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# Dynamic Stability of the Replicator Equation with Continuous Strategy Space

**Cressman, R.**

**I I A S A**

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# Dynamic Stability of the Replicator Equation with Continuous Strategy Space

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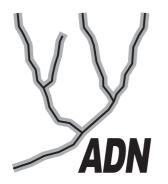
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# Contents

# Abstract

We extend previous work that analyzes the stability of dynamics on probability distributions over continuous strategy spaces. The stability concept considered is that of convergence to the equilibrium distribution in the strong topology for all initial distributions whose support is close to this equilibrium. Stability criteria involving strategy domination and local superiority are developed for equilibrium distributions that are monomorphic (i.e. the equilibrium consists of a single strategy) and for equilibrium distributions that have finite support.

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# Dynamic Stability of the Replicator Equation with Continuous Strategy Space

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## 1 Introduction

Evolutionary dynamics for continuous strategy spaces have received considerable recent attention both among theoretical biologists who are interested in the coevolution of species traits and among economists who concentrate instead on predicting rational behavior of individuals whose payoffs are given through game interactions. Most theoretical research on dynamic stability for coevolutionary models (e.g. Abrams, 2001; Doebeli and Dieckmann, 2000; Marrow et al., 1996 and the references therein) make the simplifying assumption each species is monomorphic (or homogeneous) and remains so during the course of evolution (i.e. all individuals present in a given species exhibit the same behavior).[1] This leads to the stability analysis of what are known as adaptive or strategy dynamics. Although there has been much less research in this area from the economic or game-theoretic perspective (e.g. Bomze, 1990, 1991; Seymour, 2000; Oechssler and Riedel, 2001, 2002), this literature typically considers the full dynamical system where aggregate behavior is described by a distribution on the space $S$ of individual strategy choices and assume individual payoffs are defined in terms of a function $f$ on $S \times S$.

In this paper, we follow the latter approach applied to a symmetric game with a continuous strategy space.[2] These references in the economic literature spend a great deal of time developing the evolutionary dynamic on the set of probability distributions (e.g. the replicator dynamic), proving its solutions are well-defined, and relating its properties to static equilibrium conditions (that generalize those for the case where there are a finite set of strategies that may be used by the population as in the matrix games of Section 2.1). We benefit from their work by briefly summarizing this development at the beginning of Section 2 and devoting the remainder of the paper to analyzing the dynamic stability of equilibrium distributions for the replicator dynamic.

Immediate issues that arise in this dynamic analysis are what constitutes closeness and/or convergence for probability distributions and for what initial distributions we expect this convergence. The main problem is that there are several ways to define these topological concepts that generalize the accepted approach when there are a finite set of strategies. Moreover, as clearly demonstrated by Oechssler and Riedel (2002), the conclusions related to dynamic stability depend critically on which definitions are taken. These issues are clarified in Section 2.1 by referring to well-known dynamic stability results for matrix games and, in the process, motivate our dynamic stability concept (at the beginning of Section 3).

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[1] Notable exceptions are Vincent et al. (1996) and Cressman and Garay (2003) where the effects of varying population size are also taken into account.

[2] In biological terms, we are then studying single-species coevolutionary models.

Our main goal is to derive conditions on $f$ that predict dynamic stability. Section 3 then completely characterizes (Theorems 1 and 2) dynamic stability of equilibrium distributions concentrated at a single strategy and relates these results to the coevolutionary literature for monomorphic populations (Remark 1). Section 4 extends these results to equilibrium distributions with finite support, giving sufficient conditions for dynamic stability (Theorems 3 and 4). Section 5 summarizes the methods and discusses their application from the economic perspective.

## 2    The Model

Evolutionary games with an arbitrary strategy space and their corresponding replicator dynamics have been developed by a number of researchers over the past fifteen years. Here we briefly summarize this development as it applies to our model. In general, individuals are assumed to play a strategy in the set $S$ and the population state is given by a probability measure $P$ with respect to a measure space $(S, \mathcal{B})$. If $B \in \mathcal{B}$, $P(B)$ is interpreted as the proportion of individuals in the population who are using strategies in the set $B$. For a given $s \in S$, $\delta_s$ denotes the Dirac delta measure that assigns unit mass to $\{s\}$. Let $\pi(s, P) = \pi(\delta_s, P)$ denote the expected payoff to an individual using strategy $s$ when the population is in state $P$. The mean payoff to a random individual in the population with state $P$ is then $\pi(P, P) \equiv \int_S \pi(\delta_s, P) P(ds)$.

For us, $S$ will be a nonempty compact subset of $\mathbf{R}^n$ and $\mathcal{B}$ will be the Borel subsets of $S$ (i.e. the $\sigma-$algebra of the Borel sets of $\mathbf{R}^n$ intersected with $S$ and so $P$ is a Borel measure). Let $\Delta(S)$ denote the set of Borel probability measures with respect to $(S, \mathcal{B})$. Since $P$ is a Borel measure, there is a unique closed subset of $S$, called the support of $P$, such that the measure of its complement is 0 but every open set that intersects it has positive measure (Royden, 1988). We will be most interested in the situation where the payoff function $\pi(s, P)$ is given through a continuous real-valued function $f : S \times S \longrightarrow \mathbf{R}$ by $\pi(s, P) = \int_S f(s, y) P(dy)$. Unless otherwise stated, we will assume the existence of such an $f$ for the remainder of the paper. In particular, standard symmetric evolutionary games that assume random pairwise interactions may be put in this form.[3]

We assume the replicator dynamic (1) describes how the population state evolves (i.e. its solutions define trajectories $P(t)$ in $\Delta(S)$).

$$\frac{dP}{dt}(B) = \int_B \left( \pi(\delta_s, P) - \pi(P, P) \right) P(ds) \qquad (1)$$

Heuristically, this dynamic increases the probability of those sets of strategies $B$ that have a higher expected payoff than the mean payoff to a random individual in the population. It has been shown (Oechssler and Riedel, 2001) that there is a unique solution that satisfies this dynamic for all positive $t$ given any initial probability measure $P(0)$ with compact support when $f$ is continuous. Here $B$ is a Borel subset of $S$ and $\frac{dP}{dt}(B)$ at time $t$ is defined to be $\lim_{h \to 0} \frac{P(t+h) - P(t)}{h}(B)$ with respect to the variational norm (see (4) below). Furthermore, the support of $P(t)$ is the same as $P(0)$ for all $t \geq 0$. A population state $P^*$ is an equilibrium of (1) if and only if $\pi(\delta_s, P^*) - \pi(P^*, P^*) = 0$ for all $s \in$supp$(P^*)$.

