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Aseev, S.M., Besov, K.O. and Kaniovski, S.

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International Institute for
Applied Systems Analysis
Schlossplatz 1
A-2361 Laxenburg, Austria

Tel: +43 2236 807 342
Fax: +43 2236 71313
E-mail: publications@iiasa.ac.at
Web: www.iiasa.ac.at

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Optimal Endogenous Growth with Exhaustible Resources

Sergey Aseev (aseev@mi.ras.ru)
Konstantin Besov (kbesov@mi.ras.ru)
Serguei Kaniovski (Serguei.Kaniovski@wifo.ac.at)

Approved by

Arkady Kryazhimskiy
Leader, Dynamic Systems Programs

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Abstract

We study optimal research and extraction policies in an endogenous growth model in which both production and research require an exhaustible resource. It is shown that optimal growth is not sustainable if the accumulation of knowledge depends on the resource as an input, or if the returns to scale in research are decreasing. The model is stated as an infinite-horizon optimal control problem with an integral constraint on the control variables. We consider the main mathematical aspects of the problem, establish an existence theorem and derive an appropriate version of the Pontryagin maximum principle. A complete characterization of the optimal transitional dynamics is given.

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About the Authors

Sergey Aseev
International Institute for Applied Systems Analysis (IIASA)
Schlossplatz 1
A-2361 Laxenburg - Austria
Steklov Mathematical Institute
Gubkina 8
119991 Moscow - Russia
Email: aseev@mi.ras.ru

Konstantin Besov
Steklov Mathematical Institute
Gubkina 8
119991 Moscow - Russia
Email: kbesov@mi.ras.ru

Serguei Kaniovski
Austrian Institute of Economic Research (WIFO)
Arsenal, Objekt 20
A-1103 Vienna - Austria
Email: serguei.kaniovski@wifo.ac.at

Optimal Endogenous Growth with Exhaustible Resources

Sergey Aseev, Konstantin Besov and Serguei Kaniovski

1 Introduction

Endogenous growth theory identifies technological progress as a means of sustaining economic growth despite the reliance on exhaustible resources as inputs to production. The supply of an exhaustible resource will limit growth in the long run, unless the economy can either substitute away from the resource or increase the efficiency of the resource's use to offset its scarcity. The question is then: Can an optimal research and extraction policy compensate for negative effects on production (consumption) which arise due to scarcity of the exhaustible resource?

Our point of departure is the model by Jones [18, 19]. In this two-sector endogenous growth model the production sector yields output that is consumed, while the research sector augments the productivity of the production means. There are constant returns to scale in production, and either weak or strong scale effects in the research sector. We assume that both the production of output and the generation of knowledge depend on an exhaustible resource as an input. For the sake of simplicity we consider a constant population (total labor supply) that can be shifted between production and research.

We show that optimal growth is sustainable only if the accumulation of knowledge has constant returns to scale (strong scale effects) and does not depend on the exhaustible resource. While the requirement of strong scale effects is well-known in the literature[1], the requirement of the research sector being independent of the exhaustible resource is a new result. We see sustainable (or continuing) growth as a welfare-maximizing output trajectory having perpetual positive growth, where the welfare is measured by a discounted logarithmic utility function.

The model is presented as an infinite-horizon optimal control problem whose solution is a welfare-maximizing dynamic research and extraction policy. Formulating the model as an optimal control problem allows us to derive model-consistent policy recommendations, as opposed to describing the economy either in a dynamic equilibrium, or on a balanced growth path only.

We offer a complete and rigorous inquiry into the existence of an optimal solution and formulate an appropriate version of Pontryagin's maximum principle. The problem

[1] Jones's critique generated academic interest in endogenous growth models that generate a steady-state with positive growth without strong scale effects. See, e.g., [10, 18, 22, 27].

is non-standard in that it involves an integral constraint on control variables. Integral constraints naturally arise in growth models in which an exhaustible resource is an input, as the amount of the resource extracted cannot exceed a finite stock of the resource initially available to an economy.

The paper is organized as follows. In Section 2 we specify the model as an infinite-horizon optimal control problem with an integral constraint on the control variables. In Section 3 we discuss some mathematical aspects of the problem. Section 4 is devoted to reducing the problem to an equivalent problem without any integral constraints. In Section 5 we establish an existence result for an auxiliary optimal control problem and derive an appropriate version of the Pontryagin maximum principle for the auxiliary problem, which automatically implies the existence of an optimal solution and a version of the Pontryagin maximum principle for the initial problem. In Section 6 we apply this result in order to determine an optimal research and extraction policy. Section 7 draws conclusions.

2 The model

At every instant $t \in [0, \infty)$, the economy produces output $Y(t) > 0$, which is assumed to be described by a Cobb–Douglas production function:

$$Y(t) = A(t)^{\varkappa}[L - L^A(t)]^{\alpha} R_1(t)^{1-\alpha} \qquad \text{where} \quad \alpha \in (0,1) \quad \text{and} \quad \varkappa > 0. \tag{1}$$

Here $A(t) > 0$ is the current knowledge stock and $R_1(t) > 0$ is the quantity of the exhaustible resource used in production. The population (total labor supply) is fixed at $L > 0$. Part of the labor $L - L^A(t)$ is employed in production, while the other part $L^A(t) \in [0, L)$ is allocated to research.

The amount of new knowledge produced at time t depends on the hitherto accumulated knowledge, the number of researchers and the portion of the exhaustible resource used in research:

$$\dot{A}(t) = A(t)^{\theta}[L^A(t)]^{\eta} R_2(t)^{1-\eta} \qquad \text{where} \quad \eta \in (0,1] \quad \text{and} \quad \theta \in (0,1]. \tag{2}$$

Here $R_2(t) \geq 0$ is the quantity of the exhaustible resource used in research; typically $R_2(t)$ is small compared to $R_1(t)$. The initial knowledge stock is given by $A(0) = A_0 > 0$. If $\theta \in (0,1)$, then growth rate of the knowledge stock decreases while the knowledge stock expands. The case of $\theta < 0$—when the expansion of knowledge is progressively more difficult—has also been considered in the literature (see, e.g., [19]). Empirical evidence supports the idea of weak scale effects, i.e. $\theta < 1$, in the production of knowledge. We retain $\theta = 1$ as a special case of strong scale effects.

The fact that the stock of the exhaustible resource is finite imposes the following integral constraint on the controls $R_1(\cdot)$ and $R_2(\cdot)$:

$$\int_0^{\infty} \big[R_1(t) + R_2(t)\big]\, dt \leq S_0, \tag{3}$$

where $S_0 > 0$ is the initial supply of the exhaustible resource.

We take a discounted logarithmic utility function of the output as a measure of *welfare*. This leads to the following objective functional for the economy (see (1)):

$$
\begin{aligned}
J(A(\cdot), L^A(\cdot), R_1(\cdot)) &= \int_0^\infty e^{-\rho t}\{\ln[Y(t)]\}\,dt \\
&= \int_0^\infty e^{-\rho t}\Big\{\varkappa \ln A(t) + \alpha \ln[L - L^A(t)] + (1-\alpha)\ln R_1(t)\Big\}\,dt,
\end{aligned}
$$

where $\rho > 0$ is a subjective discount rate. Maximizing the discounted output is equivalent to maximizing the discounted output per capita, as the population is constant.

Given the parameters $\theta \in (0,1]$, $\alpha \in (0,1)$, $\varkappa > 0$, $\eta \in (0,1]$, $\rho > 0$, $L > 0$ and $S_0 > 0$, the optimization problem $J(A(\cdot), L^A(\cdot), R_1(\cdot)) \to \max$, subject to equation (2) and the resource constraint (3), can be formulated as the following infinite-horizon optimal control problem (P):

$$\dot{A}(t) = A(t)^\theta [L^A(t)]^\eta R_2(t)^{1-\eta}, \tag{4}$$

$$L^A(t) \in [0, L), \qquad R_1(t) > 0, \qquad R_2(t) \geq 0, \qquad \int_0^\infty \big[R_1(t) + R_2(t)\big]\,dt \leq S_0, \tag{5}$$

$$A(0) = A_0 > 0, \tag{6}$$

$$J(A(\cdot), L^A(\cdot), R_1(\cdot)) = \int_0^\infty e^{-\rho t}\Big\{\varkappa \ln A(t) + \alpha \ln[L - L^A(t)] + (1-\alpha)\ln R_1(t)\Big\}\,dt \to \max. \tag{7}$$

By an *admissible control* $w(\cdot)\colon [0,\infty) \to \mathbb{R}^3$ in problem (P) we mean a triple $w(\cdot) = (L^A(\cdot), R_1(\cdot), R_2(\cdot))$, $t \geq 0$, of (locally) bounded measurable functions $L^A(\cdot)$, $R_1(\cdot)$ and $R_2(\cdot)$ each of which is defined on the infinite half-open time interval $[0,\infty)$ and satisfies the respective constraints in (5).

An *admissible trajectory* $A(\cdot)\colon [0,\tau) \to \mathbb{R}^1$, $\tau > 0$, corresponding to an admissible control $w(\cdot)$ is a (locally) absolutely continuous function $A(\cdot)$ which is a (Carathéodory) solution (see [11]) of the differential equation (4) on some (finite or infinite) time interval $[0,\tau)$, subject to the initial condition (6).

Due to (4) and the integral constraint in (5), for any admissible control $w(\cdot) = (L^A(\cdot), R_1(\cdot), R_2(\cdot))$ the corresponding admissible trajectory $A(\cdot)$ can be extended to the whole infinite interval $[0,\infty)$. Consequently, in what follows, without loss of generality, we always assume that any admissible trajectory $A(\cdot)$ is defined on $[0,\infty)$.

A pair $(A(\cdot), w(\cdot))$, where $w(\cdot)$ is an admissible control and $A(\cdot)$ is the corresponding admissible trajectory, is called an *admissible pair* (or a *process*) in problem (P).

For any admissible pair $(A(\cdot), w(\cdot))$ the improper integral in (7) converges either to $-\infty$ or to a finite real. Moreover, it is uniformly bounded from above; i.e., there is a number $M \geq 0$ such that

$$\sup_{(A(\cdot), w(\cdot))} \int_0^\infty e^{-\rho t}\Big\{\varkappa \ln A(t) + \alpha \ln[L - L^A(t)] + (1-\alpha)\ln R_1(t)\Big\}\,dt \leq M, \tag{8}$$

where the supremum is taken over all admissible pairs $(A(\cdot), w(\cdot))$.

Indeed, due to the integral constraint in (5), for any admissible control $w(\cdot)$ we have

$$\int_0^\infty e^{-\rho t}\ln R_1(t)\,dt < \int_0^\infty e^{-\rho t} R_1(t)\,dt < S_0. \tag{9}$$

Further, for an arbitrary admissible trajectory $A(\cdot)$ we have

$$A(t)^\theta \le A(t)+1, \qquad t \ge 0.$$

Then, due to (4), we obtain

$$\frac{d}{dt}\ln(A(t)+1) = \frac{\dot{A}(t)}{A(t)+1} \le L^\eta R_2(t)^{1-\eta}, \qquad t \ge 0,$$

and hence

$$\begin{aligned}\ln(A(t)+1) &\le \ln(A_0+1) + L^\eta \int_0^t R_2(s)^{1-\eta}\,ds \le \ln(A_0+1) + L^\eta \int_0^t (1+R_2(s))\,ds \\ &< \ln(A_0+1) + L^\eta(t+S_0), \qquad t \ge 0. \end{aligned} \tag{10}$$

This inequality immediately implies the following inequality for an arbitrary admissible trajectory $A(\cdot)$:

$$\int_0^\infty e^{-\rho t}\ln A(t)\,dt < \int_0^\infty e^{-\rho t}\ln(A(t)+1)\,dt < \frac{\ln(A_0+1)+L^\eta S_0}{\rho} + \frac{L^\eta}{\rho^2}. \tag{11}$$

Since $L^A(t) \in [0, L)$, $t \ge 0$ (see (5)), inequalities (9) and (11) provide the following uniform estimate for all control processes $(A(\cdot), w(\cdot))$:

$$\begin{aligned}&\int_0^\infty e^{-\rho t}\Big\{\varkappa \ln A(t) + \alpha \ln[L - L^A(t)] + (1-\alpha)\ln R_1(t)\Big\}\,dt \\ &\qquad\qquad < \varkappa\frac{\ln(A_0+1)+L^\eta S_0}{\rho} + \frac{\varkappa L^\eta}{\rho^2} + \frac{\alpha \ln L}{\rho} + (1-\alpha)S_0.\end{aligned}$$

This furnishes the proof of inequality (8).

