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Abstract

We provide further analysis of two-country endogenous growth model considered in Aseev, et.al., 2002. To solve a suitably defined infinite horizon dynamic optimization problem an appropriate version of the Pontryagin maximum principle is applied. The properties of optimal controls and the corresponding optimal trajectories are characterized by means of a qualitative analysis of the solutions of the Hamiltonian system arising through the implementation of the Pontryagin maximum principle.

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A Model of Optimal Allocation of Resources to R & D: Further Results. I

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Introduction

The present paper deals with a problem of optimal dynamic allocation of resources to R&D (Aseev, et.al., 2002) which essentially follows the endogenous growth theory due to Grossman and Helpman, 1991, and approaches to modelling knowledge-absorbing economies (see Hutschenreiter, et.al., 1995). In the paper by Aseev, et. al., 2002, the analysis is restricted to the case when the total labor force in the follower country is strictly smaller when the amount of labor allocated in R&D in the leading country. The goal of the present paper is to consider other cases which were not covered by Aseev, et.al., 2002.

We focus on formal analysis and do not give any economic interpretations. The analysis is carried out within the framework of mathematical control theory (Pontryagin, et. al., 1969). An important feature of the problem under consideration is that the goal functional in it is defined on an infinite time interval. In Tarasyev and Watanabe, 1999; Reshmin, 1999; and Borisov et. al, 2000, applications of the Pontryagin maximum principle have led to ultimate solutions of nonlinear problems of optimal control for dynamic models of economic systems with infinite time horizons. Technically, the present paper adjoins these publications. Key elements of the technique suggested here are a qualitative analysis of the solutions of the Hamiltonian system arising through the implementation of the Pontryagin maximum principle. We find that the global optimizers are characterized by the exceptional qualitative behavior; this allows us to select the unique optimal regime in the pool of all local extremals.

In section 1 we formulate the problem and reduce it to a simplified form.

In section 2 we deal with the first case of relation between the parameters. We implement the Pontryagin maximum principle, construct the associated Hamiltonian system, classify behaviors of the solutions of the Hamiltonian system and focus on the solutions of the Hamiltonian system which exhibit exceptional behavior (we call them equilibrium solutions). We show that a global optimizer is described by an equilibrium solution and state the uniqueness of this solution. Basing on these results, we give the final description of an optimal process and prove its uniqueness. Finally in this section we consider the family of the original problems parameterized by the initial state and describe an optimal synthesis for this family i.e., define a feedback which solves the problem with any initial state.

Section 3 is organized as the previous one and the second case of relations between the parameters is considered there.

1 Problem formulation, existence of optimal control and the Pontryagin maximum principle

1.1 Problem formulation

We deal with the following problem of optimal control:

$$\text{maximize } J(x(\cdot),u(\cdot)) = \int_0^\infty e^{-\rho t}(\ln x(t) + \ln(b - u(t)))dt, \tag{1.1}$$

$$\dot{x}(t) = u(t)(x(t) + \gamma y(t)), \tag{1.2}$$

$$\dot{y}(t) = \nu y(t), \tag{1.3}$$

$$x(0) = x_0, \quad y(0) = y_0, \tag{1.4}$$

$$u(t) \in [0, b). \tag{1.5}$$

Here ν, b γ and ρ are positive parameters, and x_0 and y_0 are positive initial values for the state variables. Here, we study the situations

$$\nu < b < \nu + \rho. \tag{1.6}$$

and

$$b > \nu + \rho. \tag{1.7}$$

The situation

$$\nu > b \tag{1.8}$$

was considered in Aseev, et.al., 2002.

Let us remind several standard definitions of theory of optimal control in the context of problem (1.1) – (1.5). A *control* is identified with any measurable function $u(\cdot) : [0,\infty) \mapsto [0, b)$. The *motion* of system (1.2), (1.3) under a control $u(\cdot)$ (with the initial state (x_0, y_0)) is the (unique) Caratheodory solution $(x(\cdot), y(\cdot))$ on $[0,\infty)$ of equation (1.2), (1.3) with the initial condition (1.4). A *control process* for system (1.2), (1.3) is a triple $(x(\cdot), y(\cdot), u(\cdot))$ where $u(\cdot)$ is a control and $x(\cdot)$ is the motion of system (1.2), (1.3) under $u(\cdot)$.

A more accurate formulation of problem (1.1) – (1.5) is as follows: maximize $J(x(\cdot), u(\cdot))$ over the set of all control processes $(x(\cdot), y(\cdot), u(\cdot))$ for system (1.2), (1.3). An *optimal control* in problem (1.1) – (1.5) is defined to be a control $u_0(\cdot)$ for system (1.2), (1.3) such that the associated control process $(x_0(\cdot), y_0(\cdot), u_0(\cdot))$ satisfies $J(x_0(\cdot), u_0(\cdot)) = J_0$ where J_0 is the maximal (*optimal*) value in problem (1.1) – (1.5).

Remark 1.1 Obviously, there is a constant K such that $J(x(\cdot), u(\cdot)) < K$ for every control process $(x(\cdot), y(\cdot), u(\cdot))$; hence, $J_0 \leq K$. For every control process $(x(\cdot), y(\cdot), u(\cdot))$ with $u(\cdot)$ taking values in $[0, b - \varepsilon]$ for some $\varepsilon > 0$ we have $J(x(\cdot), u(\cdot)) > -\infty$. However, if $u(t)$ is sufficiently close to b, the term $\ln(b - u(t))$ in (1.1) is arbitrarily close to $-\infty$. Continuing this argument, we easily find that for some control processes $(x(\cdot), y(\cdot), u(\cdot))$ we have $J(x(\cdot), u(\cdot)) = -\infty$. Therefore, in problem (1.1) – (1.5) the situation $J(x(\cdot), u(\cdot)) = -\infty$ is formally admissible.

Here we simplify the system using the transformation from Aseev et. al., 2002. Set

$$z(t) = \frac{x(t)}{y(t)}, \quad z_0 = \frac{x_0}{y_0}.$$

If $(x(\cdot), y(\cdot), u(\cdot))$ is a control process, then (see (1.2) (1.3))

$$\begin{aligned}\dot{z}(t) &= \frac{\dot{x}(t)}{y(t)} - \frac{x(t)}{y^2(t)}\dot{y}(t) \\ &= \frac{u(t)(x(t)+\gamma y(t))}{y(t)} - \frac{x(t)}{y^2(t)}\nu y(t) \\ &= u(t)(z(t)+\gamma) - \nu z(t)\end{aligned}$$

and (see (1.1))

$$\begin{aligned}J(x(\cdot), u(\cdot)) &= \int_0^\infty e^{-\rho t}(\ln(z(t)y(t)) + \ln(b-u(t)))dt \\ &= \int_0^\infty e^{-\rho t}(\ln z(t) + \ln y(t) + \ln(b-u(t)))dt \\ &= \int_0^\infty e^{-\rho t}(\ln z(t) + \ln\left(y_0 e^{at}\right) + \ln(b-u(t)))dt \\ &= \int_0^\infty e^{-\rho t}(\ln z(t) + \ln(b-u(t)))dt + K_0\end{aligned}$$

where

$$K_0 = \int_0^\infty e^{-\rho t} \ln\left(y_0 e^{at}\right) dt.$$

The next theorem was proved in Aseev et. al., 2002.

Theorem 1.1 *Problem (1.1) – (1.5) is equivalent to the optimal control problem*

$$\text{maximize } J(z(\cdot), u(\cdot)) = \int_0^\infty e^{-\rho t}(\ln z(t) + \ln(b-u(t)))dt, \tag{1.9}$$

$$\dot{z}(t) = u(t)(z(t)+\gamma) - \nu z(t), \tag{1.10}$$

$$z(0) = z_0, \tag{1.11}$$

$$u(t) \in [0, b) \tag{1.12}$$

in the following sense:

(i) $u_0(\cdot)$ is an optimal control in problem (1.9) – (1.12) if and only if it is an optimal control in problem (1.1) – (1.5),

(ii) the optimal values J_{00} and J_0 in problems (1.9) – (1.12) and (1.1) – (1.5) are related to each other through $J_{00} = J_0 + K_0$.

Problem (1.9) – (1.12) introduced in Theorem 1.1 is understood similarly to problem (1.1) – (1.5). Namely, the *motion* (of system (1.10); briefly, a *motion*) under a control $u(\cdot)$ (with the initial state z_0) is the (unique) Caratheodory solution $z(\cdot)$ on $[0, \infty)$ of equation (1.10) with the initial condition (1.11); a *control process* (for system (1.10); briefly, a *control process*) is a pair $(z(\cdot), u(\cdot))$ where $u(\cdot)$ is a control and $z(\cdot)$ is the motion under $u(\cdot)$.

The accurate formulation of problem (1.9) – (1.12) is as follows: maximize $J(z(\cdot), u(\cdot))$ over the set of all control processes $(z(\cdot), u(\cdot))$. An *optimal control* (in problem (1.9) – (1.12)) is defined to be a control $u_0(\cdot)$ such that the associated control process $(z_0(\cdot), u_0(\cdot))$ satisfies $J(z_0(\cdot), u_0(\cdot)) = J_{00}$ (recall that J_{00} is the maximal (*optimal*) value in problem (1.9) – (1.12)); the control process $(z_0(\cdot), u_0(\cdot))$ is called *optimal* (in problem (1.9) – (1.12)).

In what follows, we analyze the reduced problem (1.9) – (1.12).

We also consider the family of problems (1.9) – (1.12) parametrized by the initial state $z_0 > 0$ and describe an optimal synthesis for this family (see Pontryagin, et. al., 1969, p. 51), i.e., define a feedback which solves problem (1.9) – (1.12) with arbitrary z_0.

In this paper we define a *feedback* to be an arbitrary continuous function $U(\cdot) : z \mapsto U(z) : (0, \infty) \mapsto [0, b)$ such that for every $z_0 > 0$ the equation

$$\dot{z}(t) = U(z(t))(z(t) + \gamma) - \nu z(t) \tag{1.13}$$

has the unique solution $z(\cdot)$ defined on $[0, \infty)$ and satisfying $z(0) = z_0$; we call $z(\cdot)$ the *motion* (of system (1.10)) under feedback $U(\cdot)$ with the initial state z_0.

Remark 1.2 Equation (1.13) represents the original control system (1.10) with control values $u(t)$ formed on the basis of current states $z(t)$ via feedback $U(\cdot)$: $u(t) = U(z(t))$ $(t \geq 0)$. According to a terminology often used in control theory (1.13) is the control system (1.10) *closed* with feedback $U(\cdot)$.

Given a feedback $U(\cdot)$ and a $z_0 > 0$, we define the *control process* under $U(\cdot)$ with the initial state z_0 to be the pair $(z(\cdot), u(\cdot))$ where $z(\cdot)$ is the motion under feedback $U(\cdot)$ with the initial state z_0 and $u(\cdot) : t \mapsto u(t) = U(t, z(t)) : [0, \infty) \mapsto [0, b)$; obviously, $(z(\cdot), u(\cdot))$ is a control process for system (1.10). We call a feedback $U(\cdot)$ an *optimal synthesis* if for every $z_0 > 0$ the control process under $U(\cdot)$ with the initial state z_0 is an optimal control process in problem (1.9) – (1.12).

1.2 Existence of optimal control and the Pontryagin maximum principle

The next existence statement follows from Aseev et. al., 2002 (see Theorem 3.1).

Theorem 1.2 *There exists an optimal control in problem (1.9) – (1.12).*

Introduce the Pontryagin function $\mathcal{H}(\cdot)$ (see Pontryagin, et. al., 1969):

$$\mathcal{H}(z, u, \psi) = \psi[u(z + \gamma) - az] + e^{-\rho t}(\ln z + \ln(b - u)) \tag{1.14}$$

$$(\psi, z \in R^1, u \in [0, b)).$$

Let us introduce now a new adjoint variable $p(t) = e^{\rho t}\psi(t)y(t)$. The next theorem follows from Aseev et. al., 2002 (see Theorem 4.1).

Theorem 1.3 *Let $u_*(t)$ be an optimal control process in problem (1.9) – (1.12). Then there exists an absolutely continuous strictly positive function $\psi(t)$ defined on $[0, \infty)$ such that following conditions hold:*

1) The function $p(t)$ is a solution to the adjoint system

$$\dot{p}(t) = -(u_*(t) - \nu - \rho)p(t) - \frac{1}{z_*(t)} \tag{1.15}$$

2) For almost all $t \in [0, \infty)$ the maximum condition takes place:

$$u_*(t)p(t)(z_*(t) + \gamma) + \ln(b - u_*(t)) = \sup_{u \in [0,b)} u(t)p(t)(z_*(t) + \gamma) + \ln(b - u) \tag{1.16}$$

3) The boundedness condition is valid:

$$p(t)z_*(t) \leq \frac{1}{\rho}, \forall t \geq 0$$

Taking into account the form of the Pontryagin function (1.14), introduce the function $g(\cdot) : [0,\infty) \mapsto [0,\infty)$,

$$g(z) = \frac{1}{b(z+\gamma)} \tag{1.17}$$

and the sets

$$\begin{aligned} G_1 &= \{(z,p) \in R^2 : z > 0,\ p \geq g(z)\}, && (1.18)\\ G_2 &= \{(z,p) \in R^2 : z > 0,\ 0 \leq p < g(z)\}. && (1.19) \end{aligned}$$

Obviously, $G_1 \cup G_2 = G$ where

$$G = (0,\infty) \times [0,\infty). \tag{1.20}$$

Define functions $r(\cdot) : G \mapsto R^1$ and $s(\cdot) : G \mapsto R^1$,

$$r(z,p) = \begin{cases} (b-\nu)z + b\gamma - \dfrac{1}{p} & \text{if} \quad (z,p) \in G_1, \\ -\nu z & \text{if} \quad (z,p) \in G_2, \end{cases} \tag{1.21}$$

$$s(z,p) = \begin{cases} (\nu - b + \rho)p - \dfrac{\gamma}{(z+\gamma)z} & \text{if} \quad (z,p) \in G_1, \\ (\nu+\rho)p - \dfrac{1}{z} & \text{if} \quad (z,p) \in G_2. \end{cases} \tag{1.22}$$

Remark 1.3 One can easily check that $r(\cdot)$ and $s(\cdot)$ are continuous.

