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IIASA Interim Report
March 2000

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Interim Report **IR-00-018**

Central Path Dynamics and a Model of Competition. II

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March 24, 2000

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Abstract

Growth – the change in number or size – and adaptation – the change in quality or structure – are key attributes of global processes in natural communities, society and economics (see, e.g., Hofbauer and Sigmund, 1988; Freedman, 1991; Young, 1993). In this paper we describe a model with explicit growth-adaptation feedbacks. We treat it in the form of an economic model of competition of two firms (with several departments) on the market. Their size is measured by their capital, and their quality by their productive power (production complexity). It is assumed that the production complexity of a department or firm is a simple function (that is more general than the one considered in Kryazhimskii and Stoer, 1999) of its capital. The model works on both the firm level (competition among the departments) and the market level (competition among the firms).

The model shows some empirically observable phenomena. Typically, one of the firms will finally cover the market. The winner is not necessarily the firm with the potentially higher maximum productivity. A long-term coexistence of firms may arise in exceptional situations occurring only when the maximum potential productivities (not the actual productivities) are equal. The analysis is also based on the concept of central paths from the interior point optimization theory (see Sonnevend, 1985; and, e.g., Ye, 1997).

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Introduction

This paper continues Kryazhimskii and Stoer, 1999, where a two-level growth-adaptation ODE model has been analyzed. Here we study a more general model. The main qualitative results remain the same. For convenience, we reproduce the introduction to the above paper in the next two paragraphs.

Growth – the change in number – and adaptation – the change in structure – are key attributes of global processes in natural communities, society and economics. The idea of the interplay between growth and adaptation has given rise to game-evolutionary models of bioevolution (Hofbauer and Sigmund, 1988). Restructuring of a biological population (its adaptation) is driven by the abilities of the phenotypes to produce offsprings (thus, ensuring the population's growth). Similar views have led to several models of evolutionary processes in economics (see, e.g., Freedman, 1991; Young, 1993). Game-evolutionary modeling implies a focus on inner interactions, restructuring and adaptation. In contrast, the theory of endogenous economic growth concentrates on the dynamics of growth for constantly-structurized countries, firms, etc. (see, e.g., Grossman and Helpman, 1991).

In this paper we suggest an ODE model with explicit growth-adaptation feedbacks. We treat it as a model of competition of two firms on the market. The model works on both the firm and market levels. To win on the market through a better productivity each firm is dynamically restructuring. In turn, the shares of firms' products on the market determine proportions in firms' capitals, and – via them – the relative speeds of firms' restructuring. The model shows some empirically observable phenomena. Typically, one of the firms covers the whole market and the other dies out. A winner is not necessarily the firm with a potentially higher maximum productivity. The long-term coexistence of the firms on the market may arise in exceptional situations implying, in particular, the equality of the firms' maximum productivities. The analysis is essentially based on the method of central paths from the interior point optimization theory (see Sonnevend, 1985; and, e.g., Ye, 1997).

1 Model of firm

Let us imagine a firm working on new products. Let $p(t)$ be the firm's capital at time t, and $r(t)$ the total output produced by the firm up to time t (we set $t \geq 0$).

Assuming that the price for a unit of the output is 1, we set $r(t) = p(t)$. Let the firm consist of n structural units, departments. We number them $1, \ldots, n$. Let $p_i(t)$ be the capital of department i at time t and $r_i(t)$ the total output of department i up to time t. We define the *production complexity* of department i as a monotonically increasing function of its capital, $\sigma(p_i(t))$, positive for $p_i(t) > 0$. For example, $\sigma(p_i(t))$ can be the number of all interconnections between the researchers. If the number of researchers is proportional to $p_i(t)$, then $\sigma(p_i(t))$ is proportional to

$$\frac{p_i(t)(p_i(t)-1)}{2} = (p_i(t)-1) + (p_i(t)-2) + \cdots + 1$$

or, approximately (for $p_i(t)$ large), $p_i^2(t)/2$.

We assume that the production rate of department i, $\dot{r}_i(t)$, is proportional to its current complexity,

$$\dot{r}_i(t) = a_i \sigma(p_i(t)); \tag{1.1}$$

here a_i is a positive *productivity coefficient* of department i. The sum

$$\dot{r}(t) = \sum_{k=1}^{n} \dot{r}_k(t) = \sum_{k=1}^{n} a_k \sigma(p_k(t))$$

gives the total production rate of the firm.

The ratio

$$\rho_i(t) = \frac{\sigma(p_i(t))}{\sum_{k=1}^{n} \sigma(p_k(t))}$$

represents the *relative complexity* of department i in the firm, and $\rho_i(t)\dot{r}(t)$ the *expected* production rate of department i. The difference

$$h_i(t) = \dot{r}_i(t) - \rho_i(t)\dot{r}(t),$$

showing for how much the actual production rate of department i is higher than the expected one, estimates the relative efficiency of department i in the firm. We call $h_i(t)$ the *relative efficiency* of department i. Let

$$x_i(t) = \frac{p_i(t)}{p(t)}$$

be the current share of the capital of department i in the firm. A fair distribution of the incoming capital among the departments implies that the share of the capital of department i grows proportionally to its relative efficiency,

$$\dot{x}_i(t) = \mu h_i(t);$$

here μ is a positive coefficient. We call this regulation rule the *fairness principle*. Note that the fairness principle is feasible. Indeed, due to the fairness principle

$$\sum_{i=1}^{n} \dot{x}_i(t) = \mu \sum_{i=1}^{n} h_i(t) = \mu \sum_{i=1}^{n} [\dot{r}_i(t) - \rho_i(t)\dot{r}(t)] = \mu \sum_{i=1}^{n} \dot{r}_i(t) - \mu \sum_{i=1}^{n} \rho_i(t) \sum_{k=1}^{n} \dot{r}_k(t) = 0,$$

hence, the sum of the capital shares, $\sum_{i=1}^{n} x_i(t)$, is always 1. In what follows we assume that the fairness principle is adopted.

We specify the fairness principle as follows:

$$\begin{aligned}\dot{x}_i(t) &= \mu(\dot{r}_i(t) - \rho_i(t)\dot{r}(t)) \\ &= \mu\left(a_i\sigma(p_i(t)) - \frac{\sigma(p_i(t))}{\sum_{k=1}^n \sigma(p_k(t))}\sum_{k=1}^n a_k\sigma(p_k(t))\right) \\ &= \mu\sigma(p_i(t))\left(a_i - \frac{\sum_{k=1}^n a_k\sigma(p_k(t))}{\sum_{k=1}^n \sigma(p_k(t))}\right) \\ &= \mu\sigma(x_i(t)p(t))\left(a_i - \frac{\varphi(x(t),p(t))}{\varphi_0(x(t),p(t))}\right),\end{aligned}$$

where

$$\varphi(x(t),p(t)) = \frac{\sum_{k=1}^n a_k\sigma(x_k(t)p(t))}{\sigma(p(t))}, \qquad (1.2)$$

$$\varphi_0(x(t),p(t)) = \frac{\sum_{k=1}^n \sigma(x_k(t)p(t))}{\sigma(p(t))}. \qquad (1.3)$$

Here and in what follows, $x(t)$ stands for an n-dimensional vector with the coordinates $x_1(t), \ldots, x_n(t)$; $x(t)$ characterizes the distribution of the firm's capital among the departments. For the rate of the firm's capital, we have

$$\dot{p}(t) = \dot{r}(t) = \sum_{k=1}^n \dot{r}_k(t) = \sum_{k=1}^n a_k\sigma(p_k(t)) = \sum_{k=1}^n a_k\frac{\sigma(p_k(t))}{\sigma(p(t))}\sigma(p(t)) = \varphi(x(t),p(t))\sigma(p(t)).$$

We arrive at a system of differential equations,

$$\dot{x}_i(t) = \mu\sigma(x_i(t)p(t))\left(a_i - \frac{\varphi(x(t),p(t))}{\varphi_0(x(t),p(t))}\right) \quad (i = 1, \ldots n), \qquad (1.4)$$

$$\dot{p}(t) = \varphi(x(t),p(t))\sigma(p(t)). \qquad (1.5)$$

Equation (1.4) describes the dynamics of the capital shares within the firm and equation (1.5) the growth of the firm's total capital, or the firm's total output on the market. Note that $\sigma(p(t))$ is the *complexity* of the firm, and recall that the productivity rate of the firm, $\dot{r}(t)$, equals $\dot{p}(t)$. Thus, equation (1.5) shows that the productivity rate of the firm, $\dot{r}(t)$, is proportional to its complexity, $\sigma(p(t))$ with the *productivity coefficient* $\varphi(x(t),p(t))$,

$$\dot{r}(t) = \varphi(x(t),p(t))\sigma(p(t)).$$

Comparing with (1.1), we find that the total firm's output and output of each firm's department grow similarly. A single difference is that the productivity coefficients of the departments, a_i, are constant (we assume this for the sake of simplicity), and the productivity coefficient of the firm, $\varphi(x(t),p(t))$, depends on the distribution of the firm's capital among the departments.

2 Model of competition

Now assume that two firms, firm 1 and firm 2, compete on the market. The dynamics of firm 1 is described by the equations (1.4), (1.5), and the dynamics of firm 2 by similar equations,

$$\dot{y}_i(t) = \nu\sigma(y_i(t)q(t))\left(b_i - \frac{\psi(y(t), q(t))}{\psi_0(y(t), q(t))}\right) \quad (i = 1, \dots m), \tag{2.1}$$

$$\dot{q}(t) = \psi(y(t), q(t))\sigma(q(t)). \tag{2.2}$$

Here ν is a positive coefficient, $y(t)$ is the m-dimensional vector with the coordinates $y_1(t), \dots, y_m(t)$, which describes the distribution of the capital of firm 2 among its departments $1, \dots, m$; b_i is the productivity coefficient of department i in firm 2;

$$\psi(y(t), q(t)) = \frac{\sum_{k=1}^{m} b_k\sigma(y_k(t)q(t))}{\sigma(q(t))}, \tag{2.3}$$

is the productivity coefficient of firm 2;

$$\psi_0(y(t), q(t)) = \frac{\sum_{k=1}^{n} \sigma(y_k(t)p(t))}{\sigma(q(t))}; \tag{2.4}$$

and $q(t)$ is the total capital/output of firm 2.

