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Optimal Pollution and Optimal Population

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Optimal Pollution and Optimal Population

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September 2007

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Abstract

There is emerging evidence that environmental degradation increases human mortality. This paper provides a long-run consumer maximization model where population growth is endogenous to emissions that are generated in production. There is a trade-off between consumption and population growth; large consumption calls for large production, thus leading to high environmental mortality and low population growth. It may be optimal to end up with negative population growth implying that demographic sustainability fails as consumption increases excessively. We provide a theoretical model and suggest its calibrated version using European air pollution data. Our exercise illustrates the functioning of the theoretical model and discusses related methodological problems.

Journal of Economic Literature: Q01, Q53, Q54, J11

Keywords: Optimal growth, Optimal pollution, Sustainable development, Calibrations, Europe

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Optimal Pollution and Optimal Population

Ulla Lehmijoki
Elena Rovenskaya

1 Introduction

There is emerging evidence of the adverse affects of environmental degradation on human health as several branches of science have provided new results concerning air pollution, climate change, salination of ground waters and pollution od oceans. The reported air pollution, for example, increases both morbidity and mortality to a marked degree. Many concerns deal with short-term political and financial questions but the long-term concern is whether it is possible that environmental mortality will increase so much that demographic sustainability comes under pressure?

Sustainable development has been intensely researched since the 1970s. This research has concentrated on exhaustion of non-renewable and renewable resources as well as on pollution, but has paid little attention to the questions of environmental mortality and population growth. Some papers have assumed that population is constant, whereas others have suggested that population grows at a constant rate. However the demographic fall – a necessary outcome of resource scarcity and heavy pollution – has not been considered earlier in the framework of sustainable growth. Actually the concept of sustainable growth itself needs redefinition to take the demographic aspects into account. In this paper we give a long-run consumer optimization model where population growth, through increases in mortality, is endogenous to environmental degradation. We define demographic sustainability and ask whether it should be taken seriously by discussing the European air pollution mortality in the light of a calibrated version of the proposed model. Our emphasis is on pollution but with some re-definition of concepts the results of our paper can be applied to resource models as well.

In order to view decision process involving a long-time horizon effects as human's health and economic development the apparatus of optimal growth theory has been widely used over the past few decades. An overview of the history of the optimal growth theory can be found, e.g., in Barro and Sala-I-Martin (1995). Neoclassical growth theory allows to involve various environmental phenomena related to economic development. The well-known DICE models, for example, have studied the effects of climate change-caused damages to the economy (Nordhaus (1994)). In this paper we are considering the negative impact of pollution on the economy through population. We are looking for an optimal balance between the size of population and welfare. For this purpose we use the technique of optimal control theory (Pontryagin et al. (1962)).

It is well known that a path which maximizes a consumer's utility is not necessarily sustainable. Earlier this question has been considered by Dasgupta and Heal (1974 and 1979), Solow (1974), Stiglitz (1974), Krautkraemer (1985), Pezzey and Withagen (1998), Valente (2004), Greiner (2007). Dasgupta and Heal (1974), Pezzey and Withagen (1998), and Valente (2004) show, for example, that a typical optimal consumption path initially raises and then falls, indicating that the current generation pollutes or uses resources in excessive amounts. Hence, the consumption rates decrease over time and "...the later generations ... suffer incredibly as a result of the initial profligacy" (Dasgupta and Heal (1979)). We show, however, that a model with endogenous mortality may generate a variety of outcomes, depending upon the values of the parameters. Most importantly, if the steady state population growth is negative and the demographic sustainability fails, then the optimal consumption path keeps increasing. In this case, the future generations do not suffer in the absence of consumption but rather because of greater number of premature deaths.

To illustrate the features of our model and to discuss demographic sustainability, we calibrate it with European air pollution data. Even though air pollution in Europe is rather low in global perspective, population growth in Europe is low as well, so that a modest increase in environmental mortality may push population growth below zero. Therefore, Europe serves as an interesting example of demographic sustainability.

Calibration of a highly theoretical model is a demanding task for several reasons. Because the number of parameters is usually large and each parameter needs its own data, the overall data requirements are large. Further, the simplicity which is necessary in theoretical models, limits the empirical aspects which can be considered without depart too far from the ideas of the original model, the functioning of which is usually also exemplified by the calibrated model. In this paper we seek solutions for these problems by presenting alternative calibration strategies. The data limitations are solved by focusing on fine particulate matters because data on this pollutant is available, and also because particulate matters are considered as the most detrimental to human health (WHO (2004a)). The first-shot calibrations provided in this paper, however, should be considered as an methodological exercise rather than as a final estimate for the demographic sustainability in Europe.

The paper is organized as follows: Section 2 reviews the recent evidence on air pollution mortality. Section 3 introduces the model and its solution, Section 4 provides a calibrated version on the model and discusses its implications. Section 5 concentrates on sustainability at the European level. Section 6 discusses the results and closes the paper. In Appendix one can find all technical details related to the methodological background.

2 Air pollution mortality

There seems to be a consensus that air pollution is the greatest environmental risk to the human health in industrial countries. Air pollution mortality was first reported in the Meuse Valley, Belgium (1930) and London (1952), where smog took lives of 60 and 4000 people respectively (Nemery et al. (2001) and Logan (1953)). Air pollution consists of several components, of which particulate matter (PM) and ozone are the most dangerous (WHO (2004a)). Air pollution raises mortality mainly through an increase in respiratory and cardiovascular diseases and lung cancer (Samet et al. (2000)), but an increase in skin cancer has also been reported (Brunekreef and Holgate (2002)). All age groups are affected, but unborn, young children and the elderly are the most vulnerable.

The Clean Air for Europe program ($CAFE$) and WHO have recently provided a summary estimate for mortality caused by short-term exposure in Europe by collecting 629 time-series studies and 160 individual or panel studies up to February 2003 (WHO (2004a)). In these studies, daily adult mortality was regressed against daily changes in air pollution; the summary estimates indicate that there is a statistically significant 0.6% increase in mortality for each $10\mu g/m^3$ increase in PM and 0.5% increase in mortality for each $10ppb$ increase in Ozone.

Pope et al. (2002) analyzed the effects of long-term PM exposure in the United States in a study in which a questionnaire from 1982 provided data on sex, race, smoking, alcohol consumption, so that controlling for alternative risk sources was possible. The mortality data was collected until 1998 and regressed against local pollution data to derive 4%, 6%, and 8% increases in all-cause, respiratory, and lung cancer mortality respectively for each $10\mu g/m^3$ increase in PM. $CAFE$ and WHO applied these estimates to the European data, calculating that the short-term and long-term exposures together were responsible for 370 000 premature deaths in 2000 (WHO (2004a)).

WHO has also provided a synthesis on air pollution and child mortality (WHO (2004b)) that is based on several original studies. The conclusion is on a four-level scale from "sufficient" to "no association", showing that there is sufficient evidence of an increase in child mortality, mainly due to PM exposure. Currie and Neidell (2005) have estimated the infant mortality risk in California during the 1990s. Several covariates were applied and fetal deaths were controlled for to exclude the selection bias. The pollutants were particulate matter, ozone, carbon monoxide, and nitrogen dioxide. Single pollutant models supplied significant estimates, but when all four were included, only carbon monoxide was significant. Chay and Greenstone (2003a and 2003b) have shown that the air quality improvement under the Clean Air Act of 1970 in the United States saved more than 1,300 infants annually, and that the 1981-82 recession that led to considerable decreases in PM concentrations also had a positive effect on infant mortality.

3 The Model

In this paper we propose a model of economic growth which is extended by a negative feedback of growth and population via increasing pollution and its impact on mortality. It is a one-sector model aimed at finding optimal consumption under the idea that the society can invest in production of tangible capital goods depriving itself from the consumption today in order to increase the consumption in future. The society is guided by the objective to maximize both the size of the population and the accumulated per capita consumption.

3.1 Environmental Constraints

We consider a model where mortality depends on pollution which may raise mortality via either acute emissions or concentrated stocks. In this paper, we concentrate on emissions since they can be easily incorporated into a one-sector model. Let L stand for the size of the population, and $n = \dot{L}/L$ stand for the population growth rate, that is the difference between fertility and mortality. Here and in what follows $\dot{L}$ stands for the derivative of L with respect to the time variable; a similar notation is used for other time-dependent quantities.