The next theorem follows from Aseev et. al., 2002 (see Lemma 4.2).

Theorem 1.4 *Let $(z(\cdot), u(\cdot))$ be an optimal control process in problem (1.9) – (1.12). Then*

(i) there exists strictly positive function $p(\cdot)$ defined on $[0,\infty)$ such that $(z(\cdot), p(\cdot))$ solves the equation

$$\begin{aligned} \dot{z}(t) &= r(z(t), p(t)), && (1.23)\\ \dot{p}(t) &= s(z(t), p(t)), && (1.24) \end{aligned}$$

(in G) on $[0,\infty)$,

(ii) for a. a. $t \geq 0$

$$u(t) = \begin{cases} b - \dfrac{1}{p(t)(z(t)+\gamma)} & \textit{if} \quad (z(t),p(t)) \in G_1, \\ 0 & \textit{if} \quad (z(t),p(t)) \in G_2, \end{cases} \tag{1.25}$$

(iii) for a. a. $t \geq 0$

$$p(t)z(t) \leq \frac{1}{\rho} \tag{1.26}$$

Remark 1.4 Equation (1.23), (1.24) represents the stationary *Hamiltonian* system for problem (1.9) – (1.12). In our further analysis we will for convenience split (1.23), (1.24) in two parts:

$$\begin{aligned} \dot{z}(t) &= (b-\nu)z(t) + b\gamma - \frac{1}{p(t)}, && (1.27)\\ \dot{p}(t) &= (\nu - b + \rho)p(t) - \frac{\gamma}{(z(t)+\gamma)z(t)} && (1.28) \end{aligned}$$

$$((z(t), p(t)) \in G_1),$$

and

$$\begin{aligned} \dot{z}(t) &= -\nu z(t), && (1.29)\\ \dot{p}(t) &= (\nu+\rho)p(t) - \frac{1}{z(t))} && (1.30) \end{aligned}$$

$$((z(t), p(t)) \in G_2).$$

We will call equation (1.27), (1.28) *nondegenerate* and equation (1.29), (1.30) *degenerate.*

Now we introduce 2 variables to simplify the analysis in situation 2 (1.7).

$$(q = pz, p)$$

In the new variables the function $g(\cdot) : [0, \infty) \mapsto [0, \infty)$, (1.17) takes the form:

$$g(q) = \frac{1 - qb}{b\gamma} \tag{1.31}$$

and the domains G_1 (1.18), G_2 (1.19) take the form:

$$\begin{aligned} G_1 &= \{(q, p) \in R^2 : q > 0,\ p \geq g(q)\}, && (1.32)\\ G_2 &= \{(q, p) \in R^2 : z > 0,\ 0 \leq p < g(q)\}. && (1.33) \end{aligned}$$

The Hamiltonian system (1.27), (1.28), (1.29), (1.30) is transformed into:

$$\begin{aligned} \dot{q}(t) &= \rho q(t) - \frac{\gamma p(t)}{q(t) + \gamma p(t)} + b\gamma p(t) - 1, && (1.34)\\ \dot{p}(t) &= (\nu - b + \rho)p(t) - \frac{\gamma p^2(t)}{(q(t) + \gamma p(t))q(t)} && (1.35) \end{aligned}$$

$$((q(t), p(t)) \in G_1),$$

and

$$\begin{aligned} \dot{q}(t) &= \rho q(t) - 1, && (1.36)\\ \dot{p}(t) &= (\nu + \rho - \frac{1}{q(t)})p(t) && (1.37) \end{aligned}$$

$$((q(t), p(t)) \in G_2).$$

and the transversality condition (1.26) takes the form:
for a.a. t

$$q(t) \leq \frac{1}{\rho}$$

Define functions $r(\cdot) : G \mapsto R^1$ and $s(\cdot) : G \mapsto R^1$,

$$r(q, p) = \begin{cases} \rho q(t) - \dfrac{\gamma p(t)}{q(t) + \gamma p(t)} + b\gamma p(t) - 1 & \text{if} \quad (q, p) \in G_1, \\ \rho q(t) - 1 & \text{if} \quad (q, p) \in G_2, \end{cases} \tag{1.38}$$

$$s(q, p) = \begin{cases} (\nu - b + \rho)p(t) - \dfrac{\gamma p^2(t)}{(q(t) + \gamma p(t))q(t)} & \text{if} \quad (q, p) \in G_1, \\ (\nu + \rho - \dfrac{1}{q(t)})p(t) & \text{if} \quad (q, p) \in G_2. \end{cases} \tag{1.39}$$

Now we reformulate Lemma 1.4 in terms of the new variables.

Lemma 1.1 *Let $(z(\cdot), u(\cdot))$ be an optimal control process in problem (1.9) – (1.12). Then*

(i) there exists a nonnegative function $p(\cdot)$ defined on $[0,\infty)$ such that $(q(\cdot), p(\cdot))$, where $q(\cdot) = p(\cdot)z(\cdot)$ solves the equation

$$\begin{aligned} \dot q(t) &= r(q(t), p(t)), && (1.40)\\ \dot p(t) &= s(q(t), p(t)), && (1.41) \end{aligned}$$

(in G) on $[0,\infty)$,

(ii) for a. a. $t \ge 0$

$$u(t) = \begin{cases} b - \dfrac{1}{q(t) + p(t)\gamma} & \text{if} \quad (q(t), p(t)) \in G_1, \\ 0 & \text{if} \quad x(q(t), p(t)) \in G_2. \end{cases} \tag{1.42}$$

(iii) for a. a. $t \ge 0$

$$q(t) \le \frac{1}{\rho} \tag{1.43}$$

2 Design of optimal control. Situation 1

2.1 Assumption

In this section we consider situation 1, $\nu < b < \nu + \rho$ (see (1.6).

2.2 Qualitative analysis of Hamiltonian system

The vector field of the Hamiltonian system (1.27), (1.28) in G (see (1.20)) is the union of the vector fields of the nondegenerate equation (1.27), (1.28) in G_1 (see (1.18) and the degenerate equation (1.29), (1.30) in G_2 (see (1.19).

The vector field of the nondegenerate equation (1.27), (1.28) in G_1 has the following structure. Define $h_1(\cdot) : (0,\infty) \mapsto (0,\infty)$, and $h_2(\cdot) : (0,\infty) \mapsto (0,\infty)$,

$$h_1(z) = \frac{1}{b\gamma - (\nu - b)z}, \tag{2.1}$$

$$h_2(z) = \frac{\gamma}{(\nu - b + \rho)(z + \gamma)z}. \tag{2.2}$$

Note that $h_1(\cdot)$ is strictly decreasing on $[0,\infty)$,

$$h_1(z) \to +0 \quad \text{as} \quad z \to \infty, \tag{2.3}$$

$$h_1(z) > g(z) \quad (z \in (0,\infty)) \tag{2.4}$$

(see (1.31)), function $h_2(\cdot)$ is strictly decreasing on $(0,\infty)$, and

$$h_2(z) \to \infty \quad \text{as} \quad z \to +0. \tag{2.5}$$

The right hand side of equation (1.27) (for $z(\cdot)$) is zero on the curve

$$V_z^0 = \{(z,p) \in G_1 : p = h_1(z)\}, \tag{2.6}$$

positive in the domain

$$V_z^+ = \{(z,p) \in G_1 : p > h_1(z)\}$$

and negative in the domain

$$V_z^- = \{(z,p) \in G_1 : p < h_1(z)\}.$$

The right hand side of equation (1.28) (for $p(\cdot)$) is zero on the curve

$$V_p^0 = \{(z,p) \in G_1 : p = h_2(z)\}, \tag{2.7}$$

positive in the domain

$$V_p^+ = \{(z,p) \in G_1 : p > h_2(z)\}$$

and negative in the domain

$$V_p^- = \{(z,p) \in G_1 : p < h_2(z)\}.$$

Thus, the vector field of the nondegenerate equation (1.27), (1.28) is

(i) positive in both coordinates in the domain

$$V^{++} = V_z^+ \times V_p^+ = \{(z,p) \in G_1 : p > h_1(z),\ p > h_2(z)\}, \tag{2.8}$$

(ii) negative in both coordinates in the domain

$$V^{--} = V_z^- \times V_p^- = \{(z,p) \in G_1 : p < h_1(z),\ p < h_2(z)\}, \tag{2.9}$$

(iii) positive in the z coordinate and negative in the p coordinate in the domain

$$V^{+-} = V_z^+ \times V_p^- = \{(z,p) \in G_1 : p > h_1(z),\ p < h_2(z)\}, \tag{2.10}$$

(iv) negative in the z coordinate and positive in the p coordinate in the domain

$$V^{-+} = V_z^- \times V_p^+ = \{(z,p) \in G_1 : p < h_1(z),\ p > h_2(z)\}. \tag{2.11}$$

The rest points of equation (1.27), (1.28) in G_1 are the solutions of the next system of algebraic equations

$$p = h_1(z), \quad p = h_2(z). \tag{2.12}$$

Relations (2.4), (2.3), (2.5) and the fact that $h_2(z)$ intersects $g(z)$ imply that (2.12) has a solution in G_1. Using definitions $h_1(\cdot)$ and $h_2(\cdot)$ (see (2.1) and (2.2)), we find the single solution (z^*, p^*) to the system (2.12) through the next series of equivalent transformations:

$$\frac{\gamma}{(\nu - b + \rho)(z^* + \gamma)z^*} = \frac{1}{b\gamma - (\nu - b)z^*},$$

$$\gamma(b\gamma - (\nu - b)z^*) = (\nu - b + \rho)(z^* + \gamma)z^*,$$

$$\gamma^2 b - \gamma(\nu - b)z^* = (\nu - b + \rho)z^{*2} + (\nu - b + \rho)\gamma z^*,$$

$$(\nu - b + \rho)z^{*2} + 2(\nu - b + \rho/2)\gamma z^* - \gamma^2 b = 0,$$

finally, we get

$$z^* \in \{z_1^*, z_2^*\}$$

where

$$z_1^* = \frac{-(\nu - b + \rho/2)\gamma + [(\nu - b + \rho/2)^2\gamma^2 + (\nu - b + \rho)\gamma^2 b]^{1/2}}{\nu - b + \rho},$$

$$z_2^* = \frac{-(\nu - b + \rho/2)\gamma - [(\nu - b + \rho/2)^2\gamma^2 + (\nu - b + \rho)\gamma^2 b]^{1/2}}{\nu - b + \rho}.$$

We have $z^* = z_1^*$, for $z_2^* < 0$. Employing the first equation in (2.12), we provide the final formulas for the unigue rest point of the nondegenerate equation (1.27), (1.28) in domain G_1:

$$z^* = \frac{-(\nu - b + \rho/2)\gamma + [(\nu - b + \rho/2)^2\gamma^2 + (\nu - b + \rho)\gamma^2 b]^{1/2}}{\nu - b + \rho}, \tag{2.13}$$

$$p^* = h_1(z^*) = \frac{1}{(b\gamma - (\nu - b)z^*}. \tag{2.14}$$

Note that due to (2.4) we have

$$p^* = h_1(z^*) > g(z^*), \tag{2.15}$$

i.e., (z^*, p^*) lies in the interior of G_1 (see (1.20)).

Now let us analyze the vector field of equation (1.29), (1.30).

