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Accounting for household heterogeneity in general equilibrium models

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Abstract

The paper investigates differences in total consumption and demand according to how heterogeneity is incorporated into the model of the general equilibrium type. The sensitivity analysis for a static case with CES utilities and production functions demonstrates that the relative differences in total consumption can be considerable when a model with several heterogeneous consumer groups is compared to the one with a representative consumer. In a dynamic model, investment is proved to depend both on the production and consumption sides even in the case with one-sector production. By using the first-order optimality conditions to the multi-sector case it is shown that a model has enough capability to represent household heterogeneity to be applied for integrated assessment of carbon cycle emissions and energy demand.

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Key words: competitive equilibrium, Pareto optimality, aggregation, preferences, elasticity, CES functions, representative consumer, energy demand.

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Introduction

Change in demographic factors such as age, urbanization, and household size has a considerable impact on consumption, energy demand and carbon dioxide emissions (Dalton *et al.*, 2007, 2008). However, it is unclear when and how much compositional change might affect aggregate consumption and energy demand. The research question of the present paper is to investigate when total consumption of energy intensive goods differs according to how heterogeneity is incorporated into the model.

Most energy-economic dynamic models used for climate change policy analysis have either the representative household or overlapping generations structure (Howarth 1996, Manne 1999, Gerald and Zwaan, 2000, Leach 2004). The PET-model adopted by Dalton *et al.* (2007, 2008) was the first one to account for heterogeneity through grouping the households (for a brief description see Appendix A). Our goal is to use a simpler model, with similar key features to the PET-model, to get insights into why and under what conditions the total consumption of several heterogeneous agents differs from consumption of the representative consumer.

To start with simplest case first, Section 1 deals with the one-period static model in both exchange and production economies. First, we review conditions for aggregation of consumption and uniqueness of the equilibrium. Then we show through a series of numerical experiments with CES-type productions functions and utilities that the unique equilibrium in the heterogenous case is indeed different from the one of the representative consumer. Possible reasons for these differences are discussed.

In Section 2, we consider a dynamic general equilibrium model with autonomous multi-sector production and time-additive preferences with a common discount factor. Applying the welfare theorems (Debreu, 1954; Prescott and Lucas, 1972), we relate an equilibrium to the Pareto optimum, and prove the uniqueness of the latter using the dynamic programming approach (Stokey and Lucas, 1987). Contrary to the conclusion of Lucas and Stokey (1984), we show that even in the one-sector case investment depends on both production and consumption sides. Using the first order conditions, we demonstrate that it is even more dependent in the multi-sector case. A similar analysis of equilibria and Pareto optimum in the multi-sector case with a different production correspondence and more general recursive preferences has been done by Dana and Le Van (1990, 1991). Our production is similar to the one used in PET, thus the results of the present paper suggest that potentially the PET model can exhibit a heterogeneity effect. Further work is underway to carry out a sensitivity analysis, similar to the one we do in the static case, with the PET model to investigate the factors that add to the heterogeneity most.

1 One-period static models

1.1 Pure exchange economy

In an exchange economy, we define the notion of the market equilibrium and overview the conditions when (a) exact aggregation is possible, and (b) the equilibrium is unique.

Consider an economy with $I > 1$ consumers; each solves the problem

$$\max_{x^i \geq 0} \; u^i(x^i) \quad \text{s.t.}$$
$$p \cdot x^i = p \cdot w^i,$$

where prices $p = (p_1, \ldots, p_L) \gg 0$, i.e. $p_i > 0$, all i, and donations $w^i = (w^i_1, \ldots, w^i_L) > 0$ of L goods, i.e. $w_i \geq 0$, all i, and $w \neq 0$, are given. Dot stands for the Euclidean scalar product: $p \cdot x = \sum_{j=1}^{L} p_j x_j$.

Consider an individual demand $x^i(p, m^i)$ as a function of the given price vector p and wealth $m^i \equiv p \cdot w^i$. Define the aggregate demand $x \equiv \sum_i x^i(p, m^i)$ and aggregate wealth $m \equiv \sum_i m^i$. We point out two obvious properties. First is the homogeneity of degree zero: $x(\lambda p, \lambda m) = x(p, m)$, $\lambda > 0$, and second is Walras's law: $p \cdot (x(p, m) - w) = 0$.

When can the aggregate demand be expressed as a function of prices and the aggregate wealth only:

$$\sum_i x^i(p, m^i) = x(p, m), \tag{1}$$

for all possible individual wealth distributions? The answer is a well-known

Proposition *Aggregate demand is a function of prices and the aggregate wealth if and only if the individual demands are linear functions of wealth with a common slope, i.e. relation (1) holds for all $m^i > 0$, $m \equiv \sum_i m^i$, if and only if*

$$x^i(p, m^i) = a^i(p) + b(p)\, m^i. \tag{2}$$

Proof. The aggregate demand is a function of prices and aggregate wealth if and only if $dx = 0$ for all m such that $dm = 0$, i.e.

$$\sum_i \frac{\partial x^i(p, m^i)}{\partial m^i}\, dm^i = 0, \qquad \sum_i dm^i = 0.$$

The latter is true if and only if

$$\frac{\partial x^i(p, m^i)}{\partial m^i} = b(p).$$

□

Sufficient conditions, meaningful from the economics point of view, are as follows.

Corollary [Gorman, 1953] *If the individual utilities are identical $u^i(x) = u(x)$, and $u(x)$ is homogeneous of degree one, then the condition (2) is fulfilled. Moreover, the aggregate demand $x(p, m)$ coincides with the solution to the utility $u(x)$ maximization problem given the values m and p.*

Lemma 1. *If the utility $u(x)$ is homogeneous of degree one then the indirect utility function is linear in wealth: $v(p, m) = \tilde{v}(p)\, m$.*

Proof of the lemma. From the definition of the indirect utility function, we easily obtain

$$\begin{aligned} v(p,m) &= \max_x \{u(x) : p\cdot x \le m,\, x\ge 0\} \\ &= \max_x \{m\, u(x/m) : p\cdot (x/m) \le 1,\, (x/m)\ge 0\} \\ &= m \max_\xi \{u(\xi) : p\cdot \xi \le 1,\, \xi \ge 0\} = m\,\tilde v(p). \end{aligned}$$

□

Proof of the corollary. From the above lemma, we have $v_i(p,m) = v(p,m) = \tilde v(p)\, m$. Roy's identity (see, e.g., Mas-Collel *et. al* (1995)) yields

$$x^i(p,m^i) = \left(\frac{\partial v(p,m^i)}{\partial p}\right) \Big/ \left(\frac{\partial v(p,m^i)}{\partial m^i}\right) = \left(\frac{1}{\tilde v(p)}\right)\frac{\partial \tilde v(p)}{\partial p}\, m^i = \frac{\partial\,(\log \tilde v(p))}{\partial p} m^i.$$

□

Remark The result of the corollary remains valid for any homothetic preference given by the utility function $u(x) = f(\varphi(x))$, where f is monotone increasing and positive, and φ is homogeneous of degree 1.

As long as the distribution of initial endowment vectors is unrestricted, homotheticity of individual preferences alone imposes no restrictions on aggregate behavior. More precisely

Theorem [Mantel, 1976] *Let $f(p)$ be a twice differentiable function defined on a compact convex subset of the price simplex. Suppose that the second derivatives of $f(p)$ are uniformly bounded and $f(p)$ satisfies Walras' law. Then there exists a collection of agents, each endowed with a strictly concave, homogeneous utility function—and thus homothetic preferences—such that the aggregate excess demand function $x(p)$ coincides with $f(p)$.*

However, if the individual wealth distribution is fixed, i.e. independent of prices and net wealth, then we have the following

Theorem [Eisenberg, 1961] *Let the individual utilities $u^i(x)$ be homogeneous of degree one, and $m^i = \alpha^i m$, $\sum_i \alpha^i = 1$, $0 < \alpha^i < 1$, then $\sum_i x^i(p,m^i) = x(p,m)$. Moreover, $x(p,m)$ can be obtained through the utility $\sum_i m^i \log u^i$ with p and m given.*

Remark The condition $m^i = \alpha^i m$, $\sum_i \alpha^i = 1$, $0 < \alpha^i < 1$, is clearly satisfied if the endowments themselves are collinear: $w^i = \alpha^i w$. Since the wealth distribution is price independent, the two conditions are equivalent. For a regular economy with at least three goods: $L > 2$, a complementary result is true.