Since our emphasis is on mortality, we assume the population growth rate to consist of two components: a constant basic rate, ν, and an additional mortality associated with emissions. Thus n depends on the size of emissions, E, and we define it as a demographic response function

$$n = n(E): \quad n(0) = \nu > 0, \quad n'(E) < 0 \quad \text{for all} \quad E > 0, \tag{1}$$

indicating that under for zero level of emission the population growth rate is positive and it decreases as the level of emissions grows. The population growth function $n = n(E)$ can adopt several types of functional expressions (Medows et al.(1972)). Line A in Figure 1 argues that mortality steadily increases as emission increases. Line B refers to a threshold in mortality regard to emission; with approaching this threshold the acute increase in mortality via human resilience decreases. Several subsequent thresholds may exist. Line C refers to an ever increasing sensitivity of mortality indicating that, ultimately, the population collapses as its growth rate goes to minus infinity. The theoretic model below allows all variants, but the numerical analysis presented in Section 4 concentrates on line A. Figure 1 also shows a critical emission level, $\bar{E}$, for line A. Once emission reach this critical level, population starts to decrease.

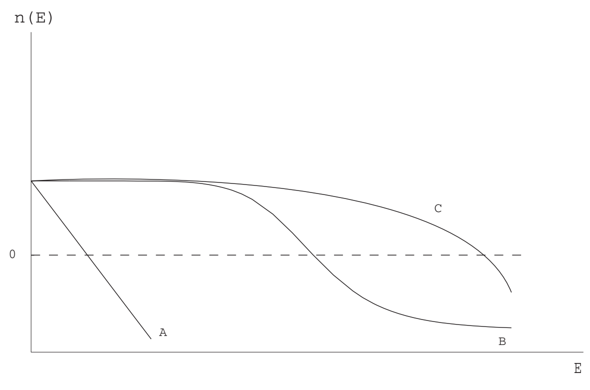

Figure 1: Population growth as a function of emission, Meadows *et al.*1972.

Normalizing dividing by the initial population size and redenoting we get

$$L(t) = \exp \int_0^t n\left(E\left(\tau\right)\right) d\tau; \tag{2}$$

here are in what follows t stands for the time variable; the initial time is set to be zero.

3.2 Economic Constraints

We assume that the productive capital K is the only input, and the production output Y is given by a production function of a standard Cobb-Douglas form

$$Y = F(K) = K^\alpha$$

where the positive parameter $\alpha < 1$ is the elasticity of the output with respect to capital.

We consider one-sector economy in which a part C of the output Y is consumed and the rest of the output, $Y - C$ is invested. Then the productive capital accumulates according to

$$\dot{K} = K^\alpha - C - \delta K, \quad K(0) = K_0, \tag{3}$$

where $\delta > 0$ refers to depreciation and K_0 is a given positive initial value for capital. From (3) one can easily get the result of the following Lemma which is important for further analysis.

Lemma 1 *For all consumption strategies* $C(t)$ *such that* $0 \leq C(t) \leq K^\alpha(t)$ *the corresponding capital trajectories* $K(t)$ *are bounded; namely,* $0 \leq K(t) \leq \tilde{K}$, *where* $\tilde{K} = \delta^{1/(\alpha-1)}$.

Both consumption and production may generate emissions. Cars pollute when used for driving but magazines pollute when the paper is produced. We assume that all emissions are born in production and that the ratio of emissions to production is constant.[1] Therefore, the emissions E become

$$E = \phi Y = \phi K^\alpha. \tag{4}$$

3.3 Utility

Consider an infinitely living central planner who wants to maximize the Benthamian societal utility defined to be a product of per capita consumption, $u(\cdot)$, and the number of people, L, in which $u(\cdot)$ adopts the $CIES$ formula

$$u(C/L) = \frac{(C/L)^{1-\theta}}{1-\theta}, \quad \theta \neq 1.^2 \tag{5}$$

Here the elasticity of the marginal utility, $\theta > 0$, acts as a measure of the social valuation of different levels of consumption reflecting the extend to which the society is willing to reduce the welfare of high-income generations in order to improve the welfare of low-income generations. Such an approach to choosing the goal function of the society is often used in modeling environmental-economical processes (see,

[1] Several authors argue that the relation of emissions to *per capita* output adopts the shape of an inverted U. The paper can be generalized to follow this Environmental-Kuznets-Curve hypothesis by assuming ϕ decreases as output increases.

for example, Nordhaus (1994)). Given a discount factor $\rho > 0$, the central planner chooses consumption $C(t)$ so as to maximize a utility index

$$\begin{aligned} U &= \int_0^\infty u(C(t)/L(t)) \cdot L(t) \cdot e^{-\rho t}\, dt \\ &= \int_0^\infty \frac{C(t)^{1-\theta}}{1-\theta} \cdot e^{-\int_0^t [\rho - \theta n(E(\tau))] d\tau}\, dt \end{aligned} \tag{6}$$

subject to the economical constraints (3), (4); the last equality in (6) follows from (5) and (2).

To keep the objective functional bounded, we introduce the following assumption.

Assumption 1 For any emission trajectory $E(t) > 0$ which may occur while functioning the economy, it holds that

$$\rho - \theta n(E(t)) \geq w > 0,$$

where w is a positive constant.

Let us notice that in a view of (1) Assumption 1 holds for sure if

$$\rho - \theta \nu \geq w.$$

3.4 Problem formulation

Note that in his intertemporal choice, the central planner faces a trade-off between consumption and population growth. If consumption is too large emissions are too large too, which leads to high environmental mortality and low societal utility. On the other hand, if consumption is too low, environmental mortality is low, which is favorable, but in this case low consumption curbs the utility. Hence, optimal consumption refers to an optimal emission regime and, in the long run, to an optimal population growth rate. Thus the central planner accepts some degree of environmental mortality as an exchange for higher consumption.

A summarizing optimal control problem is:

$$\begin{aligned} \text{maximize}_{C(\cdot)} \quad U &= \int_0^\infty \frac{C(t)^{1-\theta}}{1-\theta}\, e^{-\int_0^t \{\rho - \theta n(E(\tau))\} d\tau}\, dt, \\ \text{subject to:} \quad E(t) &= \phi K(t)^\alpha, \\ \dot{K}(t) &= K(t)^\alpha - C(t) - \delta K(t), \quad K(0) = K_0, \\ 0 &< C(t) \leq K(t)^\alpha. \end{aligned} \tag{7}$$

3.5 Equilibrium Analysis

The fact that the discount factor $\Delta(t) = \int_0^t \{\rho - \theta n(E(\tau))\} d\tau$ in (6) is not constant provides a difficulty for analysis of the problem (7). To eliminate this difficulty we apply a virtual time technique (see Uzawa (1968)). Thanks to Assumption 1 $\Delta(t)$ has the following properties:

(i) $\Delta(0) = 0$,

(ii) $\Delta(\infty) = \infty$,
(iii) $\Delta(t)$ is monotonically increasing with $\dot{\Delta}(t) = \rho - n(E(t)) > 0$.

Properties (i) – (iii) imply that $\Delta(t)$ satisfies the regularity conditions suggested by Uzawa (1968) and thus can be used as an alternative independent variable (a virtual time) Therefore we set

$$C = C(\Delta), \quad K = K(\Delta), \quad E = E(\Delta).$$

Furthermore,

$$dt = \frac{d\Delta(t)}{\rho - \theta n(E(t))} = \frac{d\Delta}{\rho - \theta n(E(\Delta))}. \tag{8}$$

Applying (8) to (3) and (6), we turn problem (7) into the following one:

$$\begin{aligned}
\text{maximize}_{C(\cdot)} \quad U &= \int_0^{\infty} \frac{C^{1-\theta(\Delta)}}{(1-\theta)(\rho - \theta n(E(\Delta)))} e^{-\Delta} d\Delta, \\
\text{subject to:} \quad E &= \phi K^{\alpha}, \\
\mathring{K} &= \frac{dK}{d\Delta} = \frac{K^{\alpha} - C - \delta K}{\rho - \theta n(E)}, \quad K(0) = K_0, \\
0 &< C(\Delta) \leq K^{\alpha}(\Delta).
\end{aligned} \tag{9}$$