The uniform bound (8) allows us to define an *optimal control* $w_*(\cdot)\colon [0,\infty) \to \mathbb{R}^3$ in problem (P) as a welfare-maximizing triple $w_*(\cdot) = (L^A_*(\cdot), R_{1*}(\cdot), R_{2*}(\cdot))$ of dynamic labor and extraction policies adopted in the research and production sectors. The corresponding trajectory $A_*(\cdot)$ is an *optimal admissible trajectory.*

When the Hamiltonian is concave, some problems of this type can be examined by means of a well-known set of sufficient conditions [1,23]. However, the success of such an approach crucially depends on the analytical tractability of the conditions of the maximum principle.

In our analysis we follow the approach based on *necessary conditions* and an *existence theorem.* This approach is more systematic. It applies to a wider range of problems for which the existence of a solution can be shown. It should be stressed that without an existence theorem one cannot be sure that a path satisfying the necessary conditions exists, or that one of the paths satisfying the necessary conditions is indeed a solution (see the discussion in [21]). We follow this general approach by first establishing an existence result and then deriving appropriate necessary conditions—a version of the Pontryagin maximum principle—for the problem under study. As a result we obtain a rigorous characterization of all optimal processes in (P).

To the best of our knowledge, this is the first rigorous study of an endogenous growth model with integral constraints on control variables which is based on the application of necessary optimality conditions, although integral constraints on control variables are typical of a class of models in the resource and growth literature. Examples include models in the tradition of [17], [9] and [25]. [2]

3 Preliminary discussion

The formulated optimal control problem (P) (see (4)–(7)) is nonstandard in the sense that it is not completely embedded in the framework of the modern optimal control theory.

Problem (P) is formulated on the infinite time interval $[0,\infty)$. The infinite time horizon gives rise to specific mathematical features of the Pontryagin maximum principle. The most characteristic feature is that the adjoint variables (shadow prices) may exhibit pathological behavior in the long run (see examples of this phenomenon in [5,13,24]). This fact prevents us from applying "naïve" infinite-horizon analogs of the classical Pontryagin maximum principle [20] designed for processes of finite duration.

There exist modifications of the Pontryagin maximum principle for infinite-horizon optimal control problems that pay attention to the above-mentioned possible pathological behavior [4–7, 23]. Yet problem (P) fails to satisfy the assumptions imposed in them due to an integral constraint on the controls $R_1(\cdot)$ and $R_2(\cdot)$. To the best of our knowledge, a version of the Pontryagin maximum principle for infinite-horizon optimal control problems with integral constraints on control variables has not yet been established.

Note that we can lift the integral constraint in problem (P) by introducing an additional state variable subject to a state constraint. Indeed, consider the following optimal control problem (Q):

$$\dot{A}(t) = A(t)^{\theta}[L^A(t)]^{\eta} R_2(t)^{1-\eta}, \qquad \dot{S}(t) = -R_1(t) - R_2(t),$$

$$L^A(t) \in [0, L), \qquad R_1(t) > 0, \qquad R_2(t) \geq 0,$$

$$S(t) \geq 0 \qquad \text{for all} \quad t \geq 0, \tag{12}$$

$$A(0) = A_0 > 0, \qquad S(0) = S_0 > 0,$$

$$J(A(\cdot), L^A(\cdot), R_1(\cdot)) = \int_0^{\infty} e^{-\rho t} \Big\{ \varkappa \ln A(t) + \alpha \ln[L - L^A(t)] + (1-\alpha) \ln R_1(t) \Big\}\, dt \to \max.$$

In the above problem $S(\cdot)$ is an additional state variable representing the current stock of the exhaustible resource. Obviously problem (Q) is equivalent to the original problem (P). Problem (Q) involves state constraint (12) instead of an integral constraint. However, this

[2] For a survey, see [12]. Basic models are discussed in an easily accessible style in [26].

fact does not make the formal treatment of problem (Q) simpler than that of (P). Even in the case of a finite horizon, general versions of the Pontryagin maximum principle for problems with state constraints (see, for example, review [15]) involve measures or functions of bounded variation as adjoint variables. This general nature of adjoint variables considerably complicates the application of the maximum principle and may lead to additional difficulties (see, for example, [2]). As regards infinite-horizon optimal control problems with state constraints, an appropriate version of the Pontryagin maximum principle for such problems has not been established.

Thus, regardless of whether it is treated as problem (P) with an integral constraint on the control variables or as problem (Q) with a state constraint, our model is not completely embedded in the framework of modern optimal control theory.

Also note that, since the range of the controls $R_1(\cdot)$ and $R_2(\cdot)$ in the statements of both problems (P) and (Q) is unbounded, we cannot directly appeal to the standard results on existence of an optimal control in the class of locally bounded measurable functions (such results usually rely on pointwise boundedness conditions; see, for example, [8]). When the admissible control set is unbounded, the integral constraint can, under appropriate conditions, guarantee the existence of an optimal control, but only in a more general class of impulse controls. This is the case even if the welfare functional is bounded on the set of admissible pairs. We illustrate this phenomenon with the following simple example:

Example. Consider the optimal control problem

$$\dot{Y}(t) = R(t), \qquad Y(0) = Y_0 > 0, \tag{13}$$

$$R(t) \geq 0, \qquad \int_0^\infty R(t)\, dt \leq S_0, \tag{14}$$

$$J(Y(\cdot)) = \int_0^\infty e^{-\rho t} \ln Y(t)\, dt \to \max. \tag{15}$$

Here $S_0 > 0$ is the initial supply of a resource. Since the aim is to optimally exhaust a finite resource S_0, we may think of the above problem as an infinite-horizon cake-eating problem for an increasing and concave utility function of consumption. As intuition would suggest, costless extraction and a positive discount rate must lead to an instantaneous exhaustion. Below, we show that the solution is indeed an impulse $R_*(t) = S_0\delta_t(0)$, where $\delta_t(0)$ is the Dirac delta (the Dirac measure concentrated at 0). The character of the solution precludes the application of Pontryagin's maximum principle for infinite-horizon optimal control problems, which requires the solution to be a locally bounded measurable function (see, for example, [3–5]).

To see why there is no optimal control in the class of locally bounded measurable functions in problem (13)–(15), note that if $R(\cdot)$ were such a control, then for the corresponding admissible trajectory $Y(\cdot)$ we would have $Y(t) \leq Y_0 + S_0$ for all $t \geq 0$ and $Y(t) < Y_0 + S_0$ on a set $\mathsf{M} \subset [0, \infty)$ of positive measure. Hence,

$$\int_0^\infty e^{-\rho t} \ln Y(t)\, dt < \frac{1}{\rho} \ln(Y_0 + S_0).$$

On the other hand, the sequence of admissible controls $\{R_k(\cdot)\}$, with $R_k(t) = kS_0$ for

$t \in [0, 1/k]$ and $R_k(t) = 0$ for $t > 1/k$, $k = 1, 2, \dots$, is a maximizing one, as

$$\int_0^\infty e^{-\rho t} \ln Y_k(t)\, dt = \int_0^{1/k} e^{-\rho t} \ln\bigl(Y_0 + kS_0 t\bigr)\, dt + \frac{e^{-\rho/k}}{\rho} \ln(Y_0 + S_0) \to \frac{1}{\rho} \ln(Y_0 + S_0)$$

as $k \to \infty$. Hence, there is no optimal admissible control in the class of locally bounded measurable functions in problem (13)–(15), while an impulse control $R_*(t) = S_0 \delta_t(0)$, where $\delta_t(0)$ is the Dirac delta, is an optimal impulse control in problem (13)–(15).

In the next two sections we overcome the difficulties arising from the presence of an integral constraint on the control variables and unboundedness of the admissible control set in problem (P) by reducing problem (P) to an auxiliary problem (P1), and then to (P1′), without integral (and state) constraints. This allows us to prove an existence result and apply a version of the Pontryagin maximum principle developed in [4, 5] for problems with dominating discount.

4 Reduction to a one-dimensional problem without integral constraints

Let us introduce a new state variable $x(\cdot)\colon [0, \infty) \to \mathbb{R}^1$ and new control variables $u(\cdot)\colon [0, \infty) \to (0, \infty)$ and $v(\cdot)\colon [0, \infty) \to [0, \infty)$ as follows:

$$x(t) = \frac{S(t)^{1-\eta}}{A(t)^{1-\theta}}, \qquad u(t) = \frac{R_1(t)}{S(t)}, \qquad v(t) = \frac{R_2(t)}{S(t)}, \qquad t > 0. \tag{16}$$

Here, as in problem (Q) above, the state variable $S(\cdot)$ represents the current supply of the exhaustible resource. This variable is a (Carathéodory) solution to the following Cauchy problem (for given admissible controls $R_1(\cdot)$ and $R_2(\cdot)$) on $[0, \infty)$:

$$\dot{S}(t) = -R_1(t) - R_2(t), \qquad S(0) = S_0. \tag{17}$$

Note that the case $\eta = \theta = 1$ is not excluded, although in this case the new variable $x(\cdot)$ degenerates into a constant. This case can easily be analyzed directly, but we include it in our general scheme to save the space. Below we show that for $\eta = \theta = 1$ the problem reduces to a zero-dimensional problem, i.e. to a problem in which the utility function depends only on the controls and does not depend on the state variables (hence the control variables take constant values maximizing the utility function at each moment in time).

Note also that $S(t) > 0$ for all $t > 0$, so the quantities $u(t)$ and $v(t)$ are well defined for all $t > 0$. Indeed, if $S(\tau) = 0$ for some $\tau > 0$, then $S(t) = 0$ for all $t > \tau$ and hence $R_1(t) = R_2(t) = 0$ for $t > \tau$, which is precluded by (5). Moreover, $u(\cdot)$ and $v(\cdot)$ are locally bounded measurable functions since $R_i(\cdot)$, $i = 1, 2$, is locally bounded and measurable and $S(\cdot)$ is positive and continuous.

Since $x(\cdot)$ is a (locally) absolutely continuous function, we can calculate its derivative a.e. on $[0, \infty)$:

$$\begin{aligned}\dot{x}(t) &= (1-\eta)\frac{\dot{S}(t)}{A(t)^{1-\theta}S(t)^{\eta}} - (1-\theta)\frac{\dot{A}(t)S(t)^{1-\eta}}{A(t)^{2-\theta}} \\ &= -(1-\eta)[u(t)+v(t)]x(t) - (1-\theta)\frac{A(t)^{\theta}[L^A(t)]^{\eta}R_2(t)^{1-\eta}S(t)^{1-\eta}}{A(t)^{2-\theta}} \\ &= -(1-\eta)[u(t)+v(t)]x(t) - (1-\theta)[L^A(t)]^{\eta}v(t)^{1-\eta}x(t)^2.\end{aligned}$$

Thus, $x(\cdot)$ is a Carathéodory solution of the differential equation

$$\dot{x}(t) = -(1-\eta)[u(t)+v(t)]x(t) - (1-\theta)[L^A(t)]^{\eta}v(t)^{1-\eta}x(t)^2, \qquad t > 0, \tag{18}$$

satisfying the initial condition

$$x(0) = x_0 = \frac{S_0^{1-\eta}}{A_0^{1-\theta}}. \tag{19}$$

Now we express the functional $J(A(\cdot), L^A(\cdot), R_1(\cdot))$ (see (7)) in terms of the new variables $x(\cdot)$, $u(\cdot)$ and $v(\cdot)$. Consider the first term in the integrand in (7):

$$\int_0^{\infty} e^{-\rho t}\ln A(t)\,dt = \frac{\ln A_0}{\rho} + \frac{1}{\rho}\int_0^{\infty} e^{-\rho t}\frac{\dot{A}(t)}{A(t)}\,dt. \tag{20}$$

This formula is valid for any admissible trajectory $A(\cdot)$ of problem (P). To show this, it suffices first to integrate by parts on a finite time interval $[0, T]$ and then pass to the limit as $T \to \infty$:

$$\int_0^{T} e^{-\rho t}\ln A(t)\,dt = \frac{\ln A_0 - e^{-\rho T}\ln A(T)}{\rho} + \frac{1}{\rho}\int_0^{T} e^{-\rho t}\frac{\dot{A}(t)}{A(t)}\,dt. \tag{21}$$

Due to (10) the integral on the left-hand side and the first term on the right-hand side tend to the corresponding terms in (20). Further, $\dot{A}(t) \geq 0$, $t > 0$; therefore, $e^{-\rho t}\dot{A}(t)/A(t)$ is integrable on $[0, +\infty)$ and the last term in (21) tends to the last term in (20).

