# Thermodynamic Laws, Economic Methods and the Productive Power of Energy[†]

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## Abstract

Energy plays only a minor role in orthodox theories of economic growth, because standard economic equilibrium conditions say that the output elasticity of a production factor, which measures the factor's productive power, is equal to the factor's share in total factor cost. Having commanded only a tiny cost share of about 5 percent so far, energy is often neglected altogether. On the other hand, energy conversion in the machines of the capital stock has been the basis of industrial growth. How can the physically obvious economic importance of energy be reconciled with the conditions for economic equilibrium, which result from the maximization of profit or overall welfare? We show that these equilibrium conditions no longer yield the equality of cost shares and output elasticities, if the optimization calculus takes technological constraints on the combinations of capital, labor and energy into account. New econometric analyses of economic growth in Germany, Japan and the USA yield output elasticities that are for energy much larger and for labor much smaller than their cost shares. Social consequences are discussed.

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## 1. Introduction

Economic ideas wield great power on our lives, because they shape the laws that regulate the economic activities in our societies, with repercussions on the social set of values. The necessity of continuous exponential economic growth is such an idea. The Stern Review Report on the Economics of Climate Change assumes an annual global growth rate of 1.9 percent for 200 years in the future [1, 2]. On the other hand, the theory of economic growth has been a frontier of research for quite some time without having been completed so far, despite of recent attempts on "endogeneous" growth theory [3]-[5].

Politicians in OECD countries aspire after annual growth rates not below 3 percent, because under the present economic frame conditions only sufficiently high growth rates can provide new jobs to replace the ones that haven fallen prey to increasing automation and globalization, and only strong economic growth can reduce the growth of state indebtedness, which is a major problem today. The developing countries strive for even higher rates of economic growth in order to catch up with the standard of living in the highly industrialized countries. Unfortunately, perpetual exponential economic growth, desirable as it may be from a social point of view, collides with the energetic and environmental restrictions established in a finite world by the first two laws of thermodynamics.

The First Law of Thermodynamic states that energy is conserved.[1] In general, energy consists of a valuable part, called exergy, which can be converted into physical work, and a useless part (sometimes called anergy), which is, for instance, heat at the temperature of the environment. The principal carriers of primary energy – the fossil and nuclear fuels, and solar radiation as well – are practically 100 percent exergy.

From an engineering point of view it is obvious that there is no industrial production without the performance of physical work on matter and the associated information processing. Therefore, energy or, more precisely, exergy is a prime mover of modern industrial economies, and restrictions on its use will restrict economic growth. This, however, is by no means the view of mainstream economic theory, which involves a fundamental methodological argument for disregarding or undervaluing energy as a factor of production. This argument becomes invalid, if engineering constraints on the combination of energy with machines are taken into account. To show this, and estimate energy's productive power econometrically, is the purpose of this paper.

## 2. Thermodynamics and Economics

An economic system – a national market economy or a sub-sector of it – consists, roughly speaking, of a physical basis that produces goods and services, and a market superstructure, where economic actors trade the products of the basis. Price signals from supply and demand provide the feed-back between the physical basis and the market superstructure.[2] Since economics understands itself as a social science, in fact as the queen of social sciences [6], it is mainly concerned with the behavior of the actors in the market superstructure and rather little, or not at all, with the engineering mechanisms of production in the realm of the physical basis, where the laws of thermodynamics reign.

The production process in the physical basis of an industrial economy requires three inputs: 1) Energy conversion devices and information processors together with all buildings and installations necessary for their protection and operation. They represent the production factor capital $K$. 2) The capital stock $K$ is manipulated and supervised by people, who

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[1]Strictly speaking: Energy, including the energy equivalent of mass, is conserved.
[2]In bygone "socialist" economies the market was replaced by the planning bureaucracy.

constitute the production factor labor $L$. 3) The machines of the capital stock are activated by energy (more precisely: exergy), which is the production factor $E$.

The energy-converting and information-processing machines are open thermodynamic systems, subject to the laws of nature when they produce the output $Y$. The output is the sum of all goods and services produced within an economic system. Its measure is the gross domestic product (GDP), or parts thereof. The natural environment, in which all economic systems are embedded, serves as the reservoir of temperature and pressure for the heat engines, transistors and all other energy conversion devices of the capital stock. It also contains the energetic and material resources.

In preindustrial agrarian economies the productive basis was the photosynthetic collection of solar energy by plants, which provided food, fuel and timber. Since plants grow on land, traditional economics has considered land as the third basic factor of production. In feudal agrarian societies it gave economic and political power to its owners. But this power was owed to photosynthesis and the people and animals tilling the soil. The production site itself is not an active factor that performs work or processes information. Therefore, "land", or its three-dimensional extension "space", is rather part of natural restrictions. Furthermore, the quantity of tolerable emissions of pollutants is limited by the absorption capacity of the biosphere, which encompasses the land.

This takes us to the Second Law of Thermodynamics, which is behind emissions from production processes in the physical basis. Consider a non-equilibrium system of arbitrary volume $V$ with surface $\Sigma$. One obtains the entropy balance equation by specifying the general balance equation for any time-changing quantity to entropy $S$:

$$\frac{dS}{dt} = -\int_{\Sigma} \vec{J}_S(\vec{r},t)d\vec{\Sigma} + \int_V \sigma_S(\vec{r},t)dV. \tag{1}$$

In words: the time change of system entropy, $dS/dt$, equals the entropy transported per unit time through the surface by the entropy current density $\vec{J}_S(\vec{r},t)$ plus the entropy produced per unit time within the volume by the source term $\sigma_S(\vec{r},t)$ – the entropy production density. In the first integral the vector of the surface element $d\vec{\Sigma}$ points out of the volume, by definition. The Second Law of Thermodynamics states that for all irreversible processes the second integral, $\int_V \sigma_S(\vec{r},t)dV$, is positive; consequently, since the volume $V$ can be arbitrarily small, entropy production density $\sigma_S(\vec{r},t)$ is positive. Total entropy balance $dS/dt$ is positive and entropy increases in $V$, if less entropy is transported out of $V$ by $\vec{J}_S$ than is being produced in $V$ by $\sigma_S$; otherwise, $dS/dt$ is negative or vanishes.

In non-equilibrium systems containing $N$ different sorts of molecules $k$ that are *locally* in thermodynamic equilibrium, and which do not undergo chemical reactions,[3] entropy production density $\sigma_S$ is equal to the "dissipative" entropy production density $\sigma_{S,dis}$ [8] :

$$\sigma_S(\vec{r},t) = \sigma_{S,dis}(\vec{r},t) = \vec{j}_Q \vec{\nabla}\frac{1}{T} + \sum_{k=1}^{N} \vec{j}_k[-\vec{\nabla}\frac{\mu_k}{T} + \frac{\vec{f}_k}{T}] > 0 \quad . \tag{2}$$

In words: positive entropy production density in the space-time point $(\vec{r},t)$ consists of the heat current density $\vec{j}_Q$ (i.e. the conductive current density of internal energy), driven by

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[3]If chemical reactions occur, too, there is also the entropy production density $\sigma_{S,chem}$. Then total entropy production density consists of two components: $\sigma_{S,dis}$ and $\sigma_{S,chem}$, where the latter involves products of scalar currents and forces, which cannot interfere with the vectorial products in $\sigma_{S,dis}$ of Eq. (2). Thus, both $\sigma_{S,dis}$ and $\sigma_{S,chem}$ are positive individually. [7]

the gradient ($\vec{\nabla}$) of temperature $T$, and diffusion current densities $\vec{j}_k$ (i.e. conductive mass current densities), driven by gradients of chemical potentials $\mu_k$ divided by $T$ and by specific external forces $\vec{f}_k$. The conductive current densities $\vec{j}_Q$ and $\vec{j}_k$ are the respective current densities minus the barycentric velocity $\vec{v}(\vec{r}, t)$ multiplied by the appropriate densities of internal energy and mass. Thus, Eq. (2) describes entropy production in the macroscopically small and microscopically large volume elements of many-particle non-equilibrium systems with mass end energy flows. These systems are an essential part of the productive physical basis of an economy with its manifold energy conversion processes. Furthermore, whenever heat is generated, valuable exergy is converted into useless anergy. This is what "energy consumption" really means. Exergy dissipation increases with the rate at which the industrial process is run [9].[4]

Besides devaluating energy, the irreversible processes of industrial production also generate emissions of molecules and heat according to Eq. (2). These emissions change the composition of and the energy flows through the biosphere to which the living species and their populations have adapted in the course of evolution. If these changes are so big that they cannot be balanced by the biological and anorganic processes that are driven by the exergy input from the Sun and the radiation of heat into space, and if they occur so rapidly that biological and social adaptation deficits develop, the emissions are perceived as environmental pollution. ($CO_2$ is a typical example of how quantities and emission rates of a sort of molecules determine its environmental impact. Practically nobody worried about $CO_2$ emissions before the 1970s – except Svante Arrhenius –, and now they are feared as a driver of climate change.) As long as heat emissions are considered as environmentally more benign than material emissions, one can transform the latter into the former by appropriate technologies like desulphurization, denitrification and, perhaps, carbon capture and storage. In such cases the second term on the r.h.s of Eq. (2) decreases and the first term must increase so much that $\sigma_S > 0$ always holds. However, waste-heat emissions, presently about $1.4 \cdot 10^{13}$ Watts, are likely to cause climate problems even without the anthropogeneous greenhouse effect, once they become comparable (in some sense) to the power of $1.7 \cdot 10^{17}$ Watts the Earth receives from the Sun [10]. One concludes that from observed local climate changes in areas where heat emissions reach a few per mill of solar insolation.