Define $h(\cdot) : (0, \infty) \mapsto (0, \infty)$,

$$h(z) = \frac{1}{(\nu + \rho)z}. \tag{2.16}$$

The vector field of the degenerate equation (1.29), (1.30) is

(i) negative in the z coordinate and zero in the p coordinate in the domain

$$W_p^0 = \{(z, p) \in G_2 : p = h(z)\}, \tag{2.17}$$

(ii) negative in both coordinates in the domain

$$W^{--} = \{(z, p) \in G_2 : p < h(z)\}, \tag{2.18}$$

(ii) negative in the z coordinate and positive in the p coordinate in the domain

$$W^{-+} = \{(z, p) \in G_2 : p > h(z)\}. \tag{2.19}$$

Let us analyze how the vector fields of the nondegenerate equation (in G_1) and degenerate equation (in G_2) are pasted together. Note that G_1 and G_2 are separated by the curve

$$G^0 = \{(z, p) \in R^2 : z > 0,\ p = g(z)\}$$

(see (1.18) and (1.19)). Inequality (2.4) shows that curve V_z^0 (2.6) does not intersect G^0. Curve V_p^0 intersects G^0 at point $(z_*, g(z_*))$ and lies above G^0 (on the (z, p) plane) in the stripe $\{(z, p) : 0 < z < z_*,\ p \geq 0\}$; more accurately,

$$h_2(z) > g(z) \ \ (z < z_*), \quad h_2(z_*) = g(z_*), \quad h_2(z) > g(z) \ \ (z > z_*) \tag{2.20}$$

where

$$z_* = \frac{b\gamma}{\nu - b + \rho}.$$

Indeed, using (2.2) and (1.31), we get the next sequence of equivalent transformations:

$$h_2(z) \geq g(z),$$

$$\frac{\gamma}{(\nu - b + \rho)(z+\gamma)z} \geq \frac{1}{b(z+\gamma)},$$
$$\frac{\gamma}{(\nu - b + \rho)z} \geq \frac{1}{b},$$
$$(\nu - b + \rho)z \leq b\gamma,$$
$$z \leq z_*.$$

Note that (2.15) implies $p^* = h_2(z^*) > g(z^*)$; consequently by (2.20)

$$z_* > z^*. \tag{2.21}$$

Formulas (1.31) and (2.16) show that curve W_p^0 (2.17) intersects G^0 at point $(z_*, g(z_*))$ and lies below it in the stripe $\{(z,p) : z > z_*,\ p \geq 0\}$; more accurately,

$$h(z) > g(z) \ \ (z < z_*), \quad h(z_0) = g(z_0), \quad h(z) < g(z) \ \ (z > z_*). \tag{2.22}$$

Indeed,

$$h(z) \leq g(z_0)$$

is equivalently transformed as follows:

$$\frac{1}{z(\nu+\rho)} \leq \frac{1}{b(z+\gamma)},$$
$$b(z+\gamma) \leq z(\nu+\rho),$$
$$(\nu - b + \rho)z \geq b\gamma,$$
$$z \geq z_*.$$

Therefore,

$$\inf\{z : (z,p) \in W^{+-}\} = z_*. \tag{2.23}$$

Relations (2.22) show that the vector field of the entire Hamiltonian system (1.23), (1.24) (in G (1.20)) changes the sign in the p coordinate on the (continuous) curve

$$L_p^0 = \{(z,p) : p = h_2(p),\ 0 < z \leq z_*\} \cup \{(z,p) : p = h(p),\ z > z_*\}. \tag{2.24}$$

We end up with the next description of the vector field of (1.23), (1.24).

Lemma 2.1 *The vector field of the Hamiltonian system (1.23), (1.24) (in G) is*

(i) positive in both coordinates in domain V^{++} (see (2.8)),

(ii) negative in both coordinates in domain $V^{--} \cup W^{--}$ (see (2.9) and (2.18)),

(iii) positive in the z coordinate and negative in the p coordinate in domain V^{+-} (see (2.10)),

(iv) negative in the z coordinate and positive in the p coordinate in domain $V^{-+} \cup W^{-+}$ (see (2.11) and (2.19)),

(v) zero in the z coordinate on curve V_z^0 (see (2.6)), and

(v) zero in the p coordinate on curve L_z^0 (see (2.24)).

The rest point (z^*, p^*) of (1.23), (1.24) in G is unique; it is defined by (2.13), (2.14) and lies in the interior of G_1.

The vector field of system (1.23), (1.24) is shown in Fig. 1.

Lemma 2.1 allows us to give a full classification of the qualitative behaviors of the solutions of the Hamiltonian system (1.23), (1.24) in G (see also Fig. 1). In what follows, cl$E$ denotes the closure of a set $E \subset R^2$.

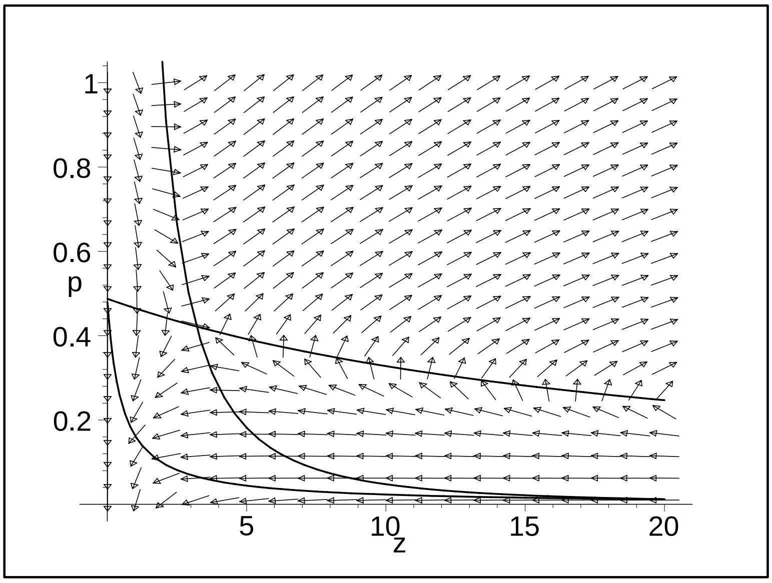

Figure 1: The vector field of the Hamiltonian system (1.23), (1.24) for $\nu = 4$, $b = 4.1$, $\rho = 0.2$, $\gamma = 0.5$ (a Mapple simulation).

Lemma 2.2 *Let $(z(\cdot), p(\cdot))$ be a solution of (1.23), (1.24) in G, which is nonextendable to the right, Δ be the interval of its definition, $t_* \in \Delta$, and $(z(t_*), p(t_*)) \neq (z^*, p^*)$.*

The following statements are true.

1. If $(z(t_*), p(t_*)) \in \mathrm{cl}(V^{--} \cup W^{--})$, then Δ is bounded, $(z(t), p(t)) \in V^{--} \cup W^{--}$ for all $t \in \Delta \cap (t_*, \infty)$, and $p(\vartheta) = 0$ where $\vartheta = \sup \Delta$.

2. If $(z(t_*), p(t_*)) \in \mathrm{cl}V^{++}$, then Δ is unbounded, $(z(t), p(t)) \in V^{++}$ for all $t \in (t_* \infty)$ and

$$\lim_{t\to\infty} z(t) = \infty, \tag{2.25}$$

$$\lim_{t\to\infty} p(t) = \infty. \tag{2.26}$$

3. If $(z(t_*), p(t_*)) \in V^{+-}$, then one of the next cases (i), (ii), (iii) takes place:

(i) Δ is bounded and $(z(t), p(t)) \in \mathrm{cl}(V^{--} \cup W^{--})$ for all $t \in \Delta \cap [t^* \infty)$ with some $t^* \in \Delta \cup [t_*, \infty)$;

(ii) Δ is unbounded, $(z(t), p(t)) \in \mathrm{cl}V^{++}$ for all $t \in [t_* \infty)$ with some $t^* \in \Delta \cup [t_*, \infty)$ and relations (2.25) and (2.26) hold;

(iii) Δ is unbounded, $(z(t), p(t)) \in \mathrm{cl}V^{+-}$ for all $t \in [t_* \infty)$ and

$$\lim_{t\to\infty} z(t) = z^*, \tag{2.27}$$

$$\lim_{t\to\infty} p(t) = p^*. \tag{2.28}$$

4. If $(z(t_*), p(t_*)) \in V^{-+} \cup W^{-+}$, then one of the next cases (i), (ii), (iii) takes place:

(i) Δ is bounded and $(z(t), p(t)) \in \mathrm{cl}(V^{--} \cup W^{--})$ for all $t \in \Delta \cap [t^* \infty)$ with some $t^* \in \Delta \cup [t_*, \infty)$;

(ii) Δ is unbounded, $(z(t), p(t)) \in \mathrm{cl}V^{++}$ for all $t \in \Delta \cap [t^*, \infty)$ for some $t^* \in \Delta \cup [t_*, \infty)$ and relations (2.25) and (2.26) hold;

(iii) Δ is unbounded, $(z(t), p(t)) \in \mathrm{cl}(V^{-+} \cup W^{-+})$ for all $t \in \Delta \cap [t^*, \infty)$ for some $t^* \in \Delta \cup [t_*, \infty)$ and relations (2.27) and (2.28) hold.

Proof. 1. Let $(z(t_*), p(t_*)) \in \mathrm{cl}(V^{--} \cup W^{--})$. The fact that the vector field of (1.23), (1.24) is negative in both coordinates in $V^{--} \cup W^{--}$ (Lemma 2.1) and the locations of V^{--} and W^{--} in G (see (2.9) and (2.18)) imply that the set $\mathrm{cl}(V^{--} \cup W^{--}) \cap \{(z,p) \in G : p \leq p(t_*)\}$ is invariant for (1.23), (1.24); moreover, $(z(t), p(t)) \in V^{--} \cup W^{--}$ for all $t \in \Delta \cap (t_*, \infty)$ and there are a $\delta > 0$ and a $t^* \in \Delta \cap [t^*, \infty)$ such that $\dot{p}(t) \leq -\delta$ for all $t \geq \Delta \cap [t^*, \infty)$. Hence, $p(\vartheta) = 0$ for some finite ϑ, i.e., $(z(\cdot, p(\cdot))$ is nonextendabe to the right in G and $\vartheta = \sup \Delta$.

2. Let $(z(t_*), p(t_*)) \in \mathrm{cl}V^{++}$. The fact that the vector field of (1.23), (1.24) is positive in both coordinates in V^{++} (Lemma 2.1) and the location of V^{++} in G (see (2.8) imply that the set $\mathrm{cl}V^{++} \cap \{(z,p) \in G : p \geq p(t_*)\}$ is invariant for (1.23), (1.24); moreover, $(z(t), p(t)) \in V^{++}$ for all $t \in \Delta \cap (t_*, \infty)$ and there are a $\delta > 0$ and a $t^* \in \Delta \cap [t^*, \infty)$ such that $\dot{p}(t) \geq \delta$ and $\dot{z}(t) \geq \delta$ for all $t \geq \Delta \cap [t^*, \infty)$. Therefore, Δ is unbounded and (2.26) holds. Now (2.3) and $\dot{z}(t) > \delta$ for all $t \geq \Delta \cap [t^*, \infty)$ imply (2.25).

3. Let $(z(t_*), p(t_*)) \in V^{+-}$, Due to the definitions of V^{+-}, V_z^0 and V_p^0 (see (2.10, (2.6 and (2.7), three cases are admissible: $(z(t^*), p(t^*)) \in V_z^0$ for some $t^* \geq t_*$ (case 1), $(z(t^*), p(t^*)) \in V_p^0$ for some $t^* \geq t_*$ (case 2), and $(z(t^*), p(t^*)) \in V^{+-}$ for all $t \in \Delta$ (case 3). Note that assumption $(z(t_*), p(t_*)) \neq (z^*, p^*)$ implies that $(z(t), p(t)) \neq (z^*, p^*)$ for all $t \in \Delta$ (we refer to the theorem of the uniqueness of the solution of a Cauchy problem for a differential equation with a Lipschitz right hand side). Therefore, in case 1 we have the situation described in statement 1 (with t_* replaced by t^*); hence, (i) holds. Smilarly, in case 2 (ii) holds due to statement 2. Let case 3 take place. If $\vartheta = \sup \Delta < \infty$, then $(z(\vartheta), p(\vartheta))$ belongs to the interior of G; hence, $(z(\cdot), p(\cdot))$ is extendable to the right, which contradicts the assumption that $(z(\cdot), p(\cdot))$ is nonextendable to the right. Therefore, Δ is unbounded. Functions $z(\cdot)$ is increasing and limited, function $p(\cdot)$ is decreaing and limited; thus,

$$z(t) \to z_1 \quad \text{as} \quad t \to \infty, \tag{2.29}$$

$$z(t) \leq z_1 \quad (t \in \Delta), \tag{2.30}$$

$$p(t) \to p_1 \quad \text{as} \quad t \to \infty,$$

$$p(t) \geq p_1 \quad (t \in \Delta).$$

Suppose $(z_1, p_1) \neq (z^*, p^*)$. Then one of the right hand sides $r(z_1, p_1)$, $s(z_1, p_1)$ of the Hamiltonian system (1.23), (1.24) is positive at point (z_1, p_1). Let, for example $r(z_1, p_1) > \delta > 0$. By (2.29) $\dot{z}(t) = r(z(t), p(t)) > \delta/2$ for all sudfficiently large t. Then, referring to (2.29) again, we find that $z(t) > z_1$ for all sufficiently large t, which contradicts (2.30). Similarly, we arrive at a contradiction if we assume $s(z_1, p_1) < \delta < 0$. Thus, $(z_1, p_1) = (z^*, p^*)$. and we get (2.27) and (2.28). Statement 3 is proved.

4. A justification of statement 4 is similar to that of statement 3.

The proof is finished.

2.3 Optimal control process

In this section we give an entire description of a solution of problem (1.9) – (1.12) and state its uniqueness.

The core of the analysis is Lemma 2.4 which selects solutions of the Hamiltonian system (1.23), (1.24) (we call them equilibrium solutions) whose qualitative behavior agrees with the Pontryagin maximum principle and also acts as a necessary condition for the global optimality in problem (1.9) – (1.12).

We call a solution $(z(\cdot), p(\cdot))$ (in G) of the Hamiltonian system (1.23) (1.24) an *equilibrium solution* if it is defined on $[0, \infty)$ and converges to the rest point (z^*, p^*), i.e. satisfies

(2.28) and (2.27). Let us formulate additional properties of equilibrium solutions basing on Lemma 2.2.