Substituting $\dot{A}(t)$ from (4) into (20), we obtain

$$\begin{aligned}\int_0^{\infty} e^{-\rho t}\ln A(t)\,dt &= \frac{\ln A_0}{\rho} + \frac{1}{\rho}\int_0^{\infty} e^{-\rho t}\frac{A(t)^{\theta}[L^A(t)]^{\eta}v(t)^{1-\eta}S(t)^{1-\eta}}{A(t)}\,dt \\ &= \frac{\ln A_0}{\rho} + \frac{1}{\rho}\int_0^{\infty} e^{-\rho t}[L^A(t)]^{\eta}v(t)^{1-\eta}x(t)\,dt.\end{aligned}$$

Similarly,

$$\begin{aligned}\int_0^{T} e^{-\rho t}\ln R_1(t)\,dt &= \int_0^{T} e^{-\rho t}\big[\ln u(t) + \ln S(t)\big]\,dt \\ &= \int_0^{T} e^{-\rho t}\ln u(t)\,dt + \frac{\ln S_0 - e^{-\rho T}\ln S(T)}{\rho} + \frac{1}{\rho}\int_0^{T} e^{-\rho t}\frac{\dot{S}(t)}{S(t)}\,dt \\ &= \frac{\ln S_0 - e^{-\rho T}\ln S(T)}{\rho} + \int_0^{T} e^{-\rho t}\left[\ln u(t) - \frac{u(t)+v(t)}{\rho}\right]dt.\end{aligned}$$

Passing to the limit as $T \to \infty$, we see that

$$\int_0^\infty e^{-\rho t} \ln R_1(t)\, dt = \frac{\ln S_0}{\rho} + \int_0^\infty e^{-\rho t} \left[\ln u(t) - \frac{u(t) + v(t)}{\rho} \right] dt,$$

where both sides may be $-\infty$.

Thus, multiplying $J(A(\cdot), L^A(\cdot), R_1(\cdot))$ by ρ and neglecting constant terms, we arrive at the functional

$$J_1(x(\cdot), L^A(\cdot), u(\cdot), v(\cdot)) = \int_0^\infty e^{-\rho t} \Big\{ \varkappa [L^A(t)]^\eta v(t)^{1-\eta} x(t) + \alpha\rho \ln[L - L^A(t)] + (1-\alpha)\rho \ln u(t) - (1-\alpha)[u(t) + v(t)] \Big\}\, dt. \quad (22)$$

Now consider the following optimal control problem (P1) (see (18), (19) and (22)):

$$\dot{x}(t) = -(1-\eta)[u(t) + v(t)]x(t) - (1-\theta)[L^A(t)]^\eta v(t)^{1-\eta} x(t)^2, \quad (23)$$

$$v(t) \in [0, \infty), \qquad L^A(t) \in [0, L), \qquad u(t) \in (0, \infty), \quad (24)$$

$$x(0) = x_0,$$

$$J_1(x(\cdot), L^A(\cdot), u(\cdot), v(\cdot)) = \int_0^\infty e^{-\rho t} \Big\{ \varkappa [L^A(t)]^\eta v(t)^{1-\eta} x(t) + \alpha\rho \ln[L - L^A(t)] + (1-\alpha)\rho \ln u(t) - (1-\alpha)[u(t) + v(t)] \Big\}\, dt \to \max. \quad (25)$$

We say that a control $\tilde{w}(\cdot) = (L^A(\cdot), u(\cdot), v(\cdot))\colon [0, \infty) \to [0, L) \times (0, \infty) \times [0, \infty)$ (which is a triple of measurable functions) is admissible in problem (P1) if the functions $u(\cdot)$ and $v(\cdot)$ are locally bounded. The corresponding trajectory $x(\cdot)\colon [0, \tau) \to \mathbb{R}^1$, $\tau > 0$, can obviously be extended to the whole infinite time interval $[0, \infty)$. So, without loss of generality, we assume that any admissible trajectory $x(\cdot)$ is defined on $[0, \infty)$. A pair $(x(\cdot), w(\cdot))$ where $w(\cdot)$ is an admissible control and $x(\cdot)$ is the corresponding trajectory is called an admissible pair or a process in problem (P1).

Note that, structurally, problem (P1) is simpler than both problems (P) and (Q) because problem (P1) contains neither integral constraints on the control variables, nor state constraints. Problem (P1) is equivalent to problem (P) in the following sense:

Lemma 1. *For fixed A_0 and S_0, there is a one-to-one correspondence between processes $(A(\cdot), w(\cdot))$ in problem* (P) *and $(x(\cdot), \tilde{w}(\cdot))$ in problem* (P1). *Moreover, the corresponding values of the objective functionals $J(A(\cdot), L^A(\cdot), R_1(\cdot))$ and $J_1(x(\cdot), L^A(\cdot), u(\cdot), v(\cdot))$ are related by a liner transformation of the form*

$$J_1(x(\cdot), L^A(\cdot), u(\cdot), v(\cdot)) = \rho J(A(\cdot), L^A(\cdot), R_1(\cdot)) + C, \quad (26)$$

where C depends only on ρ, A_0 and S_0.

Proof. As shown above, any process $(A(\cdot), w(\cdot)) = (A(\cdot), L^A(\cdot), R_1(\cdot), R_2(\cdot))$ in problem (P) generates a process $(x(\cdot), \tilde{w}(\cdot)) = (x(\cdot), L^A(\cdot), u(\cdot), v(\cdot))$ in problem (P1), and relation (26) is valid for these processes.

Now, we show that any control process $(x(\cdot), \tilde{w}(\cdot)) = (x(\cdot), L^A(\cdot), u(\cdot), v(\cdot))$ in problem (P1) corresponds to a control process $(A(\cdot), w(\cdot)) = (A(\cdot), L^A(\cdot), R_1(\cdot), R_2(\cdot))$ in problem (P). First, using the controls $u(\cdot)$ and $v(\cdot)$, we determine $S(\cdot)$ as a unique solution to the Cauchy problem

$$\dot{S}(t) = -[u(t) + v(t)]S(t), \qquad S(0) = S_0.$$

Since $u(\cdot) + v(\cdot)$ is positive and locally bounded, we obtain a positive monotonically decreasing function $S(\cdot)$ defined on $[0, \infty)$. Then we define $R_1(t) = u(t)S(t)$ and $R_2(t) = v(t)S(t)$, $t \geq 0$, which are locally bounded and satisfy the integral constraint in (5). Finally, we find $A(\cdot)$ as a unique solution to the Cauchy problem

$$\frac{d}{dt}\big[A(t)^{1-\theta}\big] = (1-\theta)[L^A(t)]^\eta v(t)^{1-\eta} S(t)^{1-\eta}, \qquad A(0) = A_0$$

if $\theta < 1$, or as a unique solution to the Cauchy problem

$$\frac{d}{dt}\big[\ln A(t)\big] = [L^A(t)]^\eta v(t)^{1-\eta} S(t)^{1-\eta}, \qquad A(0) = A_0$$

if $\theta = 1$. This is certainly possible because the right-hand side of each of these equations is positive and locally bounded.

We thus have a process $(A(\cdot), w(\cdot)) = (A(\cdot), L^A(\cdot), R_1(\cdot), R_2(\cdot))$ in problem (P). Passing from this process $(A(\cdot), w(\cdot))$ in problem (P) back to some process $(x_1(\cdot), \tilde{w}_1(\cdot))$ in problem (P1) along the scheme described at the beginning of this section, we see that $\tilde{w}_1(\cdot) = \tilde{w}(\cdot)$ and $x_1(\cdot)$ satisfies the same Cauchy problem (18), (19) as $x(\cdot)$. Therefore, by the uniqueness theorem for solutions of differential equations, $x_1(\cdot) = x(\cdot)$. This proves the required one-to-one correspondence between the admissible processes in problems (P) and (P1). Since (26) holds for the processes $(A(\cdot), w(\cdot))$ and $(x_1(\cdot), \tilde{w}_1(\cdot))$, and $(x_1(\cdot), \tilde{w}_1(\cdot)) = (x(\cdot), \tilde{w}(\cdot))$, we conclude that (26) is valid for $(A(\cdot), w(\cdot))$ and $(x(\cdot), \tilde{w}(\cdot))$. □

As a direct consequence of Lemma 1 and estimate (8) we arrive at

Lemma 2. *There exists a constant $M_1 > 0$ depending only on ρ, L, A_0 and S_0 such that*

$$\sup_{(x(\cdot),\tilde{w}(\cdot))} \int_0^\infty e^{-\rho t}\Big\{\varkappa[L^A(t)]^\eta v(t)^{1-\eta} x(t) + \alpha\rho \ln[L - L^A(t)] + (1-\alpha)\rho \ln u(t) - (1-\alpha)[u(t) + v(t)]\Big\}\, dt \leq M_1,$$

where the supremum is taken over all admissible pairs $(x(\cdot), \tilde{w}(\cdot))$ in problem (P1).

Lemma 2 allows us to define an optimal control $\tilde{w}_*(\cdot)\colon [0, \infty) \to \mathbb{R}^3$ in problem (P1) as a welfare-maximizing triple $\tilde{w}_*(\cdot) = (L^A_*(\cdot), u_*(\cdot), v_*(\cdot))$. The corresponding admissible trajectory $x_*(\cdot)$ is an optimal one in problem (P1).

To recapitulate, we have established that a process $(A(\cdot), w(\cdot))$ is optimal in problem (P) if and only if the corresponding process $(x(\cdot), \tilde{w}(\cdot))$ is optimal in problem (P1). In the next section we formulate and prove two main theoretical results on which the subsequent solution of the problem is based.

5 Existence of an optimal control and Pontryagin's maximum principle

Denote

$$
\begin{aligned}
f(x,\ell,u,v) &= -(1-\eta)(u+v)x - (1-\theta)\ell^{\eta}v^{1-\eta}x^2,\\
g(x,\ell,u,v) &= \varkappa\ell^{\eta}v^{1-\eta}x + \alpha\rho\ln(L-\ell) + (1-\alpha)\rho\ln u - (1-\alpha)(u+v), \qquad (27)\\
&x>0, \qquad \ell\in[0,L), \qquad u>0, \qquad v\geq 0,
\end{aligned}
$$

so that (23) and (25) become

$$
\dot{x}(t) = f(x(t),L^A(t),u(t),v(t)),
$$

$$
J_1(x(\cdot),L^A(\cdot),u(\cdot),v(\cdot)) = \int_0^{\infty} e^{-\rho t} g(x(t),L^A(t),u(t),v(t))\,dt \to \max.
$$

Let $\mathcal{M}(x,u,v,p)$ and $M(x,p)$ be the current value Hamilton–Pontryagin function and the current value Hamiltonian for problem (P1) in the normal form:

$$
\begin{aligned}
\mathcal{M}(x,\ell,u,v,p) &= f(x,\ell,u,v)p + g(x,\ell,u,v)\\
&= -(1-\eta)(u+v)xp - (1-\theta)\ell^{\eta}v^{1-\eta}x^2p + \varkappa\ell^{\eta}v^{1-\eta}x\\
&\quad + \alpha\rho\ln(L-\ell) + (1-\alpha)\rho\ln u - (1-\alpha)(u+v), \qquad (28)\\
M(x,p) &= \sup_{\ell\in[0,L),\ u>0,\ v\geq 0} \mathcal{M}(x,\ell,u,v,p).
\end{aligned}
$$

Here $x>0$, $\ell\in[0,L)$, $u>0$, $v\geq 0$ and $p\in\mathbb{R}^1$.

Next, we formulate two important theorems (an existence theorem and a version of the Pontryagin maximum principle for problem (P1)) that allow us to perform a qualitative analysis of the solution to problem (P) (in Section 6). The proofs of these theorems (together with all necessary auxiliary statements) constitute the rest of this section.

Theorem 1 (existence)**.** *There exists an optimal process* $(x_*(\cdot),\tilde{w}_*(\cdot))$ *in problem* (P1). *The process* $(A_*(\cdot),w_*(\cdot))$ *corresponding to* $(x_*(\cdot),\tilde{w}_*(\cdot))$ (*in the sense of Lemma* 1) *is optimal in problem* (P).