Let $u(t)$ and $v(t)$ be the market shares of firms 1 and 2, respectively,

$$u(t) = \frac{p(t)}{p(t) + q(t)}, \quad v(t) = \frac{q(t)}{p(t) + q(t)}.$$

The equations (1.5) and (2.2) describe the rates of the capitals/outputs of firms 1 and 2, respectively. In section 1 we noticed that these rates are subject to the same law as the output rates of firm's departments. Assume that the fairness principle holds on the market with, generally, another measure of complexity. Let $\tau(p(t))$ and $\tau(q(t))$ be the *market complexities* of firms 1 and 2, respectively.

Then we arrive at differential equations for the market shares $u(t)$ and $v(t)$, which have the same structure as the equations (1.4) and (2.1) for the departments' shares within the firms,

$$\dot{u}(t) = \rho\tau(u(t)(p(t) + q(t)))[\varphi(x(t), p(t)) - \gamma(x(t), y(t), p(t), q(t), u(t), v(t))] \tag{2.5}$$

$$\dot{v}(t) = \rho\tau(u(t)(p(t) + q(t)))[\psi(y(t), q(t)) - \gamma(x(t), y(t), p(t), q(t), u(t), v(t))] \tag{2.6}$$

where

$$\gamma(x(t), y(t), p(t), q(t), u(t), v(t)) =$$
$$\frac{\varphi(x(t), p(t))\tau(u(t)(p(t) + q(t))) + \psi(y(t), q(t))\tau(v(t)(p(t) + q(t)))}{\tau(u(t)(p(t) + q(t))) + \tau(v(t)(p(t) + q(t)))},$$

and ρ is a positive coefficient. The entire process involving internal restructuring (adaptation), growth in products, and external (market) competition is described by the system of equations (1.4), (1.5), (2.1), (2.2), (2.5), (2.6).

3 Model of competition. Specification

Let us introduce a simplifying assumption: $\sigma(xp) = \sigma_1(x)\sigma_2(p)$ for all positive x and p. Then necessarily $\sigma_1(x) = ax^c$ and $\sigma_2(p) = bp^c$ for some constants a, b, and c (see Aczél, 1966).

From now on we fix $c > 1$ and set $\sigma(p) = p^c$. In section 1 we noticed that $c = 2$ occurs when the production complexity is estimatecd as the number of all interconnections between the researchers. This definition implies entire coperation in production; assuming some reasonable degree of cooperation, we get $c < 2$. Note that $c = 1$ implies no cooperation in production, and this motivates the restriction $c > 1$.

We assume that the market complexity has the same form, $\tau(p) = p^c$. Thus, there is a "cooperation" between the units of firm's products. This "cooperation" can be understood as the interdepenence of the product units whose combinations, high-tech meta-products, go to the market. It is assumed that the degree of interdependence of the product units on the market, is the same as the degree of cooperation in production. Now (1.2), (1.3), (2.3) and (2.4) are specified as

$$\varphi(x(t), p(t)) = \varphi(x(t)) = \sum_{k=1}^{n} a_k x_k^c(t), \quad \varphi_0(x(t), p(t)) = \varphi_0(x(t)) = \sum_{k=1}^{n} x_k^c(t),$$

$$\psi(y(t), q(t)) = \psi(y(t)) = \sum_{k=1}^{m} b_k y_k^c(t), \quad \psi_0(y(t), q(t)) = \psi_0(x(t)) = \sum_{k=1}^{m} y_k^c(t),$$

and the model equations (1.4), (1.5), (2.1), (2.2), (2.5), (2.6) take the form

$$\dot{x}_i(t) = \mu x_i^c(t) u^c(t) \left(a_i - \frac{\varphi(x(t))}{\varphi_0(x(t))} \right) (p(t) + q(t))^c \quad (i = 1, \ldots n), \tag{3.1}$$

$$\dot{p}(t) = \mu \varphi(x(t)) p^c(t), \tag{3.2}$$

$$\dot{y}_i(t) = \nu y_i^c(t) v^c(t) \left(b_i - \frac{\psi(y(t))}{\psi_0(y(t))} \right) (p(t) + q(t))^c \quad (i = 1, \ldots m), \tag{3.3}$$

$$\dot{q}(t) = \nu \psi(y(t)) q^c(t), \tag{3.4}$$

$$\dot{u}(t) = \rho u^c(t) \left(\varphi(x(t)) - \frac{u^c(t)\varphi(x(t)) + v^c(t)\psi(y(t))}{u^c(t) + v^c(t)} \right) (p(t) + q(t))^c, \tag{3.5}$$

$$\dot{v}(t) = \rho v^c(t) \left(\psi(x(t)) - \frac{u^c(t)\varphi(x(t)) + v^c(t)\psi(y(t))}{u^c(t) + v^c(t)} \right) (p(t) + q(t))^c. \tag{3.6}$$

Notice that

$$\varphi(x(t)) \geq \varepsilon_0$$

with a positive constant ε_0. By (3.2)

$$\dot{p}(t) \geq \mu \varepsilon_0 p^c(t).$$

Hence, $p(t) \geq p_0(t)$ where $p_0(t)$ solves the equation

$$\dot{p}_0(t) = \mu \varepsilon_0 p_0^c(t)$$

with the initial condidion $p_0(0) = p(0)$. Assuming $p(0) > 0$ we find

$$p_0^{c-1}(t) = ((c-1)(c_0 - \mu\varepsilon_0 t))^{-1}$$

where $c_0 = ((c-1)p^{c-1}(0))^{-1}$. We see that $p_0(t) \to \infty$ as $t \to c_0/\mu\varepsilon_0$. Consequently $p(t) \to \infty$ as $t \to t_0 \le c_0/\mu\varepsilon_0$. Thus, the total output $p(t) + q(t)$ approaches infinity as t approaches a finite time. In other words, the market is saturated within a finite period of time. Our goal is to classify admissible limit distributions of the firms' market shares and capital shares within the firms by the time when the market is saturated.

Note that the right hand sides in (3.1), (3.3), (3.5), and (3.6) have a common multiplier $(p(t) + q(t))^c$. We omit this multiplier, which is equivalent to time rescaling, and reduce the system (3.1), (3.3), (3.5), (3.6) to

$$\dot{x}_i(t) = \mu x_i^c(t) u^c(t) \left(a_i - \frac{\varphi(x(t))}{\varphi_0(x(t))} \right) \quad (i = 1, \dots n), \tag{3.7}$$

$$\dot{y}_i(t) = \nu y_i^c(t) v^c(t) \left(b_i - \frac{\psi(y(t))}{\psi_0(y(t))} \right) \quad (i = 1, \dots m), \tag{3.8}$$

$$\dot{u}(t) = \rho u^c(t) \left(\varphi(x(t)) - \frac{u^c(t)\varphi(x(t)) + v^c(t)\psi(y(t))}{u^c(t) + v^c(t)} \right), \tag{3.9}$$

$$\dot{v}(t) = \rho v^c(t) \left(\psi(x(t)) - \frac{u^c(t)\varphi(x(t)) + v^c(t)\psi(y(t))}{u^c(t) + v^c(t)} \right). \tag{3.10}$$

4 Central path dynamics

The integration of the equations (3.7), (3.8), (3.9) and (3.10) yields

$$\frac{1}{x_i(t)^{c-1}} - \frac{1}{x_j(t)^{c-1}} = (c-1)\mu(a_j - a_i) \int_0^t u^c(\tau)d\tau + \frac{1}{x_i(0)^{c-1}} - \frac{1}{x_j(0)^{c-1}} \quad (i, j = 1 \dots, n), \tag{4.1}$$

$$\frac{1}{y_i(t)^{c-1}} - \frac{1}{y_j(t)^{c-1}} = (c-1)\nu(b_j - b_i) \int_0^t v^c(\tau)d\tau + \frac{1}{y_i(0)^{c-1}} - \frac{1}{y_j(0)^{c-1}} \quad (i, j = 1 \dots, m), \tag{4.2}$$

$$\frac{1}{u(t)^{c-1}} - \frac{1}{v(t)^{c-1}} = (c-1)\rho \left[\int_0^t \psi(y(\tau))d\tau - \int_0^t \varphi(x(\tau))d\tau \right] + \frac{1}{u(0)^{c-1}} - \frac{1}{v(0)^{c-1}}. \tag{4.3}$$

Let us rearrange (4.1) as $(i, j = 1, \dots, n)$

$$(c-1)\mu a_i \int_0^t u^c(\tau)d\tau + \frac{1}{x_i(t)^{c-1}} - \frac{1}{x_i(0)^{c-1}} = (c-1)\mu a_j \int_0^t u^c(\tau)d\tau + \frac{1}{x_j(t)^{c-1}} - \frac{1}{x_j(0)^{c-1}}. \tag{4.4}$$

Let

$$\Phi(t, x) = \left(\mu \int_0^t u^c(\tau)d\tau \right) \sum_{k=1}^n a_k x_k - \sum_{k=1}^n \frac{1}{(c-1)(c-2)} x_i^{2-c} - \sum_{k=1}^n \frac{1}{(c-1)} \frac{x_k}{x_k(0)^{c-1}}$$

$$(c \neq 2),$$

$$\Phi(t,x) = \left(\mu \int_0^t u^2(\tau)d\tau\right) \sum_{k=1}^n a_k x_k + \sum_{k=1}^n \log x_k - \sum_{k=1}^n \frac{x_k}{x_k(0)^{c-1}} \tag{4.5}$$

$$(c = 2).$$

Notice that the left- and right-hand sides in (4.4) represent, respectively, the ith and jth coordinates of $\mathrm{grad}_x \Phi(t, x(t))$, the gradient of Φ, with respect to x at point $x(t)$. The relation (4.4) shows that $\mathrm{grad}_x \Phi(t, x(t))$ is orthogonal to the affine hull of the n-dimensional simplex

$$S_n = \{x \in R^n : x_1 \geq 0, \ldots, x_n \geq 0,\ x_1 + \ldots + x_n = 1\}.$$

This orthogonality condition, together with the fact that $\Phi(t, \cdot)$ is strictly concave, imply that if $\bar{x}(t)$, a (unique) maximizer of $\Phi(t, \cdot)$ in S_n is not on the boundary of S_n, then $\bar{x}(t) = x(t)$, or

$$x(t) = \mathrm{argmax}\{\Phi(t,x) : x \in S_n\}. \tag{4.6}$$

If $c \geq 2$, then $\Phi(t,x) \to -\infty$ as x approaches the boundary of S_n. If $c < 2$, then for any point ξ on the boundary of S_n we have $\xi_i = 0$ for some i, and $\partial\Phi(t,x)/\partial x_i \to \infty$ as $x \to \xi$. Hence, the maximizer $\bar{x}(t)$ cannot lie on the boundary of S_n, and (4.6) holds true.