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[3]In fact, Bomze and Pötscher (1989) argue that the existence of such an $f$ means the evolutionary game can be interpreted as being based on pairwise interactions. It is only the form of $\pi(s, P)$ that is important to us, not whether players are competing pairwise.

The main purpose of this paper is to analyze the dynamic stability of an equilibrium state $P^*$. Heuristically, dynamic stability of $P^*$ refers to the question of whether $P(t)$ stays close and/or evolves to $P^*$ if the initial $P(0)$ is chosen appropriately in $\Delta(S)$. From Oechssler and Riedel (2002), it is clear that the answers to the stability question depend critically on the concept of closeness of probability measures (i.e. on the topology used for the space of Borel probability measures), especially when the strategy space is not a discrete subset of $\mathbf{R}^n$. We will return to this issue after the following section that begins with a finite strategy space.

## 2.1  Matrix Games

Standard matrix games emerge when there are a finite number of possible strategies individuals may use. If there are $m$ such strategies, they are often thought of as "pure strategies" and then identified with the unit coordinate vectors $e_i = (0, ..., 0, 1, 0, ..., 0)$ in $\mathbf{R}^m$ that have 1 in the $i^{th}$ component and 0 everywhere else. In our notation, $P(0)$ then has finite support contained in $\{e_1, ..., e_m\} = S$ where $S$ is now a subset of $\mathbf{R}^m$. Then $P(t) = \sum_{i=1}^m p_i(t)\delta_{e_i}$ where $p_i(t)$ is the proportion (or frequency) of individuals in the population using strategy $e_i$ at time $t$. From (1), the replicator equation becomes

$$\dot{p}_i = p_i \left( \sum_{k=1}^m f(e_i, e_k)p_k - \sum_{j,k=1}^m f(e_j, e_k)p_j p_k \right).$$

This dynamic can be rewritten in matrix form where $A$ is the $m \times m$ payoff matrix with entries $A_{ij} = f(e_i, e_j)$ as

$$\dot{p}_i = p_i (e_i - p) \cdot Ap. \tag{2}$$

Here $u \cdot Av$ is the standard inner product of column vectors $u$ and $Av$ in $\mathbf{R}^m$ (i.e. $u \cdot Av = \sum_{j,k=1}^m u_j A_{jk} v_k$) and $p$ is the frequency vector in the $m-1$ dimensional simplex $\Delta^m = \{(p_1, ..., p_m) \mid \sum p_i = 1, p_i \geq 0\}$. Since there is a $1-1$ correspondence between $\Delta(S)$ and $\Delta^m$, dynamic stability of $P^*$ becomes the stability of $p^* \in \Delta^m$ with respect to the dynamical system (2) on $\Delta^m$. It is well known that a "matrix" ESS[4] is a locally asymptotically stable equilibrium of (2) and that the converse is not true for all matrix games when there are more than two pure strategies (i.e. $m \geq 2$).

The matrix ESS concept may be developed through conditions of dynamic stability of a strategy $p^*$ when invaded by a mutant strategy $q$. This involves the extension of the above "pure-strategy" matrix game to the so-called "mixed strategy" model with a continuous strategy space. For a matrix game with $m$ pure strategies and $m \times m$ payoff matrix $A$, an individual is now allowed to play a mixed strategy $q \in \Delta^m$ where $q_i$ is then interpreted as the probability this individual will play strategy $e_i$ in a given contest. Thus $S$ becomes all of $\Delta^m$ and $P$ is a Borel probability measure on this continuous strategy space. Let $\overline{p} \in \Delta^m$ be the mean strategy $\int_{\Delta^m} qP(dq)$ of the population state $P$. In this mixed strategy model, it is assumed that $\pi(e_i, P) = \sum_k A_{ik}\overline{p}_k$ and that $\pi(q, P) = \sum_i q_i \pi(e_i, P)$.

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[4]Since the term ESS is overused in the literature, it may have several meanings for some readers. We restrict its use in this paper to that of an evolutionarily stable strategy $p^* \in \Delta^m$ of an $m \times m$ payoff matrix $A$ (hence a "matrix" ESS) as developed by Maynard Smith (1982) where there is one universally accepted meaning. Such a $p^*$ is defined to be a Nash equilibrium (i.e. it satisfies $p \cdot Ap^* \leq p^* \cdot Ap^*$ for all $p \in \Delta^m$) that also fulfills the stability condition $p \cdot Ap < p^* \cdot Ap$ whenever $p \cdot Ap^* = p^* \cdot Ap^*$ and $p \neq p^*$. This ESS concept is then equivalent to the requirement that $p \cdot Ap < p^* \cdot Ap$ for all $p \in \Delta^m$ sufficiently close (but not equal) to $p^*$ (Hofbauer and Sigmund, 1998; Cressman, 2003).

In particular, individual payoffs depend linearly on both the mean population state and on the components of the individual's mean strategy.