Thus, there are limits to growth drawn by the Second Law of Thermodynamics. But they are not the subject of the present paper. We do emphasize, however, that – because of the coupling of energy conversion to entropy production – an increasing need of energy for economic growth accelerates the approach to these limits. Because of the indicated engineering and environmental reasons the physical laws on energy and entropy must be incorporated into economic models [11].

There are some economists who betimes realized that energy and entropy matter in economics [12] – [14]. After the 1972 publication of "The Limits to Growth" [15] and the first oil-price shock in 1973-1975 a growing number of natural scientists and economists has ventured into interdisciplinary research on the relevance of energy conversion and entropy production for economic evolution [9], [16]-[29]. Ref. [9], investigating the economically-efficient level of thermodynamic effectiveness, tellingly describes the problems of interdisciplinary research in thermodynamics and economics. Ref. [23] presents a geometric view of losses in a system with optimizing behavior that arise from non-instantaneous responses to exogeneous shocks; in thermodynamic systems such losses are measured by entropy production or exergy losses,

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[4] "Haste makes waste."

and in economic systems the losses refer to welfare. There are also analyses of extremal principles and the limiting capabilities of open thermodynamic and economic macrosystems [28] and mathematical models of equilibrium in irreversible economics [29].

Mainstream economics, however, does not worry about the First and the Second Law of Thermodynamics, despite of their governing all energetic and material processes of wealth production. Of course, one realizes the problem of pollution and climate change [1, 2], but does not really believe in limits to growth in finite systems such as planet Earth. Even a Nobel Laureate in Economics stated that the " world can, in effect, get along without natural resources" [30].

When it comes to the question, how wealth is produced, standard production theory usually takes only capital $K$ and labor $L$ into account. The focus widened somewhat, when the oil price shocks 1973-1975 and 1979-1981 and the accompanying recessions known as the first and the second energy crisis prompted investigations that considered energy $E$, sometimes in combination with materials $M$, as an additional factor of production [31] -[37]. But in most ($KLE$ or $KLEM$) models energy was given only a tiny output elasticity of about 5 percent [38]. The output elasticity of a production factor measures the productive power of the factor in the sense that (roughly speaking) it gives the percentage of output change when the inputted factor changes by 1 percent while the other factors stay constant. The exact definition is in Eq. (4). For reasons indicated in Section 3 orthodox economics considers the output elasticity of a production factor as equal to its cost share. We call this the "cost share theorem". This theorem is based on a single sector, single product model of the economy, where the production factors are rented and where they are substitutable for one another without limit. On an OECD average the cost share of capital has been about 25 percent, that of labor roughly 70 percent and that of energy about 5 percent during the last decades.

The cost share theorem facilitates a Legendre transformation, given in Eq. (14), from factor quantities to factor prices. This may have been seen as a justification of restricting growth and production analyses to the market superstructure without observing the thermodynamic laws that reign in the physical basis.

According to the cost share theorem, reductions of energy inputs by up to 7 percent, observed during the first energy crisis 1973-1975, could have only caused output reductions of 0.35 percent, whereas the observed reductions of output in industrial economies were up to an order of magnitude larger. Thus, from this perspective the recessions of the energy crises are hard to understand. In addition, cost-share weighting of production factors has the problem of the Solow residual. The Solow residual accounts for that part of output growth that cannot be explained by the input growth rates weighted by the factor cost shares. It amounts to more than 50 percent of total growth in many cases. Standard neoclassical economics attributes this difference formally to what is being called "technological progress". This, however, "has lead to a criticism of the neoclassical model: it is a theory of growth that leaves the main factor in economic growth unexplained" [39], as the founder of neoclassical growth theory, Robert A. Solow, admitted himself. Recent endogeneous growth theories [5] and quantitative economic climate-change assessments [40] also employ cost-share weighting of production factors.

The objective of the present paper is a critique of the low importance attributed to energy by standard economic theory. This critique is presented in two steps. The first step, taken in Section 3, derives economic equilibrium conditions subject to technological constraints on factor combinations. This modifies the standard equilibrium conditions in the

sense that shadow prices add to factor prices, destroying the cost share theorem. Shadow prices translate technological constraints into monetary terms. The second step, Section 4, summarizes the results of heterodox growth analyses that determine the output elasticities of capital, labor and energy independently from any equilibrium conditions and cost share considerations. They turn out to be for labor much smaller and for energy much larger than the cost shares of these factors. Social consequences are indicated briefly in Section 5.

## 3. Economic Equilibrium

The output of an economic system is generated by the production factors of the physical basis. Economic actors select the combinations of production factors, observing price signals from the market superstructure. When modeling their actions, economics uses extremum principles, which have proven so successful in physics.

When the variables of a system adjust within given constraints so that a system-specific objective becomes an extremum, the system is said to be in equilibrium. There are equilibrium conditions that are time independent and such that depend on time $t$. We call them static and dynamic. In this sense the equilibrium condition "Gibbs free energy must be minimum" for a thermodynamic system in contact with a temperature and pressure reservoir is static, while Hamilton's principle of least action, from which the Lagrange equations of motion can be derived in classical mechanics, is dynamic. In *formal* one-to-one correspondence to these physical examples, economic equilibria are defined by the maximum of either profit or time-integrated utility. Needless to say that this involves assumptions on the behavior of economic actors. Thus, the economic equilibrium conditions have the character of *postulates*. The consequence of rejecting them is discussed below.

The mathematical derivation of both types of economic equilibrium starts from the assumption that the output $Y$ of an economic system is produced by three production factors $X_1, X_2, X_3$, the combinations of which are subject to technological constraints. Then, identifying the three factors with capital $K$, labor $L$, and energy (exergy) $E$, we will show that their combinations are constrained technologically by limits to the degree of automation and to the degree of capacity utilization. The corresponding shadow prices are derived.

### 3.1. Growth equation

We follow standard economic theory and assume that output $Y$ is a twice differentiable function of the (time-dependent) production factors. Furthermore, it may also depend explicitly on time $t$. Thus, the production function $Y(X_1, X_2, X_3; t)$ describes economic evolution. (It is a state function of the factors in the same sense as Gibbs free energy is a state function of temperature and pressure.) Taking its total differential $dY$ and dividing it by $Y$ yields the growth equation, which relates the growth rate $dY/Y$ of output to the growth rates $dX_i/X_i$ of the inputs, and to the change of time since an initial time $t_0$:

$$\frac{\mathrm{d}Y}{Y} = \epsilon_1 \frac{\mathrm{d}X_1}{X_1} + \epsilon_2 \frac{\mathrm{d}X_2}{X_2} + \epsilon_3 \frac{\mathrm{d}X_3}{X_3} + \delta \frac{\mathrm{d}t}{t - t_0} \quad . \tag{3}$$

Here the output elasticity $\epsilon_i$ of the factor $X_i$ is defined as

$$\epsilon_i \equiv \frac{X_i}{Y} \frac{\partial Y}{\partial X_i}, \quad i = 1, 2, 3 \quad . \tag{4}$$

The specific human contribution to growth (ideas, inventions and value decisions) that cannot be captured by changes of the $X_i$ manifests itself in the explicit time dependence of the

production function with the weight

$$\delta \equiv \frac{t - t_0}{Y} \frac{\partial Y}{\partial t} \quad . \tag{5}$$

At any *fixed* time $t$ an increase of *all* inputs by the same factor $\lambda$ must increase output by $\lambda$, because at the fixed state of technology that exists at the given time $t$ a, say, doubling of the production system doubles output; in other words: two identical factories with identical inputs of capital, labor and energy produce twice as much output as one factory. Thus, the production function must be linearly homogeneous in $(X_1, X_2, X_3)$, which means that $Y(\lambda X_1, \lambda X_2, \lambda X_3) = \lambda Y(X_1, X_2, X_3)$ for all $\lambda > 0$ and all possible factor combinations. Differentiating this equation with respect to $\lambda$ according to the chain rule and then putting $\lambda = 1$ one obtains the Euler relation $X_1(\partial Y/\partial X_1) + X_2(\partial Y/\partial X_2) + X_3(\partial Y/\partial X_3) = Y$. Dividing this by $Y$ yields $(X_1/Y)(\partial Y/\partial X_1) + (X_2/Y)(\partial Y/\partial X_2) + (X_3/Y)(\partial Y/\partial X_3) = 1$. With Eq. (4) this becomes the so-called "constant returns to scale" relation:

$$\sum_{i=1}^{3} \epsilon_i = 1 \quad . \tag{6}$$

This relation is fundamental for the cost share theorem. Often-used production functions of standard economics like Cobb-Douglas and constant-elasticity-of-substitution (CES) production functions are linearly homogeneous. Occasionally one also considers increasing or decreasing returns to scale, which would correspond to homogeneous production functions with the property $Y(\lambda X_1, \lambda X_2, \lambda X_3) = \lambda^{\nu} Y(X_1, X_2, X_3)$, with $\nu > 1(< 1)$ for increasing (decreasing) returns to scale. This, however, implies that changes of inputs by $\lambda$ are associated with alterations of the state of technology. The state of technology changes, for instance, when the thermodynamic effectiveness [9] of the production process changes. Such time-related changes are excluded by the condition "at fixed time $t$" in the derivation of Eq. (6). Linearly homogeneous production functions can take care of alterations of the state of technology by their explicit time dependence. This may manifest itself in time-changing technology parameters. An example is given in Section 4.