Referring to (4.2), we similarly find that

$$y(t) = \mathrm{argmax}\{\Psi(t,y) : y \in S_m\}, \tag{4.7}$$

where

$$\Psi(t,y) = \left(\nu \int_0^t v^c(\tau)d\tau\right) \sum_{k=1}^m b_k y_k - \sum_{k=1}^m \frac{1}{(c-1)(c-2)} y_i^{2-c} - \sum_{k=1}^n \frac{1}{(c-1)} \frac{y_k}{y_k(0)^{c-1}}$$

$$(c \neq 2),$$

$$\Psi(t,y) = \left(\nu \int_0^t v^2(\tau)d\tau\right) \sum_{k=1}^m b_k y_k + \sum_{k=1}^m \log y_k - \sum_{k=1}^n \frac{y_k}{y_k(0)}, \tag{4.8}$$

$$(c = 2),$$

and

$$S_m = \{y \in R^m : y_1 \geq 0, \ldots, y_m \geq 0,\ y_1 + \ldots + y_m = 1\}.$$

Finally, (4.3) yields

$$(u(t), v(t)) = \mathrm{argmax}\{W(t,u,v) : (u,v) \in S_2\}, \tag{4.9}$$

where

$$\begin{aligned} W(t,u,v) &= \rho u \int_0^t \varphi(x(\tau))d\tau + \rho v \int_0^t \psi(x(\tau))d\tau \\ &\quad - \frac{1}{(c-1)(c-2)}(u^{2-c} + v^{2-c}) - \frac{1}{(c-1)}\left(\frac{u}{u(0)^{c-1}} + \frac{v}{v(0)^{c-1}}\right) \end{aligned}$$

$$(c \neq 2),$$

$$\begin{aligned} W(t,u,v) &= \rho u \int_0^t \varphi(x(\tau))d\tau + \rho v \int_0^t \psi(x(\tau))d\tau \\ &\quad - \log u + \log v - \left(\frac{u}{u(0)} + \frac{v}{v(0)} \right) \end{aligned}$$

$$(c = 2),$$

and

$$S_2 = \{(u,v) \in R^2 : u \geq 0,\ v \geq 0,\ u + v = 1\}.$$

Relations of the type (4.6), (4.7) and (4.9) lie in the base of the central path homotopy methods for problems of convex optimization (see, e.g., Ye, 1997). The differential equations (3.7), (3.8), (3.9), (3.10) are counterparts of the central path equations describing optimum-approaching trajectories.

5 Limit distributions within firms

In what follows, I^+ is the set of all maximally productive departments in firm 1, i.e., the set of all $i \in \{1,\dots,n\}$ such that $a_i = a^+$ where

$$a^+ = \max\{a_i : i = 1,\dots n\}.$$

Similarly, J^+ is the set of all maximally productive departments in firm 2, i.e., the set of all $j \in \{1,\dots,m\}$ such that $b_j = b^+$ where

$$b^+ = \max\{b_j : j = 1,\dots n\}.$$

We assume that not all departments are equally productive in firms 1 and 2,

$$I^+ \neq \{1,\dots n\}, \quad J^+ \neq \{1,\dots m\}. \tag{5.1}$$

We set

$$X^+ = \{x \in S_n : x_i = 0 \text{ for all } i \notin I^+\}, \quad Y^+ = \{y \in S_m : y_j = 0 \text{ for all } j \notin J^+\}.$$

Obviously, firms 1 and 2 reach their maximal productivities at the distributions from X^+ and Y^+, respectively:

$$\varphi(x) = \varphi^+ \text{ iff } x \in X^+, \quad \psi(y) = \psi^+ \text{ iff } y \in Y^+,$$

where

$$\varphi^+ = \max\{\varphi(x) : x \in S_n\} = a^+, \quad \psi^+ = \max\{\psi(x) : y \in S_m\} = b^+.$$

Let $(x(\cdot), y(\cdot), u(\cdot), v(\cdot))$ be a solution to (3.7) – (3.10), which starts from a point (x_0, y_0, u_0, v_0) with nonzero coordinates:

$$\begin{aligned} x_{0i} &= x_i(0) > 0 \quad (i = 1,\dots n), \\ y_{0j} &= y_j(0) > 0 \quad (j = 1,\dots m), \\ u_0 &= u(0) > 0, \\ v_0 &= v(0) > 0. \end{aligned} \tag{5.2}$$

We will use the notations

$$\eta(t) = \int_0^t u^c(\tau)d\tau, \quad \eta_* = \int_0^\infty u^c(\tau)d\tau,$$

$$\zeta(t) = \int_0^t v^c(\tau)d\tau, \quad \zeta_* = \int_0^\infty v^c(\tau)d\tau.$$

$\mathrm{dist}(x(t), X^+) = \min\{|x(t)-x^+| : x^+ \in X^+\}, \quad \mathrm{dist}(y(t), Y^+) = \min\{|y(t)-y^+| : y^+ \in Y^+\};$
here $|\cdot|$ stands for the Euclidean norm.

Lemma 5.1 *1. If $\eta_* < \infty$, then there is a limit*

$$x_* = \lim_{t\to\infty} x(t),$$

and $x_* \notin X^+$.
2. If $\eta_* = \infty$, then

$$\lim_{t\to\infty} \mathrm{dist}(x(t), X^+) = 0. \tag{5.3}$$

Lemma 5.2 *1. If $\zeta_* < \infty$, then there is a limit*

$$y_* = \lim_{t\to\infty} x(t),$$

and $y_* \notin Y^+$.
2. If $\zeta_* = \infty$, then

$$\lim_{t\to\infty} \mathrm{dist}(y(t), Y^+) = 0.$$

We prove Lemma 5.1 only.

Proof of Lemma 5.1. 1. Let $\xi(t)$ be a solution to the equation (3.7) from which the multiplier $u^c(t)$ is removed,

$$\dot{\xi}_i(t) = \mu\xi_i^c(t)\left(a_i - \frac{\varphi(\xi(t))}{\varphi_0(\xi(t))}\right) \quad (i = 1, \dots n),$$

and $\xi(0) = x(0)$. Obviously,

$$x(t) = \xi\left(\int_0^t u^c(\tau)d\tau\right). \tag{5.4}$$

Let $\eta_* < \infty$. By (5.4)

$$\lim_{t\to\infty} x(t) = \xi(\eta_*) = x_*.$$

Due to the central path equality (4.6) and the definition of the function $\Phi(t, x)$ (see (4.5)) x_* maximizes for $c \neq 2$

$$\lim_{t\to\infty} \frac{\Phi(t,x)}{\eta(t)} = \mu\sum_{k=1}^n a_k x_k - \frac{1}{\eta_*}\sum_{k=1}^n \frac{1}{(c-1)(c-2)} x_i^{2-c} - \frac{1}{\eta_*}\sum_{k=1}^n \frac{1}{(c-1)}\frac{x_k}{x_k(0)^{c-1}}$$

over S_n, and for $c = 2$

$$\lim_{t\to\infty} \frac{\Phi(t,x)}{\eta(t)} = \mu \sum_{k=1}^{n} a_k x_k + \frac{1}{\eta_*} \sum_{k=1}^{n} \log x_k - \frac{1}{\eta_*} \sum_{k=1}^{n} \frac{x_k}{x_k(0)}.$$

Arguing as in section 4, we find that all coordinates of x_* are positive. Since $x_{*i} > 0$ for $i \notin I^+$ (which exists by (5.1)), we have $x_* \notin X^+$.

2. Let $\eta_* = \infty$. By (4.6) $x(t)$ maximizes for $c \neq 2$

$$\frac{\Phi(t,x)}{\eta(t)} = \mu \sum_{k=1}^{n} a_k x_k - \frac{1}{\eta(t)} \sum_{k=1}^{n} \frac{1}{(c-1)(c-2)} x_i^{2-c} - \frac{1}{\eta(t)} \sum_{k=1}^{n} \frac{1}{(c-1)} \frac{x_k}{x_k(0)^{c-1}}$$

over S_n, and for $c = 2$

$$\frac{\Phi(t,x)}{\eta(t)} = \mu \sum_{k=1}^{n} a_k x_k + \frac{1}{\eta(t)} \sum_{k=1}^{n} \log x_k - \frac{1}{\eta(t)} \sum_{k=1}^{n} \frac{x_k}{x_k(0)}.$$

Since $\lim_{t\to\infty} \eta(t) = \infty$, we have (5.3). □

Lemmas 5.1 and 5.2 imply the next statements.

Lemma 5.3 *1. There are limits*

$$\varphi_* = \lim_{t\to\infty} \varphi(x(t)) \le \varphi^+, \quad \psi_* = \lim_{t\to\infty} \psi(y(t)) \le \psi^+.$$

2. One has $\varphi_* = \varphi^+$ if and only if $\eta_* = \infty$.
3. One has $\psi_* = \psi^+$ if and only if $\zeta_* = \infty$.

Proof. Statement 1 follows from Lemmas 5.1 and 5.2 straightforwardly. If $\eta_* < \infty$ then by 5.1, 1, $x_* \notin X^+$; hence, $\varphi_* = \varphi(x_*) < \varphi^+$. If $\eta_* = \infty$, then by 5.1, 2, we have (5.3); hence, $\varphi_* = \varphi^+$. This proves statement 2. Statement 3 is proved similarly. □

The values φ_* and ψ_* characterize the limit productivities of the firms.

6 Limit distributions of market shares

Let us characterize the admissible limit distributions of the market shares, $u(t)$ and $v(t)$. We consider three basic relations between the limit productivities of the firms, $\varphi_* > \psi_*$, $\varphi_* < \psi_*$, and $\varphi_* = \psi_*$.

Theorem 6.1 *Let $\varphi_* > \psi_*$. Then*

$$\lim_{t\to\infty} u(t) = 1, \quad \lim_{t\to\infty} v(t) = 0,$$

and $\varphi_* = \varphi^+$.

Theorem 6.2 *Let $\varphi_* < \psi_*$. Then*

$$\lim_{t\to\infty} u(t) = 0, \quad \lim_{t\to\infty} v(t) = 1,$$

and $\psi_* = \psi^+$.

We prove Theorem 6.1 only.