To analyze stability in our nonlinear models, it is instructive to consider it first in this mixed strategy model when $p^*$ is invaded by a mutant strategy $q$. To this end, suppose that $p^*$ is an ESS of the payoff matrix $A$ and $q$ is some other mixed strategy sufficiently close to $p^*$. If $\text{supp}(P(0)) = \{p^*, q\}$ then the mixed strategy replicator dynamic that results from (1) is

$$\frac{dP}{dt}(\{p^*\}) = P(\{p^*\}) \, (p^* - \overline{p}) \cdot A\overline{p}$$

where $\overline{p} = P(\{p^*\})p^* + P(\{q\})q$. Since $\overline{p}$ is on the line segment from $q$ to $p^*$ (in particular, $\overline{p}$ is closer to $p^*$ than $q$ is) and $p^*$ is a matrix ESS, $\frac{dP}{dt}(\{p^*\}) > 0$ for all $0 < P(\{p^*\}) < 1$. Thus $p^*$ is globally asymptotically stable.[5]

In fact, $p^*$ is a matrix ESS if and only if $p^*$ is globally asymptotically stable for all these two-strategy dynamics with $\text{supp}(P(0)) = \{p^*, q\}$ and $q$ sufficiently close (but not equal) to $p^*$ (Cressman, 1992). It is the generalization of this result to non-matrix symmetric games that motivates our dynamic stability concept introduced in the following section.

We will also need to refer to the general mixed-strategy matrix game model with arbitrary $P(0)$. Here, the replicator dynamic (1) becomes

$$\frac{dP}{dt}(B) = \int_B (q - \overline{p}) \cdot A\overline{p}P(dq). \tag{3}$$

Akin (1982) shows that the evolution of the strategy distribution $P(t)$ is completely determined by the initial distribution $P(0)$ and the evolution of the mean strategy $\overline{p(t)}$. In particular, $\overline{p}$ converges to an ESS $p^*$ whenever $p^*$ is in the convex hull of $\text{supp}(P(0))$ and $\text{supp}(P(0))$ is sufficiently close to $p^*$.

# 3 Stability of Monomorphic Populations

Our concept of dynamic stability requires a topological notion of closeness on the set of probability measures. There are several topologies on $\Delta(S)$ that are all equivalent to the Euclidean topology on $\Delta^m$ when $S$ is a finite set with $m$ strategies. The most important for us is the strong topology based on the variational norm (Bomze, 1990, 1991; Oechssler and Riedel, 2001) defined by

$$\| P - Q \| \equiv 2 \sup_{B \in \mathcal{B}} | P(B) - Q(B) | \tag{4}$$

for $P, Q \in \Delta(S)$. That is, an open set in the strong topology is one for which every $P$ in it contains a ball of positive radius with respect to this norm centred at $P$.

If $Q$ in (4) is the monomorphic population $\delta_s$ where all individuals in the population use the same strategy $s$, then

$$\| P - \delta_s \| \equiv 2 \max \left\{ \sup_{s \in B} | P(B) - 1 |, \sup_{s \notin B} P(B) \right\}$$
$$= 2 \max \{ | P(\{s\}) - 1 |, P(S \backslash \{s\}) \}$$
$$= 2(1 - P(\{s\})).$$

---

[5]This is a slight abuse of notation that should not cause the readers undue confusion. Formally, in terms of the measure $P$, $P(\{p^*\})$ monotonically increases to 1 (as $P(\{q\})$ decreases to 0) and so $P(t)$ evolves to $\delta_{p^*}$ in the strong topology (see Section 3) under (1).

Thus, $P(t)$ evolves to $\delta_s$ in the strong topology if and only if $P(\{s\})$ converges to 1. By default, every $\delta_s$ is an equilibrium of (1). The question in this section then becomes which, if any, $s^* \in S$ corresponds to a dynamically stable equilibrium $\delta_{s^*}$ of (1).

Generalizing the result for the matrix game model of Section 2.1 when $p^*$ is invaded by a mutant strategy $q$, our dynamic stability concept requires $P(t)$ to converge to $\delta_{s^*}$ in the strong topology if $\mathrm{supp}(P(0)) = \{s^*, s\}$ for all $s$ sufficiently close (but not equal) to $s^*$. In biological terms, the monomorphic population $\delta_{s^*}$ resists invasion by any mutant strategy $s$ sufficiently close to $s^*$ (whether this mutant is rare or not). From Section 2, $\pi(\delta_{s^*}, P) = f(s^*, s^*)P(\{s^*\}) + f(s^*, s)P(\{s\})$, $\pi(\delta_s, P) = f(s, s^*)P(\{s^*\}) + f(s, s)P(\{s\})$ and $\pi(P, P) = P(\{s^*\})\pi(\delta_{s^*}, P) + P(\{s\})\pi(\delta_s, P)$. Thus

$$\frac{dP}{dt}(\{s^*\}) = P(\{s^*\})P(\{s\})\left[(f(s^*, s^*) - f(s, s^*))P(\{s^*\}) + (f(s^*, s) - f(s, s))P(\{s\})\right]$$

and so $\frac{dP}{dt}(\{s^*\}) > 0$ for all $0 < P(\{s^*\}) < 1$ if and only if

$$f(s^*, s^*) \geq f(s, s^*) \text{ and } f(s^*, s) \geq f(s, s) \tag{5}$$

with strict inequality in at least one of these for all $s$ sufficiently close (but not equal) to $s^*$. These inequalities state that, for all $s$ sufficiently close (but not equal) to $s^*$, $s^*$ weakly dominates $s$ in the two-strategy game between $s^*$ and $s$ with payoff matrix

$$A = \left[\begin{array}{cc} f(s^*, s^*) & f(s^*, s) \\ f(s, s^*) & f(s, s) \end{array}\right].$$

The above discussion proves the result summarized in the following statement.

Suppose $s^*$ weakly dominates $s$ in the two-strategy game between $s^*$ and $s$ for all $s$ sufficiently close (but not equal) to $s^*$. Then, for all such $s$, $P(t)$ converges to $\delta_{s^*}$ in the strong topology if $\mathrm{supp}(P(0)) = \{s^*, s\}$. The converse is also true.

Theorem 3 is the analogue of the matrix ESS concept developed as a strategy that resists invasion by a mutant. As noted at the end of Section 2.1, a matrix ESS $p^*$ is also related to dynamic stability in the mixed-strategy matrix game model. The corresponding question here becomes how the conditions of Theorem 1 relate to dynamic stability of the measure-theoretic replicator equation (1). In the remainder of this section, we consider dynamic stability of a homogeneous population $s^*$ in the interior of the strategy space $S$.