### 3.2. Optimization subject to constraints

The standard assumptions for computing macroeconomic equilibrium are that the actions of all economic actors result in the maximization of either profit or overall welfare. Profit is output $Y$ minus factor cost, and overall welfare is the time integral of a utility function. One may question these assumptions, e.g. because of game-theoretical findings and the experiences of the 2008 financial market crash. If one rejects them, one also rejects the cost share theorem, and there is no reason to believe in the tiny output elasticity of energy. If one accepts them, we will show that the cost share theorem is killed by hitherto neglected technological constraints. In either case alternative methods of computing output elasticities are required. The results of one method are presented in Section 4.

### 3.2.1. Profit maximization

We assume that the three production factors $(X_1, X_2, X_3) \equiv \vec{X}$ have the exogeneously given prices per factor unit $(p_1, p_2, p_3) \equiv \vec{p}$, so that total factor cost is $\vec{p}(t) \cdot \vec{X}(t) = \sum_{i=1}^{3} p_i(t) X_i(t)$. The factors can vary independently within technological constraints until profit $Y - \vec{p} \cdot \vec{X}$ becomes maximum. The technological constraints are labeled by $a$. They can be brought into the form of equations,

$$f_a(\vec{X}, t) = 0, \tag{7}$$

with the help of slack variables. Slack variables change inequalities into equalities. They define the range in factor space within which the factors can vary independently at time $t$. They are given explicitly in the Appendix for the factors capital, labor and energy. There are two technological constraints. Thus, $a$ is either, say, $A$ or $B$.

The necessary condition for a maximum of profit $G \equiv Y - \vec{p} \cdot \vec{X}$, subject to the technological constraints (7), is:

$$\vec{\nabla} \left[ Y(\vec{X}; t) - \sum_{i=1}^{3} p_i(t) X_i(t) + \sum_{a} \mu_a f_a(\vec{X}, t) \right] = 0 \quad , \tag{8}$$

where $\vec{\nabla} \equiv (\partial/\partial X_1, \partial/\partial X_2, \partial/\partial X_3)$ is the gradient in factor space, and the $\mu_a$ are Lagrange multipliers. (The sufficient condition for profit maximum involves a sum of second-order derivatives of $Y(\vec{X}; t) - \vec{p} \cdot \vec{X} + \sum_a \mu_a f_a(\vec{X}, t)$. It is assumed that the extremum of profit at finite $X_i$ *is* the maximum.) This yields the three equilibrium conditions

$$\frac{\partial Y}{\partial X_i} - p_i + \sum_{a} \mu_a \frac{\partial f_a}{\partial X_i} = 0, \quad i = 1, 2, 3 \quad . \tag{9}$$

Multiplication of Eq. (9) with $\frac{X_i}{Y}$, and observing Eq. (4) brings the equilibrium conditions into the form

$$\epsilon_i \equiv \frac{X_i}{Y} \frac{\partial Y}{\partial X_i} = \frac{X_i}{Y} \left[ p_i - \sum_{a} \mu_a \frac{\partial f_a}{\partial X_i} \right], \quad i = 1, 2, 3 \quad . \tag{10}$$

Combining Eqs. (6) and (10) yields

$$Y = \sum_{i=1}^{3} X_i \left[ p_i - \sum_{a} \mu_a \frac{\partial f_a}{\partial X_i} \right] \quad . \tag{11}$$

Inserting this $Y$ into Eq. (10) results in the equilibrium conditions

$$\epsilon_i = \frac{X_i \left[ p_i - \sum_a \mu_a \frac{\partial f_a}{\partial X_i} \right]}{\sum_{i=1}^{3} X_i \left[ p_i - \sum_a \mu_a \frac{\partial f_a}{\partial X_i} \right]} \equiv \frac{X_i \left[ p_i + s_i \right]}{\sum_{i=1}^{3} X_i \left[ p_i + s_i \right]} \quad . \tag{12}$$

Here $s_i$, defined as

$$s_i \equiv -\mu_A \frac{\partial f_A}{\partial X_i} - \mu_B \frac{\partial f_B}{\partial X_i}, \tag{13}$$

is the shadow price of the production factor $X_i$.

In the absence of technological constraints the Lagrange multipliers $\mu_a$ (with $a = A, B$) and the shadow prices $s_i$ would be zero, and the equilibrium conditions (12) would turn into the cost share theorem: the numerator would be the cost $p_i X_i$ of the factor $X_i$, the denominator would be the sum of all factor costs, and the quotient would represent the cost share of $X_i$ in total factor cost. This would also justify the neoclassical duality of production factors and factor prices, which is often used in orthodox growth analyses. This duality is a consequence of the Legendre transformation that results from the requirement that profit $G(\vec{X}, \vec{p}) = Y(\vec{X}) - \vec{p} \cdot \vec{X}$ is maximum *without any constraints on* $X_1, X_2, X_3$. Then Eq. (9) would hold with $\mu_a = 0$ and yield equilibrium values $X_{1M}(\vec{p}), X_{2M}(\vec{p}), X_{3M}(\vec{p})$. With $\vec{X}_M(\vec{p})$ the profit function turns into the price function

$$G(\vec{X}_M(\vec{p}), \vec{p}) = Y(\vec{X}_M(\vec{p})) - \vec{p} \cdot \vec{X}_M(\vec{p}) \equiv g(\vec{p}). \tag{14}$$

The price function $g(\vec{p})$ is the Legendre transform of the production function $Y(\vec{X})$. This is in formal analogy to the Hamilton function being the Legendre transform of the Lagrange function in classical mechanics, or to enthalpy and free energy being Legendre transforms of internal energy in thermodynamics. However, in the presence of technological constraints and the resulting shadow prices the cost share theorem and Eq. (14) are not valid.

In order to indicate a framework for the calculation of Lagrange multipliers, shadow prices and the equilibrium factor vector $\vec{X}_{eq}$ in the presence of technological constraints we define

$$f_{Ai} \equiv \frac{\partial f_A}{\partial X_i}, \quad f_{Bi} \equiv \frac{\partial f_B}{\partial X_i}, \quad i = 1, 2, 3 \quad , \tag{15}$$

and write the equilibrium conditions, Eq. (9), as

$$\frac{\partial Y}{\partial X_i} - p_i + \mu_A f_{Ai} + \mu_B f_{Bi} = 0, \quad i = 1, 2, 3 \quad . \tag{16}$$

Simple algebra eliminates $\mu_B$ from these conditions for $i = 1$ and $i = 2$. One obtains

$$\mu_A = \frac{f_{B1}(p_2 - \frac{\partial Y}{\partial X_2}) - f_{B2}(p_1 - \frac{\partial Y}{\partial X_1})}{f_{A2}f_{B1} - f_{A1}f_{B2}} \quad . \tag{17}$$

Inserting this $\mu_A$ into Eq. (16) for $i = 1$ yields

$$\mu_B = \frac{p_1 - \frac{\partial Y}{\partial X_1}}{f_{B1}} - \frac{f_{A1}[f_{B1}(p_2 - \frac{\partial Y}{\partial X_2}) - f_{B2}(p_1 - \frac{\partial Y}{\partial X_1})]}{f_{B1}(f_{A2}f_{B1} - f_{A1}f_{B2})}. \tag{18}$$

Thus, all quantities entering the shadow prices (13) in the equilibrium conditions (12) are known in principle, if one knows the production function $Y(\vec{X}; t)$, the prices per factor unit $p_i$, and the constraint equations $f_A(\vec{X}, t) = 0$, $f_B(\vec{X}, t) = 0$.

Inserting $\mu_A$ from Eq. (17) and $\mu_B$ from Eq. (18) into Eq. (16) for $i = 3$ yields one equation for the equilibrium vector $\vec{X}_{eq}$. Furthermore there are the two constraint equations. It remains to be seen, whether the factor magnitudes that result from these three equations lead in a straightforward manner to the absolute profit maximum for an appropriate set of slack variables, or whether other methods of constrained nonlinear optimization, e.g. the Levenberg-Marquardt method [41], employed in minimizing the sum of squared errors (56) subject to the constraints (57), are better suited for computing $\vec{X}_{eq}$. For our purpose of elucidating the destruction of the cost share theorem by technological constraints Eqs. (12) - (18) are sufficient.