Problem (9) equivalent to problem (7) is an infinite time horizon optimal control problem with mixed constraints on the state and control variables. The basic control-theoretic technique – the Pontryagin maximum principle – was originally developed for problems with finite time horizons (Pontryagin *et al* (1962)). It is known that for problems with infinite time horizon the necessary optimality conditions may not be valid (e.g., Aseev and Kryazhimsky (2004), (2007)). The latter work suggest a justified version of the Pontryagin maximum principle for optimal control problems with infinite time horizons. In Appendix A we discuss in detail an application of this technique to problem (9). The principle scheme is the following. First we rewrite problem (9) in a standard form with controls lying in a compact set. Namely, we introduce a lower bound for consumption assuming $C \geq \epsilon Y$ instead of $C > 0$; where $\epsilon > 0$ is a small parameter (a modified formulation of problem (9) is given in (13)). Then we make sure that this problem satisfies to the conditions (A1) – (A7) sufficient for the validity of the infinite-horizon Pontryagin maximum principle for problems with infinite time horizon which are suggested in Aseev and Kryazhimsky (2007). In this manner we provide a formal basis for the use of the maximum principle. Based on that, we write out the necessary conditions for optimality, (14) – (18). We state (see Statements 4, 5) that the extreme consumption strategies $C(t)$ which touch the bounds for their admissible values, i.e. such that $C(t) = \epsilon K^{\alpha}(t)$ or $C(t) = K^{\alpha}(t)$ for some $t > 0$, cannot be optimal in problem (7). Based on that in our further analysis we concentrate only on consumption paths lying entirely inside the admissible interval. The Hamiltonian system supplying the optimal paths for problem (7) has a form

$$\frac{\dot{C}}{C} = \frac{1}{\theta}\left\{\frac{\theta n' \phi \alpha K^{\alpha-1}}{\rho - \theta n(\phi K^{\alpha})}\left(\frac{\theta C}{1-\theta} + K^{\alpha} - \delta K\right) + \left[\left(\alpha K^{\alpha-1} - \delta\right) - (\rho - \theta n(\phi K^{\alpha}))\right]\right\}, \tag{10}$$

$$\dot{K} = K^{\alpha} - C - \delta K, \tag{11}$$

the zero curves for its vector field are given by

$$\frac{\dot{C}}{C} = 0 \Leftrightarrow C = \frac{\theta-1}{\theta}\left\{K^\alpha - \delta K + \frac{\rho-\theta n(\phi K^\alpha)}{\theta n'(\phi K^\alpha)\phi\alpha K^{\alpha-1}}\left[\left(\alpha K^{\alpha-1}-\delta\right)-(\rho-\theta n(\phi K^\alpha))\right]\right\},$$

$$\dot{K} = 0 \Leftrightarrow C = K^\alpha - \delta K.$$

The curve $\dot{K} = 0$ depicted on the (K, C) plane is strictly concave, reaches its maximum at $\hat{K} = (\alpha/\delta)^{1/(1-\alpha)}$ and hits the vertical axis at $K = 0$ and at $\tilde{K} = \delta^{1/(\alpha-1)}$ (Figure 2). The shape of the curve $\dot{C}/C = 0$ depends on the value of θ. Hall (1988) has argued that empirical elasticities tend to be large, therefore we assume $\theta > 1$. The following lemma provides a sufficient condition for the existence of an interior steady state for system (10), (11).

Lemma 2 *Let Assumption 1 be satisfied. Then the system (10), (11). has at least one steady state* (K^*, C^*), *where* $K^* > 0$, $C^* > 0$ *and* $K^* < \tilde{K}) = \delta^{1/(\alpha-1)}$.

Proof. Let $K = 0$. For a point (K, C) on the curve $\dot{K} = 0$ we have $C = K^\alpha - \delta K = 0$ and for a point (K, C) on the curve $\dot{C} = 0$ we have

$$C = \frac{\theta-1}{\theta}\frac{\rho-\theta n(0)}{\theta n'(0)\phi} < 0$$

due to (1) and Assumption 1.

Let $K = \tilde{K}$. For a point (K, C) on the curve $\dot{K} = 0$ we have $C = 0$ and for a point (K, C) on the curve $\dot{C} = 0$ we have

$$C = \frac{\theta-1}{\theta}\frac{\rho-\theta n(\phi K^\alpha)}{\theta n'(\phi K^\alpha)\phi\alpha\tilde{K}^{\alpha-1}}\left[\left(\alpha\tilde{K}^{\alpha-1}-\delta\right)-(\rho-\theta n(\phi K^\alpha))\right] > 0.$$

Indeed, $\rho - \theta n(\phi K^\alpha) > 0$ by Assumption 1 and a slope of the curve $C = K^\alpha - \delta K$ in negative $K = \tilde{K}$ (the latter fact leads to a negative value of the derivative, i.e., $\alpha\tilde{K}^{\alpha-1} - \delta$).

Hence, the curve $\dot{C}/C = 0$ lies below the curve $\dot{K} = 0$ for $K = 0$ and above it for $K = \tilde{K}$. Since both curves are continuous, curve $\dot{C}/C = 0$ intersects curve $\dot{K} = 0$ at some point (K^*, C^*), where $K^* > 0$, $C^* > 0$ and $K^* < \tilde{K}$.

In what follows, we consider problem (7) under the following assumption.

Assumption 2 The Hamiltonian system (10) – (11) has a single steady state (K^*, C^*) and

$$K_0 \leq K^* \leq \hat{K}$$

where $\hat{K}$ is the maximum point for $K^\alpha - \delta K$.

A standard local stability analysis carried out in Appendix B shows that the steady state (K^*, C^*) is a saddle and stable saddle paths approach it from the South-West and North-East (Fig. 2). Typically, saddle paths solve optimal control problems with infinite horizons. Therefore we claim the following:

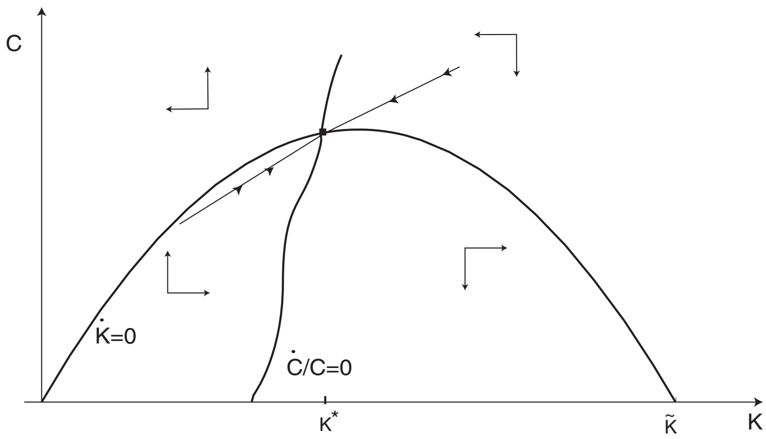

Figure 2: The phase diagram of the model.

Statement 1 *The stable saddle path $(K(t), C(t))$ of the Hamiltonian system (10) – (11) such that $K(0) = K_0$ is uniquely defined and solves the optimal control problem (7).*

The formal validation of Statement 1 is discussed in Appendix C.

4 Calibrated Model

In this section we illustrate the use of Statement 1 for a calibrated version of the model. Table 4 provides the functional forms and the calibrated benchmark values for the model's parameters. The functions and some values are standard[3]. For example, Barro and Sala-i-Martin (1995), argue that the elasticity of the output with respect to capital in Cobb-Douglas production function $Y = K^\alpha$, the discount factor, the elasticity of marginal utility, and the rate of depreciation should be close to $\alpha = 0.7$, $\rho = 0.05$, $\theta = 3$, and $\delta = 0.05$, respectively (these values are given in Table 4). A more complete formula for the production function would be $Y = AK^\alpha$, where A refers to the level of technology. However, because this level greatly varies across countries, it is hard to give any general estimate for A. In this paper, we only provide results (growth rates and normalized time paths), which are independent of the value of A. One can also think that the formula $Y = K^\alpha$, applied here, is identical to $Y = AK^\alpha$ with normalization $A = 1$.