To begin with, we require any initial $P(0)$, with $\mathrm{supp}(P(0))$ sufficiently close to $s^*$ and $P(\{s^*\}) > 0$ initially, to converge to $\delta_{s^*}$ in the strong topology.[6] Weak domination is no longer sufficient in general (although it remains a necessary condition since the converse of Theorem 3 must still hold). To see this, consider the mixed-strategy matrix model with $\mathrm{supp}(P(0)) = \{p^*, p^* + \epsilon(p - p^*), p^* - \epsilon(p - p^*)\}$ and $p^*$ an ESS of $A$. Then $\bar{p} = p^*$ whenever $P(\{p^* + \epsilon(p - p^*)\}) = P(\{p^* - \epsilon(p - p^*)\})$ and $\epsilon$ is a small nonzero number with $p^* \pm \epsilon(p - p^*) \in \Delta^m$. Thus, every such state with $\epsilon$ small is a rest point of the replicator dynamic and so $P(t)$ does not converge to $\delta_{p^*}$. At best $\delta_{p^*}$ may be neutrally stable; although, in this mixed strategy model the population mean strategy does converge to $p^*$ (see (3)).

To avoid this type of neutral stability, let us assume that, for all $s$ sufficiently close (but not equal) to $s^*$, $s^*$ strongly dominates $s$ in the two-strategy game between $s^*$ and $s$. That is, for all such $s$,

$$\begin{array}{l} f(s^*, s^*) > f(s, s^*) \\ f(s^*, s) > f(s, s). \end{array} \tag{6}$$

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[6]In measure theory, a measure $P$ is *absolutely continuous* with respect to $Q$ if, for all $B \in \mathcal{B}$, $P(B) > 0$ implies $Q(B) > 0$. Thus, the assumption $P(\{s^*\}) > 0$ states that $\delta_{s^*}$ is absolutely continuous with respect to $P$ as a measure.

In particular, from the first inequality, $s^*$ is locally a strict NE. For technical reasons (see Remark 1 below), we will also assume that this domination is determined by the second order Taylor expansion of $f(x,y)$ and that $s^*$ is in the interior of $S$. In particular, the function $f(x,y)$ has continuous partial derivatives up to second order. For example, when $S$ is a subset of $\mathbf{R}$, the Taylor expansion is

$$f(s,y) = f(s^*,s^*) + f_1\,(s - s^*) + f_2\,(y - s^*) \tag{7}$$
$$+ \frac{1}{2}\left[ f_{11}\,(s - s^*)^2 + 2f_{12}\,(s - s^*)\,(y - s^*) + f_{22}\,(y - s^*)^2 + \text{ h.o.t.}\right]$$

where $f_1$ is the first order partial derivative of $f(x,y)$ evaluated at $(s^*,s^*)$ with respect to the first variable, etc. Since $s^*$ is in the interior of $S$, $f_1 = 0$. By ignoring the higher order terms, the two inequalities in (6) become

$$f_{11} < 0 \text{ and } f_{11} + 2f_{12} < 0 \tag{8}$$

respectively.

With these assumptions, we then have the following result.

Suppose $s^*$ is in the interior of $S \subset \mathbf{R}^n$ and that domination in the two-strategy game between $s^*$ and $s$ for all $s$ sufficiently close (but not equal) to $s^*$ is determined by the second order Taylor expansion of $f(x,y)$ about $x = y = s^*$. If $P(\{s^*\}) > 0$ initially, $\text{supp}(P(0))$ is sufficiently close to $s^*$ and $s^*$ satisfies (6) for all $s$ in a neighborhood of $s^*$, then $P(t)$ converges to $\delta_{s^*}$ in the strong topology.

**Proof.** We restrict the proof here to the case of a one dimensional strategy space (i.e. $S$ is a compact subset of $\mathbf{R}$). The general proof is in the Appendix. From (1),

$$\frac{dP}{dt}(\{s^*\}) = P(\{s^*\})\left(\pi(\delta_{s^*}, P) - \pi(P,P)\right)$$
$$= P(\{s^*\})\left( \int_S f(s^*, y)P(dy) - \int_S \int_S f(s,y)P(dy)P(ds)\right)$$
$$= P(\{s^*\}) \int_S \int_S \left(f(s^*,y) - f(s,y)\right)P(dy)P(ds).$$

From (7),

$$f(s^*,y) - f(s,y) = -\frac{1}{2}f_{11}\,(s - s^*)^2 - f_{12}\,(s - s^*)\,(y - s^*) + \text{ h.o.t.}$$
$$\cong -\frac{1}{2}\left[ f_{11}\left((s - s^*)^2 - (s - s^*)\,(y - s^*)\right) + (f_{11} + 2f_{12})\,(s - s^*)\,(y - s^*)\right].$$

Now $\int_S \int_S \left((s - s^*)^2 - (s - s^*)\,(y - s^*)\right)P(dy)P(ds) = \int_S (s - \overline{s})^2 P(ds)$ where $\overline{s} = \int_S sP(ds)$ is the mean strategy of the population and $\int_S \int_S (s - s^*)\,(y - s^*)\,P(dy)P(ds) = (\overline{s} - s^*)^2$. Thus, from (8),

$$\int_S \int_S \left(f(s^*,y) - f(s,y)\right)P(dy)P(ds) \cong -\frac{1}{2}\left[ f_{11}\left(\int_S (s - \overline{s})^2 P(ds)\right) + (f_{11} + 2f_{12})\,(\overline{s} - s^*)^2\right]$$
$$\geq 0$$

with strict inequality whenever $\text{supp}(P)$ is sufficiently close to $s^*$, unless $\overline{s} = s^*$ and $P = \delta_{\overline{s}}$. Since $P(\{s^*\}) > 0$, $\frac{dP}{dt}(\{s^*\}) > 0$ unless $P = \delta_{s^*}$ if $\text{supp}(P(0))$ is sufficiently close to $s^*$. Thus $P(\{s^*\})$ converges to 1.