### 3.2.2. Overall welfare maximization

The following derivation of economic equilibrium by overall welfare maximization tests the sensitivity of the equilibrium conditions to modified behavioral assumptions, and illustrates how extremum principles of classical mechanics work in economic optimization.

Like Samuelson and Solow [42] we assume that "... society maximizes the (undiscounted)[5] integral of all future utilities of consumption subject to the fact that the sum of current consumption and of current capital formation is limited by what the current capital stock

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[5]Discounting the future is controversial [1, 2], [30].

can produce." The formalism of Ref. [42] is used with the following modifications: 1. There is not one variable production factor but three: $X_1$, $X_2, X_3$. 2. There are contraints on magnitudes and combinations of these factors. 3. As in Hellwig et al. [43], optimization is done within finite time horizons.

Thus, we integrate utility $U$ of consumption $C$ between the times $t_0$ and $t_1$ and maximize this (undiscounted)[6] integral, which is overall welfare

$$W[s] = \int_{t_0}^{t_1} U[C]dt, \tag{19}$$

subject to the technological constraints of Eq. (7). In addition, there is an economic constraint: the total cost $\vec{p} \cdot \vec{X}$ of producing consumption $C$, Eq. (21), by means of the factors $(X_1, X_2, X_3) \equiv \vec{X}$ must not diverge but has finite magnitudes $c_f(t)$ at all times $t$, where each price per factor unit, $p_i$, is exogeneously given:

$$c_f(t) - \sum_{i=1}^{3} p_i(t)X_i(t) = 0 \quad . \tag{20}$$

The following variational formalism of welfare optimization is the same as that of deriving the Lagrange equations of motion from Hamilton's principle of least action in classical mechanics.

$W[s]$ is a functional of the curve $[s]$ along which the production factors evolve; $[s] = \{t, \vec{X} : \vec{X} = \vec{X}(t), \quad t_0 \le t \le t_1\}$. It depends on the variables that enter consumption $C$. In general, utility may depend on many variables. In the present case the utility function $U[C]$ depends on output minus capital formation. Output (per unit time) is described by the macroeconomic production function $Y(\vec{X};t)$. Part of $Y$ goes into consumption $C$ and the rest into new capital formation $\dot{X}_1 \equiv \frac{dX_1}{dt}$ plus replacement of depreciated capital. As usual we approximate the annual replacement rate by $\delta^d X_1$, where $\delta^d$ is the depreciation rate. $Y(\vec{X};t)$ and $\dot{X}_1 + \delta^d X_1$ are annual output and annual capital formation, respectively. Then annual consumption is $C = Y(\vec{X};t) - \dot{X}_1 - \delta^d X_1$. Economic research institutions provide the price of capital utilization $p_1$ as the sum of net interest, depreciation and state influences. Furtheron we use this price. Since it already includes depreciation, we can omit explicit reference to the depreciation rate, [7] so that consumption is given by

$$C = Y(\vec{X};t) - \dot{X}_1 \quad . \tag{21}$$

Including the constraint equations (7) and (20) in the maximization of welfare (19) with the help of the Lagrange multipliers $\mu_a$ and $\mu$ we have the optimization problem: Maximize

$$W[s] = \int_{t_0}^{t_1} dt \left\{ U[C(\vec{X}, \dot{X}_1)] + \mu(c_f(t) - \vec{p} \cdot \vec{X}) + \sum_a \mu_a f_a(\vec{X}, t) \right\} \quad . \tag{22}$$

Varying $\vec{X} \to \vec{X} + \vec{h}$ and $\dot{X}_1 \to \dot{X}_1 + \dot{h}_1$ in the integrand of Eq. (22), where $h$ is small, and zero in $t_0$ and $t_1$, one obtains the following conditions for $W[s, \vec{h}] - W[s] = 0$, so that $W$ is extremal:

$$\frac{dU}{dC}\frac{\partial C}{\partial X_1} - \frac{d}{dt}\left(\frac{dU}{dC}\frac{\partial C}{\partial \dot{X}_1}\right) - \mu p_1 + \sum_a \mu_a \frac{\partial f_a}{\partial X_1} = 0 \tag{23}$$

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[6]If one multiplied $U[C]$ in the integrand of Eq. (19) by $\exp(-\delta t)$, $\delta$ being the pure time discount rate, one would have to subtract $\delta \frac{dU}{dC}$ from $\frac{d}{dt}(\frac{dU}{dC})$ in Eqs. (27) and (28).

[7]If we kept $\delta^d X_1$, we would get a term proportional to $\delta^d X_1$ added to $p_1$ everywhere.

and

$$\frac{dU}{dC}\frac{\partial C}{\partial X_i} - \mu p_i + \sum_a \mu_a \frac{\partial f_a}{\partial X_i} = 0, \quad i = 2, 3. \tag{24}$$

With $C$ from Eq. (21) these three equilibrium conditions turn into

$$\frac{dU}{dC}\frac{\partial Y}{\partial X_i} - \mu p_i + \sum_a \mu_a \frac{\partial f_a}{\partial X_i} = -\frac{d}{dt}\left(\frac{dU}{dC}\right)\delta_{i,1}, \quad i = 1, 2, 3 \quad ; \tag{25}$$

the Kronecker symbol $\delta_{i,1}$ is 1 for $i = 1$ and 0 otherwise.

Dividing Eqs. (25) by $dU/dC$ and multiplying them by $X_i/Y$ changes them to

$$\frac{X_i}{Y}\frac{\partial Y}{\partial X_i} = \frac{\mu X_i}{Y\frac{dU}{dC}}\left[p_i - \sum_a \frac{\mu_a}{\mu}\frac{\partial f_a}{\partial X_i} - \delta_{i,1}\frac{1}{\mu}\frac{d}{dt}(\frac{dU}{dC})\right], \quad i = 1, 2, 3. \tag{26}$$

The left-hand side of this equation is the output elasticity $\epsilon_i$. Inserting the right-hand side into Eq. (6) yields

$$\mu = \frac{Y\frac{dU}{dC}}{\sum_{i=1}^{3} X_i \left[p_i - \sum_a \frac{\mu_a}{\mu}\frac{\partial f_a}{\partial X_i} - \delta_{i,1}\frac{1}{\mu}\frac{d}{dt}(\frac{dU}{dC})\right]} \quad . \tag{27}$$

With that the equilibrium conditions (26) become

$$\epsilon_i = \frac{X_i \left[p_i - \sum_a \frac{\mu_a}{\mu}\frac{\partial f_a}{\partial X_i} - \delta_{i,1}\frac{1}{\mu}\frac{d}{dt}(\frac{dU}{dC})\right]}{\sum_{i=1}^{3} X_i \left[p_i - \sum_a \frac{\mu_a}{\mu}\frac{\partial f_a}{\partial X_i} - \delta_{i,1}\frac{1}{\mu}\frac{d}{dt}(\frac{dU}{dC})\right]}, \quad i = 1, 2, 3. \tag{28}$$

This can be written in the form of the last term on the r.h.s. of Eq. (12), where the shadow prices are now

$$s_i \equiv -\sum_{a=A}^{B} \frac{\mu_a}{\mu}\frac{\partial f_a}{\partial X_i} - \delta_{i,1}\frac{1}{\mu}\frac{d}{dt}(\frac{dU}{dC}), \quad i = 1, 2, 3. \tag{29}$$

This set of equations differs from the equilibrium conditions (12) and the shadow prices (13) in two aspects. First, there is the term $\delta_{i,1}\frac{1}{\mu}\frac{d}{dt}(\frac{dU}{dC})$. It originates from Eq. (21) and partial integration of the term with $\dot{h}_1$ in the variation of $W$. It is due to taking capital formation into account in intertemporal utility optimization, whereas capital formation is no issue in profit optimization. Second, ratios $\mu_a/\mu$ of Lagrange multipliers take the positions of the Lagrange multipliers $\mu_a$ in Eqs. (12) and (13). If one does profit maximization subject to the additional constraint $\vec{p} \cdot \vec{X} = c_f$, which fixes factor cost, one gets $\mu_a/\mu$ instead of $\mu_a$ in the equations that then replace Eqs. (12) and (13). Thus, the second difference is rather a formal one.

The first difference vanishes, if one can disregard decreasing marginal utility and approximate the utility function $U(C)$ by a linear function in $C$. For instance, if the function of decreasing marginal utility [1] is $U(C) = C_0 \ln \frac{C}{C_0} + U_0$, and if it can be approximated by its Taylor expansion up to first order in $\frac{C}{C_0} - 1$, one has [8]

$$U(C) \approx C - C_0 + U_0. \tag{30}$$

Then $\frac{d}{dt}\left(\frac{dU}{dC}\right) = 0$.

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[8] A linear approximation of $\ln x$ is acceptable for $x < 4$.

