Capital vs Education: Assessment of Economic Growth from Two Perspectives ⋆

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Abstract: The paper is devoted to construction of optimal trajectories in the model which balances growth trends of investments in capital and labor efficiency. The model is constructed within the framework of classical approaches of the growth theory. It is based on three production factors: capital, educated labor and useful work. It is assumed that capital and educated labor are invested endogenously, and useful work is an exogenous flow. The level of GDP is described by an exponential production function of the Cobb-Douglas type. The utility function of the growth process is given by an integral consumption index discounted on the infinite horizon. The optimal control problem is posed to balance investments in capital and labor efficiency. The problem is solved on the basis of dynamic programming principles. Series of Hamiltonian systems are examined including analysis of steady states, properties of trajectories and their growth rates. A novelty of the solution consists in constructing nonlinear stabilizers based on the feedback principle which lead the system from any current position to an equilibrium steady state. Growth and decline trends of the model trajectories are studied for all components of the system and their proportions including: dynamics of GDP, consumption, capital, labor efficiency, investments in capital and labor efficiency.

Keywords: optimal control, nonlinear control system, nonlinear stabilizer, economic systems.

1. INTRODUCTION

The research is focused on analysis and construction of optimal growth trajectories in the model of searching right proportions for investments in capital and labor efficiency. The model is based on classical constructions of growth theory ((12), (11)). Also it is relied on ideas of a SEDIM model (10) which describes the role of different economic factors, including the demographic ones in a country's economic development. Another technique in the background ((2), (5) and (6)) considers capital and useful work as the key drivers of economic growth and uses optimal control theory to design past and future growth trajectories.

Investigated model is similar to the one suggested in (5). Three production factors, such as capital, educated labor, and useful work, stimulate growth of production. In the first statement it is supposed to consider investments in capital and educated labor (human capital) as endogenous control factors, and useful work is considered to be an exogenous flow subject to growth dynamics of the logistic type. It is assumed that the level of GDP is generated by an exponential production function of the Cobb-Douglas type.

Starting from the three-factor Cobb-Douglas production function and arguing like in (5) and (6) we construct a per capita production function, in which efficiency of labor, (see (10), page 4), acts as one of the main production factors. Following (6), we pose an optimal control problem to optimize investments in capital and labor efficiency on the model trajectories maximizing the utility function of per capita consumption index. To solve the problem we use methods of the optimal control theory (9). Specifically, we base the research on the existence results, Pontryagin's maximum principle and transversality conditions in optimal control problems with infinite horizon (1).

We investigate properties of the maximized Hamiltonian function and provide analysis of existence of steady states in domains of specific control regimes and focus attention on the domain corresponding to the transient control regimes of investment. Also we consider linearized Hamiltonian system in this domain. Special attention is given to the Jacobi matrix which has two negative and two positive eigenvalues that is the steady state has the saddle character. According to the results of the qualitative theory of differential equations (4) the trajectory of the nonlinear Hamiltonian dynamics converges to the

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steady state tangentially to the plane which is generated by eigenvectors corresponding to negative eigenvalues of the Jacobi matrix. This analysis provides the important information about the growth rates of optimal synthetic trajectories and outlines saturation levels of per capita variables.

A novelty of the proposed solution based on the idea of creating of nonlinear stabilizers built on the feedback principle ((7), (8)) which lead the system from any current position to a steady state. Constructed nonlinear stabilizer generates the dynamic system that is closed on the phase variables and has a property of a local stability.

2. DYNAMIC GROWTH MODEL OF CAPITAL VS EDUCATION

The model
The model is based on the following assumptions

1. *Labor Efficiency.*
Labor input $L(t)$ is generated from the total size of the *labor force* $P(t)$ and the driving force of generation is measured by index $E(t)$ of the *labor efficiency* (10).

$$L(t) = P(t)E(t). \tag{1}$$

Following formula (3) in Sanderson (2004), we assume that parameter E is essentially determined by the educational level of the population. One can call E the *efficiency of one worker.*

2. *Production Function.*
Let $Y(t)$ be the country's GDP at time t. We introduce the production function F in which GDP Y is the output and production factors like capital K, labor input L, and useful work U are input parameters, i.e. $Y = F(K, L, U)$.

