negative). Functions $f : [0,\infty) \times G \times U \mapsto R^n$ and $\tilde{g} : [0,\infty) \times G \times U \mapsto R^1$ are assumed to satisfy condition $(A1)$. All other data in problem $(\tilde{P})$ are as in problem (P).

Obviously, problem $(\tilde{P})$ is a particular case of problem (P) with function $g(t,x,u) = e^{-\rho t}\tilde{g}(t,x,u)$, $t \geq 0$, $x \in G$, $u \in U$ (actually the two problem are equivalent). The only difference between $(\tilde{P})$ and (P) is that the integrand in (5.3) contains the discount factor $e^{-\rho t}$ explicitly.

Assume that $u_*(\cdot)$ is a LWOO control in problem $(\tilde{P})$ and $x_*(\cdot)$ – the corresponding admissible trajectory.

Now let us specify some sufficient conditions for validity of $(A2)$ (for problem (P) with function $g(t,x,u) = e^{-\rho t}\tilde{g}(t,x,u)$, $t \geq 0$, $x \in G$, $u \in U$) in terms of the discount rate ρ and the growth parameters of problem $(\tilde{P})$. For this end let us introduce the corresponding growth parameters $\lambda \in R^1$, $\mu \geq 0$ and $r \geq 0$ (see analogous conditions $(A2)$ and $(A3)$ in [**7**]).

Assumption (A3): *There exist numbers $\lambda \in R^1$, $\mu \geq 0$, $r \geq 0$, $\kappa \geq 0$, $\gamma > 0$, $c_3 \geq 0$ and $c_4 > 0$ such that*

(i) $\|x_*(t)\| \leq c_3 e^{\mu t}$ *for every* $t \geq 0$,

(ii) *for every $\zeta \in G$ with $\|\zeta - x_0\| < \gamma$ equation* (2.3) *with $u(\cdot) = u_*(\cdot)$ and initial condition $x(0) = \zeta$ (instead of $x(0) = x_0$) has a solution $x(\zeta;\cdot)$ on $[0,\infty)$ in G and the following estimations hold:*

$$\|x(\zeta;t) - x_*(t)\| \leq c_4\|\zeta - x_0\| e^{\lambda t} \quad \textit{for every} \quad t \geq 0,$$

$$\|\tilde{g}_x(t,\theta,u_*(t))\| \overset{a.e.}{\leq} \kappa\left(1 + \|\theta\|^r\right) \qquad \textit{for every} \;\; \theta \in [x(\zeta;t), x_*(t)], \quad t \geq 0.$$

(The number λ should not be confused with the function $\lambda(\cdot)$ in (A2).)

The following inequality gives a sufficient condition for the validity of $(A2)$ in terms of the growth parameters of problem $(\tilde{P})$ (see similar dominating discount conditions $(A6)$ in [**3**], $(A7)$ in [**5**] and $(A4)$ in [**7**]).

Assumption (A4):

$$\rho > \lambda + r\max\{\mu, \lambda\}.$$

Lemma 5.1. *Conditions* $(A3)$ *and* $(A4)$ *imply the validity of condition* $(A2)$ *for problem* (P) *with function* $g(t,x,u) = e^{-\rho t}\tilde{g}(t,x,u)$, $t \geq 0$, $x \in G$, $u \in U$.

Proof. Indeed, due to $(A3)$ (ii) for every $\zeta \in G$ with $\|\zeta - x_0\| < \gamma$, equation (2.3) with $u(\cdot) = u_*(\cdot)$ and initial condition $x(0) = \zeta$ (instead of $x(0) = x_0$) has a solution $x(\zeta;\cdot)$ on $[0,\infty)$ in G. Further, due to estimations (i) and (ii) of $(A3)$, we get

$$\begin{aligned} &\max_{\theta\in[x(\zeta;t),x_*(t)]}\left|\langle e^{-\rho t}\tilde{g}_x(t,\theta,u_*(t)), x(\zeta;t) - x_*(t)\rangle\right| \\ &\overset{a.e.}{\leq} \kappa c_3 c_4\|\zeta - x_0\| e^{-\rho t}\left(1 + e^{r\max\{\mu,\lambda\}t}\right)e^{\lambda t}, \qquad t \geq 0. \end{aligned}$$

Due to $(A4)$, the last estimation implies the validity of $(A2)$ for problem (P) with function $g(t,x,u) = e^{-\rho t}\tilde{g}(t,x,u)$, $t \geq 0$, $x \in G$, $u \in U$ with integrable function

$$\lambda(t) = \kappa c_3 c_4 e^{-(\rho-\lambda)t}\left(1 + e^{r\max\{\mu,\lambda\}t}\right), \qquad t \geq 0.$$

$\square$

Lemma 5.1 together with Theorem 4.1 implies the following version of the maximum principle for non-autunomous infinite-horizon problem $(\tilde{P})$ with dominating discount which is similar to [**7**, Theorem 1].