In order to compute the ratios of Lagrange multipliers in case that Eq. (30) holds, we define

$$\mu_1 \equiv \frac{\mu_A}{\mu}, \quad \mu_2 \equiv \frac{\mu_B}{\mu}, \tag{31}$$

and abbreviate the partial derivatives of the two constraint equations as in Eq. (15). With that the equilibrum conditions (28) become

$$\epsilon_i = \frac{X_i \left[ p_i - \mu_1 f_{Ai} - \mu_2 f_{Bi} \right]}{\sum_{i=1}^3 X_i \left[ p_i - \mu_1 f_{Ai} - \mu_2 f_{Bi} \right]}, \quad i = 1, 2, 3. \tag{32}$$

If one resolves the two independent ratios

$$\frac{\epsilon_1}{\epsilon_2} = \frac{X_1 \left[ p_1 - \mu_1 f_{A1} - \mu_2 f_{B1} \right]}{X_2 \left[ p_2 - \mu_1 f_{A2} - \mu_2 f_{B2} \right]} \tag{33}$$

and

$$\frac{\epsilon_1}{\epsilon_3} = \frac{X_1 \left[ p_1 - \mu_1 f_{A1} - \mu_2 f_{B1} \right]}{X_3 \left[ p_3 - \mu_1 f_{A3} - \mu_2 f_{B3} \right]}, \tag{34}$$

with respect to $\mu_1$ and $\mu_2$, using the definitions

$$R_{21} \equiv \frac{X_2 \epsilon_1}{X_1 \epsilon_2}, \quad R_{31} \equiv \frac{X_3 \epsilon_1}{X_1 \epsilon_3} \quad, \tag{35}$$

and performs some algebraic manipulations, one obtains

$$\mu_1 = \frac{(p_1 - p_2 R_{21})}{f_{A1} - f_{A2} R_{21}} + \frac{f_{B2} R_{21} - f_{B1}}{f_{A1} - f_{A2} R_{21}} \cdot \mu_2 \quad, \tag{36}$$

and

$$\mu_2 = \frac{(p_1 - p_3 R_{31})(f_{A1} - f_{A2} R_{21}) - (p_1 - p_2 R_{21})(f_{A1} - f_{A3} R_{31})}{(f_{B2} R_{21} - f_{B1})(f_{A1} - f_{A3} R_{31}) - (f_{B3} R_{31} - f_{B1})(f_{A1} - f_{A2} R_{21})} \quad. \tag{37}$$

In summary, equilibrium conditions derived from profit or overall welfare optimization do no longer support the cost share theorem, if technological constraints on factor combinations are taken into account.

### 3.3. Technological constraints on capital, labor and energy

Technological constraints are part of production models. In a world, where the capital stock consists of simple tools and sheds to house them, output results from the combination of human and animal muscle power with these tools. In such a world the disregard of technological constraints is not unreasonable. On the other hand, and in contrast to the preindustrial situation, the physical basis of an industrial production system *is* subject to binding technological constraints.

In order to identify these constraints we recall that the capital stock $K(t)$ at time $t$ consists of all energy-converting and information-processing machines together with all buildings and installations necessary for their protection and operation. Output $Y$ results from work performance and information processing by the combination of such capital with (routine) labor $L(t)$ and energy $E(t)$.[9]

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[9]Some models like [31] and [34] take materials into account as a fourth factor of production. Since materials are passive partners of the production process, which do not contribute actively to output – their atoms and molecules are merely arranged in orderly patterns by capital, labor and energy when value added is created – we do not include them in the model. Like other models the present model also disregards land as a factor of production for the reasons indicated in Section 2.

By definition, routine labor can be substituted by some combination of capital and energy. In addition there is the specific human contribution to production and growth that cannot be provided by any machine, even a sophisticated computer capable of learning from experience. We call it *creativity*. It includes ideas, inventions, valuations, and (especially) interactive decisions depending on human reactions and characteristics. It is important to recognize that the non-routine component of human labor may decline over time, but it is never zero. The ultimate lower limit of routine labor inputs is probably un-knowable, because it depends to some degree on the limits of artificial intelligence. But we need not concern ourselves with the ultimate limit. At any given time, with a given technology and state of automation, there is a limit to the extent that routine labor can increase output. In other words, the model postulates the possibility of a combination of capital and exergy such that adding one more unskilled worker adds nothing to gross economic output. (In some manufacturing sectors of industrialized countries this point does not seem to be far away.)

There is another fairly obvious technological constraint on the combinations of factors. In brief, machines are designed and built for specific exergy inputs. In some cases (e.g. for some electric motors) there is a modest overload capability. Buildings can be over-heated or over-cooled, to be sure, but this does not contribute to productivity. On average the maximum exergy input is fixed by design. Thus, the ratio of exergy to capital must not exceed a definite upper limit.

The bottom line of the above considerations is that the use of capital, labor and energy in industrial systems is subject to technological constraints that are the consequence of limits to capacity utilization and to the substitution of capital and energy for labor. This substitution changes what we call "degree of automation".

For quantitative analyses, output and inputs must be specified by measurement prescriptions. Output $Y$ and capital stock $K$ are measured in deflated monetary units, as reported by the national accounts. The time series of these monetary units can be related to time series of physical units that aggregate output and capital in terms of work performance and information processing [18]. Routine labor $L$ is measured in man hours worked per year, as given by the national labor statistics, and energy $E$ is measured in petajoules (or tons of oil equivalents, or quads) per year, as shown by the national energy balances.

Of course, the theory must be independent from the choice of units. Therefore, in our three-factor model, with $X_1 \equiv K, X_2 \equiv L$, and $X_3 \equiv E$, it is convenient to introduce *new, dimensionless variables, for which we use lower case letters,* by writing inputs and output as multiples of their quantities $K_0, L_0, E_0$, and $Y_0$ in a base year $t_0$. The transformation to the dimensionless time series of capital, $k(t)$, labor, $l(t)$, and energy, $e(t)$, is given by

$$k(t) \equiv K(t)/K_0, \quad l(t) \equiv L(t)/L_0, \quad e(t) \equiv E(t)/E_0 \quad , \tag{38}$$

and the dimensionless production function is

$$y[k, l, e; t] \equiv Y(kK_0, lL_0, eE_0; t)/Y_0 \quad ; \tag{39}$$

for the sake of notational simplicity we do not always indicate the time dependence of $k, l, e$ explicitly. From here on we work in the "space" of the dimensionless inputs and outputs, defined by Eqs. (38) and (39).

The *degree of automation* $\rho$ of a production system is proportional to the actual capital stock $k$ of the system divided by the capital stock $k_m(y)$ that would be required for *m*aximally automated production of actual output $y$; in the state of maximally automated

production the output elasticity of routine labor would be vanishingly small by definition. The proportionality factor is the *degree of capacity utilization of the capital stock*, $\eta$. Thus, the degree of automation is given by [16]

$$\rho = \eta \frac{k}{k_m(y)} \quad . \tag{40}$$

Entrepreneurial decisions, aiming at producing a certain quantity of output $y$ within existing technology, determine the absolute magnitude of the total capital stock $k$, its degree of capacity utilization $\eta$, and its degree of automation $\rho$. Obviously, $\rho$ and $\eta$ are functions of capital $k$, labor $l$, and energy $e$. They are definitely constrained by $\rho(k, l, e) \leq 1$ and $\eta(k, l, e) \leq 1$, i.e. the maximum degree of automation (at a given time) cannot be exceeded, and a production system cannot operate above design capacity.[10]

However, there is a technical limit to the degree of automation at time $t$ that lies below 1. We call it $\rho_T(t)$. It depends on mass, volume and exergy requirements of the machines, especially information processors, in the capital stock. Imagine the vacuum-tube computers of the 1960s, when the tiny transistor, invented in 1947 by Bardeen, Brattain and Shockley, had not yet diffused into the capital stock. A vacuum-tube computer with the computing power of a 2009 notebook computer would have had a volume of many thousands of cubic meters. In 1960 a degree of automation, that is standard 40 years later in the highly industrialized countries, would have resulted in factories many orders of magnitude bigger than today, probably exceeding the available land area. In the course of time, the technical limit to automation, $\rho_T(t)$, moves towards the theoretical limit 1. This is facilitated by the density increase of information processors (transitors) on a microchip. According to "Moore's Law" transistor density has doubled every 18 months during the last four decades. It may continue like that for a while, thanks to nano-technological progress. But there is a thermodynamic limit to transistor density, because the electricity required for information processing eventually ends up in heat. If this heat can no longer escape sufficiently rapidly out of the microchip because of too densely packed transistors, it will melt down the conducting elements and destroy the chip. We do not know exactly, how far the technical limit to automation can be pushed. For our purposes, however, it is sufficient to know that at any time $t$ such a limit $\rho_T(t)$ exists.

Since the technical properties of the capital stock do not change with $\eta$, the constraint on automation applies to the situation of maximum capacity utilization. With $\eta = 1$ in Eq. (40) the (inequality) formulation of an upper limit to automation is: $k/k_m(y) \leq \rho_T(t)$. It is brought into the form of a constraint *equation*, as required by the method of the Lagrange multipliers, with the help of the slack variable $k_\rho$ :

$$f_A(K, L, E, t) \equiv \frac{k + k_\rho}{k_m(y)} - \rho_T(t) = 0 \quad ; \tag{41}$$

$k_\rho$ is the capital that has to be added to $k$ so that the total capital stock $k + k_\rho$, working at full capacity, exhausts the technologically possible automation potential $\rho_T(t)$.