Theorem [Polemarchakis, 1983] *Under the assumption that agents are endowed with a fixed share of the aggregate endowment vector, homotheticity of preferences is necessary, as well as sufficient, for aggregation.*

Definition 1. *The equilibrium in the exchange economy is the tuple $(p, x^1, \cdots, x^I)$ such that the excess demand is zero:*[1]

$$X(p) \equiv \sum_i x^i(p) - \sum_i w^i = 0.$$

Existence is due to Arrow-Debreu, under pretty mild conditions. The most important is that sets $\{x : u_i(x) \ge c\}$ must be convex.

[1] If there is one consumer, then the equilibrium price vector is undefined.

A sufficient condition for uniqueness here is the gross substitutability condition (GS) of the excess demand function:

$$\frac{\partial X_k}{\partial p_l} > 0, \quad k \neq l, \quad k,l = \overline{1,L}, \quad p \gg 0.$$

For CES utility: $u^i(x^i) = (\sum_l a_l (x^i_l)^\rho)^{1/\rho}$, it is straightforward to check that $x_i(p)$ satisfies GS if $0 < \rho < 1$. Since GS property aggregates from individual consumer's $x^i(p)$ to $X(p)$, the equilibrium in the exchange economy with u^i of CES type and $0 < \rho_i < 1$ is unique.

A necessary condition for uniqueness is the weak axiom of revealed preference (WA):

$$\sum_{ll'} \xi_l \frac{\partial X_l}{\partial p_{l'}} \xi_{l'} \leq 0 \qquad \text{for all } \xi \quad \text{s.t.} \quad \xi \cdot X = 0,$$

where $\xi = (\xi_1, \ldots, \xi_L)$. WA is also "almost always" sufficient. Namely, it guarantees that the set of equilibria is convex. Convexity yields uniqueness of the equilibrium for a regular economy. Usually it can be shown that a generic economy is regular (Kehoe, 1991). Therefore, one should not bother about the non-regular ones except for a situation when a whole family of economies is considered depending continuously on a set of parameters, say endowments. In this case one can inevitably pass through a non-regular economy as parameters change.

Most important, WA is clearly satisfied in the case when there is only one consumer, i.e. roughly aggregation implies uniqueness. Unlike GS, in general WA does not aggregate from individual excess demand functions. It it is easy to show that in general WA does not imply GS. However, there is one exceptional case of the two good economy when GS and WA are equivalent [Kehoe, Mas-Colell, 1984].

Recent studies (Codenotti et al., 2005) showed that the equilibrium in the exchange economy is unique if u^i-s are of CES type with $-1 < \rho_i < 0$ (also true for nested CES). Furthermore, the above authors proved that this is a polynomial-time problem Scarf's algorithms lacked this property (see Kehoe, (1991)). For $\rho_i < -1$, Gjerstad (1996) proved that there can be multiple equilibria for consumers with arbitrarily close preferences.

Note that the total consumption $\sum x^i_l$ of the l-th good is independent of consumer's individual consumption choices and is equal to the net donation $\sum_i w^i_l$, which is exogenous. Thus, the equilibrium in the exchange economy is rather a welfare issue. To allow the total consumption to change with prices, we need to consider the closed-loop system with production.

1.2 Economy with production

In the economy with production, the same as before, we define the notion of the market equilibrium and overview the conditions for exact aggregation and uniqueness of the equilibrium.

Let there be J producers, each transforms an M-tuple of production factors z to an amount $f_j(z)$ of jth consumer good. Assume that *production factors* are not consumed and thus *do not enter consumers utilities.* Then market equilibrium is described as follows.

- The j-th producer maximizes the profit taking material prices q and output prices p as given:

$$\max_{z_j \geq 0} (p_j f_j(z_j) - q \cdot z_j) = 0, \tag{3}$$

under the assumption of *constant returns to scale*.[2] Optimality conditions yield

$$p_j \nabla f_j(z_j) = q$$

($J \cdot M$ relations for all producers).

- Materials are initially owned by consumers in the form of donations w^i, so that

$$\sum_j z_j = \sum_i w^i \tag{4}$$

(M relations).

- The i-th consumer maximizes utility:

$$\begin{aligned} \max_{x^i \geq 0} \; & u^i(x^i) \quad \text{s.t.} \\ & p \cdot x^i = q \cdot w^i. \end{aligned} \tag{5}$$

(Optimality conditions give $J \cdot I$ relations.) Constant returns imply zero profits at the equilibrium. Thus, the production ownership shares $0 < \theta_{ij} < 1$, $\sum_i \theta_{ij} = 1$, used in the general case (see, e.g., Mas-Collel *et. al* (1995)), are irrelevant here.

- Supply equals demand:

$$f_j = \sum_i x_j^i, \qquad j = \overline{1, J} \tag{6}$$

(J relations).

In the end, we have a nonlinear system of $J(M + I + 1) + M$ equations and the same number of variables (the dimension of the system can be reduced by one if we normalize the prices, e.g. take one of the prices as a numeraire). Given the endowments $w^1,\ldots,w^I$, an equilibrium is the tuple $(p, q, x^1, \ldots, x^I, z_1, \ldots, z_M)$ satisfying (3)–(6).

It is readily seen that, once the prices are given, the production does not depend explicitly on individual consumers distributions of materials as well as consumption shares, but only on the net resources. Similarly, an individual's consumption does not depend explicitly on the distribution of material use, but only on the individual wealth. However, production and consumption are implicitly linked through prices.

Suppose now that the consumers have identical homogeneous preferences or collinear endowments. Then one aggregate consumer can be considered instead. The last three equilibrium conditions (4)–(6) can be then rewritten as follows.

- The aggregate consumer maximizes utility:

$$\begin{aligned} \max_{x \geq 0} \; & u(x) \quad \text{s.t.} \\ & p \cdot x = q \cdot w, \end{aligned}$$

(optimality conditions give J relations).

[2]This yields that the production set

$$Y = \{(-z, y) \in \mathbb{R}_+^{M+J} : f_j(z_j) - y_j \leq 0,\, j = 1, \ldots, J,\, z = \sum_j z_j\}$$

is a cone.

- Supply equals demand:

$$\sum_j z_j = w, \qquad f_j = x_j, \quad j = \overline{1, J}$$

  ($J + M$ relations).

An equilibrium $(p, q, x, z_1, \ldots, z_J)$ is now the solution to the nonlinear system of $J(M+2)+M$ equations.

Note that, when the endowments w^i are collinear, the consumption side (5) can be always replaced by one consumer with the utility $u = \sum_i (q \cdot w^i) \log u^i$. However, the utility then depends on the prices q of the production factors.

GS is not a sufficient condition for uniqueness in the production economy any more. However, WA is still a necessary condition, and the equilibrium with one consumer is unique in a regular economy analogously to the pure exchange case. Thus in either Gorman (identical homothetic preferences) or Eisenberg (collinear endowments) case, we can obtain the same unique equilibrium with the aggregate consumer. In general this is almost the only meaningful case of unique equilibrium in the economy with (homogeneous) production. Another one applies when the equilibrium is fully defined by the production sector (Kehoe, 1983), and thus of less interest for us. Mas-Colell (1989) introduced a class of super Cobb-Douglas economies with restrictions on both production and consumption for which he proved uniqueness (see Kehoe, 1991). This includes (nested) CES case with $0 < \rho_i < 1$. In the particular case of homogeneous utilities, Jain et al, (2005) obtained uniqueness results for (nested) CES with $-1 \leq \rho_i \leq 0$, and provided poly-time algorithms.

The bottom line here is that there are two cases when exact aggregation applies: either identical homothetic preferences and arbitrary endowments or collinear endowments with different homothetic preferences. Then there exists a unique equilibrium under mild concavity assumptions. Aside of those, uniqueness results have been proven for economies with (possibly nested) CES production and utility functions when the absolute value of the elasticity of substitution is less or equal than one (excluding the linear case).

1.3 Sensitivity analysis in a two good production economy

As we have seen, the conditions for exact aggregation are very stringent. In this section, we compare the total consumption of two heterogeneous consumers with consumption of the representative consumer when those assumptions are not fulfilled. The production side is kept the same in both cases.