Theorem 2 (maximum principle)**.** *Let* $(x_*(\cdot),\tilde{w}_*(\cdot)) = (x_*(\cdot),L_*^A(\cdot),u_*(\cdot),v_*(\cdot))$ *be an optimal process in problem* (P1) *and* $(A_*(\cdot),w_*(\cdot))$ *be the corresponding* (*in the sense of Lemma* 1) *optimal process in problem* (P). *Then there exists a current value adjoint variable* $p(\cdot)$ *such that the following conditions hold*:

(i) *The process* $(x_*(\cdot),\tilde{w}_*(\cdot))$, *together with the current value adjoint variable* $p(\cdot)$, *satisfies the core relations of the Pontryagin maximum principle in the normal form on the infinite time interval* $[0,\infty)$:

$$
\dot{p}(t) = \rho p(t) - \frac{\partial\mathcal{M}(x_*(t),L_*^A(t),u_*(t),v_*(t),p(t))}{\partial x} \qquad \textit{for a.e. } t>0, \qquad (29)
$$

$$
\mathcal{M}(x_*(t),L_*^A(t),u_*(t),v_*(t),p(t)) = M(x_*(t),p(t)) \qquad \textit{for a.e. } t>0. \qquad (30)
$$

(ii) *The process* $(x_*(\cdot), \tilde{w}_*(\cdot))$*, together with the current value adjoint variable* $p(\cdot)$*, satisfies the normal-form stationarity condition*

$$M(x_*(t), p(t)) = \rho e^{\rho t} \int_t^\infty e^{-\rho s} g(x_*(s), L_*^A(s), u_*(s), v_*(s))\, ds \qquad \text{for all} \quad t \geq 0.$$

(iii) *For any* $t \geq 0$

$$p(t) = e^{\rho t} e^{-y(t)} \int_t^\infty e^{-\rho s} e^{y(s)} \frac{\partial g(x_*(s), L_*^A(s), u_*(s), v_*(s))}{\partial x}\, ds, \tag{31}$$

where $y(t) = \int_0^t \frac{\partial f(x_*(s), L_*^A(s), u_*(s), v_*(s))}{\partial x}\, ds \leq 0$.

Let us outline the scheme of proofs of these two theorems. First, we show that it suffices to consider only bounded controls in problem (P1). Then we introduce the problem with a slightly modified objective functional, which is defined for controls that take values in the compact closure of the admissible control set. We show that the optimal processes in these two problems coincide. Finally, using standard results of optimal control theory, we prove analogs of Theorems 1 and 2 for the modified problem, which automatically implies the assertions of Theorems 1 and 2.

This scheme is implemented below as a series of auxiliary lemmas. The rigorous derivation of the theorems from the lemmas is presented at the end of this section.

Denote

$$V_0 = \left(\frac{(1-\eta)\varkappa L^\eta x_0}{1-\alpha} \right)^{1/\eta} \tag{32}$$

and consider the following optimal control problem (P1′) with bounded controls:

$$\dot{x}(t) = -(1-\eta)[u(t) + v(t)]x(t) - (1-\theta)[L^A(t)]^\eta v(t)^{1-\eta} x(t)^2, \tag{33}$$

$$L^A(t) \in [0, L), \qquad u(t) \in (0, \rho], \qquad v(t) \in [0, V_0], \tag{34}$$

$$x(0) = x_0, \tag{35}$$

$$J_1(x(\cdot), L^A(\cdot), u(\cdot), v(\cdot)) = \int_0^\infty e^{-\rho t} \Big\{ \varkappa [L^A(t)]^\eta v(t)^{1-\eta} x(t) + \alpha\rho \ln[L - L^A(t)] + (1-\alpha)\rho \ln u(t) - (1-\alpha)[u(t) + v(t)] \Big\}\, dt \to \max. \tag{36}$$

Lemma 3. *If* $\tilde{w}_*(\cdot) = (L_*^A(\cdot), u_*(\cdot), v_*(\cdot))$ *is an optimal admissible control in problem* (P1)*, then*

$$u_*(t) \leq \rho \qquad \text{and} \qquad v_*(t) \leq V_0 = \left(\frac{(1-\eta)\varkappa L^\eta x_0}{1-\alpha} \right)^{1/\eta} \qquad \text{for a.e. } t > 0,$$

and so $\tilde{w}_*(\cdot)$ *is also an optimal admissible control in problem* (P1′)*. Conversely, if* $\hat{\tilde{w}}_*(\cdot)$ *is an optimal admissible control in problem* (P1′)*, then it is also an optimal admissible control in problem* (P1).

Before proving the lemma, we point out a corollary to this lemma and formula (31).

Corollary 1. *The current value adjoint variable $p(\cdot)$ satisfying the conditions of Theorem 2 is bounded:*

$$0 \le p(t) \le \frac{\varkappa L^\eta V_0^{1-\eta}}{\rho} \qquad \textit{for all } t > 0$$

(if $\eta = 1$, then $V_0 = 0$ and we consider $V_0^{1-\eta}$ to be 1). In particular, the transversality condition

$$\lim_{t\to\infty} e^{-\rho t} x_*(t) p(t) = 0$$

holds for any optimal process $(x_*(\cdot), \tilde{w}_*(\cdot))$ in problem (P1).

Proof. Indeed, since $\frac{\partial f}{\partial x}(x, \ell, u, v) \le 0$ for all $x > 0$, $\ell \in [0, L)$, $u > 0$ and $v \ge 0$, it follows that $y(\cdot)$ is a monotonically decreasing function, and so

$$0 \le p(t) \le e^{\rho t} \int_t^\infty e^{-\rho s} \varkappa L_*^A(s)^\eta v_*(s)^{1-\eta}\, ds \le \frac{\varkappa L^\eta V_0^{1-\eta}}{\rho} \qquad \text{for all } t > 0.$$

This implies the transversality condition, as $0 < x_*(t) \le x_0$ for $t > 0$. □

Proof of Lemma 3. Let $\tilde{w}(\cdot) = (L^A(\cdot), u(\cdot), v(\cdot))$ be an admissible control in problem (P1) such that $\operatorname{ess\,sup}_{t>0} u(t) > \rho$ or $\operatorname{ess\,sup}_{t>0} v(t) > V_0$. Define a new admissible bounded control $\overline{w}(\cdot) = (L^A(\cdot), \bar{u}(\cdot), \bar{v}(\cdot))$ with $\bar{u}(t) = \min\{u(t), \rho\}$ and $\bar{v}(t) = \min\{v(t), V_0\}$, $t \ge 0$. Note that $\overline{w}(\cdot)$ is also an admissible control in problem (P1′).

Let $x(\cdot)$ and $\bar{x}(\cdot)$ be the trajectories of problem (P1) (with the same initial condition x_0) that correspond to $\tilde{w}(\cdot)$ and $\overline{w}(\cdot)$, respectively ($\bar{x}(\cdot)$ is also a trajectory of problem (P1′)). Then we have

$$\bar{u}(t) \le u(t), \qquad \bar{v}(t) \le v(t) \qquad \text{and} \qquad x_0 \ge \bar{x}(t) \ge x(t) > 0 \qquad \text{for all } t > 0$$

by virtue of equation (23). Therefore,

$$\begin{aligned} J_1(x(\cdot), L^A(\cdot), u(\cdot), v(\cdot)) &\le \int_0^\infty e^{-\rho t}\Big\{\varkappa [L^A(t)]^\eta v(t)^{1-\eta} \bar{x}(t) + \alpha\rho \ln[L - L^A(t)] \\ &\qquad + (1-\alpha)\rho \ln u(t) - (1-\alpha)[u(t) + v(t)]\Big\}\, dt \\ &< \int_0^\infty e^{-\rho t}\Big\{\varkappa [L^A(t)]^\eta \bar{v}(t)^{1-\eta} \bar{x}(t) + \alpha\rho \ln[L - L^A(t)] \\ &\qquad + (1-\alpha)\rho \ln \bar{u}(t) - (1-\alpha)[\bar{u}(t) + \bar{v}(t)]\Big\}\, dt \\ &= J_1(\bar{x}(\cdot), L^A(\cdot), \bar{u}(\cdot), \bar{v}(\cdot)), \end{aligned}$$

where we applied the fact that

$$\frac{d}{du}\big((1-\alpha)\rho \ln u - (1-\alpha)u\big) < 0, \qquad \frac{d}{dv}\big(\varkappa [L^A(t)]^\eta v^{1-\eta} \bar{x}(t) - (1-\alpha)v\big) < 0$$

for all $t > 0$ and $u > \rho$, $v > V_0$.

Thus, we see that if $\operatorname{ess\,sup}_{t>0} u(t) > \rho$ or $\operatorname{ess\,sup}_{t>0} v(t) > V_0$, then the control $\tilde{w}(\cdot)$ cannot be optimal. This proves the first part of the lemma.

Conversely, if $(x_*(\cdot), \tilde{w}_*(\cdot)) = (x_*(\cdot), L_*^A(\cdot), u_*(\cdot), v_*(\cdot))$ is an optimal process in problem (P1′) and $(x(\cdot), \tilde{w}(\cdot)) = (x(\cdot), L^A(\cdot), u(\cdot), v(\cdot))$ is any process in problem (P1), then, again, introducing a new bounded control $\overline{w}(\cdot) = (L^A(\cdot), \bar{u}(\cdot), \bar{v}(\cdot))$ with $\bar{u}(t) = \min\{u(t), \rho\}$ and $\bar{v}(t) = \min\{v(t), V_0\}$, $t \geq 0$, we see that

$$J_1(x(\cdot), L^A(\cdot), u(\cdot), v(\cdot)) \leq J_1(\bar{x}(\cdot), L^A(\cdot), \bar{u}(\cdot), \bar{v}(\cdot)) \leq J_1(x_*(\cdot), L_*^A(\cdot), u_*(\cdot), v_*(\cdot)),$$

where $\bar{x}(\cdot)$ is the trajectory of problem (P1) (as well as of (P1′)) corresponding to the control $\overline{w}(\cdot)$. □

Our next goal is to establish the existence of an optimal admissible control $\tilde{w}_*(\cdot)$ in problem (P1′). To apply a standard existence theorem of optimal control theory, we need to compactify the range of values of the control variables. For this purpose, we introduce the function

$$\mathcal{L}_\varepsilon(\xi) = \begin{cases} \ln\varepsilon + \dfrac{1}{\varepsilon}(\xi - \varepsilon) & \text{for } \ 0 \leq \xi \leq \varepsilon, \\ \ln\xi & \text{for } \ \xi > \varepsilon, \end{cases} \tag{37}$$

where $\varepsilon < 1$ is a small positive constant, to the utility functional $J_1(x(\cdot), L^A(\cdot), u(\cdot), v(\cdot))$. Obviously, $\mathcal{L}_\varepsilon(\cdot)$ is a continuously differentiable concave function on $[0, \infty)$ and $\mathcal{L}_\varepsilon(\xi) \geq \ln\xi$ for $\xi \in (0, \infty)$.

Now consider an auxiliary problem (P_ε):

$$\dot{x}(t) = -(1-\eta)[u(t) + v(t)]x(t) - (1-\theta)[L^A(t)]^\eta v(t)^{1-\eta} x(t)^2, \tag{38}$$

$$L^A(t) \in [0, L], \qquad u(t) \in [0, \rho], \qquad v(t) \in [0, V_0], \tag{39}$$

$$x(0) = x_0,$$

$$J_\varepsilon(x(\cdot), L^A(\cdot), u(\cdot), v(\cdot)) = \int_0^\infty e^{-\rho t} \Big\{ \varkappa [L^A(t)]^\eta v(t)^{1-\eta} x(t) + \alpha\rho \mathcal{L}_\varepsilon\big(L - L^A(t)\big) + (1-\alpha)\rho\mathcal{L}_\varepsilon(u(t)) - (1-\alpha)[u(t) + v(t)] \Big\}\, dt \to \max, \tag{40}$$

where x_0 is the same as in (35). Clearly, any process $(x(\cdot), \tilde{w}(\cdot)) = (x(\cdot), L^A(\cdot), u(\cdot), v(\cdot))$ in problem (P1′) is also an admissible process in problem (P_ε).

Lemma 4. *If there is an optimal process $(x_*(\cdot), \tilde{w}_*(\cdot)) = (x_*(\cdot), L_*^A(\cdot), u_*(\cdot), v_*(\cdot))$ in problem (P_ε) such that $L_*^A(t) \leq L - \varepsilon$ and $u_*(t) \geq \varepsilon$ for a.e. $t \in (0, \infty)$, then*

(i) *this process is also optimal in problem* (P1′);

(ii) *any other optimal process $(\hat{x}_*(\cdot), \hat{\tilde{w}}_*(\cdot)) = (\hat{x}_*(\cdot), \hat{L}_*^A(\cdot), \hat{u}_*(\cdot), \hat{v}_*(\cdot))$ (if it exists) in problem* (P1′) *is such that $\hat{L}_*^A(t) \leq L - \varepsilon$ and $\hat{u}_*(t) \geq \varepsilon$ for a.e. $t \in (0, \infty)$ and so it is also optimal in problem (P_ε).*

Proof. Assertion (i) is valid because $J_\varepsilon(x(\cdot), L^A(\cdot), u(\cdot), v(\cdot)) \geq J_1(x(\cdot), L^A(\cdot), u(\cdot), v(\cdot))$ for any admissible process $(x(\cdot), \tilde{w}(\cdot)) = (x(\cdot), L^A(\cdot), u(\cdot), v(\cdot))$ in problem (P1′), while $J_\varepsilon(x_*(\cdot), L^A_*(\cdot), u_*(\cdot), v_*(\cdot)) = J_1(x_*(\cdot), L^A_*(\cdot), u_*(\cdot), v_*(\cdot))$.