Lemma 2.3 *Let $(z(\cdot), p(\cdot))$ be an equilibrium solution of the Hamiltonian system (1.23) (1.24). Then*

(i) $z(0) = z^*$ implies that $(z(t), p(t)) = (z^*, p^*)$ for all $t \geq 0$,

(ii) $z(0) < z^*$ implies that $(z(t), p(t)) \in V^{+-}$ for all $t \geq 0$,

(iii) $z(0) > z^*$ implies that $(z(t), p(t)) \in V^{-+} \cup W^{-+}$ for all $t \geq 0$.

Proof. Prove (i). Let $z(0) = z^*$. If $p(0) < p^*$, then $(z(0), p(0)) \in \mathrm{cl}V^{--} \cup \mathrm{cl}W^{--}$ (see (2.9)). Hence, by Lemma 2.2, (statement 1) the interval of definition of $(z(\cdot, p(\cdot))$ is bounded, which is not the case. If $p(0) > p^*$, then $(z(0), p(0)) \in V^{++}$ (see (2.8)). Hence, by Lemma 2.2 (statement 2) (2.25), (2.26), hold, which contradicts (2.27), (2.28). Thus, $p(0) = p^*$. Due to the uniqueness of the solution of a Cauchy problem for system (1.23), (1.24) we have $(z(t), p(t)) = (z^*, p^*)$ for all $t \geq 0$.

Prove (ii). Let $z(0) < z^*$. Then

$$(z(0), p(0)) \notin \mathrm{cl}V^{-+} \cup \mathrm{cl}W^{-+}. \tag{2.31}$$

Indeed, the definition of V^{-+} (2.11), the facts that $h_1(\cdot)$ and $h_2(\cdot)$ intersect in point $p^* = h_1(z^*) = h_2(z^*)$ imply that $z \geq z^*$ for every $(z, p) \in \mathrm{cl}V^{-+}$. Furthermore, by (2.23) and (2.21) $z(0) \notin W^{-+}$ In view of (2.31) three cases are admissible: $(z(0), p(0)) \in \mathrm{cl}V^{--} \cup \mathrm{cl}W^{--}$ (case 1), $(z(0), p(0)) \in \mathrm{cl}V^{++}$ (case 2), and $(z(0), p(0)) \in V^{+-}$ (case 3). In case 1 by Lemma 2.2 (statement 1) the interval of definition of $(z(\cdot, p(\cdot))$ is bounded, which is a contradiction. In case 2 by Lemma 2.2 (statement 2) we have (2.25), (2.26), which contradicts (2.27), (2.28). Therefore, case 3 takes place. For this case statement 3 of Lemma 2.2 holds. Situations (i) and (ii) of this statement do not take place (see the above argument). Therefore, we have situation (iii) of this statement, which proves (ii) in the present lemma.

Statement (iii) is proved similarly.

The proof is finished.

Lemma 2.4 *Let $(z(\cdot), u(\cdot))$ be an optimal control process in problem (1.9) – (1.12). Then*

(i) there exists a (nonnegative) function $p(\cdot)$ such that $(z(\cdot), p(\cdot))$ is an equilibrium solution of the Hamiltonian system (1.23) (1.24),

(ii) for a. a. $t \geq 0$ (1.42) holds.

Proof. By Lemma 1.4 there exists a nonnegative function $p(\cdot)$ defined on $[0, \infty)$ such that $(z(\cdot), p(\cdot))$ solves (1.23) (1.24) (in G) on $[0, \infty)$. According to Lemma 2.2 three cases are admissible:

Case 1: the interval of definition of $(z(\cdot), p(\cdot))$ is bounded (Lemma 2.2, statement 1, statement 3, (i), and statement 4, (i)).

Case 2: relations (2.25), (2.26) hold (Lemma 2.2, statement 2, statement 3, (ii), and statement 4, (ii)).

Case 3: relations (2.27), (2.28) hold, i.e., $(z(\cdot), p(\cdot))$ is equilibrium (Lemma 2.2, statement 3, (iii), and statement 4, (iii)).

Case 1 is not possible since $(z(\cdot), p(\cdot))$ is defined on $[0, \infty)$.

Let us show that case 2 is not possible either. Suppose, case 2 takes place. Then $p(t)z(t) \to \infty, t \to \infty$, which contradicts transversality condition (1.26) Therefore, $(z(\cdot), u(\cdot))$ is not optimal, which contradicts the assumption. Thus, case 2 is not possible. By excluding cases 1 and 2 we state that case 3 takes place. By Lemma 1.4 (ii) is true. The lemma is proved.

Lemma 2.5 *For every $z^0 > 0$ there exists the unique equilibrium solution $(z(\cdot), p(\cdot))$ of the Hamiltonian system (1.23), (1.24), which satisfies $z(0) = z^0$.*

Proof. Suppose $z^0 = z^*$. By Lemma 2.3 for any equilibrium solution $(z(\cdot), p(\cdot))$ of (1.23), (1.24) such that $z(0) = z^0 = z^*$ we have $(z(t), p(t)) = (z^*, p^*)$ $(t \geq 0)$, which completes the proof.

Let $z^0 < z^*$. The existence of a desired equilibrium solution follows from the existence of an optimal control process. Indeed, by Lemma 1.2 there exists an optimal control process $(z(\cdot), u(\cdot))$ in problem (1.9) – (1.12). Setting $z_0 = z^0$ in problem (1.9) – (1.12), we get $z(0) = z^0$. By Lemma 2.4, (i), there exists a function $p(\cdot)$ such that $(z(\cdot), p(\cdot))$ is an equilibrium solution of the Hamiltonian system (1.23) (1.24). (Note that the existence of a desired equilibrium solution can also be proved explicitly).

Let us state the uniqueness of the considered equilibrium solution. Suppose there are two different equilibrium solutions of (1.23), (1.24), $(z_1(\cdot), p_1(\cdot))$ and $(z_2(\cdot), p_2(\cdot))$, such that $z_1(0) = z_2(0) = z^0$. Then

$$\lim_{t\to\infty} z_i(t) = z^*, \quad \lim_{t\to\infty} p_i(t) = p^*, \tag{2.32}$$

$i = 1, 2$, and $p_2(0) \neq p_1(0)$ (otherwise $(z_2(\cdot), p_2(\cdot))$ and $(z_1(\cdot), p_1(\cdot))$ coincide due to the uniqueness of the solution of a Cauchy problem for equation (1.23), (1.24)). Denote $p_i^0 = p_i(0)$, $i = 1, 2$. With no loss of generality assume

$$p_2^0 > p_1^0. \tag{2.33}$$

By Lemma 2.3

$$(z_i(t), p_i(t)) \in V^{+-} \quad (t \geq 0), \tag{2.34}$$

$i = 1, 2$. Hence, $\dot{z}_i(t) > 0$ $(t > 0)$, $i = 1, 2$. Define $\bar{p}_i(\cdot) : [z^0, z^*) \mapsto [p_i(0), \infty)$ by $\bar{p}_i(\zeta) = p_i(z_i^{-1}(\zeta))$ Due to (2.32) $\lim_{z\to z_*} \bar{p}_i(z) = p^*$, $i = 1, 2$, in particular,

$$\lim_{z\to z_*} (\bar{p}_2(z) - \bar{p}_1(z)) = 0. \tag{2.35}$$

We have

$$\frac{d}{dz}\bar{p}_i(z) = f(z, \bar{p}_i(z)), \quad (z \in [z^0, z^*)), \quad p(z^0) = p_i(0), \tag{2.36}$$

$i = 1, 2$, where

$$f(z, p) = \frac{s(z, p)}{r(z, p)} \tag{2.37}$$

(recall that $r(\cdot)$ and $s(\cdot)$ determine the right hand side of the Hamiltonian system (1.23), (1.24)). Due to (2.33)

$$\bar{p}_2(z) > \bar{p}_1(z) \quad (z \in [z^0, z^*)). \tag{2.38}$$

For $(z, p) \in V^{+-} \subset G_1$ (see (2.10), (1.18), (2.4)), (1.21), (1.22)) we have

$$r(z, p) = (b - \nu)z + b\gamma - \frac{1}{p} > 0,$$

$$s(z, p) = (\nu - b + \rho)p - \frac{\gamma}{(z + \gamma)z} < 0;$$

hence,

$$\begin{aligned} \frac{\partial f(z, p)}{\partial p} &= \left(\frac{\partial s(z, p)}{\partial p} r(z, p) - \frac{\partial r(z, p)}{\partial p} s(z, p) \right) \frac{1}{r^2(z, p)} \\ &= \left((\nu - b + \rho) r(z, p) - \frac{1}{\partial p^2} s(z, p) \right) \frac{1}{r^2(z, p)} > 0. \end{aligned}$$

Then, in view of (2.38) and (2.36),

$$\frac{d}{dz}\bar{p}_2(z) - \frac{d}{dz}\bar{p}_1(z) \geq 0 \quad (z \in [z^0, z^*)).$$

Hence (see (2.36) again),

$$\bar{p}_2(z) - \bar{p}_1(z) \geq p_2^0 - p_1^0 \quad (z \in [z^0, z^*)),$$

which contradicts (2.35). The contradiction completes the proof for $z^0 < z^*$. The case $z^0 > z^*$ is treated similarly. The lemma is proved.

Given a $z^0 > 0$, the equilibrium solution $(z(\cdot), p(\cdot))$ of the Hamiltonian system (1.23), (1.24), which satisfies $z(0) = z^0$ (and whose uniqueness has been stated in Lemma 2.5) will further be said to be *determined by* z^0.

Lemmas 2.4 and 2.5 yield the next characterization of a solution of problem (1.9) – (1.12).

Theorem 2.1 *Let $(z(\cdot), p(\cdot))$ be the equilibrium solution of the Hamiltonian system (1.23), (1.24) which is determined by z_0. A control process $(z_0(\cdot), u(\cdot))$ is optimal in problem (1.9) – (1.12) if and only if $z_0(\cdot) = z(\cdot)$ and (1.42) holds for a. a. $t \geq 0$.*

Proof. Necessity. Let a control process $(z_0(\cdot), u(\cdot))$ be optimal in problem (1.9) – (1.12). By Lemmas 2.4 and 2.5 $z_0(\cdot) = z(\cdot)$ and (1.42) holds for a. a. $t \geq 0$.

Sufficiency. Let a control process $(z_0(\cdot), u(\cdot))$ satisfy $z_0(\cdot) = z(\cdot)$ and (1.42) hold for a. a. $t \geq 0$. Suppose $(z_0(\cdot), u(\cdot))$ is not optimal in problem (1.9) – (1.12). By Lemma 1.2 there exists an optimal control process $(z_*(\cdot), u_*(\cdot))$. By Lemmas 2.4 and 2.5 $z_*(\cdot) = z(\cdot)$ and (1.42), where $u(t)$ is replaced by $u_*(t)$, holds for a. a. $t \geq 0$. Hence, $z_*(\cdot) = z_0(\cdot)$ and $u_*(t) = u(t)$ for a. a. $t \geq 0$. Therefore, $(z_0(\cdot), u(\cdot))$ is optimal, which contradicts the assumption. The contradiction completes the proof.

Theorem 2.1 and Lemma 2.5 imply the next uniqueness result.

Corollary 2.1 *The optimal control process in problem (1.9) – (1.12) is unique in the following sense: if $(z_1(\cdot), u_1(\cdot))$ and $(z_2(\cdot), u_2(\cdot))$ are optimal control processes in problem (1.9) – (1.12), then $z_1(\cdot) = z_2(\cdot)$ and $u_1(t) = u_2(t)$ for a. a. $t \geq 0$.*

Theorem 2.1 provides the next solution algorithm for problem (1.9) – (1.12).

Algorithm of constructing the optimal control process $(z_0(\cdot), u(\cdot))$ in problem (1.9) – (1.12).

1. Find the equilibrium solution $(z(\cdot), p(\cdot))$ of the Hamiltonian system (1.23), (1.24) which is determined by z_0.
2. Set $z_0(\cdot) = z(\cdot)$ and define $u(\cdot)$ by (1.42) ($t \geq 0$).

2.4 Optimal synthesis

In this section we consider the family of problems (1.9) – (1.12) parametrized by the initial state $z_0 > 0$ and describe an optimal synthesis for this family (see Pontryagin, et. al., 1969), i.e., define a feedback which solves problem (1.9) – (1.12) with arbitrary z_0.

In the construction of an optimal feedback, our main instrument will be one-dimensional representations of the equilibrium solutions of the Hamiltonian system (1.23), (1.24). These are functions $z \mapsto \bar{p}(z)$ solving the one-dimensional equation

$$\frac{d}{dz}\bar{p}(z) = f(z, \bar{p}(z)) \tag{2.39}$$

which is derived from (1.23), (1.24) by deviding its second component by the first one. Thus, in (2.39) and in what follows $f(\cdot) : (z,p) \mapsto f(z,p)$ is defined by (2.37). Note that the domain of definition of $f(\cdot)$ is $\mathrm{dom} f(\cdot) == G \setminus V_z^0$ (see (1.20) and (2.6)); therefore, solutions of (2.39) are understood as those in $\mathrm{dom} f(\cdot) = G \setminus V_z^0$ (i.e., by definition every solution $\bar{p}(\cdot)$ of (2.39) satisfies $(z, \bar{p}(z)) \in G \setminus V_z^0$ for any z from the domain of its definition). A positive solution $\bar{p}(\cdot)$ of (2.39) (in $G \setminus V_z^0$) will be called

(i) a *left equilibrium solution* if $\bar{p}(\cdot)$ is defined on $(0, z^*)$ and

$$\lim_{z \to z^*} \bar{p}(z) = p^*, \tag{2.40}$$

(ii) a *right equilibrium solution* if $\bar{p}(\cdot)$ is defined on (z^*, ∞) and (2.40) holds.