THEOREM 5.2. *Let $u_*(\cdot)$ be an admissible LWOO control in problem $(\tilde{P})$ and let $x_*(\cdot)$ be the corresponding trajectory. Assume that $(A3)$ and $(A4)$ hold. Then*

(i) *For any $t \geq 0$ the integral*

$$I_*(t) = \int_t^\infty e^{-\rho s}[Z_*(s)]^{-1}\tilde{g}_x(s, x_*(s), u_*(s))\,ds$$

converges absolutely.

(ii) *The vector function $\psi : [0,\infty) \mapsto R^n$ defined by*

$$\psi(t) = Z_*(t)I_*(t), \quad t \geq 0$$

is (locally) absolutely continuous and satisfies the conditions of the normal form maximum principle, i.e. $\psi(\cdot)$ is a solution of the adjoint system

$$\dot{\psi}(t) = -\mathcal{H}_x(t, x_*(t), u_*(t), \psi(t))$$

and the maximum condition holds:

$$\mathcal{H}(t, x_*(t), u_*(t), \psi(t)) \overset{a.e.}{=} \sup_{u\in U}\mathcal{H}(t, x_*(t), u, \psi(t)).$$

Notice that in the case of problem $(P1)$ considered in Example 2.4 the dominating discount condition $(A4)$ is not satisfied. Indeed, it is easy to see that $\rho = 0$, $\lambda = 1$, $\mu = 0$ and $r = 0$ (see $(A3)$) in this case. Thus $(A4)$ fails. Thus Theorem 5.2 is not applicable to problem $(P1)$ while Theorem 4.1 gives a complete description of all strongly optimal solutions in this problem.

3. Now we consider another example that shows the advantage of a certain invariance property of our assumption $(A2)$.

Consider the following problem $(P2)$:

$$J(x(\cdot), u(\cdot)) = \int_0^\infty \left(1 - \frac{1}{x(t)}\right) dt \to \max,$$

$$\dot{x}(t) = u(t)\,x(t), \qquad u(t) \in [0,1],$$

$$x(0) = 1.$$

Here $x \in R^1$ and $G = (0,\infty)$. Obviously condition $(A1)$ is satisfied.

Let $u(\cdot)$ be an admissible control in problem $(P2)$. Then $x(t) = e^{\int_0^t u(s)\,ds}$, $t \geq 0$, is the corresponding admissible trajectory and

$$J(x(\cdot), u(\cdot)) = \int_0^\infty \left(1 - e^{-\int_0^t u(s)\,ds}\right) dt \tag{5.4}$$

is the corresponding utility value. Obviously, if $\operatorname{meas}\{t \geq 0 : u(t) \neq 0\} > 0$ then the corresponding utility value (5.4) is ∞, and hence the concept of strong optimality is not applicable here. It is easy to see that $u_*(t) \stackrel{a.e.}{=} 1$, $x_*(t) \equiv e^t$, $t \geq 0$ is a unique LWOO pair in $(P2)$.

For problem $(P2)$ the dominating discount condition $(A4)$ in Theorem 5.2 (as far as similar condition in [**7**, Theorem 1]) is read as $\rho > \lambda$, where ρ is the discount rate (in our case $\rho = 0$) and the scalar λ (see $(A3)$) should be such that $\|x(\zeta; t) - x_*(t)\| \leq c_4\|\zeta - x(0)\|\, e^{\lambda t}$ is satisfied with some constant $c_4 \geq 0$ and all $t \geq 0$. Clearly in our case $\lambda = 1$, this dominating discount conditions that formulated in terms of growth parameters of problem $(\tilde{P})$ is violated.

On the other hand, if we introduce the new state variable $\tilde{x}(t) = e^{-\rho t}x(t)$, $t \geq 0$ with a $\rho \in (0, 1)$ then in terms of the state variable $\tilde{x}(\cdot)$ we obtain the following (equivalent to $(P2)$) optimal control problem $(P3)$:

$$J(\tilde{x}(\cdot), u(\cdot)) = \int_0^\infty e^{-\rho t}\Big(e^{\rho t} - \frac{1}{\tilde{x}(t)}\Big)\, dt \to \max,$$

$$\dot{\tilde{x}}(t) = (u(t) - \rho)\, \tilde{x}(t), \qquad u(t) \in [0, 1],$$

$$\tilde{x}(0) = 1.$$

Set $G = (0, \infty)$. Obviously condition $(A1)$ is satisfied.