By Theorem 3, if $P(t)$ converges to $\delta_{s^*}$ in the strong topology for all initial $P(0)$ with support sufficiently close to $s^*$ and $P(\{s^*\}) > 0$, then $s^*$ weakly dominates in the two-strategy game between $s^*$ and $s$ for all $s$ sufficiently close (but not equal) to $s^*$. This gives a partial converse of Theorem 3. In the special case when there are no terms higher than quadratic in the Taylor expansion of $f(x, y)$, the proof of Theorem 3 immediately shows the following result on global convergence.

If $f(x, y)$ is a quadratic polynomial and $s^*$ strongly dominates $s$ in the two-strategy game between $s^*$ and $s$ for all $s$ sufficiently close (but not equal) to $s^*$, then $P(t)$ converges to $\delta_{s^*}$ in the strong topology whenever $s^*$ is in the interior of $S \subset \mathbf{R}^n$ and $P(\{s^*\}) > 0$.

The negativity conditions on $f_{11}$ and $f_{11} + 2f_{12}$ in (8), that follow from the Taylor expansion of (6) when $S$ is one dimensional, have received considerable attention in the coevolutionary literature. The first is often called the ESS criterion for $s^*$ when the payoff function is nonlinear in its first variable since it implies that a monomorphic population $s^*$ cannot be invaded by a rare mutant strategy $s$, paralleling the intuition developed by Maynard Smith (1982). However, as remarked earlier, this term is potentially ambiguous and so the phrase "local strict NE" is preferred.

The second negativity condition is that $s^*$ is a NIS (*neighborhood invader strategy*) (McKelvey and Apaloo, 1995; Apaloo, 1997). It is interesting to note that together these two inequalities imply that $f_{11} + f_{12} < 0$, which is one of the early criteria (Eshel, 1983) for stability of homogeneous populations called CSS (*continuously stable strategy*).

Situations where the Taylor expansion is not valid have also been considered. For example, Seymour (2000) has strengthened the strictness concept to that of a super-strict NE whereby, for some $\varepsilon > 0$, $f(s^*, s^*) > f(s, s^*) + \varepsilon$ whenever $s \neq s^*$. In particular, $f$ is not continuous in its first variable. However, if $f$ is still continuous in its second first variable, $f(s^*, y) - f(s, y) > 0$ for all $(s, y)$ sufficiently close (but not equal) to $(s^*, s^*)$ with $s$ different than $s^*$. The above method of proof then shows Theorem 3 remains valid in these circumstances, a result that also follows from Seymour's analysis.

Theorems 3 and 3 cannot be extended to initial $P(0)$ that have most of their support near $s^*$. To see this, consider the following example taken from Oechssler and Riedel (2002) who considered only one dimensional continuous strategy spaces. Let $f(x, y) = -x^2 + x^2 y^2$. It is straightforward to confirm that $s^* = 0$ strongly dominates $s$ in the two-strategy game between $s^*$ and $s$ for all $0 < |s| < 1$. However, $\frac{dP}{dt}(\{s^*\}) < 0$ if $\text{supp}(P(0)) = \{s^*, s\}$ and $P(\{s\})s^4 - s^2 > 0$, and so $P(t)$ does not converge to $\delta_{s^*}$ in the strong topology for all initial $P(0)$ if $|s| > 1$. The problem here is that initial states can be close to $\delta_{s^*}$ in the strong topology without the Hausdorff distance (see Section 4 below) between their compact supports being small.

This problem led Oechssler and Riedel (2002) to consider other topologies[7] on the set of probability measures and introduce other stronger static conditions that potentially imply dynamic stability with respect to this topology. Their most promising static concept (see evolutionary robustness at the end of Section 4) is based on intuitive dynamic stability for initial $P(0)$ that allow for "both a large change of strategic play by a small fraction of players as well as a small change of strategic play by a large fraction of the population" (which is an informal description of the weak topology). Unfortunately, they were unable to prove a general dynamic stability result which gives a main impetus for the approach adopted in this paper.

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[7]Specifically, they consider the weak topology whereby $P(t)$ converges to $P$ in the weak topology if $\int_S g(s)dP(t)(s)$ converges to $\int_S g(s)dP(s)$ for all continuous functions $g$ on $S$. Then $P(t)$ converging to $P$ in the strong topology implies weak convergence but not conversely.

Oechssler and Riedel (2002) also provide an example to show that the local strict NE condition, $f_{11} < 0$, is not sufficient for even "local" dynamic stability with respect to the weak topology when there is a continuous strategy space. Specifically, with $f(x, y)$ the quadratic polynomial $-x^2 + 4xy$, $\delta_{s^*}$ with $s^* = 0$ is a local strict NE but nearby (with respect to the weak topology) initial distributions $P(0)$ do not converge to $\delta_{s^*}$. This contrasts to the situation for finite strategy spaces where the local asymptotic stability of a (local) strict NE is one of the main results of the Folk Theorem of Evolutionary Game Theory (Hofbauer and Sigmund, 1998; Cressman, 2003).

## 4 Stability of Dimorphic Populations

In contrast to the study of stability for monomorphic populations $P^* = \delta_{s^*}$, very little research has been done that analyzes the dynamic stability of a general equilibrium of the replicator dynamic. Although the main result (Theorem 4 below) applies to all $P^*$ with finite support, we concentrate here on the most elementary extension; namely, when $\text{supp}(P^*)$ has two strategies $\{s^*, r^*\}$ (hence, a dimorphism). To generalize Theorem 3, we want conditions for which $P(t)$ converges to $P^*$ in the strong topology if $P(\{s^*\})P(\{r^*\}) > 0$ initially and $\text{supp}(P(0))$ is sufficiently close to $\{s^*, r^*\}$.[8]

In particular, when $\text{supp}(P(0)) = \{s^*, r^*\}$, $P^*$ must be globally asymptotically stable for the one-dimensional replicator dynamic which is

$$\frac{dP}{dt}(\{s^*\}) = P(\{s^*\})P(\{r^*\})\left[(f(s^*, s^*) - f(s^*, r^*))P(\{s^*\}) + (f(r^*, s^*) - f(r^*, r^*))P(\{r^*\})\right].$$

from (1). This is the replicator dynamic for the two-strategy matrix game with payoff matrix

$$A = \left[ \begin{array}{cc} f(s^*, s^*) & f(s^*, r^*) \\ f(r^*, s^*) & f(r^*, r^*) \end{array} \right].$$