Proof of Theorem 6.1. The inequality $\varphi_* > \psi_*$ implies that the right-hand side in (4.3) approaches $-\infty$ as $t \to \infty$. Then (4.3) implies $\lim_{t\to\infty} v(t) = 0$. Hence, $\lim_{t\to\infty} u(t) = 1$. Therefore, $\eta_* = \infty$. By Lemma 5.3, 2, $\varphi_* = \varphi^+$. □

We study the case $\varphi_* = \psi_*$ under the assumption that the maximal productivities of firms 1 and 2, $\varphi^+ = a^+$ and $\psi^+ = b^+$, are different. To be particular, assume $a^+ > b^+$.

Theorem 6.3 *Let $a^+ > b^+$ and $\varphi_* = \psi_*$. Then*

$$\lim_{t\to\infty} u(t) = 0, \quad \lim_{t\to\infty} v(t) = 1,$$

and $\psi_* = \psi^+$.

Proof. Since $\varphi^+ = a^+ > b^+ = \psi^+$ and $\varphi_* = \psi_* \le \psi^+$, we have $\varphi_* < \varphi^+$. Then by Lemma 5.3, 2, $\eta_* < \infty$. It is sufficient to show that $u(t) \to 0$ as $t \to 0$. Assume this is not so. Then $u(\xi_k) > \delta$ for some $\delta > 0$ and some $\xi_k \to \infty$. Note that $\eta_* < \infty$ implies $u(t_k) \to 0$ for some $t_k \to \infty$. With no loss of generality, assume $t_k < \xi_k$. Next, we consider only large i, for which $u(t_k) < \delta/2$ and $u(\xi_k) > \delta$. Let

$$\tau_k = \max\{t \in [t_k, \xi_k] : u(t) \le \delta/2\}.$$

We have $u(\tau_k) = \delta/2$ and $u(t) \ge \delta/2$ for all $t \in [\tau_k, \xi_k]$. The right-hand side of the equation (3.9) is bounded. Hence, there is $c_0 > 0$ such that $|\dot{u}(t)| < c_0$ for all $t \ge 0$. Therefore $\xi_k - \tau_k \ge \delta/2c_0$. Consequently,

$$\eta_* \ge \sum_{i=1}^{\infty} \int_{\tau_k}^{\xi_k} u^c(t)dt \ge \sum_{i=1}^{\infty} \left(\frac{\delta}{2}\right)^c (\xi_k - \tau_k) \ge \sum_{i=1}^{\infty} \left(\frac{\delta}{2}\right)^c \frac{\delta}{2c_0}.$$

Thus, $\eta_* = \infty$. We arrived at a contradiction. The theorem is proved. □

Theorems 6.1 – 6.3 show that only three types of solutions, $(x(\cdot), y(\cdot), u(\cdot), v(\cdot))$, of the equations (3.7) – (3.10) may exist. We call the solutions described in Theorem 6.1 *favourable for firm* 1, solutions described in Theorem 6.2 *favourable for firm* 2, and solutions which are favourable neither for firm 1, nor for firm 2, *critical.* The critical solutions are characterized in Theorem 6.3 under the assumption that $a^+ > b^+$ (a symmetric characterization holds if $a^+ < b^+$). For the case $a^+ = b^+$ the critical solutions will be studied in section 7.

7 Feasibility of limit distributions

In this section we prove the existence of the solutions of all three types under the assumption that $a^+ > b^-$ and7.te $a^- < b^+$, where

$$a^- = \min\{a_i : i = 1, \dots n\}, \quad b^- = \min\{b_j : j = 1, \dots n\}.$$

Let

$$\sigma_1 = \min_{x\in S_n} \frac{\sum_{k=1}^n a_k x_k^{2c-1}}{\sum_{k=1}^n x_k^c}, \quad \sigma_2 = \min_{y\in S_m} \frac{\sum_{k=1}^m b_k y_k^{2c-1}}{\sum_{k=1}^m y_k^c}.$$

Obviously, σ_1 and σ_2 are positive.

We base our analysis on the next technical lemmas.

Lemma 7.1 *Let* $t_0 \geq 0,\ \alpha > 0,\ \delta > 0,\ \beta > 0,$

$$\sum_{i \in I^+} x_i^c(t_0) > 1 - \alpha,$$

$$\psi(y(t_0)) + 2\delta < a^+(1-\alpha),$$

$$\left(\psi(y(t_0)) - \frac{(b^+)^2}{\sigma_2}\right) e^{-c\nu\sigma_2\beta} + \frac{(b^+)^2}{\sigma_2} < \psi(y(t_0)) + \delta,$$

$$\frac{(c-1)v(t_0)}{\delta(1-v(t_0))^c} < \frac{\beta}{2}.$$

Then the solution $(x(\cdot), y(\cdot), u(\cdot), v(\cdot))$ *is favourable for firm 1.*

Lemma 7.2 *Let* $t_0 \geq 0,\ \alpha > 0,\ \delta > 0,\ \beta > 0,$

$$\sum_{j \in J^+} y_j^c(t_0) > 1 - \alpha, \tag{7.1}$$

$$\varphi(x(t_0)) + 2\delta < b^+(1-\alpha), \tag{7.2}$$

$$\left(\varphi(x(t_0)) - \frac{(a^+)^2}{\sigma_1}\right) e^{-c\mu\sigma_1\beta} + \frac{(a^+)^2}{\sigma_1} < \varphi(x(t_0)) + \delta, \tag{7.3}$$

$$\frac{(c-1)u(t_0)}{\delta(1-u(t_0))^c} < \frac{\beta}{2}. \tag{7.4}$$

Then the solution $(x(\cdot), y(\cdot), u(\cdot), v(\cdot))$ *is favourable for firm 2.*

We prove Lemma 7.2 only.

Proof of Lemma 7.2. Let us estimate the derivative of

$$\varphi(t) = \varphi(x(t)) = \sum_{k=1}^{n} a_k x_k^c(t).$$

We have

$$\begin{aligned}
\dot\varphi(t) &= c\sum_{k=1}^{n} a_k x_k^{c-1}(t)\dot x_k(t) = c\sum_{k=1}^{n} a_k x_k^{c-1}(t)\mu u^c(t) x_k^c(t)\left(a_k - \frac{\varphi(t)}{\varphi_0(x(t))}\right) \\
&= c\mu\sum_{k=1}^{n} a_k u^c(t) x_k^{2c-1}(t)\left(a_k - \frac{\varphi(t)}{\varphi_0(x(t))}\right) \\
&= c\mu u^c(t)\left(\sum_{k=1}^{n} a_k^2 x_k^{2c-1} - \frac{\sum_{k=1}^{n} a_k x_k^{2c-1}(t)}{\sum_{k=1}^{n} x_k^c(t)}\varphi(t)\right) \\
&\leq c\mu u^c(t)((a^+)^2 - \sigma_1\varphi(t)).
\end{aligned}$$

Let $\bar\varphi(t)$ solve the Cauchy problem

$$\dot{\bar\varphi}(t) = c\mu((a^+)^2 - \sigma_1\bar\varphi(t)), \quad \bar\varphi(t_0) = \varphi(t_0).$$

Evidently,

$$\varphi(t) \leq \bar\varphi\left(\int_{t_0}^{t} u^c(\tau)d\tau\right) = \bar\varphi(\eta(t) - \eta(t_0)). \tag{7.5}$$

We have

$$\begin{aligned}
\bar{\varphi}(t) &= e^{-c\mu\sigma_1(t-t_0)}\varphi(t_0) + \int_{t_0}^{t} e^{-c\mu\sigma_1(t-s)} c\mu(a^+)^2 ds \\
&= e^{-c\mu\sigma_1(t-t_0)}\varphi(t_0) + e^{-c\mu\sigma_1 t}\int_{t_0}^{t} e^{c\mu\sigma_1 s} c\mu(a^+)^2 ds \\
&= e^{-c\mu\sigma_1(t-t_0)}\varphi(t_0) + \frac{(a^+)^2}{\sigma_1} e^{-c\mu\sigma_1 t}\left(e^{c\mu\sigma_1 t} - e^{c\mu\sigma_1 t_0}\right) \\
&= e^{-c\mu\sigma_1(t-t_0)}\varphi(t_0) + \frac{(a^+)^2}{\sigma_1}\left(1 - e^{-c\mu\sigma_1(t-t_0)}\right) \\
&= \left(\varphi(t_0) - \frac{(a^+)^2}{\sigma_1}\right) e^{-c\mu\sigma_1(t-t_0)} + \frac{(a^+)^2}{\sigma_1}.
\end{aligned}$$

Hence, by (7.5)

$$\varphi(t) \leq \left(\varphi(t_0) - \frac{(a^+)^2}{\sigma_1}\right) e^{-c\mu\sigma_1(\eta(t)-\eta(t_0))} + \frac{(a^+)^2}{\sigma_1}. \tag{7.6}$$

Note that by the definition of σ_1

$$\sigma_1 \leq \frac{\sum_{k=1}^n a_k x_k^{2c-1}(t_0)}{\sum_{k=1}^n x_k^c(t_0)}.$$

Hence,

$$\begin{aligned}
\varphi(t_0) - \frac{(a^+)^2}{\sigma_1} &= \sum_{k=1}^n a_k x_k^{2c-1}(t_0) - \frac{(a^+)^2}{\sigma_1} \\
&\leq \sum_{k=1}^n a_k x_k^c(t_0) - (a^+)^2 \frac{\sum_{k=1}^n x_k^c(t_0)}{\sum_{k=1}^n a_k x_k^{2c-1}(t_0)} \\
&\leq a^+ \sum_{k=1}^n x_k^c(t_0)\left(1 - \frac{a^+}{\sum_{k=1}^n a_k x_k^{2c-1}(t_0)}\right) && (7.7) \\
&\leq 0 && (7.8)
\end{aligned}$$

because of $\sum_{k=1}^n a_k x_k^{2c-1}(t_0) \leq a^+$. Due to (3.8)

$$\sum_{j\in J^+} \dot{y}_j(t) > 0.$$

Hence, for $t > t_0$

$$\psi(t) = \psi(y(t)) \geq \sum_{j\in J^+} b^+ y_j^c(t) > b^+ \sum_{j\in J^+} y_j^c(t_0) > b^+(1-\alpha) > \varphi(t_0) + 2\delta. \tag{7.9}$$