We assume that function F is homogeneous with the unitary degree of homogeneity, i.e.
$F(\alpha K, \alpha L, \alpha U) = \alpha F(K, L, U), \quad \alpha \geq 0.$

3. *Dynamics of Labor Force.*
Let us suppose that labor force $P(t)$ is growing proportionally to population growth

$$\dot{P}(t) = \rho P(t) \tag{2}$$

with the given initial condition for labor force $P(t_0) = P^0$. Here parameter ρ is a growth rate of labor force. In this version of the model, we assume that rate ρ is constant.

4. *Dynamics of Capital.*
Following to classical models by Solow we introduce dynamics of capital $K(t)$ proportional to capital investment $S(t) = s(t)Y(t)$ with depreciation effect $\delta K(t)$

$$\dot{K}(t) = S(t) - \delta K(t) = s(t)Y(t) - \delta K(t). \tag{3}$$

The initial condition for capital is given by relation $K(t_0) = K^0$. It is assumed that the investment share $s(t)$ is a control parameter restricted by constraints
$0 \leq s(t) \leq a_s < 1$. Here parameter a_s stands for an upper bound of capital investment.

5. *Dynamics of Educated Labor Force.*
Let us introduce dynamics for educated labor force $L(t)$. We assume that an increment in educated force is proportional to investment $R(t) = r(t)Y(t)$ in human capital

$$\dot{L}(t) = bR(t) = br(t)Y(t). \tag{4}$$

Here parameter b $(b \geq 0)$, stands for the marginal effectiveness of investment in human capital. The initial condition for the educated labor force is fixed $L(t_0) = L^0$. The investment share $r(t)$ is a control parameter with the following restrictions $0 \leq r(t) \leq a_r < 1$. Here parameter a_r denotes an upper bound of investment in human capital.

6. *Balance Equations.*
The investments $S(t)$, $R(t)$, and consumption $C(t)$ should not exceed the total value of GDP, i.e.
$0 \leq S(t) + R(t) = (s(t) + r(t))Y(t) \leq Y(t)$.
More accurately, we assume that the following balance relations take place: $0 \leq s(t) \leq a_s < 1$,
$0 \leq r(t) \leq a_r < 1$ and $0 \leq s(t) + r(t) \leq a_s + a_r < 1$.

Further, we will use per capita variables normalizing GDP Y, consumption C, capital K, educated labor force L, and useful work U, with respect to labor force P:
$y = \frac{Y}{P}, \quad c = \frac{C}{P}, \quad k = \frac{K}{P}, \quad l = \frac{L}{P}, \quad u = \frac{U}{P}.$

Based on assumption (1) per capita variable l coincides with labor efficiency E, $l = E$.

In current model version we assume the level of per capita useful work is constant $u = \widehat{u}$, where parameter $\widehat{u}$ is equal to average value of per capita useful work. Hence, the production function in per capita variables looks like:
$y(t) = f(k(t), l(t)) = F(k(t), l(t), \widehat{u})$.

Relying on introduced differential equations for capital K (3), educated labor force L (4), and labor force P (2), dynamics of per capita variables and initial conditions can be written as follows

$$\begin{aligned} \dot{k}(t) &= s(t)y(t) - (\delta + \rho)k(t), \; k(t_0) = k^0 = \frac{K^0}{P^0} \\ \dot{l}(t) &= \quad br(t)y(t) - \rho l(t), \qquad l(t_0) = l^0 = \frac{L^0}{P^0} \end{aligned}$$

Utility Function
Let us consider the utility function of the model as the integrated logarithmic consumption index

$$J = \int_0^{+\infty} e^{-\lambda t} \ln c(t)\, dt. \tag{5}$$

Here parameter $c(t)$ stands for per capita consumption. Since we assume an economic system to be closed, where the gain $Y(t)$ is spent on consumption $C(t)$, on savings $S(t)$, and on investments in raising the efficiency of labor input $R(t)$, we get that the consumption part of the gain is defined by relation: $C(t) = Y(t) - S(t) - R(t)$.