Similarly, the formulation of an upper limit to capacity utilization, $\eta(k, l, e) \leq 1$, is brought into the required form of a constraint equation with the help of the slack variables $e_\eta(t)$ and

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[10]Strictly speaking, the limit 1 for $\eta$ is a sharp technological limit only, when "working at full capacity" means working 24 hours per day and 365 days per year. There are branches of business, where machines have to run less time per day and year in order to be considered as working at full capacity. To keep things simple we disregard these "soft" limits to capacity utilization.

$l_\eta(t)$ :

$$f_B(K, L, E, t) \equiv \eta(k, l + l_\eta, e + e_\eta) - 1 = 0 \quad ; \tag{42}$$

$l + l_\eta$ and $e + e_\eta$ are the quantities of labor and energy required for full capacity utilization of the capital stock $k$ at time $t$. In a rough approximation one may assume that $e_\eta(t)$ and $l_\eta(t)$ are related by $e_\eta(t) = d(t) \cdot l_\eta(t)$. Here, the technological state of the capital stock determines the labor-energy-coupling parameter $d(t)$. The explicit relations between the factors $k, l, e$ and the slack variables $k_\rho$, $l_\eta$ and $e_\eta$ are given by Eqs. (63), (64) and (71) of the Appendix.

We need an explicit functional form for the degree of capacity utilization $\eta$. Since $\eta$ does not change, if $k, l$ and $e$ all change by the same factor, it is a homogeneous function of degree zero: $\eta = \eta(l/k, e/k)$. A trial form can be derived from a Taylor expansion of $\ln \eta[\ln(l/k), \ln(e/k)]$ around some point '0'$\equiv (\ln(l/k)_0, \ln(e/k)_0)$, up to first order in $\ln(l/k) - \ln(l/k)_0$ and $\ln(e/k) - \ln(e/k)_0$. This approximation yields

$$\eta = \eta_0 \left( \frac{l}{k} \right)^\lambda \left( \frac{e}{k} \right)^\nu , \tag{43}$$

where $\lambda$ and $\nu$ are the derivatives of $\ln \eta$ with respect to $\ln l/k$ and $\ln e/k$ in the point '0'. The parameters $\eta_0, \lambda$ and $\nu$ can be determined from empirical data on capacity utilization. Then, combining Eq. (42) with Eq. (43), one has the complete equation describing constrained capacity utilization. This equation and Eq. (41) determine the shadow prices (13) and (29). In Eqs. (13) and (29) one has to identify $X_1 = K_0 k, X_2 = L_0 l, X_3 = E_0 e$ and replace subscripts 1,2,3 by $K, L, E$.

For instance, in the case of profit maximization the shadow price of capital becomes

$$s_K = -\frac{1}{K_0} \left[ \mu_A \frac{\partial f_A}{\partial k} + \mu_B \frac{\partial f_B}{\partial k} \right] . \tag{44}$$

In the shadow prices of labor, $s_L$, and energy, $s_E$, the derivatives are with respect to $l$ and $e$, and $K_0$ is replaced by $L_0$ and $E_0$. The Lagrange multipliers $\mu_A$ and $\mu_B$ are given by Eqs. (17) and (18).

## 4. Computing Output Elasticities

The destruction of the cost share theorem by technological constraints obliterates the standard method of determining output elasticities simply from factor prices and factor quantities. More information is needed for computing the productive powers of capital, labor and energy. In order to obtain this information we develop a model that tries to map the essential production processes that run in the physical basis of the economy into the production function $y[k, l, e; t]$, introduced in Eq. (39). In terms of the dimensionless variables the growth equation (3) becomes

$$\frac{\mathrm{d}y}{y} = \alpha \frac{\mathrm{d}k}{k} + \beta \frac{\mathrm{d}l}{l} + \gamma \frac{\mathrm{d}e}{e} + \delta \frac{\mathrm{d}t}{t - t_0} \quad . \tag{45}$$

The output elasticities $\epsilon_i$, defined in Eq. (4 ), are specified for capital $k$, labor $l$ and energy $e$ by $\alpha, \beta$ and $\gamma$. In terms of the dimensionless variables they and $\delta$ are given by

$$\alpha \equiv \frac{k}{y} \frac{\partial y}{\partial k}, \quad \beta \equiv \frac{l}{y} \frac{\partial y}{\partial l}, \quad \gamma \equiv \frac{e}{y} \frac{\partial y}{\partial e}, \quad \delta \equiv \frac{t - t_0}{y} \frac{\partial y}{\partial t}. \tag{46}$$

They are known, once $y[k, l, e; t]$ is known. In order to obtain both output elasticities and production functions we first calculate functional forms of the output elasticities that are consistent with the standard mathematical requirement on production functions, namely that $y[k, l, e; t]$ must be twice-differentiable with respect to the variables $k, l, e$. This means that its second-order mixed derivatives must be equal. The requirement of twice differentiablity, when applied to thermodynamic potentials like internal energy, free energy etc., leads to the Maxwell relations in thermodynamics. When applied to the production function $y[k, l, e; t]$ it leads to the partial differential equations

$$l\frac{\partial\alpha}{\partial l} = k\frac{\partial\beta}{\partial k}, \quad e\frac{\partial\beta}{\partial e} = l\frac{\partial\gamma}{\partial l}, \quad k\frac{\partial\gamma}{\partial k} = e\frac{\partial\alpha}{\partial e}. \tag{47}$$

If the production function is linearly homogeneous, so that according to Eq. (6) one can write $\gamma = 1 - \alpha - \beta$, these equations become [16, 18, 20]

$$l\frac{\partial\alpha}{\partial l} = k\frac{\partial\beta}{\partial k}, \quad k\frac{\partial\alpha}{\partial k} + l\frac{\partial\alpha}{\partial l} + e\frac{\partial\alpha}{\partial e} = 0, \quad k\frac{\partial\beta}{\partial k} + l\frac{\partial\beta}{\partial l} + e\frac{\partial\beta}{\partial e} = 0. \tag{48}$$

The most general solutions are

$$\alpha = A\left(\frac{l}{k}, \frac{e}{k}\right), \qquad \beta = \int \frac{l}{k}\frac{\partial A}{\partial l}\mathrm{d}k + J\left(\frac{l}{e}\right), \tag{49}$$

where $A$ and $J$ are any differentiable functions of their arguments.

The trivial solutions of Eqs. (48) are constants: $\alpha = \alpha_0$, $\beta = \beta_0$. If one inserts them into Eq. (45) at fixed $t$, observes $\gamma_0 = 1 - \alpha_0 - \beta_0$, and integrates $y$ from $y_0$ to $y_{CDE}$ and the factors from $(1, 1, 1)$ to $(k, l, e)$, one obtains the energy-dependent Cobb-Douglas function

$$y_{CDE} = y_0 k^{\alpha_0} l^{\beta_0} e^{1-\alpha_0-\beta_0}. \tag{50}$$

This function has been often used in quantitative analyses, where mainstream economics identifies $\alpha_0$, $\beta_0$, and $1 - \alpha_0 - \beta_0$ with the cost shares of capital, labor, and energy; these shares have happened to be approximately constant until recently.

The simplest non-trivial solutions of the partial diffferential equations (48) are [18]

$$\alpha = a\frac{l+e}{k}, \quad \beta = al\left(\frac{c}{e} - \frac{1}{k}\right), \quad \gamma = 1 - \alpha - \beta. \tag{51}$$

They satisfy the asymptotic boundary conditions $\alpha \to 0$, if $(l+e)/k \to 0$, and $\beta \to 0$, if $k \to k_m(y)$ and $e \to ck_m(y)$. Here, $e_m \equiv ck_m(y)$ is the energy input into the maximally automated capital stock $k_m(y)$ working at full capacity. The asymptotic boundary condition for $\alpha$ incorporates the law of diminshing returns: machines don't run without energy and (still) require people for handling them; thus, if the ratio of labor and energy to capital decreases, the output of an additional unit of capital decreases, too. The asymptotic boundary condition for $\beta$ describes the effect of energy and capital substituting for labor.

If one inserts the output elasticities (51) into Eq. (45) at fixed $t$ and integrates $y$ from $y_0$ to $y_{L1}$ and the factors from $(1, 1, 1)$ to $(k, l, e)$, one obtains the (first) Linex production function

$$y_{L1} = y_0 e \exp\left[a\left(2 - \frac{l+e}{k}\right) + ac\left(\frac{l}{e} - 1\right)\right], \tag{52}$$

which depends *lin*early on energy and *ex*ponentially on factor quotients [18].

The Linex function contains the technology parameters $a$ (capital effectiveness), $c$ (energy demand of the fully utilized capital stock), and $y_0$. Some or all of these parameters become time dependent, when creativity alters efficiencies and production structures. Then the Linex function acquires an explicit time dependence: $y_{L1} = y_{L1}[k, l, e; t]$. For instance, $c(t)$ decreases when investments in energy conservation reduce the energy demand of the capital stock. (This occurred quite noticeably in response to the oil price shocks and is an example for the – thermodynamically limited – substitution of capital for energy.)