If $(\hat{x}(\cdot), \hat{\tilde{w}}(\cdot)) = (\hat{x}(\cdot), \hat{L}^A(\cdot), \hat{u}(\cdot), \hat{v}(\cdot))$ is a process in problem (P1′) such that $\hat{L}^A(t) > L - \varepsilon$ or $\hat{u}(t) < \varepsilon$ on a positive measure set of values of t, then

$$\begin{aligned} J_1(\hat{x}(\cdot), \hat{L}^A(\cdot), \hat{u}(\cdot), \hat{v}(\cdot)) &< J_\varepsilon(\hat{x}(\cdot), \hat{L}^A(\cdot), \hat{u}(\cdot), \hat{v}(\cdot)) \leq J_\varepsilon(x_*(\cdot), L^A_*(\cdot), u_*(\cdot), v_*(\cdot)) \\ &= J_1(x_*(\cdot), L^A_*(\cdot), u_*(\cdot), v_*(\cdot)) \end{aligned}$$

and hence this process cannot be optimal in problem (P1′). This implies (ii). □

Denote

$$W = [0, L] \times [0, \rho] \times [0, V_0]$$

and

$$\begin{gathered} g_\varepsilon(x, \ell, u, v) = \varkappa \ell^\eta v^{1-\eta} x + \alpha\rho\mathcal{L}_\varepsilon(L - \ell) + (1-\alpha)\rho\mathcal{L}_\varepsilon(u) - (1-\alpha)(u + v), \\ x > 0, \qquad (\ell, u, v) \in W, \end{gathered} \tag{41}$$

so that (38) and (40) become

$$\begin{gathered} \dot{x}(t) = f(x(t), L^A(t), u(t), v(t)), \\ J_\varepsilon(x(\cdot), L^A(\cdot), u(\cdot), v(\cdot)) = \int_0^\infty e^{-\rho t} g_\varepsilon(x(t), L^A(t), u(t), v(t))\, dt \to \max \end{gathered}$$

(see (27)).

For every $x > 0$, consider the following set, which is standard in optimal control theory:

$$Q(x) = \big\{(z^0, z) \in \mathbb{R}^2 : \ z^0 \leq g_\varepsilon(x, \ell, u, v), \ z = f(x, \ell, u, v), \ (\ell, u, v) \in W\big\}.$$

Lemma 5. *For every $x > 0$, the set $Q(x)$ is convex.*

Proof. It suffices to show that for any two points $(z_1^0, z_1), (z_2^0, z_2) \in Q(x)$ the midpoint of the segment joining (z_1^0, z_1) to (z_2^0, z_2) also lies in $Q(x)$. Let $z_i = f(x, \ell_i, u_i, v_i)$ and $z_i^0 \leq g_\varepsilon(x, \ell_i, u_i, v_i)$ for some $(\ell_i, u_i, v_i) \in W$ $(i = 1, 2)$. We need to show that there exists $(\bar{\ell}, \bar{u}, \bar{v}) \in W$ such that

$$f(x, \bar{\ell}, \bar{u}, \bar{v}) = \bar{z} = \frac{z_1 + z_2}{2} \qquad \text{and} \qquad g_\varepsilon(x, \bar{\ell}, \bar{u}, \bar{v}) \geq \bar{z}^0 = \frac{z_1^0 + z_2^0}{2}.$$

We will seek $(\bar{\ell}, \bar{u}, \bar{v})$ in the form

$$\bar{\ell} = \bar{\ell}(\epsilon) = \frac{\ell_1 + \ell_2}{2} - \epsilon, \qquad \bar{u} = \frac{u_1 + u_2}{2}, \qquad \bar{v} = \frac{v_1 + v_2}{2}$$

with $0 \leq \epsilon \leq \frac{\ell_1 + \ell_2}{2}$. It is obvious that such a triple belongs to W.

Note that

$$\left(\frac{\ell_1+\ell_2}{2}\right)^{\eta}\left(\frac{v_1+v_2}{2}\right)^{1-\eta} \geq \frac{\ell_1^{\eta}v_1^{1-\eta}+\ell_2^{\eta}v_2^{1-\eta}}{2}, \qquad 0\leq\eta\leq 1$$

(see, e.g., [14, Theorem 38]). Therefore,

$$f(x,0,\bar{u},\bar{v}) \geq \bar{z} \qquad \text{and} \qquad f(x,\bar{\ell}(0),\bar{u},\bar{v}) \leq \bar{z}.$$

Since $f(x,\bar{\ell}(\cdot),\bar{u},\bar{v})$ is a continuous function of ϵ, there indeed exists an ϵ, $0\leq\epsilon\leq\frac{\ell_1+\ell_2}{2}$, such that

$$f(x,\bar{\ell}(\epsilon),\bar{u},\bar{v}) = \bar{z}. \tag{42}$$

We fix such an ϵ and write simply $\bar{\ell}$ instead of $\bar{\ell}(\epsilon)$ in what follows.

Now let us show that $g_\varepsilon(x,\bar{\ell},\bar{u},\bar{v}) \geq \bar{z}^0$. Note that due to (42), for $\theta<1$,

$$\begin{aligned}\bar{\ell}^{\eta}\bar{v}^{1-\eta}x &= \frac{-(1-\eta)(\bar{u}+\bar{v})x-\bar{z}}{(1-\theta)x} = \frac{-(1-\eta)(u_1+u_2+v_1+v_2)x-(z_1+z_2)}{2(1-\theta)x}\\ &= \frac{\ell_1^{\eta}v_1^{1-\eta}x+\ell_2^{\eta}v_2^{1-\eta}x}{2}.\end{aligned} \tag{43}$$

If $\theta=1$, then $f(\cdot)$ does not depend on ℓ and so (42) holds for all ϵ. Therefore, choosing an appropriate ϵ, we can achieve the equality of the first and last expressions in the chain (43) in this case as well.

Since $\mathcal{L}_\varepsilon(\cdot)$ is a concave increasing function, we have $\mathcal{L}_\varepsilon(L-\bar{\ell}) \geq \mathcal{L}_\varepsilon(L-\bar{\ell}(0))$ and in view of (43) find that

$$g_\varepsilon(x,\bar{\ell},\bar{u},\bar{v}) \geq \frac{g_\varepsilon(x,\ell_1,u_1,v_1)+g_\varepsilon(x,\ell_2,u_2,v_2)}{2} \geq \bar{z}^0.$$

This completes the proof of Lemma 5. □

Lemma 6. *For any ε, $0<\varepsilon<1$, there exists an optimal control in problem* (P_ε). *Moreover, if ε is small enough, then any optimal control $\tilde{w}(\cdot) = (L_*^A(\cdot),u_*(\cdot),v_*(\cdot))$ in problem* (P_ε) *is such that $L_*^A(t)\leq L-\varepsilon$ and $u_*(t)\geq\varepsilon$ for a.e. $t\in(0,\infty)$.*

Proof. The existence follows from Theorem 2.1 in [5] and Lemma 5.

Note that problem (P_ε) falls within the case of dominating discount (see [5, Section 12]), so we can apply the version of Pontryagin's maximum principle formulated in [5, Theorem 12.1] to this problem. To this end, define the current value Hamilton–Pontryagin function $\mathcal{M}_\varepsilon(x,u,v,p)$ and the current value Hamiltonian $M_\varepsilon(x,p)$ in problem (P_ε) in the normal form:

$$\begin{aligned}\mathcal{M}_\varepsilon(x,\ell,u,v,p) &= f(x,\ell,u,v)p+g_\varepsilon(x,\ell,u,v)\\ &= -(1-\eta)(u+v)xp-(1-\theta)\ell^{\eta}v^{1-\eta}x^2p+\varkappa\ell^{\eta}v^{1-\eta}x\\ &\quad +\alpha\rho\mathcal{L}_\varepsilon(L-\ell)+(1-\alpha)\rho\mathcal{L}_\varepsilon(u)-(1-\alpha)(u+v),\end{aligned} \tag{44}$$

$$M_\varepsilon(x,p) = \sup_{(\ell,u,v)\in W}\mathcal{M}_\varepsilon(x,\ell,u,v,p). \tag{45}$$

Here $x > 0$, $(\ell, u, v) \in W$ and $p \in \mathbb{R}^1$.

Let $(x_*(\cdot), \tilde{w}_*(\cdot)) = (x_*(\cdot), L_*^A(\cdot), u_*(\cdot), v_*(\cdot))$ be an optimal process in problem (P_ε). Then, by Theorem 12.1 from [5], we have

$$\mathcal{M}_\varepsilon(x_*(t), L_*^A(t), u_*(t), v_*(t), p(t)) = M_\varepsilon(x_*(t), p(t)) \qquad \text{for a.e. } t > 0, \tag{46}$$

where

$$p(t) = e^{\rho t} e^{-y(t)} \int_t^\infty e^{-\rho s} e^{y(s)} \frac{\partial g_\varepsilon(x_*(s), L_*^A(s), u_*(s), v_*(s))}{\partial x}\, ds \tag{47}$$

with the same $y(\cdot)$ as in Theorem 2. As shown in the proof of Corollary 1, $y(\cdot)$ is a monotonically decreasing function, and so

$$0 \le p(t) \le \frac{1}{\rho} \sup_{x>0,\ (\ell,u,v)\in W} \frac{\partial g_\varepsilon(x, \ell, u, v)}{\partial x} = \frac{\varkappa L^\eta V_0^{1-\eta}}{\rho} \qquad \text{for all } t > 0.$$

We also have $0 < x_*(\cdot) \le x_0$. However, it is easy to show that if ε is sufficiently small,[3] then the maximum of the function $\mathcal{M}_\varepsilon(x, \cdot, \cdot, \cdot, p)$ with respect to $(\ell, u, v) \in W$ for fixed $x \in (0, x_0]$ and $p \in [0, \varkappa L^\eta V_0^{1-\eta}/\rho]$ cannot be attained at a point (ℓ, u, v) such that $\ell > L - \varepsilon$ or $u < \varepsilon$. Indeed, it suffices to calculate the partial derivatives of $\mathcal{M}_\varepsilon$ with respect to ℓ and u.

This fact, together with the maximum condition (46), completes the proof of the lemma. □

Proof of Theorem 1. Above we have shown that the auxiliary problem (P_ε) has a solution, i.e. an optimal process $(x_*(\cdot), \tilde{w}_*(\cdot)) = (x_*(\cdot), L_*^A(\cdot), u_*(\cdot), v_*(\cdot))$, and proved certain estimates for the corresponding optimal control (Lemma 6). These estimates show (Lemma 4) that any such solution is also an optimal process in problem (P1′), and so is an optimal process in problem (P1) (Lemma 3), which is equivalent to the original problem (P) (Lemma 1). Thus, we obtain the existence of an optimal control in problem (P). □

Proof of Theorem 2. Fix a sufficiently small ε. By Lemmas 6 and 4(ii), $L_*^A(t) \le L - \varepsilon$ and $u_*(t) \ge \varepsilon$ for a.e. $t \in (0, \infty)$, and $(x_*(\cdot), \tilde{w}_*(\cdot))$ is an optimal process in problem (P_ε).

By Theorem 12.1 in [5], such an adjoint variable $p(\cdot)$ satisfying properties (i)–(iii) of Theorem 2 (with $g_\varepsilon(\cdot)$, $M_\varepsilon(\cdot)$ and $\mathcal{M}_\varepsilon(\cdot)$ instead of $g(\cdot)$, $M(\cdot)$ and $\mathcal{M}(\cdot)$, respectively) exists for the optimal process $(x_*(\cdot), \tilde{w}_*(\cdot))$ in problem (P_ε). Since $L_*^A(t) \le L - \varepsilon$ and $u_*(t) \ge \varepsilon$ for a.e. $t > 0$, we have $g(x_*(t), L_*^A(t), u_*(t), v_*(t)) = g_\varepsilon(x_*(t), L_*^A(t), u_*(t), v_*(t))$ and $\mathcal{M}(x_*(t), L_*^A(t), u_*(t), v_*(t), p(t)) = \mathcal{M}_\varepsilon(x_*(t), L_*^A(t), u_*(t), v_*(t), p(t))$ for a.e. $t > 0$. Moreover, since $\mathcal{M}(x, \ell, u, v, p) \le \mathcal{M}_\varepsilon(x, \ell, u, v, p)$ for all $x > 0$, $p > 0$ and $(\ell, u, v) \in W$, we also have $M(x_*(t), p(t)) = M_\varepsilon(x_*(t), p(t))$.