To evaluate the value for the emission rate ϕ we make use of the fact that air pollution is the most acute environmental killer in industrial countries and fine particulates $PM_{2.5}$ are its most dangerous component (WHO (2004a)). Therefore, we alleviate our data requirements by working with the $PM_{2.5}$ data in this exercise. We regress the $PM_{2.5}$ emissions (tons) against the GDP data (in thousand international dollars) from 25 European countries in the year 2000 to derive estimate $\phi = 0.11208$ kt per each trillion USD of the GDP. The data on the GDP comes from Heston et

[3]In advance we emphasize the fact that for the parameter values given in Table 4 all technical assumptions made both earlier and in the subsequent text are not violated which justifies the formal validity of the results presented in this section. We discuss issues related to those assumptions in Appendix D.

$Y = K^{\alpha}$	Cobb-Douglas production function
$\alpha = 0.7$	Elasticity of output with respect to capital
$\rho = 0.05$ 1/year	Discount factor
$u(C) = \frac{C^{1-\theta}}{1-\theta}$	*CIES* utility function
$\theta = 3$	The elasticity of marginal utility
$\delta = 0.05$ 1/year	Depreciation rate
$E = \phi Y$	Emission
$\phi = 0.112086$ kt/tril USD	Emission rate
$n(E) = \nu - \beta E$	Demographic response function
$\nu = 0.0049$ 1/year	Basic population growth rate
$\beta = 0.00038072211$ kt	Benchmark environmental mortality

Table 1: The functional forms and calibrated values for the parameters.

al. (2006) and that on emissions comes from the *RAINS* database (Atmospheric Pollution Program, IIASA).

Consider the demographic response function $n = n(E)$ discussed in Section 3.1 and depicted in Figure 1. Figure 1 shows that several alternatives to describe $n = n(E)$ are possible, each telling us a particular story about pollution and population growth. The medical studies discussed above (Samet et al. (2000), Brunekreef and Holgate (2002), Pope et al. (2002), Currie and Neidell (2005), Chay and Greenstone (2003a and 2003b), WHO (2004a and 2004b)) assume that the relationship between pollution and mortality is linear and we keep this assumption here. Hence we set

$$n = n(E) = \nu - \beta E = \nu - \beta\phi Y = \nu - \beta\phi K^{\alpha}, \tag{12}$$

where $\nu = n(0) > 0$ and $\beta > 0$. This formula corresponds to curve A in Figure 1. To calibrate ν we calculate the aggregate population for 25 European countries for the years 1950 – 2004 and find that the average annual population growth rate in Europe has been 0.49%. Naturally, air pollution has already some effect on this number but given the long time span of the data, we can assume that this effect is negligibly small. Therefore, we attribute the value $\nu = 0.0049$ per year as the autonomous population growth rate in equation (12).

In our model, we theoretize in terms of environmental mortality, thus the value of the mortality parameter β in (12) is of special interest for us. In this section we give a rough calibration for it and carry out the sensitivity analysis of the optimal behavior of the economy to its value assuming that β varies in a neibourhood of a benchmark value β_0. Note that (12) implies that if the capital stock equals $\bar{K} = \left(\frac{\nu}{\beta\phi}\right)^{1/\alpha}$, then the population size is stabilized. On the other hand, equations (12) and (12) imply that the steady-state capital K^* can take several values depending upon the parameters' values. Therefore, we find three types three types of the steady state dynamics are possible:

(i) if $K^* > \bar{K}$, then $n^* > 0$;
(ii) if $K^* = \bar{K}$, then $n^* = 0$;
(iii) if $K^* < \bar{K}$, then $n^* < 0$,

denoting by n^* a steady population growth rate (i.e., $n^* = n(E^*)$, $E^* = \phi K^{*\alpha}$). Both K^* and $\bar{K}$ depend on β. We calculate the benchmark value $\beta = \beta_0 = 0.00038072211$ such that $K^*(\beta_0) = \bar{K}(\beta_0)$ implying the stable population in the steady state. Because environmental mortality increases together with β, any value for β which is either higher or lower than the benchmark value β_0 leads to a either a negative or a positive population growth in the steady state. [4]

To calculate the time paths for variables, we apply the time elimination method, in which the stable saddle path is calculated by taking the steady state (K^*, C^*) as a starting point and continuing the phase trajectory of the Hamiltonian system (10) – (11) until reaching $K = K_0$. We assume zero initial conditions for phase variables aiming at calculating entire optimal trajectories and identify the year 2000 as the initial time moment (see Figure 2). Since the slope of the saddle path in the $K, C-$space is given as $dC/dK = \dot{C}/\dot{K}$, the time paths for capital, emission, population, and consumption – $K(t)$, $E(t)$, $L(t)$, $C(t)$ – are calculated by applying (2) – (4). For details of the time elimination method, see Mulligan and Sala-i-Martin (1991).

Let us illustrate the role of β providing sensitivity analysis of the optimal respond of the economy to varying it. Figure 3 shows the time paths for the population growth rate $n(t)$ and population size $L(t)$ for the benchmark value $\beta = \beta_0 = 0.00038072211$ (the central path) and for its values perturbed for ± 0.0000025 and ± 0.00005. The five time paths show that for $\beta = \beta_0$, population size first increases and then levels-off, whereas it keeps increasing or decreasing for the perturbed values $\beta < \beta_0$ or $\beta > \beta_0$ respectively. Thus Figure 3 shows that the demographic role of β is qualitative and critical in a sense that the mode of population growth changes if β is perturbed. However, β has only quantitative effect on other features of the model. For example the steady state values for capital, K^*, and consumption, C^*, decrease if β increases, indicating that a right tendency would be to accumulate and consume less because higher economic activity would lead to demographic losses. But once we standardize all the steady states to unity, the optimal time paths for capital, $K(t)$, and consumption, $C(t)$, show almost no variation responding to changes in β, as depicted in Figure 4. The constant character of of capital K^* and consumption C^* in the steady state implies that the growth rate of the per capita numbers $k = K/L$ and $c = C/L$ react to the steady state population growth rate, being respectively negative, zero, or positive if this rate is respectively positive, zero or negative as is depicted in Figure 5. Therefore, the central planner faces a trade-off between population and per capita consumption since it is impossible to keep them both increasing.

5 Demographic Sustainability

In 1987, the Brundtland Comission defined sustainable development as a development that "meets the needs of the present without compromising the ability of future generations to meet their own needs" (WCED (1987)). This traditional definition refers to non-decreasing consumption or non-decreasing utility, concepts that are

[4] Programming was performed by using Mathematica 5.2. The program is available from the authors on request.

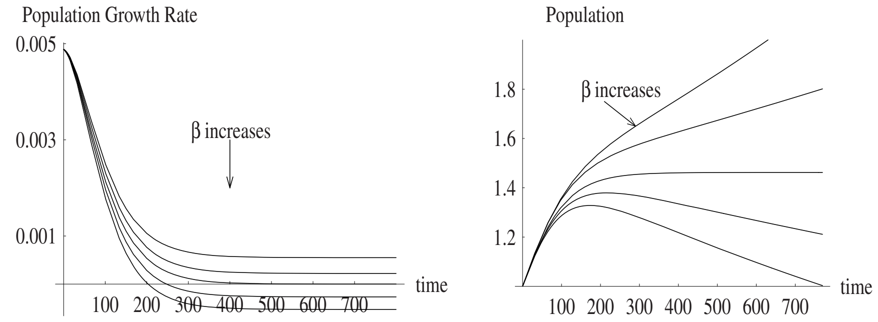

Figure 3: The time paths for the population growth rate $n(t)$ and population size $L(t)$ normalized by the initial value L_0 for the benchmark value $\beta_0 = 0.00038072211$ (the central path) and for its values perturbed for ± 0.0000025 and ± 0.00005.

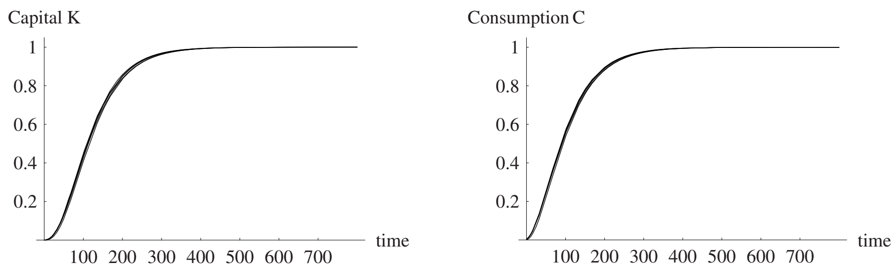

Figure 4: The time paths for capital $K(t)$ and consumption $C(t)$ for the benchmark value $\beta_0 = 0.00038072211$ (the central path) and for its values perturbed for ± 0.0000025 and ± 0.00005.