The last two inequalities hold due to (7.1) and (7.2). Sequentially using (7.6), (7.8), (7.3) and (7.9), we obtain the next estimates for all $t \geq t_0$ such that $\eta(t) - \eta(t_0) < \beta$:

$$\begin{aligned}
\varphi(t) &\leq \left(\varphi(t_0) - \frac{(a^+)^2}{\sigma_1}\right) e^{-2\mu\sigma_1(\eta(t)-\eta(t_0))} + \frac{(a^+)^2}{\sigma_1} \\
&\leq \left(\varphi(t_0) - \frac{(a^+)^2}{\sigma_1}\right) e^{-2\mu\sigma_1\beta} + \frac{(a^+)^2}{\sigma_1} \\
&\leq \varphi(t_0) + \delta \\
&\leq \psi(t) - \delta.
\end{aligned}$$

Let

$$\xi = \sup\{t > t_0 : \eta(t) - \eta(t_0) < \beta\}. \tag{7.10}$$

We will show that $\xi = \infty$. For all $t \in [t_0, \xi)$ due to (3.9) we have

$$\begin{aligned} \dot{u}(t) &= \frac{u^c(t)v^c(t)}{u^c(t)+v^c(t)}(\varphi(t)-\psi(t)) \le -\delta\frac{u^c(t)v^c(t)}{u^c(t)+v^c(t)} \le -\delta u^c(t)v^c(t) \\ &= -\delta u^c(t)(1-u(t))^c. \end{aligned}$$

Hence, for $t \in [t_0, \xi)$, $\dot{u}(t) < 0$, which implies

$$\dot{u}(t) \le -\delta_0 u^c(t)$$

where

$$\delta_0 = \delta(1-u(t_0))^c.$$

Then for $t \in [t_0, \xi)$

$$u(t) \le \bar{u}(t)$$

where $\bar{u}(t)$ solves the Cauchy problem

$$\dot{\bar{u}}(t) = -\delta_0 \bar{u}^c(t), \quad \bar{u}(t_0) = u(t_0).$$

We have

$$\frac{\dot{\bar{u}}(t)}{\bar{u}^c(t)} = -\delta_0,$$

$$-\frac{1}{(c-1)\bar{u}^{c-1}(t)} = -\delta_0(t-t_0) - c_0, \quad c_0 = \frac{1}{(c-1)u^{c-1}(t_0)},$$

$$\bar{u}(t) = \Big(\frac{1}{(c-1)(\delta_0(t-t_0)+c_0)}\Big)^{1/(c-1)}.$$

Thus,

$$u(t) \le \alpha\left(\frac{1}{\delta_0(t-t_0)+c_0}\right)^{1/(c-1)}, \quad \text{where } \alpha := (c-1)^{-1/(c-1)},$$

for $t \in [t_0, \xi)$. Then for $t \in [t_0, \xi)$

$$\begin{aligned} \eta(t) - \eta(t_0) &= \int_{t_0}^t u^c(\tau)d\tau \le \alpha\int_{t_0}^t \frac{d\tau}{(\delta_0(\tau-t_0)+c_0)^{c/(c-1)}} \\ &= \alpha\int_0^{t-t_0} \frac{d\tau}{(\delta_0\tau+c_0)^{c/(c-1)}} \\ &= -\alpha(c-1)\frac{1}{\delta_0}\left(\frac{1}{(\delta_0\tau+c_0)^{c/(c-1)-1}}\ \Big|_0^{t-t_0}\right) \\ &\le \frac{\alpha(c-1)}{\delta_0 c_0^{1/(c-1)}} = \frac{(c-1)u(t_0)}{\delta(1-u(t_0))^c} < \frac{\beta}{2} \end{aligned}$$

because of $c_0 = (u^{c-1}(t_0)(c-1))^{-1}$.

The last inequality follows from (7.4). If we assume $\xi < \infty$, we get

$$\eta(\xi) - \eta(t_0) \le \frac{\beta}{2},$$

whereas (7.10) implies

$$\eta(\xi) - \eta(t_0) = \beta.$$

Hence, $\xi = \infty$, i.e., $\eta(t) - \eta(t_0) < \beta$ for all $t \geq t_0$. Then, as stated above, $\varphi(t) < \psi(t) - \delta$ for all $t \geq t_0$. Consequently,

$$\varphi_* = \lim_{t\to\infty}(\varphi(t)) < \lim_{t\to\infty}(\psi(t)) = \psi_*.$$

By definition (see also Theorem 6.2) the solution $x(t)$, $y(t)$, $u(t)$, $v(t)$ is favourable for firm 2. □

In what follows, we denote Z^0 the set of all initial points $(x_0, y_0, u_0, v_0) \in S_n \times S_m \times S_2$ satisfying (5.2). We denote Z_1^0 the set of all initial states from Z^0 such that the solution originating from this state is favourable for firm 1. Symmetrically, we denote Z_2^0 the set of all initial states from Z^0 such that the solution originating from this state is favourable for firm 2.

Lemmas 7.1 and 7.2 yield the following.

Lemma 7.3 *The sets Z_1^0 and Z_2^0 are open.*

Proof. We prove the openness of Z_2^0 only. Let $(x_0, y_0, u_0, v_0) \in Z_2^0$ and $(x(\cdot), y(\cdot), u(\cdot), v(\cdot))$ be the solution with the initial condition (5.2). Since it is favourable for firm 2, by Theorem 6.2 we have $\varphi_* < \psi_* = \psi^+ = b^+$ and

$$\lim_{t\to\infty} u(t) = 0 \tag{7.11}$$

By Lemma 5.3, 3, $\zeta_* = \infty$ and hence, by Lemma 5.2, 2, for all large t_0 the relation (7.1) holds. Take positive α and δ so that for all large t_0 the relation (7.2) holds. Such a choice is possible due to the inequality $\varphi_* < b^+$. Let $\beta > 0$ be such that for all large t_0 (7.3) is satisfied. By (7.11) for all large t_0 the inequality (7.4) holds. Thus, there is a (large) t_0 for which the estimates (7.1) – (7.4) of Lemma 7.2 are satisfied. Then (7.1) – (7.4) hold if point $x(t_0), y(t_0), u(t_0), v(t_0)$ is replaced by an arbitrary point from certain neighborhood, V, of $x(t_0), y(t_0), u(t_0), v(t_0)$. Let $\hat{\mathcal{Z}}$ be the set of all solutions $(\hat{x}(\cdot), \hat{y}(\cdot), \hat{u}(\cdot), \hat{v}(\cdot))$ such that $(\hat{x}(t_0), \hat{y}(t_0), \hat{u}(t_0), \hat{v}(t_0)) \in V$. By Lemma 7.2 all solutions from $\mathcal{Z}$ are favourable for firm 2. Let $\hat{Z} = \{(\hat{x}(0), \hat{y}(0), \hat{u}(0), \hat{v}(0)) : (\hat{x}(\cdot), \hat{y}(\cdot), \hat{u}(\cdot), \hat{v}(\cdot)) \in \hat{\mathcal{Z}}\}$. The set $\hat{Z}$ obviously contains a neighborhood of the point (x_0, y_0, u_0, v_0). Thus, Z_2^0 is open. □

Theorem 7.1 *If $b^- < a^+$, then there exist a solution favourable for firm 1. If $a^- < b^+$, then there exist a solution favourable for firm 2.*

Proof. We prove the second statement only. Let $t_0 = 0$. In view of $a^- < b^+$, there exists an initial point (5.2) in Z^0 such that the relations (7.1) – (7.4) hold with some positive α, δ and β. Then by Lemma 7.2 the solution originating from this initial state is favourable for firm 2. □

Theorem 7.2 *There exists a critical solution.*

Proof. Let I be a segment with the endpoints in Z_1^0 and Z_2^0. We have $I \subset Z^0$ due to the convexity of Z^0. By definition the sets Z_1^0 and Z_2^0 do not intersect. By Lemma 7.3 they are open. Then I cannot be covered by the union of Z_1^0 and Z_2^0. Therefore the set $Z = Z^0 \setminus (Z_1^0 \cup Z_2^0)$ is nonempty. A solution originating from a point in Z is favourable neither for firm 1, nor for firm 2. By definition this solution is critical. □

8 Firms with equal maximal productivities: existence of balanced solutions

In section 6 we showed that if the firms have different maximal productivities, say, $a^+ > b^+$, then every critical solution is such that the limit market share of the firm, which has the higher maximal productivity (firm 1), is 0, and that of the firm, which has the lower maximal productivity (firm 2), is 1 (Theorem 6.3). Thus, a single chance for the firms to coexist in the long run, i.e., to have nonzero limits of their market shares as time apprioaches infinity, arises when the firms have equal maximal productivities, $a^+ = b^+$.

In this section we assume that $a^+ = b^+$. We shall call a solution $(x(\cdot), y(\cdot), u(\cdot), v(\cdot))$ to (3.7) – (3.10), (5.2) *balanced* if there are nonzero limits $\lim_{t\to\infty} u(t)$ and $\lim_{t\to\infty} v(t)$. Note that a balanced solution is necessarily critical. In this section we focus on a simplest situation where each firm has only two departments, which are not equal in productivity.

Theorem 8.1 *Let $0 < c \le 2$, $m = n = 2$, $a^+ = b^+$, $a^- < a^+$, and $b^- < b^+$. Then there exists a balanced solution.*

In this section, when writing out differential equations, we, for brievity, omit the time argument in the notation of the sought functions. A proof of Theorem 8.1 is given in the end of this section. It is based on the next theorem.

Theorem 8.2 [Hartman, 1964, p. 294]. *Let*

(i) a system of finite-dimensional differential equations have the form

$$p' = Pp + F_1(\tau, p, q), \quad q' = Qq + F_2(\tau, p, q), \tag{8.1}$$

(ii) the real parts of all eigenvalues of the matrix P be not greater than ω, and the real parts of all eigenvalues of the matrix Q be strictly greater than ω,

(iii) the function $F = (F_1, F_2)$ be continuous and

$$|F(\tau, \xi)| \le l(\tau)|\xi| \quad (\xi = (p, q)) \tag{8.2}$$

hold for all $\tau \ge 0$ and all ξ from a neighborhood of the origin,

(iv) l be continuous and

$$\lim_{\tau\to\infty} \sup_{s\ge\tau} \frac{1}{1+s-\tau} \int_\tau^s l(\zeta)d\zeta = 0. \tag{8.3}$$

Then there exist $\tau_* \ge 0$ and $\delta_1 > 0$ such that for every $\tau_0 \ge \tau_*$ and every p_0 satisfying $|p_0| < \delta_1$ there is a q_0 with the property that the Cauchy problem for the system (8.1) with the initial condition $p(\tau_0) = p_0$, $q(\tau_0) = q_0$ has a solution $\xi(\cdot) = ((p(\cdot), q(\cdot))$ on $[\tau_0, \infty)$, which satisfies either $(p(\cdot), q(\cdot)) = 0$ or $p(\tau) \ne 0$ for all $\tau \ge \tau_0$, and

$$|q(\tau)| = o(|p(\tau)|) \quad \text{as} \quad \tau \to \infty,$$

$$\limsup_{\tau\to\infty} \frac{\log|\xi(\tau)|}{\tau} \le \omega.$$

We shall use the next corollary of Theorem 8.2.