Thus, properties (i)–(iii) of Theorem 2 with $g(\cdot)$, $M(\cdot)$ and $\mathcal{M}(\cdot)$ follow from the same properties with $g_\varepsilon(\cdot)$, $M_\varepsilon(\cdot)$ and $\mathcal{M}_\varepsilon(\cdot)$. In particular, (46) and (47) become (30) and (31). □

Theorem 2 allows us to explicitly write the Hamiltonian system of the Pontryagin maximum principle for problem (P1). In the next section, we will analyze the qualitative behavior of solutions to this system and single out all optimal regimes.

[3] Of course, the upper bound for ε that guarantees the validity of this statement depends on x_0, but x_0 is fixed from the onset.

6 Analysis of the Hamiltonian system

We know from Theorem 1 that an optimal process $(x_*(\cdot), \tilde{w}_*(\cdot))$ in problem (P1) exists and satisfies the relations of Theorem 2. Using Theorem 2, we can construct the Hamiltonian system of the Pontryagin maximum principle for problem (P1) in the variables $x(\cdot)$ and $p(\cdot)$ directly. However, to simplify the further analysis, we pass from the variable $p(\cdot)$ to a new variable $\phi(\cdot)$ defined as $\phi(t) = x(t)p(t)$, $t > 0$. Then we write and analyze the relations of the Hamiltonian system of the Pontryagin maximum principle for problem (P1) in the variables $x(\cdot)$ and $\phi(\cdot)$.

In terms of the variable $\phi(\cdot)$, the adjoint system (see (29)) and the maximum condition (see (30)) take the forms

$$\dot{\phi}(t) = \dot{x}(t)p(t) + x(t)\dot{p}(t) = \rho\phi(t) + L^A(t)^\eta v(t)^{1-\eta} x(t)\big[(1-\theta)\phi(t) - \varkappa\big] \tag{48}$$

and

$$\tilde{\mathcal{M}}(x, \ell, u, v, \phi) \to \max_{\ell\in[0,L), u>0, v\geq 0}, \tag{49}$$

respectively. Here the function $\tilde{\mathcal{M}}(\cdot)$ is defined by the equality (see (28))

$$\begin{aligned} \tilde{\mathcal{M}}(x, \ell, u, v, \phi) = &-\big[1 - \alpha + (1-\eta)\phi\big](u + v) \\ &+ \big[\varkappa - (1-\theta)\phi\big]\ell^\eta v^{1-\eta} x + \alpha\rho\ln(L - \ell) + (1-\alpha)\rho\ln u, \end{aligned} \tag{50}$$

for all $x > 0$, $\phi \geq 0$, $u > 0$, $v \geq 0$ and $0 \leq \ell < L$.

Our first aim is to write the Hamiltonian system of the maximum principle for problem (P1) in terms of the variables $x(\cdot)$ and $\phi(\cdot)$ by combining equations (23) and (48) (and using maximum condition (49)). To this end, we first express the quantities $L^A(x, \phi)$, $u(x, \phi)$ and $v(x, \phi)$ as functions of x and ϕ that are (unique) maximizers of $\tilde{\mathcal{M}}(\cdot)$ with respect to ℓ, u and v, respectively (see maximum condition (49)), for all $x > 0$ and $\phi \geq 0$. Then, substituting these maximizers into equations (23) and (48), we get the Hamiltonian system of the maximum principle for problem (P1) in the form

$$\begin{aligned} \dot{x}(t) = &-(1-\eta)[u(x(t), \phi(t)) + v(x(t), \phi(t))]x(t) \\ &- (1-\theta)L^A(x(t), \phi(t))^\eta v(x(t), \phi(t))^{1-\eta} x(t)^2, \\ \dot{\phi}(t) = &\ \rho\phi(t) + L^A(x(t), \phi(t))^\eta v(x(t), \phi(t))^{1-\eta} x(t)\big[(1-\theta)\phi(t) - \varkappa\big]. \end{aligned} \tag{51}$$

The value $u(x, \phi)$ at which the maximum of $\tilde{\mathcal{M}}(\cdot)$ with respect to u is attained can easily be found by means of differentiation (see (50)):

$$u(x, \phi) = \frac{(1-\alpha)\rho}{1 - \alpha + (1-\eta)\phi}. \tag{52}$$

If $\varkappa \leq (1-\theta)\phi$, then the maximum of $\tilde{\mathcal{M}}(\cdot)$ with respect to ℓ and v is attained for $v(x, \phi) = L^A(x, \phi) = 0$.

Suppose that $\varkappa > (1-\theta)\phi$. If $\eta = 1$, then $v(x,\phi) = 0$ simply because of the constraint $0 \le v \le V_0 = 0$ (see (39) and (32)), and $u(x,\phi) = \rho$ (see (52)). In this case it is obvious that the maximum point of $\tilde{\mathcal{M}}(\cdot)$ as a function of ℓ is given by

$$L^A(x,\phi) = L - \frac{\alpha\rho}{(\varkappa - (1-\theta)\phi)x}. \tag{53}$$

Finally, consider the case when $\varkappa > (1-\theta)\phi$ and $\eta < 1$. Note that $\tilde{\mathcal{M}}(x,\ell,u,v,\phi) \to -\infty$ as $v \to \infty$ or $\ell \to L - 0$. On the other hand, if one of the variables, v or ℓ, is zero, then the maximum with respect to the other variable is attained at zero. Therefore, the maximum of $\tilde{\mathcal{M}}(\cdot)$ with respect to ℓ and v is attained either at the point $v(x,\phi) = L^A(x,\phi) = 0$ or at an interior point, in which case this point can be found by equating the partial derivatives of $\tilde{\mathcal{M}}(\cdot)$ with respect to ℓ and v to zero:

$$\eta\big[\varkappa - (1-\theta)\phi\big]\Big(\frac{v}{\ell}\Big)^{1-\eta} x = \frac{\alpha\rho}{L-\ell}, \tag{54}$$

$$(1-\eta)\big[\varkappa - (1-\theta)\phi\big]\Big(\frac{\ell}{v}\Big)^{\eta} x = 1 - \alpha + (1-\eta)\phi. \tag{55}$$

Denoting

$$h(x,\phi) = \frac{1-\alpha+(1-\eta)\phi}{(1-\eta)x[\varkappa - (1-\theta)\phi]}, \qquad x > 0, \quad 0 \le \phi < \frac{\varkappa}{1-\theta},$$

we find

$$\frac{\ell}{v} = h(x,\phi)^{\frac{1}{\eta}} \tag{56}$$

and

$$\ell = L - \frac{\alpha\rho h(x,\phi)^{\frac{1-\eta}{\eta}}}{\eta[\varkappa - (1-\theta)\phi]x} = L - \frac{\alpha\rho(1-\alpha+(1-\eta)\phi)^{\frac{1-\eta}{\eta}}}{\eta(1-\eta)^{\frac{1-\eta}{\eta}}(x[\varkappa-(1-\theta)\phi])^{\frac{1}{\eta}}}, \tag{57}$$

$$v = \frac{L}{h(x,\phi)^{\frac{1}{\eta}}} - \frac{\alpha\rho h(x,\phi)^{-1}}{\eta[\varkappa-(1-\theta)\phi]x} = \frac{L((1-\eta)x[\varkappa-(1-\theta)\phi])^{\frac{1}{\eta}}}{(1-\alpha+(1-\eta)\phi)^{\frac{1}{\eta}}} - \frac{\alpha\rho(1-\eta)}{\eta(1-\alpha+(1-\eta)\phi)}. \tag{58}$$

If these formulas yield positive values $v(x,\phi)$ and $L^A(x,\phi)$ of v and ℓ, then this is the maximum point of $\tilde{\mathcal{M}}(\cdot)$ with respect to v and ℓ. Otherwise, the maximum point is $v(x,\phi) = L^A(x,\phi) = 0$.

Note that (57) and (58) for $\eta = 1$ turn into (53) and $v(x,\phi) = 0$, respectively, if we consider $(1-\eta)^{1-\eta}$ to be 1 for $\eta = 1$.

Set

$$h_1(\phi) = \frac{\alpha^\eta\rho^\eta(1-\alpha+(1-\eta)\phi)^{1-\eta}}{L^\eta\eta^\eta(1-\eta)^{1-\eta}[\varkappa-(1-\theta)\phi]}, \qquad 0 \le (1-\theta)\phi < \varkappa,$$

and introduce the following sets (see Fig. 1):

$$\Gamma = \{(x,\phi) \in \mathbb{R}^2\colon\ x > 0,\ \phi \ge 0\},$$

$$\Gamma_0 = \big\{(x,\phi) \in \Gamma\colon\ (1-\theta)\phi \ge \varkappa \ \text{ or } \ \{(1-\theta)\phi < \varkappa,\ x < h_1(\phi)\}\big\}, \qquad \Gamma_1 = \Gamma \setminus \Gamma_0.$$

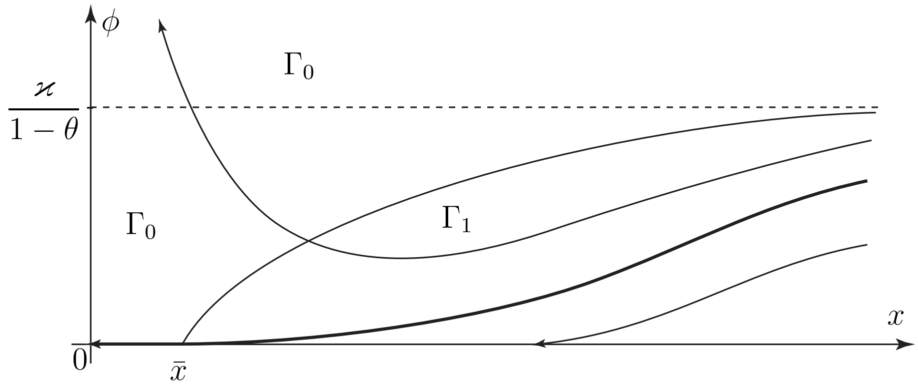

Figure 1: The sets Γ_0 and Γ_1 and the optimal trajectory (thick line). All trajectories lying above the optimal one tend to infinity along the ϕ-axis. All trajectories lying below the optimal one transversally intersect the x-axis.

According to the above analysis, in Γ_0 both $L^A(x,\phi)$ and $v(x,\phi)$ vanish, and so our Hamiltonian system (51) in Γ_0 has the form

$$\dot{x}(t) = -\frac{(1-\eta)(1-\alpha)\rho}{1-\alpha+(1-\eta)\phi(t)}x(t),$$
$$\dot{\phi}(t) = \rho\phi(t).$$

Note that $h_1(\cdot)$ is a monotonically increasing function of ϕ (except for the case $\eta = \theta = 1$, in which $h_1(\cdot) \equiv \text{const}$). Therefore, any trajectory of our system that reaches the set Γ_0 cannot leave this set afterwards. (Indeed, at every point of Γ_0 we have $\dot{x}(\cdot) \le 0$ and $\dot{\phi}(\cdot) \ge 0$.) However, we know that $\phi(\cdot)$ is bounded along an optimal trajectory (e.g., by Corollary 1); hence the only candidate for an optimal trajectory in Γ_0 lies on the x-axis and looks like

$$x(t) = \bar{x}e^{-(1-\eta)\rho(t-\tau)}, \qquad \phi(t) = 0 \qquad \text{for} \quad t \ge \tau, \tag{59}$$

where

$$\bar{x} = h_1(0) = \frac{\rho^\eta\alpha^\eta(1-\alpha)^{1-\eta}}{L^\eta\eta^\eta(1-\eta)^{1-\eta}\varkappa}. \tag{60}$$

On the other hand, since $\dot{x}(t) \le 0$, any bounded trajectory must tend to a fixed point. If $\eta < 1$, then $\dot{x}(\cdot) < 0$ in the interior of Γ_1 and consequently any trajectory of our system starting in Γ_1 eventually enters the set Γ_0. This shows that there is a unique bounded trajectory of our system, and hence the optimal process in problem (P1) is also unique. The tail of this trajectory is described by (59).

If $\eta = 1$ and $\theta < 1$, then for similar reasons any bounded trajectory starting in Γ_1 tends to the point $(\bar{x}, 0)$ on the boundary of Γ_1. Let us show that there is only one such trajectory $(\tilde{x}(\cdot), \tilde{\phi}(\cdot))$ in Γ_1. Indeed, if there were two trajectories lying in Γ_1 and tending to $(\bar{x}, 0)$, then any trajectory lying between these two would also tend to $(\bar{x}, 0)$ (because $\dot{x}(\cdot) \le 0$). However, this is impossible, as we can show, for example, by considering the linearization of the Hamiltonian system of the maximum principle in Γ_1 at the point $(\bar{x}, 0)$ and applying the Grobman–Hartman theorem (see [16]).