Consider a static production economy with two consumers ($I = 2$) and two consumer goods ($J = 2$). Consumers' utility is assumed to be of CES type:

$$u_1(x_1, x_2) = (a x_1^\rho + (1-a) x_2^\rho)^{1/\rho}, \qquad 0 \leq a \leq 1, \quad 0 \leq \rho \leq 1,$$

$$u_2(y_1, y_2) = (b y_1^\sigma + (1-b) y_2^\sigma)^{1/\sigma}, \qquad 0 \leq b \leq 1, \quad 0 \leq \sigma \leq 1.$$

Define the utility of the representative consumer as follows:

$$u(z_1, z_2) = (c z_1^\gamma + (1-c) z_2^\gamma)^{1/\gamma},$$

where the coefficients c and γ are set to be the averages of the ones for the heterogeneous consumers:

$$c = \frac{1}{2}(a+b), \quad \gamma = \frac{1}{2}(\rho + \sigma). \tag{7}$$

There are two production factors owned by the consumers. The endowments of the first and the second consumers are set to be $(w_1, 0)$ and $(0, w_2)$ respectively, so that the aggregate endowment is $w = (w_1, w_2)$. Thus, the 1st consumer solves the following optimization problems:

$$u_1(x_1, x_2) \to \max, \qquad p_1 x_1 + p_2 x_2 = q_1 w_1, \tag{8}$$

and the 2nd consumer solves the following one:

$$u_2(y_1, y_2) \to \max, \qquad p_1 y_1 + p_2 y_2 = q_2 w_2. \tag{9}$$

One can think of the 1st consumer as "younger households" and of the 2nd as "elderly households", and label factors of production as "labor" and "capital". Then the above assumption on endowments describe an idealized situation where the "younger households" own all of the labor, while the "elderly households" all of the capital.

Each consumption good is supplied by one producer with the CES production function:

$$\begin{aligned} f_1(z_1, z_2) &= (k z_1^\delta + (1-k) z_2^\delta)^{1/\delta}, \quad 0 \le k \le 1, \\ f_2(v_1, v_2) &= (l v_1^\delta + (1-l) v_2^\delta)^{1/\delta}, \quad 0 \le l \le 1, \end{aligned} \tag{10}$$

where $0 \le \delta \le 1$ is the common substitution coefficient. The producers' profit maximization problems read

$$\begin{aligned} p_1 f_1(z_1, z_2) - q_1 z_1 - q_2 z_2 &\to \max, \\ p_2 f_2(v_1, v_2) - q_1 v_1 - q_2 v_2 &\to \max. \end{aligned} \tag{11}$$

Supply-equals-demand conditions are

$$z_1 + v_1 = w_1, \qquad z_2 + v_2 = w_2, \tag{12}$$

$$x_1 + y_1 = f_1, \qquad x_2 + y_2 = f_2. \tag{13}$$

We make a sensitivity analysis of the equilibrium total consumptions of the 1st good $x_1 + y_1$ and 2nd good $x_2 + y_2$ with respect to (i) preference coefficients a and b, and (ii) substitution coefficients ρ and σ. In both cases, we fix $k = 0.9$ and $l = 0.1$ to make the production asymmetric in the demand for production factors, i.e. the first sector is "labor intensive" and the other one is "capital intensive". We normalize the endowments so that their product is constant and equal to unity: $w_1 = d$ and $w_2 = 1/d$, which is equivalent to normalizing the total endowment (sum) of the goods to unity. This implies that it is sufficient to consider only $0 < d \le 1$. Indeed, the case $1 \le d$ reduces to the previous one by relabeling z_1 to z_2 and v_1 to v_2, respectively.

First, we set ρ and σ to be equal, and consider the parametrization $b = 1-a$, $0 \le a \le 1$. Clearly, in this case the average preference (7) is constant, so that the equilibrium of the representative consumer is independent of a. At $a = 0.5$ the utilities are identical, and thus the disaggregated equilibrium coincides with the one of the representative consumer.

Note that, if the endowments of the two goods are equal: $d = 1$, the disaggregated equilibrium coincide with the aggregated one for all a. Indeed, with $b = 1 - a$, we have $u_1(x_1, x_2) = u_2(x_2, x_1)$, all a, and the endowments are also symmetric. Thus, with the transform

$$x_1 \leftrightarrow y_2, \quad x_2 \leftrightarrow y_1, \quad p_1 \leftrightarrow p_2, \quad q_1 \leftrightarrow q_2 \tag{14}$$

the optimization problem (8) maps into (9) and vice versa. Due to (13), the values of the output f_1 and f_2 must also change places. Since $l = 1 - k$ in (10), the swap $f_1 \leftrightarrow f_2$ can be achieved by the transform

$$z_1 \leftrightarrow v_2, \quad z_2 \leftrightarrow v_1. \tag{15}$$

Therefore, if (x_1, x_2, y_1, y_2) are the equilibrium consumptions, (z_1, z_2, v_1, v_2) are the equilibrium inputs, and (q_1, q_2, p_1, p_2) are the equilibrium prices, then the tuples (y_2, y_1, x_2, x_1), (v_2, v_1, z_2, z_1) and (q_2, q_1, p_2, p_1) also form an equilibrium. Due to uniqueness, the two must coincide, and thus, at the equilibrium with $d = 1$, we have

$$x_1 = y_2, \quad x_2 = y_1, \quad z_1 = v_2, \quad z_2 = v_1, \quad p_1 = p_2 = p, \quad q_1 = q_2 = q.$$

This yields that the total consumption of both goods are constant and equal to each other, all a and $b = 1 - a$. However, in general, when the endowments of the two goods are different, the relative equilibrium values of the total consumptions differ from the ones of the representative consumer at $a \neq 0.5$ as well (see Figures 1–2).

Second, we set $a = b = 0.5$, and consider the parametrization $\sigma = 1 - \rho$, all $0 \leq \rho \leq 1$. Again, the average substitution (7) is constant, so that the equilibrium of the representative consumer is independent of ρ. At $\rho = 0.5$ the utilities are identical, and thus the disaggregated equilibrium coincides with the one of the representative consumer. As in the previous case, if the endowments of the two goods are equal: $d = 1$, the disaggregated equilibrium coincide with the aggregated one for all ρ. Indeed, in that case $u_i(x_1, x_2) = u_i(x_2, x_1)$, $i = 1, 2$, imply $x_1 = x_2 = x$ and $y_1 = y_2 = y$, so that the total consumption of the both goods is equal to $x + y$. Figure 3 shows that, same as in the previous case, for nonequal endowments of the two goods and $\rho \neq 0.5$, the relative equilibrium values of the total consumptions differ from the ones of the representative consumer.

From Figures 1–3 (top panel), we see that the consumption of the good that is more dependant on a scarce resource is more sensitive, i.e. its relative change is greater. Moreover, when the substitution coefficient is positive, for $a \in [0, 0.5]$ the relative change is larger than for $a \in [0.5, 1]$ (with negative substitution coefficient the opposite is observed). One can try to describe the situation that corresponds to $a \in [0, 0.5]$ as "consumers prefer the opposite to what they own factors for". For the prices of the consumer goods the situation is the opposite (see the middle panel in Fig. 1–3). Namely, the price of the good whose output depends on a more abundant resource has a larger relative change. Clearly the magnitude of the both effects depend on the substitution factor (Fig. 1–2). Also, the larger is the difference in the endowments, the larger are the relative changes (Fig. 3).

The bottom panel in Figures 1–3 shows the consumers' wealth: $p_1x_1 + p_2x_2$ and $p_1y_1 + p_2y_2$. One can see that wealth of the consumer that owns a larger endowment has a more visible relative change. The relative change is larger for $\rho \in [0.5, 1]$.

1.4 Economy with taxes and government spending

The PET-model includes government spending and various taxes, in particular a CO_2-emission tax. The notion of the equilibrium can be extended to this situation as well. However, due to the externalities the connection between the equilibrium and Pareto optimum in general fails, and already existence of the equilibrium poses a problem (Kehoe, 1991). Therefore, this paper is confined mainly to the models without taxes. Here we give a brief account of the static economy with production and imperfect markets.

The equilibrium is the tuple $(p, q, r, x^1, \dots, x^I, z_1, \dots, z_M)$, where r is the amount of government spending, such that the following conditions are satisfied.