It is well-known there is an interior equilibrium $(p_1^*, p_2^*) = (P(\{s^*\}), P(\{t^*\}))$ given by

$$p_1^* = \frac{f(s^*, r^*) - f(r^*, r^*)}{f(s^*, r^*) - f(r^*, r^*) + f(r^*, s^*) - f(s^*, s^*)}$$

$$p_2^* = \frac{f(r^*, s^*) - f(s^*, s^*)}{f(s^*, r^*) - f(r^*, r^*) + f(r^*, s^*) - f(s^*, s^*)}$$

if and only if $(f(s^*, r^*) - f(r^*, r^*))(f(r^*, s^*) - f(s^*, s^*)) > 0$. Moreover, $(p_1^*, p_2^*)$ is globally asymptotically stable if and only if $f(s^*, r^*) > f(r^*, r^*)$ and $f(r^*, s^*) > f(s^*, s^*)$. These inequalities, which we assume are valid throughout the remainder of this section, assert $(p_1^*, p_2^*)$ is the unique ESS of the $2 \times 2$ payoff matrix $A$.

Suppose this dimorphism is invaded by a mutant strategy $s$ near $s^*$ or $r^*$. The replicator dynamic is now with respect to the $3 \times 3$ payoff matrix

$$\left[ \begin{array}{ccc} f(s^*, s^*) & f(s^*, r^*) & f(s^*, s) \\ f(r^*, s^*) & f(r^*, r^*) & f(r^*, s) \\ f(s, s^*) & f(s, r^*) & f(s, s) \end{array} \right]. \tag{9}$$

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[8]We take the Hausdorff distance (Gulick, 1992) as a measure of closeness between two compact sets. Since $\{s^*, r^*\}$ is contained in $\text{supp}(P(0))$, the Hausdorff distance between these sets is $\max_{s \in \text{supp}(P(0))} (\min\{|s - s^*|, |s - r^*|\})$. For the homogeneous equilibria of Section 3, the Hausdorff distance between the supports of $P(0)$ and $\delta_{s^*}$ is $\max_{s \in \text{supp}(P(0))}\{|s - s^*|\}$.

Conditions for local asymptotic stability of $(p_1^*, p_2^*, 0)$ are reasonably well understood. For example, we need that the invading strategy has no higher payoff than the two resident strategies at equilibrium (i.e. $p_1^* f(s, s^*) + p_2^* f(s, r^*) \leq p_1^* f(s^*, s^*) + p_2^* f(s^*, r^*) = p_1^* f(r^*, s^*) + p_2^* f(r^*, r^*)$). We will in fact assume $p_1^* f(s, s^*) + p_2^* f(s, r^*) \neq p_1^* f(s^*, s^*) + p_2^* f(s^*, r^*)$ if $s \notin \{s^*, r^*\}$ to avoid technical complications.[9] Then, $(p_1^*, p_2^*, 0)$ is locally asymptotically stable if and only if

$$p_1^* f(s, s^*) + p_2^* f(s, r^*) < p_1^* f(s^*, s^*) + p_2^* f(s^*, r^*) = p_1^* f(r^*, s^*) + p_2^* f(r^*, r^*). \qquad (10)$$

These conditions state that $(p_1^*, p_2^*, 0)$ is a *local quasi-strict NE* (i.e. a quasi-strict NE of (9) for all $s$ near $s^*$ or $r^*$) and correspond to the first inequality in (6)

The inequality condition in (10) is also the frequency version (Vincent and Cressman, 2000) of the *ESS maximum principle* for a coalition of two developed for coevolutionary population dynamics by Vincent and co-workers (see, for example, Vincent et al. (1996) and the references therein). We do not use this designation to avoid possible confusion with the matrix ESS concept. Heuristically, the maximum principle states that the "fitness generating function" as a function of $s$, $p_1^* f(s, s^*) + p_2^* f(s, r^*)$, corresponding to this dynamic attains its local maximum value at precisely $s^*$ or $r^*$.

Conditions for global asymptotic stability of $(p_1^*, p_2^*, 0)$ are not so precise. Here, we concentrate on the following concept which implies global asymptotic stability of $(p_1^*, p_2^*, 0)$ by applying Theorem 4 below to the dimorphic case. At this point, it should also be emphasized again that the following theory (especially Theorem 4) is developed for any $P^*$ with finite support and not only for the dimorphic (or monomorphic) model.

The probability measure $P^* \in \Delta(S)$ is *locally superior* if, for all other $P$ with support sufficiently close to the support of $P^*$, $\pi(P^*, P) > \pi(P, P)$.

Weibull (1995) defined the concept of locally superior for matrix game models and showed a $p^* \in \Delta^m$ is locally superior if and only if $p^*$ is a matrix ESS. His definition (that $\pi(\overline{p^*}, \overline{p}) > \pi(\overline{p}, \overline{p})$ for all $\overline{p}$ sufficiently close (but not equal) to $\overline{p^*}$) is equivalent to Definition 4 in the mixed strategy matrix model if we only consider those $P$ for which the mean strategy $\overline{p}$ is different from that of $P^*$ since $\pi(P, Q) = \pi(\overline{p}, \overline{q})$.