For per capita variable of consumption $c(t)$ we obtain the following balance equation

$$c(t) = \frac{C(t)}{P(t)} = (1 - s(t) - r(t))y(t).$$

Let us make an assumption that the term $s(t)r(t)$ is much smaller than $s(t)$ and $r(t)$ and, hence, can be neglected. In this case, we calculate $c(t)$ as follows

$$c(t) = (1 - r(t))(1 - s(t))f(k(t), l(t)). \tag{6}$$

Substituting per capita consumption (6) into the utility functional J (5) we obtain the next its representation

$$J = \int_0^{+\infty} e^{-\lambda t} \left(\ln(1 - s(t)) + \ln(1 - r(t)) + \right.$$
$$\left. + \ \ln f(k(t), l(t)) \right) dt \qquad (7)$$

Optimal Control Problem
The optimal control problem consists in the maximization of the utility function (7) over trajectory $(k(\cdot), l(\cdot), s(\cdot), r(\cdot))$ of the system

$$\begin{cases} \dot{k}(t) = s(t)f(k(t), l(t)) - (\delta + \rho)k(t) \\ \dot{l}(t) = br(t)f(k(t), l(t)) - \rho l(t) \end{cases} \qquad (8)$$

with control parameters $(s(\cdot), r(\cdot))$ subject to constraints

$$0 \le s(t) \le a_s, \quad 0 \le r(t) \le a_r, \quad 0 \le a_s + a_r < 1, \qquad (9)$$

and phase variables $(k(\cdot), l(\cdot))$ satisfying initial conditions

$$k(t_0) = k^0, \quad l(t_0) = l^0. \qquad (10)$$

Let us note the problem (7)-(10), can be solved within the optimal control theory for problems with infinite horizon ((1), (6)).

3. MODEL ANALYSIS

Hamilton function
The Hamiltonian function of the optimal control problem (7)-(10) is defined by the following relation

$$\widetilde{H}(t, k, l; s, r; \widetilde{\psi_1}, \widetilde{\psi_2}) = e^{-\lambda t} \left(\ln(1 - s) + \ln(1 - r) + \right.$$
$$\left. \ln f(k, l) \right) + \widetilde{\psi_1} \left(sf(k, l) - (\delta + \rho)k \right) + \widetilde{\psi_2} \left(brf(k, l) - \rho l \right).$$

Here adjoint variables $\widetilde{\psi_1}$, $\widetilde{\psi_2}$ stand for "shadow prices" (model prices) for capital k, and labor efficiency l, respectively. Let us make the change of the Hamiltonian and adjoint variables

$$H = \widetilde{H} e^{\lambda t}, \quad z_1 = k\widetilde{\psi_1} e^{\lambda t}, \quad z_2 = l\widetilde{\psi_2} e^{\lambda t}.$$

In new variables the Hamiltonian has the following form

$$\hat{H}(k, l; s, r; z_1, z_2) = \ln(1 - s)(1 - r)f(k, l) \qquad (11)$$
$$+ \frac{z_1}{k} \left(sf(k, l) - (\delta + \rho)k \right) + \frac{z_2}{l} \left(brf(k, l) - \rho l \right).$$

Lemma 1. The Hamiltonian $\hat{H}(k, l; s, r; z_1, z_2)$ (11) is a strictly concave function in control variables s and r.

The proof follows immediately from strict negativity of second derivatives of the Hamiltonian (11) in variables s and r.

Solving the problem of maximization of Hamiltonian $\hat{H}$ (11) over control parameters s, and r subject to constraints (9), one can find the structure of optimal control as

$$s^0 = \begin{cases} 0, & z_1 f(k, l) \le k; \\ 1 - \dfrac{k}{z_1 f(k, l)}, & k \le z_1 f(k, l) \le \dfrac{k}{1 - a_s}; \\ a_s, & z_1 f(k, l) \ge \dfrac{k}{1 - a_s}; \end{cases},$$
$$r^0 = \begin{cases} 0, & z_2 f(k, l) \le \dfrac{l}{b}; \\ 1 - \dfrac{l}{bz_2 f(k, l)}, & \dfrac{l}{b} \le z_2 f(k, l) \le \dfrac{l}{b(1 - a_r)}; \\ a_r, & z_2 f(k, l) \ge \dfrac{l}{b(1 - a_r)}. \end{cases} \qquad (12)$$

It is important to note that the symmetric properties arise in expressions (11) and (12) for control variables s and r due to conversion of the model into per capita variables. This fact helps us to explore the model analytically and to develop methods for computing optimal trajectories.