Lemma 2.6 *1. Let $\bar{p}(\cdot)$ be a left equilibrium solution of (2.39). Then*

$$(z, \bar{p}(z)) \in V^{+-} \quad (z \in (0, z^*)). \tag{2.41}$$

2. Let $\bar{p}(\cdot)$ be a right equilibrium solution of (2.39). Then

$$(z, \bar{p}(z)) \in V^{-+} \quad (z \in (z^*, \infty)). \tag{2.42}$$

Proof. We will prove statement 1 only (the proof of statement 2 is similar). Suppose statement 1 is not true, i.e., $(z^0, \bar{p}(z^0)) \notin V^{+-}$ for some $z^0 \in (0, z^*)$. Let $(z(\cdot), p(\cdot))$ be the nonextendable solution of the Hamiltonian system (1.23), (1.24) (in G), which satisfies $(z(0), p(0)) = (z^0, \bar{p}(z^0))$. The definition of V^{-+} (2.11) shows that $z^0 < z^*$ yields $(z(0), p(0)) \notin V^{-+}$. Therefore,

$$(z(0), p(0)) = (z^0, \bar{p}(z^0)) \in \mathrm{cl}(V^{--} \cup W^{--}) \cup \mathrm{cl} V^{++}.$$

Note that $(z^0, \bar{p}(z^0))$ lies in $\mathrm{dom} f(\cdot) = G \setminus V_z^0$ of $f(\cdot)$ (see (2.37)), i.e., $(z^0, \bar{p}(z^0)) \notin V_z^0$. We consider separetely the cases

$$(z(0), p(0)) = (z^0, \bar{p}(z^0)) \in \mathrm{cl}(V^{--} \cup W^{--}) \tag{2.43}$$

and

$$(z(0), p(0)) = (z^0, \bar{p}(z^0)) \in \mathrm{cl} V^{++}. \tag{2.44}$$

Let (2.43) hold. Then by Lemma 2.2

$$(z(t), p(t)) \in V^{--} \cup W^{--} \quad (t \in \Delta \cap (0, \infty)) \tag{2.45}$$

where Δ is the domain of definition of $(z(\cdot), p(\cdot))$; moreover, Δ is bounded and

$$p(\vartheta) = 0 \tag{2.46}$$

where $\vartheta = \sup \Delta$. By (2.45) $(z(t), p(t))$ lies in $\mathrm{dom} f(\cdot) = G \setminus V_z^0$ for all $t \in \Delta$ and $z(\cdot)$ is strictly decreasing. Hence, $\hat{p}(\cdot) : (z(\vartheta), z^0) \mapsto [0, \infty)$ defined by $\hat{p}(\zeta) = p(z^{-1}(\zeta))$ solves equation (2.39). Since $\hat{p}(z^0) = p(0) = \bar{p}(z^0)$, and due to the uniqueness of the solution of a Cauchy problem for equation (2.39), we get $\bar{p}(z) = \hat{p}(z)$ for all $z \in (z(\vartheta), z^0)$. In particular,

$$\bar{p}(z(\vartheta)) = \hat{p}(z(\vartheta)) = p(z^{-1}(z(\vartheta)) = p(\vartheta) = 0$$

(see (2.46)). By the definition of $f(\cdot)$ (see (2.37), (1.21), (1.22)) we have $f(z(\vartheta), 0) > 0$. Hence, solution $\bar{p}(\cdot)$ of (2.39) is nonextendable to the left of $z(\vartheta) > 0$ in G (see (1.20)), which contradicts the assumption that $\bar{p}(\cdot)$ is defined on $(0, z^*)$. Thus, (2.43) is untrue.

Suppose (2.44) holds. Then by Lemma 2.2

$$(z(t), p(t)) \in V^{++} \quad (t > 0) \tag{2.47}$$

and relations (2.25), (2.26) hold. By (2.47) $(z(t), p(t))$ lies in $\mathrm{dom} f(\cdot) = G \setminus V_z^0$ for all $t \geq 0$ and $z(\cdot)$ is strictly increasing. Hence, $\hat{p}(\cdot) : (z(\vartheta), z^0) \mapsto [0, \infty)$ defined by $\hat{p}(\zeta) = p(z^{-1}(\zeta))$ solves equation (2.39). Since $\hat{p}(z^0) = p(0) = \bar{p}(z^0)$, and due to the uniqueness of the solution of a Cauchy problem for equation (2.39), we get $\bar{p}(z) = \hat{p}(z)$ for all $z \in (z^0, z^*)$. Then by (2.25) and (2.26) $\bar{p}(z)) = \hat{p}(z) \to \infty$ as $z \to z^*$, which in not possible, for the left equilibrium solution $\bar{p}(\cdot)$ of (2.39) satisfies (2.40). The contradiction eliminates case (2.44) and completes the proof.

We use Lemma 2.6 for proving the uniqueness part of the next existence and uniqueness theorem.

Theorem 2.2 *There exist the unique left equilibrium solution of (2.39) and the unique right equilibrium solution of (2.39).*

Proof. We will prove the existence and uniqueness of the left equilibrium solution of (2.39) (the existence and uniqueness of the right equilibrium solution is stated similarly). Take a $z^0 \in (0, z^*)$. By Lemma 2.5 there exists an equilibrium solution $(z(\cdot), p(\cdot))$ of the Hamiltonian system (1.23), (1.24), which satisfies $z(0) = z^0$. By Lemma 2.3, (ii),

$$(z(t), p(t)) \in V^{+-} \quad (t \geq 0). \tag{2.48}$$

Hence, $\dot{z}(t) = r(z(t), p(t)) > 0$ $(t > 0)$ and $\bar{p}(\cdot) : [z^0, z^*) \mapsto [p(0), \infty)$ defined by $\bar{p}(z) = p(z^{-1}(z))$ solves (2.39). By definition the equilibrium solution $(z(\cdot), p(\cdot))$ satisfies (2.27), (2.28), which implies (2.40). By (2.48) $(z, \bar{p}(z)) \in V^{+-}$ for all $z \in [z^0, z^*)$. Now consider a solution $\hat{p}(\cdot)$ of (2.39) in V^{+-}, which is nonextendable to the left and satisfies $\hat{p}(z) = \bar{p}(z)$ for all $z \in [z^0, z^*)$. Let us fix the fact that

$$(z, \hat{p}(z)) \in V^{+-} \quad (z \in [z^0, z^*)). \tag{2.49}$$

In order to state that $\hat{p}(\cdot)$ is a left equilibrium solution of (2.39), it is sufficient to show that its domain of definition is $(0, z^*)$. Suppose the domain of definition of $\hat{p}(\cdot)$ is (ζ, z^*) where $\zeta > 0$. For all $z \in (\zeta, z^*)$, we have $(z, \hat{p}(z)) \in V^{+-}$ and hence, $f(z, \hat{p}(z)) > 0$. Therefore, $\hat{p}(\cdot)$ is decreasing and there is the limit

$$\pi = \lim_{z \to \zeta} \hat{p}(z)$$

satisfying

$$\pi > p^*. \tag{2.50}$$

By the definition of V^{+-} (see (2.10)) the set $\{(z, p) \in V^{+-} : z \geq \zeta\}$ is bounded. Consequently, π is finite and (ζ, π) lies on the boundary of V^{+-}. Two cases are admissible:

$$(\zeta, \pi) \in V_z^0 \tag{2.51}$$

(see (2.6)) and

$$(\zeta, \pi) \in V_p^0 \tag{2.52}$$

(see (2.7). If (2.51) holds, then $\pi = h_1(\zeta) < h_1(z^*) = p^*$ (recall that $\zeta < z^*$ and $h_1(\cdot)$ is strictly increasing); we get a contradiction with (2.50). Thus, (2.51) is not possible.

Suppose (2.52) holds. Then $\pi = h_2(\zeta)$ and $s(\zeta,\pi) = 0$; the latter implies $f(\zeta,\pi) = 0$ (see (2.37)). Take an $\varepsilon > 0$. There is a $\delta > 0$ such that

$$\left|\frac{d}{dz}\bar{p}(z)\right| < \varepsilon \quad (z \in (\zeta, \zeta+\delta]).$$

Let

$$2\varepsilon < 1 = \inf_{z\in[\zeta,z^*]} |h_2'(z)|$$

(recall that $h_2(\cdot)$ is strictly decreasing) and

$$\zeta_1 \in (\zeta, \zeta + \varepsilon\delta/2]$$

satisfy $\zeta_1 < z^* - \delta$ (with no loss of generality we assume that δ is small enough, for example, $\delta < (z^* - \zeta)/2$) and

$$|\hat{p}(\zeta_1) - \pi| < \varepsilon\delta/2.$$

Then using the fact that $h_2(\cdot)$ is decreasing, we get

$$\begin{aligned} \hat{p}(\zeta_1 + \delta/2) &> \hat{p}(\zeta_1) - \varepsilon\delta/2 > \pi - \varepsilon\delta \\ &= h_2(\zeta) - \varepsilon\delta > h_2(\zeta) - 1\delta/2 > h_2(\zeta + \delta/2) \\ &> h_2(\zeta_1 + \delta/2). \end{aligned}$$

Hence, $\hat{p}(\zeta_1 + \delta/2) \notin V^{+-}$ (see (2.10)), which contradicts (2.49). Thus, (2.52) is not possible. We have proved that $\hat{p}(\cdot)$ is defined on $(0, z^*)$. Consequently, $\hat{p}(\cdot)$ is a left equilibrium solution of (2.39).

It remains to prove that the left equilibrium solution of (2.39) is unique. Suppose there are two left equilibrium solutions of (2.39), $\hat{p}_1(\cdot)$ and $\hat{p}_2(\cdot)$. So,

$$\hat{p}_1^0 = \hat{p}_1(z^0) \neq \hat{p}_2(z^0) = \hat{p}_2^0 \tag{2.53}$$

for some $z^0 \in (0, z^*)$. By Lemma 2.6

$$\hat{p}_i(z) \in V^{+-} \quad (z \in (0, z^*)), \tag{2.54}$$

$i = 1, 2$. Let $(z_i(\cdot), p_i(\cdot))$ be the nonextendable solution of the Hamiltonian system (1.23), (1.24) (in G), which satisfies

$$(z_i(0), p_i(0)) = (z^0, \hat{p}_i^0), \tag{2.55}$$

$i = 1, 2$. Take an $i \in \{1, 2\}$. Point $(z_i(0), p_i(0)) \in V^{-+}$ lies in $\mathrm{dom} f(\cdot) = G \setminus V_z^0$; therefore, $(z_i(t), p_i(t)) \in G \setminus V_z^0$ for all t from a right neighborhood of 0. Let ϑ_i be the supremum of all $\tau \geq 0$ such that $(z_i(t), p_i(t)) \in \mathrm{dom} f(\cdot)$ for every $t \in [0, \tau]$, $i = 1, 2$. Then necessarily

$$\dot{z}_i(t) = r(z_i(t), p_i(t)) > 0 \quad (t \in [0, \vartheta_i)); \tag{2.56}$$

hence, setting

$$\xi_i = \lim_{t\to\vartheta_i} z_i(t),$$

we find that $\bar{p}_i(\cdot) : [z^0, \xi_i) \mapsto [0, \infty)$ defined by

$$\bar{p}_i(\zeta) = p_i(z_i^{-1}(\zeta)) \tag{2.57}$$

solves (2.39) Consequently, $\bar{p}_i(z) = \hat{p}_i(z)$ for all $z \in [z^0, \min\{z^*, \xi_i\})$.

Suppose $\xi_i < z^*$. By the definition of ϑ_i

$$\lim_{\zeta\to\xi_i}(\zeta,\bar{p}_i(z_i(\zeta)) = \lim_{\zeta\to\xi_i}(\zeta,\hat{p}_i(z_i(\zeta)) = (\xi_i,\hat{p}_i(\xi_i)) \notin \mathrm{dom} f(\cdot),$$

which is not possible, for $(z,\hat{p}_i(z)) \in \mathrm{dom} f(\cdot)$ for all $z \in (0,z^*)$. Thus,

$$\xi_i \geq z^*. \tag{2.58}$$

As soon as $\hat{p}_i(\cdot)$ is a left equilibrium solution of (2.39), we have

$$\hat{p}_i(\zeta) = \bar{p}_i(\zeta) \to p^* \quad \text{as} \quad \zeta \to z^*. \tag{2.59}$$

Suppose inequality (2.58) is strict, i.e., $\xi_i > z^*$. Then $\bar{p}_i(z^*) = p_*$ and $(z_i(\tau_i),p_i(\tau_i)) = (z^*,p^*)$ where $\tau_i = z_i^{-1}(z^*)$; consequently, $(z_i(\cdot),p_i(\cdot))$ is the stationary solution of the Hamiltonian system (1.23), (1.24), i.e., $(z_i(t),p_i(t)) = (z^*,p^*)$ for all t from its domain of definition, which contradicts (2.55) (recall that $z^0 < z^*$. Thus, (2.58) is in fact the equality, $\xi_i = z^*$. Then referring to (2.59), (2.56), (2.57), we find that

$$\lim_{t\to\vartheta_i} p_i(t) = \lim_{z\to z^*} \bar{p}_i(\zeta) = p^* \tag{2.60}$$

Recall that $(z_i(\cdot),p_i(\cdot))$ is not the stationary solution of the Hamiltonian system (1.23), (1.24), i.e., $(z_i(t),p_i(t)) \neq (z^*,p^*)$ for all t. Then by Lemma 2.2 (2.60) yields $\vartheta_i = \infty$. Therefore, $(z_i(\cdot),p_i(\cdot))$ is an equilibrium solution of the Hamiltonian system (1.23), (1.24) which satisfies $z_i(0) = z^0$ (see (2.55)). This holds for $i = 1,2$. Hence, by the uniqueness Lemma 2.5 $(z_1(\cdot),p_1(\cdot)) = (z_2(\cdot),p_2(\cdot))$. However, (2.53) and (2.55) show that $p_1(0) \neq p_2(0)$. The contradiction completes the proof of the theorem.