Problem $(P3)$ has a unique LWOO solution $u_*(t) \stackrel{a.e.}{=} 1$, $\tilde{x}_*(t) \equiv e^{(1-\rho)t}$, $t \geq 0$. Here we have a discount rate $\rho \in (0, 1)$ and $\lambda = 1 - \rho$. Therefore the dominating discount condition $\rho > \lambda$ (see (A4)) holds in problem $(P3)$ if $\rho > 1/2$.

The above example shows that the possibility to apply the results based on the dominating discount conditions which are formulated in terms of the discount rate and growth parameters of problem $(\tilde{P})$ (such as Theorem 5.2, [**3**, Lemma 4], [**4**, Theorem 4], [**5**, Theorem 12.1] and [**7**, Theorem 1])) crucially depends on the particular reformulation of the problem out of many possible.

Now let us consider condition $(A2)$ in Theorem 4.1 for the first formulation of the above problem (see $(P2)$). Here $g_x(t, \theta, u_*(t)) = 1/\theta^2$ and $x(\zeta; t) = \zeta\, e^t$, $t \geq 0$. Then the inequality in (A2) with $\zeta = 1 + \beta$, $|\beta| < 1/2$, reads as

$$\sup_{s \in [0,1]} \frac{1}{e^{2t}(1 + s\beta)^2}\, e^t \beta \leq \beta\lambda(t), \quad t \geq 0,$$

which is obviously satisfied with the integrable function $\lambda(\cdot)$: $\lambda(t) = 4\, e^{-t}$, $t \geq 0$. Thus condition $(A2)$ holds and hence Theorem 4.1 is applicable in this case.

It is easy to verify that condition $(A2)$ holds also for the second version of the problem (see $(P3)$) with the same function $\lambda(\cdot)$.

Note also that in this example the alternative variant of the maximum principle with "invariant" dominating discount condition [**3**, Theorem 4] is not applicable because the optimal utility value is equal to ∞.

References

[1] V. M. Alexeev, V. M. Tikhomirov, S. V. Fomin, *Optimal control*, Nauka, Moscow, 1979 (Plenum, New York, 1987).

[2] A. V. Arutyunov, *The Pontryagin maximum principle and sufficient optimality conditions for nonlinear problems*, Differential Equations, **39** (2003) 1671-1679.

[3] S. M. Aseev, K. O. Besov, and A. V. Kryazhimskii, *Infinite-horizon optimal control problems in economics*, Russ. Math. Surv., **67**:2 (2012) 195-253.

[4] S. M. Aseev and A. V. Kryazhimskii, *T*he Pontryagin Maximum principle and transversality conditions for a class optimal control problems with infinite time horizons, SIAM J. Control Optim., **43** (2004) 1094-1119.

[5] S. M. Aseev and A. V. Kryazhimskii, *The Pontryagin maximum principle and optimal economic growth problems*, Proc. Steklov Inst. Math., **257** (2007) 1-255.

[6] S. M. Aseev and A. V. Kryazhimskii, *On a class of optimal control problems arising in mathematical economics*, Proc. Steklov Inst. Math., **262** (2008) 1-16.

[7] S. M. Aseev, V. M. Veliov, *Maximum principle for infinite-horizon optimal control problems with dominating discount*, Dynamics of Continuous, Discrete and Impulsive Systems, Ser. B: Applications & Algorithms, **19** (2012), 43–63.

[8] J.-P. Aubin and F. H. Clarke, *Shadow prices and duality for a class of optimal control problems*, SIAM J. Control Optim., **17**, 567-586 (1979).

[9] R.J. Barro, X. Sala-i-Martin, *Economic growth*, McGraw Hill, New York, 1995.

[10] D. A. Carlson, A. B. Haurie, A. Leizarowitz, *Infinite horizon optimal control. Deterministic and stochastic systems*, Springer, Berlin, 1991.

[11] A. F. Filippov, *Differential equations with discontinuous right-hand sides*, Nauka, Moscow, 1985 (Kluwer, Dordrecht, 1988).

[12] H. Halkin, Necessary conditions for optimal control problems with infinite horizons, *Econometrica*, **42** (1974) 267-272.

[13] P. Hartman, *Ordinary differential equations*, J. Wiley & Sons, New York, 1964.

[14] P. Michel, *On the transversality condition in infinite horizon optimal problems*, Econometrica, **50** (1982) 975-985.

[15] I. P. Natanson, *Theory of functions of a real variable*, Frederick Ungar Publishing Co., New York, 1955.

[16] L. S. Pontryagin, V. G. Boltyanskij, R. V. Gamkrelidze, E. F. Mishchenko, *The mathematical theory of optimal processes*, Fizmatgiz, Moscow, 1961 (Pergamon, Oxford, 1964).

[17] K. Shell, *Applications of Pontryagin's maximum principle to economics*, Mathematical Systems Theory and Economics 1, Springer, Berlin, 1969, 241-292 (Lect. Notes Oper. Res. Math. Econ. **11**).

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