Corollary 8.1 *Let*

(i) a finite-dimensional differential equation have the form

$$r' = \frac{h_1(s,r)}{s} + h_2(s,r), \tag{8.4}$$

(ii) h_1 and h_2 be differentiable at a point $(0, r_*)$ satisfying $h_1(0, r_*) = 0$, and there be $K > 0$ and $d > 1$ such that

$$\left| h_1(s,r) - \left(\frac{\partial h_1}{\partial s}(0,r_*)s + \frac{\partial h_1}{\partial r}(0,r_*)(r - r_*) \right) \right| \le K(|s| + |r - r_*|)^d,$$

$$\left| sh_2(s,r) - s\frac{\partial h_2}{\partial r}(0,r_*)(r - r_*) \right| \le K(|s| + |r - r_*|)^d$$

for all (s,r) from a neighborhood of $(0, r_*)$,

(iii) the matrix

$$H = \begin{pmatrix} -1 & 0 \\ -\left[\frac{\partial h_1}{\partial s}(0,r_*) + h_2(0,r_*)\right] & -\frac{\partial h_1}{\partial r}(0,r_*) \end{pmatrix} \tag{8.5}$$

have an eigenvector $(\bar{s}, \bar{r})$ with the eigenvalue -1, for which $\bar{s} \neq 0$.

Then for some $\delta > 0$ the equation (8.4) has a solution $r(\cdot)$ defined on $(0, \delta)$ satisfying $\lim_{s \to +0} r(s) = r_*$.

Proof. Without loss of generality assume $r_* = 0$. Introduce a new independent variable, $\tau = -\log s$. Then $s = e^{-\tau}$, $s' = -s$, and the equation (8.4) takes the form

$$s' = -s, \quad r' = -h_1(s,r) - sh_2(s,r). \tag{8.6}$$

It is sufficient to prove that (8.6) has a solution $(s(\cdot), r(\cdot))$ on $[\bar{\tau}_1, \infty)$, $\bar{\tau}_1 > 0$, such that $s(\tau) > 0$ for all $\tau \ge \bar{\tau}_1$ and

$$\lim_{\tau \to \infty} r(\tau) = 0. \tag{8.7}$$

Setting $y = (s, r)$, we represent (8.6) as

$$y' = Hy + G(y), \tag{8.8}$$

where H is given in (8.5) and G is continuous and

$$|G(y)| \le K|y|^d$$

for all y from a neigborhood of 0.

We shall make two linear transformations of the state variables, which will bring H to a (P, Q)-block form indicated in Theorem 8.2. The first linear transformation, $z = T_1 y$, corresponds to passing from the original basis, $e_1, e_2, \ldots, e_{k+1}$, in the ($(k+1)$-dimensional) state space of the system (8.8) (here e_i is the ith unit coordinate vector) to the basis $\bar{y}, e_2, \ldots, e_{k+1}$, where $\bar{y} = (\bar{s}, \bar{r})$ (see (iii) in the formulation of the Corollary). Since $\bar{s} \neq 0$, the system $\bar{y}, e_2, \ldots, e_{k+1}$ is a basis indeed. The matrix H is transformed into the two-block matrix

$$H_1 = T_1 H T_1^{-1} = \begin{pmatrix} -1 & 0 \\ 0 & \bar{H}_1 \end{pmatrix}.$$

Let $z = (z_1, \eta^{(1)})$, $\eta^{(1)} = (z_2, \ldots, z_{k+1})$. The second linear transformation, $\xi = T_2 z$, does not involve z_1, so, $\xi = (z_1, \eta^{(2)})$ and

$$T_2 = \begin{pmatrix} 1 & 0 \\ 0 & \bar{T}_2 \end{pmatrix}.$$

The transformation $\eta^{(2)} = \bar{T}_2 \eta^{(1)}$ transforms the matrix $\bar{H}_1$ to

$$\bar{H}_2 = T_2 \bar{H}_1 T_2^{-1} = \begin{pmatrix} D_1 & 0 \\ 0 & D_2 \end{pmatrix},$$

where the real parts of all eigenvalues of D_1 are not greater than -1, and the real parts of all eigenvalues of D_2 are strictly greater than -1 (see Hirsch and Smale, 1974, p. 129); with no loss of generality we assume that $\bar{H}_1$ has an eigenvalue whose real part is strictly greater than -1 (one can always extend $\bar{H}_1$ for an extra zero row and an extra zero column). The resulting transformation, $\xi = Ty = T_2 T_1 y$, transforms the original matrix H to

$$E = THT^{-1} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & D_1 & 0 \\ 0 & 0 & D_2 \end{pmatrix} = \begin{pmatrix} P & 0 \\ 0 & Q \end{pmatrix},$$

where

$$P = \begin{pmatrix} -1 & 0 \\ 0 & D_1 \end{pmatrix} \tag{8.9}$$

and $Q = D_2$. The real parts of all eigenvalues of P are not greater than -1 and the real parts of all eigenvalues of Q are strictly greater than -1.

Note that for any vector ξ with $\xi_1 \neq 0$ its preimage $y = (s, r) = T^{-1}\xi = T_2^{-1} T_1^{-1} \xi$ satisfies $s \neq 0$. Indeed, due to the block structure of T_2 for $z = T_1 y = T_2^{-1}\xi$ we have $z_1 = \xi_1 \neq 0$. Recall that the first transformation of the variables, which is given by the matrix T_1, corresponds to passing from the original basis $e_1, e_2, \ldots, e_{k+1}$, to the basis $\bar{y}, e_2, \ldots, e_{k+1}$, where $\bar{y} = (\bar{s}, \bar{r})$. Therefore, z_1 is the first coefficient, λ_1, in the (unique) representation $y = \lambda_1 \bar{y} + \lambda_2 e_2 + \ldots + \lambda_{k+1} e_{k+1}$. The projection of this equality to the first coordinate gives $s = \lambda_1 \bar{s}$. Since $\lambda_1 \neq 0$ and $\bar{s} \neq 0$ (see the assumptions of the Corollary), we get $s \neq 0$.

Let $\xi = (p, q)$, where $p = (\xi_1, \ldots, \xi_{k_1})$, $q = (\xi_{k_1+1}, \ldots, \xi_{k+1})$, and k_1 and $k+1-k_1$ are the dimensions of the matrices P and Q, respectively. With respect to the variables p and q, the equation (8.8) has the form

$$p' = Pp + G_1^*(p, q), \quad q' = Qq + G_2^*(p, q), \tag{8.10}$$

where $G^* = (G_1^*, G_2^*)^T$ is continuous and satisfies

$$|G^*(\xi)| \leq K|\xi|^d \quad (\xi = (p, q))$$

for all ξ from a neighborhood of the origin (without loss of generality we assume that G^* is defined and continuous on the whole $(k+1)$-dimensional Euclidean space). Let $G_1^* = (G_{11}^*, \ldots, G_{1k_1}^*)$. The evolution of the first coordinate, $\xi_1 = p_1$, of ξ (or p) will play an exceptional role. In this context we introduce a modified system,

$$p' = Pp + G_1^{**}(p,q), \quad q' = Qq + G_2^{**}(p,q). \tag{8.11}$$

Here

$$G_2^{**} = G_2^*, \quad G_1^{**} = (G_{11}^{**}, G_{12}^*, \ldots, G_{1k_1}^*)^T, \tag{8.12}$$

and

$$G_{11}^{**}(p,q) = \min\{|G_{11}^*(p,q)|, |\varepsilon p_1|\} \text{ sign } G_{11}^*(p,q). \tag{8.13}$$

A positive ε is chosen so that there is $\delta > 0$ such that

$$\delta d > 1 + \varepsilon, \quad \delta < 1. \tag{8.14}$$

Obviously, $G^{**} = (G_1^{**}, G_2^{**})$ is continuous and satisfies

$$|G^{**}(\xi)| \le K|\xi|^d \quad (\xi = (p,q)) \tag{8.15}$$

for all ξ from a neighborhood of the origin.

Set

$$F(\tau,\xi) = \min\{|\xi|^{d-1}, e^{-\varepsilon\tau}\}\frac{G^{**}(\xi)}{|\xi|^{d-1}}. \tag{8.16}$$

Obviously, F is continuous, and (8.2) holds with $l(\tau) = Ke^{-\varepsilon\tau}$ for all $\tau \ge 0$ and all ξ from a neighborhood of the origin. Clearly, $l(\cdot)$ satisfies (8.3). Thus, the system (8.1) with $(F_1, F_2) = F$ meets all the assumptions of Theorem 8.2 for $\omega = -1$. By Theorem 8.2 there exist $\tau_0 > 0$ and $\delta_1 > 0$ such that for $p_0 = (\pi, 0, \ldots, 0)$ with $0 < |\pi| < \delta_1$ there is q_0 with the property that the Cauchy problem for the system (8.1) with the initial condition $p(\tau_0) = p_0$, $q(\tau_0) = q_0$ has a solution $\xi(\cdot) = ((p(\cdot), q(\cdot))$ on $[\tau_0, \infty)$, for which

$$\limsup_{\tau\to\infty} \frac{\log|\xi(\tau)|}{\tau} \le -1. \tag{8.17}$$

Since $\pi \ne 0$, the preimage of $\xi_0 = (p_0, q_0)$, $y_0 = (s_0, r_0) = T^{-1}\xi_0$, satisfies $s_0 \ne 0$. We choose the sign of π so that $s_0 > 0$. Assume that $\pi > 0$ (the opposite case is treated similarly).