Various methods have been used in order to determine the parameters $a, c$ and $y_0$ [11, 20, 27]. Already the simplest case of fitting the Linex function with three constant parameters to the empirical time series of output reproduces the general trend of economic growth, and one obtains output elasticities for capital, labor and energy that are of the same order of magnitude as the ones in Table 2. However, the Durbin-Watson coefficients $d_W$ of autocorrelation have been mostly below 1. The best $d_W$ value, indicating the absence of autocorrelations, is 2. The closer one comes to $d_W = 2$, the more confident one can be that one has taken all relevant factors into account.

In order to see, whether a reduction of autocorrelations has a significant impact on the output elasticities of capital, labor and energy, we allow for time dependencies of the technology parameters and model them by logistic functions, which are typical for growth in complex systems and innovation diffusion. Let $p(t)$ represent either the capital effectiveness parameter $a(t)$ or the energy demand parameter $c(t)$. Its logistic differential equation

$$\frac{d}{dt}\left(p(t) - p_2\right) = p_3 \left(p(t) - p_2\right)\left(1 - \frac{p(t) - p_2}{p_1 - p_2}\right) \tag{53}$$

has the solution [44]

$$p(t) = \frac{p_1 - p_2}{1 + \exp\left[-p_3\left(t - t_0 - p_4\right)\right]} + p_2, \tag{54}$$

with the free (characteristic) coefficients $p_1, \ldots, p_4 \geq 0$; the variable $t$ is dimensionless and given by "time interval divided by one year". As an alternative to logistics we have also looked into Taylor expansions of $a(t)$ and $c(t)$ in terms of $t - t_0$ with a minimum of free coefficients. With that the output elasticity of creativity, $\delta$, defined in Eq. (46), is given by

$$\delta = \frac{(t - t_0)}{y_{L1}}\left[\frac{\partial y_{L1}}{\partial a}\frac{da}{dt} + \frac{\partial y_{L1}}{\partial c}\frac{dc}{dt} + \frac{\partial y_{L1}}{\partial y_0}\frac{dy_0}{dt}\right]. \tag{55}$$

The free coefficients of the logistic functions, or of the Taylor expansions, are determined by minimizing the sum of squared errors (SSE),

$$\sum_i \left[y_{empirical}(t_i) - y_{L1}(t_i)\right]^2 \quad . \tag{56}$$

The sum goes over all years $t_i$ between the initial and the final observation time. It contains the empiricial time series of output $y_{empirical}(t_i)$, and the Linex function $y_{Lt}(t_i)$ with the empirical time series of $k$, $l$ and $e$ as inputs at times $t_i$. Minimization is subject to the constraints that the output elasticities of Eq. (51) must be non-negative, because otherwise the increase of an input would decrease output – a situation the economic actors will avoid. These constraints turn into the constraints on $a(t)$ and $c(t)$, or on $k, l, e$ for given $a$ and $c$ :

$$0 \leq a(t) \leq a_{max}(t) \equiv \frac{k(t)}{l(t) + e(t)},$$

$$\frac{e(t)}{k(t)} \equiv c_{min}(t) \le c(t), \quad 0 \le a(t) \left[\frac{e}{k} + c(t)\frac{l}{e}\right] \le 1. \tag{57}$$

SSE minimization has been done with the Levenberg-Marquardt method of nonlinear optimization [41] in combination with a new, self-consistent iteration procedure that helps avoid divergencies in the fitting procedure or convergence in a side minimum; details are in [45] and [47].

German reunification on October 3, 1990 provides an interesting test of the model. The sudden merger of the planned economy of the former German Democratic Republic with the market economy of the Federal Republic of Germany (FRG) into what continues to be the Federal Republic of Germany was a result of political, social and economic decisions with far-reaching consequences. As it turns out, it is possible to model this working of "creativity" phenomenologically by just five free coefficients that enter the Taylor series expansion for $a(t)$ and the combination of step functions[11] for $c(t)$ in the model for Germany's total economy: $a(t) = 0.34 - 8.9 \cdot 10^{-3}(t - t_0) + 4.7 \cdot 10^{-3}(t - t_0)^2, c(t) = 1 \cdot \theta(1990 - t) +$ $1.51 \cdot \theta(t - 1991)$, where the step function $\theta(x)$ is 1 for $x \ge 0$ and 0 otherwise. For the other considered systems $a$ and $c$ are given by the logistics function (54), with $p_j \equiv a_j$ for $a(t)$ and $p_j \equiv c_j$ for $c(t)$, $j = 1, 2, 3, 4$. The $\{a_j\}$ and $\{c_j\}$ listed in Table 1 result from SSE minimization with an iteration procedure that observes that the proper starting values for the numerical iteration (with up to 32 000 iteration steps) are crucial for convergence in the global minimum. The growth curves obtained from the Linex function with the $a(t)$ and $c(t)$ reproduce well the empirical time series of output [46, 47]. The technology parameters

Table 1: The free coefficients of $a(t)$ and $c(t)$ in the logistics for the Federal Republic of Germany's industrial sector "Warenproduzierendes Gewerbe" (FRG I), the Japanese sector "Industries" (Japan I), and the total economy of the USA (USA TE).

| System | $a_1$ | $a_2$ | $a_3$ | $a_4$ | $c_1$ | $c_2$ | $c_3$ | $c_4$ |
|--------|-------|-------|-------|-------|-------|-------|-------|-------|
| FRG I | 0.33 | 0.67 | 0.19 | 32 | 1 | 1.46 | 19.1 | 31 |
| Japan I | 0.16 | 0.2 | 1.87 | 20.1 | 2.75 | 0.45 | 0.86 | 14.61 |
| USA TE | 0.21 | 0.49 | 0.97 | 22.64 | 2.63 | 0.81 | 0.81 | 17.24 |

$a(t)$ and $c(t)$ and the time series of $k$, $l$, $e$ are inserted into $\alpha$, $\beta$ and $\gamma$ of Eq. (51) and $\delta$ of Eq. (55). (The law of error propagation produces the largest errors for $\delta$ according to Eq. (55).) Then the time-averaged output elasticities of capital, $\bar{\alpha}$, labor, $\bar{\beta}$, energy, $\bar{\gamma}$, and creativity, $\bar{\delta}$, are computed. The results are shown in Table 2, together with the adjusted coefficient of determination $R^2$ and the Durbin-Watson coefficient $d_W$, for the economic systems: FRG TE (total economy of the Federal Republic of Germany before and after reunification), FRG I (German industrial sector "Warenproduzierendes Gewerbe"), Japan I (Japanese sector "Industries", which produces about 90% of Japanese GDP), USA TE (total economy of the USA). The statistical quality measures $R^2$ and $d_W$ show that there are only marginal deviations of the theoretical growth curves from the empirical ones.[12] We note that

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[11]Step function results from the logistic (54) for $p_3 \to \infty$.

[12]If one wonders, whether $R^2 > 0.99$ is not too good to be true, all data and the new, selfconsistent iteration procedure of nonlinear optimization, outlined in [45] and [47], can be provided. This method has produced similar $R^2$ before [44].

Table 2: Time-averaged Linex output elasticities and statistical quality measures.

| System | FRG TE 1960-2000 | FRG I 1960-99 | Japan I 1965-92 | USA TE 1960-96 |
|---|---|---|---|---|
| $\bar{\alpha}$ | 0.38±0.09 | 0.37± 0.09 | 0.18± 0.07 | 0.51± 0.15 |
| $\bar{\beta}$ | 0.15±0.05 | 0.11±0.07 | 0.09±0.09 | 0.14±0.14 |
| $\bar{\gamma}$ | 0.47±0.1 | 0.52±0.09 | 0.73±0.16 | 0.35±0.11 |
| $\bar{\delta}$ | 0.19±0.2 | 0.12±0.13 | 0.14±0.19 | 0.10±0.17 |
| $R^2$ | >0.999 | 0.996 | 0.999 | 0.999 |
| $d_W$ | 1.64 | 1.9 | 1.71 | 1.46 |

improving the modeling of the explicit ("creativity-induced") time dependence of the Linex production function by Taylor series and/or logistics for the technology parameters $a(t)$ and $c(t)$ does not change the findings from less sophisticated fitting procedures [11], [20, 27], [48]: the output elasticity of energy is much larger than its small cost share of roughly 5 percent. Cointegration analysis of output, capital, labor, and energy [49] confirms the output elasticities of $k, l, e$ in Table 2. Cobb-Douglas production functions, Eq. (50), in which the $\alpha_0$ and $\beta_0$ are of the order of magnitude of the $\bar{\alpha}$ and $\bar{\beta}$ in Table 2, also reproduce economic growth with relatively small residuals, albeit with worse statistical quality measures [49]. (Service production functions similar to the Linex production function have modeled the evolution of German market-determined services between 1960 and 1989 satisfactorily; their time-averaged output elasticities are for labor 0.31 until 1977 and 0.26 after 1978, and for energy they are 0.15 and 0.21, respectively [50]. Thus, even in the labor-dominated sector of the economy, and during times of much less computerized information processing than at present, energy's economic weight exceeds energy's cost share substantially.)