Finally, if $\eta = \theta = 1$, then $x(t) \equiv 1$ (see (16)) and $\dot{\phi}(t) = \rho\phi(t) - \ell\varkappa$, where $\ell = \max\{0, L - \frac{\alpha\rho}{\varkappa}\}$. Thus, the only bounded trajectory is the fixed point $x = 1$, $\phi = \max\{0, \frac{L\varkappa}{\rho} - \alpha\}$. Recall that in this case the optimal controls are $u(t) \equiv \rho$, $v(t) \equiv 0$ and $L^A(t) \equiv \max\{0, L - \frac{\alpha\rho}{\varkappa}\}$.

Let us now examine the initial part of the optimal trajectory lying in Γ_1, for $\eta < 1$. Using formulas (56) and (58), we find

$$\ell^\eta v^{1-\eta} = h(x,\phi)v = \frac{L}{h(x,\phi)^{\frac{1-\eta}{\eta}}} - \frac{\alpha\rho}{\eta x[\varkappa - (1-\theta)\phi]}.$$

Similarly, due to (52) and (58), we obtain

$$u + v = \frac{(\eta-\alpha)\rho}{\eta(1-\alpha+(1-\eta)\phi)} + \frac{L}{h(x,\phi)^{\frac{1}{\eta}}}.$$

Thus, our system (51) in Γ_1 has the form

$$\begin{aligned} \dot{x}(t) &= -(1-\eta)\left[\frac{(\eta-\alpha)\rho}{\eta(1-\alpha+(1-\eta)\phi(t))} + \frac{L}{h(x(t),\phi(t))^{\frac{1}{\eta}}}\right]x(t) \\ &\quad - (1-\theta)\left[\frac{L}{h(x(t),\phi(t))^{\frac{1-\eta}{\eta}}} - \frac{\alpha\rho}{\eta x(t)[\varkappa - (1-\theta)\phi(t)]}\right]x(t)^2, \\ \dot{\phi}(t) &= \rho\phi(t) - \frac{L(1-\alpha+(1-\eta)\phi(t))}{(1-\eta)h(x(t),\phi(t))^{\frac{1}{\eta}}} + \frac{\alpha\rho}{\eta}, \end{aligned} \tag{61}$$

and we are interested in the trajectory $(\tilde{x}(\cdot), \tilde{\phi}(\cdot))$ that passes through the point $(\bar{x}, 0)$. It would be difficult to solve this system analytically, but for numerical simulations it suffices to know that the sought trajectory $(\tilde{x}(\cdot), \tilde{\phi}(\cdot))$ is a solution to the Cauchy problem for system (61) in reverse time (i.e., with the right-hand side taken with the opposite sign) under the initial condition $\tilde{x}(0) = \bar{x}$, $\tilde{\phi}(0) = 0$.

Moreover, since $\dot{\tilde{x}}(t) < 0$ for all $t > 0$, we can express $\tilde{\phi}(\cdot)$ as a function of $\tilde{x}(\cdot)$ along this trajectory, $\tilde{\phi} = \phi_*(x)$.

If $\eta = 1$ and $\theta < 1$, we can also express $\tilde{\phi}(\cdot)$ as a (continuous) function of $\tilde{x}(\cdot)$ along this trajectory, $\tilde{\phi} = \phi_*(x)$ (with $\phi_*(x) = 0$ for $x \leq \bar{x}$). However, this trajectory cannot be found as a solution of the Cauchy problem, as described above, because $(\bar{x}, 0)$ is a fixed point of the Hamiltonian system for $\eta = 1$.

Thus, for $\eta\theta < 1$ we obtain a unique optimal feedback control $u_*(x) = u(x, \phi_*(x))$, $v_*(x) = v(x, \phi_*(x))$, $L_*^A(x) = L^A(x, \phi_*(x))$ according to formulas (52), (58) and (53), (57).

Let us summarize the above analysis of the Hamiltonian system as follows:

Theorem 3. (a) *If* $\eta = 1$ *and* $\theta = 1$, *then there is a unique optimal control* $\tilde{w}(\cdot) = (L_*^A(\cdot), u_*(\cdot), v_*(\cdot))$ *in problem* (P1), *with*

$$L_*^A(t) \equiv \max\Big\{0, L - \frac{\alpha\rho}{\varkappa}\Big\}, \qquad u_*(t) \equiv \rho, \qquad v_*(t) \equiv 0 \qquad \textit{for all} \quad t \in [0,\infty).$$

In this case $x(t) \equiv x_0 = 1$, $t \geq 0$ *is a unique admissible trajectory* (*see* (16)).

(b) *If* $\eta\theta < 1$, *then there is a unique optimal feedback control* (*optimal synthesis*) $\tilde{w}_*(x) = (L_*^A(x), u_*(x), v_*(x))$ *in problem* (P1), *with* $L_*^A(x) = L^A(x, \phi_*(x))$, $u_*(x) = u(x, \phi_*(x))$ *and* $v_*(x) = v(x, \phi_*(x))$ *determined by formulas* (53), (57), (52) *and* (58). *Here the feedback* $\phi_*(x)$ *is generated by a unique solution* $(\tilde{x}(\cdot), \tilde{\phi}(\cdot))$ *of the Hamiltonian system* (61) *that reaches* (*or tends to*) *the point* $(\bar{x}, 0)$ *from the right, where* (*see* (60))

$$\bar{x} = \frac{\rho^\eta \alpha^\eta (1-\alpha)^{1-\eta}}{L^\eta \eta^\eta (1-\eta)^{1-\eta} \varkappa}.$$

Namely,

(b.1) *If* $\eta\theta < 1$ *and* $x \leq \bar{x}$, *then*

$$L_*^A(x) = 0, \qquad u_*(x) = \rho, \qquad v_*(x) = 0.$$

(b.2) *If* $\eta = 1$, $\theta < 1$ *and* $x > \bar{x}$, *then* (*see* (53), (52) *and* (58))

$$L_*^A(x) = L - \frac{\alpha\rho}{(\varkappa - (1-\theta)\phi_*(x))x}, \qquad u_*(x) = \rho, \qquad v_*(x) = 0.$$

In the case of $\eta = 1$ *and* $\theta < 1$, *for any initial state* $x_0 \leq \bar{x}$ *the corresponding optimal trajectory* $x_*(\cdot)$ *is* $x_*(t) \equiv x_0$, $t \geq 0$, *while for any initial state* $x_0 > \bar{x}$ *the corresponding optimal trajectory* $x_*(\cdot)$ *monotonically tends to the point* $\bar{x}$ *from the right as* $t \to \infty$.

(b.3) *If* $\eta < 1$, $\theta \leq 1$ *and* $x > \bar{x}$, *then* (*see* (57), (52) *and* (58))

$$L_*^A(x) = L - \frac{\alpha\rho(1-\alpha+(1-\eta)\phi_*(x))^{\frac{1-\eta}{\eta}}}{\eta(1-\eta)^{\frac{1-\eta}{\eta}}(x[\varkappa-(1-\theta)\phi_*(x)])^{\frac{1}{\eta}}},$$

$$u_*(x) = \frac{(1-\alpha)\rho}{1-\alpha+(1-\eta)\phi_*(x)},$$

$$v_*(x) = \frac{L((1-\eta)x[\varkappa-(1-\theta)\phi_*(x)])^{\frac{1}{\eta}}}{(1-\alpha+(1-\eta)\phi_*(x))^{\frac{1}{\eta}}} - \frac{\alpha\rho(1-\eta)}{\eta(1-\alpha+(1-\eta)\phi_*(x))}.$$

In the case of $\eta < 1$ *and* $\theta \leq 1$, *for any initial state* $x_0 > 0$, *the corresponding optimal trajectory* $x_*(\cdot)$ *monotonically decreases to* 0 *as* $t \to \infty$.

Finally let us analyze the dynamics of the output $Y(\cdot)$ and the knowledge stock $A(\cdot)$ along the optimal trajectory.

If $\eta = \theta = 1$, then (Theorem 3(a)) the optimal controls are $u(t) \equiv \rho$, $v(t) \equiv 0$ and $L^A(t) \equiv \max\{0, L - \frac{\alpha\rho}{\varkappa}\}$. In the case of $L\varkappa \leq \alpha\rho$, we have stagnation of the knowledge stock ($\dot{A}(t) \equiv 0$) and depletion of the output ($Y(t) \to 0$ as $t \to \infty$). For $L\varkappa > \alpha\rho$, the knowledge stock grows exponentially, while the output still depletes to zero for $L\varkappa < \rho(\alpha + \varkappa(1-\alpha))$, is constant for $L\varkappa = \rho(\alpha + \varkappa(1-\alpha))$, and grows exponentially for $L\varkappa > \rho(\alpha + \varkappa(1-\alpha))$.

Let us consider the case $\eta\theta < 1$ in more detail. If $x_0 \leq \bar{x}$, then we again have stagnation of the knowledge stock and depletion of the output. If $x_0 > \bar{x}$, then the knowledge stock grows in the beginning, but the growth either terminates at a certain instant ($\eta < 1$) or decelerates ($\eta = 1$), so that the knowledge stock never exceeds a certain level determined by the parameters of the system. The output falls to zero in the long run. However, the following proposition shows that it may grow on some initial time interval.

Proposition 1. *Let $\eta\theta < 1$. Then, for sufficiently large initial values x_0 (i.e., for a relatively large initial stock of the exhaustible resource S_0 and/or for a relatively small initial knowledge stock A_0; see (19)), the output $Y(\cdot)$ as a function of t increases on some initial time interval $0 < t < \tau$, $\tau > 0$.*

Proof. For large x_0 the initial part of the optimal trajectory lies in Γ_1 and hence $Y(\cdot)$ is continuously differentiable for the corresponding values of t. Let us show that $\dot{Y}(t) > 0$ on the initial time interval $0 < t < \tau$, $\tau > 0$, of the optimal trajectory. We have

$$\begin{aligned}\dot{Y}(t) &= Y(t)\left[\varkappa\frac{\dot{A}(t)}{A(t)} - \alpha\frac{\dot{L}^A(t)}{L - L^A(t)} + (1-\alpha)\frac{\dot{u}(t)}{u(t)} + (1-\alpha)\frac{\dot{S}(t)}{S(t)}\right] \\ &= Y(t)\left[\varkappa L^A(t)^\eta v(t)^{1-\eta}x(t) - \alpha\frac{\dot{L}^A(t)}{L - L^A(t)} + (1-\alpha)\frac{\dot{u}(t)}{u(t)} - (1-\alpha)(u(t)+v(t))\right] \qquad (62)\end{aligned}$$

(see (1), (2), (17) and (16)), where $u(t) = u_*(x(t))$, $v(t) = v_*(x(t))$ and $L^A(t) = L^A_*(x(t))$.

Let us show that $\dot{\phi}(t) < 0$ along the optimal trajectory in Γ_1. To see this, note that the curve on which $\dot{\phi}(t) = 0$ in Γ_1 is described by the equation

$$\rho\phi + \frac{\alpha\rho}{\eta} = \frac{L(1-\alpha+(1-\eta)\phi)}{(1-\eta)h(x,\phi)^{\frac{1}{\eta}}} = \frac{L(1-\eta)^{\frac{1-\eta}{\eta}}(x[\varkappa-(1-\theta)\phi])^{\frac{1}{\eta}}}{(1-\alpha+(1-\eta)\phi)^{\frac{1-\eta}{\eta}}}. \qquad (63)$$

This equation defines x as a monotonically increasing function of ϕ. So any trajectory of our system that intersects this curve at some instant τ (at a point different from $(\bar{x}, 0)$) acquires a positive derivative of the ϕ-coordinate and later enters the set Γ_0 (at a point different from $(\bar{x}, \phi)$). Such a trajectory tends to infinity and so it is not optimal. Hence our optimal trajectory lies in Γ_1 completely below the above curve, and $\dot{\phi}(t) < 0$ on it. This immediately implies that $\dot{u}(t) \geq 0$ in (62) (see (52)).