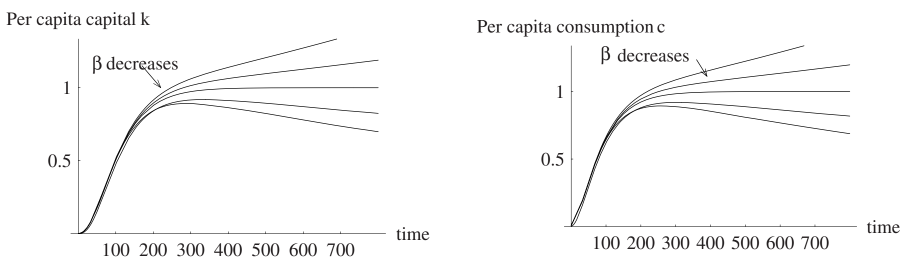

Figure 5: The time paths for per capita capital $k(t)$ and consumption $k(t)$ normalizes on the benchmark steady-state values k_* and c_* respectively for the benchmark value $\beta_0 = 0.00038072211$ (the central path) and for its values perturbed for ± 0.0000025 and ± 0.00005.

also used by most economists (for a review, see Pezzey (1992)). The traditional definition above concentrates on intergenerational equity but pays no attention to the demographic aspect (Lehmijoki (2006)). Figure 5 shows, however, that rising consumption paths lead to ever-decreasing population, a situation that can hardly be considered as sustainable even if the requirement of intergenerational equity is satisfied. Therefore, we define demographic sustainability here as follows:

Definition 1 *A path is demographically sustainable if population is non-decreasing.*

This definition claims that a path leading to decreasing population growth can not be demographically sustainable. In a steady state, the situation is particularly simple because the steady state consumption C^* is constant. The steady state per capita consumption $c^* = C^*/L^*$ grows at the rate $-n = -\dot{L}/L$ and the single steady state keeping both intergenerational equity and demographic sustainability is that at which the population growth rate is zero.[5]

The model suggested here can end up with several outcomes depending upon the values of the parameters. The steady state population growth rate may be negative, implying that the size of the population steadily decreases and, ultimately, goes to zero. In some cases, however, it is optimal to increase consumption to such an extend that demographic sustainability fails. The conflict between demographic sustainability and optimality is particularly striking, because it implies that mankind may go to a deliberate extinction, at least asymptotically.

Should we take demographic sustainability seriously? Can it fail in some observable economies or only in some theoretical cases? It seems possible that environmental degradation can endanger demographic sustainability if it is serious or unexpected, as has been discussed in some recent scenarios of climate change (IPPC (2007)). But demographic sustainability can also fail if population growth is already at a very low level, as it is the case in Europe, where a modest increase in air pollution can increase mortality enough to push population growth below zero.

To see whether Europe is following a demographically sustainable path, we calibrate the environmental mortality for the European data to see how the obtained value for β compares with the benchmark value $\beta = \beta_0$. To this end, note that all mortality estimates discussed in the introduction of this paper are partial in nature, and refer usually to a single pollutant without giving any estimate for pollution in general. Therefore, in this simple exercise we concentrate on one pollutant alone, namely on fine particulate matters $PM_{2.5}$ suggested as most detrimental to human health (WHO (2004a)).

Another difficulty arises because the medical estimates reported in Introduction give mortality numbers in terms of concentrations of pollutants, but not in terms of their emissions as required in (12). Although concentrations have their origins in emissions, the association between these two is not clearly understood yet as

[5]This result is not generic. It follows from the simplicity of our model; if technical progress were included, then the steady state per capita consumption and population could grow together. Moreover, it is possible that complicated models exhibit more complicated off-steady state behavior than that observed here because it is possible that population temporarily decreases but then ultimately levels-off. Because the economy approaches the steady state, we concentrate on steady state situations in this paper.

local weather conditions dictate this association to a great extend (Amann et al. (2007)). We apply two alternative calibration strategies to overcome this difficulty. In the first, we try to change emissions to concentrations. Although we do not know exactly, how emissions turn to concentrations in the nature, we can estimate their average association by regressing the observed concentrations against the observed emissions. Suitable data is available in Amann et al (2007), reporting the local $PM_{2.5}$ emissions and $PM_{2.5}$ concentrations for 470 European cities in 2000. The derived OLS estimate shows that an increase of emissions by one kilo increases its annual mean concentration for $0.00000135 \mu g/m^3$. On the other hand, Pope et al. (2002) have estimated that that there is an 0.006 increase in mortality for each unit (in $\mu g/m^3$) increase in the $PM_{2.5}$ concentration.[6] Multiplying these numbers, we end up with an estimate $\beta = 0.81239E - 8$ which, however, is in magnitude smaller than the calculated benchmark value $\beta_0 = 0.00038072211$.

Several factors increase this basic estimate. First, we have concentrated on one pollutant only, but mortality effects for other pollutants have been reported too (WHO (2004a)). Kappos et al. (2004) suggest that the magnitude of PM_{10} emissions is approximately one and a half times more than the magnitude of $PM_{2.5}$.

Assuming that concentration of PM_{10} emissions follow the same pattern as that of $PM_{2.5}$ emissions,[7] and given the approximately same mortality response, the estimated β can be approximately 1.5 times greater.

Second, it is possible that mortality reacts to peak concentration values rather than to annual averages, reported in Amann et al. (2007). Third, the population in cities may be distributed in such a way that the greatest densities appear in areas which are the most heavily polluted, e.g., by traffic emissions. But even if all these effects are taken into account, the estimate for β seems to hardly reach the benchmark value $\beta = \beta_0$.

Now we turn to an alternative estimation strategy in which we bypass the concentration-emission link by connecting mortality directly to emissions. $CAFE$ and WHO have reported that, at the European level the number of premature deaths due to air pollution was as large as 370, 000 in th eyear 2000, implying that for the 25 European countries, this number was 230, 680 (WHO (2004a)). Given that the annual total $PM_{2.5}$ emission for this area was $1,744,000000$ kilos (Amann et al. (2007)), the equation (12) implies the estimate $\beta = 0.00013227$. This estimate is much bigger than the previous one but still well below the benchmark value $\beta_0 = 0.00038072211$. Note, however, that this calibration strategy also suffers from being based on $PM_{2.5}$ emissions alone. If we correct this estimate upwards in the same way as we corrected the previous one, its value already approaches the critical value β_0.

The big difference between the estimated values for β shows that it is necessary to develop both data and estimation methods to increase the reliability of estimates. Estimation of theoretical models is particularly demanding because the number of parameters may be large. Since each parameter needs its own data, the overall data limitations may be serious. In this paper we have solved these limitations

[6] Equivalently, an 0.06 increase in mortality for each $10 \mu g/m^3$ increase in $PM_{2.5}$, see the Introduction of this paper.

[7] To our knowledge, emission-concentration data is not available for PM_{10}.

by focusing on fine particulate matters $PM_{2.5}$, because for this pollutant we can find data on the emissions, the emission-concentration data, and the concentration-mortality data, all of which are necessary to follow the first estimation strategy. The data limitations in the second estimation strategy are less severe because it bypasses the concentrations by connecting the mortality numbers directly to emissions. But challenges appear here as well because emissions should be considered as aggregates. Therefore, the first-shot calibrations provided in this paper should be considered as methodological exercises rather than as final answers about the demographic sustainability in Europe.

6 Conclusions

This paper provides an infinite-horizon consumer maximization model with population growth endogenous to emissions that are generated in production. There is a trade-off between consumption and population growth because high consumption calls for high production, which leads to high environmental mortality. The model developed here may end up with positive, zero, or negative population growth rate. If population growth is positive, then the optimal path for per capita consumption initially raises but then falls leading ultimately to very low consumption rates. But if population growth is negative, then consumption keeps raising implying that people live in greater and greater affluence but pay for this by increasing numbers of premature deaths.