- The j-th producer faces an *ad valorem* taxes τ_j on products and maximizes the after-tax profit taking material prices q and output prices p as given:

$$\max_{z_j \geq 0} \left(p_j(1 - \tau_j)\, f_j(z_j) - q \cdot z_j\right) = 0.$$

- The i-th consumer maximizes utility with a modified budget constraint:

$$\max_{x^i \geq 0} \; u^i(x^i) \quad \text{s.t.}$$
$$\sum_j p_j \cdot x_j^i = \sum_m q_m(1 - t_m) w_m^i + \gamma_i r.$$

  Here t_m is the *ad valorem* tax on the material good m and $\gamma_i \geq 0$, $\sum_i \gamma_i = 1$, is the share of government revenue r received by the consumers i.

- Supply equals demand conditions:

$$\sum_j z_j = \sum_i w^i, \qquad f_j = \sum_i x_j^i, \quad j = \overline{1, J}.$$

Walras's law takes the form

$$\sum_j p_j \tau_j f_j + \sum_m q_m t_m w_m = r.$$

Uniqueness of the equilibrium can be studied using the index theory along the lines of Kehoe (1983, 1985). Two remarks are in place here. First, as was shown by Kehoe (1985), in the economy with the Leontieff production sector even the WA can be insufficient for the uniqueness. Second, as often happens in the economies with taxes and externalities, the equilibrium can be not Pareto optimal.

2 Dynamic general equilibrium model

2.1 Consumption

Consider a finite number of infinitely lived agents indexed by $i = \overline{1, n}$. Each agent's consumption c_i is a sequence $c_i = \{c_{it}\}_{t=1}^{\infty}$. Consumption goods are indexed by $j = \overline{1, m}$, so that each time c_{it} is a vector $c_{it} = (c_{i1t}, \dots, c_{imt})$. Consumptions are nonnegative and uniformly bounded, i.e. each c_i is an elements of l_+^m, the nonnegative orthant of l_∞^m, with the norm

$$\|c_i\| = \sup_{j,t} |c_{ijt}|.$$

Remark 1. *The choice of l_∞ is an important technical requirement. It is easy to check that the positive orthant of spaces l_p, $1 \leq p < \infty$, and of the spaces $\{x : \sum_t \beta^t |x_t| < \infty\}$, $0 < \beta < 1$, has no interior points. Nonempty interior allows to apply the Hahn-Banach separation theorem. Following L&S (1984), we omit the infinity subscript in the notation.*

The preference of an agent at time t is given by the single-period utility function $U(c_t)$ (index i is omitted here), and the overall preference has the time-additive form:

$$u(c) = \sum_t \beta^t U(c_t), \quad 0 < \beta < 1.$$

Definition 2. *The utility function u is strictly concave if $c, c' \in l_+^m$ and $u(c) > u(c')$ imply $u(\theta c + (1-\theta)c') > u(c')$ for all $\theta \in (0, 1)$.*

Let U be (strictly) concave and bounded, so that u is well-defined, i.e.

$$\|u\| = \sup_c |u(c)| < \infty.$$

Given $c = (c_1, c_2, \dots)$, denote $c^T = (c_1, \dots, c_T, 0, \dots)$.

Lemma 1 (Truncation property in consumption)**.** *If $c, c' \in l_+^m$ and $u(c) > u(c')$, then $u(c^T) > u(c')$, for all T sufficiently large.*

Proof. Indeed, for every $c \in l_+$, we have $u(c) = \lim_{T\to\infty} u(c^T)$. □

To secure non-zero capital shares over the the whole time span a common time preference coefficient β must be taken for all consumers (Coles, 1986, Lucas & Stokey 1984 and Becker, 2006). Further on we will need the following

Definition 3. *The consumption $c \in l_+$ is a saturation point if $c' \in l_+$ implies $u(c') \le u(c)$.*

2.2 Production

There is a multi-sector one capital production. Each consumption good is produced by one producer. The capital sequence $k = \{k_t\}_{t=1}^{\infty}$ is assumed to be non-negative and bounded: $k \in l_+$ with the norm $\|k\| = \sup_t |k_t|$. The technology is assumed to be stationary, i.e. each time the admissible total consumption c_t, investment x_t and the next time capital k_{t+1} are determined by the current capital k_t only. Denote the set of all admissible tuples (c_t, k_{t+1}) as $\Gamma(k_t)$, so that the technology is described by the production correspondence (set-valued map) $\Gamma : \mathbb{R}_+ \to \mathbb{R}_+^m \times \mathbb{R}_+ \times \mathbb{R}_+$. The production correspondence Γ is assumed to be continuous and have the following properties (for discussion see L&S, 1984):

(G1) for each k, the set $\Gamma(k)$ is compact and convex;

(G2) $(c, x, y) \in \Gamma(k)$ and $0 \le (c', y') \le (c, y)$ imply $(c', y') \in \Gamma(k)$;

(G3) $k' \le k$ implies $\Gamma(k') \in \Gamma(k)$;

(G4) $(c, y) \in \Gamma(k)$ and $(c', y') \in \Gamma(k')$ imply

$$(\theta c + (1-\theta)c', \theta y + (1-\theta)y') \in \Gamma(\theta k + (1-\theta)k')$$

for all $\theta \in [0, 1]$;

(G5) the set $M \equiv \{k \in \mathbb{R}_+ : \ (0, k) \in \Gamma(k)\}$ has a nonempty interior;

(G6) if k is an interior point of M, then $(c, k) \in \Gamma(k)$ for some $c \gg 0$.

Given the initial capital k_0, the set of all admissible consumption sequences is defined as follows:

$$Y(k_0) \equiv \{c : \exists\, k \in l_+ \text{ s.t. } (c_t, k_{t+1}) \in \Gamma(k_t)\ \forall\, t = 0, 1, 2, \dots\}. \tag{16}$$

Lemma 2 (L&S, 1984)**.** *Let the production correspondence Γ satisfy the assumptions (G1)–(G6) and let the production set be defined by (16). Then for each k_0 the set $Y(k_0)$ is closed and convex, and if k_0 is an interior point of M, $Y(k_0)$ has an nonempty interior.*[3]

Note that in the Stokey-Lucas setup the production set consists of the output only, not $(c, x, -k)$ as it should in the classical Arrow-Debreu theory. Due to this difference just free disposal assumption is insufficient to provide a nonempty interior, as in Debreu (1954). Only assumptions (G2),(G5) and (G6) together guarantee nonempty interior in the Lucas setting. Assumption (G2) also yields the following

Lemma 3 (Truncation property in production)**.** *Given k_0, $c \in Y(k_0)$ implies $c^T \in Y(k_0)$ for all T sufficiently large.*

[3] $Y(k_0)$ is not compact in the sup-norm, but compact in the product topology due to Tikhonov theorem (see Dana and Le Van, 1990).

2.3 Equilibrium and Pareto optimum

Following Debreu (1954) and L&S (1984), we give the following definitions.

Definition 4. *Given the initial capital k_0, an allocation $(c_i) \in l_+^{mn}$ is feasible if $\bar{c} = \sum_i c_i \in Y(k_0)$.*

Definition 5. *The price system is an m-tuple $p \in l_+^m$ such that $p \cdot c = \sum_{jt} p_{jt} c_{jt} < \infty$ for all $c \in l_+^m$.*

Definition 6. *Given k_0, a competitive equilibrium is a feasible allocation (c_i) together with the price system p such that*

(i) $p \cdot c_i' \le p \cdot c_i$ implies $u^i(c_i') \le u^i(c_i)$ for all i and for all $(c_i') \in l_+^{mn}$;

(ii) $p \cdot \bar{c}' \le p \cdot \bar{c}$ for all $\bar{c}' \in Y(k_0)$.