By appropriate choices of $P$, Definition 4 includes our development so far of stability in the dimorphic model with $\mathrm{supp}(P^*) = \{s^*, r^*\}$. First, by taking $\mathrm{supp}(P) = \{s^*, r^*\}$, we find local superiority implies that $f(s^*, r^*) > f(r^*, r^*)$ and $f(r^*, s^*) > f(s^*, s^*)$ (i.e. $(p_1^*, p_2^*)$ is globally asymptotically stable for the two-strategy game between $s^*$ and $r^*$). Also, if $P$ is of the form $(1 - \varepsilon)P^* + \varepsilon\delta_s$ for some $0 < \varepsilon < 1$, local superiority implies that $P^*$ is a locally quasi-strict NE for the three-strategy game with payoff matrix given by (9). Moreover, when Definition 4 is applied to a monomorphic $P^* = \delta_{s^*}$, $P^*$ is locally superior if and only if $s^*$ strictly dominates $s$ in the two person game between $s$ and $s^*$ whenever $s$ is sufficiently close to $s^*$ and domination is determined by the second order Taylor expansion of $f(x, y)$. This is essentially what is proven in Theorem 3.

Suppose $P^*$ is a locally superior probability measure with finite support. If $\mathrm{supp}(P(0))$ is sufficiently close to $\mathrm{supp}(P^*)$ and $P(\{s^*\}) > 0$ initially for all $s^* \in \mathrm{supp}(P^*)$, then $P(t)$ converges to $P^*$ in the strong topology.

**Proof.** We only provide the proof for the case where $\mathrm{supp}(P^*) = \{s^*, r^*\}$. The general proof follows analogously. Consider the function $V : \Delta(S) \to \mathbf{R}$ given by $V(P) \equiv P(\{s^*\})^{p_1^*} P(\{r^*\})^{p_2^*}$. A straightforward calculation shows that, under the assumptions in the statement of the theorem, $\frac{1}{V}\dot{V} = \pi(P^*, P) - \pi(P, P) > 0$ if $P \neq P^*$. Thus

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[9]This assumption has the unfortunate consequence that our method is not directly applicable to the mixed strategy matrix model of Section 2.1 since payoff linearity there implies $p_1^* f(s, s^*) + p_2^* f(s, t^*) = p_1^* f(s^*, s^*) + p_2^* f(s^*, t^*)$ if $s$ is on the line segment from $s^*$ to $t^*$.

$V(P)$ is strictly increasing toward its unique maximum attained when $P(\{s^*\}) = p_1^*$ and $P(\{r^*\}) = p_2^*$ (i.e. when $P = P^*$). Since $\parallel P - P^* \parallel \equiv 2\max\{1 - P(\{s^*, r^*\}), P^*(\{s^*\}) - P(\{s^*\}), P^*(\{r^*\}) - P(\{r^*\})\}$ from (4), $P$ converges to $P^*$ in the strong topology.

Domination conditions may be used in place of local superiority to show $P(t)$ converges to $P^*$ in the strong topology. However, we can no longer expect $s^*$ to dominate all strategies in supp$(P(0))$ as in Theorems 3 and 3 since it does not dominate $r^*$ in the two-strategy game between $s^*$ and $r^*$.[10] Instead, we look for dominance of nearby strategies as follows. Suppose supp$(P^*) = \{s^*, r^*\}$ and, in the game with strategy set supp$(P(0))$, $s^*$ strictly dominates $s$ for all other $s$ near $s^*$ (i.e. $f(s^*, y) > f(s, y)$ for all $y \in$supp$(P(0))$) and $r^*$ strictly dominates $r$ for all other $r$ near $r^*$. Then, if $B$ is a compact set sufficiently close to and disjoint from $s^*$ with $P(B) > 0$,

$$\frac{d}{dt}\left(\frac{P(B)}{P(\{s^*\})}\right) = \frac{P(\{s^*\})\int_B (\pi(s, P) - \pi(P, P))\, P(ds) - P(\{s^*\})(\pi(s^*, P) - \pi(P, P))\, P(B)}{(P(\{s^*\}))^2}$$

$$= \frac{1}{(P(\{s^*\}))^2}\int_B (\pi(s, P) - \pi(s^*, P))\, P(ds)$$

$$= \frac{1}{(P(\{s^*\}))^2}\int_B \int_S (f(s, y) - f(s^*, y))\, P(ds)P(dy)$$

$$< 0.$$

Thus $\frac{P(B)}{P(\{s^*\})}$ is monotone decreasing and so $P(B)$ must converge to 0. Similarly, $P(C)$ converges to 0 for all compact sets sufficiently close to and disjoint from $r^*$. Thus $P(\{s^*, r^*\})$ converges to 1 and we already know that, for this two strategy game, we have convergence to $P^*$ in the strong topology. That is, we have shown the following result.

Suppose there are $m$ strategies in the support of $P^*$ and $P^*$ is a globally asymptotically stable equilibrium in the corresponding $m-$strategy game. Furthermore, suppose supp$(P^*)$ is covered by $m$ disjoint open sets each containing one element $s^*$ of this support and $s^*$ strictly dominates every other nearby $s$ in this cover (in the game whose strategy set is the entire cover). If supp$(P(0))$ is sufficiently close to supp$(P^*)$ and $P(\{s^*\}) > 0$ initially for all $s^* \in$supp$(P^*)$, then $P(t)$ converges to $P^*$ in the strong topology.

Theorem 4 demonstrates how difficult it is to get dynamic stability results for monomorphic $\delta_{s^*}$ if the supp$(P(0))$ is not restricted to be close to $s^*$. In particular, by Theorems 3 and 3, the assumptions of Theorem 4 also imply that each monomorphic $\delta_{s^*}$ with $s^* \in$supp$(P^*)$ is dynamically stable for all initial $P(0)$ with support sufficiently close to $s^*$ and $P(\{s^*\}) > 0$. That is, a small perturbation far away from a dynamically stable $\delta_{s^*}$ has the potential to destabilize the monomorphism. This led Oechssler and Riedel (2002) to generalize Definition 4 by calling $P^*$ *evolutionarily robust* if $\pi(P^*, P) > \pi(P, P)$ for all other $P$ sufficiently close to $P^*$ in the weak topology. They then proved an analogous result to Theorem 4. However, as we have just argued, evolutionary robustness is a very strong assumption that will be difficult to satisfy for most interesting payoff functions (besides those that are linear as in the matrix game model).