The maximized Hamiltonian is the function by variables k, l, z_1, z_2 which is defined as maximum of the original Hamilton function by control parameters.

$$H(k, l; z_1, z_2) = \max_{s, r} \hat{H}(k, l; s, r; z_1, z_2), \qquad (13)$$
$$s \in [0, a_s], \quad r \in [0, a_r].$$

Basing on the Pontryagin maximum principle differential equations for adjoint variables z_1 and z_2 look as follows

$$\dot{z}_1(t) = \lambda z_1(t) - k\frac{\partial H}{\partial k}(k(t), l(t); z_1(t), z_2(t)) + \dot{k}\frac{z_1}{k},$$
$$\dot{z}_2(t) = \lambda z_2(t) - l\frac{\partial H}{\partial l}(k(t), l(t); z_1(t), z_2(t)) + \dot{l}\frac{z_2}{l}.$$

Departing from the structure of optimal controls s^0, r^0 (12) we can write nine domains of definition of the maximized Hamiltonian H (13).

s^0, r^0	0	$1 - \dfrac{l}{bz_2 f(k, l)}$	a_r
0	D_{11}	D_{12}	D_{13}
$1 - \dfrac{k}{z_1 f(k, l)}$	D_{21}	D_{22}	D_{23}
a_s	D_{31}	D_{32}	D_{33}

On Fig. 1 sections of domains are depicted in phase variables k, l under fixed cost variables z_1, z_2.

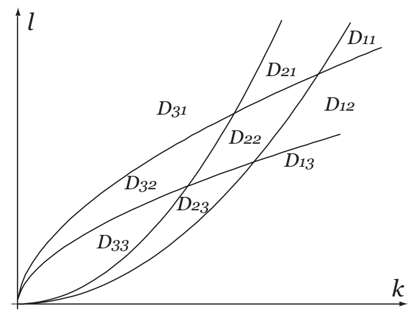

Fig. 1. Structure of domains

Properties of maximized Hamiltonian function (13) will be considered below.

Lemma 2. The maximized Hamiltonian function (13) $H(k, l; z_1, z_2)$ is smoothly pasted out of branches $H_{ij}(k, l; z_1, z_2)$, $i, j = \overline{1, 3}$, in variables $(k, l; z_1, z_2)$ on borders of domains D_{ij}, $i, j = \overline{1, 3}$.

The result of Lemma 2 is proved by direct calculations of derivatives of the Hamiltonian on domain borders.

Lemma 3. If the condition (14) is satisfied for all $k > 0$ and $l > 0$

$$\left(\frac{\partial f(k, l)}{\partial k} \right)^2 + f(k, l)\frac{\partial^2 f(k, l)}{\partial k^2} < 0 \qquad (14)$$

$$\left(\frac{\partial f(k,l)}{\partial l}\right)^2 + f(k,l)\frac{\partial^2 f(k,l)}{\partial l^2} < 0,$$

then the maximized Hamiltonian function $H(k,l;z_1,z_2)$ is a strictly concave function in variables (k,l) for all $z_1 > 0$ and $z_2 > 0$.

The proof of Lemma 3 consists in the checking of inequalities $\frac{\partial^2 H(k,l,z_1,z_2)}{\partial k^2} < 0$ and $\frac{\partial^2 H(k,l,z_1,z_2)}{\partial l^2} < 0$ in all domains D_{ij}, $i,j=\overline{1,3}$. Conditions (14) are necessary in domain D_{22}. In another domains the proposition of Lemma 3 holds without these restrictions (14).

Due to above-listed properties of the Hamilton function (Lemmas 1 - 3), the Pontryagin maximum principle ensures sufficient optimality conditions in the problem (7) - (10). The proof of this statement can be found in (6).

Further we will consider the domain in which optimal controls are not constant. This domain is denoted D_{22}.