Definition 7. *Given k_0, an allocation (c_i) is Pareto optimum if it is feasible, and there is no other feasible allocation (c_i') such that $(c_i') > (c_i)$, i.e.*

$$u^i(c_i') \ge u^i(c_i) \qquad \text{for all } i,$$

and

$$u^i(c_i') > u^i(c_i) \quad \text{for some } i.$$

Due to Debreu, 1954 and Stokey and Lucas, 1987, 1987, we have the following

Theorem 1 (First Welfare Theorem)**.** *Suppose for each i, the utility $u^i : l_+^m \to \mathbb{R}_+$ is strictly convex. Let (c_i, p) be a competitive equilibrium, where none of the c_i is a saturation point. Then (c_i) is a Pareto-optimum allocation.*

Proof. The proof closely follows the one of Debreu (1954). First, $u^i(c_i') > u(c_i)$ implies $p \cdot c_i' > p \cdot c_i$. This follows directly from condition (i) of Definition 6. Second, $u^i(c_i') = u(c_i)$ implies $p \cdot c_i' \ge p \cdot c_i$. This is a consequence of strict convexity of u^i. Now, let $(c_i') > (c_i)$ then $\sum_i p \cdot c_i' > \sum_i p \cdot c_i$, or equivalently, $p \cdot \bar{c}' = p \cdot \sum_i c_i' > p \cdot \sum_i c_i = p \cdot \bar{c}$. This contradicts to condition (ii) of Definition 6. □

S&L, 1987 replace the strong convexity assumption by a similar one: for every $c \in l_+$ there exists a sequence $\{c_{(n)}\} \subset l_+$ such that $c_{(n)} \to c$ and $u^i(c_{(n)}) > u^i(c)$, all i.

The converse statement says that the supporting price system for the optimum allocation can be constructed (Prescott & Lucas, 1982). This theorem requires additional convexity and interior point assumptions, since the argument draws on the Hahn-Banach Theorem (see KF and Luenberger, 1969). The truncation property (Theorem 1) guarantees that the linear functional in the Hahn-Banach theorem represents a price system and does not have "weight at infinity", i.e. the functional belongs to l_1^m.

Theorem 2 (Second Welfare Theorem)**.** *Suppose for each i, the utility $u^i : l_+^m \to \mathbb{R}_+$ is strictly convex. Let assumptions (G1)–(G6) be satisfied, let (c_i) be a Pareto-optimal allocation, given k_0, and assume that some c_i is not a saturation point. Then there exists a price system p such that*

(i) $u^i(c_i') \le u^i(c_i)$ implies $p \cdot c_i' \le p \cdot c_i$ for all i and for all $(c_i') \in l_+^{mn}$;

(ii) $p \cdot \bar{c}' \le p \cdot \bar{c}$ for all $\bar{c}' \in Y(k_0)$.

Remark 2. *Condition (i) in Theorem 2 coincides with the one in Definition 6 if none of the c_i is a corner solution, i.e. there exists $c_i' \in l_+$ such that $u^i(c_i') < u^i(c_i)$, all i.*

The two welfare theorems enables us to study a Pareto-optimum instead of an equilibrium (including the questions of existence and uniqueness).

2.4 Value function

Define the set of all possible utility combinations, given the value of the capital $k \in \mathbb{R}_+$ in the current period:

$$Z(k) = \{z \in \mathbb{R}^n_+ : \, z_i = u^i(c_i), \ i = \overline{1,n}, \text{ for some feasible } (c_i) \in l^{mn}_+\}.$$

Lemma 4 (Stokey and Lucas, 1984)**.** *Given k, the set $Z(k)$ is convex and compact, and possesses the "free disposal" property: $0 < z' < z$ and $z \in Z(k)$ imply $z' \in Z(k)$. Moreover, the correspondence $k \to Z(k)$ is continuous.*

The ordering on consumption space l_+ corresponds to the usual ordering on $\mathbb{R}^m_+$: given $z, z' \in Z(k)$, the relation $z' \leq z$ is equivalent to $z'_i \leq z_i$, all i. The "north-western boundary" of the set $Z(k)$ corresponds to Pareto-optimum allocations. More precisely, given k, an allocation (c_i) is Pareto-optimal if and only if the point $z = (u^1(c_1), \dots, u^n(c_n))$ is the unique intersection point of the set $Z(k)$ with the shifted non-negative orthant $\{z\}+\mathbb{R}^n_+$. This allows to reduce the search for Pareto-optimal allocations to a maximization problem.

Given the vector θ that belongs to the unit simplex $\Delta^{n-1} = \{\theta \in \mathbb{R}^n_+ : \sum_i \theta_i = 1\}$, define the value function as the support function of the set $Z(k)$:

$$v(k, \theta) \equiv \sup_{z \in Z(k)} \sum_i \theta_i z_i. \tag{17}$$

As a consequence of lemma 4 the supremum in (17) can be replaced by maximum.

Lemma 5. *Given k, the allocation (c_i) is Pareto-optimal if and only if there exists a nonnegative vector $\theta \in \mathbb{R}^n_+$ such that (c_i) attains supremum in (17).*

Proof. Both $Z(k)$ and $\{z\} + \mathbb{R}^n_+$ are convex sets in $\mathbb{R}^n$. Hence there exits a separating hypersurface $\{w \in \mathbb{R}^n : \sum_i \theta_i(w_i - z_i) = 0\}$ such that $\sum_i \theta_i z'_i \leq \sum_i \theta_i z_i$ for all $z' \in Z(k)$, and $\sum_i \theta_i w_i \geq 0$ for all $w \in \mathbb{R}^n_+$. The latter yields $\theta_i \geq 0$, all i. □

From Lemma 4 we immediately obtain

Lemma 6. *The value function $v(k, \theta)$ is bounded and continuous.*

2.5 Optimality principle

Next we show that the value function $v(k, \theta)$ is the unique solution to the Bellman equation

$$V(k, \theta) = \max_{(\bar{c}, y) \in \Gamma(k)} \left(\sum_i \theta_i U_i(c_i) + \beta V(y, \theta) \right). \tag{18}$$

In the case when β's are different, the Bellman equation generalizes as follows:

$$V(k, \theta) = \max_{\substack{(\bar{c}, y) \in \Gamma(k) \\ w \geq 0}} \sum_i \theta_i (U_i(c_i) + \beta_i w_i) \quad \text{s.t.}$$
$$\min_{\lambda \in \Delta^{n-1}} V(y, \lambda) \geq \lambda \cdot w.$$

In this case, λ is the next step guess for the weights θ.

Lemma 7. *Bellman equation (18) has the unique solution $V(k, \theta)$. This function $V(k, \theta)$ is increasing and (strictly) concave in k, and convex in θ.*

Proof. Standard reasoning (e.g. S&L, 1989). Define the operator

$$Tf(k,\theta) = \max_{(\bar{c},y)\in\Gamma(k)} \left(\sum_i \theta_i U_i(c_i) + \beta f(k,\theta) \right).$$

on the space B of all bounded and continuous functions over $\mathbb{R}_+ \times \Delta^{n-1}$. Since Γ is compact and f is continuous, Tf is correctly defined (maximum is attained) and bounded; since Γ and f are continuous, so is Tf. Blackwell's sufficient conditions for contraction are satisfied:

$$f \le \tilde{f} \;\Rightarrow\; Tf \le T\tilde{f}, \quad \forall\, f, \tilde{f} \in B.$$

and

$$T(f+a) \le Tf + \beta a, \quad \forall\, f \in B \text{ and } \forall\, a > 0.$$

Thus, there exist the unique fixed point V such that $\lim_{n\to\infty} \|Tf_0 - V\| = 0$, for all $f_0 \in B$.

The fact that $V(k,\theta)$ is increasing in k follows from (G3). If f is concave in k and convex in θ, so is Tf. Indeed, let $k \neq k'$, and (c,y) and (c',y') attain $\Gamma(k)$ and $\Gamma(k')$, respectively. By (G4), $(c^\lambda, y^\lambda) \in \Gamma(k^\lambda)$. From concavity of U_i it follows

$$\begin{aligned}(Tf)(k^\lambda,\theta) &\ge \sum_i \theta_i U_i(c_i^\lambda) + \beta f(k^\lambda,\theta) \\ &\ge \lambda \left[\sum_i \theta_i U_i(c_i) + \beta f(k,\theta) \right] + (1-\lambda) \left[\sum_i \theta_i U_i(c_i') + \beta f(k',\theta) \right] \\ &= \lambda (Tf)(k,\theta) + (1-\lambda)(Tf)(k',\theta)\end{aligned}$$

(the second inequality is strict if U_i are strictly concave). □

It is easy to show that $v(k,\theta)$ is the solution to (18). Given $V(k,\theta)$ the r.h.s. of equation (18) determines the (non-empty and upper semi-continuous) solution correspondence $G : \mathbb{R}_+ \to \mathbb{R}_+^m \times \mathbb{R}_+$ that maps k to $(c,y) = (G^c(k), G^y(k))$. If $\Gamma(k)$ and U_i, all i, are strictly convex, then so is $V(k,\theta)$. Once $V(k,\theta)$ is strictly convex in k, G is single-valued and continuous.