In Fig. 2 the left and right equilibrium solutions of equation (2.39) are shown.

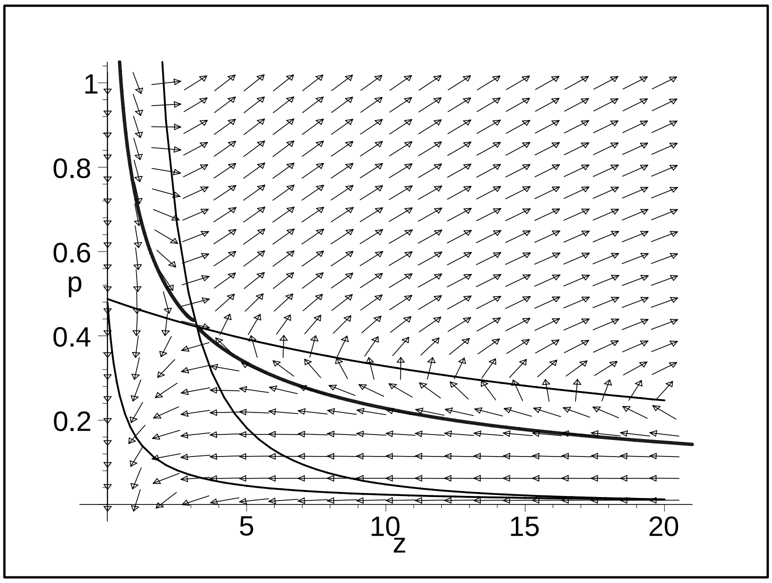

Figure 2: The left and right equilibrium solutions of (2.39) for $\nu = 4$, $b = 4.1$, $\rho = 0.2$, $\gamma = 0.5$ (a Mapple simulation).

In what follows we denote the unique left equilibrium solution of (2.39) by $\bar{p}_-(\cdot)$ and the unique left equilibrium solution of (2.39) by $\bar{p}_+(\cdot)$.

Now we are ready to construct a desired optimal synthesis $U(\cdot)$. The idea is the following. In the expression (1.42) for an optimal control $u(t)$ we replace $z(t)$ by a free z and replace $p(t)$ by $\bar{p}_-(z)$ if $z < z^*$, on $\bar{p}_+(z)$ if $z > z^*$, and by p^* if $z = z^*$. Thus, we define $U(\cdot) : (0, \infty) \mapsto [0, b)$ by

$$U(z) = \begin{cases} b - \dfrac{1}{p^*(z^* + \gamma)}, & \text{if} \quad z = z^*, \\ b - \dfrac{1}{\bar{p}_-(z)(z + \gamma)}, & \text{if} \quad z \in (0, z^*), \\ b - \dfrac{1}{\bar{p}_+(z)(z + \gamma)}, & \text{if} \quad z \in (0, z^*), \ (z, \bar{p}_+(z)) \in G_1, \\ 0 & \text{if} \quad z \in (0, z^*), \ (z, \bar{p}_+(z)) \in G_2; \end{cases} \tag{2.61}$$

note that by Lemma 2.6 we have $(z, \bar{p}_-(z)) \in V^{+-}$ for $z \in (0, z^*)$; as soon as $V^{+-} \subset G_1$ (see (2.10), (1.18), (2.4)), $U(z)$ is given by the single formula for $z \in (0, z^*)$.

Lemma 2.7 *Function $U(\cdot)$ (2.61) is a feedback.*

Proof. Obviously, $U(\cdot)$ is continuous at every $z \neq z^*$. The fact that $\bar{p}_-(\cdot)$ is the left equilibrium solution and $\bar{p}_+(\cdot)$ is the right equilibrium solution of (2.39) implies that $U(\cdot)$ is continuous at z^* as well. Moreover, the right hand side of equation (1.13) (for the "closed" system) is, obviously, Lipschitz on every bounded interval in $(0, \infty)$ which does not intersect a neighborhood of z^*. If it is also Lipschitz in a nerighborhood of z^*, then for every $z_0 > 0$, equation (1.13) has the unique solution $z(\cdot)$ defined on $[0, \infty)$ and satisfying $z(0) = z_0$, which proves that $U(\cdot)$ is a feedback. Now we will state that the right hand side of (1.13) is Lipschitz in a neighborhood of z^*. It is sufficient to show that $U(\cdot)$ is Lipschitz in a neighborhood of z^*; this is so if, in turn, $\bar{p}_-(\cdot)$ and $\bar{p}_+(\cdot)$ are Lipschitz in a neighborhood of z^* (see formula (2.61)). To prove the Lipschitz character of $\bar{p}_-(\cdot)$ and $\bar{p}_+(\cdot)$ in a neighborhood of z^* it is enough to verify that

$$\limsup_{z \to z^*} \frac{d}{dz} \bar{p}_-(z) < \infty, \tag{2.62}$$

$$\liminf_{z \to z^*} \frac{d}{dz} \bar{p}_-(z) > -\infty, \tag{2.63}$$

$$\limsup_{z \to z^*} \frac{d}{dz} \bar{p}_+(z) < \infty, \tag{2.64}$$

$$\liminf_{z \to z^*} \frac{d}{dz} \bar{p}_+(z) > -\infty. \tag{2.65}$$

By Lemma 2.6

$$\bar{p}_-(z) \in V^{+-} \quad (z \in (0, z^*), \tag{2.66}$$

$$\bar{p}_+(z) \in V^{-+} \quad (z \in (z^*, \infty). \tag{2.67}$$

Hence, $d\bar{p}_-(z)/dz < 0$ $(z \in (0, z^*)$ and $d\bar{p}_+(z)/dz < 0$ $(z \in (z^*, \infty)$. Thus, (2.62) and (2.64) hold.

Let us show (2.63). Take a $z \in (0, z^*)$. We have $(z, \bar{p}_-(z)) \in G_1$ and (see (2.39), (2.37), (1.21), (1.22))

$$\frac{d}{dz} \bar{p}_-(z) = f(z, \bar{p}_-(z)) = \frac{s(z, \bar{p}_-(z))}{r(z, \bar{p}_-(z))},$$

$$s(z, \bar{p}_-(z)) < 0, \quad r(z, \bar{p}_-(z)) > 0.$$

By (2.66)

$$\bar{p}_-(z) > \lim_{z \to z^*} \bar{p}_-(z) = p^*.$$

Then

$$0 > s(z, \bar{p}_-(z)) = (\nu - b + \rho)\bar{p}_-(z) - \frac{\gamma}{(z+\gamma)z} > (\nu - b + \rho)p^* - \frac{\gamma}{(z+\gamma)z},$$

$$r(z, \bar{p}_-(z)) = (b - \nu)z + b\gamma - \frac{1}{\bar{p}_-(z)} > (b - \nu)z + b\gamma - \frac{1}{p^*} > 0.$$

Using the Lopital theorem, we get

$$\begin{aligned} \liminf_{z \to z^*} \frac{d}{dz}\bar{p}_-(z) &\geq \lim_{z \to z^*} \frac{(\nu - b + \rho)p^* - \frac{\gamma}{(z+\gamma)z}}{(b-\nu)z + b\gamma - \frac{1}{p^*}} \\ &= \frac{\gamma}{(z^*+\gamma)^2 z^{*2}} \frac{2z^* + \gamma}{b - \nu} \end{aligned}$$

Inequality (2.63) is proved.

Let us show (2.65). Take a $z > z^*$ sufficiently close to z^*. Since (z^*, p^*) lies in the interior of G_1, we have $(z, \bar{p}(z)_+) \in G_1$.

$$\frac{d}{dz}\bar{p}_+(z) = f(z, \bar{p}_-(z)) = \frac{s(z, \bar{p}_+(z))}{r(z, \bar{p}_+(z))},$$

$$s(z, \bar{p}_+(z)) > 0, \quad r(z, \bar{p}_-(z)) < 0.$$

By (2.66)

$$\bar{p}_+(z) < \lim_{z \to z^*} \bar{p}_+(z) = p^*.$$

Then

$$0 > s(z, \bar{p}_+(z)) = (\nu - b + \rho)\bar{p}_+(z) - \frac{\gamma}{(z+\gamma)z} > (\nu - b + \rho)p^* - \frac{\gamma}{(z+\gamma)z},$$

$$r(z, \bar{p}_+(z)) = (b - \nu)z + b\gamma - \frac{1}{\bar{p}_+(z)} < (b - \nu)z + b\gamma - \frac{1}{p^*} < 0.$$

Using the Lopital theorem, we get

$$\begin{aligned} \liminf_{z \to z^*} \frac{d}{dz}\bar{p}_+(z) &\geq \lim_{z \to z^*} \frac{(\nu - b + \rho)p^* - \frac{\gamma}{(z+\gamma)z}}{(b-\nu)z + b\gamma - \frac{1}{p^*}} \\ &= \frac{\gamma}{(z^*+\gamma)^2 z^{*2}} \frac{2z^* + \gamma}{b - \nu} \end{aligned}$$

which proves (2.65).

The proof is finished.

The next theorem presents our final result.

Theorem 2.3 *Feedback $U(\cdot)$ (2.61) is an optimal synthesis.*

Proof. Take a $z_0 > 0$. We must show that the control process under feedback $U(\cdot)$ with the initial state z_0 is an optimal control process in problem (1.9) – (1.12).

Consider the equilibrium solution $(z(\cdot), p(\cdot))$ of the Hamiltonian system (1.23), (1.24) which satisfies $z(0) = z_0$. By Lemma 2.5 this solution is unique and by Theorem 2.1 the pair $z(\cdot), u(\cdot))$ where $u(\cdot)$ is given by (1.42) is an optimal control process in problem (1.9) – (1.12). Therefore, in order to complete the proof of the theorem, it is sufficient to state that $z(\cdot), u(\cdot))$ is the control process under feedback $U(\cdot)$ with the initial state z_0.

Suppose $z_0 = z^*$. Then by Lemma 2.2 $(z(t), p(t)) = (z^*, p^*)$ $(t \geq 0)$ and by (1.42) and (2.61) $u(t) = U(z^*)$ $(t \geq 0)$. For $z(t) = z^*$ the right hand side of equation (1.13) for the "closed" system is zero; Thus, $z(\cdot), u(\cdot))$ is the control process under feedback $U(\cdot)$ with the initial state $z_0 = z^*$.

Consider the case $z_0 < z^*$ (the case $z_0 > z^*$ is treated similarly). By Lemma 2.3 we have

$$(z(t), p(t)) \in V^{+-} \quad (t \geq 0). \tag{2.68}$$

Consequently, $(z(t), p(t)) \in \mathrm{dom} f(\cdot) = G \setminus V_z^0$ for all $t \geq 0$ and $z(\cdot)$ is strictly increasing. Since $(z(\cdot), p(\cdot))$ is an equilibrium solution of (1.23), (1.24), the limit relations (2.27), (2.28) hold. Hence, the function $\bar{p}(\cdot) : \zeta \mapsto \bar{p}(\zeta) = p(z^{-1}(\zeta))$ is defined on $[z_0, z^*)$ and solves equation (2.39) on this interval. Due to (2.27), (2.28) $\bar{p}(\cdot)$ satisfies the limit relation (2.40). Therefore, $\bar{p}(\cdot)$ is the restriction to $[z^0, z^*)$ of the (unique) left equilibrium solution $\bar{p}_-(\cdot)$ of (2.39), and we have

$$p(t) = \bar{p}(z(t)) = \bar{p}_-(z(t)) \quad (t \geq 0). \tag{2.69}$$

By Theorem 2.1 $(z(\cdot), u(\cdot))$ where $u(\cdot)$ is defined by (1.42) is an optimal control process in problem (1.9) – (1.12). Now we replace $p(t)$ in (1.42) by $\bar{p}(z(t))$ (see (2.69)). Comparing with (2.61), we find that $u(t) = U(z(t))$ $(t \geq 0)$. Then

$$\dot{z}(t) = r(z(t), p(t)) = r(z(t), \bar{p}(z(t))) = U(z(t))(z(t) + \gamma) - \nu z(t)$$

$(t \geq 0)$, i.e., $z(\cdot)$ solves equation (1.13) for the "closed" system on $[0, \infty)$. Hence, $(z(\cdot), u(\cdot))$ is the control process under feedback $U(\cdot)$ with the initial state z_0. The proof is completed.

In Fig. 3 the shape of the optimal synthesis $U(\cdot)$ is illustrated.