Qualitative analysis of the Hamiltonian system
In domain D_{22} values of optimal control are defined in the transient regime

$$s^0 = 1 - \frac{k}{z_1 f(k,l)}, \quad r^0 = 1 - \frac{l}{bz_2 f(k,l)}. \tag{15}$$

The maximized Hamiltonian H is defined by relation

$$H(k,l;z_1,z_2) = z_1\frac{f(k,l)}{k} - \ln\left(z_1\frac{f(k,l)}{k}\right) + \ln f(k,l) + bz_2\frac{f(k,l)}{l} - \ln\left(bz_2\frac{f(k,l)}{l}\right) - (\delta+\rho)z_1 - \rho z_2 - 2.$$

Domain D_{22} is described by inequalities

$$D_{22} = \left\{(k,l,z_1,z_2) : k \le z_1 f(k,l) \le \frac{k}{(1-a_s)} \wedge \frac{l}{b} \le z_2 f(k,l) \le \frac{l}{b(1-a_r)}\right\}.$$

In domain D_{22} the Hamiltonian dynamics is given by differential equations

$$\begin{cases} \dot{k} = & f(k,l) - (\delta+\rho)k - \dfrac{k}{z_1} = H_1, \\ \dot{l} = & bf(k,l) - \rho l - \dfrac{l}{z_2} = H_2, \\ \dot{z_1} = & \left(\lambda - \dfrac{\partial f(k,l)}{\partial k} + \dfrac{f(k,l)}{k}\right)z_1 - b\dfrac{k}{l}\dfrac{\partial f(k,l)}{\partial k}z_2 \\ & + \dfrac{k}{f(k,l)}\dfrac{\partial f(k,l)}{\partial k} - 1 = H_3, \\ \dot{z_2} = & -\dfrac{l}{k}\dfrac{\partial f(k,l)}{\partial l}z_1 + \left(\lambda - b\dfrac{\partial f(k,l)}{\partial l} + b\dfrac{f(k,l)}{l}\right)z_2 \\ & + \dfrac{l}{f(k,l)}\dfrac{\partial f(k,l)}{\partial l} - 1 = H_4, \end{cases} \tag{16}$$

here $H_i = H_i(k,l,z_1,z_2)$, $i=\overline{1,4}$.

The question about steady state existence is very important. In order to analyze the existence and representation of Hamilton system steady states in the domain D_{22} it is convenient to introduce the following notations: $a_k = f(k,l)\,/\,k$ and $a_l = b\,f(k,l)\,/\,l$.

Here and further in this paper we assume that production function f is a power function of the Cobb-Douglas type: $f(k,l) = \mu k^\alpha l^\beta$. Here parameter μ, $\mu > 0$, is a given scale parameter. Elasticity coefficients α, β satisfy conditions

$$\alpha \ge 0, \quad \beta \ge 0, \quad \alpha+\beta \le 1. \tag{17}$$

Based on the power production function one can obtain, the following relations are useful in analysis of steady states equations for derivatives $\frac{\partial f(k,l)}{\partial k} = \alpha a_k$, $\frac{\partial f(k,l)}{\partial l} = \frac{\beta}{b}a_l$; and ratios $bka_k = la_l$.

Let us provide solutions of steady states equations in domain D_{22}.

In this area two steady states are possible. In both equilibrium points coordinates of conjugate variables z_1^* and z_2^* depend on values a_k^* and a_l^* the same way:

$$z_1^* = \frac{1}{a_k^* - (\delta+\rho)}, \quad z_2^* = \frac{1}{a_l^* - \rho}. \tag{18}$$

The first steady state has following values of variables a_k and a_l:

$$a_k^* = \frac{\lambda(1-\beta)}{\alpha-1}, \quad a_l^* = \frac{\lambda(\alpha-1)}{(1-\beta)}. \tag{19}$$

The second steady state is defined as follows

$$a_k^* = \frac{(\lambda+(1-\beta)\rho)(\delta+\lambda+\rho)}{\alpha(\lambda+\rho)}, \tag{20}$$
$$a_l^* = \frac{(\lambda+\rho)(\lambda+(1-\alpha)(\delta+\rho))}{\beta(\delta+\lambda+\rho)}.$$

Coordinates of steady states in domain D_{22} should satisfy conditions

$$a_k^* > \delta+\rho > 0,\ a_l^* > \rho > 0, \quad a_s \ge \frac{\delta+\rho}{a_k^*},\ a_r \ge \frac{\rho}{a_l^*}. \tag{21}$$

Coordinates of the first steady state (19) do not satisfy conditions (21) since coefficients of elasticity α and β meet restrictions (17). Therefore we will consider the second steady state (20) below.