2.6 Dynamics

Next we investigate the dependence of the trajectory $(c,y) = (G^c(k), G^y(k))$ on the parameter θ. To do that we need to impose additional smoothness assumption on both production functions and utilities, and the Inada condition on utilities: $\lim_{c_j\to 0} U_{i,j}(c) = \infty$.

2.6.1 One-sector case

In the one-sector case, the constraint $(\bar{c}, y) \in \Gamma(k)$ can be replaced by a simpler one: $0 \le \bar{c} \le F(k,y)$, where the function $F : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+^m$ is defined as

$$F(k,y) = \max_{(c,y)\in\Gamma(k)} c.$$

Lemma 8 (L&S, 1984)**.** *Suppose F and U_i, $i = \overline{1,n}$ differentiable. If $\theta \gg 0$ then for all $k \ge 0$ the function $V(k,\theta)$ is differentiable with*

$$V_k(k,\theta) = \theta_i U_i'(G_i^c(k,\theta)) F_k(k, G^y(k,\theta)), \quad i = \overline{1,n}. \tag{19}$$

The first order conditions for the r.h.s. of (18) are obtained from the Lagrange function[4]

$$\mathcal{L} = \sum_i \theta_i U_i(c_i) + \beta V(y,\theta) + \lambda \left(F(k,y) - \sum_i c_i \right).$$

If all U_i satisfy the Inada condition, the inequality $c \geq 0$ can be dismissed. Then the interior solution is described by

$$0 = \theta_i U_i'(c_i) - \lambda, \quad i = \overline{1,n}, \tag{20}$$

$$0 = \beta V_k(y,\theta) + \lambda F_y(k,y). \tag{21}$$

Substituting (19) into (21) and rearranging, we come to

$$\beta U_i'(G_i^c(y,\theta)) F_k(y, G^k(y,\theta)) + U_i'(G_i^c(k,\theta)) F_y(k,y) = 0, \quad i = \overline{1,n}. \tag{22}$$

In particular, for the standard one-sector model with $F(k,y) = f(k) + (1-\delta)k - y$, we have

$$\beta U_i'(G_i^c(y,\theta))(f'(k) + (1-\delta)) + U_i'(G_i^c(k,\theta)) = 0, \quad i = \overline{1,n}.$$

At the steady state equations (22) can be reduced to

$$(\beta F_k + F_y)_{y=k} = 0.$$

For example, for the standard one-sector model with $F(k,y) = f(k) + (1-\delta)k - y$, this yields the "golden rule" steady state: $\beta(f'(k) + (1-\delta)) - 1 = 0$.

2.6.2 Multi-sector case

The standard one-sector production correspondence

$$\Gamma(k_t) \equiv \left\{ (c_t, k_{t+1}) \in \mathbb{R}_+^2 \, : c_t + x_t \leq f(k_t), \; k_{t+1} = (1-\delta)k_t + x_t \right\},$$

in the multi-sector case, can be generalized as follows:

$$\Gamma(k_t) \equiv \left\{ (c_t, k_{t+1}) \in \mathbb{R}_+^m \times \mathbb{R}_+ \; : \begin{array}{c} c_{jt} \leq f_j(k_{jt}), \; j = \overline{1,m}, \\ x_t \leq f_{m+1}(k_{m+1t}), \; \sum_{j=1}^{m+1} k_{jt} = k_t, \\ k_{t+1} = (1-\delta)k_t + x_t \end{array} \right\}.$$

If the production functions f_j are concave, then assumptions (G1)–(G6) are satisfied.

The Lagrange function takes the form:

$$\mathcal{L} = \sum_i \theta_i U_i(c_i) + \beta V(y,\theta) + \sum_j \lambda_j \cdot \left(f(k_j) - \sum_i c_{ij} \right) + \lambda_{m+1}(f_{m+1}(k_{m+1}) + (1-\delta)k - y) + \nu \left(k - \sum_j^{m+1} k_j \right). \tag{23}$$

[4] In the homogeneous case one can plug in $c = F(k,y)$ directly into the utility function and solve the unconstrained problem $\max\{U(F(k,y)) + \beta V(y) : \, y \in G(k)\}$.

The first order conditions are as follows:

$$0 = \theta_i U_{i,j}(c_i) - \lambda_j, \qquad i = \overline{1,n}, \qquad j = \overline{1,m} \tag{24}$$

$$0 = \beta V_k(y,\theta) - \lambda_{m+1}, \tag{25}$$

$$0 = \lambda_j f_j'(k_j) - \nu, \qquad j = \overline{1,m+1}. \tag{26}$$

Eliminating λ_j from (25) and (26), we get

$$\theta_i U_{i,j}(c_i) f_j'(k_j) = \nu, \qquad i = \overline{1,n}, \qquad j = \overline{1,m}$$

and

$$\beta V_k(y,\theta) f_{m+1}'(k_{m+1}) = \nu.$$

Though the Pareto optimum is shown to be unique, the equilibrium might not be. However, as in the static case (the same duality argument now in the Banach space), we have

Theorem 3. *The equilibrium is unique if the consumers have a common homothetic preference.*

In general, a regular (generic) economy has a finite number of equilibria (Kehoe, 1991).

3 Discussion

Numerical experiments done in Section 1 show that an equilibrium in a static model with several heterogeneous consumers each having different CES utilities can differ substantially from the one with a representative consumer having the CES utility with averaged preferences, if preferences across consumers differ enough and endowments are not co-linear. This conclusion applies equally to the case when the elasticities are averaged instead of the preferences.

From section 2, we conclude that the transitional dynamics of the total consumption depends on both the production side and individual preferences. The steady state equilibrium, if it exists, does not depend on the weights θ's, which allows the turnpike property under stronger assumptions (Bewley, 1982; Yano, 1982). Note that the PET model has additional potential for representing heterogeneity as the labor supply and preference coefficients are time-dependent.

The work by Alvarez and Stokey (1998) suggests that our analysis can be extended to the constant-returns-to-scale production. From the results surveyed in Section 1 it is then natural to conjecture that restricting the production and consumption to CES type one can prove uniqueness of the equilibrium in the dynamic model as well. This is important for numerical calculations since iterative algorithms, like Gauss-Seidel method, are most often applied.

A straightforward application of neither dynamic programming, nor welfare theorems is possible for economies with taxes. Some ideas on dynamic model with taxes can be found in Kehoe (1991).

Appendix A. PET model: synopsis

PET is a dynamic general equilibrium fully calibrated model, which accounts for heterogeneity through grouping the households (Dalton *et al.*, 2008; Dalton and Goulder, 2001). To be specific, consider a version of the PET model that accounts for the age heterogeneity. The households are grouped into dynasties indexed by i (see Appendix B). Each dynasty owns the capital k_t^i in per capita units, which is perfectly substitutable in production. Households buy investment goods x_t^i at price q_t adding it to the capital stock, which depreciate at the common rate $\delta > 0$:

$$k_{t+1}^i = (1-\delta)k_t^i + x_t^i. \tag{27}$$

There are consumer goods indexed by j. Per capita consumption for a household in dynasty i, of good j, at date t is denoted by c_{jt}^i. The instantaneous budget constraint in dynasty i is written as

$$\sum_j p_{jt}c_{jt}^i + q_t x_t^i = (1-\theta_t^i)w_t l_t^i + (1-\varphi_t^i)r_t k_t^i + g_t^i \tag{28}$$

where l_t^i is the labor supply, $w_t = 1$ is wages (taken as a numeraire, by way of Walras's law) θ_t^i and φ_t^i are lump-sum tax rates (on labor and capital income), r_t is the interest rate, and g_t^i is the lump-sum transfers from the government.