To estimate the second term in the square brackets in (62), we first denote $\zeta(t) = x(t)[\varkappa - (1-\theta)\phi(t)]$, $\zeta_*(x) = x[\varkappa - (1-\theta)\phi_*(x)]$, and calculate (along the optimal trajectory in Γ_1)

$$\begin{aligned}\dot{\zeta}(t) &= \frac{d}{dt}\big(x(t)\big[\varkappa - (1-\theta)\phi(t)\big]\big) = \dot{x}(t)\big[\varkappa - (1-\theta)\phi(t)\big] - (1-\theta)x(t)\dot{\phi}(t) \\ &= -(1-\eta)\big[u_*(x(t)) + v_*(x(t))\big]\zeta(t) - (1-\theta)\rho x(t)\phi(t) < 0, \qquad (64)\end{aligned}$$

because $\zeta(t) > 0$ for $(x(t), \phi(t)) \in \Gamma_1$. Then, after some calculations, we find from (53) for $\eta = 1$, from (57) for $\eta < 1$, and from (48), (64) that

$$-\frac{\dot{L}^A(t)}{L - L^A(t)} = \frac{(1-\eta)^2\dot{\phi}(t)}{\eta\big(1-\alpha+(1-\eta)\phi(t)\big)} - \frac{1}{\eta}\frac{\dot{\zeta}(t)}{\zeta(t)}$$

$$> -\frac{(1-\eta)^2 L^A(t)^\eta v(t)^{1-\eta}\zeta(t)}{\eta\big(1-\alpha+(1-\eta)\phi(t)\big)} + \frac{(1-\theta)\rho x(t)\phi(t)}{\zeta(t)}. \tag{65}$$

If $\eta = 1$ and $\theta < 1$, then the right-hand side of (65) is positive; hence $\frac{dL_*^A(x)}{dx} > 0$ and

$$L_*^A(x)^\eta v_*(x)^{1-\eta} x \to +\infty \qquad \text{as} \;\; x \to +\infty. \tag{66}$$

This obviously implies that $\dot{Y}(t) > 0$ for large $x(t)$ along the optimal trajectory, as the second and third terms in the square brackets in (62) are nonnegative, while the last term is bounded due to the restrictions $u(t) \le \rho$ and $v(t) = 0$.

If $\eta < 1$ and $\theta \le 1$, then $\phi_*(x) < \varkappa/(1-\theta)$ in Γ_1. Let us show that $\phi_*(x) \to \varkappa/(1-\theta)$ as $x \to \infty$. Indeed, suppose the contrary. Then it follows from (57) that $L_*^A(x) \to L$ as $x \to \infty$, and due to (56) $v_*(x) \sim x^{1/\eta}$ as $x \to \infty$. Therefore,

$$\frac{d\phi_*(x)}{dx} = \frac{\dot{\phi}(t)}{\dot{x}(t)} = \frac{L_*^A(x)^\eta v_*(x)^{1-\eta} x[\varkappa - (1-\theta)\phi_*(x)] - \rho\phi_*(x)}{(1-\eta)[u_*(x)+v_*(x)]x + (1-\theta)L_*^A(x)^\eta v_*(x)^{1-\eta}x^2} \sim \frac{1}{x}, \tag{67}$$

which contradicts the boundedness of $\phi_*(\cdot)$. Thus, $\phi_*(x) \to \varkappa/(1-\theta)$ as $x \to \infty$.

If $\zeta_*(\cdot)$ is unbounded, then by (57) $L_*^A(x) \to L$ as $x \to \infty$, and by (56) $v_*(x) \sim \zeta_*(x)^{1/\eta} = o(x^{1/\eta})$ and $v_*(x) \to \infty$ as $x \to \infty$. This shows that the first term in the square brackets in (65) dominates all the negative terms there, and so $\dot{Y}(t) > 0$ for large $x(t)$ along the optimal trajectory.

If $\zeta_*(\cdot)$ is bounded, then $v_*(\cdot)$ is bounded by (55). Hence the right-hand side of (65) is positive for large $x(t)$ and, in particular, $\frac{dL_*^A(x)}{dx} > 0$ for large x. Therefore, again by (55), $v_*(x)$ is bounded away from zero for large x. We see that (66) holds in this case as well, which again implies that $\dot{Y}(t) > 0$ for large $x(t)$ along the optimal trajectory.

Finally, consider the case of $\eta < 1$ and $\theta = 1$. In this case $\zeta(t) = \varkappa x(t)$. Multiplying equation (54) raised to the power η by equation (55) raised to the power $1-\eta$, we find that

$$\eta^\eta(1-\eta)^{1-\eta}\varkappa x = \frac{\alpha^\eta\rho^\eta\big(1-\alpha+(1-\eta)\phi_*(x)\big)^{1-\eta}}{(L - L_*^A(x))^\eta}$$

Recall that $\phi_*(\cdot)$ is a monotonically increasing function of x. If it were bounded, then we would have $L_*^A(x) \to L$ as $x \to \infty$, $v_*(x)^\eta \sim x$ by (55), and hence (67) would be valid, which is impossible for a bounded $\phi_*(\cdot)$. Thus, $\phi_*(x) \to \infty$ as $x \to \infty$.

On the other hand, $\phi_*(x) = O(x)$ because the optimal trajectory lies below the curve described by (63). Therefore, $L_*^A(x) \to L$ as $x \to \infty$ by (57) and $v_*(x) \ge v_0$ for some $v_0 > 0$ and for all sufficiently large x by (58). At the same time, $v_*(x)^\eta = o(x)$ by (58). This shows that the first term in the square brackets in (65) dominates all the negative terms there, and so $\dot{Y}(t) > 0$ for large $x(t)$ along the optimal trajectory.

We showed that for $\eta\theta < 1$ the output $Y(t)$ increases on some initial time interval provided that the initial supply of exhaustible resource S_0 is large and/or the initial knowledge stock A_0 is small. $\square$

7 Summary

The above analysis of the Hamiltonian system yields a complete characterization of the optimal transitional dynamics of the model. The results of this analysis are summarized in Theorem 3.

The dynamics of the output $Y(\cdot)$ is illustrated in Fig. 2. Optimal growth is only sustainable if the following three conditions hold simultaneously:

(i) the accumulation of knowledge has strong scale effects;

(ii) the exhaustible resource is not an input to the production of knowledge;

(iii) the population is not too small.

In this scenario the growth of output is exponential. The resulting dynamics correspond to the optimal balanced growth path. While the requirement of a strong scale effect in the accumulation of knowledge is well-known in the economics literature, the requirement of the independence of the research sector from the exhaustible resource is a new result. Fortunately, the research sector does not depend excessively on narrow-sense exhaustible resources, such as minerals.

Condition (iii) says that a sufficiently small economy will not grow, even under strong scale effects and even if the accumulation of knowledge does not depend on the exhaustible resource. This minimum size condition is the least restrictive of all conditions and can be assumed to hold a priori. In the typical case $\varkappa = 1$, we have $L > \rho$. This inequality can be maintained in all cases of interest since L is the size of the labor force and ρ is the discount rate. The opposite case $L \leq \rho$ is included for completeness.

In the sustainable growth scenario $\eta = \theta = 1$, for a sufficiently large population size L, a constant fraction of labor is allocated to research. The lower the discount rate ρ, the higher this fraction. The fraction also depends on the elasticity of substitution in the production function. The optimal extraction policy implies an exponential depletion of the stock of the exhaustible resource, with the rate equal to the discount rate. This is the well-known Hotelling rule for the optimal depletion of exhaustible resources. In sum this implies an exponential growth of the knowledge stock $A(\cdot)$. The dynamics of the knowledge stock $A(\cdot)$ are illustrated in Fig. 3.

Given Jones' critique, we hold Scenario *3* in Fig. 2 to be the most realistic of the four scenarios. In this case ($\eta\theta < 1$) we may have two qualitatively different optimal policies depending on whether the accumulation of knowledge requires the resource:

(i) When the accumulation of knowledge is independent of the resource ($\eta = 1$), the fraction of labor employed in research tends from an initially positive value to zero. This means that the research effort becomes successively smaller. The extraction policy is identical to that in the case of optimal sustainable growth described above. The stock of the exhaustible resource depletes exponentially with the rate equal to the discount rate. The policy described above is optimal provided the initial knowledge stock is not too large ($x_0 > \bar{x}$). Otherwise it is optimal to allocate the entire labor to production from the onset.

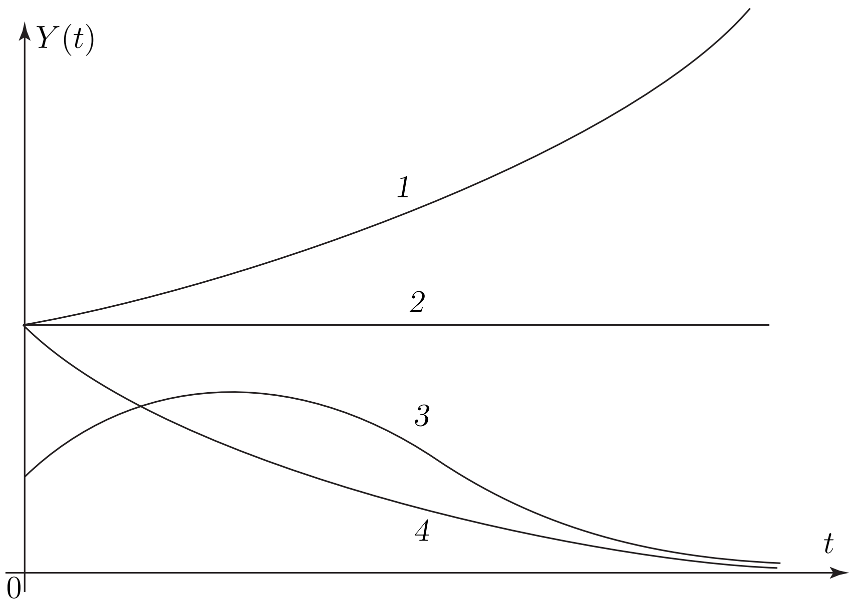

Figure 2: Dynamics of the output $Y(\cdot)$ under optimal resource allocation: (*1*) $\eta = \theta = 1$, $L\varkappa > \rho(\alpha + \varkappa(1 - \alpha))$; (*2*) $\eta = \theta = 1$, $L\varkappa = \rho(\alpha + \varkappa(1 - \alpha))$; (*3*) $\eta\theta < 1$; (*4*) $\eta = \theta = 1$, $L\varkappa < \rho(\alpha + \varkappa(1 - \alpha))$.

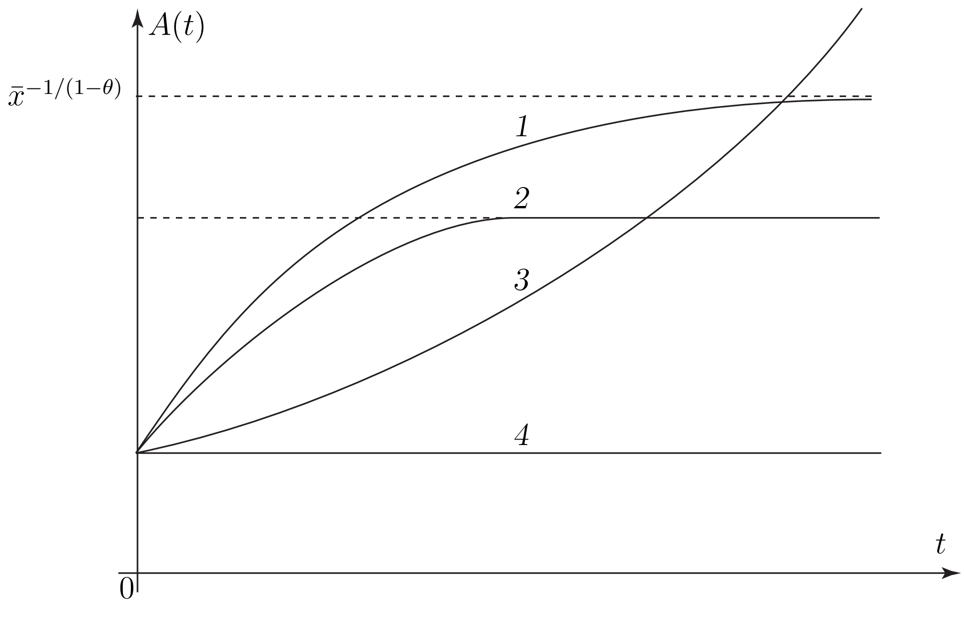

Figure 3: Dynamics of the knowledge stock $A(\cdot)$ under optimal resource allocation: (*1*) $\eta = 1$, $\theta < 1$; (*2*) $\eta < 1$; (*3*) $\eta = \theta = 1$, $L\varkappa > \alpha\rho$; (*4*) $\eta = \theta = 1$, $L\varkappa \leq \alpha\rho$.

(ii) When the accumulation of knowledge requires the resource ($\eta < 1$), it is optimal to conduct research until a certain ratio (given by (60)) between the knowledge stock and the current supply of the resource is reached. In this case the labor and resource allocated to research gradually decrease and ultimately vanish at the moment of reaching the above-mentioned ratio. Afterwards the research effort stops and the stock of knowledge remains at its maximum level. This policy is optimal when $x_0 > \bar{x}$. For $x_0 < \bar{x}$ it is optimal not to invest in research as the initial knowledge stock is sufficiently large.

On a final note we would like to emphasize that in the most realistic scenario, in which technological progress is subject to weak scale effects or does depend on an exhaustible resource, growth will eventually cease. This suggests that sustainable growth along one technological trajectory is impossible and there is a need for a transition to a new technological trajectory based on an alternative resource in production.

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