The coordinates of phase variables (k^*,l^*) of the steady state are recalculated according to the following relations

$$k^* = (AB^\beta)^\gamma, \quad l^* = (AB^{(1-\alpha)})^\gamma, \tag{22}$$

where $A = \mu/a_k^*$, $B = ba_k^*/a_l^*$, $\gamma = 1/(1-\alpha-\beta)$.

One can define also values of optimal control at steady states $s^* = (\delta+\rho)/a_k^* \le a_s < 1$ and $r^* = \rho/a_l^* \le a_r < 1$.

Coefficients of the Jacobian matrix I at the steady state (20) are evaluated by formulas

$$\frac{\partial H_1}{\partial k} = -(1-\alpha)a_k^*, \quad \frac{\partial H_1}{\partial l} = \frac{\beta a_l^*}{b},$$
$$\frac{\partial H_1}{\partial z_1} = k^*(a_k^* - \delta - \rho)^2, \quad \frac{\partial H_1}{\partial z_2} = 0,$$
$$\frac{\partial H_2}{\partial k} = \alpha b a_k, \quad \frac{\partial H_2}{\partial l} = -(1-\beta)a_l^*,$$
$$\frac{\partial H_2}{\partial z_1} = 0, \quad \frac{\partial H_2}{\partial z_2} = l^*(a_l^* - \rho)^2,$$

$$\frac{\partial H_3}{\partial k} = -\frac{(1-\alpha)^2 a_k^*}{k^*(a_k^* - \delta - \rho)} - \frac{\alpha^2 a_l^*}{k^*(a_l^* - \rho)},$$
$$\frac{\partial H_3}{\partial l} = \frac{(1-\alpha)\beta a_k^*}{l^*(a_k^* - \delta - \rho)} + \frac{\alpha(1-\beta) a_l^*}{l^*(a_l^* - \rho)},$$
$$\frac{\partial H_3}{\partial z_1} = \lambda + (1-\alpha)a_k^*, \quad \frac{\partial H_3}{\partial z_2} = -\alpha a_l^*,$$
$$\frac{\partial H_4}{\partial k} = \frac{(1-\alpha)\beta a_k^*}{k^*(a_k^* - \delta - \rho)} + \frac{\alpha(1-\beta) a_l^*}{k^*(a_l^* - \rho)},$$
$$\frac{\partial H_4}{\partial l} = -\frac{\beta^2 a_k^*}{l^*(a_k^* - \delta - \rho)} - \frac{(1-\beta)^2 a_l^*}{l^*(a_l^* - \rho)},$$
$$\frac{\partial H_4}{\partial z_1} = -\beta a_k^*, \quad \frac{\partial H_4}{\partial z_2} = \lambda + (1-\beta)a_l^*.$$

Unfortunately it is not possible to calculate eigenvalues of the Jacobian matrix I analytically. We do this for the following range of calibrated model parameters based on a real macroeconomic data of the US economy from 1900 to 2005. These data were reduced to the values of 1900. Values of parameters are equal $\mu = 2.19941$, $\alpha = 0.3$, $\beta = 0.1$, $\lambda = 0.03$, $\delta = 0.2$, $\rho = 0.013$, $b = 0.31$, $a_s = 0.3$, $a_r = 0.2$. Initial point is $(k^0, l^0) = (1, 1)$. The steady state (18), (22) of the Hamiltonian system (16) has following coordinates: $k^* = 5.75$, $l^* = 5.2$, $z_1^* = 1.758$, $z_2^* = 3.822$. Values of control variables at the steady state are given as follows: $s^* = 0.2795$, $r^* = 0.0379$. With these parameters, the Jacobian matrix I has two negative and two positive eigenvalues: $\lambda_1 = -0.094$, $\lambda_2 = -0.268$, $\lambda_3 = 0.124$, $\lambda_4 = 0.298$.

4. NONLINEAR STABILIZERS

Consider the case when linearized Hamilton system in domain D_{22} has two real negative λ_1, λ_2 and two real positive λ_3, λ_4 eigenvalues.