Under the above mentioned constraints, each dynasty chooses its consumption $\{c_{jt}^i\}$, for all j, and investment x_t^i to maximize its lifetime utility

$$\frac{1}{\rho}\sum_{t=1}^{\infty}\beta^t n_t^i \left(\sum_j \mu_{jt}^i (c_{jt}^i)^\sigma\right)^{\frac{\rho}{\sigma}}, \tag{29}$$

where $0 < \beta < 1$ is a common discount factor, $\rho < 1$ is the intertemporal substitution parameter, $\sigma < 1$ is the consumption substitution parameter, n_t^i is the size of the dynasty at time t, and μ_{jt}^i are preference parameters calibrated to the expenditure shares in the base year. Households are assumed to take prices as given and have a perfect foresight of future values of the variables, such as r_t and q_t, and future assets of other households. The Euler equation can be obtained in closed form. Together with an imposed transversality condition it provides a solution sequence.

Producers also take prices as given and assign their cost-minimizing production shares according to a CES production function (we omit the index j here):

$$X = \gamma_X\left(\alpha_K(G_K K)^{\rho_X} + \alpha_L(G_L L)^{\rho_X} + \alpha_{\bar{E}}(G_{\bar{E}}\bar{E})^{\rho_X} + \alpha_M(G_M M)^{\rho_X}\right)^{1/\rho_X}$$

with capital K, labor L, energy composite $\bar{E}$ and materials M as inputs. The technology of government output is also represented by a Cobb-Douglas production function; government's purchases are constant in real terms.

The equilibrium is defined by market clearing conditions (supply equals demand) and governmental budget. Equilibrium prices are determined iteratively under certain regularity assumptions.

Appendix B. Dynasty approach

A dynasty is a chain of households representing different age groups: the elderly of age ω, their descendants of age $\omega - \tau$, descendants of their descendants of age $\omega - 2\tau$, etc. Once the period τ and the age resolution of o cohort $L \leq \tau$ are fixed, there are $N = \tau/L$ dynasties numbered by i. The grouping is assumed to be time consistent, i. e. members of the dynasties do not mingle over time (ways to relax this assumption are discussed at the end). Each dynasty maximizes its own intertemporal utility adapting accordingly its consumption and investment sequences. The principal purpose of arranging the model that way is to investigate the impact of age and other forms of heterogeneity on consumption and investment patterns.

The population n_{it} of the dynasty, as well as its labor supply l_{it}, are simply the total population and work force of all contributing households. This data comes from household projections and is available in advance for the the whole model run period (Dalton *et al.*, 2008).

Consumption preferences μ_{ijt} of the dynasty i toward the good j are calculated as the average of households' preferences in the dynasty, weighted according to their population share. This way the dynastic utility accounts for different consumption patterns of different age groups.

The initial capital k_{0i} is the weighted average of the statistical data on capital over different age groups within the dynasty. Since generally the elderly have more capital through savings, the initial capital of the dynasty will be greater than just the statistical capital of the youngest household. That serves as a proxy for the intergenerational (bequest) transfers linking the oldest and youngest households. The latter was the main obstacle in using the OLG type of models.[5] The representative consumer framework is obtained as a particular case with one dynasty. The values of n_t, l_t, μ_{jt} and k_0 in this case are clearly independent of $L = \tau$.

In principal, an aging population has two major off-setting effects on capital. On the one hand, the labor supply decreases, thus diminishing the capital. On the other hand, elderly people have more capital saved during the previous years. According to the model description, for each dynasty aging will result in (a) diminishing the labor force l_{it}, (b) shifting the consumption preference coefficients μ_{ijt}, and (c) changing the value of the initial capital k_{0i}.

Note that adopting the above framework does not modify the structure of the dynasty's utility, say to incorporate the consumption of other members of the dynasty. As a consequence, the fact that a member of one dynasty never becomes a member of another one is not important any more. Thus, the same approach can be used to study the impact of urbanization or household size. Grouping in general could be worthwhile if (a) there is a shift in the category (e.g. aging, urbanization, etc.), and (b) data shows that preferences vary considerably across the category.

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[5]The continuous time analogue of the OLG model used by Marini and Scaramozzino (1995) is not particularly convenient for practical implementation because of unrealistic demographic assumptions and absence of the intergenerational transfers (on the latter see Melnikov and Sanderson, 2007).

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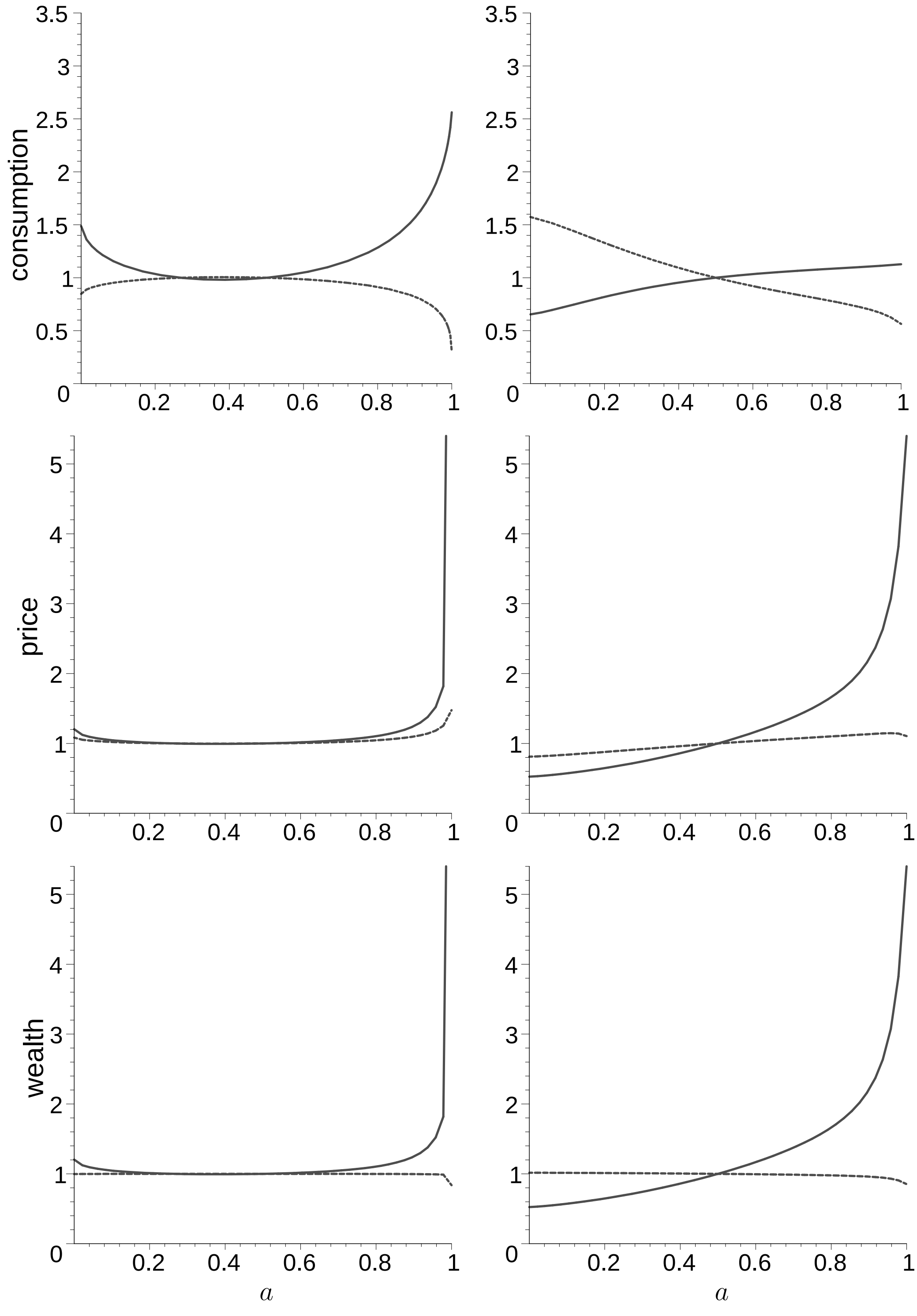

Figure 1: Equilibrium characteristics relative to their values at $a = 0.5$ in case with $d = 10$ and $\rho = \sigma = -0.5$ (left) and $\rho = \sigma = 0.2$ (right): total consumption of the 1st good (solid) and 2nd good (dashed); price of the 1st good (solid) and 2nd good (dashed); wealth of the 1st household (solid) and 2nd household (dashed).