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[10]Furthermore, $p_1^* s^* + p_2^* r^*$ does not dominate $s^*$ or $r^*$ either, so there is no elementary method to confirm

$\pi(P^*, P) - \pi(P, P) = \int_S \int_S (p_1^* f(s^*, y) + p_2^* f(r^*, y) - f(s, y))\, P(dy)P(ds)$

is positive (i.e. $P^*$ is locally superior).

# 5    Conclusion

Predicting the behaviors of rational individuals involved in game interactions through analyzing stable equilibria of evolutionary dynamics is a well-accepted approach (Weibull, 1995; Hofbauer and Sigmund, 1998; Cressman, 2003), especially when the game has a finite number of strategies. When there is a continuous strategy space, conditions on the game's payoffs for dynamic stability are not so well known, perhaps because there is no general consensus on what constitutes such stability. In this paper, we have taken the concept to mean that all distributions (with respect to which $P^*$ is absolutely continuous in the sense of measure theory) with support close to the equilibrium distribution $P^*$ converge to $P^*$ in the strong topology.

From this perspective, Section 2 demonstrates that our concept is a natural extension of the finite strategy model. Sections 3 and 4 then develop the stability conditions. Here it is seen that conditions on $f(x, y)$ for stability in terms of strategy domination (Theorems 3, 3 and 4) emerge naturally by considering finite strategy games contained in the continuous model. These domination conditions (e.g.(8)) are relatively elementary inequalities to confirm for a given function $f$. On the other hand, the intuitive local superiority condition of Theorem 4 is not as easy to verify but does have the potential to predict rational behavior when the equilibrium does not have finite (or even discrete) support. For instance, the candidate stable equilibrium $P^*$ for the standard War of Attrition Game (Oechssler and Riedel, 2001; Cressman, 2003) with continuous strategy space a compact interval has the entire first half of this interval contained in its support. Oechssler and Riedel [16] show that this $P^*$ is locally superior[11] and are able to prove this implies convergence in the weak topology.

The techniques developed in this paper are more directly applicable to an equilibrium $P^*$ with finite support. They are particularly relevant for the stability analysis of homogeneous populations, a topic closely connected to recent work on coevolutionary models among theoretical biologists. This suggests that the emerging theory of evolutionary dynamics on continuous strategy spaces will continue the tradition of evolutionary game theory that fosters corresponding models and methods in predicting behavior both of human and of other species.

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[11]They actually show global superiority.

# 6  Appendix

**Proof of the remainder of Theorem 3.**  From the assumptions that $s^*$ is in the interior of $S \subset \mathbf{R}^n$ and that domination in the two-strategy game between $s^*$ and $s$ for all $s$ sufficiently close (but not equal) to $s^*$ is determined by the second order Taylor expansion of $f(x, y)$ about $x = y = s^*$, we have that the inequalities in (8) must hold where the second-order partial derivatives are now in the direction from $s^*$ to $s$. The first inequality implies that

$$\sum_{i,j} \frac{\partial^2 f(x,y)}{\partial x_i \partial x_j} \left( s_i - s_i^* \right) \left( s_j - s_j^* \right) < 0$$

for all $s$ in a neighborhood of $s^*$ (here the second-order partial derivatives are evaluated at $(s^*, s^*)$). That is, the $n \times n$ matrix with $ij$ entry $\frac{\partial^2 f}{\partial x_i \partial x_j} \equiv f_{x_i x_j}$ (i.e. the matrix $\left[ \frac{\partial^2 f}{\partial x_i \partial x_j} \right]$) is negative definite. Similarly, from the second inequality in (8), the $n \times n$ matrix $\left[ \frac{\partial^2 f}{\partial x_i \partial x_j} + \frac{\partial^2 f}{\partial x_i \partial y_j} + \frac{\partial^2 f}{\partial y_i \partial x_j} \right]$ is also negative definite.[12]

From the second order Taylor expansion, $\int_S \int_S \left( f(s^*, y) - f(s, y) \right) P(dy) P(ds)$ is now given by

$$-\frac{1}{2} \left[ \begin{array}{c} \int_S \int_S \sum_{i,j} f_{x_i x_j} \left( \left( s_i - s_i^* \right) \left( s_j - s_j^* \right) - \left( s_i - s_i^* \right) \left( y_j - s_j^* \right) \right) P(dy) P(ds) \\ + \int_S \int_S \sum_{i,j} \left( f_{x_i x_j} + 2 f_{x_i y_j} \right) \left( s_i - s_i^* \right) \left( y_j - s_j^* \right) P(dy) P(ds) \end{array} \right]$$

$$= -\frac{1}{2} \left[ \int_S \sum_{i,j} f_{x_i x_j} \left( s_i - \overline{s_i} \right) \left( s_j - \overline{s_j} \right) P(ds) + \sum_{i,j} \left( f_{x_i x_j} + f_{x_i y_j} + f_{y_i x_j} \right) \left( \overline{s_i} - s_i^* \right) \left( \overline{s_j} - s_j^* \right) \right].$$

Since $\left[ f_{x_i x_j} \right]$ is negative definite, $\int_S \sum_{i,j} f_{x_i x_j} \left( s_i - \overline{s_i} \right) \left( s_j - \overline{s_j} \right) P(ds)$ is negative unless $s_i = \overline{s_i}$ for all $i$ whenever $s_i \in \mathrm{supp}(P)$. Similarly, the negative definiteness of $\left[ f_{x_i x_j} + f_{x_i y_j} + f_{y_i x_j} \right]$ implies $\sum_{i,j} \left( f_{x_i x_j} + f_{x_i y_j} + f_{y_i x_j} \right) \left( \overline{s_i} - s_i^* \right) \left( \overline{s_j} - s_j^* \right)$ is negative unless $\overline{s_i} = s_i^*$ for all $i$. The remainder of the proof follows as in the main text.

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[12]The use of negative definiteness in the proof is similar to the method used by Meszéna et al. 2001 (see also Bomze, 1990; Apaloo, 1997). where $\int_S \left( s_i - \overline{s_i} \right) \left( s_j - \overline{s_j} \right) P(ds)$ give entries of the covariance of the population mean strategy.

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