In order to construct a nonlinear stabilizer, do the following
1. Build the plane by two eigenvectors h_1, h_2 corresponding to negative eigenvalues λ_1, λ_2 and holding the steady state

$$\begin{cases} h_{31}(k-k^*) + h_{32}(l-l^*) & + \\ h_{33}(z_1 - z_1^*) + h_{34}(z_2 - z_2^*) & = 0, \\ h_{41}(k-k^*) + h_{42}(l-l^*) & + \\ h_{43}(z_1 - z_1^*) + h_{44}(z_2 - z_2^*) & = 0. \end{cases} \tag{23}$$

Here coefficients $h_{i,j}$, $i, j = \overline{1,4}$ are coordinates of eigenvectors h_1, h_2, h_3, h_4 and (k^*, l^*, z_1^*, z_2^*) coordinates of the steady state.

2. Try to find conjugate variables trough the phase ones from the equations of the plane (23):

$$z_1(k,l) = z_1^* + \gamma_{11}(k-k^*) + \gamma_{12}(l-l^*), \tag{24}$$
$$z_2(k,l) = z_2^* + \gamma_{21}(k-k^*) + \gamma_{22}(l-l^*),$$

where coefficients γ_{ij}, $(i, j = \overline{1,2})$ can be found as follows:

$$\gamma_{11} = \frac{h_{41}h_{34} - h_{31}h_{44}}{h_{44}h_{33} - h_{43}h_{34}}, \quad \gamma_{12} = \frac{h_{42}h_{34} - h_{44}h_{32}}{h_{44}h_{33} - h_{43}h_{34}},$$
$$\gamma_{21} = \frac{h_{31}h_{43} - h_{41}h_{33}}{h_{44}h_{33} - h_{43}h_{34}}, \quad \gamma_{22} = \frac{h_{43}h_{32} - h_{42}h_{33}}{h_{44}h_{33} - h_{43}h_{34}}.$$

Specified presentation for adjoint variables exists when the condition is fulfilled: $h_{33}h_{44} - h_{34}h_{43} \neq 0$.
Important property of the functions $z_1(k,l)$ and $z_2(k,l)$ can be expressed by equalities:

$$z_1(k^*, l^*) = z_1^*, \quad z_2(k^*, l^*) = z_2^*. \tag{25}$$

3. Substitute conjugate variables (24) to the Hamiltonian dynamics (16) and obtain the closed system of differential equations with respect to phase variables k and l

$$\begin{cases} \dot{k} = f(k,l) - (\delta+\rho)k - \dfrac{k}{z_1(k,l)}, \\ \dot{l} = bf(k,l) - \rho l - \dfrac{l}{z_2(k,l)}. \end{cases} \tag{26}$$

Nonlinear stabilizers in dynamics (26) are defined by relations

$$\widehat{s} = 1 - \frac{k}{z_1(k,l)f(k,l)}, \quad \widehat{r} = 1 - \frac{l}{bz_2(k,l)f(k,l)}. \tag{27}$$

Due to the properties (25) of the functions $z_1(k,l)$ and $z_2(k,l)$ (24), nonlinear stabilized system (26) has the stationary point (k^*, l^*) with the same two first coordinates as the steady state (k^*, l^*, z_1^*, z_2^*) of the Hamiltonian system (16). Moreover, linearized system for this dynamics has the same negative eigenvalues λ_1, λ_2. It means that the stationary point (k^*, l^*) is locally stable.

5. RESULTS OF NUMERICAL EXPERIMENTS

Trajectories of the system (26) generalized by the nonlinear stabilizer (27) are constructed numerically by the Runge-Kutta method. They can be considered as the first approximation of optimal trajectories and used for preliminary assessment of growth trends. They can serve also as the basis of the algorithm for constructing optimal trajectories. The lack of initial data for variables z_1, z_2 impedes the solving of original Hamilton system. That is why we try to find solution of dynamic system in reverse time. According to the results of the qualitative theory of differential equations (4) the trajectory of the nonlinear Hamiltonian dynamics converges to the steady state (k^*, l^*, z_1^*, z_2^*) tangentially to the plane which is generated by eigenvectors corresponding to negative eigenvalues of the Jacobi matrix. Approximate values of start position (k^T, l^T, z_1^T, z_2^T) for reverse time system can be taken from the vicinity of points which are situated on the trajectories of stabilized Hamilton system. If the computed solution comes in the original initial point (k^0, l^0) (10), then one can assume that the original Hamilton system is solved.