To illustrate the results of the theoretical model, we provide a calibrated version of it. This calibration shows that European countries proceed on a demographically sustainable path. However, more reliable calibration methods and more reliable data are needed to narrow the gap between several estimates for air pollution data. Therefore, the first-shot calibration provided in this paper should be considered as a methodological exercise. Furthermore, the model itself can be developed in several ways. Above all, it does not contain technical progress which is an essential element in modern production and also in environmental mortality, which may decrease due to such progress in the future. The economy also typically contains both polluting and cleaning industries, whose share is important in evaluating total emissions. Hence, a multi-sector approach should be a step toward more realistic models.

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A Appendix: Pontryagin Maximum Principle

Here we discuss a way to solve problem (9) that can be attributed to the class of optimal control problems with infinite time horizons and mixed constraints. We consider $c = K^\alpha/C \in (0,1]$ as a new control variable. Let us note that zero consumption brings the value of the utility U minus infinity. Obviously such a strategy cannot be optimal. in order to make the utility bounded we set $c = c(\Delta) \in [\epsilon, 1]$, where $\epsilon > 0$ is a small parameter. Then problem (9) takes the form

$$\begin{aligned}
\text{maximize}_{c(\cdot)} \quad U &= \int_0^\infty K(\Delta)^{\alpha(1-\theta)} \frac{c(\Delta)^{1-\theta}}{(1-\theta)(\rho - \theta n(E(\Delta)))} e^{-\Delta} d\Delta, \\
\text{subject to:} \quad E(\Delta) &= \phi K^\alpha(\Delta), \\
\mathring{K}(\Delta) &= \frac{(1-c(\Delta))K(\Delta)^\alpha - \delta K(\Delta)}{\rho - \theta n(E(\Delta))}, \quad K(0) = K_0, \\
c(\Delta) &\in [\epsilon, 1]. \qquad (13)
\end{aligned}$$

In further consideration we accept the following assumption on the population growth rate:

Assumption 3 For all $E > 0$ which appear while functioning the economy

$$n'(E) \geq -r_0, \quad \text{where} \quad r_0 > 0.$$

Aiming at providing necessary conditions of optimality for problem (13) we address to Theorem 10.1 and Corollary 10.3 in Aseev and Kryazhimsky (2007), which require a few formal assumptions. Let us rewrite these assumptions in terms of problem (13) and make sure that they are satisfied. Based on Lemma 1 we get

(A1) there exists $M = Const \geq 0$ such that

$$\frac{K[(1-c)K^\alpha - \delta K]}{\rho - \theta n(E(K))} \leq M(1 + K^2) \quad \text{for all} \quad c \in [\epsilon, 1], \quad K \in [0, \tilde{K}];$$

(A2) for each $K \in [0, \tilde{K}]$ the control-dependent term in the state equation for K, i.e., the function

$$c \mapsto \frac{(1-c)K^\alpha - \delta K}{\rho - \theta n(E)}$$

is affine;

(A3) for each $K \in [0, \tilde{K}]$ the integrand in the utility, i.e., the function

$$c \mapsto K^{\alpha(1-\theta)} \frac{c^{1-\theta}}{(1-\theta)(\rho - \theta n(E))}$$

is concave;

(A4) there exist positive-valued functions $\mu(\cdot)$ and $w(\cdot)$ such that $\mu(\Delta) \to 0$, $w(\Delta) \to 0$ as $\Delta \to 0$, and for any admissible pair (K, c) in problem (13) it holds that

$$\begin{aligned}
e^{-\Delta} \max_{c \in [\epsilon, 1]} K(\Delta)^{\alpha(1-\theta)} \frac{c^{1-\theta}}{(1-\theta)(\rho - \theta n(E(\Delta)))} &\leq \mu(\Delta) \quad \text{for all} \quad \Delta > 0, \\
\int_T^\infty e^{-\Delta} K(\Delta)^{\alpha(1-\theta)} \frac{c(\Delta)^{1-\theta}}{(1-\theta)(\rho - \theta n(E(\Delta)))} &\leq w(T) \quad \text{for all} \quad T > 0;
\end{aligned}$$

(A5) for every admissible pair (K, c) one has

$$\frac{\partial}{\partial K}\left(e^{-\Delta}K(\Delta)^{\alpha(1-\theta)}\frac{c(\Delta)^{1-\theta}}{(1-\theta)(\rho-\theta n(E(\Delta)))}\right) > 0 \quad \text{for all} \quad \Delta \geq 0;$$

(A6) there exists a $c_0 \in [\epsilon, 1]$ such that the corresponding initial velocity of K

$$\dot{K}_0 = \frac{(1-c_0)K_0^{\alpha} - \delta K_0}{\rho - \theta n(E(K_0))}$$

is positive;

(A7) for the optimal admissible pair (K^*, c^*) it holds that

$$\frac{(1-c_*(\Delta))K^*(\Delta)^{\alpha} - \delta K^*(\Delta)}{\rho - \theta n(E(K^*(\Delta)))} \geq a_0 > 0 \quad \text{for a.a. } \Delta \geq \Delta_0,$$

with some $\Delta_0 > 0$.

Lemma 3 *Let Assumptions 1 and 2 hold. Then problem (13) satisfies to assumptions (A1) – (A6).*

Proof. The validity of Assumptions (A1) and (A4) obviously follows from Lemma 1. The "linear-convex" structure of problem (13) with respect to control (Assumptions (A2) and (A3)) is stated directly. We also state directly that directly that the utility's integrand grows, i.e., Assumption (A5) holds; here we use the fact that the population growth rate decreases as capital grows.

By Assumption 2 K_0 lies on the increasing branch of curve $C = K^{\alpha} - \delta K$. Using Assumption 1 and letting $\epsilon > 0$ be sufficiently small, for $c_0 = \epsilon$ we get $(1-c_0)K_0^{\alpha} - \delta K_0 > 0$ and hence (A6) is satisfied.

To satisfy (A7) we introduce the following assumption:

Assumption 4 The optimal capital grows.

Now having made ourselves sure about Assumptions (A1) – (A7) write the necessary conditions of optimality for the problem (13). Let $\lambda = \lambda(\Delta)$ be an adjoint variable. Then the Hamiltonian becomes

$$H(K, c, \lambda) = \frac{1}{\rho - \theta n(E)}\left(\frac{K^{\alpha(1-\theta)}c^{1-\theta}}{1-\theta} + \lambda((1-c)K^{\alpha} - \delta K)\right). \tag{14}$$

The dynamics of the adjoined variable is given by

$$\mathring{\lambda} = -\frac{\partial H}{\partial K} + \lambda$$

or

$$\mathring{\lambda} = -\frac{\theta\alpha\phi n' K^{\alpha-1}}{(\rho-\theta n)^2}H(K, c, \lambda) + \frac{1}{\rho - \theta n}\left(\alpha c^{1-\theta}K^{\alpha(1-\theta)-1} + \lambda(\alpha(1-c)K^{\alpha-1} - \delta)\right). \tag{15}$$

The last equation together with

$$\overset{\circ}{K} = \frac{(1-c)K^{\alpha} - \delta K}{\rho - \theta n} \tag{16}$$

forms the Hamiltonian system for problem (13). The function $c \mapsto H(K, c, \lambda)$ is convex and approaches its maximum if

$$\frac{\partial H}{\partial c} = \frac{1}{\rho - \theta n}(c^{-\theta} K^{\alpha(1-\theta)} - \lambda K^{\alpha}) = 0 \quad \Longleftrightarrow \quad c^{-\theta} = \lambda K^{\alpha\theta}.$$

Recalling that admissible controls lie in the interval $[\epsilon, 1]$, the condition of maximality of the Hamiltonian becomes
(i) if $\lambda < K^{-\alpha\theta}$ then $c \equiv 1$;
(ii) if $\lambda > \epsilon^{-\theta} K^{-\alpha\theta}$ then $c \equiv \epsilon$;
(iii) otherwise $c = \lambda^{-1/\theta} K^{-\alpha}$.
The transversality condition are

$$e^{\Delta}\lambda(\Delta) \to 0 \quad \text{as} \quad \Delta \to \infty \tag{17}$$

and besides

$$\lambda(\Delta) > 0 \quad \text{for all} \quad \Delta \geq 0. \tag{18}$$

In accordance with the Theorem 10.1 and the Corollary 10.3 in Aseev and Kryazhimsky (2007) the equations (14) – (18) together with the maximality conditions (i) – (iii) supply the solution of the problem (13). Let us analyze the behavior of the Hamilton system (15), (16) in three zones corresponding to optimality conditions (i) – (iii). The Figure 6 illustrates these zones on the phase plane of variables λ and K.