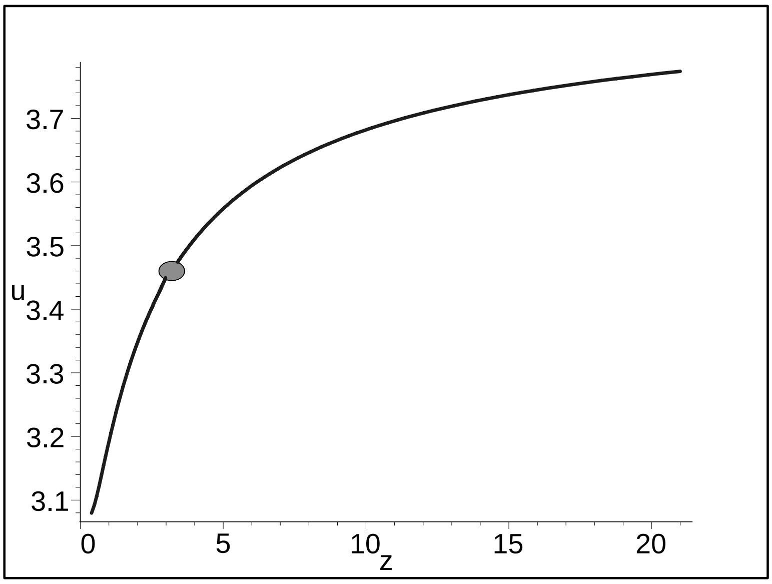

Figure 3: The optimal synthesis for $\nu = 4$, $b = 4.1$, $\rho = 0.2$, $\gamma = 0.5$ (a Mapple simulation).

Theorem 2.3 provides the next algorithm for the construction of solutions in the family of problems (1.9) – (1.12) parametrized by the initial state.

Algorithm for the construction of the optimal control processes in the family of problems (1.9) – (1.12) parametrized by the initial state z_0.

1. Find the left equilibrium solution $\bar{p}_-(\cdot)$ and the right equilibrium solution $\bar{p}_+(\cdot)$ of equation (2.39).

2. Given a $z_0 > 0$, find the optimal control process $(z(\cdot), u(\cdot))$ in problem (1.9) – (1.12) as the control process under feedback $U(\cdot)$ (2.61) with the initial state z_0.

3 Design of optimal control. Situation 2

3.1 Assumption

In what follows we consider situation 2, $b > \nu + \rho$ (1.7).

3.2 Qualitative analysis of Hamiltonian system

The vector field of the Hamiltonian system (1.34), (1.35) in G (see (1.20)) is the union of the vector fields of the nondegenerate equation (1.34), (1.35) in G_1 (see (1.32) and the degenerate equation (1.36), (1.37) in G_2 (see (1.33).

The vector field of the nondegenerate equation (1.34), (1.35) in G_1 has the following structure.

Define

$$h(q,p) = \rho q(t) - \frac{\gamma p(t)}{q(t) + \gamma p(t)} + b\gamma p(t) - 1. \tag{3.1}$$

The right hand side of equation (1.34) (for $q(\cdot)$) is zero on the curve

$$V_q^0 = \{(q,p) \in G_1 : h(q,p) = 0\}, \tag{3.2}$$

positive in the domain

$$V_q^+ = \{(q,p) \in G_1 : h(q,p) > 0\}$$

and negative in the domain

$$V_q^- = \{(q,p) \in G_1 : h(q,p) < 0\}.$$

The right hand side of equation (1.35) (for $p(\cdot)$) is always negative in the domain G_1. Thus, the vector field of the nondegenerate equation (1.34), (1.35) is

(i) positive in the q coordinate and negative in the p coordinate in the domain

$$V^{+-} = V_q^+ \tag{3.3}$$

(ii) negative in both coordinates in the domain

$$V^{--} = V_q^-. \tag{3.4}$$

In what follows, cl$E$ denotes the closure of a set $E \subset R^2$.

There exists unique rest point of equation (1.34), (1.35)

$$q^* = \frac{1}{\rho} \tag{3.5}$$

$$p^* = 0 \tag{3.6}$$

in clG_1.

The vector field of the degenerate equation (1.36), (1.37) is negative in both coordinates in the domain G_2.

Let us analyze how the vector fields of the nondegenerate equation (in G_1) and degenerate equation (in G_2) are pasted together. Note that G_1 and G_2 are separated by the curve

$$G^0 = \{(q,p) \in R^2 : q > 0,\ p = \frac{1-qb}{b\gamma}\}$$

(see (1.32) and (1.33)).

Subsitute

$$p(q) = \frac{1-qb}{b\gamma}$$

into (3.1) and solve this equation with respect to q The unique solution is

$$q = \frac{1}{\rho}.$$

Due to (1.7)

$$p(\frac{1}{\rho}) = \frac{\rho - b}{b^2\gamma}$$

is negative. Thus the curve V_q^0 (3.2) does not intersect G^0. We end up with the next description of the vector field of (1.40), (1.41).

Lemma 3.1 *The vector field of the Hamiltonian system (1.40), (1.41) (in G) is*

(i) positive in the q coordinate and negative in the p coordinate in domain V^{+-} (see (3.3)),

(ii) negative in both coordinates in domain $V^{--} \cup G_2$ (see (3.4) and (1.33)),

The rest point (q^*, p^*) of (1.40), (1.41) in cl G is unique; it is defined by (3.5), (3.6)

The vector field of system (1.40), (1.41) is shown in Fig. 4.

Lemma 3.1 allows us to give a full classification of the qualitative behaviors of the solutions of the Hamiltonian system (1.40), (1.41) in G (see also Fig. 4).

Lemma 3.2 *Let $(q(\cdot), p(\cdot))$ be a solution of (1.40), (1.41) in G, which is nonextendable to the right, Δ be the interval of its definition, $t_* \in \Delta$, and $(q(t_*), p(t_*)) \neq (q^*, p^*)$.*

The following statements are true.

1. If $(q(t_*), p(t_*)) \in \mathrm{cl}(V^{--} \cup G_2)$, then Δ is bounded, $(q(t), p(t)) \in V^{--} \cup G_2$ for all $t \in \Delta \cap (t_*, \infty)$, and $p(\vartheta) = 0$ where $\vartheta = \sup \Delta$.

2. If $(q(t_*), p(t_*)) \in V^{+-}$, then one of the next cases (i), (ii), (iii) takes place:

(i) Δ is bounded and $(q(t), p(t)) \in \mathrm{cl}(V^{--} \cup G_2)$ for all $t \in \Delta \cap [t^* \infty)$ with some $t^* \in \Delta \cup [t_*, \infty)$;

(ii) Δ is unbounded, $(z(t), p(t)) \in \mathrm{cl} V^{+-}$ for all $t \in [t_* \infty)$ with some $t^* \in \Delta \cup [t_*, \infty)$ and relations

$$\lim_{t \to \infty} q(t) = \infty, \tag{3.7}$$

and

$$\lim_{t \to \infty} p(t) = 0 \tag{3.8}$$

hold;

(iii) Δ *is unbounded,* $(q(t),p(t)) \in \mathrm{cl}V^{+-}$ *for all* $t \in [t_*\infty)$ *and*

$$\lim_{t\to\infty} q(t) = q^*, \tag{3.9}$$

$$\lim_{t\to\infty} p(t) = p^*. \tag{3.10}$$

Proof. 1. Let $(q(t_*),p(t_*)) \in \mathrm{cl}(V^{--}\cup G_2)$. The fact that the vector field of (1.40), (1.41) is negative in both coordinates in $V^{--}\cup G_2$ (Lemma 3.1) and the locations of V^{--} and G_2 in G (see (3.4) and (1.33)) imply that the set $\mathrm{cl}(V^{--}\cup G_2)\cap\{(q,p)\in G : p \le p(t_*)\}$ is invariant for (1.40), (1.41); moreover, $(q(t),p(t)) \in V^{--}\cup G_2$ for all $t \in \Delta\cap(t_*,\infty)$ and there are a $\delta > 0$ and a $t^* \in \Delta\cap[t^*,\infty)$ such that $\dot p(t) \le -\delta$ for all $t \ge \Delta\cap[t^*,\infty)$. Hence, $p(\vartheta) = 0$ for some finite ϑ, i.e., $(q(\cdot,p(\cdot))$ is nonextendabe to the right in G and $\vartheta = \sup\Delta$.

2. Let $(q(t_*),p(t_*)) \in V^{+-}$. Due to the definitions of V^{+-} and V_q^0 (see (3.3), (3.2), cases

1. $\exists t^* \ge t_*$, such that $(q(t^*),p(t^*)) \in V^{--}$
2. $\exists t^* \ge t_*$, such that

   $q(t^*) > q_*$
3. $q(t) \to q_*$, $p(t) \to p_*$

are admissible. Let case 1 take place. Then we have the situation described in statement 1 (with t_* replaced by t^*); hence, (i) holds.

In case 2 there is a $\delta > 0$

such that $\dot q(t) \ge \delta$ and $\dot p(t) \to 0$ for all $t \ge \Delta\cap[t^*,\infty)$. Therefore, Δ is unbounded and (3.7) holds.

Let case 3 take place. Then $q(t) \to q_*$, $p(t) \to p_*$, $\dot q(t) \to 0$, $\dot p(t) \to 0$, Thus (iii) holds.

The proof is finished.

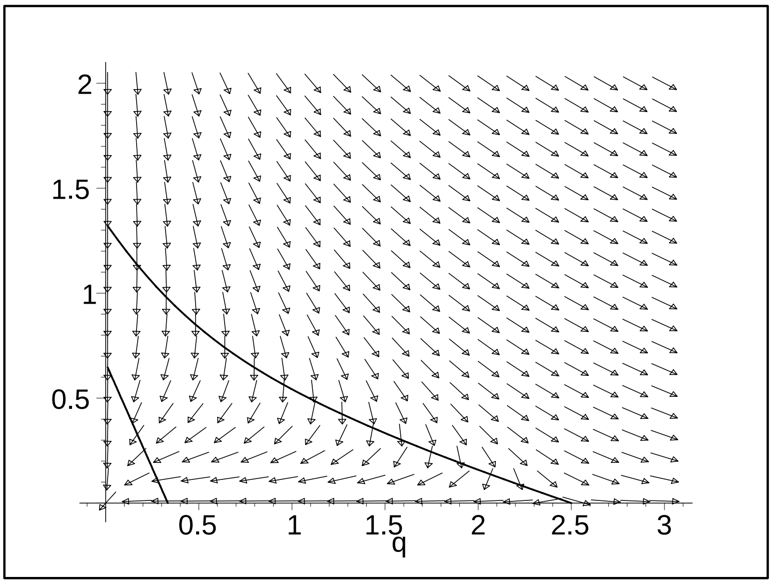

Figure 4: The vector field of the Hamiltonian system (1.40), (1.41) for $\nu = 2$, $b = 3$, $\rho = 0.4$, $\gamma = 0.5$ (a Mapple simulation).

3.3 Optimal control process

In this section we give an entire description of a solution of problem (1.9) – (1.12) and state its uniqueness.

The core of the analysis is Lemma 3.3, which selects solutions of the Hamiltonian system (1.40), (1.41) (we call them equilibrium solutions) whose qualitative behavior agrees with the Pontryagin maximum principle and also acts as a necessary condition for the global optimality in problem (1.9) – (1.12).

We call a solution $(q(\cdot), p(\cdot))$ (in G) of the Hamiltonian system (1.40) (1.41) an *equilibrium solution* if it is defined on $[0, \infty)$ and converges to the rest point (q^*, p^*), i.e. satisfies (3.10) and (3.9).

Lemma 3.3 *Let $(z(\cdot), u(\cdot))$ be an optimal control process in problem (1.9) – (1.12). Then*

(i) there exists a (nonnegative) function $p(\cdot)$ such that $(q(\cdot), p(\cdot))$ where $q(\cdot) = p(\cdot)z(\cdot)$ is an equilibrium solution of the Hamiltonian system (1.40) (1.41),

(ii) for a. a. $t \geq 0$ (1.42) holds.

Proof. By Lemma 1.1 there exists a nonnegative function $p(\cdot)$ defined on $[0, \infty)$ such that $(q(\cdot), p(\cdot))$ solves (1.40) (1.41) (in G) on $[0, \infty)$. According to Lemma 3.2 three cases are admissible:

Case 1: the interval of definition of $(q(\cdot), p(\cdot))$ is bounded (Lemma 3.2, statement 1, statement 2, (i)).

Case 2: relations (3.7), (3.8) hold (Lemma 3.2, statement 2, (ii)).

Case 3: relations (3.9), (3.10) hold, i.e., $(q(\cdot), p(\cdot))$ is equilibrium (Lemma 3.2, statement 2, (iii)).

Case 1 is not possible since $(q(\cdot), p(\cdot))$ is defined on $[0, \infty)$.

Case 2 contradicts the transversality condition $q(t) \leq \frac{1}{\rho}$ (see Lemma 1.1)

Thus, Case 2 is not possible. By excluding Cases 1 and 2 we state that Case 3 takes place. By Lemma 1.1 (ii) is true. The lemma is proved.

Lemma 3.4 *For every z^0 there exists the unique equilibrium solution $(q(\cdot), p(\cdot))$ of the Hamiltonian system (1.40), (1.41), which satisfies*
$\frac{q(0)}{p(0)} = z^0$.

Proof. The existence of a desired equilibrium solution follows from the existence of an optimal control process. Indeed, by Lemma 1.2 there exists an optimal control process $(z(\cdot), u(\cdot))$ in problem (1.9) – (1.12). Setting $z_0 = z^0$ in problem (1.9) – (1.12), we get $z(0) = z^0$. By Lemma 3.3, (i), there exists a function $p(\cdot)$ such that $(q(\cdot), p(\cdot))$ is an equilibrium solution of the Hamiltonian system (1.40) (1.41). (Note that the existence of a desired equilibrium solution can also be proved explicitly).