Solutions of the system (26) are depicted on the figures which demonstrate growth and decline trends of capital $k(t)$, educated labor $l(t)$, production $y(t)$, consumption $c(t)$, and investments $s(t)$, $r(t)$.
Figures 2.*a* and 2.*b* show growth trends of per capita values of capital $k(t)$ and labor efficiency $l(t)$. One can see that both graphs demonstrate growth with the saturation property.
The model trajectories of capital stock $K(t)$, labour force $L(t)$ and statistic data in a period since 1900 to 2005 are depicted on the figure 3.*a,b*.
Figure 2.*d* presents decline trends of relative values of

investment in capital $s(t)$ and labor efficiency $r(t)$ in the share structure of GDP. One can observe that the share of investment in capital $s(t)$ rapidly drops down from the level of 30% to the level of 27.27% of GDP at the steady state. The share of investment in labor efficiency $r(t)$ smoothly decreases from the level of 20% to the saturation level of 3.79% around the steady state.

An important note is that all graphs demonstrate the convex or concave properties without periods of increasing return and inflection points. These trends are definitely explained by concave properties of the Cobb-Douglas production function. It is worth to mention that the model ((2), (5), (6)) with the linear-exponential (LINEX) production function catches both periods of increasing and decreasing return, as well as inclination points and saturation levels.

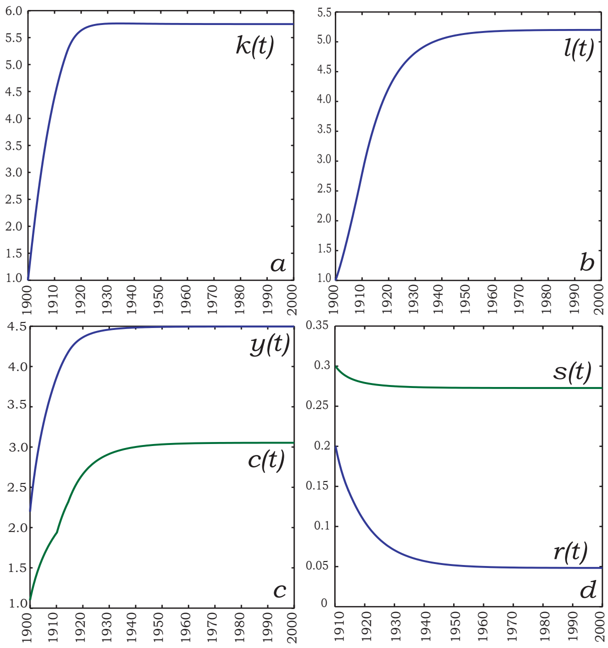

Fig. 2. *a)* Trends of capital per capita $k(t)$; *b)* Trends of labour efficiency $l(t)$; *c)* Trends of GDP $y(t)$ and consumption $c(t)$; *d)* Investments share in human capital $r(t)$ and capital $s(t)$.

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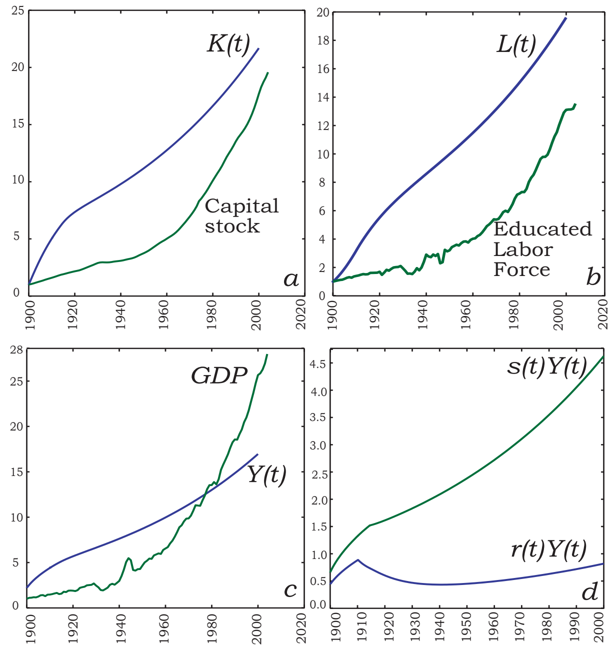

Fig. 3. *a)* Capital $K(t)$ and statistic data; *b)* Trends of educated labour force $L(t)$ and statistic data; *c)* The GDP $Y(t)$ and statistic data; *d)* Investments in human capital and stock capital.

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