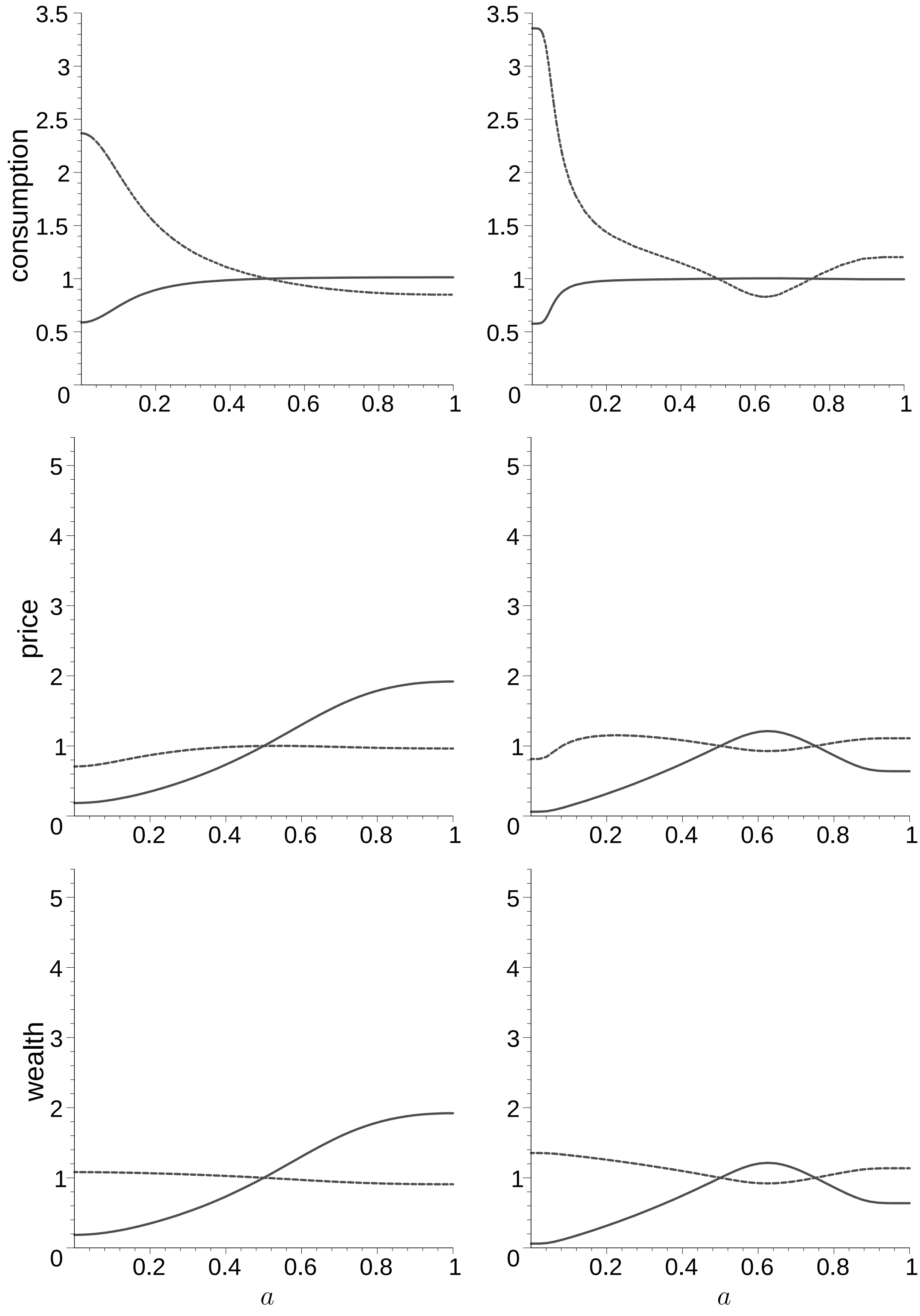

Figure 2: Equilibrium characteristics relative to their values at $a = 0.5$ in case with $d = 10$ and $\rho = \sigma = 0.5$ (left) and $\rho = \sigma = 0.8$ (right): total consumption of the 1st good (solid) and 2nd good (dashed); price of the 1st good (solid) and 2nd good (dashed); wealth of the 1st household (solid) and 2nd household (dashed).

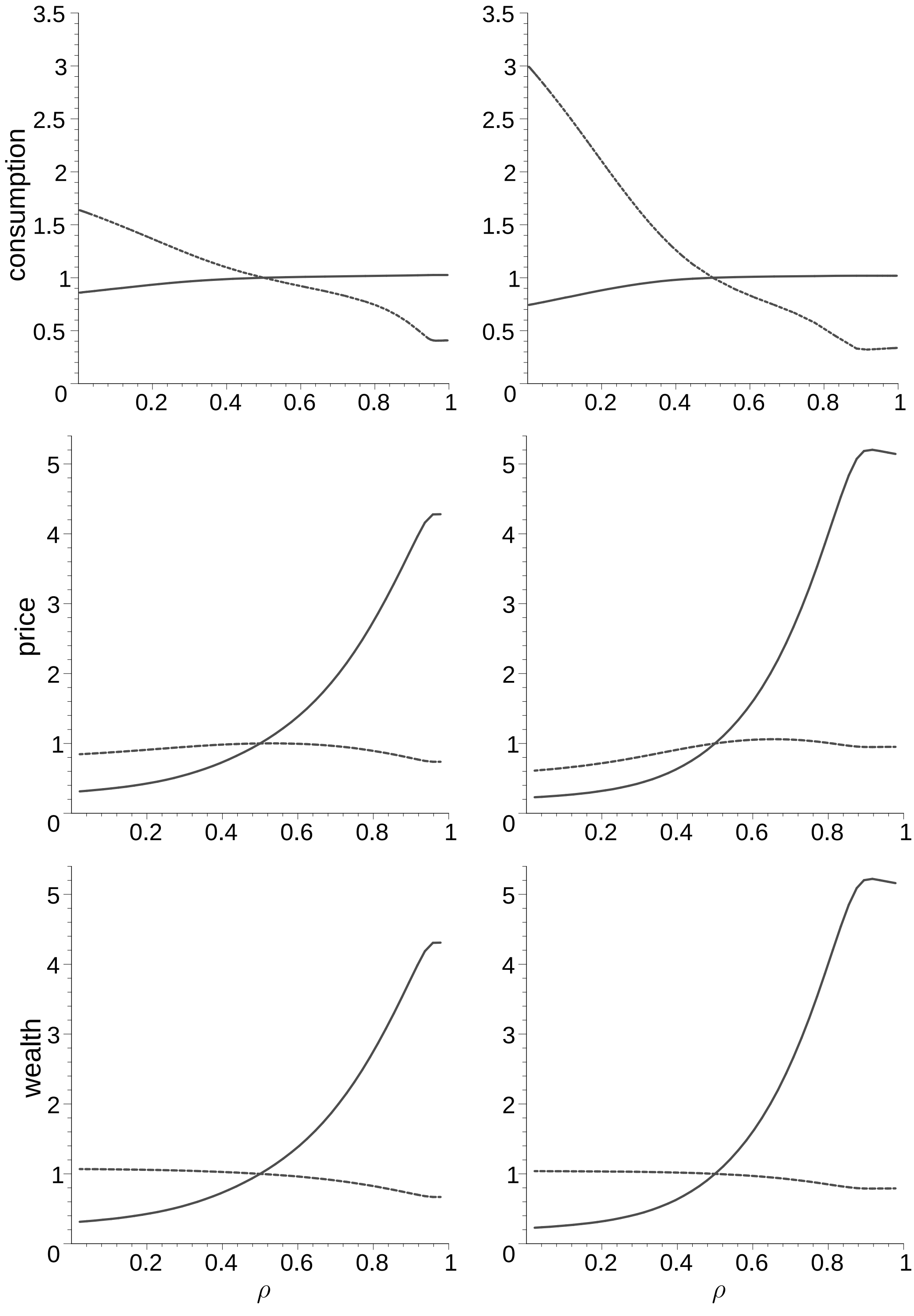

Figure 3: Equilibrium characteristics relative to their values at $\rho = 0.5$ in case with $a = b = 0.5$, and $d = 10$ (left) and $d = 20$ (right): total consumption of the 1st good (solid) and 2nd good (dashed); price of the 1st good (solid) and 2nd good (dashed); wealth of the 1st household (solid) and 2nd household (dashed).