Let us state the uniqueness of the considered equilibrium solution. Suppose there are two different equilibrium solutions of (1.40), (1.41), $(q_1(\cdot), p_1(\cdot))$ and $(q_2(\cdot), p_2(\cdot))$, We consider case $q_1(0) = q_2(0) = q^0$, case $q_1(0) \neq q_2(0)$ could be treated similarly. Then

$$\lim_{t\to\infty} q_i(t) = q^*, \quad \lim_{t\to\infty} p_i(t) = p^*, \tag{3.11}$$

$i = 1, 2$, and $p_2(0) \neq p_1(0)$ (otherwise $(q_2(\cdot), p_2(\cdot))$ and $(q_1(\cdot), p_1(\cdot))$ coincide due to the uniqueness of the solution of a Cauchy problem for equation (1.40), (1.41)). Denote $p_i^0 = p_i(0)$, $i = 1, 2$. With no loss of generality assume

$$p_2^0 > p_1^0. \tag{3.12}$$

Thus,

$$(q_i(t), p_i(t)) \in V^{+-} \quad (t \geq 0), \tag{3.13}$$

$i = 1,2$. Hence, $\dot{q}_i(t) > 0$ $(t > 0)$, $i = 1,2$. Define $\bar{p}_i(\cdot) : [q^0, q^*) \mapsto [p_i(0), \infty)$ by $\bar{p}_i(\zeta) = p_i(q_i^{-1}(\zeta))$ Due to (3.11) $\lim_{q \to q_*} \bar{p}_i(q) = p^*$, $i = 1,2$, in particular,

$$\lim_{q \to q_*} (\bar{p}_2(q) - \bar{p}_1(q)) = 0. \tag{3.14}$$

We have

$$\frac{d}{dq}\bar{p}_i(q) = f(q, \bar{p}_i(q)), \quad (q \in [q^0, q^*)), \quad p(q^0) = p_i(0), \tag{3.15}$$

$i = 1,2$, where

$$f(q,p) = \frac{s(q,p)}{r(q,p)} \tag{3.16}$$

(recall that $r(\cdot)$ and $s(\cdot)$ determine the right hand side of the Hamiltonian system (1.40), (1.41)). Due to (3.12)

$$\bar{p}_2(q) > \bar{p}_1(q) \quad (q \in [q^0, q^*)). \tag{3.17}$$

For $(q,p) \in V^{+-} \subset G_1$ (see (3.3), (1.32), (3.1)), (1.38), (1.39)) we have

$$r(q,p) = \rho q(t) - \frac{\gamma p(t)}{q(t) + \gamma p(t)} > 0,$$

$$s(z,p) = (\nu - b + \rho)p(t) - \frac{\gamma p^2(t)}{(q(t) + \gamma p(t))q(t)} < 0;$$

hence,

$$\frac{\partial f(q,p)}{\partial p} \quad = \quad \left(\frac{\partial s(q,p)}{\partial p} r(q,p) - \frac{\partial r(q,p)}{\partial p} s(q,p)\right) \frac{1}{r^2(q,p)}.$$

If

$$p \to 0$$

and

$$q \to \frac{1}{\rho} - 0$$

then

$$\frac{\partial f(q,p)}{\partial p} \to (b - \nu - \rho)q^3(1 - \rho q)\frac{1}{r^2(q,0)} > 0,$$

due to (1.7)

Then, in view of (3.17) and (3.15),

$$\frac{d}{dq}\bar{p}_2(q) - \frac{d}{dq}\bar{p}_1(q) \geq 0 \quad (q \in [q, q^*)),$$

for some $q < q^*$. Hence (see (3.15) again),

$$\bar{p}_2(q) - \bar{p}_1(q) \geq \bar{p}_2(\bar{q}) - \bar{p}_1(\bar{q}) \geq 0 \quad (q \in [q, q^*)),$$

due to (3.17) which contradicts (3.14). The contradiction completes the proof.

Given a

$$z^0 = \frac{q^0}{p^0} > 0,$$

the equilibrium solution $(q(\cdot), p(\cdot))$ of the Hamiltonian system (1.40), (1.41), which satisfies $q(0)/p(0) = z^0$ (and whose uniqueness has been stated in Lemma 3.4) will further be said to be *determined by* z^0.

Lemmas 3.3 and 3.4 yield the next characterization of a solution of problem (1.9) – (1.12).

Theorem 3.1 *Let $(q(\cdot), p(\cdot))$ be the equilibrium solution of the Hamiltonian system (1.40), (1.41), which is determined by z_0. A control process $(z_0(\cdot), u(\cdot))$ is optimal in problem (1.9) – (1.12) if and only if*

$$z_0(\cdot) = \frac{q(\cdot)}{p(\cdot)}$$

and (1.42) holds for a. a. $t \geq 0$.

Proof. Necessity. Let a control process $(z_0(\cdot), u(\cdot))$ be optimal in problem (1.9) – (1.12). By Lemmas 3.3 and 3.4

$$z_0(\cdot) = \frac{q(\cdot)}{p(\cdot)}$$

and (1.42) holds for a. a. $t \geq 0$.

Sufficiency. Let a control process $(z_0(\cdot), u(\cdot))$ satisfy

$$z_0(\cdot) = \frac{q(\cdot)}{p(\cdot)}$$

and (1.42) hold for a. a. $t \geq 0$. Suppose $(z_0(\cdot), u(\cdot))$ is not optimal in problem (1.9) – (1.12). By Lemma 1.2 there exists an optimal control process $(z_*(\cdot), u_*(\cdot))$. By Lemmas 3.3 and 3.4

$$z_*(\cdot) = \frac{q(\cdot)}{p(\cdot)}$$

and (1.42) where $u(t)$ is replaced by $u_*(t)$, holds for a. a. $t \geq 0$. Hence, $z_*(\cdot) = z_0(\cdot)$ and $u_*(t) = u(t)$ for a. a. $t \geq 0$. Therefore, $(z_0(\cdot), u(\cdot))$ is optimal, which contradicts the assumption. The contradiction completes the proof.

Theorem 3.1 and Lemma 3.4 imply the next uniqueness result.

Corollary 3.1 *The optimal control process in problem (1.9) – (1.12) is unique in the following sense: if $(z_1(\cdot), u_1(\cdot))$ and $(z_2(\cdot), u_2(\cdot))$ are optimal control processes in problem (1.9) – (1.12), then $z_1(\cdot) = z_2(\cdot)$ and $u_1(t) = u_2(t)$ for a. a. $t \geq 0$.*

Theorem 3.1 provides the next solution algorithm for problem (1.9) – (1.12).

Algorithm of constructing the optimal control process $(z_0(\cdot), u(\cdot))$ in problem (1.9) – (1.12).

1. Find the equilibrium solution $(q(\cdot), p(\cdot))$ of the Hamiltonian system (1.40), (1.41) which is determined by

$$z_0 = \frac{q_0}{p_0}.$$

2. Set

$$z_0(\cdot) = \frac{q(\cdot)}{p(\cdot)}$$

and define $u(\cdot)$ by (1.42) $(t \geq 0)$.

3.4 Optimal synthesis

In the construction of an optimal feedback, our main instrument will be a one-dimensional representations of the equilibrium solutions of the Hamiltonian system (1.40), (1.41). These are functions $q \mapsto \bar{p}(q)$ solving the one-dimensional equation

$$\frac{d}{dq}\bar{p}(q) = f(q, \bar{p}(q)) \tag{3.18}$$

which is derived from (1.40), (1.41) by dividing its second component by the first one. Thus, in (3.18) and in what follows $f(\cdot) : (q, p) \mapsto f(q, p)$ is defined by (3.16). Note that the domain of definition of $f(\cdot)$ is $\mathrm{dom} f(\cdot) = G \setminus V_q^0$ (see (1.20) and (3.2)); therefore, solutions of (3.18) are understood as those in $\mathrm{dom} f(\cdot) = G \setminus V_q^0$ (i.e., by definition every solution $\bar{p}(\cdot)$ of (3.18) satisfies $(q, \bar{p}(q)) \in G \setminus V_q^0$ for any q from the domain of its definition).

A positive solution $\bar{p}(\cdot)$ of (3.18) (in $G \setminus V_q^0$) will be called an *equilibrium solution* if $\bar{p}(\cdot)$ is defined on $(0, q^*)$ and

$$\lim_{q \to q^*} \bar{p}(q) = p^*, \tag{3.19}$$

Lemma 3.5 *Let $\bar{p}(\cdot)$ be an equilibrium solution of (3.18). Set $p_1(q) = qz_0$. Then for each z_0 ¿ 0 curves $P_1(z_0) = \{(p_1(q), q) : q \in (0, q^*)\}$ and $P_2 = \{(\bar{p}(q), q) : q \in (0, q^*)\}$ intersect at the unique point $(p_0(z_0), q_0(z_0))$*

Proof. Considering curve P_1 we note

$$\frac{d}{dq}p_1(q) > 0, p_1(0) = 0, p_1(q^*) > 0$$

Considering curve P_2 by (3.15) we get

$$\frac{d}{dq}\bar{p}(q) < 0$$

for all $q \in (0, q^*), \bar{p}(0) \to \infty, \bar{p}(q^*) = 0$.

Thus $P_1 \cap P_2 = (p_0, q_0)$ intersect at the unique point.

In Fig. 5 the equilibrium solution of equation (3.18) is shown.

Now we are ready to construct a desired optimal synthesis $U(\cdot)$.

In the expression (1.42) for an optimal control $u(t)$ we replace $z(t)$ by a free z and replace $q(t)$ and $p(t)$ by $q_0(z)$ and $p_0(z)$ Thus, we define $U(\cdot) : (0, \infty) \mapsto [0, b)$ by

$$U(z) = \begin{cases} b - \dfrac{1}{q_0(z) + p_0(z)\gamma} & \text{if} \quad z \in G_1, \\ 0 & \text{if} \quad z \in G_2; \end{cases} \tag{3.20}$$

In Fig. 6 the shape of the optimal synthesis $U(\cdot)$ is illustrated.

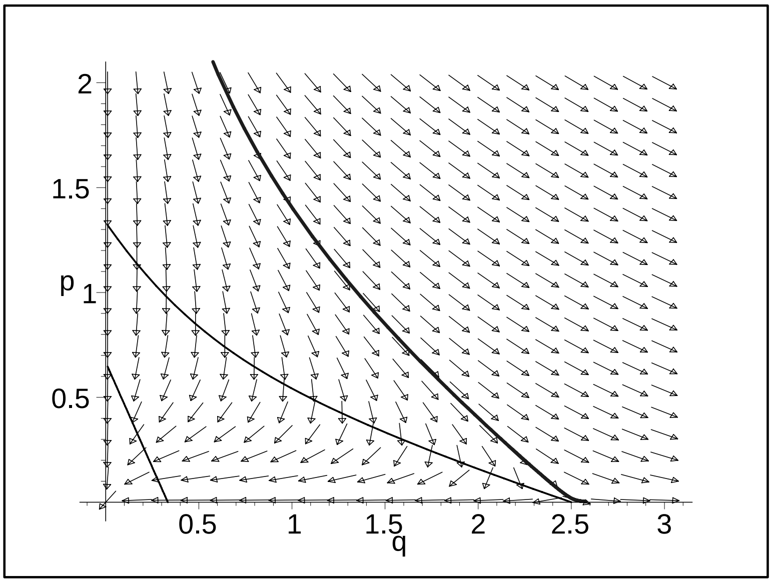

Figure 5: The equilibrium solution of (3.18) for $\nu = 2$, $b = 3$, $\rho = 0.4$, $\gamma = 0.5$ (a Mapple simulation).

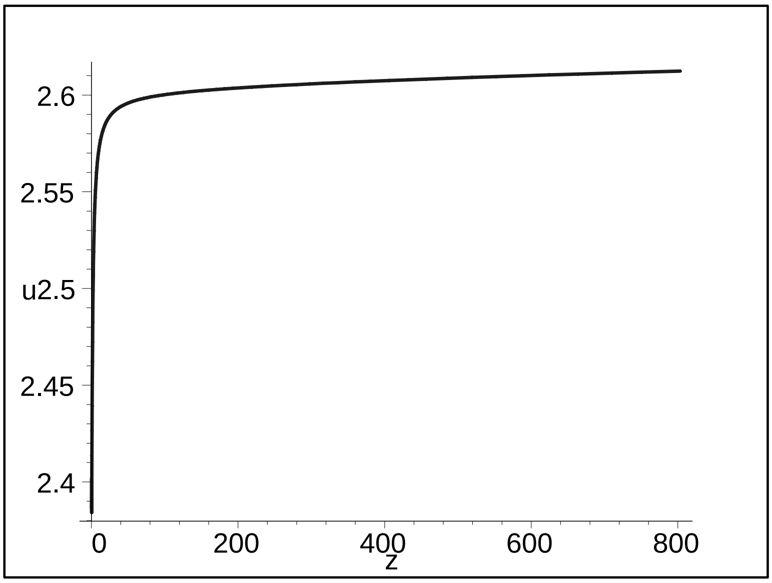

Figure 6: The optimal synthesis for $\nu = 2$, $b = 3$, $\rho = 0.4$, $\gamma = 0.5$ (a Mapple simulation).

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