The relation (8.17) and the second inequality in (8.14) imply

$$|\xi(\tau)| \le e^{-\delta\tau} \tag{8.18}$$

for all sufficiently large τ. Therefore, for all sufficiently large τ we have $F(\tau, \xi(\tau)) = G^{**}(\xi(\tau))$ (see (8.16)). Hence, there is $\bar\tau > 0$ such that $\xi(\cdot)$ solves (8.11) on $[\bar\tau, \infty)$. On $[\bar\tau, \infty)$, due to (8.12), $q(\cdot)$ satisfies the second equation in (8.10), and in the first (vector) equation in (8.10), the scalar equations for the coordinates $2, \ldots, k_1$ are satisfied by $p_2(\cdot), \ldots, p_{k_1}(\cdot)$. We shall prove that in a neighborhood of infinity $p_1(\cdot)$ satisfies the first coordinate equation in (8.10), which, as (8.9) shows, has the form

$$p_1' = -p_1 + G_{11}^*(p,q). \tag{8.19}$$

First, however, we shall state that $p_1(\tau) > 0$ for all $\tau \geq \tau_0$. For all $\tau > \tau_0$ we have (see (8.16))

$$p_1'(\tau) = -p_1(\tau) + \lambda(\tau) G_{11}^{**}(p(\tau), q(\tau)),$$

where

$$\lambda(\tau) = \min\left\{1, \frac{e^{-\varepsilon\tau}}{|\xi(\tau)|^{d-1}}\right\}.$$

Due to (8.13)

$$p_1'(\tau) \geq -\lambda(\tau)(1+\varepsilon)p_1(\tau)$$

for all $\tau > \tau_0$. Since $p_1(\tau_0) = \pi > 0$,

$$p_1(\tau) \geq \pi e^{-(1+\varepsilon)\int_{\tau_0}^{\tau} \lambda(r)dr} > 0$$

for all $\tau \geq \tau_0$.

Recall that for all $\tau \geq \bar{\tau}$

$$p_1'(\tau) = -p_1(\tau) + G_{11}^{**}(p(\tau), q(\tau)) \geq -(1+\varepsilon)p_1(\tau)$$

(the inequality follows from (8.13)), and hence

$$p_1(\tau) \geq p_1(\bar{\tau}) e^{-(1+\varepsilon)(\tau-\bar{\tau})}.$$

By (8.18) and (8.15)

$$|G_{11}^{**}(p(\tau), q(\tau))| \leq K e^{-\delta d\tau}$$

for all large τ. Due to the first inequality in (8.14) there is $\bar{\tau}_1 > \tau_0$ such that

$$|G_{11}^{**}(p(\tau), q(\tau))| \leq |\varepsilon p_1(\tau)|$$

for all $\tau \geq \bar{\tau}_1 > 0$. Then by (8.13) for all $\tau \geq \bar{\tau}_1 > 0$

$$G_{11}^{**}(p(\tau), q(\tau)) = G_{11}^{*}(p(\tau), q(\tau)).$$

We stated that $p_1(\cdot)$ satisfies (8.19) on $[\bar{\tau}_1, \infty)$. Thus, $\xi(\cdot) = (p(\cdot), q(\cdot))$ is a solution to (8.11) on $[\bar{\tau}_1, \infty)$. The preimage of $\xi(\cdot)$, $\tau \mapsto y(\tau) = (s(\tau), r(\tau)) = T^{-1}\xi(\tau)$, solves the original equation (8.6) on $[\bar{\tau}_1, \infty)$. As soon as $p_1(\tau) = \xi_1(\tau) \neq 0$ for all $\tau \geq \tau_0$, we have $s(\tau) \neq 0$ for all $\tau \geq \tau_0$; since $s(\tau_0) > 0$, we find that $s(\tau) > 0$ for all $\tau \geq \tau_0$. Finally, (8.18) implies (8.7). □ **Proof of Theorem 8.1.** 1. With no loss of generality assume $a_1 = a^+$ and $b_1 = b^+ = a^+$. Set $x(t) = x_1(t)$ and $y(t) = y_1(t)$. The system of equations (3.7) – (3.10) is reduced to

$$\dot{x} = \alpha u^c \frac{x^c(1-x)^c}{x^c + (1-x)^c}, \tag{8.20}$$

$$\dot{y} = \beta(1-u)^c \frac{y^c(1-y)^c}{y^c + (1-y)^c}, \tag{8.21}$$

$$\dot{u} = \lambda(u)[a^+(x^c - y^c) + a^-(1-x)^c - b^-(1-y)^c] \tag{8.22}$$

with

$$\alpha = \mu(a^+ - a^-), \quad \beta = \nu(a^+ - b^-), \quad \lambda(u) = \rho \frac{u^c(1-u)^c}{u^c + (1-u)^c}. \tag{8.23}$$

Our goal is to show that there is a solution $(x(\cdot), y(\cdot), u(\cdot))$ to (8.20) – (8.22) such that

$$x(t) \in (0,1), \quad y(t) \in (0,1), \quad u(t) \in (0,1) \quad (t \geq 0) \tag{8.24}$$

and

$$u_* = \lim_{t\to\infty} u(t) \in (0,1). \tag{8.25}$$

Introduce new time and state variables,

$$s^{c-1} = \frac{1}{t}, \quad w = \frac{s^{c-1}}{(1-x)^{c-1}}, \quad z = \frac{s^{c-1}}{(1-y)^{c-1}}.$$

We have

$$\begin{aligned} w' &= \frac{(c-1)s^{c-2}}{(1-x)^{c-1}} + \frac{(c-1)s^{c-1}}{(1-x)^c} x' \\ &= \frac{(c-1)w}{s} + \frac{(c-1)w^{c/(c-1)}}{s} \left(-\alpha u^c \frac{x^c(1-x)^c}{x^c + (1-x)^c} \right) \frac{c-1}{s^c} \\ &= (c-1)\frac{w}{s} - (c-1)^2 \frac{w^{c/(c-1)}}{s^{c+1}} \alpha u^c \frac{(1 - s/w^{1/(c-1)})^c (s/w^{1/(c-1)})^c}{(1 - s/w^{1/(c-1)})^c + (s/w^{1/(c-1)})^c} \\ &= (c-1)\frac{w}{s} - (c-1)^2 \frac{w^{c/(c-1)}}{s^{c+1}} \alpha u^c \frac{(w^{1/(c-1)} - s)^c s^c / w^{2c/(c-1)}}{[(w^{1/(c-1)} - s)^c + s^c]/w^{c/(c-1)}} \\ &= (c-1)\frac{w}{s} - (c-1)^2 \frac{\alpha u^c}{s} \frac{(w^{1/(c-1)} - s)^c}{(w^{1/(c-1)} - s)^c + s^c}. \end{aligned}$$

Thus,

$$w' = \frac{c-1}{s} \left[w - (c-1)\alpha u^c \left(1 - \frac{s^c}{(w^{1/(c-1)} - s)^c + s^c} \right) \right]. \tag{8.26}$$

Similarly, we get a differential equation for z:

$$z' = \frac{c-1}{s} \left[z - (c-1)\beta (1-u)^c \left(1 - \frac{s^c}{(z^{1/(c-1)} - s)^c + s^c} \right) \right]. \tag{8.27}$$

Using (8.22), we find:

$$\begin{aligned} u' &= -(c-1)\frac{\lambda(u)}{s^c}[a^+(x^c - y^c) + a^-(1-x)^c - b^-(1-y)^c] \\ &= -(c-1)\frac{\lambda(u)}{s^c}[a^+(x-y)^{c-1} g(x,y) + a^-(1-x)^c - b^-(1-y)^c] \\ &= -(c-1)\frac{\lambda(u)}{s^c}[a^+((1-y) - (1-x))^{c-1} g(x,y) + a^-(1-x)^c - b^-(1-y)^c], \end{aligned}$$

where

$$g(x,y) = g_0(s,w,z) = \frac{x^c - y^c}{(x-y)^{c-1}}. \tag{8.28}$$

Continue as follows:

$$
\begin{aligned}
u' &= -(c-1)\frac{\lambda(u)a^+}{s^c}\left(\frac{s}{z^{1/(c-1)}}-\frac{s}{w^{1/(c-1)}}\right)^{c-1} g_0(s,w,z)\\
&\quad -(c-1)\frac{\lambda(u)a^+}{s^c}\left[a^-\left(\frac{s}{w^{1/(c-1)}}\right)^c - b^-\left(\frac{s}{z^{1/(c-1)}}\right)^c\right]\\
&= -\frac{1}{s}(c-1)\frac{\lambda(u)a^+g_0(s,w,z)(w^{1/(c-1)}-z^{1/(c-1)})^{c-1}}{zw}\\
&\quad -(c-1)\lambda(u)a^+\left(\frac{a^-}{w^{c/(c-1)}}-\frac{b^-}{z^{c/(c-1)}}\right).
\end{aligned}
$$

Hence the differential equation for u takes the form

$$
\begin{aligned}
u' &= -\frac{1}{s}(c-1)\frac{\lambda(u)a^+g_0(s,w,z)(w^{1/(c-1)}-z^{1/(c-1)})^{c-1}}{zw}\\
&\quad -(c-1)\lambda(u)a^+\left(\frac{a^-}{w^{c/(c-1)}}-\frac{b^-}{z^{c/(c-1)}}\right).
\end{aligned}
\tag{8.29}
$$

Rewrite (8.26) – (8.29) as

$$
w' = (c-1)\frac{w-(c-1)\alpha u^c f(s,w)}{s}, \tag{8.30}
$$

$$
z' = (c-1)\frac{z-(c-1)\beta(1-u)^c f(s,z)}{s}, \tag{8.31}
$$

$$
u' = \frac{g_1(s,w,z,u)}{s} + g_2(w,z,u), \tag{8.32}
$$

where

$$
\begin{aligned}
f(s,p) &= 1-\frac{s^c}{(p^{1/(c-1)}-s)^c+s^c},\\
g_1(s,w,z,u) &= -(c-1)\frac{\lambda(u)a^+g_0(s,w,z)(w^{1/(c-1)}-z^{1/(c-1)})^{c-1}}{zw},\\
g_2(w,z,u) &= -(c-1)\lambda(u)a^+\left(\frac{a^-}{w^{c/(c-1)}}-\frac{b^-}{z^{c/(c-1)}}\right).
\end{aligned}
\tag{8.33}
$$

Note that

$$
\frac{\partial f(s,p)}{\partial s}(0,p)=0, \quad \frac{\partial f(s,p)}{\partial p}(0,p)=0. \tag{8.34}
$$

If the system of equations (8.30) – (8.32) has a solution $(w(\cdot),z(\cdot),u(\cdot))$ defined on $(0,\delta)$, where $\delta>0$, and satisfying

$$
\lim_{s\to+0} w(s)=w_*, \quad \lim_{s\to+0} z(s)=z_*, \quad \lim_{s\to+0} u(s)=u_* \tag{8.35}
$$

with $w_*>0$, $z_*>0$, and $u_*\in(0,1)$, then the system of equations (8.20) – (8.22) has a solution $(x(\cdot),y(\cdot),u(\cdot))$ defined on $[0,\infty)$ and satisfying (8.24) and (8.25), and Theorem 8.1 is proved.