If "useful work", defined as exergy, multiplied by appropriate conversion efficiencies, plus physical work by animals, is used as energy variable in the Linex function, two constant technology parameters suffice to reproduce well the gross domestic product of the US economy between 1900 and 1998 [51, 52]; the time-averaged output elasticities are similar to the ones in Table 2. The "useful work" data [53] already include most of the efficiency improvements that have occurred in energy converting systems during the 20th century. If, on the other hand, one uses primary energy input as energy variable, as it is done in the Linex functions that yield the output elasticities of Table 2, one needs the time-dependent technology parameters. A noticeable time dependence is induced by changes in energy conversion efficiencies. Outsourcing the production of energy-intensive intermediate goods, and limiting the generation of value added to importing and upgrading them, changes the energy demand parameter, too.

The figures that show the empirical and theoretical growth curves are presented in Refs. [46], [47], [51], and [52]. The Solow residual is absent in them, and the theoretical outputs $y_{L1}$ closely follow the empirical ones, including the ups and downs of the energy inputs during the energy crises 1973-1975 and 1979-1981. Energy conversion takes the place of what neoclassical economics calls "technological progress".

## 5. Summary and Conclusions

The cost share theorem of standard economics is not valid in modern industrial economies,

where capital, labor and energy are the main factors of production. Maximization of profit or overall welfare subject to the technological constraints "limits to automation" and "limits to capacity utilization" yields new conditions for economic equilibrium. According to them output elasticities, which measure the factors' economic weights or productive powers, are not equal to the factors' cost shares but rather to "shadowed" cost shares, where shadow prices due to the constraints add to factor prices.

Consequently, output elasticities must be computed independently from equilibrium conditions. They are obtained as solutions of a set of partial differential equations that result from the standard requirement that production functions must be twice differentiable; two technology parameters are estimated econometrically. The numbers in Table 2 and Ref. [51] show that cheap energy has a high productive power, while expensive labor has a low productive power.

This explains the pressure to increase automation, substituting cheap energy/capital combinations for expensive labor. It also reinforces the trend towards globalization, because goods and services produced in low-wage countries can be transported cheaply to high-wage countries. Thus, if the differences between productive powers and cost shares of labor and energy are too pronounced, there is the danger that newly emerging and expanding business sectors cannot generate enough new jobs that can compensate for the ones lost to progress in automation and globalization. This, then, will result in the net loss of routine jobs in high-wage countries and increasing unemployment in the less qualified part of the labor force. A slow-down of economic growth, as natural constraints may cause, or economic recessions for whatever reasons, will aggravate the problem of unemployment. Whether this can be alleviated by shifting the burden of taxes and levies from labor to energy is discussed in Ref. [46].

The finding that energy (exergy) has high productive power in industrial economies exemplifies the importance of thermodynamics for all energy-converting systems, whether they are inanimate, as in the field of physics, or involve people and machines, as in economics. Energy's high productive power is in line with the view that the First and the Second Law of Thermodynamics represent the "Constitution of the Universe" [54]. Furthermore, while the differential equations (47) for output elasticities correspond to the Maxwell relations in equilibrium thermodynamics, we feel that also methods of non-equilibrium thermodynamics may prove useful in the modeling of economic fluctuations, disregarded in this paper. Spontaneous parameter fluctuations in thermodynamic equilibrium of dissipative physical systems result in Nyquist noise of electric circuits, Brownian motion in fluids, and pressure fluctuations in a gas [55]. Behavioral fluctuations of economic actors, triggered by shocks [23], irrational expectations, pursuit of market dominance irrespective of cost, and other deviations from the idealized behavior of the "homo oeconomicus", perturb macroeconomic equilibrium in the (constrained or unconstrained) maximum of profit or overall welfare. Adaptation of the statistical methods used in the derivation of the general fluctuation-dissipation theorem [56] to fluctuations about economic equilibrium may contribute to progress in non-equilibrium economics and demonstrate the power of non-equilibrium thermodynamics. This is a subject of future research.

## Appendix: Explicit constraint equations

The capital stock $k_m(y)$ for maximally automated production of output $y$ at time $t$, required in constraint equation (41), can be calculated from the generally time-dependent Linex function by demanding that

$$y_L[k, l, e; t] = y_L[k_m, l_m, e_m = ck_m; t]. \tag{58}$$

The routine labor $l_m$ that remains in the state of maximum automation is certainly much smaller than $k_m$. If one neglects $l_m/k_m \ll 1$, Eq. (58) becomes

$$y_0 e \exp\left[ a(t)(2 - \frac{l+e}{k}) + a(t)c(t)(\frac{l}{e} - 1) \right]$$
$$= y_0 c(t)k_m(y) \exp\left[ a(t)(2 - c(t)) - a(t)c(t) \right]. \tag{59}$$

This yields the capital stock for the maximally automated production of an output $y$ that at time $t$ is produced by the factors $k(t)$, $l(t)$ and $e(t)$:

$$k_m(y) = \frac{e(t)}{c(t)} \exp\left[ a(t)c(t) \left( 1 + \frac{l(t)}{e(t)} \right) - a(t)\frac{l(t) + e(t)}{k(t)} \right]. \tag{60}$$

Inserting $k_m(y)$ into Eq. (41), where the technical limit to automation $\rho_T(t)$ and the slack variable $k_\rho$ model the technological constraint, we obtain

$$f_A(K, L, E, t) \equiv \frac{(k + k_\rho)}{k_m(y)} - \rho_T(t)$$
$$= (k + k_\rho)\frac{c}{e} \exp\left[ -ac(1 + \frac{l}{e}) + a\frac{l+e}{k} \right] - \rho_T(t) = 0. \tag{61}$$

Here, and in the following, we drop the time argument of factors and parameters for the sake of simplicity.

The equation for the constraint on capacity utilization results from Eqs. (42) and (43) as

$$f_B(K, L, E, t) \equiv \eta_0 \left( \frac{l + l_\eta(t)}{k} \right)^\lambda \left( \frac{e + e_\eta(t)}{k} \right)^\nu - 1 = 0. \tag{62}$$

Eqs. (61) and (62) yield the slack-variable relations

$$k + k_\rho = k_m(y)\rho_T(t) \tag{63}$$

and

$$e + e_\eta = \frac{k}{\eta_0^{1/\nu} \left( \frac{l+l_\eta}{k} \right)^{\lambda/\nu}} \quad . \tag{64}$$

The derivatives of $f_A$ and $f_B$ are calculated by observing Eqs. (38) and the chain rule so that $\partial f_A/\partial K = (1/K_0)(\partial f_A/\partial k)$ etc. From Eqs. (61)-(64) we obtain

$$\frac{\partial f_A}{\partial k} = \frac{1}{k_m(y)} - a\frac{l+e}{k^2}\rho_T \tag{65}$$

$$\frac{\partial f_B}{\partial k} = -\frac{\lambda + \nu}{k} \tag{66}$$

$$\frac{\partial f_A}{\partial l} = -a\left(\frac{c}{e} - \frac{1}{k}\right)\rho_T \tag{67}$$

$$\frac{\partial f_B}{\partial l} = \frac{\lambda}{l + l_\eta} \tag{68}$$

$$\frac{\partial f_A}{\partial e} = \left(\frac{a}{k} + \frac{acl}{e^2} - \frac{1}{e}\right)\rho_T \tag{69}$$

$$\frac{\partial f_B}{\partial e} = \frac{\nu}{e + e_\eta} = \frac{\nu}{k}\eta_0^{1/\nu}\left(\frac{l + l_\eta}{k}\right)^{\lambda/\nu}. \tag{70}$$

Inserting them into the shadow price equation for capital, Eq. (44), and the corresponding equations for the shadow prices of labor and energy one gets the explicit equations for all shadow prices.

In order to compute the shadow prices from the general theoretical framework for an existing economic system one has to take the following steps. 1) The technology parameters $a$ and $c$ have to be determined econometrically for the system. 2) In a rough approximation one may assume proportionality between the slack variables in the constraint on capacity utilization:

$$e_\eta(t) = d(t) \cdot l_\eta(t); \tag{71}$$

here $d(t)$ is the second constraint parameter besides $\rho_T(t)$. We call it the "labor-energy-coupling parameter at full capacity". Ideally, one should be able to determine it from measurements of the energy and labor increases required in order to go from any degree $\eta$ of capacity utilization to 1. With that Eq. (64) becomes the relation between $l_\eta$ (or $e_\eta$) and $k, l, e$. 3) The multiplier $\eta_0$ and the exponents $\lambda$ and $\nu$ may be obtained by fitting the phenomenological $\eta$ of Eq. (43) to empirical time series of $\eta$, which are available from economic research institutions. 4) The technical limit $\rho_T(t)$ to the degree of automation can be any number between 0 and 1. General business inquiries should give clues to it.

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