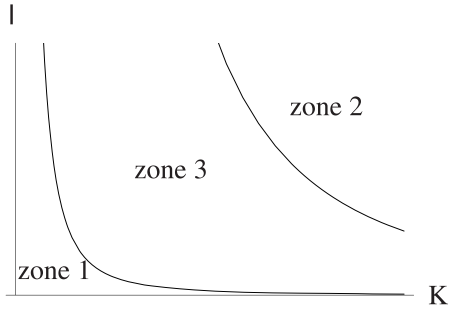

Figure 6: Zones 1-3 on the K-λ plane, corresponding to different maximizators of the Hamiltonian.

Zone 1: $c \equiv 1$.
The Hamiltonian system (15), (16) becomes

$$\overset{\circ}{\lambda} = -\frac{\theta\alpha\phi n' K^{\alpha-1}}{(\rho-\theta n)^2}\left(\frac{K^{\alpha(1-\theta)}}{1-\theta} - \delta K\lambda\right) - \frac{1}{\rho-\theta n}\left(\alpha K^{\alpha(1-\theta)-1} - \delta\lambda\right) + \lambda \tag{19}$$

$$\overset{\circ}{K} = -\frac{\delta K}{\rho - \theta n} < 0. \tag{20}$$

Lemma 4 *Let Assumptions 3 and 4 hold, and the initial capital* K_0 *satisfy to*

$$K_0^{1-\alpha} \geq \frac{\alpha r}{(\delta + r)\rho}. \tag{21}$$

Then trajectories starting from the zone 1 violate the condition (18) and that is why cannot be optimal for the problem (13).

Proof.

Let us estimate the derivative $\overset{\circ}{\lambda}$. Note first that from (20) it follows that in the zone 1 consumption decreases and hence $K \leq K_0$. From this fact and (19) under assumptions of the Statement we have

$$\overset{\circ}{\lambda} \leq \lambda \left(\frac{\delta}{\rho - \theta n(E)} + 1 \right) - \frac{\alpha K_0^{\alpha(1-\theta)-1}}{\rho - \theta n(E)}.$$

Since the expression in the round brackets is positive and in the zone 1 $\lambda < K^{-\alpha\theta} \leq K_0^{-\alpha\theta}$

$$\overset{\circ}{\lambda} \leq K_0^{-\alpha\theta} \left(\frac{\delta}{\rho - \theta n(E)} + 1 \right) - \frac{\alpha K_0^{\alpha(1-\theta)-1}}{\rho - \theta n(E)} = K_0^{-\alpha\theta} \left(\frac{\delta}{\rho - \theta n(E)} + 1 - \frac{\alpha K_0^{\alpha-1}}{\rho - \theta n(E)} \right).$$

The last inequality together with the Assumption 3 and (21) lead to

$$\overset{\circ}{\lambda} \leq K_0^{-\alpha\theta} \left(\frac{\delta}{r} + 1 - \frac{\alpha K_0^{\alpha-1}}{\rho} \right) < 0.$$

Consequently in the considered zone 1 $\overset{\circ}{\lambda} < 0$ and $\overset{\circ}{K} < 0$. It can be shown that λ approach zero for the finite time which violate condition (18).

Zone 2: $c \equiv \epsilon$**.** The Hamiltonian system (15), (16) becomes

$$\begin{aligned} \overset{\circ}{\lambda} &= -\frac{\theta\alpha\phi n' K^{\alpha-1}}{(\rho - \theta n)^2} \left(\frac{K^{\alpha(1-\theta)} \epsilon^{1-\theta}}{1-\theta} + \lambda((1-\epsilon)K^\alpha - \delta K) \right) \\ &\quad - \frac{1}{\rho - \theta n(E)} \left(\alpha \epsilon^{1-\theta} K^{\alpha(1-\theta)-1} + \lambda(\alpha(1-\epsilon)K^{\alpha-1} - \delta) \right) + \lambda \end{aligned} \tag{22}$$

$$\overset{\circ}{K} = \frac{(1-\epsilon)K^\alpha - \delta K}{\rho - \theta n}. \tag{23}$$

Lemma 5 *Let Assumptions 3 and 4 hold, and the initial capital* K_0 *satisfy to*

$$K_0^{1-\alpha} \geq \frac{\alpha}{\rho + \delta}, \tag{24}$$

and besides let

$$(\rho + \delta)\theta\alpha\phi \max_E |n'(E)|(1-\alpha) > \rho^2. \tag{25}$$

Then the trajectories starting from the zone 2 violate transversality condition (17) and cannot be optimal for problem (13).

Proof. From (22) we have

$$\begin{aligned}\overset{\circ}{\lambda} &= \lambda\left(1-\frac{\alpha(1-\epsilon)K^{\alpha-1}-\delta}{\rho-\theta n}-\frac{\theta\alpha\phi n' K^{\alpha-1}[(1-\epsilon)K^{\alpha}-\delta K]}{(\rho-\theta n)^2}\right)\\ &- \frac{\alpha\epsilon^{1-\theta}K^{\alpha(1-\theta)-1}}{\rho-\theta n}-\frac{\theta\alpha\phi n' K^{\alpha-1}K^{\alpha(1-\theta)}\epsilon^{1-\theta}}{1-\theta}.\end{aligned}$$

Let us estimate the multiplier appearing at λ in the right side of the last equation. Taking into account (24), Assumption 3 and Lemma 1 we get

$$1-\frac{\alpha(1-\epsilon)K^{\alpha-1}-\delta}{\rho-\theta n}-\frac{\theta\alpha\phi n' K^{\alpha-1}[(1-\epsilon)K^{\alpha}-\delta K]}{(\rho-\theta n)^2} \geq 1-\frac{\alpha(1-\epsilon)K_0^{\alpha-1}-\delta}{\rho} > 0.$$

The last inequality implies that λ increases exponentially. Let us show that it increases with a rate greater then unit. In order to that we should prove that

$$\frac{\alpha(1-\epsilon)K^{\alpha-1}-\delta}{\rho-\theta n} < -\frac{\theta\alpha\phi n' K^{\alpha-1}[(1-\epsilon)K^{\alpha}-\delta K]}{(\rho-\theta n)^2}$$

or

$$\alpha(1-\epsilon)K^{\alpha-1}-\delta < \frac{\theta\alpha\phi|n'|K^{\alpha-1}[(1-\epsilon)K^{\alpha}-\delta K]}{\rho-\theta n} \tag{26}$$

The right side of (26) is always positive whereas the left side takes the positive value along the increasing branch of the curve $(1-\epsilon)K^{\alpha-1}-\delta K$ and the negative value on the decreasing branch. In the latter case inequality (26) holds automatically. It can be easily shown that condition (25) guarantees (26) in the case positive $(1-\epsilon)K^{\alpha-1}-\delta K$. Thus λ increases exponentially with the rate greater then 1 and hence $\lambda e^{-\Delta}\to\infty$ which violate (17). It means that the paths starting in the zone 2 lie entirely there and cannot be optimal.

Zone 3: $c=\lambda^{-1/\theta}K^{-\alpha}$. It follows from above that controls optimal for problem (13) do not touch the bounds of its admissible values lying strictly inside the interval $[\epsilon,1]$ whereas the corresponding trajectories of K and λ lie necesseraly in the zone 3. Which means that in our further analysis we consider only this case without special stress on that in formulas and figures. The Hamiltonian system turns into

$$\overset{\circ}{\lambda} = -\frac{\theta\alpha\phi n'(E)K^{\alpha-1}}{(\rho-\theta n(E))^2}\left(\frac{\theta}{1-\theta}\lambda^{1-1/\theta}+\lambda(K^{\alpha}-\delta K)\right)-\frac{1}{\rho-\theta n(E)}\lambda(\alpha K^{\alpha-1}-\delta)+\lambda \tag{27}$$

$$\overset{\circ}{K} = \frac{K^{\alpha}-\delta K-\lambda^{-1/\theta}}{\rho-\theta n(E)}. \tag{28}$$

Or in terms of c equation (27) turns into

$$\begin{aligned}\frac{\overset{\circ}{c}}{c} &= \frac{1}{\rho-\theta n(E)}\left(\frac{\alpha\phi n'(E)K^{\alpha-1}}{\rho-\theta n(E)}\frac{K^{\alpha}-\delta(1-\theta)K}{1-\theta}+\frac{1}{\theta}(\alpha K^{\alpha-1}-\delta)-\frac{1}{\theta}(\rho-\theta n(E))\right.\\ &\left.+\alpha\delta-(1-c)K^{\alpha}\left(\frac{\theta}{1-\theta}+\frac{\alpha}{K}\right)\right)\end{aligned} \tag{29}$$

Further analysis (the existence of a steady state, its local stability and optimality) is done in real time t; so we rewrite equations (28), (29) by applying (8).