Fix positive w_* and z_* and $u_* \in (0,1)$ such that the numerators of the singular ratios in the right hand sides of (8.30) – (8.32) vanish at $(0, w_*, z_*, u_*)$:

$$w_* = (c-1)\alpha u_*^c = (c-1)\beta(1-u_*)^c = z_*. \tag{8.36}$$

We consider the cases $c = 2$ and $c < 2$ separately.

2. Let $c = 2$. Then

$$f(s,p) = 1 - \frac{s^2}{(p-s)^2 + s^2},$$

$$g_0(s,w,z) = x + y = 2 - \frac{s}{w} - \frac{s}{z}$$

(see (8.28)), g_1 is given by (8.33), and

$$g_2(w,z,u) = -\lambda(u)a^+\left(\frac{a^-}{w^2} - \frac{b^-}{z^2}\right).$$

The functions $(s,w,z,u) \to f(s,w)$, $(s,w,z,u) \to f(s,z)$, $(s,w,z,u) \to g_1(s,w,z,u)$, and $(s,w,z,u) \to g_2(w,z,u)$ are twice continuously differentiable in a neighborhood of $(0, w_*, z_*, u_*)$. The system of equations (8.30) – (8.32) satisfies conditions (i) and (ii) of Corollary 8.1 with $d = 2$. Taking into account (8.34), we find that the matrix (8.5) has the form

$$H = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 2\alpha u_* \\ 0 & 0 & -1 & -2\beta(1-u_*) \\ -g_{2*} & -g_{1*} & g_{1*} & 0 \end{pmatrix};$$

here

$$g_{1*} = 2\frac{a^+\lambda(u_*)}{w_* z_*} > 0$$

and $g_{2*} = g_2(w_*, z_*, u_*)$. For the matrix H the eigenvectors (s,w,z,u) with the eigenvalue -1 are the solutions to

$$-s = -s,$$

$$-w + 2\alpha u_* u = -w,$$

$$-z - 2\beta(1-u_*)u = -z,$$

$$-g_{2*}s - g_{1*}(w - z) = u.$$

The first coordinate, $\bar{s}$, of the eigenvector $(\bar{s}, \bar{w}, \bar{z}, \bar{u}) = (1, -g_{2*}/g_{1*}, 0, 0)$ is nonzero. Condition (iii) of Corollary 8.1 is satisfied. By Corollary 8.1 the system of equations (8.30) – (8.32) has a solution $(w(\cdot), z(\cdot), u(\cdot))$ defined on $(0, \delta)$, where $\delta > 0$, and satisfying (8.35).

3. Let $c < 2$. In the original variables (x, y, u) the point (w_*, z_*, u_*) has the coordinates $(1, 1, u_*)$. Since the function $p \mapsto p^c$ is Lipschitzian in a neighborhood of 1, (8.28) implies

$$|g_0(s,w,z)| = |g(x,y)| \le K_1|x - y|^{2-c}$$

for all x, y from a neighborhood of 1 with certain constant K_1. Since

$$x = 1 - \frac{s}{w^{1/(c-1)}}, \quad y = 1 - \frac{s}{z^{1/(c-1)}},$$

we have

$$|g_0(s, w, z)| \leq K_1 s^{2-c} \frac{|w^{1/(c-1)} - z^{1/(c-1)}|^{2-c}}{w^{(2-c)/(c-1)} z^{(2-c)/(c-1)}}.$$

Then by (8.33)

$$\begin{aligned} |g_1(s, w, z, u)| &\leq K_2 s^{2-c} |w^{1/(c-1)} - z^{1/(c-1)}|^{2-c} |w^{1/(c-1)} - z^{1/(c-1)}|^{c-1} \\ &= K_2 s^{2-c} |w^{1/(c-1)} - z^{1/(c-1)}| \end{aligned}$$

for all (s, w, z, u) from a neighborhood of $(0, w_*, z_*, u_*)$ with certain constant K_2. Since the function $p \mapsto p^{1/(c-1)}$ is Lipschitzian in a neighborhood of $w_* = z_*$, we find that

$$|g_1(s, w, z, u)| \leq K s^{2-c} (|w - w_*| + |z - z_*|) \tag{8.37}$$

for all (s, w, z, u) from a neighborhood of $(0, w_*, z_*, u_*)$ with certain constant K. Hence,

$$\frac{\partial g_1}{\partial s}(0, w_*, z_*, u_*) = \frac{\partial g_1}{\partial w}(0, w_*, z_*, u_*) = \frac{\partial g_1}{\partial z}(0, w_*, z_*, u_*) = \frac{\partial g_1}{\partial u}(0, w_*, z_*, u_*) = 0. \tag{8.38}$$

Let us show that g_1 is differentiable at $(0, w_*, z_*, u_*)$, moreover, there is a $d > 1$ such that

$$|g_1(s, w, z, u)| \leq (s + |w - w_*| + |z - z_*|)^d \tag{8.39}$$

for all (s, w, z, u) from a neighborhood of $(0, w_*, z_*, u_*)$. Let

$$\gamma = 2 - c, \quad d = 1 + \gamma/2. \tag{8.40}$$

Suppose there is no neighborhood of $(0, w_*, z_*, u_*)$ such that for all its elements (s, w, z, u) (8.39) holds. Then one can find a sequence $(s_i, w_i, z_i, u_i) \to (0, w_*, z_*, u_*)$ such that

$$|g_1(s_i, w_i, z_i, u_i)| > (s_i + |w_i - w_*| + |z_i - z_*|)^d \quad (i = 1, 2, \ldots).$$

In view of (8.37)

$$s_i > 0, \quad p_i = |w_i - w_*| + |z_i - z_*| > 0 \quad (i = 1, 2, \ldots)$$

and

$$K s_i^{2-c} p_i > (s_i + p_i)^d \quad (i = 1, 2, \ldots),$$

or, in notations (8.40),

$$K s_i^{\gamma} p_i > (s_i + p_i)^{1+\gamma/2} \quad (i = 1, 2, \ldots).$$

Then

$$\liminf \frac{s_i^{\gamma} p_i}{(s_i + p_i)^{1+\gamma/2}} \geq \frac{1}{K}. \tag{8.41}$$

With no loss of generality, we assume that there is the limit

$$\lim \frac{p_i}{s_i^{1+\gamma/2}} = a.$$

Let $a < \infty$. Then

$$\begin{aligned} \frac{s_i^\gamma p_i}{(s_i + p_i)^{1+\gamma/2}} &= \frac{s_i^\gamma}{(s_i + p_i)^{1+\gamma/2} s_i^{-(1+\gamma/2)}} \frac{p_i}{s_i^{1+\gamma/2}} \\ &= \frac{s_i^\gamma}{(1 + p_i/s_i)^{1+\gamma/2}} \frac{p_i}{s_i^{1+\gamma/2}} \\ &\leq s_i^\gamma \frac{p_i}{s_i^{1+\gamma/2}} \to 0, \end{aligned}$$

which contradicts (8.41). Let $a = \infty$. Then

$$\lim \frac{s_i^{(1+\gamma/2)\gamma/2}}{p_i^{\gamma/2}} = 0.$$

Since $\gamma/2 < 1$ (see (8.40)), we have

$$\zeta = \gamma - (1 + \gamma/2)\gamma/2 > 0,$$

$$\lim \frac{s_i^\gamma}{p_i^{\gamma/2}} = \lim s_i^\zeta \lim \frac{s_i^{(1+\gamma/2)\gamma/2}}{p_i^{\gamma/2}} = 0.$$

Hence,

$$\frac{s_i^\gamma p_i}{(s_i + p_i)^{1+\gamma/2}} \leq \frac{s_i^\gamma p_i}{p_i^{1+\gamma/2}} = \frac{s_i^\gamma}{p_i^{\gamma/2}} \to 0,$$

which again contradicts (8.41). The contradictions prove that the estimate (8.39) holds for all (s, w, z, u) from a neighborhood of $(0, w_*, z_*, u_*)$. The estimate (8.39) and equalities (8.38) imply the differentiability of g_1 at $(0, w_*, z_*, u_*)$. Taking into account that the functions $(s, w, z, u) \to f(s, w)$, $(s, w, z, u) \to f(s, z)$, and $(s, w, z, u) \to g_2(w, z, u)$ are twice continuously differentiable in a neighborhood of $(0, w_*, z_*, u_*)$, we find that the system of equations (8.30) – (8.32) satisfies conditions (i) and (ii) of Corollary 8.1 with d given in (8.40). Using (8.34), we arrive at the next form of the matrix (8.5):

$$H = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -(c-1) & 0 & c(c-1)\alpha u_*^{c-1} \\ 0 & 0 & -(c-1) & -c(c-1)\beta(1-u_*)^{c-1} \\ -g_{2*} & 0 & 0 & 0 \end{pmatrix},$$

where $g_{*2} = g_2(w_*, z_*, u_*)$. For the matrix H the eigenvectors (s, w, z, u) with the eigenvalue -1 are the solutions to

$$-s = -s,$$

$$(1 - c)w + c(c-1)\alpha u_*^{c-1} u = -w,$$

$$(1-c)z - c(c-1)\beta(1-u_*)^{c-1}u = -z,$$
$$-g_{2*}s = -u,$$

or

$$(2-c)w = -c(c-1)\alpha u_*^{c-1}u,$$
$$(2-c)z = c(c-1)\beta(1-u_*)^{c-1}u,$$
$$g_{2*}s = u,$$

The first coordinate, $\bar{s}$, of the eigenvector $(\bar{s}, \bar{w}, \bar{z}, \bar{u})$,

$$\bar{s} = 1, \quad \bar{w} = -\frac{c(c-1)\alpha u_*^{c-1} g_{2*}}{2-c}, \quad \bar{z} = \frac{c(c-1)\beta(1-u_*)^{c-1} g_{2*}}{2-c}, \quad \bar{u} = g_{2*},$$

is nonzero. Condition (iii) of Corollary 8.1 is satisfied. By Corollary 8.1 the system of equations (8.30) – (8.32) has a solution $(w(\cdot), z(\cdot), u(\cdot))$ defined on $(0, \delta)$, where $\delta > 0$, and satisfying (8.35). □

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