B Appendix: Local Stability of the Steady State

Lemma 6 *The steady state is a saddle.*

Proof. Let us write $\dot{K} = \varphi(K, C)$ and $\dot{C}/C = \psi(K, C)$. The Jacobian of the model is

$$J = \begin{bmatrix} \varphi_K & \varphi_C \\ \psi_K & \psi_C \end{bmatrix}.$$

As evaluated around the steady state, its elements become

$$\begin{aligned}
\varphi_K &= \alpha K^{\alpha-1} - \delta, \\
\varphi_C &= -1, \\
\psi_K &= \frac{1}{\theta}\left\{\frac{d\left[\theta n'\phi\alpha K^{\alpha-1}/(\rho-\theta n)\right]}{dK}\left(\frac{\theta C}{1-\theta} + K^{\alpha} - \delta K\right) + \frac{\theta n'\phi\alpha K^{\alpha-1}}{\rho-\theta n}\left(\alpha K^{\alpha-1} - \delta\right)\right. \\
&\quad \left. + \alpha\left(\alpha-1\right)K^{\alpha-2} - \theta n'\right\}, \\
\psi_C &= \frac{1}{1-\theta}\frac{\theta n'\phi\alpha K^{\alpha-1}}{\rho-\theta n}
\end{aligned}$$

Because ψ_K contains the undefined second derivative of $n(E)$, we write

$$\begin{aligned}
DET\ J &= \varphi_K \cdot \psi_C - \psi_K \cdot \varphi_C \\
&= \left[\left(-\frac{\varphi_K}{\varphi_C}\right) - \left(-\frac{\psi_K}{\psi_C}\right)\right](-\varphi_C)\cdot\psi_C.
\end{aligned}$$

The expression $(-\varphi_C)\cdot\psi_C = \frac{1}{1-\theta}\frac{\theta n'\phi\alpha K^{\alpha-1}}{\rho-\theta n}$ is positive. The expression in the square brackets is the difference in the slopes of the phase lines $\dot{K} = 0$ and $\dot{C}/C = 0$. In the steady state, the $\dot{C}/C = 0-$line hits the $\dot{K} = 0-$line from below and this expression is negative, implying $DET\ J < 0$. Therefore, the steady state is a saddle.

The dynamics outside the steady state is the following: because $\varphi_C = -1$, the capital stock increases (decreases) below (above) the $\dot{K} = 0-$line. The behavior of consumption is given by $\psi_C = \frac{1}{1-\theta}\frac{\theta n'\phi\alpha K^{\alpha-1}}{\rho-\theta n} > 0$. Therefore, consumption increases (decreases) above (below) the $\dot{C} = 0-$line. Hence, the stable saddle paths approach the steady state from the South-West and North-East (see Figure 2).

C Appendix: Optimality of the Saddle Paths

Lemma 7 *Let Assumptions 1 and 2 be satisfied. Then the stable saddle paths are the single optimal paths for problem (7).*

Proof. To demonstrate that the stable saddle paths dominate all other candidates for optimal paths, consider the transversality condition written in real time

$$\lim_{t\to\infty} \lambda(t)e^{-\Delta(t)} = 0.$$

A stable saddle path leading to an interior steady state by definition satisfies to

$$\lim_{t\to\infty} K(t) = K^* \in (0,\infty) \quad \text{and} \quad \lim_{t\to\infty} C(t) = C^* \in (0,\infty).$$

The first order conditions, the transversality condition turns into

$$\lim_{t\to\infty} C(t)^{-\theta} e^{-\Delta(t)} = 0.$$

Since from the property (ii) of the function $\Delta(\cdot)$ it holds that $\lim_{t\to\infty} e^{-\Delta(t)} = 0$, we state that any stable path satisfies to the transversality condition.

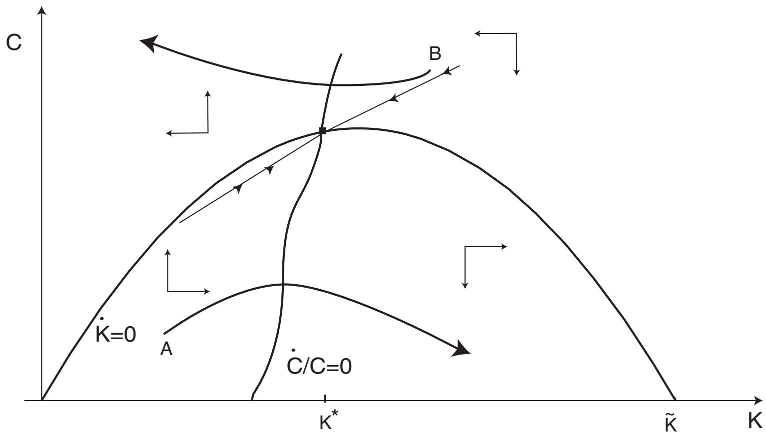

Figure 7: Optimal paths.

The candidate paths which lie in the South-East (see Figure 7) cannot be optimal since Assumption 2 doesn't hold for them. Consider the candidate paths which lie in the North-West. In this area $\dot{C} > 0$ and consumption increases unless it touches the upper limit K^α. It was shown (see Lemma 4 of Appendix A) that such paths cannot be optimal. Therefore, the stable saddle paths are the single optimal paths.

D Appendix: Assumptions

Analyzing we see that all the theory and calculations presented in this paper has been carried out under a number of assumptions. The reason for introducing them was rather technical. Here we summarize them, discuss their compatibility and physical meaning.

Assumption 1 guarantees the boundedness of the integral utility which is a necessary condition of the correctness of the formulation of any optimal control problem. Here it turns into the constraint on population growth rate values, namely, we treat it not to exceed some level during functioning the economy. What will happen otherwise? Exponentially growing term $e^{-\int_0^t \{\rho - \theta n(E(\tau))\} d\tau}$ will lead instantenous utility to minus infinity – hence such trajectories can not be optimal.

Then we reinforce Assumption 1 moving to the Assumption 3. It strengthens the upper limit for population growth rate values and besides prohibit it to decrease too fast. In this paper we consider the linear respond of population growth rate to emissions (curve A at Figure 1) so Assumption 3 turns into the constraint on β and steady capital K^* (see the item (a1) below).

In order to prove the fact that the optimal path doesn't touch the extreme consumption modes we introduced two constraints on the initial capital K_0 in Lemmas 4 and 5. Both conditions (21) and (2) can be replaced by a weaker sufficient condition $K_0^{1-\alpha} \geq \alpha/\rho$ which we keep as a next assumption.

Besides Statement 5 requires condition (25) which supplements the lower constraint on the derivative $n'(E)$ made in Assumption 3 by an upper constraint. Due to r_0 is an arbitrary finite number this bilateral constraints is correct.

Finally we come to assumptions concerning the optimal path (Assumptions 2 and 4) which tell that the optimal capital grows such that asymptotically it achieves its steady value not greater then $(\alpha/\delta)^{\frac{1}{1-\alpha}}$.

Summarizing we come to the following number of assumptions:

(a1) there exist such γ_0, γ_1 and γ_2 that for all E appearing while functioning the economy $n(E) \leq \gamma_0$, $\gamma_1 \leq n'(E) \leq \gamma_2$;

(a2) $K_0^{1-\alpha} \geq \alpha/\rho$;

(a3) the optimal capital grows such that asymptotically it achieves its steady value not greater then $(\alpha/\delta)^{\frac{1}{1-\alpha}}$.

For the illustrative numerical calculations we present in this paper conditions (a1) – (a3) are satisfied which confirms non-contradictiveness of them.